FLOW-PASSAGE-GEOMETRY OPTIMIZATION INSIDE A MODEL COMPRESSOR ROTOR

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This is to certify that the

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ABSTRACT

FLOW-PASSAGE-GEOMETRY OPTIMIZATION
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The goal of this work is to determine optimal internal flow passages for compressor rotors. This work should be regarded as a first approach to the problem of designing optimal internal flow passages. A great many other phenomena such as shocks, boundary-layers, etc. need to be considered if a realistic compressor rotor is to be designed. Steady isentropic flow of a fluid, which obeys the ideal gas relations, is assumed throughout this work. The equations developed in this work may be applied to axial-flow, mix-flow, and radial-flow rotors.

The problem of describing the motion of the fluid continuum is formulated as a minimum problem of Variational Calculus, and the equation which results from this formulation is called "the fluid particle minimum principle". Basically, this minimum principle states that of all the possible motions the fluid will travel along the one family of pathlines (or streamlines) which causes the kinetic energy minus the potential and enthalpy energies of each fluid particle to be a minimum. A fluid particle is defined as an infinitesimal volume of fluid whose surface is impervious to the flow of matter.

The "optimal flow passage geometry" is defined as the geometry for which the entire flow region satisfies the fluid particle minimum principle, continuity equation, boundary conditions, and various "optimal" constraints. An optimal constraint is any side condition which is imposed on the problem in an effort to produce desirable or optimal results. Constraints are imposed in order to control the pressure and energy increase of the fluid inside a rotor and they often simplify the problem.

The flow problem is treated as a boundary value problem. One of the boundary conditions is that the pathlines of the flow region must coincide with the walls of the passage. When the flow is rotational, the following procedure is employed to determine the optimal flow passage geometry. A family of pathlines is determined which satisfy all the equations and boundary conditions except the above mentioned boundary condition. The passage geometry is then selected to coincide with any set of pathlines, belonging to the given family of pathlines, and the boundary value problem is thus completely determined. When the flow is irrotational, the boundary value problem is determined by the velocity potential function. In general, a rotational flow, inside a rotor, can satisfy the fluid particle minimum principle at only one set of operating conditions. Whereas, an irrotational flow will satisfy the fluid particle minimum principle over a wide range of operating conditions.

In order to demonstrate the optimization procedure, a "maximum kinetic energy increase" centrifugal rotor is investigated. Application of the optimization procedure to axial-flow rotors is also discussed.

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NOMENCLATURE

S	entropy
p	pressure
T	temperature
ρ	density
$h = C_p T$	enthalpy of ideal gas
$H \equiv \frac{1}{2} V^2 + h$	total enthalpy
C_p	specific heat at constant pressure
C_V	specific heat at constant volume
$K = C_p / C_V$	ratio of specific heats
R_c	ideal gas constant
H_R	total relative enthalpy
$\nabla x(\ )$	curl operator
$\nabla(\ )$	gradient operator
$\nabla \cdot (\ )$	divergence operator
$\frac{dy}{dt} \equiv \dot{y}$	time derivative
$\frac{\partial F}{\partial y} \equiv F_y$	partial derivative
$\delta(\ )$	variational derivative operator
$\Delta(H) = H_2 - H_1$	difference operator
$\mathcal{V}$	volume
A	cross sectional area
s	arc length
$\vec{R}$	position vector

$\vec{R}_o$	position vector at $t = 0$
t	time
$\frac{d\vec{R}}{dt} = \dot{\vec{R}} = \vec{V}$	velocity
$\vec{R}_1$	position vector drawn from the relative reference frame
$\frac{d\vec{R}_1}{dt} = \dot{\vec{R}}_1 = \vec{W}$	relative velocity
$\vec{f}$	body forces per unit mass
$\vec{f}_N$	non-conservative body forces
$\vec{f}_c$	conservative body forces
G	force potential function $(\nabla G = -\vec{f})$
g	velocity potential function $(\nabla g = \vec{V})$
$\int F dt$	functional
F	integrand of functional
$E_1(a,\vec{y})$, $E_2(b,\vec{y})$	functions of the coordinates of the end points of the pathline
$\vec{y} = y_1, y_2, y_3$	space variables
$\dot{\vec{y}} = \dot{y}_1, \dot{y}_2, \dot{y}_3$	time derivative of space variables
u_1, u_2, u_3	orthogonal curvilinear coordinates
x, y, z	Cartesian coordinates
r, α, z	cylindrical coordinates
$\hat{\imath}, \hat{\jmath}, \hat{k}$	Cartesian unit vectors
$\hat{\imath}_r, \hat{\imath}_\alpha, \hat{\imath}_z$	cylindrical unit vectors
$\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1$	Cartesian unit vectors of rotating reference frame
$\hat{\imath}_r, \hat{\imath}_\theta, \hat{\imath}_z$	cylindrical unit vectors of rotating reference frame

ω	angular speed of rotor (ω is constant)
$\dot{\alpha} = \dot{\theta} + \omega$	absolute angular speed of a fluid particle
$\dot{\theta}$	angular speed of a fluid particle with respect to rotor
$a_* = \sqrt{C_p(K-1)T_*}$	speed of sound of an ideal gas
$M_* = V/a_*$	Mach number
$U = mu$	internal energy
w	work
$K.E. \equiv \frac{1}{2} mV^2$	kinetic energy
m	mass

Subscripts

$*$	denotes sonic conditions
o	denotes stagnation conditions
i	denotes intake point
d	denotes discharge point
$K \equiv C_p/C_v$	ratio of specific heats

INTRODUCTION

The Lagrangian method of describing the motion of a continuum is employed in this work. The Eulerian description of motion is the traditional method used to describe the motion of a fluid continuum. However, the Lagrangian description was chosen because the methods of Variational Calculus are considerably less complicated for functions of one independent variable. The problem of describing the motion of the fluid continuum is formulated as a minimum problem of Variational Calculus, and the equation which results from this formulation is called "the fluid particle minimum principle". Basically, the fluid particle minimum principle states that of all the possible motions the fluid will travel along the one family of pathlines which causes the kinetic energy minus the potential and enthalpy energies of each fluid particle to be a minimum. The energy equation is of primary importance in the present formulation, while the momentum equation is always satisfied. The energy equation is the first integral of the momentum equation for the case of isentropic flow. This approach differs from the traditional Eulerian method, wherein the momentum equation is the equation primarily operated upon.

Steady isentropic flow of a fluid, which obeys the ideal gas relations, is assumed throughout this work. The path of a fluid particle is called a pathline. For the case of steady flow pathlines and streamlines coincide. In section-1 the Lagrangian description of motion of a fluid continuum is explained and a fluid particle is defined. In section-2 the Lagrangian form of the momentum and continuity equations are listed. A special form of the Lagrangian continuity equation is derived for the case of steady flow. Some of the results of Variational Calculus, that are employed in later sections, are listed in section-3. In section-4, the problem of describing the motion of a fluid continuum is formulated as a minimum problem of Variational Calculus. The fluid particle minimum principle is then developed. In section-5 some classical fluid problems are investigated using the previously developed theory. All (irrotational) potential flow problems are shown to satisfy the fluid particle minimum principle. The conditions under which (one-dimensional) isentropic compressible flow satisfies the fluid particle minimum principle are also investigated. In section-6 the fluid particle minimum principle of section-4 is adapted to the flow inside a rotating reference frame. The energy equation is then derived. In section-7 it is shown that there exists only one trivial case for which the flow inside a rotating passage is irrotational. Thus this case is excluded from the following investigations. In section-8 the continuity equation and energy equation are combined and the boundary value problem is described for the case of rotational flow. In section-9 the optimal constraints are selected. The system of equations and boundary conditions, which are employed to determine the optimal compressor passages, are summarized at the end of section-9. Section-10 contains a demonstration of how an optimal flow passage may be determined. A "maximum kinetic energy increase" radial blade centrifugal rotor is considered in this section. In section-11 flows which are irrotational in the relative reference frame are investigated. Section-12 contains some concluding remarks.

1. KINEMATICS

The Lagrangian method of describing the motion of a continuum will be employed in this thesis. In the Lagrangian description of motion, the path of each particle is described by the locus of points traced out by the end point of a position vector, $\vec{R}[\vec{R}_o, x(t), y(t), z(t)]$, with respect to a fixed (Newtonian) reference frame [8]. The reference position of each particle is given by the constant position vector, $\vec{R}_o$, which is the position of the particle at time, $t = 0$. The coordinates of the particle at $t = 0$ are known as the material coordinates of the particle.

A fluid particle is defined as a differential volume of fluid which may change shape, volume, and density but must always contain the same molecules of the fluid [10]. A fluid particle is an infinitesimal closed system since no mass may cross its boundary. When the Lagrangian description of motion is employed to describe the motion of a fluid continuum, the trajectory (or pathline) of each fluid particle is described by the locus of points traced out by the end point of the position vector, $\vec{R}[\vec{R}_o, x(t), y(t), z(t)]$. An infinite number of position vectors is needed to describe the motion of the fluid continuum. For the present time we assume that the pathlines traced out by the position vectors do not intersect in the flow region under consideration. For the case of steady flow, pathlines and streamlines coincide [10]. Only steady flow is considered in this work.

2. LAGRANGIAN FORM OF THE MOMENTUM AND CONTINUITY EQUATIONS

The Lagrangian form of the momentum equation, for a non-viscous fluid, that will be employed in this work is

$$\frac{d^2\vec{R}}{dt^2} + \frac{\nabla p}{\rho} - \vec{f} = 0 \ , \tag{2.1}$$

where ρ is the density of the fluid, p is the pressure, $\vec{f}$ represents the body forces per unit mass acting on the fluid particle, and $\vec{R}$ is the position vector of the fluid particle [10].

The Lagrangian form of the continuity equation is often written in the below form

$$\rho d\mathcal{V} = \rho_1 d\mathcal{V}_1 = \text{constant}. \tag{2.2}$$

A second form is

$$\nabla \cdot \frac{d\vec{R}}{dt} = \frac{1}{d\mathcal{V}} \frac{d(d\mathcal{V})}{dt} = -\frac{1}{\rho}\frac{d\rho}{dt} \ , \tag{2.3}$$

where t denotes time, ρ is density, $\mathcal{V}$ is the volume of the fluid, and $\vec{R}$ is the position vector of the fluid particle [10].

We now seek a more convenient form of the continuity equation for the case of steady flow. We select the material coordinates to be a set of orthogonal curvilinear coordinates, (u_1,u_2,u_3), and the u_1 curve is selected to coincide with the pathlines of the fluid particles. That is, at time $t = 0$ the fluid continuum is described by the curves:

$$u_1(x,y,z) = C_1 = \text{constant along a pathline,} \tag{2.4a}$$

$$u_2(x,y,z) = C_2 = \text{constant,} \tag{2.4b}$$

$$u_3(x,y,z) = C_3 = \text{constant.} \tag{2.4c}$$

For the case of steady flow, the path of the fluid particles will remain coincident with the u_1 curves. A volume element, $d\mathcal{V}$, about any point, P_B, for a moving orthogonal curvilinear coordinate system (u_1,u_2,u_3) is defined as [12]

$$d\mathcal{V} = \left|\frac{\partial\vec{B}}{\partial u_1} du_1 \cdot \left(\frac{\partial\vec{B}}{\partial u_2} du_2 \times \frac{\partial\vec{B}}{\partial u_3} du_3\right)\right| , \tag{2.5}$$

where $\vec{B}(u_1,u_2,u_3)$ is a position vector drawn from the origin to the fluid particle at P_B, see figure 2.1.

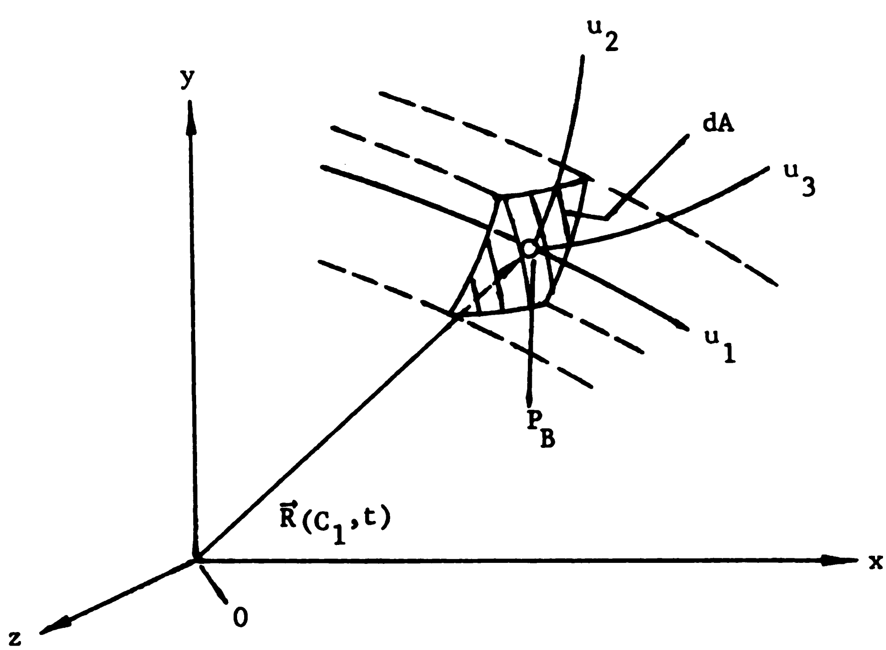

Figure 2.1

Curvilinear Curves

The cross product in (2.5) may be interpreted as the change in cross sectional area, dA, normal to the u_1 curve. And the term $\frac{\partial \vec{B}}{\partial u_1} du_1$ is the change in arc length, ds_1, along the u_1 curve. Thus (2.5) may be rewritten as

$$dV = ds_1 dA \;, \tag{2.6a}$$

where
$$dA \equiv \left| \frac{\partial \vec{B}}{\partial u_2} du_2 \times \frac{\partial \vec{B}}{\partial u_3} du_3 \right| \equiv ds_2 ds_3 \;, \tag{2.6b}$$

and dividing (2.6a) by dt yields

$$\frac{dV}{dt} = \frac{ds_1}{dt} dA \;. \tag{2.7}$$

Since the ratio of the change in arc length to the change in time is a measure of the speed of a fluid particle along the u_1 curve, equation (2.7) can be rewritten as

$$\frac{dV}{dt} = \frac{d\vec{R}}{dt} \cdot d\vec{A} = \vec{V} \cdot d\vec{A} \;. \tag{2.8}$$

Dividing equation (2.2) by dt and then substituting (2.8) into the resulting equation yields

$$\rho\vec{V} \cdot d\vec{A} = \rho_1 \vec{V}_1 \cdot d\vec{A}_1 \;, \tag{2.9a}$$

or

$$\frac{\rho}{\rho_1} = \frac{\vec{V}_1 \cdot d\vec{A}_1}{\vec{V} \cdot d\vec{A}} = \frac{V_1 dA_1}{VdA} \;. \tag{2.9b}$$

Equation (2.9) is the Lagrangian form of the continuity equation that is employed in this work. It is only valid for the flow of a fluid particle along a time independent pathline.

3. VARIATIONAL CALCULUS

We shall be concerned with the following problem from The Calculus of Variations. Consider the variable end point problem for the functional

$$\int_{t=a}^{t=b} F(\vec{y},\dot{\vec{y}})dt + E_1(a,\vec{y}) + E_2(b,\vec{y}) \;, \qquad (3.1)$$

where E_1 and E_2 are functions of the coordinates of the end points of the path along which the functional is considered, $\vec{y} = y_1,y_2,y_3$ represents the space variables, and $\dot{\vec{y}} = \dot{y}_1,\dot{y}_2,\dot{y}_3$ represents the components of velocity. Calculating the variation of the functional, (3.1), and setting the result equal to zero we obtain the well known Euler-Lagrange equations

$$F_{y_i} - \frac{d}{dt} F_{\dot{y}_i} = 0 \quad (i = 1,2,3) \;, \qquad (3.2)$$

and the boundary conditions;

$$(F_{\dot{y}_i} - E_{1_{y_i}})\big|_{t=a} = 0 \quad (i = 1,2,3) \;, \qquad (3.3a)$$

$$(F_{\dot{y}_i} - E_{2_{y_i}})\big|_{t=b} = 0 \quad (i = 1,2,3) \;, \qquad (3.3b)$$

where the subscripts y_i and $\dot{y}_i$ denote partial differentiation, i.e., $\frac{\partial F}{\partial y_i} \equiv F_{y_i}$ [2]. The solution to the Euler-Lagrange equations, (3.2), is called an extremum or an extremal curve. The family of curves satisfying, (3.2), is called a family of extremal curves.

We shall be concerned with minimizing the functional, (3.1). Consider the functional,

$$\int_a^b F(\vec{y},\dot{\vec{y}})dt \quad , \tag{3.4}$$

where the end points of each extremal curve is specified. Equation (3.4) will be called the fixed end point functional. Calculating the variation of (3.4) and setting the result equal to zero yields the Euler-Lagrange equations, (3.2). Thus the problem of minimizing the variable end point functional, (3.1), is equivalent to minimizing the fixed end point functional, (3.4), subject to the boundary conditions (or side conditions), (3.3). From Variational Calculus it is known that (3.4) is a minimum when the following conditions are satisfied, see Gelfand [2], pg. 146-148.

I. The Euler-Lagrange equations, (3.2), are satisfied.

II. The matrix $\|F_{\dot{y}_i\dot{y}_j}\|$ is positive definite along the extremal.

III. The interval $[a,b]$ contains no conjugate points. A conjugate point is a point of intersection of the neighboring extremals.

IV. The value of the functional, (3.4), is independent of the path of integration. Or, more precisely, the Weierstrass E-function is ≥ 0 along the extremal curve.

And for the functional, (3.1), we also impose the following additional condition.

V. The boundary conditions, (3.3), must be satisfied.

When conditions I-V are satisfied, the minimum is said to be a "strong minimum". When all the conditions except IV are satisfied, the minimum is said to be a "weak minimum". For the case of a weak minimum, the family of extermals always possess conjugate points and the functional is a minimum only in local regions which are free of conjugate points.

The family of extremal curves is obtained by integrating the Euler-Lagrange equations, (3.2). The Euler-Lagrange equations, (3.2), may be integrated in the below manner. Multiplying (3.2) by $\dot{y}_i$ and then adding and subtracting the term, $F_{\dot{y}_i}\ddot{y}_i$, yields

$$F_{y_i}\dot{y}_i + F_{\dot{y}_i}\ddot{y}_i - F_{\dot{y}_i}\ddot{y}_i - \dot{y}_i\frac{d}{dt}F_{\dot{y}_i} = 0, \quad (i = 1,2,3). \tag{3.5}$$

Adding the above equations yields

$$\sum_{i=1}^{3}\{(F_{y_i}\dot{y}_i + F_{\dot{y}_i}\ddot{y}_i) - (F_{\dot{y}_i}\ddot{y}_i + \dot{y}_i\frac{d}{dt}F_{\dot{y}_i})\} = 0 . \tag{3.6}$$

Employing the chain-rule of Calculus, we observe that

$$\frac{d}{dt}F(y_1,y_2,y_3,\dot{y}_1,\dot{y}_2,\dot{y}_3) = \sum_{i=1}^{3}[F_{y_i}\dot{y}_i + F_{\dot{y}_i}\ddot{y}_i] . \tag{3.7}$$

Substituting (3.7) into (3.6) yeilds

$$\frac{dF}{dt} - \sum_{i=1}^{3}(F_{\dot{y}_i}\ddot{y}_i + \dot{y}_i\frac{d}{dt}F_{\dot{y}_i}) = 0 , \tag{3.8a}$$

or

$$\frac{d}{dt}[F - \sum_{i=1}^{3}\dot{y}_i F_{\dot{y}_i}] = 0 . \tag{3.8b}$$

Integrating (3.8b) yields

$$F - \sum_{i=1}^{3}\dot{y}_i F_{\dot{y}_i} = -H = \text{constant along each extremal curve} . \tag{3.9}$$

Equation (3.9) is called the first integral of the Euler-Lagrange equations, (3.2). Since the first integral of a system of differential equations is a function which has a constant value along each integral curve of the system, we see that the function, H, is a constant along each integral curve determined by (3.2).

4. FLUID PARTICLE MINIMUM PRINCIPLE

In this section, the problem of describing the motion of a fluid continuum will be formulated as a minimum problem of The Calculus of Variation. Consider one fluid particle moving along one pathline during time $t = a$ to $t = b$. The motion of the fluid particle is described by the locus of points traced out by the end of the position vector, $\vec{R}\left[\vec{R}_o, y_1(t), y_2(t), y_3(t)\right]$. As explained in section-1, the motion of the fluid continuum is described by an infinite number of position vectors which trace out a family of pathlines. The motion of each fluid particle must obey the Lagrangian form of the momentum equation, (2.1). To derive the fluid particle minimum principle, the dot product of the momentum equation, (2.1), times the variation of the position vector, $\delta\vec{R}$, is taken and the result is integrated with respect to time from $t = a$ to $t = b$, which yields

$$\int_{t=a}^{t=b} [\ddot{\vec{R}} \cdot \delta\vec{R} + \frac{\nabla p}{\rho} \cdot \delta\vec{R} - \vec{f} \cdot \delta\vec{R}]dt = 0 , \tag{4.1}$$

where $\ddot{\vec{R}} \equiv \frac{d^2\vec{R}}{dt^2}$. The first term of (4.1) may be integrated by parts in the below manner

$$\int_a^b \ddot{\vec{R}} \cdot \delta\vec{R}\, dt = \dot{\vec{R}} \cdot \delta\vec{R}\Big]_a^b - \int_a^b \dot{\vec{R}} \cdot \delta\dot{\vec{R}}\, dt \tag{4.2}$$

$$= \dot{\vec{R}} \cdot \delta\vec{R}\Big]_a^b - \int_a^b \frac{1}{2}\, \delta[(\dot{\vec{R}})^2]dt .$$

When the end points of the pathline are specified, the boundary conditions are

$$\delta\vec{R}]_a = 0, \text{ and } \delta\vec{R}]_b = 0 \ ; \tag{4.3}$$

and (4.2) becomes

$$\int_a^b \ddot{\vec{R}} \cdot \delta\vec{R}\, dt = -\int_a^b \frac{1}{2}\, \delta[(\dot{\vec{R}})^2]dt \ . \tag{4.4}$$

The second term of (4.1) may be expanded in the below manner

$$\frac{1}{\rho}\nabla p \cdot \delta\vec{R} = \frac{1}{\rho}\sum_{i=1}^{3} \frac{\partial p}{\partial y_i}\,\delta y_i = \frac{\delta p}{\rho} \ . \tag{4.5}$$

From Thermodynamics it is known that for the isentropic flow of fluid obeying the perfect gas laws that

$$TdS = 0 = C_p dT - \frac{dp}{\rho} \ , \tag{4.6a}$$

or

$$C_p dT = \frac{dp}{\rho} \ , \tag{4.6b}$$

where T is temperature, S is entropy, C_p is the specific heat of the fluid at constant pressure, p is pressure, and ρ is density [10]. Replacing the total derivatives in (4.6b) by variational derivatives and equating the resulting equation with (4.5) yields

$$\frac{\nabla p}{\rho} \cdot \delta\vec{R} = \frac{\delta p}{\rho} = C_p\ \delta T \ . \tag{4.7}$$

When the body forces, $\vec{f}$, acting on the fluid particle are conservative, the third term of (4.1) may be replaced by a force potential, i.e. $\nabla G = -\vec{f}_c$, and

$$\delta G = -\vec{f}_c \cdot \delta\vec{R} \quad (\text{when } \vec{f} \text{ is conservative, } \nabla \times \vec{f} = 0). \tag{4.8}$$

Substituting (4.8), (4.7), and (4.4) into (4.1) yields

$$\int_a^b \left[-\frac{1}{2}\,\delta[(\dot{\vec{R}})^2] + C_p \delta T + \delta G \right] dt = 0 \ . \tag{4.9}$$

Factoring out the variational derivative operator, δ, and multiplying by a minus one yields

$$\delta \int_a^b \left[\frac{1}{2}(\dot{\vec{R}})^2 - C_p T - G\right] dt = 0 \ . \tag{4.10}$$

In words, equation (4.10) states that the isentropic flow of an ideal fluid particle moves between two specified points in a conservative force field in such a way that the functional, (4.10), is a minimum. A result which is equivalent to (4.10), but written in a more general form, was published by Nantanson in a series of papers from 1896 to 1902 [6]. In most fluid flow problems, the end points of the pathlines are unknown. We therefore consider the variable end point functional

$$\int_a^b \left[\frac{1}{2}(\dot{\vec{R}})^2 - C_p T(\vec{R}) - G(\vec{R})\right] dt + E_1(a,\vec{R}) + E_2(b,\vec{R}) \ , \tag{4.11}$$

where E_1 and E_2 are known functions of the coordinates of the end points of the pathline along which the functional is considered. Calculating the variation of (4.11), and remembering that boundary condition (4.3) no longer applies, yields

$$\delta \int_a^b \left[\frac{1}{2}(\dot{\vec{R}})^2 - C_p T(\vec{R}) - G(\vec{R})\right] dt +$$

$$[(\nabla E_1 - \dot{\vec{R}}) \cdot \delta\vec{R}]_a + [(\nabla E_2 - \dot{\vec{R}}) \cdot \delta\vec{R}]_b = 0 \ , \tag{4.12}$$

where $T(\vec{R})$ and $G(\vec{R})$ are functions of the space variables, i.e.

$T(\vec{R}) \equiv T(y_1,y_2,y_3)$, also $(\dot{\vec{R}})^2 = \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2$. In order to describe the motion of a fluid continuum, the functional must be solved along each pathline of the flow region. However, the functions ∇E_1 and ∇E_2 will be chosen so that they specify the velocity, $\dot{\vec{R}}$, along every pathline at the cross sections, 1 and 2. Then equation (4.12) will apply to every pathline in the flow region. And the solution of (4.12) will be a family of pathlines which describe the motion of the fluid continuum. Equation (4.12) will be called the "fluid particle minimum principle". In words, (4.12) states that the isentropic flow of a fluid, obeying the ideal gas laws in a conservative force field, will travel along the one family of pathlines which causes the variable end point functional, (4.12), to be a minimum.

The functional, (4.12), is a minimum when the five conditions, (I-V), of section-3 are satisfied. We will now discuss these conditions for the special case of (4.11). The Euler-Lagrange equation of (4.11) is identical to the momentum equation, (2.1). This statement is easily verified by substituting the integrand, F, of (4.11) into (3.2) which yields

$$\ddot{y}_i + C_p \frac{\partial T}{\partial y_i} + \frac{\partial G}{\partial y_i} = 0 \qquad (i = 1,2,3) \ , \tag{4.13}$$

or in vector form

$$\ddot{\vec{R}} + C_p \nabla T + \nabla G = 0 \ , \tag{4.14}$$

and substituting $(-\vec{f} \equiv \nabla G)$ and $(\frac{\nabla p}{\rho} = C_p \nabla T)$ into (4.14) yields

$$\ddot{\vec{R}} + \frac{\nabla p}{\rho} - \vec{f} = 0 \ , \tag{4.15}$$

which is identical to (2.1). The first integral of the Euler-Lagrange equation of (4.11) is obtained by substituting the integrand, F, of (4.11) into (3.9) which yields

$$\frac{1}{2}(\dot{\vec{R}})^2 - C_pT - G - \sum_{i=1}^{3} \dot{y}_i\dot{y}_i = -H \; ; \tag{4.17a}$$

and since $(\dot{\vec{R}})^2 = \Sigma(\dot{y}_i)^2$, (4.17a) becomes

$$\frac{1}{2}(\dot{\vec{R}})^2 + C_pT + G = H = \text{constant along each pathline.} \tag{4.17b}$$

Equation (4.17b) is equivalent to Bernoulli's equation. We shall call (4.17b) the energy equation, and it is shown in appendix-A that the First Law of Thermodynamics agrees with (4.17b). Since the energy equation, (4.17b), is derived by integrating the Euler-Lagrange (or momentum) equation, we conclude that condition I of section-3 is satisfied when (4.17b) is satisfied.

Condition II of section-3 is always satisfied for the functional (4.11) since

$$\|F_{\dot{y}_i\dot{y}_j}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{4.18}$$

is always positive. Let us now consider condition IV. In appendix-B it is shown that condition IV is satisfied when the flow is irrotational. When the flow is irrotational there exists a velocity potential function, g, such that

$$\dot{y}_i = \frac{\partial g}{\partial y_i} \qquad (i = 1,2,3) \; ; \tag{4.19a}$$

or in vector form

$$\dot{\vec{R}} = \nabla g \; . \tag{4.19b}$$

It is also shown in appendix-B that the integrand, F, reduces to

$$F = \frac{dg}{dt} \tag{4.20}$$

for the case of irrotational flow. Let $g = E = E_1 = E_2$ and substituting (4.20) into (4.11) yields

$$\int_a^b \frac{dg}{dt}\, dt + g(a) + g(b). \tag{4.21}$$

From (4.21) we conclude that the known functions, E_1 and E_2, and the functional, (4.11), are completely determined by the potential function, g, for the case of irrotational flow. Since the value of (4.21) does not depend on the path of integration, condition IV is satisfied. We now list the conditions for which the fluid particle minimum principle, (4.12), is satisfied. We consider two cases, rotational and irrotational flow.

A. Rotational flow (weak minimum)

4.A

4.A.1 The energy equation, (4.17b), is satisfied.

4.A.2 The boundary conditions, (3.3), or;

$$(\nabla E_1 - \dot{\vec{R}})|_a = 0 , \tag{4.22a}$$

$$(\nabla E_2 - \dot{\vec{R}})|_b = 0 \tag{4.22b}$$

are satisfied.

4.A.3 The pathlines do not intersect in the flow region, i.e., there are no conjugate points.

The following procedure from Gelfand [2] pg. 130 may be employed to test for conjugate points. Let $y = y(t,\alpha,\beta)$ be a general solution of (3.2) depending on two parameters, α and β. When the ratio

$$\frac{\partial y/\partial \alpha}{\partial y/\partial \beta}\, , \tag{4.22c}$$

is the same at two points, the points are conjugate.

B. Irrotational flow (strong minimum)

4.B
- 4.B.1 The energy equation, (4.17b), is satisfied.
- 4.B.2 The potential function, g, satisfies the boundary conditions;

$$(\nabla g - \dot{\vec{R}})\big|_a = 0 \ , \tag{4.23a}$$

$$(\nabla g - \dot{\vec{R}})\big|_b = 0 \ . \tag{4.23b}$$

- 4.B.3 The pathlines do not intersect in the flow region.

The fluid particle minimum principle, (4.12), is said to be satisfied when conditions (4.A) or (4.B) are satisfied. Whenever the fluid particle minimum principle is satisfied both the momentum equation, (4.15), and the energy equation, (4.17b), are satisfied. In fact, they are equivalent equations for the case of isentropic flow. In addition to the fluid particle minimum conditions, (4.A) or (4.B), the flow must also satisfy the continuity equation and the condition that the pathlines of the flow region coincide with the walls of the passage.

The fluid particle minimum principle may be extended to include forces, f_N, which are not derivable from a potential force function, G. Letting $f = f_C + f_N$ in equation (4.1) yields

$$\int_a^b [\ddot{\vec{R}} \cdot \delta\vec{R} + \frac{\nabla p}{\rho} \cdot \delta\vec{R} - \vec{f}_c \cdot \delta\vec{R}]dt - \int_a^b \vec{f}_N \cdot \delta\vec{R}\, dt = 0 \ , \tag{4.24}$$

where $\vec{f}_c$ represents the conservative forces. Repeating the previous steps of this section up to equation (4.10) yields

$$\delta\int_a^b [\tfrac{1}{2}(\dot{\vec{R}})^2 - C_pT - G]dt - \int_a^b \vec{f}_N \cdot \delta\vec{R}\, dt = 0 \ . \tag{4.25}$$

Calculating the variation of the first term of (4.25), (4.25) becomes

$$\int_a^b [\ddot{\vec{R}} + C_p\nabla T + \nabla G] \cdot \delta\vec{R}\, dt - \int_a^b \vec{f}_N \cdot \delta\vec{R}\, dt = 0 . \qquad (4.26)$$

Since $\delta\vec{R}$ is arbitrary, (4.26) reduces to

$$\ddot{\vec{R}} + C_p\nabla T + \nabla G - \vec{f}_N = 0 ; \qquad (4.27)$$

which is the momentum equation, see equation (2.1) (with $C_p\nabla T = \nabla p/\rho$ and $\nabla G - \vec{f}_N = \vec{f}$). The variable end point form of (4.25) is

$$\delta\int_a^b [\tfrac{1}{2}(\dot{\vec{R}})^2 - C_pT - G]dt - \int_a^b \vec{f}_N \cdot \delta\vec{R}\, dt +$$

$$[(\nabla E_1 - \dot{\vec{R}}) \cdot \delta\vec{R}]_a + [(\nabla E_2 - \dot{\vec{R}}) \cdot \delta\vec{R}]_b = 0 , \qquad (4.28)$$

which is similar to (4.12). The Euler-Lagrange equation of (4.28) is again the momentum equation, (2.1). The boundary conditions of (4.28) are the same as the boundary conditions for (4.12), i.e., the boundary conditions are given by (4.22). It should be pointed out that (4.28) is, in general, difficult to employ because the second integral in (4.28) cannot be evaluated, in practice, without additional information. Fortunately, for the case, which we shall consider, $\vec{f}_N$ is normal to $\delta\vec{R}$ and thus the second integral in (4.28) reduces to zero.

5. SOLUTION OF SOME CLASSICAL EXAMPLES

In this section, some classical example problems are investigated using the previously developed equations. One of the purposes of this section is to demonstrate that the Lagrangian description of motion may be employed to solve fluid problems, which are traditionally solved by the Eulerian method. Two examples will be considered, incompressible potential flow and isentropic compressible flow.

A. Incompressible Potential Flow

Consider the isentropic flow of a fluid in a region where the body forces, $\nabla G = -\vec{f}$, may be neglected. We assume that the flow is irrotational and that the velocity potential function, g, is known. The velocity, $\vec{V}$, of the fluid is then determined from the gradient of the potential function, i.e.,

$$\vec{V} \equiv \dot{\vec{R}} = \nabla g \ . \tag{5.1}$$

Substituting (5.1) into the energy equation (4.17b), with $G \equiv 0$, yields

$$\frac{(\nabla g)^2}{2} + C_p T = H = \text{constant along each pathline.} \tag{5.2}$$

For the case of isentropic flow,

$$dh = \frac{dp}{\rho} = C_p dT \ ; \tag{5.3}$$

and since the density ρ is constant, integration of (5.3) yields

$$C_p T = \frac{p}{\rho} + \text{constant} \ . \tag{5.4}$$

Substituting (5.4) into (5.2) yields the incompressible form of the energy equation,

$$\frac{(\nabla g)^2}{2} + \frac{p}{\rho} = \text{constant along each pathline} \ . \tag{5.5}$$

Since the flow is incompressible, from (2.2) we observe that the change in volume is constant, i.e.,

$$d\mathcal{V} = \text{constant} \ . \tag{5.6}$$

Substituting (5.6) into (2.3) yields

$$\nabla \cdot \vec{V} = 0 \ . \tag{5.7}$$

Substituting (5.1) into (5.7) yields

$$\nabla \cdot (\nabla g) = \nabla^2 g = 0 \ , \tag{5.8}$$

which is the well known Laplace equation.

Once a potential function, g, is selected which satisfies the boundary conditions and Laplace's equation, the family of pathlines is uniquely determined by the potential function, g. The pressure distribution is then determined by the energy equation, (5.5). Of course, this result is exactly the same as the results of "Classical Incompressible Potential Flow Theory" [4]. The only difference is that the classical results are derived from the Eulerian point of view, whereas, the present derivation is from the Lagrangian point of view. Since for the case of steady flow pathlines and streamlines coincide, the same equations result regardless of our point of view.

Let us now consider the fluid particle minimum principle. Condition (4.B.1) is satisfied by (5.5). Condition (4.B.2) is satisfied by (5.1) for all values of time $t = a$ and $t = b$. Condition (4.B.3) is satisfied in regions which do not contain conjugate points. Thus we conclude that all incompressible potential flow problems satisfy the fluid particle minimum principle in flow regions which exclude conjugate points.

For the case of flow around a two-dimensional airfoil (with circulation), conjugate points occur at the stagnation point and trailing edge of the air-foil as shown in figure 5.1. Thus, these two points are excluded from the flow region. The fluid particle minimum principle does not predict the nature of the flow in the neighborhood of these two points. The stagnation streamline divides the flow into two regions and the flow in these two regions, excluding the stagnation streamline, satisfy the fluid particle minimum principle at all points.

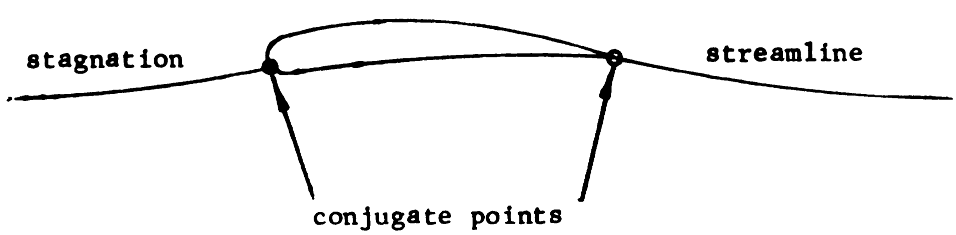

Figure 5.1

Conjugate Points

B. Isentropic Compressible Flow

Consider the isentropic flow of a fluid obeying the ideal gas relations. Assume that the flow is in a region where the body forces, $\nabla G = -\vec{f}$, may be neglected. The energy equation as given by (4.17b), with $G \equiv 0$, is

$$\frac{1}{2} V^2 + C_p T = \text{constant along each pathline}, \tag{5.9}$$

where $\vec{V} \equiv \dot{\vec{R}}$. The constant in (5.9) may be evaluated at the stagnation condition (denoted by the subscript o), i.e.,

$$\frac{1}{2} V^2 + C_p T = C_p T_o \ . \tag{5.10}$$

Or the constant can be evaluated at sonic conditions (denoted by the subscript *), i.e.,

$$\frac{1}{2} V^2 + C_p T = C_p T_* + \frac{1}{2} V_*^2 \ . \tag{5.11}$$

The definitions of the Mach number, M, and the speed of sound in an ideal gas, a, are

$$M = V/a \ , \tag{5.12}$$

$$M_* = V/V_* \equiv V/a_* \ , \tag{5.13}$$

$$a^2 = K R_c T = C_p (K-1) T \ , \tag{5.14}$$

where R_c is the ideal gas constant and $K \equiv C_p/C_V$ is the ratio of specific heats. From Thermodynamics we recall the isentropic relations

$$\frac{T}{T_1} = \left[\frac{p}{p_1}\right]^{\frac{K-1}{K}} = \left[\frac{\rho}{\rho_1}\right]^{K-1} . \tag{5.15}$$

Dividing each term of (5.10) by (5.14) yields

$$\frac{1}{2}\frac{V^2}{a^2} + \frac{1}{K-1} = \frac{1}{K-1}\frac{T_o}{T} . \tag{5.16}$$

Substituting (5.12) into (5.16) and solving for T_o/T yields

$$\frac{T_o}{T} = 1 + \frac{K-1}{2} M^2 . \tag{5.17}$$

Dividing each term of (5.11) by $\left[(5.14) \text{ with } T = T_*\right]$ yields

$$\frac{1}{2}\frac{V^2}{a_*^2} + \frac{1}{K-1}\frac{T}{T_*} = \frac{1}{K-1} + \frac{1}{2}\frac{V_*^2}{a_*^2} . \tag{5.18}$$

Substituting $V_* \equiv a_*$ and $M_* \equiv V/a_*$ into (5.18) and solving for M_*^2 yields

$$M_*^2 = \frac{-2}{K-1}\left[\frac{T}{T_*} - \frac{K+1}{2}\right] . \tag{5.19}$$

Substituting (5.15) into (5.17) and solving for $\frac{p_o}{p}$ and $\frac{\rho_o}{\rho}$ yields

$$\frac{p_o}{p} = (1 + \frac{K-1}{2} M^2)^{\frac{K}{K-1}} , \text{ and} \tag{5.20}$$

$$\frac{\rho_o}{\rho} = (1 + \frac{K-1}{2} M^2)^{\frac{1}{K-1}} . \tag{5.21}$$

At sonic conditions, $M = 1$ and (5.17), (5.20), and (5.21) reduce to:

$$\frac{T_o}{T_*} = \frac{K+1}{2} , \tag{5.22}$$

$$\frac{p_o}{p_*} = (\frac{K+1}{2})^{\frac{K}{K-1}} , \tag{5.23}$$

$$\frac{\rho_o}{\rho_*} = (\frac{K+1}{2})^{\frac{1}{K-1}} . \tag{5.24}$$

All of the above equations, (5.10-5.24), agree with the classical one-dimensional isentropic flow relations [11]. This agreement is expected since all the equations and definitions used so far are exactly the same. The only difference is that the above equations give the changes in fluid properties along a pathline, whereas in the classical one-dimensional method the above equations give the changes in properties along a "one-dimensional" streamline. However, we now introduce the Lagrangian continuity equation, (2.9), which differs from the continuity equation employed in the classical method. The Lagrangian continuity equation, (2.9a), is

$$\rho\ V\ dA = \rho_* V_*\ dA_* \ . \tag{5.25}$$

The Eulerian continuity equation employed in classical one-dimensional gas dynamics is

$$\rho\ V\ A = \rho_* V_* A_* = \text{constant} \ . \tag{5.26}$$

We will now use the Lagrangian form of the continuity equation, (5.25), to obtain dA in terms of the Mach number, M. This result will then be compared to the similar equation obtained when the Eulerian continuity equation, (5.26), is employed. Substituting (5.25) into (5.15) yields

$$\frac{T}{T_*} = \left[\frac{\rho}{\rho_*}\right]^{K-1} = \left[\frac{V_* dA_*}{V dA}\right]^{K-1} = \left[M_* \frac{dA}{dA_*}\right]^{1-K} , \tag{5.27a}$$

or

$$\frac{dA}{dA_*} = M_*^{-1}\left[\frac{T}{T_*}\right]^{\frac{1}{1-K}} . \tag{5.27b}$$

Substituting (5.19) into (5.27b) yields

$$\frac{dA}{dA_*} = \left[\frac{-2}{K-1}\left[\frac{T}{T_*} - \frac{K+1}{2}\right]\right]^{-\frac{1}{2}}\left[\frac{T}{T_*}\right]^{\frac{1}{1-K}} . \tag{5.28}$$

Substituting (5.22) into (5.28) yields

$$\frac{dA}{dA_*} = \left\{\frac{-2}{K-1}\,\frac{T}{T_*}\left[1 - \frac{T_o}{T}\right]\right\}^{-\frac{1}{2}} \left[\frac{T}{T_*}\right]^{\frac{1}{1-K}} \quad . \tag{5.29}$$

Substituting (5.17) into (5.29) yields

$$\begin{aligned}\frac{dA}{dA_*} &= \left[\frac{-2}{K-1}\,\frac{T}{T_*}\left[1 - 1 - \frac{K-1}{2}M^2\right]\right]^{-\frac{1}{2}} \left[\frac{T}{T_*}\right]^{\frac{1}{1-K}} \\ &= \frac{1}{M}\left[\frac{T}{T_*}\right]^{\frac{-(K+1)}{2(K-1)}} \quad .\end{aligned} \tag{5.30}$$

Dividing (5.22) by (5.17) yields

$$\frac{T}{T_*} = \frac{T_o}{T_*}\,\frac{T}{T_o} = \left[\frac{2}{K+1}\left[1 + \frac{K-1}{2}M^2\right]\right]^{-1} ; \tag{5.31}$$

which when substituted into (5.30) yields

$$\frac{dA}{dA_*} = \frac{1}{M}\left[\frac{2}{K+1}\left[1 + \frac{K-1}{2}M^2\right]\right]^{\frac{K+1}{2(K-1)}} \quad . \tag{5.32}$$

We now compare the above equation to the similar relation, (5.33), from classical one-dimensional gas dynamics which is given below [11]

$$\frac{A}{A_*} = \frac{1}{M}\left[\frac{2}{K+1}\left[1 + \frac{K-1}{2}M^2\right]\right]^{\frac{K+1}{2(K-1)}} \quad . \tag{5.33}$$

Notice that the right hand side of (5.32) and (5.33) are identical. We will now show that for a hypothetical case of flow bounded by radial pathlines (or streamlines), the left hand side of (5.32) and (5.33) are equivalent. Consider a family of radial pathlines as shown in figure 5.2. Let r_* locate the cross section at which sonic conditions occur.

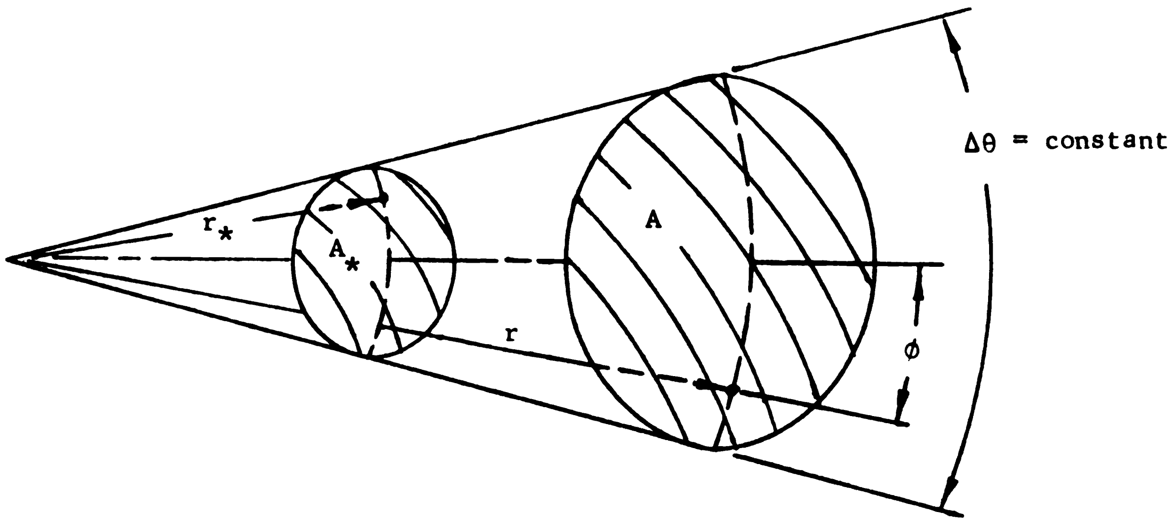

Figure 5.2

Radial Pathlines Enclosing
a Pathtube

The cross sectional area normal to the pathtube (or streamtube) is given by

$$A \simeq \pi(r\Delta\theta)^2 \ , \tag{5.34}$$

and the area ratio is

$$\frac{A}{A_*} = \frac{\pi(r\Delta\theta)^2}{\pi(r_*\Delta\theta)^2} = \frac{r^2}{r_*^2} \ . \tag{5.35}$$

The rate of change of the cross sectional area is

$$dA = r^2 \sin\theta \; d\theta \; d\phi \ ; \tag{5.36}$$

and since $d\theta$ and $d\phi$ are constant along each pathline the dA ratio is

$$\frac{dA}{dA_*} = \frac{r^2}{r_*^2} \ , \tag{5.37}$$

which is the same as (5.35). Thus we can conclude that for the case of radial pathline flow, (5.32) reduces to the classical result, (5.33). If we would analyze the flow through a straight-radial wall nozzle by the present equations and by the one-dimensional method we would obtain exactly the same results (all the equations are identical). However, when we analyze the flow through a nozzle with curved walls, the two methods would not agree because the ratio of increment of areas in (5.32) is not equal to the ratio of areas in (5.33). The greater the curvature of the walls, the greater will be the disagreement.

Let us now consider the fluid particle minimum principle. Consider the case of flow through an internal flow passage where the pressure at the intake and discharge of the passage is known. The geometry of the passage is also known. A family of pathlines is then selected so that the outer pathlines of the flow region coincide with the walls of the passage. For example, for the case of flow through a conical passage, as shown in figure 5.2, a radial family of pathlines is selected. The ratio dA/dA_* can then be calculated. We now seek the function ∇g in terms of dA/dA_*. Substituting $\left[\frac{p}{p_*}\right]^{1/K} = \frac{\rho}{\rho_*}$ (from 5.15) into (5.27a) yields

$$\left[\frac{p}{p_*}\right]^{1/K} = \frac{V_*}{V}\frac{dA_*}{dA} \quad . \tag{5.38}$$

Solving for V yields

$$V = \left[\frac{p_*}{p}\right]^{1/K} \frac{dA_*}{dA} V_* \quad . \tag{5.39}$$

Dividing (5.23) by (5.20) yields

$$\frac{p}{p_*} = \frac{p}{p_o}\frac{p_o}{p_*} = \left[\frac{K+1}{2(1+\frac{K-1}{2}M^2)}\right]^{\frac{K}{K-1}} \quad ; \tag{5.40}$$

which when substituted into (5.39) yields

$$V = \left[\frac{K+1}{2(1+\frac{K-1}{2}M^2)}\right]^{\frac{1}{K-1}} \frac{dA_*}{dA} V_* \equiv |\nabla g|, \tag{5.41}$$

where we have defined the right hand side of (5.41) to be equal to $|\nabla g|$. The Mach number M is determined implicitly in terms of dA/dA_* by (5.32). Since dA/dA_* is a function of the space variables, the function ∇g, defined by (5.41), is also determined as a function of the space variables.

Substituting the known pressures, p_d and p_i, into (5.39) determines the velocity at both ends of the flow passage, i.e.,

$$V_d = \left[\left[\frac{p_*}{p_d}\right]^{1/K} \frac{dA_*}{dA}\right]_d V_* \text{ , and} \tag{5.42a}$$

$$V_i = \left[\left[\frac{p_*}{p_i}\right]^{1/K} \frac{dA_*}{dA}\right]_i V_* \text{ .} \tag{5.42b}$$

Condition (4.B.1) is satisfied by (5.9). Condition (4.B.2) is satisfied when (5.42) is satisfied or when (5.41) is satisfied. Since (5.41) is not valid across a shock, condition (4.B.2) is satisfied in flow regions which exclude shocks. Condition (4.B.3) is satisfied in regions free of conjugate points. Thus we can conclude that the fluid particle minimum principle is satisfied for the case of flow through a passage when (5.42) is satisfied in a region free of shocks and conjugate points.

It should be pointed out that it is difficult to select a family of pathlines that coincide with the walls of some given flow passage. An iteration procedure may be necessary in order to determine this proper family of pathlines. The inverse procedure is much easier. That is, given a family of pathlines we may choose any set of pathlines to be the outer pathlines of the flow region and thus determine the geometry of the flow passage. This inverse procedure will be employed in later sections to determine optimal geometries of compressor rotor passages.

It should also be pointed out that all compressible (irrotational) potential flows satisfy the fluid particle minimum principle in regions free of shocks and conjugate points. That is, once a velocity potential function, g, is selected which satisfies the boundary conditions and the continuity equation (2.9), the family of pathlines is uniquely determined, see (5.41). The boundary conditions are (5.42) plus the condition that the family of pathlines coincide with the walls of the flow passage. For the case of rotational flows, there does not exist a potential function and the fluid particle minimum principle is often not satisfied.

6. ROTATING FLOW PASSAGE

In this section, the fluid particle minimum principles, (4.12) and (4.28), are adapted to the flow inside a rotating passage. Consider a fluid particle that is moving along a path-line which lies inside a rotating passage. The fluid particle is rotating at an angular speed, $\dot{\theta}$, relative to the passage as shown in figure 6.1. The wall of the passage is rotating at a constant angular speed, ω. The total angular speed of the particle, $\dot{\alpha}$, is

$$\dot{\alpha} = \omega + \dot{\theta} \ . \tag{6.1}$$

The absolute position vector, $\vec{R}$, of the fluid particle may be expressed with respect to the fixed reference frame (with unit vectors $\hat{i},\hat{j},\hat{k}$) in the below manner

$$\vec{R} = r \cos \alpha \ \hat{i} + r \sin \alpha \ \hat{j} + z \ \hat{k} \ . \tag{6.2}$$

Where (r,α,z) are the cylindrical coordinates related to the Cartesian coordinates (x,y,z) by:

$$x = r \cos \alpha \ , \ y = r \sin \alpha \ , \ z = z \ . \tag{6.3}$$

The first time derivative of (6.2) is

$$\frac{d\vec{R}}{dt} = \vec{V} = (\dot{r} \cos \alpha - \dot{\alpha} \ r \sin \alpha)\hat{i} + (\dot{r} \sin \alpha + \dot{\alpha} \ r \cos \alpha)\hat{j} + \dot{z} \ \hat{k} \ ; \tag{6.4}$$

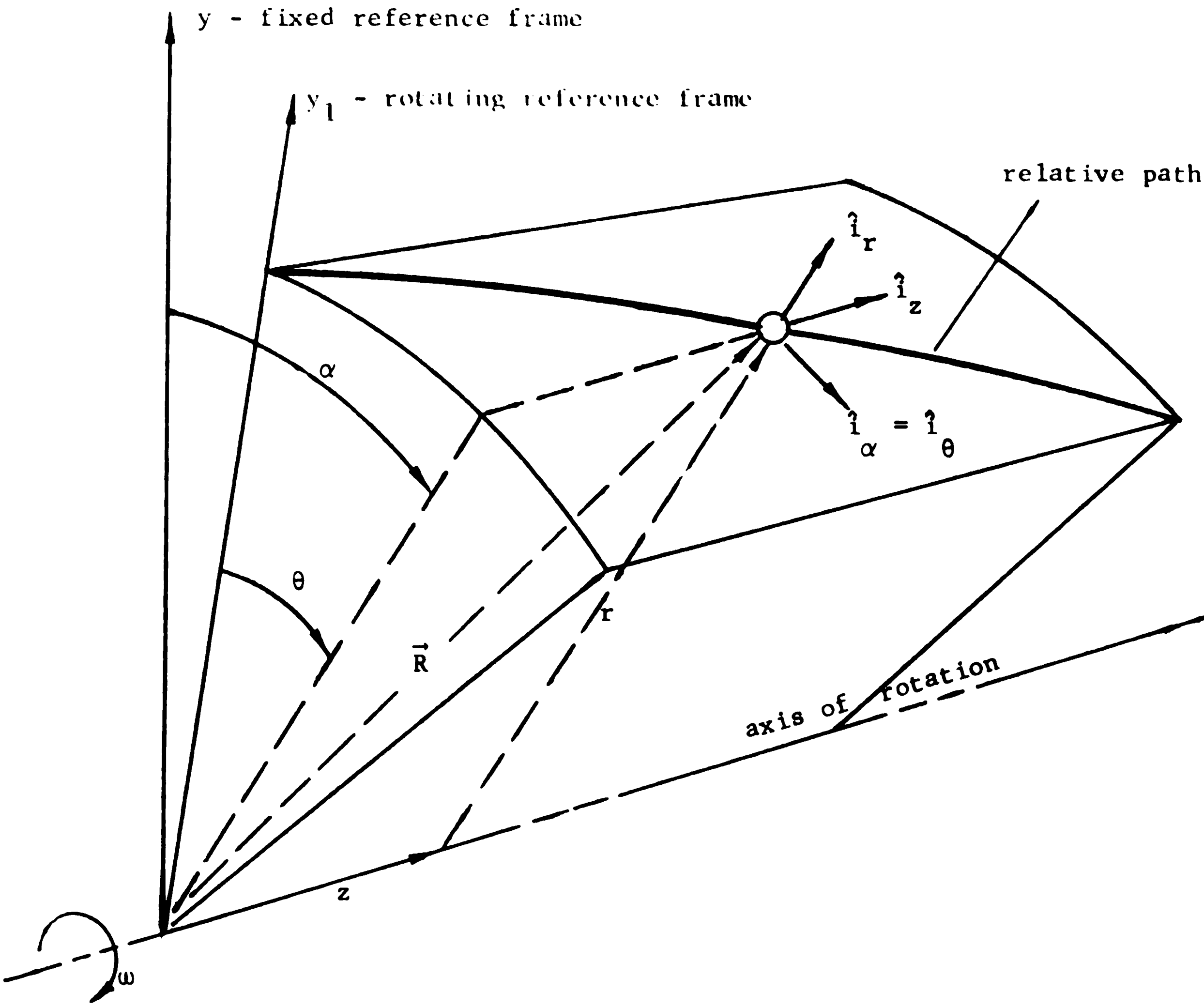

Figure 6.1

Rotating Passage

and the kinetic energy is

$$\frac{1}{2}(\dot{\vec{R}})^2 \equiv \frac{1}{2}V^2 = \frac{(\dot{r})^2}{2} + \frac{(r\dot{\alpha})^2}{2} + \frac{(\dot{z})^2}{2} . \tag{6.5}$$

Since the Euler-Lagrange equations and the functional, (3.1), are invariant under coordinate transformation, the introduction of cylindrical coordinates does not change any of the equations developed in the previous sections [2]. In order to apply the result of the previous sections, we simply replace the terms

$\vec{R}$, $\dot{\vec{R}}$, $\frac{1}{2}V^2$ by equations (6.2), (6.4), (6.5) respectively. For example, replacing $\frac{1}{2}(\dot{\vec{R}})^2$ by (6.5) in the fluid particle minimum principle, [(4.12) (with $G = 0$)], yields

$$\delta \int_a^b [\frac{(\dot{r})^2}{2} + \frac{(r\dot{\alpha})^2}{2} + \frac{(\dot{z})^2}{2} - C_p T]dt +$$

$$[(\nabla E_1 - \dot{\vec{R}}) \cdot \delta\vec{R}]_a + [(\nabla E_2 - \dot{\vec{R}}) \cdot \delta\vec{R}]_b = 0 . \qquad (6.6)$$

The Euler-Lagrange equations of the above functional are obtained by substituting the integrand of (6.6) into (3.2) (with $y_1 = r$, $y_2 = \alpha$, $y_3 = z$) which yields

$$\frac{d\dot{r}}{dt} - r\dot{\alpha}^2 + \frac{\partial h}{\partial r} = 0 , \qquad (6.7a)$$

$$\frac{d(r^2\dot{\alpha})}{dt} + \frac{\partial h}{\partial \alpha} = 0 , \qquad (6.7b)$$

$$\frac{d\dot{z}}{dt} + \frac{\partial h}{\partial z} = 0 , \qquad (6.7c)$$

where $h \equiv C_p T$. Multiplying (6.7b) by $1/r$ and substituting

$$\nabla h = \frac{\nabla p}{\rho} = \frac{1}{\rho}[\frac{\partial p}{\partial r}\hat{i}_r + \frac{1}{r}\frac{\partial p}{\partial \alpha}\hat{i}_\alpha + \frac{\partial p}{\partial z}\hat{i}_z] \qquad (6.8)$$

into (6.7) yields

$$\ddot{r} - r\dot{\alpha}^2 + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 , \qquad (6.9a)$$

$$r\ddot{\alpha} + 2\dot{r}\dot{\alpha} + \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \alpha} = 0 , \qquad (6.9b)$$

$$\ddot{z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 ; \qquad (6.9c)$$

which is the Lagrangian form of the momentum equation written in terms of cylindrical coordinates for the case of isentropic (frictionless) flow, see Owczarek [10] page 93. The functional

(6.6) depends on time, t, because the total energy of a fluid particle is increased while passing through a rotating passage. When the functional depends on time, the first integral of the Euler-Lagrange eqaution (i.e., the energy equation) is no longer given by (3.9) but is given by the below equation, see Gelfand [2] page 70

$$\frac{\partial H}{\partial t} = \frac{dH}{dt} = \frac{d}{dt}\left(\frac{V^2}{2} + h\right) . \tag{6.10}$$

We now choose a reference frame which is attached to the passage and thus is rotating at a constant angular speed, ω, see figure 6.1. This rotating reference frame will be called the relative reference frame. The position vector drawn from the relative reference frame to the fluid particle is called the relative position vector, $\vec{R}_1$, which is written in terms of the relative unit vectors, $(\hat{i}_1, \hat{j}_1, \hat{k}_1)$ as

$$\vec{R}_1 = r \cos\theta \, \hat{i}_1 + r \sin\theta \, \hat{j}_1 + z \, \hat{k}_1 . \tag{6.11}$$

The relative velocity, $\vec{W}$, is the first time derivative of (6.11), i.e.,

$$\dot{\vec{R}}_1 \equiv \vec{W} = (\dot{r} \cos\theta - \dot{\theta} \, r \sin\theta)\hat{i}_1 + (\dot{r} \sin\theta + \dot{\theta} \, r \cos\theta)\hat{j}_1 + \dot{z} \, \hat{k}_1, \tag{6.12}$$

where $\hat{i}_1, \hat{j}_1, \hat{k}_1$ are the unit vectors of the rotating reference frame. The relative kinetic energy is

$$\frac{1}{2}(\dot{\vec{R}}_1)^2 \equiv \frac{1}{2} W^2 = \frac{1}{2} [\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2] . \tag{6.13}$$

We now seek an expression for the acceleration of the particle along the rotating pathline. Drawing the absolute position vector, $\vec{R}$, from the fixed reference frame to the particle, we may differentiate $\vec{R}$ with respect to time in order to obtain the absolute velocity and acceleration of the fluid particle. The absolute position vector, $\vec{R}$, and the relative position vector, $\vec{R}_1$, coincide, i.e.,

$$\vec{R} = \vec{R}_1 \; . \tag{6.14}$$

Differentiating (6.14) yields

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{d\vec{R}_1}{dt} + \vec{\omega} \times \vec{R}_1 = \vec{W} + \omega r \, \hat{i}_\theta \; . \tag{6.15}$$

Notice that the relative position vector, $\vec{R}_1$, is changing its magnitude and direction. The term, $\vec{\omega} \times \vec{R}_1$, in (6.15), arises because $\vec{R}_1$ is rotating. In general, derivatives taken in the rotating reference frame are related to derivatives taken in the inertial reference frame according to the operator [10],

$$\left.\frac{d(\;)}{dt}\right]_{\text{INERTIAL}} = \left.\frac{d(\;)}{dt}\right]_{\text{ROTATING}} + \vec{\omega} \times (\;) \; . \tag{6.16}$$

The absolute acceleration is obtained by differentiating (6.15) and employing (6.16) which yields

$$\ddot{\vec{R}} = \ddot{\vec{R}}_1 + 2\vec{\omega} \times \dot{\vec{R}}_1 + \vec{\omega} \times (\vec{\omega} \times \vec{R}_1) \; , \tag{6.17a}$$

or

$$\dot{\vec{V}} = \dot{\vec{W}} + 2\vec{\omega} \times \vec{W} - \omega^2 r \, \hat{i}_r \; , \tag{6.17b}$$

where $\frac{d\vec{R}}{dt} \equiv \vec{V}$ and $\frac{d\vec{R}_1}{dt} \equiv \vec{W}$.

We will now formulate a fluid particle minimum principle in terms of quantities measured from the relative reference frame, i.e., in terms of $\vec{R}_1$ and $\vec{W}$. The absolute acceleration, $\dot{\vec{V}}$, is given in terms of quantities measured in the relative reference frame by the right hand side of (6.17). Substituting (6.17b) into the momentum equation, (2.1), (with $\vec{f} \equiv 0$) yields

$$\dot{\vec{W}} + 2\vec{\omega} \times \vec{W} - \omega^2 r \ \hat{i}_r + \frac{\nabla p}{\rho} = 0 \ ; \tag{6.18}$$

which is the momentum equation as viewed from the rotating reference frame. We observe that Newton's second law of motion does not retain its form $(\dot{\vec{W}} = \frac{-\nabla p}{\rho} + \vec{f})$ in the rotating reference frame [3]. Taking the dot product of (6.18) times $\delta\vec{R}_1$ and then multiplying by dt and integrating yields

$$\int_a^b [\dot{\vec{W}} \cdot \delta\vec{R}_1 + \frac{\nabla p}{\rho} \cdot \delta\vec{R}_1 + 2(\vec{\omega} \times \vec{W}) \cdot \delta\vec{R}_1 - \omega^2 r \ \hat{i}_r \cdot \delta\vec{R}_1] dt = 0 \ . \tag{6.19}$$

When the inertia accelerations are treated as forces which act on the relative reference frame, i.e., when we let

$$\vec{f}_N = -2 \ \vec{\omega} \times \vec{W} \ , \quad \text{and} \tag{6.20a}$$

$$\vec{f}_C = \omega^2 r \ \hat{i}_r \ , \tag{6.20b}$$

then (6.19) is of the same form as the general functional, (4.24), and as shown in section-4 the general form of the fluid particle minimum principle is given by (4.28).

We now seek an expression for the "potential energy", G, where we define G so that $\delta G = -\vec{f}_C \cdot \delta\vec{R}_1$. First, observe that the centrifugal acceleration, $\omega^2 r \ \hat{i}_r$, produces a conservative force field since

$$\nabla \times (\omega^2 r\, \hat{i}_r) = \begin{vmatrix} \frac{1}{r}\hat{i}_r & \hat{i}_\theta & \frac{1}{r}\hat{i}_k \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ r\,\omega^2 & 0 & 0 \end{vmatrix} = 0 \; . \tag{6.20c}$$

Thus the potential energy of a fluid particle in the rotating reference frame is

$$\delta G = -\omega^2 r\, \hat{i}_r \cdot \delta\vec{R}_1 = -\omega^2 r \delta r = -\delta\, \frac{\omega^2 r^2}{2} \; . \tag{6.21}$$

Substituting (6.21) and (6.20a) into the general form of the minimum principle, (4.28), yields

$$\delta \int_a^b [\frac{W^2}{2} + \frac{\omega^2 r^2}{2} - h]dt + \int_a^b (2\vec{\omega} \times \vec{W}) \cdot \delta\vec{R}_1 dt +$$

$$[(\nabla E_1 - \vec{W}) \cdot \delta\vec{R}_1]_a + [(\nabla E_2 - \vec{W}) \cdot \delta\vec{R}_1]_b = 0 \; , \tag{6.22}$$

where $h \equiv C_p T$ and $\vec{f}_N = 2\vec{\omega} \times \vec{W}$. The second term of (6.22) is zero since the Coriolis acceleration, $2\vec{\omega} \times \vec{W}$, is normal to $\delta\vec{R}_1$. However this term will be retained since its omission will alter the Euler-Lagrange equation. The Euler-Lagrange equation of (6.22) is given by (4.27) (with $G = \frac{-\omega^2 r^2}{2}$, $\vec{f}_N = -2\vec{\omega} \times \vec{W}$, and $\dot{\vec{R}} = \vec{W}$), i.e.,

$$\dot{\vec{W}} + C_p \nabla T - \omega^2 r\, \hat{i}_r + 2\vec{\omega} \times \vec{W} = 0 \; , \tag{6.23a}$$

or

$$\dot{\vec{W}} + \frac{\nabla p}{\rho} - \omega^2 r\, \hat{i}_r + 2\vec{\omega} \times \vec{W} = 0 \; . \tag{6.23b}$$

Substituting (6.17b) into (6.23b) yields

$$\dot{\vec{V}} + \frac{\nabla p}{\rho} = 0 \; ; \tag{6.24}$$

which is the momentum equation as viewed from the fixed reference frame. Expanding the above vector equation into its three scalar equations, in terms of cylindrical coordinates, yields (6.9). Thus we can conclude that the Euler-Lagrange equations of the functionals (6.6) and (6.22) are identical. It is known from Variational Calculus that two functionals are equivalent when their respective Euler-Lagrange equations are identical. The two fluid particle minimum principles, (6.6) and (6.22), are therefore equivalent and thus we may operate in the relative reference frame using the minimum principle given by (6.22).

The first integral of the Euler-Lagrange equation may be found by two methods. In the first method we form the dot product of (6.23b) times $d\vec{R}_1$ and then integrate. In the second method we substitute the integrand of (6.22) into (3.9). In both methods, we observe that $(2\vec{\omega} \times \vec{W}) \cdot d\vec{R}_1 = 0$. Employing one of the above methods yields

$$\frac{1}{2} W^2 - \frac{\omega^2 r^2}{2} + h = H_R = \text{constant along each relative pathline;} \tag{6.25a}$$

or

$$\frac{1}{2}[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2] - \frac{\omega^2 r^2}{2} + h = H_R \, . \tag{6.25b}$$

The relative pathline is the path of the fluid particle as viewed from the relative reference frame. Substituting $\dot{\theta} = \dot{\alpha} - \omega$ from (6.1) into (6.25b) yields

$$\frac{1}{2} [\dot{r}^2 + (r\dot{\alpha})^2 + \dot{z}^2] - r^2\dot{\alpha}\omega + h = H_R \, . \tag{6.26}$$

Substituting (6.5) into (6.26) yields

$$\frac{1}{2} V^2 - r^2 \dot{\alpha}\omega + h = H_R = \text{constant} , \qquad (6.27a)$$

or

$$\frac{V^2}{2} + h = r^2 \dot{\alpha}\omega + H_R = H \neq \text{constant} ; \qquad (6.27b)$$

where $H \equiv \frac{V^2}{2} + h$, and H is called the total absolute enthalpy. The symbol H_R is called the total relative enthalpy. Taking the time derivative of (6.27b) yields

$$\frac{d}{dt} \left(\frac{V^2}{2} + h\right) = \frac{dH}{dt} = \frac{d}{dt} (r^2 \dot{\alpha}\omega) . \qquad (6.28)$$

Equation (6.28) is called Euler's turbine equation which is valid for steady isentropic flow, see Owczarek [10] page 95. The right hand side of (6.28) represents the rate at which work is added to a fluid particle along an absolute streamline or an absolute pathline in this derivation. Comparing (6.28) and (6.10) we see that they are equivalent equations. Thus we can conclude that the same absolute energy equation results regardless of the minimum principle that is employed to describe the motion.

In later sections, the fluid particle minimum principle, (6.22), will be employed to determine the family of pathlines which describe the motion of the fluid continuum. The "relative energy equation", (6.25), is the energy equation corresponding to (6.22) which must be satisfied in order that condition (4.A.1) of section-4 is satisfied.

7. IRROTATIONAL FLOW

In this section we seek the condition at which the flow inside the rotating passage is irrotational, i.e., we seek the condition, $\nabla \times \vec{V} = 0$. Substituting (6.15) into $\nabla \times \vec{V} = 0$ yields

$$\nabla \times (\vec{W} + \omega r \, \hat{i}_\theta) = 0 \text{ , or} \tag{7.1a}$$

$$\nabla \times \vec{W} = -\hat{i}_z \frac{\omega}{r} \frac{\partial(r^2)}{\partial r} + \hat{i}_r \frac{\omega}{r} \frac{\partial(r^2)}{\partial z} = -\hat{i}_z 2\omega \text{ .} \tag{7.1b}$$

Since the curl of a velocity vector is equal to twice the angular speed of the fluid particle, the angular speed of the relative velocity vector is equal to $-\omega$ when the fluid is irrotational, i.e., the irrotational condition is

$$\nabla \times \vec{V} = 0 \quad \text{IFF} \quad \dot{\theta} = -\omega \text{ .} \tag{7.2}$$

This type of flow is called a "free-vortex" flow in turbomachinery literature [13]. When (7.2) is substituted into Euler's turbine equation, (6.28), we see that the energy level, H, of a free-vortex flow remains constant, i.e., substituting $\dot{\alpha} = \omega + \dot{\theta} = \omega - \omega = 0$ into (6.28) yields $\dot{H} = 0$. Thus we conclude that the strong minimum energy state of the system corresponds to a zero change in energy along an absolute pathline. Since the principle function of a compressor is to add energy to the fluid, the irrotational case, (7.2), is dismissed as a trivial case.

8. ROTATIONAL FLOW

In the previous section, it was shown that there exists only one trivial case for which the flow inside a rotating passage is irrotational. In this section, we investigate the conditions for which the fluid particle minimum principle, (6.22), and the continuity equation are satisfied in a rotational flow.

In section-6 the relative energy equation, (6.25), was derived, which is the first integral of the Euler-Lagrange equation corresponding to the fluid particle minimum principle, (6.22). As shown in section-6, the Euler-Lagrange equation, (6.24), of (6.22) is identical to the momentum equation. Thus the energy and momentum equations are satisfied whenever the fluid particle minimum principle, (6.22), is satisfied. The continuity equation is now combined with the relative energy equation and an expression for the change in cross sectional area, dA, of a differential pathtube is obtained in terms of (ω, r, W, h) as shown below. A precise definition of dA was given in section-2, see equation (2.6b). We proceed in a manner similar to the procedure employed in the second example of section-5. The relative energy equation, (6.25), as derived in section-6 is

$$\frac{W^2}{2} - \frac{\omega^2 r^2}{2} + C_p T = H_R = \text{constant along each relative pathline} . \tag{8.1}$$

Dividing each term of (8.1) by the speed of sound squared, $a_*^2 = C_p(k-1)T_*$, where $*$ denotes sonic conditions, yields

$$\frac{1}{2}\frac{W^2}{a_*^2} - \frac{\omega^2 r^2}{2a_*^2} + \frac{1}{K-1}\frac{T}{T_*} = \frac{H_R}{a_*^2} \quad . \tag{8.2}$$

Substituting the continuity equation, given by (5.27a with $W = V$) into (8.2) yields

$$\frac{1}{2}\frac{W^2}{2a_*^2} - \frac{\omega^2 r^2}{2a_*^2} + \frac{1}{K-1}\left[\frac{W}{a_*}\frac{dA}{dA_*}\right]^{1-K} = \frac{H_R}{a_*^2} \, . \tag{8.3}$$

Solving for dA/dA_* yields

$$\frac{dA}{dA_*} = \frac{a_*}{W}\left[(K-1)\left[\frac{H_R}{a_*^2} + \frac{\omega^2 r^2}{2a_*^2} - \frac{W^2}{2a_*^2}\right]\right]^{\frac{1}{1-K}} \quad . \tag{8.4}$$

Equation (8.4) can be written for any two points along a pathline in the below manner

$$\frac{dA_2}{dA_1} = \frac{dA_2/dA_*}{dA_1/dA_*} = \frac{W_1}{W_2}\left[\frac{H_R + \frac{1}{2}\omega^2 r_2^2 - \frac{1}{2}W_2^2}{H_R + \frac{1}{2}\omega^2 r_1^2 - \frac{1}{2}W_1^2}\right]^{\frac{1}{1-K}} \quad . \tag{8.5}$$

Multiplying (8.5) by W_2/W_1 and using the fact that the relative velocity vector, $\vec{W}$, is perpendicular to the change in area vector, $d\vec{A}$, yields

$$\frac{\vec{W}_2 \cdot d\vec{A}_2}{\vec{W}_1 \cdot d\vec{A}_1} = \frac{W_2 dA_2}{W_1 dA_1} = \left[\frac{h_1 - \frac{1}{2}\left[W_2^2 - W_1^2 - \omega^2(r_2^2 - r_1^2)\right]}{h_1}\right]^{\frac{1}{1-K}} \, , \tag{8.6}$$

where $h_1 \equiv C_p T = H_R + \frac{1}{2}\omega^2 r_1^2 - \frac{1}{2}W_1^2$ (see 8.1). The changes in the thermodynamic properties of the fluid can be found by employing the below equations after (8.6) has been evaluated. Employing the continuity equation, (2.9b) with $W = V$, yields

$$\frac{\rho_2}{\rho_1} = \frac{W_1 dA_1}{W_2 dA_2} \quad . \tag{8.7}$$

Substituting (8.7) into the isentropic gas relation, (5.15), yields

$$\frac{T_2}{T_1} = \left[\frac{\rho_2}{\rho_1}\right]^{K-1} = \left[\frac{W_1 dA_1}{W_2 dA_2}\right]^{K-1} \quad . \tag{8.8}$$

Substituting (8.7) and (8.8) into the ideal gas equation of state yields

$$\frac{P_2}{P_1} = \frac{\rho_2 R_c T_2}{\rho_1 R_c T_1} = \left[\frac{W_1 dA_1}{W_2 dA_2}\right]^{K} \quad . \tag{8.9}$$

When equation (8.5) or (8.6) is satisfied along a relative pathline, the momentum, energy, and continuity equations are satisfied. If the flow is in a region which does not contain conjugate points and if the boundary conditions, (4.22), are satisfied, then the fluid particle minimum principle is satisfied. That is, when conditions (4.A) of section-4 are satisfied, the fluid particle minimum principle is satisfied. We can now formulate a boundary value problem. Equation (8.5) must be satisfied along every pathline inside the flow region. At the ends of the flow region, boundary conditions (4.22) must hold. Along the walls of the flow passage the pathlines must coincide with the walls of the passage. The following "inverse" procedure is used to determine the optimal geometry of the flow passage. We impose "optimal" constraints on the problem. A family of pathlines is then determined which satisfy (8.4), (4.22), and the optimal constraints. The passage geometry is then selected to coincide with the pathlines, and thus the boundary value problem is completely determined. The optimal constraints are developed in the next section.

9. OPTIMAL CONSTRAINTS

In this section a few optimal constraints are selected which, in the author's opinion, are expected to produce optimal performance of a compressor rotor. The choice of these optimal constraints is supported by intuitive arguments. The author knows of no rigorous procedure for selecting the optimal constraints. Optimal constraints are imposed in order to produce desirable operating conditions. However, it is not always possible to solve the boundary value problem subject to several constraints. When this occurs, it may be necessary to remove one or more constraints.

A rotor usually has a uniform inlet pressure and back pressure imposed on the intake and discharge cross sections (see Vavra [13] page 212). However, the intake pressure, p_i, and the discharge pressure, p_d, inside the rotor is, in general, not uniform. This type of situation may produce secondary flows especially if the pressure distribution is highly non-uniform. It is therefore reasonable to constrain p_i and p_d to be approximately uniform. Examination of the momentum equation, (6.9), yields the following steady flow case in which the intake pressure, p_i, is uniform over the (r, θ)-plane of the intake section

$$\dot{r} = \text{constant}, \ \dot{\alpha}_i = \omega - \dot{\theta}_i = 0, \ \ddot{z} = \frac{1}{\rho}\frac{\partial p}{\partial z}, \ \ddot{r} = \ddot{\alpha} = 0 . \qquad (9.1)$$

We shall call (9.1) the "free-vortex intake condition". Notice that this condition can hold only at one given rotor speed, ω.

Given a uniform pressure (or approximately uniform pressure), p_i, at the intake section, the discharge pressure, p_d, will be uniform (or approximately uniform) only if the pressure increases by the same amount along every pathline between the intake and discharge sections. We will also constain the pressure to monotonically increase along each pathline. A local region of rapid pressure change could cause the boundary-layer to separate, which is undesirable. Upon examining (8.6) and (8.9), we conclude that the above "uniform pressure increase constraint" is met when

$$\frac{p}{p_i} = \left[\frac{W\ dA}{W_i dA_i}\right]^{-K} = \left[\frac{h_i - \frac{1}{2}\left[W^2 - W_i^2 - \omega^2(r^2 - r_i^2)\right]}{h_i}\right]^{\frac{-K}{1-K}}$$

$$= L(r,\theta,z) \ , \tag{9.2}$$

where $L(r,\theta,z)$ is a monotonic increasing function which has the same value when evaluated between the intake and discharge points of each pathline.

A special case of (9.2) is

$$\frac{W\ dA}{W_i dA_i} = \left[L(r,\theta,z)\right]^{-1/K} \equiv 1 \ . \tag{9.3}$$

From (8.7 - 8.9) we observe that the thermodynamic variables, (ρ,T,p), remain constant along the pathlines when constraint (9.3) is satisfied. Constraint (9.3) will be called the "maximum kinetic energy increase constraint" because all the energy being added to the fluid is being converted into kinetic energy, while the enthalpy, h, of the fluid remains constant.

Solving (8.6) for dA/dA_i yields

$$\frac{dA}{dA_i} = \frac{W_i}{W}\left[\frac{h_i - \frac{1}{2}\left[W^2 - W_i^2 - \omega^2(r^2 - r_i^2)\right]}{h_i}\right]^{\frac{1}{1-K}} \quad . \tag{9.4}$$

The above equation determines the ratio dA/dA_i along each pathline in the flow region. For a given set of design conditions, $(h_i, W_i, \omega$, etc.), equation (9.4) is employed to determine a family of pathlines. The walls of the flow passage are then selected to coincide with the pathlines of the flow region, and thus the geometry of the flow passage is also determined from (9.4). When one of the design conditions, $(h_i, W_i, \omega$, etc.), is changed, the pathlines of the flow region will, in general, no longer coincide with the walls of the flow passage. There is only one special case of (9.4) in which the pathlines coincide with the passage walls at all rotor speeds, ω, and initial conditions, (h_i, W_i). That is, if we impose the constraints

$$\frac{W}{W_i} = 1 \quad \text{and} \quad \frac{r}{r_i} = 1 \ , \tag{9.5a}$$

then (9.4) reduces to

$$\frac{dA}{dA_i} \equiv 1 \ , \tag{9.5b}$$

which is independent of the parameters (h_i, W_i, ω). The above equations, (9.5), will be called the "maximum speed range" constraint.

It is common practice to design a rotor so that the same amount of energy is added to each fluid particle which passes through the rotor [13]. The "uniform energy increase constraint" is imposed in order to reduce mixing losses after the flow leaves the rotor. Thus the change in energy, ΔH, is constrained to be a monotonic increasing function, $L(r,\theta,z)$, which has the same value when evaluated between the intake and discharge points of each pathline. The change in energy, ΔH, along a pathline is determined by integrating (6.28) with respect to time from $t = t_i$ to $t = t$. Substituting $\dot{\alpha} = \dot{\theta} + \omega$ into the resulting equation yields

$$H - H_i = \omega^2(r^2 - r_i^2) + r^2\omega\dot{\theta} - r_i^2\omega\dot{\theta}_i = L(r,\theta,z) \ . \tag{9.7}$$

The equation of a pathline can be expressed in the following parametric form;

$$r = r(t), \quad z = z(t), \quad \theta = \theta(t) \ . \tag{9.8}$$

For the case of steady flow, it is possible to eliminate time, t, from one of the above equations and express the equation of the pathline in terms of the other two space variables, i.e., we may write

$$\theta = \theta(r,z) \ . \tag{9.9}$$

Differentiating the above equation and employing the chain rule of Calculus yields

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\partial\theta}{\partial z}\frac{dz}{dt} + \frac{\partial\theta}{\partial r}\frac{dr}{dt} \ . \tag{9.10}$$

Substituting (9.10) into (9.7) yields

$$H - H_i = \omega^2(r^2 - r_i^2) + r^2\omega\dot{z}\,\frac{\partial\theta}{\partial z} - r_i^2\omega\dot{z}_i\left[\frac{\partial\theta}{\partial z}\right]_i + r^2\omega\dot{r}\,\frac{\partial\theta}{\partial r} - r_i^2\omega\dot{r}_i\left[\frac{\partial\theta}{\partial r}\right]_i \ . \tag{9.11}$$

Defining the angles γ, ξ so that

$$\tan \gamma \equiv r \frac{\partial \theta}{\partial z} \quad \text{and} \quad \tan \xi \equiv r \frac{\partial \theta}{\partial z} \tag{9.12}$$

equation (9.11) becomes

$$H - H_i = \omega^2 (r^2 - r_i^2) + r\omega \dot{z} \tan \gamma - r_i \omega \dot{z}_i \tan \gamma_i +$$
$$r\omega \dot{r} \tan \xi - r_i \omega \dot{r}_i \tan \xi_i \; . \tag{9.13}$$

We now list the system of equations and boundary conditions which will be employed to determine optimal internal flow passages.

I. The uniform pressure increase constraint, (9.2), plus the continuity, momemtum, and energy equation, (8.6), requires that

$$\left[\frac{W\,dA}{W_i dA_i}\right]^{-K} = \left[\frac{h_i - \frac{1}{2}\left[W^2 - W_i^2 - \omega^2 (r^2 - r_i^2)\right]}{h_i}\right]^{\frac{-K}{1-K}} = L(r,\theta,z), \tag{9.14}$$

where $L(r,\theta,z)$ is a monotonic increasing function which has the same value when evaluated between the intake and discharge points of each pathline. When the "free-vortex intake condition", (9.1), is satisfied, the intake pressure, p_i, and discharge pressure, p_d, are uniform over their respective cross sections. Two special cases of (9.14) are listed below.

A. "Maximum Kinetic Energy Increase" Constraint

1) $$W^2 - W_i^2 = \omega^2 (r^2 - r_i^2) \; , \tag{9.15a}$$

2) $$\frac{W\,dA}{W_i dA_i} = 1 \Rightarrow \frac{p}{p_i} = 1 \quad \text{and} \quad \frac{h}{h_i} = 1 \; . \tag{9.15b}$$

B. "Maximum Speed Range" Constraint

1) $$\frac{r}{r_i} = 1, \; \frac{W}{W_i} = 1, \; \frac{dA}{dA_i} = 1 \tag{9.16a}$$

2) $\frac{W\,dA}{W_i dA_i} = 1 \Rightarrow \frac{p}{p_i} = 1$ and $\frac{h}{h_i} = 1$ (9.16b)

II. Boundary Conditions

A. The initial or intake conditions, (p_i, W_i, h_i, etc.) are known for each pathline. The intake pressure, p_i, is uniform over the intake section when the "free-vortex intake condition", (9.1), is satisfied (i.e., when $\omega = -\dot{\theta}_i$).

B. The discharge pressure, p_d, is known for each pathline. It is constrained, by equation (9.14), to be uniform over the discharge section when p_i is uniform (i.e., when $\omega = -\dot{\theta}_i$).

C. The boundary condition, (4.22b), for the variable end point functional is determined in terms of the pressure boundary conditions by equation (8.9), i.e.,

$$W_d = W_i \frac{dA_i}{dA_d}\left[\frac{p_i}{p_d}\right]^{1/K} \equiv \nabla E_2 \ . \qquad (9.17)$$

D. The pathlines of the flow region must coincide with the walls of the flow passage. Since the walls of the flow passage are determined from (9.14), this boundary condition is, in general, satisfied throughout the entire flow region only at one set of design conditions, (ω, h_i, W_i, etc.). However, when the maximum speed range constraint, (9.16), is satisfied, this boundary condition is satisfied for all rotor speeds, ω, and intake conditions, (h_i, W_i, etc.).

When the above equations and boundary conditions are satisfied in regions which exclude conjugate points and shocks, the fluid particle minimum principle, (6.22), is satisfied. That is, conditions (4.A) of section-4 are satisfied for equation (6.22). Shocks must be excluded from the flow region since equation (9.14) does not hold across a shock. The following two constraints may also be imposed in order to reduce mixing losses after the flow leaves the rotor.

III. The "uniform energy increase constraint" requires that

$$H - H_i = \omega^2(r^2 - r_i^2) + r\omega\dot{z} \tan \gamma - r_i\omega\dot{z}_i \tan \gamma_i + r\omega\dot{r} \tan \xi - r_i\omega\dot{r}_i \tan \xi_i = K(r,\theta,z) \,, \qquad (9.18)$$

where $K(r,\theta,z)$ is a monotonic increasing function which has the same value when evaluated between the intake and discharge points of each pathline.

IV. The "uniform discharge velocity constraint" requires that

$$V_d^2 = 2(H_d - H_i) + V_i^2 - (r_d^2 - r_i^2)\omega^2 + W_d^2 - W_i^2 \qquad (9.19)$$

have a uniform value over the discharge section. Where (9.19) was obtained by integrating (6.28) with respect to time from $t = i$ to $t = d$ and then substituting h_i and h_d, which are evaluated from (6.25a), into the resulting equation.

10. A "MAXIMUM KINETIC ENERGY INCREASE" CENTRIFUGAL ROTOR PASSAGE

In this section, we seek the geometry of the flow passage of a centrifugal (mix-flow) rotor which will satisfy the "fluid particle minimum principle" and the "maximum kinetic energy increase constraint" of section-9. This example is intended to serve only as an academic demonstration of how an optimal rotor passage may be determined.

We consider only centrifugal rotors having radial blades in this section. We constrain each pathline to lie on a (r,z)-plane (radial plane) as shown in figure 10.1.

All the initial conditions of each pathline inside the rotor, (Ψ_i, $\dot{r}_i$, etc.), must be known. The pressure distribution over the intake section is then determined from the momentum equation, (6.9). It is interesting to observe that when the flow satisfies the "free-vortex intake condition", (9.1), (i.e., when we assume $\dot{\theta}_i = -\omega$), then the present example reduces to the trivial irrotational case discussed in section-7. That is, when condition (9.1) is satisfied, the pressure distribution is uniform over the intake section. And since the pressure is constrained by (9.15) to be constant along each pathline, the pressure is constant throughout the flow region. The momentum equation, (6.9), then reduces to $\dot{r}$ = constant, $\dot{z}$ = constant, and $\dot{\theta} = -\omega$ throughout the flow region. And Euler's turbine equation, (6.28), then

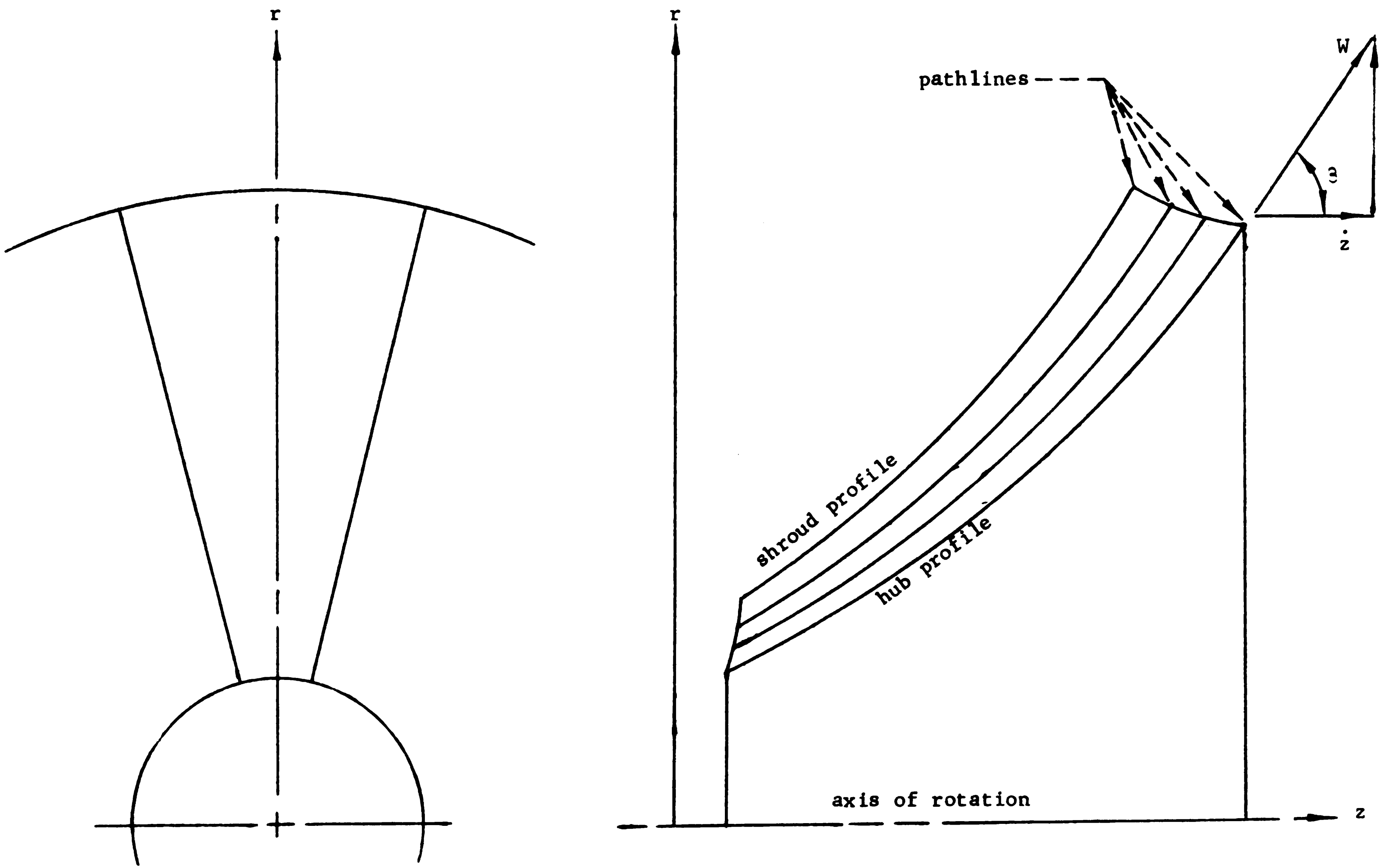

Figure 10.1

An Optimal Centrifugal Rotor Passage

reduces to $\dot{H} \equiv 0$, which is a trivial case.

For the purpose of demonstration, we shall assume that the fluid enters the rotor with zero velocity in the tangential direction, i.e., we assume $\dot{\theta}_i = 0$. The other intake conditions, ($\dot{r}_i$, $\dot{z}_i$, p_i, etc.), are assumed to be known but will not be assigned specific values. The discharge pressure, p_d, is constrained to equal the intake pressure, p_i, (see equation 9.15). Since the pathlines lie in the (r,z)-plane,

$$\dot{\theta} = \dot{\theta}_i = 0 \ . \tag{10.1}$$

In order to satisfy the "maximum kinetic energy constraint", (9.15), we let

$$\text{a)} \quad W^2 - W_i^2 = \omega^2 Q(z) \ , \tag{10.2a}$$

$$\text{b)} \quad r^2 - r_i^2 = Q(z) \ . \tag{10.2b}$$

Substituting (10.1) and (10.2) into (9.14) and (9.7) it is easily verified that

$$\frac{W\,dA}{W_i dA_i} = 1 \ , \tag{10.3}$$

and

$$H - H_i = \omega^2 Q(z) \ , \quad \text{or} \tag{10.4a}$$

$$H_d - H_i = \omega^2 Q(z_d) - \omega^2 Q(z_i) \ , \tag{10.4b}$$

where we require that $Q(z)$ be a monotonic increasing function which has the same value when evaluated between the intake and discharge points of each pathline. Equation (10.4b) then satisfies the "uniform energy increase constraint", (9.18).

Substituting (10.4b) and (10.2) into the "uniform discharge velocity constraint", (9.19), yields

$$V_d^2 = 2\omega^2[Q(z_d) - Q(z_i)] + V_i^2 . \tag{10.5}$$

From (10.5) we observe that V_d^2 is uniform when V_i^2 is uniform over the intake section. Expanding V_i^2 yields

$$V_i^2 = \dot{r}_i^2 + \dot{z}_i^2 + r_i^2\,\omega^2 . \tag{10.6}$$

or

$$\dot{r}_i = \sqrt{V_i^2 - \dot{z}_i^2 - r_i^2\,\omega^2} . \tag{10.7}$$

Letting V_i^2 = constant, the intake angle, β_i, of each pathline is

$$\beta_i \equiv \tan^{-1}\frac{\dot{r}_i}{\dot{z}_i} = \tan^{-1}\frac{\sqrt{V_i^2 - \dot{z}_i^2 - r_i^2\,\omega^2}}{\dot{z}_i} , \tag{10.8}$$

and the "uniform discharge velocity constraint", (9.19), is then satisfied at the design conditions, (ω, V_i, etc.).

We now seek an expression for the change in area, dA, (see equation 2.6b for a definition of dA). We assume that the flow may be represented by a family of pathlines, ψ, and orthogonal curves, ϕ, as shown in figure 10.2.

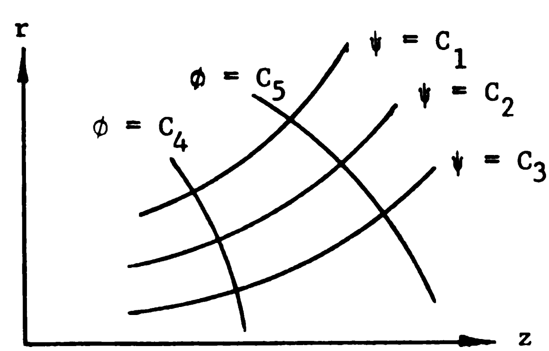

Figure 10.2

Flow Net

The equation of the pathlines is determined from (10.2b), i.e.,

$$\psi(r,z) = r^2 - Q(z) = r_i^2 = \text{constant} . \tag{10.9}$$

The slope of the pathlines is obtained by solving (10.9) for r and differentiating with respect to z, i.e.,

$$\left(\frac{dr}{dz}\right)_\psi = \frac{d}{dz}\left[\sqrt{r_i^2 + Q(Z)}\right]$$

$$= \frac{1/2\ dQ/dz}{\sqrt{r_i^2 + Q}} = \frac{Q'(z)}{2r} . \tag{10.10}$$

Since the ψ and ϕ curves are orthogonal,

$$\left(\frac{dr}{dz}\right)_\phi = -\left(\frac{dz}{dr}\right)_\psi = \frac{-2r}{Q'} , \tag{10.11}$$

where the subscripts ϕ and ψ denote the curve along which the differentiation is performed. The equation of the ϕ curves is found by integrating (10.11), i.e.,

$$r = -\int \frac{2r}{Q'}\,dz - \phi , \tag{10.12}$$

where ϕ is the constant of integration. Substituting (10.12) into (10.9) yields

$$\psi = \left[\phi + \int \frac{2r}{Q'(z)}\,dz\right]^2 - Q(z) . \tag{10.13}$$

Differentiating (10.13) with respect to z, and holding ϕ constant yields

$$\left(\frac{d\psi}{dz}\right)_\phi = 2\left[\phi + \int \frac{2r}{Q'}\,dz\right]\frac{2r}{Q'} - Q' . \tag{10.14}$$

Substituting (10.12) into (10.14) yields

$$\left(\frac{d\psi}{dz}\right)_\phi = \frac{-4r^2}{Q'} - Q' \,. \tag{10.15}$$

The differential change in area, dA, is equal to the differential change in arc length, ds_1, of the $\phi(r,z)$ = constant curve times the change in arc length, ds_2, in the r-θ plane as shown in figure 10.3.

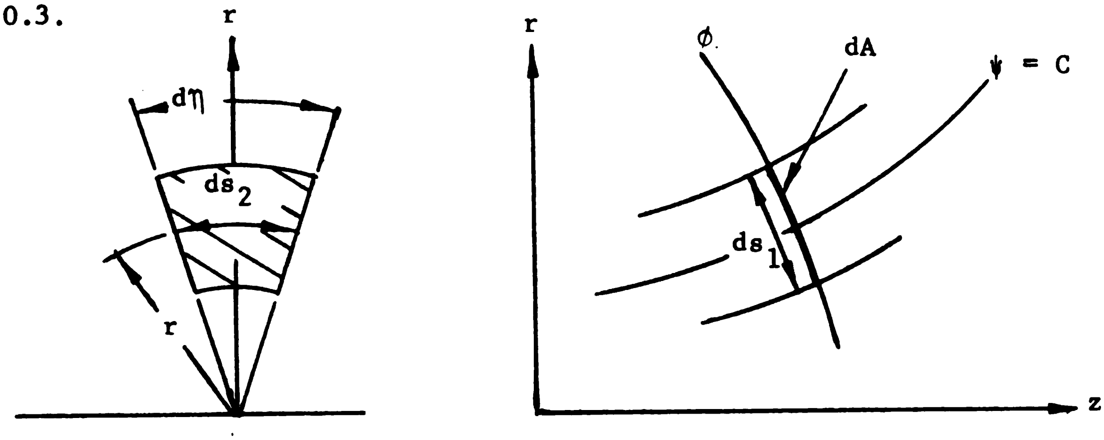

Figure 10.3

Area Increment

$$dA = ds_1 ds_2 = \pm\sqrt{1 + \left(\frac{dr}{dz}\right)_\phi^2}\; dz\; r\; d\eta \;, \text{ or} \tag{10.16a}$$

$$\frac{dA}{d\psi} = \pm\sqrt{1 + \left(\frac{dr}{dz}\right)_\phi^2}\;\left(\frac{dz}{d\psi}\right)_\phi\; r\; d\eta \,. \tag{10.16b}$$

Substituting (10.15) and (10.11) into the minus value of equation (10.16b) yields

$$\frac{dA}{d\psi} = \frac{-\sqrt{1 + 4r^2/(Q')^2}\; r\; d\eta}{-\left[Q' + 4r^2/Q'\right]} = \frac{r\; d\eta}{\sqrt{(Q')^2 + 4r^2}} \,. \tag{10.17}$$

Substituting (10.2b) into (10.17) yields

$$\frac{dA}{d\psi} = r\; d\eta\; \{[Q'(z)]^2 + 4r_i^2 + 4Q(z)\}^{-\frac{1}{2}} \,. \tag{10.18}$$

Substituting the derivative of (10.2b) with respect to z into (10.17) yields

$$\frac{dA}{d\psi} = r\ d\eta\ \{4r^2(\frac{dr}{dz})^2 + 4r^2\}^{-\frac{1}{2}}\ . \tag{10.19}$$

Substituting $\tan\beta \equiv \frac{dr}{dz}$ into (10.19) yields

$$\frac{dA}{d\psi} = r\ d\eta\ [4r^2(\tan^2\beta + 1)]^{-\frac{1}{2}} = \frac{d\eta}{2\sec\beta}\ . \tag{10.20}$$

Evaluating (10.20) at the intake point (i) and then dividing it into (10.18) yields

$$\frac{dA}{dA_i} = 2r\sec\beta_i\ \{[Q'(z)]^2 + 4r_i^2 + 4Q(z)\}^{-\frac{1}{2}}\ , \tag{10.21}$$

where $d\psi$ and $d\eta$ cancels with $d\psi_i$ and $d\eta_i$ respectively because they are constants along each pathline. Substituting (10.2a) into (10.3) and solving for dA/dA_i yields

$$\frac{dA}{dA_i} = \frac{W_i}{W} = \frac{W_i}{\sqrt{W_i^2 + \omega^2 Q}}\ . \tag{10.22}$$

Equating (10.22) and (10.21) yields

$$\frac{W_i}{\sqrt{W_i^2 + \omega^2 Q}} = \frac{2r\sec\beta_i}{[(Q')^2 + 4r_i^2 + 4Q]^{\frac{1}{2}}}\ . \tag{10.23}$$

Substituting $r = \sqrt{r_i^2 + Q}$ from (10.2b) into (10.23), then squaring both sides and solving for Q' yields

$$\frac{dQ}{dz} = \left[\frac{4\sec^2\beta_i}{W_i^2}(W_i^2 + \omega^2 Q)(r_i^2 + Q) - 4r_i^2 - 4Q\right]^{\frac{1}{2}}\ . \tag{10.24}$$

Multiplying (10.24) by $1/r_i$, and then separating variables and integrating yields

$$z = \frac{1}{r_i} \int \frac{dQ}{\sqrt{cQ^2 + bQ + a}} + \ell \ , \tag{10.25a}$$

where; $a = 4(\sec^2\beta_i - 1) = 4 \tan^2\beta_i$, (10.25b)

$$b = \frac{4 \sec^2\beta_i}{r_i^2 W_i^2} (W_i^2 + r_i^2 \omega^2) - \frac{4}{r_i^2} \ , \tag{10.25c}$$

$$c = \frac{4\omega^2 \sec^2\beta_i}{r_i^2 W_i^2} \ , \tag{10.25d}$$

ℓ = constant of integration . (10.25e)

Performing the integration of (10.25a) yields

$$z = \frac{1}{r_i \sqrt{c}} \ell n \left[\sqrt{cQ^2 + bQ + a} + Q\sqrt{c} + \frac{b}{2\sqrt{c}} \right] + \ell \ , \tag{10.26a}$$

where $c > 0$; or

$$z = \frac{1}{r_i \sqrt{c}} \sinh^{-1} \left[\frac{2cQ + b}{\sqrt{4ac - b^2}} \right] + \ell \ , \tag{10.26b}$$

where $4ac - b^2 > 0$. Only (10.26a) will be considered in detail. Substituting (10.2b) into (10.26a) yields

$$z = \frac{1}{r_i \sqrt{c}} \ell n \left[\sqrt{c(r^2 - r_i^2)^2 + b(r^2 - r_i^2) + a} + (r^2 - r_i^2)\sqrt{c} + \frac{b}{2\sqrt{c}} \right] + \ell . \tag{10.27}$$

Evaluating the above equation at (r_i, z_i) and (r_d, z_d) respectively yields

$$z_i = \frac{1}{r_i \sqrt{c}} \ell n \left[\sqrt{a} + \frac{b}{2\sqrt{c}} \right] + \ell \text{ , and} \tag{10.28}$$

$$z_d = \frac{1}{r_i \sqrt{c}} \ell n \left[\sqrt{c (Q_d)^2 + b\, Q_d + a} + Q_d c + \frac{b}{2\sqrt{c}} \right] + \ell, \tag{10.29}$$

where $Q_d = r_d^2 - r_i^2 = \frac{\Delta H}{\omega^2}$ = constant. Equation (10.29) determines the end point of each pathline so that the "uniform energy increase constraint", (9.18), is satisfied. The constant of integration, ℓ, is arbitrary and may be set equal to zero.

Equation (10.27) determines the equation of each pathline in the r-z plane. Notice that the constants, (a,b,c), vary from pathline to pathline. In order to satisfy boundary condition D of section-9, we must select the hub and shroud profiles of the centrifugal rotor to coincide with equation (10.27), see figure 10.1. Since the constants, b and c, in (10.27) depend on the design conditions, (ω and W_i), boundary condition D is, in general, satisfied only at one set of design conditions. This set of design conditions is the only operating point at which the fluid particle minimum principle is satisfied throughout the entire flow region.

Boundary condition C of section-9 is satisfied, when the discharge pressure, p_d, equals the intake pressure, p_i, i.e., when $p_d = p_i$ then

$$\frac{W_d dA_d}{W_i dA_i} = \left[\frac{p_d}{p_i} \right]^{-\frac{1}{K}} = 1 \text{ ,} \tag{10.30}$$

which is required by condition (10.3). In most flow situations the discharge pressure, p_d, will equal the pressure of the chamber into which the flow is being emitted. In these situations the pressure of the chamber must be controlled so that it is approximately equal to the discharge pressure, p_d. See Sharpiro [11] page 91 for a discussion of the effect of back pressure on flow through nozzles.

Although it is theoretically possible to satisfy all the constraints mentioned in this section, employment of all these constraints may result in an impractical rotor. If the above situation arises, the "uniform discharge velocity constraint", (10.8), may be omitted, or the condition that Q_d possess exactly the same value when evaluated between the intake and discharge points of each pathline may be relaxed.

In conclusion, we observe that the "maximum kinetic energy increase constraint", (9.15), and the "fluid particle minimum principle" are satisfied when the following five conditions are satisfied:

1. The rotor operates at the given design conditions, (ω, ΔH, β_i, W_i, etc.).
2. The hub and shroud profiles of the rotor conform to equation (10.27).
3. The discharge pressure, p_d, and intake pressure, p_i, of each pathline are equal.
4. Shocks are excluded from the flow region.
5. A flow region is selected from the family of pathlines, (10.27), which is free of conjugate points.

We also observe that it is possible to satisfy the "uniform energy increase constraint", (9.18), and the "uniform discharge velocity constraint", (9.19). The initial and end points of each pathline are partly determined by the employment of these constraints, (9.18) and (9.19).

The flow chart in figure 10.4 outlines the procedure for determining the optimal geometry of the internal flow passage for the centrifugal rotor discussed in this section. Figure 10.1 shows the geometry of the rotor which is determined by the set of design conditions listed below:

ω = 2000 RAD/SEC, V_i = 500 FT/SEC, $\dot{\theta}_i = 0$, $\Delta H = 4.16 \times 10^5$ FT^2/SEC^2.

The initial points of some representative pathlines are:

1	$r_i = 1.00''$	$\beta_i = 28^\circ$
2	$r_i = 1.166''$	$\beta_i = 29^\circ$
3	$r_i = 1.333''$	$\beta_i = 30^\circ$
4	$r_i = 1.50''$	$\beta_i = 31^\circ$

Because the parameters, b and c, depend on the operating conditions, ω and W_i, the family of pathlines, (10.27), will coincide with the hub and shroud profiles of the flow passage only at the design point, (ω, W_i). However, if the flow can be controlled so that ω = constant W_i, then the parameters, b and c, no longer depend on ω and W_i. The family of pathlines, (10.27), is then independent of all operating conditions, (p_i, T_i, W_i, ω), and the fluid particle minimum principle is satisfied at all operating points, (ω, W_i).

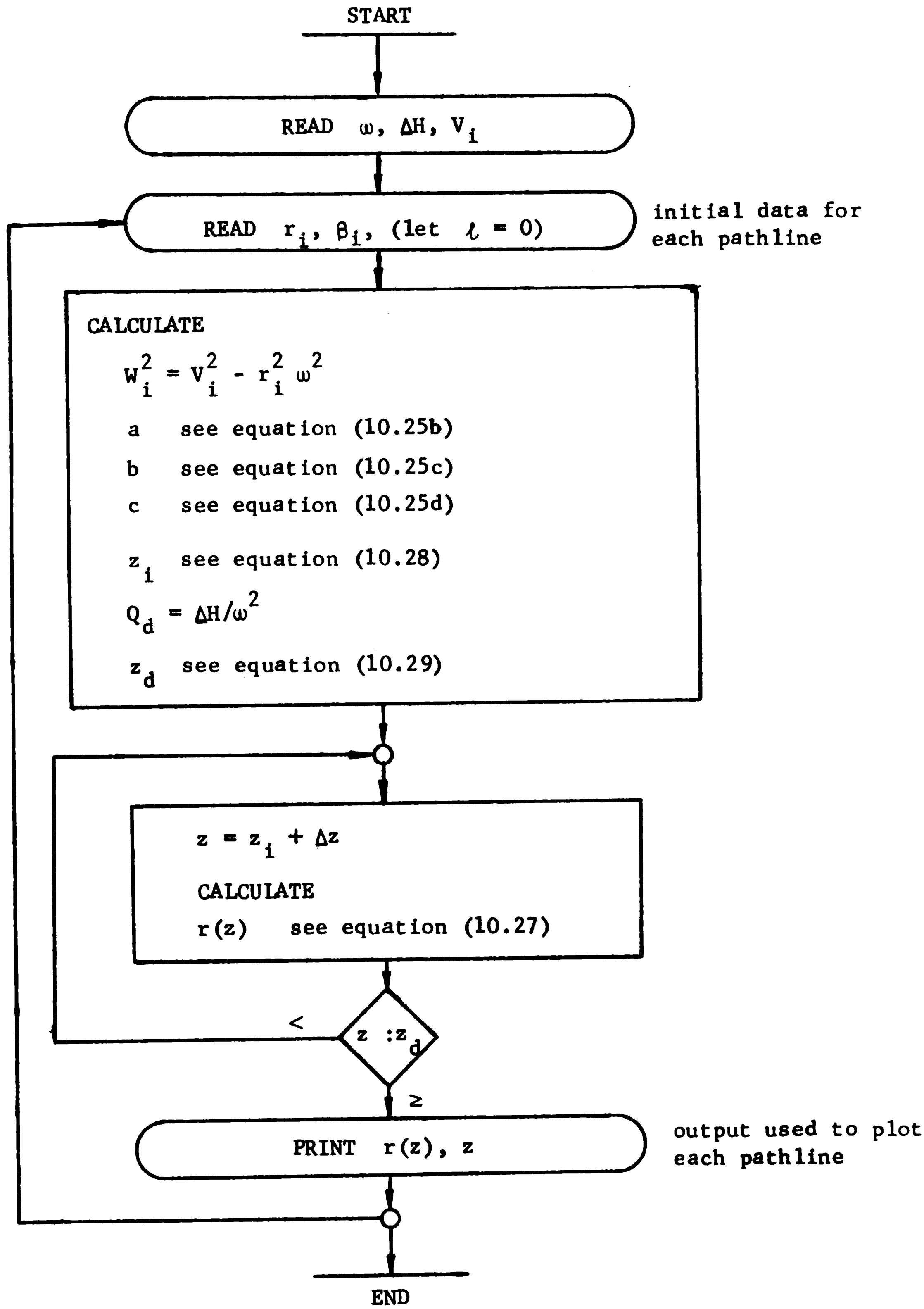

Figure 10.4

Flow Chart

11. AXIAL-FLOW ROTORS

In this section it is shown that it is possible to design "special axial-flow" rotors which satisfy the fluid particle minimum principle over a wide range of operating conditions. We shall impose the constraint,

$$r/r_i = 1 \ , \tag{11.1}$$

on the flows discussed in this section. This constraint requires each pathline to lie on a right cylindrical surface. The Coriolis acceleration is then normal to the flow as shown below

$$2\vec{\omega} \times \vec{W} = 2 \begin{vmatrix} \frac{1}{r}\hat{i}_r & \hat{i}_\theta & \frac{1}{r}\hat{i}_K \\ 0 & 0 & \omega \\ 0 & r^2\dot{\theta} & \dot{z} \end{vmatrix} = -2r\,\omega\,\dot{\theta}\,\hat{i}_r \ . \tag{11.2}$$

Thus both the Coriolis acceleration and the centrifugal acceleration, $-\omega^2 r\ \hat{i}_r$, are normal to the cylindrical surface containing the flow. We shall assume that the inertia forces and pressure gradient are in stable equilibrium in the r-direction. That is, we assume that the radial component of the momentum equation, (6.9a), is satisfied throughout the flow region. This condition is called the "radial equilibrium condition". The flow in each $(r\theta, z)$-cylindrical plane can then be treated as a two-dimensional flow which is independent of the radial component of the momentum equation, (6.9a). The fluid particle minimum principle, (6.22), then reduces to

$$\delta \int_a^b \left(\frac{W^2}{2} - h\right)dt + \left[(\nabla E_1 - \vec{W})\cdot\delta\vec{R}_1\right]_a + \left[(\nabla E_2 - \vec{W})\cdot\delta\vec{R}_1\right]_b = 0 , \qquad (11.3)$$

where $W^2 = (r\dot{\theta})^2 + \dot{z}^2$ and r = constant. The Euler-Lagrange equations corresponding to (11.3) are

$$r\ddot{\theta} + \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 , \qquad (11.4a)$$

$$\ddot{z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 . \qquad (11.4b)$$

And the "radial equilibrium condition" is

$$-r(\dot{\theta} + \omega)^2 + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 . \qquad (11.5)$$

Substituting r = constant, $\dot{\alpha} = \dot{\theta} + \omega$, and $\hat{i}_\alpha \equiv \hat{i}_\theta$ into the momentum equation, (6.9), yields (11.4) and (11.5). Thus we conclude that the Euler-Lagrange equations of the functional (11.3) plus the "radial equilibrium condition" are identical to the Euler-Lagrange equations of the functional (6.22). It is known from Variational Calculus that two functionals are equivalent when their respective Euler-Lagrange equations are identical. The fluid particle minimum principle (11.3) plus the constraint (11.5) is therefore equivalent to the fluid particle minimum principle (6.6). Thus we may operate in the relative reference frame using the fluid particle minimum principle, (11.3), and the constraints, (11.5) and (11.1). Since the inertia forces do not act in the $(r\theta,z)$-cylindrical plane (they act normal to the plane), the flow may be irrotational in each $(r\theta,z)$-plane. That is, the flow is irrotational when

$$\nabla \times \vec{W} = \begin{vmatrix} \frac{1}{r}\hat{i}_r & \hat{i}_\theta & \frac{1}{r}\hat{i}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2\dot{\theta} & \dot{z} \end{vmatrix} = \frac{1}{r}\hat{i}_r \left[\frac{\partial \dot{z}}{\partial \theta} - \frac{\partial(r^2\dot{\theta})}{\partial z}\right] = 0; \quad (11.6)$$

or the flow is irrotational in each $(r\theta, z)$-plane when

$$\frac{\partial \dot{z}}{\partial \theta} = r^2 \frac{\partial \dot{\theta}}{\partial z} \quad . \qquad (11.7)$$

However, the flow is still rotational as viewed from the fixed reference frame.

When the flow is irrotational, the fluid particle minimum principle, (11.3), reduces to the irrotational flow (strong minimum) problem (4.B) discussed in sections 4 and 5. In this case, it is possible to satisfy the fluid particle minimum principle over a wide range of operating conditions.

The traditional method of designing blades for axial-flow rotors is to first assume that the flow is two-dimensional on each $(r\theta, z)$-cylindrical surface. Next, the "radial equilibrium condition" is satisfied. Then the classical incompressible potential flow theory (or some other method) is employed to map the flow through a cascade of blades, see Vavra [13] page 312. Thus we can conclude that the assumptions employed in the traditional method are often equivalent to the constraints employed in the optimization procedure.

12. CONCLUDING REMARKS

The present optimization procedure predicts the "minimal energy configuration" of the flow field, inside a rotating passage, only when the fluid particle minimum principle is satisfied. When the optimization procedure is employed, the intake conditions of the fluid, inside the rotor, must be accurately known. The intake conditions may be difficult to determine in practice. The present work does not discuss how the intake conditions may be determined.

In order to design a practical rotor, the present optimization procedure as demonstrated in section-10, must be employed in conjunction with "other analytical methods". The author suggests the following iteration procedure for incorporating the present optimization procedure into a design program. The design conditions (W_i, ω, r_i, etc.) may be treated as unknown parameters each of which is restricted to lie within a specified range. Employment of the optimization procedure then yields the equation of a family of pathlines, which depends on the parameters, (W_i, ω, r_i, etc.). For example, the constants, (ΔH, ω, V_i, β_i, r_i), in the problem discussed in section-10 could have been treated as unknown parameters. Then, the design problem is to determine a passage geometry which coincides with one of the set of pathlines determined by the optimization procedure and which also appears to be a reasonable geometry based on the "other analytical methods" (i.e., based on the boundary-layer analysis, off-design analysis, etc.). This would involve an iteration procedure in which the "other analytical methods" are employed for each set of parameters, (ΔH, ω, V_i, β_i, r_i).

This work, in general, agrees with the well known equations and assumptions traditionally employed in turbomachinery design work. However, we will now discuss a few points in the present work which deviate from the traditional procedures. Often in turbomachinery design procedures, the one-dimensional compressible flow theory, as described in reference [11], is employed to investigate the flow inside the rotor. As mentioned in section-5, the one-dimensional theory becomes increasingly inaccurate as the curvature of the streamlines increase. Since the absolute streamlines inside a rotor are usually curved lines, this procedure may yield inaccurate results. A more accurate procedure is to employ the three-dimensional equations of section-8 in the investigation of the flow inside the rotor.

The employment of "optimal constraints" in the present work also deviates from the traditional procedures employed in turbomachinery design work. Any constraint may be imposed on the flow as long as the fluid particle minimum principle is satisfied. Thus if we impose the constraint, that the flow is two-dimensional, and then discover that the fluid particle minimum principle is not satisfied, we must conclude that the flow will not be two-dimensional. That is, we can only force (or constrain) a flow to be two-dimensional when the fluid particle minimum principle is satisfied. The traditional methods employed in turbomachinery work often contain simplifying assumptions, such as the assumption, that the flow is approximately two-dimensional. In the present optimization procedure the simplifying assumptions are often replaced by the "optimal constraints".

It was shown in sections 4 and 5 that all (irrotational) potential flow problems satisfy the fluid particle minimum principle. However, for the case of flow inside a rotating passage, the fluid particle minimum principle is seldom satisfied. In general, the flow inside a (rigid geometry) rotating passage will satisfy the fluid particle minimum principle at no more than one set of design conditions. The "maximum speed range" constraint, (9.16), and the special-rotors discussed in sections 10 and 11 represent cases in which it is, theoretically, possible to satisfy the fluid particle minimum principle over a wide range of operating conditions.

APPENDIX-A

In this section it will be shown that the energy equation, (4.17b), is equivalent to the First Law of Thermodynamics for the case of steady isentropic flow. Consider a fluid particle, i.e., an infinitesimal closed system, moving at velocity, $\vec{V} \equiv \dot{\vec{R}}$, along a pathline. The First Law of Thermodynamics for a moving closed system and for steady isentropic conditions is

$$0 = \frac{dU}{dt} + \frac{d}{dt}(K.E.) + \frac{dG^*}{dt} + \frac{dw}{dt} , \tag{A.1}$$

where: $U = mu$ internal energy of system,

$K.E. = m\frac{1}{2}V^2$ kinetic energy of system,

$G^* = mG$ potential energy of system,

w = work injected into system,

m = mass of system.

Substituting the definition of the mass of the fluid particle,

$$m \equiv \int_{\mathcal{V}} \rho d\mathcal{V} , \tag{A.2}$$

into(A.1)yields

$$0 = \frac{dw}{dt} + \frac{d}{dt}\int_{\mathcal{V}} \rho\left[u + \frac{V^2}{2} + G\right] d\mathcal{V} . \tag{A.3}$$

The thermodynamic reversible compression work for a closed system is [7]

$$w = \int_{\mathcal{V}} pd\mathcal{V} . \tag{A.4}$$

Substituting (A.4) into (A.3) and rearranging terms yields

$$0 = \frac{d}{dt} \int_{\mathcal{V}} \left[\frac{p}{\rho} + u + \frac{V^2}{2} + G \right] \rho d\mathcal{V} \ . \tag{A.5}$$

Since the size of the volume, $\mathcal{V}$, is arbitrary, the integrand of (A.5) must vanish everywhere in the flow region. Observing that the element of mass, $\rho d\mathcal{V}$, is a constant, (A.5) becomes

$$\frac{d}{dt} \left[\frac{p}{\rho} + u + \frac{V^2}{2} + G \right] = 0 \ . \tag{A.6}$$

Multiplying (A.6) by dt and integrating with respect to time yields

$$\frac{p}{\rho} + u + \frac{V^2}{2} + G = H \ , \tag{A.7a}$$

or

$$C_p T + \frac{(\dot{\vec{R}})^2}{2} + G = H = \text{constant along each pathline} \ , \tag{A.7b}$$

where $C_p T \equiv h \equiv u + \frac{p}{\rho}$ and $V \equiv \dot{R}$. The constant of integration, H, is a constant along each pathline because equation (A.1) applies to one fluid particle which is traveling along one pathline. Equation (A.7b) is identical to the energy equation, (4.17b).

APPENDIX B

In this section a brief review of a "field of a functional" is presented. All the definitions and theorems listed in this section are taken from reference [2] chapters 5 and 6. Consider a system of second order differential equations (such as the Euler-Lagrange equations)

$$\ddot{y}_i = f_i[y_i(t),\ y_2(t),\ y_3(t)],\ (i = 1,2,3)\ . \tag{B.1}$$

In order to single out a definite solution of this system, we have to specify six boundary conditions of the form

$$\dot{y}_i = \psi_i[y_1(t),\ y_2(t),\ y_3(t)],\ (i = 1,2,3) \tag{B.2}$$

for two values of time, t. The family of boundary conditions, (B.2), is called a field (of directions) for the given system (B.1) when equation (B.2) holds at all values of time, t. A necessary condition for (B.2) to be a field of a functional is that (B.2) must first be a field of the system of the Euler-Lagrange equations of the functional.

Theorem-B-1. A necessary and sufficient condition for the family of directions, (B.2), to be a field of the functional,

$$\int_{t_1}^{t_2} F(\vec{y},\ \dot{\vec{y}})dt\ , \tag{B.3}$$

is that the self-adjointness conditions (this is the irrotational condition in our application),

$$\frac{\partial P_i}{\partial y_k} = \frac{\partial P_k}{\partial y_i} \qquad \begin{matrix}(i = 1,2,3)\\(k = 1,2,3)\end{matrix} \quad , \tag{B.4}$$

and the consistency conditions (this is the momentum or Euler-Lagrange equation in our application),

$$\frac{\partial P_i}{\partial t} = \frac{\partial H}{\partial y_i} \quad (i = 1,2,3) \ , \tag{B.5}$$

be satisfied at every point t in $[t_1,t_2]$, where P_i is the "momenta" (it is the velocity in our application) defined as

$$P_i = F_{\dot{y}_i} \ , \tag{B.6}$$

and H is the Hamiltonian function (it is the total enthalpy plus a constant in our application) is defined as

$$H = \sum_{i=1}^{3} P_i \, \dot{y}_i - F + \text{constant} \ . \tag{B.7}$$

Theorem-B-2. The expression

$$\frac{\partial P_i}{\partial y_k} - \frac{\partial P_k}{\partial y_i} \qquad \begin{matrix}(i = 1,2,3)\\(k = 1,2,3)\end{matrix} \tag{B.8}$$

(this is the vorticity in our application) has a constant value along each extremal (i.e., along each pathline).

The self-adjointness condition (or irrotational condition), (B.4), implies that there exists a potential function (a velocity potential function in our application), g, such that

$$\frac{\partial g}{\partial y_i} = P_i \ . \tag{B.9}$$

Theorem-B-3. The boundary conditions (B.2) defined by (B.9) are a field of the Euler-Lagrange equations if and only if the potential function, g, satisfies the Hamilton-Jacobi equation (this is the energy equation in our application)

$$\frac{\partial g}{\partial t} + H(\vec{y}, \nabla g) = 0 . \tag{B.10}$$

We observe that the Hamilton-Jacobi equation, (B.10), and the self-adjointness conditions, (B.4), (i.e., the energy equation and irrotational condition) require that the integrand, Fdt, have an exact differential, dg. That is, since

$$\frac{\partial g}{\partial t} = -H \quad \text{and} \quad \frac{\partial g}{\partial y_i} = P_i \tag{B.11}$$

then

$$Fdt = -Hdt + \Sigma\, P_i dy_i = \frac{\partial g}{\partial t}\, dt + \sum_{i=1}^{3} \frac{\partial g}{\partial y_i}\, dy_i = dg . \tag{B.12}$$

The above equation forms the basis for Hilbert's Invariant Theorem which is formally stated in the below manner.

Theorem-B-4. Given a field of directions (B.2) of the Euler-Lagrange equation, the directions (B.2) define a field of the functional

$$\int_{t_1}^{t_2} F\, dt \tag{B.13}$$

if the Hilbert integral

$$\int_c \left[F - \sum_{i=1}^{3} \psi_i\, F_{\dot{y}_i} \right] dt + \sum_{i=1}^{3} F_{\dot{y}_i}\, dy_i \tag{B.14}$$

depends only on the end points of the curve along which it is taken and not on the curve itself. If the curve c along which the integral, (B.14), is evaluated is one of the extremals (path-lines) of the field, then

$$dy_i = \psi_i \, dt \tag{B.15}$$

along c, and hence (B.14) reduces to

$$\int_c F \, dt \; . \tag{B.16}$$

When the conditions (B.10) and (B.4) hold (i.e., when the flow is irrotational and the energy equation holds), then (B.12) may be substituted into (B.16) which yields

$$\int_c F \, dt = \int_c \frac{dg}{dt} \, dt = g_2 - g_1 \; , \tag{B.17}$$

which is independent of the path of integration.

When the potential function, g, is known, it may be used as the boundary conditions of the variable end point functional, (3.1). That is, equating (B.9) and (B.6) yields

$$g_{y_i} = F_{\dot{y}_i} \; , \tag{B.18}$$

which when substituted into the boundary conditions, (3.3), yields

$$\frac{\partial}{\partial y_i} (E_1 - g) = 0, \; \frac{\partial}{\partial y_i} (E_2 - g) = 0 \; , \quad \text{or}$$

$$E_1 = E_2 = g \; , \tag{B.19}$$

at all values of time, t.

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