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ABSTRACT

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The validity of many Limit Theorems of Probability Theory is intimately connected with the geometry of the underlying Banach Space. This is especially true of the Strong Law of Large Numbers (SLLN). In this thesis, Cotype q Banach Spaces are characterized as those in which a certain condition is necessary for the SLLN to hold. Also, Logtype p spaces are characterized as those in which another condition is sufficient for its validity. The best results are obtained for a Hilbert Space. The results on the SLLN are formulated in terms of the validity of a SLLN for real-valued random variables, necessary and sufficient conditions for which have been obtained by Nagaev. It is shown that the above results are the best of their kind.

In addition, Laws of the Iterated Logarithm are proved for certain classes of random variables taking values in an arbitrary separable Banach Space.

To the memory of my father.

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

We will consider a sequence $\{X_n\}_{n=1}^{\infty}$ of symmetric and independent (but not necessarily identically distributed) random variables defined on some probability space $(\Omega', \underline{F}', P')$ and taking values in a real separable Banach space B equipped with the norm $\| \cdot \|$. Recall that a vector valued random variable X is said to be symmetric if the probability distributions of X and -X are the same.

Consider also a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of independent random variables each assuming the values +1 and -1 with probability 1/2. Such a sequence is called a Rademacher sequence. We will assume throughout that the sequence $\{\varepsilon_n\}$ is defined on another probability space $(\Omega'', \underline{F}'', P'')$ and is independent of the sequence $\{X_n\}$. We will often consider the sequence $\{\varepsilon_n X_n\}_{n=1}^{\infty}$ defined on the product space $(\Omega, \underline{F}, P) = (\Omega' \times \Omega'', \underline{F}' \times \underline{F}'', P' \times P'')$.

Let $S_n = \sum_{j=1}^{n} X_j$, $s_n^2 = E\|S_n\|^2$ and denote the set of integers $\{2^n+1,\ldots,2^{n+1}\}$ by $I(n)$. The function $LL(\cdot)$ is defined by $LLx = \max(1,\log(\log x))$. Throughout, C will denote a generic constant whose value will usually be unspecified.

We need to introduce notation and terminology from Probability Limit Theory on the one hand and from Banach Space Theory on the other. Let us first examine the probabilistic side of the coin.

We shall say that the sequence $\{X_n\}$ satisfies

(a) <u>The Strong Law of Large Numbers</u>

($\{X_n\} \in$ SLLN) if $\lim_{n\to\infty} \|S_n\|/n = 0$ a.s.

(b) <u>The Weak Law of Large Numbers</u>

($\{X_n\} \in$ WLLN) if $\|S_n\|/n \to 0$ in probability as $n \to \infty$.

(c) <u>The Bounded Law of the Iterated Logarithm</u>

($\{X_n\} \in$ BLIL) if $\limsup_{n\to\infty} \|S_n\|/(2s_n^2 \, LLs_n^2)^{1/2} < \infty$ a.s.

(By Kolmogorov's zero or one law, $\{X_n\} \in$ BLIL iff there exists a constant $\Lambda \in [0,\infty)$ such that

$$\limsup_{n\to\infty} \|S_n\|/(2s_n^2 \, LLs_n^2)^{1/2} = \Lambda \text{ a.s.}).$$

and

(d) <u>The Compact Law of the Iterated Logarithm</u>

($\{X_n\} \in$ CLIL) if there exists a non-random, compact, symmetric and convex set $D \subset B$ such that

$$P(d(S_n/(2s_n^2 LLs_n^2)^{1/2}, D) \to 0) = 1 \tag{1.1}$$

and

$$P(C\{S_n/(2s_n^2 \, LLs_n^2)^{1/2}\} = D) = 1 \tag{1.2}$$

Here, $d(x,A) = \inf_{y \varepsilon A} \|x-y\|$ and $C(A_n(w))$ denotes the set of all cluster points of the (random) sequence $\{A_n\}$. The set D in the CLIL is called the "limit set".

The above definitions of the BLIL and CLIL differ from the original definitions of Kuelbs [22,24] who defines $s_n^2 = \sum_{j=1}^{n} E\|X_j\|^2$. We shall see that the two formulations are similar if B is a Hilbert space, or more generally, a <u>type-2</u> space.

Suppose that $\{X_n\} \varepsilon$ SLLN. It follows from the triangle inequality that $\lim_{n\to\infty} \|X_n\|/n = o$ a.s. This trivial necessary condition for the SLLN shows that one may, without loss of generality, assume that $\|X_n\| = o(n)$ a.s. while proving strong laws.

Since $\{X_n\}$ is a sequence of symmetrically distributed random variables, it is easy to see that $\{X_n\}$ and $\{\varepsilon_n X_n\}$ are equidistributed. It follows that $\{X_n\} \varepsilon$ SLLN iff $\{\varepsilon_n X_n\} \varepsilon$ SLLN. An application of Fubini's theorem shows that $\{\varepsilon_n X_n\} \varepsilon$ SLLN iff $\{\varepsilon_n(\cdot)X_n(\omega)\} \varepsilon$ SLLN for almost all $\omega \varepsilon \Omega$. This elementary device of Kahane [21] will be used repeatedly in what follows. We shall denote $X_n(\omega)$ (for a fixed $\omega \varepsilon \Omega$) by x_n whenever there is no possibility of confusion.

Let us next consider some basic notions from Banach Space Theory. Let B be an arbitrary real separable Banach space. For $p \geq 1$ we will denote by $L^p(B)$ the equivalence class of B-valued random variables X with $\|X\|_p = (E\|X\|^p)^{1/p} < \infty$. $L^o(B)$ will denote the equivalence class of

B-valued random variables, with distance $d_0(X,Y) = E(\|X-Y\|/1+\|X-Y\|)$.

Then $L^p(B)$ is a Banach space for $p \geq 1$, while $L^0(B)$ is a Fréchet space.

A B-valued random variable X is said to be <u>Gaussian</u> if $(f_1(X),\ldots,f_n(X))$ has an n-dimensional normal distribution for each $f_1,\ldots,f_n \in B^*$, $n \geq 1$.

The notions of <u>type</u> and <u>cotype</u> are fundamental to Banach space theory and were formulated by Maurey, Hoffmann-Jorgensen and Pisier in a series of papers in the early and mid seventies (see, for example [19, 33, 34, 38]. To motivate the definitions, let us begin with the parallelogram law in Hilbert spaces:

$$\|x_1+x_2\|^2/2 + \|x_1-x_2\|^2/2 = \|x_1\|^2 + \|x_2\|^2 (x_1,x_2 \in H)$$

which can be rephrased probabilistically as

$$E\|\varepsilon_1x_1+\varepsilon_2x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 \tag{1.3}$$

and thus by induction on n as

$$E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2 = \sum_{j=1}^{n} \|x_j\|^2 \tag{1.4}$$

We now generalize (1.4) by replacing the squares by p^{th} moments and the equality by an inequality: A Banach space B is said to be of <u>type</u> p ($1 \leq p \leq 2$) if there is a constant $A_p = A_p(B) \in (0,\infty)$ such that for any finite sequence $\{x_j\}_{j=1}^n$ in B,

$$E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \leq A_p \sum_{j=1}^{n} \|x_j\|^p.$$

Each Banach space is trivially of type 1, and it can be shown using the Kolmogorov three series theorem (and an alternative definition of the type of a space) that no non-trivial Banach space can be of type p for $p > 2$. A Banach space of type p is automatically of type p' for each $p' < p$, so that we may talk of $\{p \mid B \text{ is of type } p\}$ which is an interval that need not, in general, be closed above. In particular, there exist spaces of type p' for each $p' < p$ that are not of type p. (See Pisier [41] for examples of such spaces.) Among the classical Banach spaces, $C[0,1]$, c_0, L^∞ and spaces of measures are of type 1 (and no better) and the L^p spaces $(1 \leq p < \infty)$ are of type $\min(2,p)$ (and no better).

Hoffmann-Jorgensen and Pisier [19] proved an important result connecting the two sides of the aforementioned coin. They showed that a Banach space B is of type p $(1 \leq p \leq 2)$ iff each sequence $\{X_n\}$ of independent, zero mean, p-integrable B-valued random variables satisfying the condition $\sum_{j=1}^{\infty} E\|X_j\|^p/j^p < \infty$ also satisfied the SLLN. Their SLLN improved previous strong laws of Beck [3,4] and Woyczynski [47], just as the notion of type generalized previous notions of B-convexity and G_α spaces due to Beck [3,4] and Mourier and Woyczynski [35,47] respectively.

Let us turn next to the definition of cotype. A Banach space B is said to be of <u>cotype q</u> $(2 \leq q < \infty)$ if there exists a constant $A_q = A_q(B)$ $\varepsilon\ (0,\infty)$ such that for each finite sequence $\{x_j\}_{j=1}^n$ in B,

$$E\left\|\sum_{j=1}^{n} \varepsilon_j x_j\right\|^q \geq A_q \sum_{j=1}^{n} \|x_j\|^q$$

A cotype ∞ space is one in which

$$\sup_{\{\varepsilon_j\}\varepsilon\{-1,1\}^n} \|\sum_{j=1}^{n} \varepsilon_j x_j\| \geq A_\infty \sup_{1\leq j\leq n} \|x_j\|$$

for some constant A_∞.

Each Banach space is trivially of cotype ∞ and it can be shown that a non-trivial Banach space cannot be of cotype q for $q < 2$. A space of cotype q is also of cotype q' for $q' > q$, but $\{q|B \text{ is of cotype } q\}$ need not be closed below. Recently, Ledoux has generalized the examples of Pisier to construct spaces of type $2 - \varepsilon$ and cotype $2 + \varepsilon$ (for each $\varepsilon > 0$) that are not of type 2 (or cotype 2). The spaces C[0,1], c_0 and L^∞ are, predictably, of cotype ∞; the L^p spaces are of cotype max(2,p) (and no better).

While a considerable amount of research has been done relating the cotype of a space to the validity of a central limit theorem in that space, such a link has not, to the best of my knowledge, been made for the Strong Law of Large Numbers. We deal, in Chapter II, with this question.

Kwapien [26] proved that a Banach space is both of type 2 and of cotype 2 iff it is isomorphic to a Hilbert space.

We need to define another class of Banach spaces: B is said to be of <u>$(LLn)^{p-1}$-type p</u> or simply of <u>logtype p</u> ($1 \leq p \leq 2$) if there exists a constant $A_p^* = A_p^*(B)$ such that for each finite sequence $\{x_j\}_{j=1}^n$ in B,

$$E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \leq A_p^*(LLn)^{p-1} \sum_{j=1}^{n} \|x_j\|^p$$

These spaces have been studied before: Pisier [41] characterized logtype-2 spaces (he did not call them that) as those in which each sequence $\{X_n\}_{n=1}^{\infty}$ of i.i.d. random variables with $EX = 0$ and $E\|X\|^2 < \infty$ obeyed the CLIL. He also showed (Lemma 4 in [39]) that a logtype p space is of type r for each $r < p$.

We shall study the relationship between the geometry of the underlying Banach space (as manifested in its cotype, type or logtype) and the validity of a SLLN for independent symmetric random variables taking values in that space. We study necessary conditions for the SLLN in Chapter II and sufficient conditions in Chapter III (Chapter IV deals with the BLIL for certain classes of B-valued random variables; no link is made with the geometry of B). Most of the results on the SLLN are expressed in terms of the validity of a real valued SLLN, and would not be of much use unless one could find necessary and sufficient criteria for the validity of the latter. Such criteria were obtained by Nagaev [36] and later generalized by him and Volodin [37,46] to cover the case of an arbitrary stabilizing sequence $\{b_n\}$ ($b_n \to \infty$). For completeness, let us state the basic result of Nagaev.

Theorem 1.1 (Nagaev [36])

Let $\{X_n\}$ be a sequence of independent, symmetric real valued random variables. Then $\{X_n\} \in$ SLLN iff for each $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n| > \varepsilon n) < \infty \tag{1.5}$$

and

$$(1.6) \qquad \sum_{n=1}^{\infty} \exp(-\varepsilon h_n(\varepsilon) 2^{n+1}) < \infty \ (\varepsilon > 0).$$

Here $f_j(h,\varepsilon) = E[\exp(hX_j)]I(|X_j| \leq j\ \varepsilon)$ and $h_n(\varepsilon)$ is the solution of the differential equation

$$\psi_n(h,\varepsilon) = \sum_{j \varepsilon I(n)} [\frac{d}{dh} f_j(h,\varepsilon)]/f_j(h,\varepsilon) = \varepsilon 2^{n+1}$$

provided

$$\sup_h \psi_n(h,\varepsilon) \geq \varepsilon 2^{n+1}.$$

Otherwise, $h_n(\varepsilon) = \infty$. $h_n(\varepsilon)$ is well defined by the monotonicity of $\psi_n(h,\varepsilon)$ in h.

The conditions of Theorem 1.1 are, to say the least, complicated. This is only to be expected. The problem of finding necessary and sufficient conditions for the SLLN is a long-standing one (see Chung [8] for a discussion of the problems involved). Moreover, Prokhorov [42] expressed the belief that criteria in terms of the moments of the individual summands were probably impossible. Nagaev proved that this was indeed the case by exhibiting two sequences $\{X_n\}$ and $\{Y_n\}$ having the same moments up to any given order $s < \infty$ but such that $\{X_n\} \varepsilon$ SLLN and $\{Y_n\} \not\varepsilon$ SLLN.

While (1.6) is complicated, it can certainly be verified. Moreover, we shall see that the real-valued SLLN's that do arise can often be verified or disproved by other relatively simple means (such as by direct calculation). The utility of our results should not, therefore, be gauged by the fact that they might be hard to verify, but rather by the fact that they often yield a conclusion when all other SLLN's are inconclusive.

It should be pointed out that we will need to verify the generalizations of Theorem 1.1 (Nagaev and Volodin) rather than Theorem 1.1 itself. Also, Nagaev's conditions may be reexpressed more simply in terms of a standard minimization in Markov's inequality. See [36] for details.

In Chapter II, we consider necessary conditions for $\{X_n\}$ to satisfy the SLLN. We show (Theorem 2.1) that cotype q spaces ($2 \leq q < \infty$) are precisely those in which the condition $1/n^q \sum_{j=1}^{n} \|X_j\|^q \to 0$ a.s. is a necessary condition for the SLLN to hold, for each independent symmetric sequence $\{X_n\}$. We also show that the above necessary condition can be expressed in terms of the individual moments if $\|X_n\| \leq Cn$ a.s. ($n \geq 1$). Examples are given to show that the necessary condition is the best of its kind.

Sufficient conditions for the validity of a B-valued SLLN have been studied by Beck [3,4], Beck, Giesy and Warren [5], Woyczynski [47], Hoffmann-Jorgensen and Pisier [19], Kuelbs and Zinn [25] and Heinkel [15,16,17]. We first obtain an exponential inequality for Rademacher sequences in B and use it to prove a SLLN for random variables in type p spaces ($1 \leq p \leq 2$). This result is improved in Theorem 3.14, which characterizes logtype p spaces as those in which each independent symmetric sequence $\{X_n\}$ satisfying $(\mathrm{LLn})^{p-1}/n^p \sum_{j=1}^{n} \|X_j\|^p \to 0$ a.s. also satisfies the SLLN. Examples are given to show that the above result is best of its kind and that it may be used in situations where all other relevant SLLN's are inconclusive. Furthermore, the sufficient conditions of Theorem 3.14 may be expressed in terms of the individual moments if $\|X_n\| \le Cn/LLn$ a.s. $(n \ge 1)$.

For B = H, a Hilbert space, the necessary and sufficient conditions almost coincide: $\{X_n\} \in$ SLLN if $LLn/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. and only if $1/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. For B = H we also prove (Theorem 3.22) an extension of the Hoffmann-Jorgensen and Pisier Theorem.

We do not consider non-symmetric random variables, but they may easily be studied using an elementary result of Kuelbs and Zinn [25] which states that $\{X_n\} \in$ SLLN iff $\{X_n^s\} \in$ SLLN and $\{X_n\} \in$ WLLN. Here, X_n^s is the symmetrized version of X_n and is defined by

$$X_n^s(\omega,\eta) = X_n(\omega) - X_n'(\eta)$$

where X_n' is an independent copy of $X_n (n \ge 1)$. A symmetrized version must always exist, at least on the probability space $(\Omega \times \Omega, \underline{F} \times \underline{F}, P \times P)$.

In Chapter IV we treat the Bounded law of the Iterated Logarithm. A BLIL is proved for B-valued Rademacher sequences (Theorem 4.3). This is related to a theorem of Kuelbs [24]. Similarly, a BLIL is proved for independent Gaussian sequences using an inequality of Fernique [11,12]. This result is related to a theorem of Carmona and Kono [6].

CHAPTER II

NECESSARY CONDITIONS FOR THE SLLN

The following is the main result of Chapter II.

Theorem 2.1. The following are equivalent:

(2.1) B is of cotype q $(2 \leq q < \infty)$

(2.2) Each sequence $\{X_n\}$ of independent symmetric B-valued random variables satisfying the SLLN also satisfies $\sum_{j=1}^{n} \|X_j\|^q/n^q \to 0$ a.s.

Proof: We will first show that (2.1) implies (2.2). Assume that (2.1) holds and let $\{X_n\}$ be any independent symmetric sequence satisfying the SLLN. It follows that $\{\varepsilon_n x_n\} \in$ SLLN for almost all $\omega \in \Omega$. We need to prove that $\sum_{j=1}^{n} \|x_j\|/n^q \to 0$.

Kahane [21] proved that a Rademacher series $\sum_{j=1}^{\infty} \varepsilon_j x_j$ that converges in probability also satisfies $E\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|^p < \infty$ for each $p > 0$. Motivated by this result, we define the Fréchet space $(E,\|\cdot\|_E)$ and the Banach space $(F,\|\cdot\|_F)$ by

$$E = \{x = (x_1,x_2,\ldots.) \in B^\infty : \sum_{j=1}^{\infty} \varepsilon_j x_j \text{ converges in probability}\}$$

$$d_E(x,y) = E(\|\sum_{j=1}^{\infty} \varepsilon_j(x_j-y_j)\|/[1 + \|\sum_{j=1}^{\infty} \varepsilon_j(x_j-y_j)\|])$$

$$F = \{x=(x_1,x_2,\ldots) \in B^\infty : E\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|^q < \infty\}$$

$$\|x\|_F = (E\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|^q)^{1/q}.$$

Kahane's theorem implies that $E \subset F$. The injection $E \to F$ is clearly a linear operator with closed graph (since the probability and L^q limits of a sequence must coincide) and is therefore continuous (in particular at the origin)(see Theorem 4, page 57 of Dunford and Schwartz [50]). It follows that for each $\varepsilon > 0$, there is a $\delta > 0$ such that $(E\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|^q)^{1/q}$ $< \varepsilon$ whenever $E(\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|/(1+\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|)) < \delta$. Since $\{\varepsilon_n x_n\}$ satisfies the WLLN by hypothesis, we have $(1/n\|\sum_{j=1}^{n} \varepsilon_j x_j\|)/(1+1/n\|\sum_{j=1}^{n}\varepsilon_j x_j\|) \to 0$ in probability. The bounded convergence theorem now proves that

$E[(1/n\|\sum_{j=1}^{n} \varepsilon_j x_j\|)/(1+1/n\|\sum_{j=1}^{n} \varepsilon_j x_j\|)] < \delta(n \geq N)$ so that

$1/n(E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^q)^{1/q} < \varepsilon(n \geq N)$. By the cotype q inequality,

$\sum_{j=1}^{n} \|x_j\|^q/n^q \to 0$, as asserted.

Conversely, suppose that $\{x_n\} \subset B$, and define the sequence $\{X_n\}$ by $X_n = n\,\varepsilon_n x_n$ $(n \geq 1)$. By assumption, $\sum_{j=1}^{n} j^q\|x_j\|^q/n^q \to 0$ if $\sum_{j=1}^{n} j\,\varepsilon_j x_j/n \to 0$ a.s. and thus (by Kronecker's Lemma, which is valid in any Banach space) if the series $\sum_{j=1}^{\infty} \varepsilon_j x_j$ converges almost surely. Kahane [21] and Ito and Nisio [20] have shown that the a.s. convergence $\sum_{j=1}^{\infty} \varepsilon_j x_j$ is a consequence of its convergence in $L^q(B)$. With this in mind, we define the Banach spaces $(E,\|\cdot\|_E)$ and $F,\|\cdot\|_F)$ by

$$E = \{x=(x_1,x_2,\dots) \in B^\infty : \sum_{j=1}^{\infty} \varepsilon_j x_j \text{ converges in } L^q(B)\}$$

$$\|x\|_E = (E\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|^q)^{1/q}$$

$$F = \{x=(x_1 x_2,\dots) \in B^\infty : \sum_{j=1}^{n} j^q\|x_j\|^q/n^q \to 0\}$$

$$\|x\|_F = \sup_n (\sum_{j=1}^{n} j^q\|x_j\|^q/n^q)^{1/q} .$$

The above discussion shows that $E \subset F$. Assume that $\|x^n-x\|_E \to 0$ and $\|x^n-y\|_F \to 0$. It can be easily shown that $\|x_j^n-x_j\| \to 0$ and $\|x_j^n-y_j\| \to 0$, for any j. It follows that $x_j = y_j$, $j = 1,2,\dots$ so that $x = y$. The injection $E \to F$ thus has a closed graph and is, by the closed graph theorem, continuous. Hence there exists a constant $C < \infty$ such that

$$\sup_n (\sum_{j=1}^{n} j^q\|x_j\|^q/n^q)^{1/q} \le C(E\|\sum_{j=1}^{\infty} \varepsilon_j x_j\|^q)^{1/q}$$

In other words (keeping in mind that C denotes a generic constant), for any n and for $\{x_j\}_{j=1}^n$ in B,

$$\sum_{j=1}^{n} j^q \|x_j\|^q/n^q \leq C\, E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^q. \tag{2.3}$$

Fix $N \geq 1$ and define, as in Hoffmann-Jogensen and Pisier [19]

$$\begin{aligned} y_j &= 0 \quad 1 \leq j \leq N \\ &= x_{j-n} \quad N < j \leq N+n. \end{aligned}$$

We have by (2.3),

$$\sum_{j=1}^{N+n} j^q \|y_j\|^q \leq C(N+n)^q\, E\|\sum_{j=1}^{N+n} \varepsilon_j y_j\|^q$$

so that

$$\sum_{j=1}^{n} (N+j)^q \|x_j\|^q \leq C(N+n)^q E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^q$$

and thus

$$\sum_{j=1}^{n} (N+j)^q \|x_j\|^q/(N+n)^q \leq C\, E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^q (n, N \geq 1).$$

Choosing $N = n$ yields

$$(1/2)^q \sum_{j=1}^{n} \|x_j\|^q \leq C\, E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^q,$$

proving that B is of cotype q. This completes the proof.

The necessary condition in Theorem 2.1 can be verified using the criteria of Volodin and Nagaev [37,46] but is far less appealing than an individual moment condition such as $\sum_{j=1}^{n} E\|x_j\|^q/n^q \to 0$. When is such a condition necessary for the SLLN to hold? Corollary 2.3 below provides an answer to this question. We will need the following

Lemma 2.2. Let $\{X_n\}$ be a sequence of independent symmetric random variables taking values in an arbitrary separable Banach space B. Assume that $\|X_j\| \leq j$ $(j \geq 1)$ and that $\{X_n\} \in$ WLLN. Then $E\|S_n/n\|^q \to 0$ for each $q \geq 1$.

Proof: The proof is an obvious modification of Lemma 2.3 in Kuelbs and Zinn [25]. For $B = \mathbb{R}$ and $q = 2$, a proof may be found in Stout [43] (Theorem 3.4.2) or in Loeve [28] (Corollary 1, page 253).

Fix $0 < \varepsilon < 1$ and $q \geq 1$. Since $\{X_n\} \in$ WLLN, there exists an n_0 such that $\sup_{n \geq n_0} P(\|S_n\| \geq n\varepsilon) \leq 1/8 \cdot 3^q$. An application of Hoffmann-Jorgensen's inequality (Theorem 3.1 in [18]) yields, for any $A > 0$

$$\int_0^A qt^{q-1}P(\|S_n\| \geq nt)dt = q \cdot 3^q \int_0^{A/3} t^{q-1}P(\|S_n\| \geq 3nt)dt$$

$$\leq q \cdot 3^q[4\int_0^{A/3} t^{q-1}P^2(\|S_n\| \geq nt)dt + \int_0^{A/3} t^{q-1}P(N_n \geq nt)dt]$$

$$\leq q \cdot 3^q[4\varepsilon + 1/2 \cdot 3^q \int_0^A t^{q-1}P(\|S_n\| \geq nt)dt + \int_0^{A/3} t^{q-1}P(N_n \geq nt)dt$$

$$\leq 8q\ 3^q\ \varepsilon + 2q\ 3^q \int_0^{A/3} t^{q-1}P(N_n \geq nt)dt$$

for $n \geq n_0$, where $N_n = \max_{1 \leq j \leq n} \|X_j\|$. Since $N_n \leq n$ and A is arbitrary, we have

$$E\|S_n/n\|^q \leq 8q\ 3^q\ \varepsilon + 2q\ 3^q \int_0^1 P(N_n \geq nt)dt\ .$$

For any $t > 0$, however,

$$P(N_n \geq nt) \leq P(\max_{1 \leq K \leq n} \|S_K\| \geq nt/2) \leq 2P(\|S_n\| \geq nt/2) \to 0$$

(By Lévy's inequality). The dominated convergence theorem now implies the result.

<u>Corollary 2.3</u>. Let $\{X_n\}$ be a sequence of independent symmetric random variables with values in a cotype q Banach space ($2 \leq q < \infty$). Assume that

(2.4) $\|X_j\| \leq j$ a.s. ($j \geq 1$)

(2.5) $\{X_n\} \in$ WLLN

Then

$$\sum_{j=1}^{n} E\|X_j\|^q/n^q \to 0.$$

<u>Proof</u>: Immediate from Lemma 2.2 and the (alternative) definition of a cotype q space. Notice how the above conclusion was obtained merely by altering the blanket assumption $\|X_n\| = o(n)$ a.s. to $\|X_n\| \leq n$ a.s. ($n \geq 1$).

<u>Remark 2.4</u>. The proofs of Theorem 2.1 and Corollary 2.3 show that if $\{X_n\}$ is any sequence of indepdendent symmetric B-valued random variables (B is arbitrary) satisfying the SLLN, then

$$E_\varepsilon\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2/n^2 \to 0$$

for almost all $\{x_j\}$. If $\|X_n\| \leq n$ a.s. ($n \geq 1$) then $E\|S_n/n\|^2 \to 0$ as well.

Lemma 2.5. (Due to Prokhorov; for a proof, see Stout [43], page 159). Let $\{X_n\}$ be a sequence of independent symmetric B-valued random variables. Then $\{X_n\} \in$ SLLN iff $(S_{2^{n+1}}-S_{2^n})/2^n \to 0$ a.s.

Lemma 2.6. (Prokhorov [42]). Let $\{X_n\}$ be a sequence of independent real-valued random variables satisfying for some $C < \infty$,

(2.6) $|X_n| \leq Cn/LLn$ a.s. $(n \geq 1)$

(2.7) $E(X_n) = 0$ $(n \geq 1)$.

Then $\{X_n\} \in$ SLLN iff $\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) < \infty$ for each $\varepsilon > 0$, where

$$\Lambda(n) = \sum_{j\in I(n)} EX_j^2/4^n.$$

Examples 2.7 and 2.8 below show that the necessary conditions in Theorem 2.1 and Corollary 2.3 are the best possible of their kind and that they are not, in general, sufficient conditions.

Example 2.7. Let ϕ_n be any sequence of real numbers increasing to $+\infty$. Then there exists a sequence $\{X_n\}$ of independent symmetric real-valued random variables satisfying the SLLN but such that

$$\phi_n/n^q \sum_{j=1}^{n} |X_j|^q \to \infty \quad \text{a.s.}$$

and

$$\phi_n/n^q \sum_{j=1}^{n} E|X_j|^q \to \infty$$

To see this, let

$$X_{2^n} = \varepsilon_{2^n} 2^n/(\phi_{2^n})^{1/2q}$$

$$X_K = 0 \ (K \neq 2^n \text{ for any } n)$$

Then

$$\left|S_{2^{n+1}} - S_{2^n}\right|/2^n = 1/(\phi_{2^n})^{1/2q} \to 0$$

for each $\omega \in \Omega$, so that $\{X_n\} \in$ SLLN by Lemma 2.5. On the other hand,

$$\phi_{2^n}/2^{nq} \sum_{j \in I(n)} |X_j|^q = (\phi(2^n))^{1/2} \to \infty.$$

Such an example exists in any Banach space B, since $\mathbb{R} \subset B$.

Example 2.8. The condition

$$\sum_{j=1}^{n} \|X_j\|^q/n^q \to 0 \text{ a.s.}$$

is not sufficient condition for the SLLN in any Banach space. In fact, for each sequence $\phi_n = o(LLn)$ there exists a sequence of independent symmetric real random variables satisfying

$$(\phi_n/n^2) \sum_{j=1}^{n} X_j^2 \to 0 \text{ a.s.}$$

but failing the SLLN. To see this, let $X_n = \varepsilon_n (n/LLn)^{1/2}$ $(n \geq 1)$.

(2.6) and (2.7) are clearly satisfied, but $\Lambda(n) \geq C/LL2^n \geq C/\log n$.

Hence

$$\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) = \infty$$

if $\varepsilon \leq C$, so that the SLLN fails by Lemma 2.6. On the other hand,

$$(\phi_n/n^2) \sum_{j=1}^{n} X_j^2 = (\phi_n/n^2) \sum_{j=1}^{n} j/LLj \leq C\, \phi_n/LLn \to 0 \text{ a.s.}$$

CHAPTER III

SUFFICIENT CONDITIONS FOR THE SLLN

The study of SLLN's in separable Banach spaces was initiated by Mourier [35], who proved that an i.i.d. sequence $\{X_n\}$ satisfies the SLLN iff $E\|X_1\| < \infty$. Subsequent work (in the non-identically distributed case) was done by Beck [3,4], Beck, Giesy and Warren [5], Woyczinski [47], Hoffmann-Jorgensen and Pisier [19], Kuelbs and Zinn [25] and Heinkel [15,16,17]. As with all limit theorems in Banach spaces, these results fall into two natural categories, with restrictions being placed either on the probability distributions of the sequence $\{X_n\}$ or on the geometry of the underlying Banach space. We will start by considering the first class of results. The results of Kuelbs and Zinn (Theorems 3.2 and 3.3 below) fall into this category, but may easily be restated as statements about the geometry of B. Theorem 3.1 below is due to Beck, Giesy and Warren. It is a result that (a) is valid for each Banach space and (b) is in terms of the moments of the individual summands. We will see subsequently how difficult it is to prove a non-trivial result at this level of generality without making additional assumptions (Kuelbs and Zinn hypothesize, for example, that the WLLN is satisfied).

Theorem 3.1. (Beck, Giesy and Warren [5]). Let $\{X_n\}$ be a sequence of independent B-valued random variables with $EX_n = 0$ $(n \geq 1)$ and satisfying either

$$(3.1)\qquad \sum_{j=1}^{\infty} E\|X_j\|^2/j^2 < \infty \quad\text{and}\quad \lim_{n\to\infty} \sum_{j=1}^{n} (E\|X_j\|^2)^{1/2}/n = 0$$

or

$$(3.2)\qquad \lim_{n\to\infty} \sum_{j=1}^{n} \operatorname{ess\,sup} \|X_j\|/n = 0$$

Then $\{X_n\} \in$ SLLN. Furthermore, (3.1) and (3.2) are the best possible in the sense that weakening either yields a result that is no longer true for all Banach spaces.

Theorem 3.1 follows as a corollary of Theorem 3.3 below (Kuelbs and Zinn). In fact, both (3.1) and (3.2) imply that

$$\sum_{j=1}^{n} \|X_j\|/n \to 0 \text{ a.s.}$$

which is a completely trivial sufficient condition for the SLLN. (3.2) obviously implies that

$$\sum_{j=1}^{n} \|X_j\|/n \to 0 \text{ a.s.}$$

Assume that (3.1) holds. We need to prove that $\{\|X_n\|\} \in$ SLLN

It suffices, therefore, to show that $\{\|X_n\|^S\} \in$ SLLN and $\{\|X_n\|\} \in$ WLLN. Note that

$$\sum_{j=1}^{\infty} E(\|X_j\|^S)^2/j^2 \le 2\sum_{j=1}^{\infty} E\|X_j\|^2/j^2 < \infty,$$

so that $\{\|X_n\|^S\} \in$ SLLN by Kolmogorov's real-line SLLN. Also

$$P(\sum_{j=1}^{n} \|X_j\| > n\varepsilon) \le 1/n\varepsilon \sum_{j=1}^{n} E\|X_j\| \le 1/n\varepsilon \sum_{j=1}^{n} (E\|X_j\|^2)^{1/2} \to 0,$$

proving that $\{\|X_n\|\} \in$ WLLN.

<u>Theorem 3.2</u>. (Kuelbs and Zinn [25]). Let $\{X_n\}$ be a sequence of independent B-valued random variables satisfying

(3.3) $$X_n/n \to 0 \text{ a.s.}$$

(3.4) for some $p \in [1,2]$ and for some $r \in (0,\infty)$,

$$\sum_{n=1}^{\infty} \Lambda(n,p)^r < \infty \text{ , where } \Lambda(n,p) = 1/2^{np} \sum_{j\in I(n)} E\|X_j\|^p.$$

(3.5) $\{X_n\} \in$ WLLN

Then $\{X_n\} \in$ SLLN.

Kuelbs' and Zinn's next result extends Lemma 2.6 (the sufficient part) to the B-valued case:

<u>Theorem 3.3</u>. (Kuelbs and Zinn [25]). Let $\{X_n\}$ be a sequence of independent B-valued random variables such that (3.5) holds and

(3.6) $\|X_j\| \leq Cj/LLj$ a.s. for some $C < \infty$, $(j \geq 1)$.

(3.7)

$$\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) < \infty \text{ for all } \varepsilon > 0, \text{ where } \Lambda(n) = \Lambda(n,2) = 1/4^n \sum_{j\in I(n)} E\|X_j\|^2$$

Then $\{X_n\}$ $\in$ SLLN.

<u>Remarks 3.4</u>. The WLLN hypothesis of Theorem 3.3 may be difficult to verify unless one assumes, for example, that B is a type-p space in which case it may be replaced by

(3.8) $$1/n^p \sum_{j=1}^{n} E\|X_j\|^p \to 0 \ (n \to \infty)$$

which is a condition in terms of the individual moments. Notice, however, how stringent (3.8) is for $p = 1$. This just reaffirms the fact that norm and/or moment assumptions will not yield useful SLLN's if B is of type 1 (and no better). (If $B = \ell'$, for example, and one defines $X_n = a_n \varepsilon_n e_n$, where $\{a_n\} \subset \mathbb{R}$ and $\{e_n\}$ is the canonical basis of ℓ', then it is easy to see that $\{X_n\} \in$ SLLN iff $\sum_{j=1}^{n} \|X_j\|/n \to 0$ a.s.) Notice also that if B is a type-2 space, then (3.5) is automatically implied by (3.4) or (3.7) and need not be hypothesized.

If B is of type p, $1 \leq p < 2$, however, then (3.5) is crucial and may not be omitted even if $\{X_n\}$ is a symmetric sequence. In other words, the WLLN hypothesis of Theorems 3.2 and 3.3 is not merely a "desymmetrization" assumption. To see this, consider, for $p \in [1,2)$, the ℓ^p valued sequence $\{X_n\}$ defined by

$$X_n(\omega,k) = \varepsilon_n(\omega)(n/(LLn)^{\alpha})^{1/2} I_{\{n\}}(k)(n \geq 1)$$

where $\alpha > 1$ is arbitrary. We have, for any ω, $\|X_j\| = (j/(LLj)^{\alpha})^{1/2}$ so that (3.6) is satisfied. Also,

$$(LLn/n^2) \sum_{j=1}^{n} E\|X_j\|^2 = (LLn/n^2) \sum_{j=1}^{n} j/(LLj)^{\alpha} \leq C/(LLn)^{\alpha-1} \to 0$$

so that $(\log n)\, \Lambda(n) \to 0$. It follows that (3.7) holds. Notice, however, that $\{X_n\} \notin$ SLLN, since for any ω,

$$\|\sum_{j=1}^{n} X_j/n\| = n^{-1}[\sum_{j=1}^{n} (j/(LLj)^{\alpha})^{p/2}]^{1/p} \geq Cn^{-1}[n^{p/2+1}/(LLn)^{\alpha p/2}]^{1/p} \to \infty.$$

A similar example may be constructed to show that (3.5) is a crucial hypothesis in Theorem 3.2 as well.

We next consider the second class of results on the SLLN; ones in which conditions are imposed on the Banach space B. The basic result in this direction is due to Hoffmann-Jorgensen and Pisier who obtain an analog of the classical Kolmogorov-Chung SLLN (Theorem 3.5). Heinkel has obtained an improvement of Theorem 3.3 for Hilbert-space valued random variables (Theorem 3.6 below).

Theorem 3.5. (Hoffmann-Jorgensen and Pisier [19]). The following are equivalent

(3.9) B is of type p ($1 \leq p \leq 2$).

(3.10) Each sequence $\{X_n\}$ of independent zero-mean B valued random variables with $\sum_{j=1}^{\infty} E\|X_j\|^p/j^p < \infty$ satisfies the SLLN.

Theorem 3.6. (Heinkel [16]). Suppose $\{X_n\}$ is a sequence of independent centered random variables with values in a 2-uniformly smooth Banach space B (see [16] for a definition). Assume that (3.6) holds and that

$$\sum_{n=1}^{\infty} \exp(-\varepsilon/\Gamma(n)) < \infty \text{ for each } \varepsilon > 0, \tag{3.11}$$

where $\Gamma(n) = 1/4^n \sum_{j\varepsilon I(n)} \sup_{\|f\|\leq 1, f\varepsilon B^*} E(f,X_j)^2$.

$$1/n^2 \sum_{j=1}^{n} E\|X_j\|^2 \to 0. \tag{3.12}$$

Then $\{X_n\} \varepsilon$ SLLN.

Heinkel also constructs a sequence of ℓ^2-valued random variables satisfying the hypotheses of Theorem 3.6 but not of Theorem 3.3, proving that (3.7) is not a necessary condition for the SLLN to hold, even in a cotype 2 space and even if (3.6) holds. Zinn [49] has given another example.

There are three major differences between the present investigation and the work of the above authors. Most of the results are stated in terms of the validity of a real-valued SLLN. They are shown to be the best possible of their kind. Finally, no WLLN hypothesis is made in the symmetric case. Convergence in probability need only be hypothesized, therefore, to be able to conclude that $\{X_n\}$ obeys the SLLN if $\{X_n^s\}$ does. Also, the results resemble Theorems 3.2 and 3.5 in that no specific hypothesis is made on the magnitudes of the norms of the X_n's and because one obtains characterizations of certain classes of Banach spaces through the validity of a SLLN. There is also a strong similarity with Theorems 3.3 and 3.6 (See Remark 3.12).

The first group of results depend on an exponential inequality (Lemma 3.7). They are not the best possible (and are, in fact, improved later in this chapter) but are included because the nature of their proofs is quite revealing (see Remark 3.18) and also because Lemma 3.7 plays an important role in Chapter IV on the (bounded) law of the iterated logarithm.

Lemma 3.7. Let B be a separable Banach space. Consider the Rademacher series $\sum_{n=1}^{\infty} \varepsilon_n x_n$, where $\{x_n\} \subset B$. Set $S_n = \sum_{j=1}^{n} \varepsilon_j x_j$ and $s_n^2 = E\|S_n\|^2$.

Then there exists a constant $M = M(B)$ such that for each $\varepsilon > 0$,

$$P(\|S_n\|/s_n > \varepsilon) \leq 3 \exp(-\varepsilon^2/M^2). \tag{3.13}$$

Proof: We shall use a basic result of Kwapién [27] which states that for any sequence $\{x_n\}$ in an arbitrary Banach space B, the almost sure convergence of $\sum_{j=1}^{\infty} \varepsilon_j x_j$ implies that $E[\exp \alpha \| \sum_{j=1}^{\infty} \varepsilon_j x_j \|^2] < \infty$ for each $\alpha > 0$. Let the Banach space $(E, \|\cdot\|_E)$ and the Orlicz space $(F, \|\cdot\|_F)$ be defined by

$$E = \{x = (x_1, x_2, \ldots) \in B^{\infty} : \sum_{j=1}^{\infty} \varepsilon_j x_j \text{ converges in } L^2(B)\}$$

$$\|x\|_E = (E\| \sum_{j=1}^{\infty} \varepsilon_j x_j \|^2)^{1/2}$$

$$F = \{x = (x_1, x_2, \ldots) \in B^{\infty} : E \exp(\alpha \| \sum_{j=1}^{\infty} \varepsilon_j x_j \|^2) < \infty \text{ for each } \alpha > 0\}$$

$$\|x\|_F = \inf\{t > 0 : E \exp(\| \sum_{j=1}^{\infty} \varepsilon_j x_j \|^2 / t^2) \leq e\}$$

The fact that both E and F are Banach spaces is well known and may be verified by a tedious but routine calculation. Kwapien's theorem asserts that $E \subset F$. Suppose that $\|x^n - x\|_E \to 0$ and $\|x^n - y\|_F \to 0$. It then follows that $E\| \sum_{j=1}^{\infty} \varepsilon_j (x_j^n - x_j) \|^2 \to 0$ and $E\| \sum_{j=1}^{\infty} \varepsilon_j (x_j^n - y_j) \|^2 \to 0$ so that $x = y$. The closed graph theorem now implies that there is a constant $M < \infty$ such that

$$\inf\{t > 0 : E \exp(\|S\|^2 / t^2) \leq e\} \leq M(E\|S\|^2)^{1/2} \tag{3.14}$$

where $S = \sum_{j=1}^{\infty} \varepsilon_j x_j$. In other words, for each n,

$E \exp(\|S\|^2 / M^2 E\|S\|^2 + n^{-1}) \leq e$. An application of Fatou's lemma yields

(3.15) $$E \exp(\|S\|^2/M^2 E\|S\|^2) \leq e$$

so that for each n,

$$P(\|S_n\|/s_n \geq \varepsilon) = P(\|S_n\|^2/M^2 s_n^2 \geq \varepsilon^2/M^2)$$

$$\leq E(\exp \|S_n\|^2/M^2 s_n^2) \exp(-\varepsilon^2/M^2)$$

$$\leq 3 \exp(-\varepsilon^2/M^2),$$

by (3.15). This proves (3.13).

Remark 3.8. Kahane [21] showed that $E \exp(\alpha\|S\|) < \infty$ for each $\alpha \leq \alpha_0$ whenever S is an a.s. convergent series; Kwapien's theorem is obtained by using Kahane's result together with an additional argument. Marcus and Pisier [31] generalize Kwapien's result and also show how it may be proved directly. The Orlicz norm of Lemma 3.7 was first used by Pisier [40]. It must be pointed out that Lemma 3.7 is implicit in the work of the above authors and has merely been retrieved from Kwapien's theorem.

The following result of Kahane is a consequence of Kwapien's result and generalizes the classical Khinchin inequalities. It shows that the L^p norms $(1 \leq p < \infty)$ are all equivalent on the linear span generated (in B) by the Rademacher random variables. It may also be interpreted as a converse Holder inequality on this subspace.

Lemma 3.9. (Kahane's inequalities [21]). For each p and q satisfying $1 \leq p \leq q < \infty$, there exists a universal constant $K_{p,q}$ such that for each Banach space B and for each finite sequence $\{x_j\}_{j=1}^n$ in B,

$$\Big(E\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|^q\Big)^{1/q} \leq K_{p,q}\Big(E\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|^p\Big)^{1/p}.$$

The next lemma is a fundamental result of Hoffmann-Jorgensen. It provides conditions under which the almost sure and L^p-convergence of a series of independent B-valued random variables are equivalent:

Lemma 3.10. (Hoffmann-Jorgensen [18]). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent B-valued random variables so that S_n converges a.s. to S, and let $0 < p < \infty$. Then the following are equivalent.

(3.16) $$S_n \to S \text{ in } L^p(B).$$

(3.17) $$S \in L^p(B).$$

(3.18) $$M = \sup_n \|S_n\| \in L^p(\mathbb{R}).$$

(3.19) $$N = \sup_n \|X_n\| \in L^p(\mathbb{R}).$$

(3.20) $$\{S_n\}_{n=1}^{\infty} \text{ is a bounded subset of } L^p(B) \quad .$$

The next proposition gives 3 sufficient conditions for the SLLN. The first two are in terms of the a.s. convergence of a series of real random variables, while the last is in terms of the validity of a real-valued SLLN.

Proposition 3.11. Let $\{X_n\}$ be a sequence of independent symmetric B-valued random variables. Let $t_{n\omega}^2$ and $T_{n\omega}$ (or simply t_n^2 and T_n, when there is no possibility of confusion) denote the quantities

$E_\varepsilon \|\sum_{j\in I(n)} \varepsilon_j X_j(\omega)\|^2$ and $\sum_{j\in I(n)} \varepsilon_j X_j(\omega)$, respectively. Then

(a) $\{X_n\} \in$ SLLN if

$$\sum_{n=1}^{\infty} \exp(-\varepsilon 4^n/t_{n\omega}^2) < \infty \text{ a.s. for each } \varepsilon > 0. \tag{3.21}$$

If, in addition B is of type p ($1 \leq p \leq 2$) then

(b) $\{X_n\} \in$ SLLN if

$$\sum_{n=1}^{\infty} \exp(-\varepsilon 4^n/[\sum_{j\in I(n)} \|X_j(\omega)\|^p]^{2/p} < \infty \text{ a.s. for each } \varepsilon > 0 \tag{3.22}$$

and

(c) $\{X_n\} \in$ SLLN if

$$[(LLn)^{p/2}/n^p]\sum_{j=1}^{n} \|X_j\|^p \to 0 \text{ a.s.} \tag{3.23}$$

<u>Proof</u>: We will show that $\{\varepsilon_n X_n(\omega)\} \in$ SLLN for each ω satisfying (3.21). By Lemma 2.5 and the Borel Cantelli Lemma it suffices to show that

$$\sum_{n=1}^{\infty} P(\|T_n\| > 2^n\varepsilon) < \infty$$

for each $\varepsilon > 0$. Fix $\varepsilon > 0$ and apply Lemma 3.7 to get

$P(\|T_n\| \geq 2^n\varepsilon) = P(\|T_n\|/t_n \geq 2^n\varepsilon/t_n) \leq 3\exp(-4^n\varepsilon^2/t_n^2 M^2)$. By (3.21) the last series is summable for each $\varepsilon > 0$. This proves Part (a).

Part (b) follows immediately from (a), Kahane's inequality and the definition of type, since

$$t_{n\omega}^2 = E\|\sum_{j\in I(n)} \varepsilon_j X_j(\omega)\|^2 \leq K_{p,2}^2 (E\|\sum_{j\in I(n)} \varepsilon_j X_j(\omega)\|^p)^{2/p}$$

$$\leq K_{p,2}^2 A_p^{2/p} (\sum \|X_j(\omega)\|^p)^{2/p}.$$

To prove (c), notice that by (3.23)

$$\sum_{n=1}^{\infty} \exp(-\varepsilon 4^n/[\sum_{j\in I(n)} \|X_j(\omega)\|^p]^{2/p} =$$

$$\sum_{n=1}^{\infty} \exp(-\varepsilon\{2^{np}/(LL2^n)^{p/2} \sum_{j\in I(n)} \|X_j(\omega)\|^p\}^{2/p} LL2^n)$$

$$\leq C_\varepsilon \sum_{n=1}^{\infty} n^{-2} < \infty.$$

In other words 3.23 implies 3.22. This completes the proof of (c).

Remarks 3.12. The sufficient condition in part (c) above may be verified by the criteria of Volodin and Nagaev [37,46]. One may attempt similarly to check the conditions in (a) and (b) using Kolmogorov's three-series theorem, but it is more convenient to use Hoffman-Jorgensen's result (Lemma 3.10). Assume that (3.21) holds. Since (3.19) is obviously satisfied we must have (3.17). In other words,

$$(3.24) \qquad \sum_{n=1}^{\infty} E \exp(-\varepsilon 4^n/t_{n\omega}^2) < \infty \quad (\varepsilon > 0).$$

Conversely, if (3.24) holds, the series $\sum_{n=1}^{\infty} \exp(-\varepsilon 4^n/t_{n\omega}^2)$ converges in L' for each $\varepsilon > 0$ (since it is a positive term series) and thus almost surely, for each $\varepsilon > 0$ (by Lévy's theorem). (3.21) and (3.24) are thus equivalent. Similarly, (3.22) is equivalent to

$$(3.25) \qquad \sum_{n=1}^{\infty} E \exp[-\varepsilon 4^n/(\sum_{j\in I(n)} \|X_j\|^p)^{2/p}] < \infty \quad (\varepsilon > 0)$$

so that (3.21) and (3.22) hold iff just one of the series in Kolmogorov's three-series criterion converges. Unfortunately, (3.24) and (3.25) cannot be thought of as being computationally easy substitutes for (3.21) and (3.22). It is not clear, therefore, how (a) and (b) of Proposition 3.11 may be verified, even though they are formulated in terms of the almost sure convergence of a series of independent real random variables. In Proposition 3.13 below, we show when the sufficient conditions of (b) and (c) are equivalent.

The rates in (c) are not the best possible (this is painfully obvious for $p = 1$) unless $p = 2$. We shall obtain the best rates in Theorem 3.14. Assume therefore that $p = 2$. Part (b) of Proposition 3.11 states that a Rademacher sequence $\{\varepsilon_n x_n\}$ satisfies the SLLN if

$$(3.26) \qquad \sum_{n=1}^{\infty} \exp(-\varepsilon 4^n / \sum_{j \in I(n)} \|x_j\|^2) < \infty \quad (\varepsilon > 0) ,$$

which is exactly what the sufficient conditions of Kuelbs and Zinn (Theorem 3.3) and Heinkel (Theorem 3.6) reduce to for such a sequence. Suppose one were trying to prove the SLLN for an arbitrary sequence $\{X_n\}$ of independent symmetric random variables with values in a type 2 space, using the criteria of Kuelbs and Zinn or Heinkel. Suppose also that one chose to prove the SLLN for $\{\varepsilon_n X_n(\omega)\}$ (for almost all ω) instead of for $\{X_n\}$. (3.7) and (3.11) would then coincide with (3.22), the sufficient condition of Proposition 3.11(b). This is not to suggest that these 3 conditions are equivalent; they are not. The above remark provides a clue, however, as to how they might be related. Notice also that (3.22) implies the SLLN even if the boundedness hypothesis ($\|X_j\| \leq Cj/LLj$) of Kuelbs, Zinn and Heinkel is not satisfied.

Proposition 3.13. Let $\Lambda(n,p) = \sum_{j\varepsilon I(n)} |X_j|^p/2^{np}$ form a decreasing sequence of real numbers, and suppose that $\sum_{n=1}^{\infty} \exp(-\varepsilon/[\Lambda(n,p)]^{2/p}) < \infty$ for each $\varepsilon > 0$. Then $(\log n)^{p/2} \Lambda(n,p) \to 0$ (thus (b) and (c) of Proposition 3.11 are equivalent if $\Lambda(n,p)$ forms a decreasing sequence for almost all $\omega \varepsilon \Omega$, a condition that most sequences $\{X_n\}$ would satisfy).

Proof: Let $\{a_n\}$ be any decreasing sequence of real numbers. We will prove that $a_n = o(1/\log n)$ if $\sum_{n=1}^{\infty} \exp(-\varepsilon/a_n) < \infty$ for each $\varepsilon > 0$. The proof is along the lines of Proposition 6.7 in Dudley [9]. Let $a_n = \alpha_n/\log n$. We will show that $\alpha_n \to 0$. Suppose on the contrary that $\limsup_{n\to\infty} \alpha_n = 2\delta > 0$, so that $\alpha_n > \delta$ for arbitrarily large values of n. Choose such an n and suppose that $[n^{1/2}] \leq j \leq n$. Then

$$\alpha_j = a_j \log j \geq a_n \log j = (\alpha_n/\log n)\log j \geq \alpha_n/2 .$$

Thus

$$\sum_{j=[n^{1/2}]+1}^{n} \exp(-\varepsilon/a_j) = \sum_{j=[n^{1/2}]+1}^{n} j^{-\varepsilon/\alpha_j} \geq \sum_{j=[n^{1/2}]+1}^{n} n^{-2\varepsilon/\alpha_n}$$

$$\geq \sum_{j=[n^{1/2}]+1}^{n} n^{-2\varepsilon/\delta} = (n-[n^{1/2}])/n^{2\varepsilon/\delta} .$$

It is clear that $\sum_{n=1}^{\infty} \exp(-\varepsilon/a_n)$ cannot converge for each $\varepsilon > 0$. This proves the result.

The following example shows that Proposition 3.13 is false in general. Define a symmetric independent real-valued sequence $\{X_n\}$ by

$$X_{2^{2^K}+1} = \varepsilon_K\, 2^{2^K}/K^{1/2} \quad (K \geq 1)$$

$$X_j = 0 (j \neq 2^{2^K}+1 \text{ for any } K).$$

Then for any $\omega \in \Omega$ and for each K,

$$\Lambda(2^K,p) = 1/2^{2^K \cdot p} \sum_{j=2^{2^K}}^{2^{2^K+1}} |X_j|^p \geq C/K^{p/2}$$

so that $\limsup_{n \to \infty} (\log n)^{p/2} \Lambda(n,p)$ is strictly positive. Note also that $(\Lambda(2^K,p))^{2/p} \leq C/K$. Suppose now that $n \neq 2^K$ for any K. Then $2^\ell+1 \leq n \leq 2^{\ell+1}-1$ for some ℓ, so that $2^{n+1} > 2^{2^\ell+1}$ and $2^{n+1} < 2^{2^{\ell+1}+1}$. It follows that $\Lambda(n,p) = 0$ and thus

$$\sum_{n=1}^{\infty} \exp(-\varepsilon/[\Lambda(n,p)]^{2/p}) = \sum_{K=1}^{\infty} \exp(-\varepsilon/[\Lambda(2^K,p)]^{2/p}) \leq \sum_{K=1}^{\infty} \exp(-\varepsilon K/C) < \infty$$

for each $\varepsilon > 0$.

There are several directions in which one may hope to improve Proposition 3.11(c). One possibility would be to widen the domain of its validity and another would be to improve the rate in (3.23) (which is intolerably bad for p = 1). Finally one may want to obtain a characterization of certain Banach spaces through the validity of a SLLN. Some of this is accomplished in the following theorem, which is the main result of this chapter.

Theorem 3.14. Let $\{X_n\}$ be a sequence of independent symmetric random variables taking values in a real separable Banach space B. Assume that

(3.27) For some $p \in [1,2]$, $(LLn)^{p-1}/n^p \sum_{j=1}^{n} \|X_j\|^p \to 0$ a.s.

(3.28) $\{\varepsilon_n X_n(\omega)\} \in$ WLLN for almost all ω.

Then $\{X_n\} \in$ SLLN.

In particular, the following are equivalent

(3.29) B is of log type-p.

(3.30) Each sequence $\{X_n\}$ of independent symmetric B valued random variables satisfying (3.27) also satisfies the SLLN.

Proof: The proof makes use of Theorem 3.3 (Kuelbs and Zinn) and a truncation argument. Let $X_j = Y_j + Z_j$, where $Y_j = X_j I(\|X_j\| \leq j/LLj)$ and $Z_j = X_j I(\|X_j\| > j/LLj)$. Note that $\{Y_n\}$ and $\{Z_n\}$ are both independent symmetric sequences. By Lemma 2.5, $\{Z_n\} \in$ SLLN iff $\sum_{j \in I(n)} Z_j/2^n \to 0$ a.s.

We have

$$\Big\| \sum_{j \in I(n)} Z_j/2^n \Big\| \leq 1/2^n \sum_{j \in I(n)} I^{p-1}(\|X_j\| > j/LLj)\|X_j\|$$

$$\leq C(LL2^n)^{p-1}/2^{np} \sum_{j \in I(n)} \|X_j\|^p \to 0 \text{ a.s.},$$

by (3.27). Let us next consider the sequence $\{Y_n\}$. It is clear that $\{Y_n\} \in$ SLLN iff $\{\varepsilon_n x_n I(\|x_n\| \leq n/LLn)\} \in$ SLLN for almost all ω. Choose an ω for which (3.27) and (3.28) hold. We need to verify that (3.5) through (3.7) hold for the sequence $\{\varepsilon_n x_n I(\|x_n\| \leq n/LLn)\}$. (3.6) is obviously satisfied. Let y_n denote $x_n I(\|x_n\| \leq n/LLn)$. We have

$$LL2^n/4^n \sum_{j\in I(n)} \|y_j\|^2 = LL2^n/2^{np} \sum_{j\in I(n)} \|y_j\|^p \|y_j\|^{2-p}/2^{n(2-p)}$$

$$\le CLL2^n/2^{np} \cdot 1/(LL2^n)^{2-p} \sum_{j\in I(n)} \|y_j\|^p =$$

$$C(LL2^n)^{p-1}/2^{np} \sum_{j\in I(n)} \|y_j\|^p \le C(LL2^n)^{p-1}/2^{np} \sum_{j\in I(n)} \|x_j\|^p \to 0,$$

by (3.27). Hence

$$\sum_{n=1}^{\infty} \exp(-\varepsilon 4^n)/(\sum_{j\in I(n)} \|y_j\|^2) \le C_\varepsilon \sum_{n=1}^{\infty} n^{-2} < \infty$$

so that (3.7) holds. (3.27) implies that $\|x_n\| \le n/(LLn)^{p-1/p}$ if n is large enough. (3.28) asserts that $\{\varepsilon_n x_n\} \in$ WLLN. We may thus apply Lemma 2.2 to conclude that $1/n\ E\|\sum_{j=1}^{n} \varepsilon_j x_j\| \to 0$. Kahane's contraction principle [21] or Hoffmann-Jorgensen's comparison principle (Lemma 4.1 in [18]) now show that $1/n\ E\|\sum_{j=1}^{n} \varepsilon_j y_j\| \to 0$ so that (3.5) holds. Another proof of the last fact may be given by using Lévy's inequality. This proves the first part of the theorem.

If B is of logtype-p then

$$P(\|\sum_{j=1}^{n} \varepsilon_j x_j\| > n\varepsilon) \le (n\varepsilon)^{-p} E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \le C(n\varepsilon)^{-p}(LLn)^{p-1} \sum_{j=1}^{n} \|x_j\|^p \to 0$$

for each ω satisfying (3.27), so that (3.28) holds. It follows that (3.29) implies (3.30). We turn next to the converse proposition. This is proved by using yet another closed graph argument. Consider the Rademacher sequence $\{\varepsilon_n x_n\}$, $\{x_n\} \subset B$. By hypothesis, $\{\varepsilon_n x_n\} \in$ SLLN if $(LLn)^{p-1}/n^p \sum_{j=1}^{n} \|x_j\|^p \to 0$. Kronecker's lemma and an argument similar to the one in Theorem 2.1 now show that

$$1/n^p E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \to 0 \text{ if } \sum_{j=1}^{\infty}(LLj)^{p-1} \|x_j\|^p/j^p < \infty .$$

Define the spaces $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ by

$$E = \{x \in B^\infty : \sum_{j=1}^{\infty}(LLj)^{p-1}\|x_j\|^p/j^p < \infty\}$$

$$\|x\|_E = [\sum_{j=1}^{\infty}(LLj)^{p-1}\|x_j\|^p/j^p]^{1/p}$$

$$F = \{x \in B^\infty : 1/n^p \ E \|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \to 0\}$$

$$\|x\|_F = \sup_n 1/n(E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p)^{1/p}.$$

$(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ may easily be seen to be Banach spaces with $E \subset F$.

Suppose that $\|x^n-x\|_E \to 0$ and $\|x^n-y\|_F \to 0$. It follows easily that

$\|x_j^n-x_j\| \to 0$ and $\|x_j^n-y_j\| \to 0$ for each j. It follows that $x = y$. The

closed graph theorem implies that there exists a $C < \infty$ such that

$$\sup_n 1/n(E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p)^{1/p} \le C(\sum_{j=1}^{\infty}(LLj)^{p-1}\|x_j\|^p/j^p)^{1/p}.$$

In other words, for each $n \ge 1$

$$(3.31) \qquad E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \le C\ n^p(\sum_{j=1}^{n}(LLj)^{p-1}\|x_j\|^p/j^p).$$

Fix $N \ge 1$ and define

$$y_j = 0(1 \le j \le N)$$

$$= x_{j-N}(N < j \le N + n).$$

By (3.31), for each n

$$E\|\sum_{j=1}^{N+n} \varepsilon_j y_j\|^p \leq C(N+n)^p(\sum_{j=1}^{N+n} (LLj)^{p-1}\|x_j\|^p/j^p)$$

so that

$$E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \leq C(N+n)^p(\sum_{j=1}^{n} [LL(N+j)]^{p-1}\|x_j\|^p/(N+j)^p)$$

$$\leq C\{(N+n)^p[LL(N+n)]^{p-1}/(N+1)^p\}\sum_{j=1}^{n} \|x_j\|^p.$$

Setting N = n, we obtain

$$E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^p \leq C\ 2^{p+1}(LLn)^{p-1}\sum_{j=1}^{n}\|x_j\|^p,$$

which proves that B is of logtype-p. (Pisier [39] has shown that this implies that B is of type r for each r < p.)

No assumptions are made on the magnitudes of the norms of $\{X_n\}$ in Theorem 3.14. Theorems 3.2 (Kuelbs and Zinn) and 3.5 (Hoffmann-Jorgensen and Pisier) fall in this category. We would like to show next that our result may be used in situations when both the above theorems are inconclusive.

<u>Example 3.15</u>. Define the sequence $\{X_n\}$ by

$$X_{2^n} = \varepsilon_{2^n} \cdot 2^n/(LL2^n)^{3/4} (n = 1,2,\ldots)$$

$$X_j = 0 \quad (j \neq 2^n \text{ for any } n).$$

Then, $\sum_{j=1}^{2^n} E|X_j|^p/j^p = \sum_{j=1}^{n} 1/(LL2^j)^{3p/4} \to \infty$ for each $p \in [1,2]$, so that Theorem 3.5 is inconclusive. Also,

$\Lambda(n,p) = 1/2^{np} \sum_{j\in I(n)} E|X_j|^p = (LL2^n)^{-3p/4}$. It follows that $\sum_{n=1}^{\infty} \Lambda(n,p)^r = \infty$ for each $p \in [1,2]$ and $r \in (0,\infty)$. No conclusion may thus be reached using Theorem 3.2. On the other hand, for any $p \in [1,2]$ and $n \in I(K)$,

$$(LLn)^{p-1}/n^p \sum_{j=1}^{n} |X_j|^p \leq (LL2^{K+1})^{p-1}/2^{Kp} \sum_{j=1}^{2^{K+1}} |X_k|^p$$

$$= (LL2^{K+1})^{p-1}/2^{Kp} \sum_{j=1}^{K+1} 2^{jp}/(LL2^j)^{3p/4} \leq C(LL2^K)^{p/4-1} \to 0$$

a.s., so that $\{X_n\} \in$ SLLN. Theorem 3.14 does not, however, contain either of the two theorems; to see this, let

$$X_{a_n} = \varepsilon_{a_n} x_{a_n} \quad (n = 1,2,\ldots)$$

$$X_j = 0 \qquad (j \neq a_n \text{ for any } n)$$

where $a_n = 2^{2^{2^n}}$ and $x_n = a_n/n^2$. We then have

$$\sum_{j=1}^{\infty} E|X_j|^p/j^p = \sum_{n=1}^{\infty} E|X_{a_n}|^p/a_n p = \sum_{n=1}^{\infty} n^{-2p} < \infty$$

for each $p \in [1,2]$. Also it is clear that $\Lambda(n,p) = 0$ $(n \neq 2^{2^K})$ and $\Lambda(2^{2^K},p) = K^{-2p}$ so that $\sum_{n=1}^{\infty} \Lambda(n,p) < \infty$ for each $p \in [1,2]$. However,

$$(LLa_n)^{p-1}/a_n^p \sum_{j=1}^{a_n} |X_j|^p \leq C\ 2^{n(p-1)}/a_n^p \sum_{j=1}^{n} a_j^p/j^{2p} \to \infty \text{ a.s.}$$

Similar examples may be constructed to show that for $B = \mathbb{R}$, (3.27) is not comparable to any of Teicher's [45] sufficient conditions for the SLLN.

Theorems 2.1 and 3.14 may now be combined (with $p = q = 2$) to obtain the following.

Corollary 3.16: A sequence $\{X_n\}$ of independent symmetric random variables with values in a Hilbert space H (or more generally a cotype 2, logtype-2 space) satisfies the SLLN if $LLn/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. and only if $1/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. Furthermore, the condition $\phi(n)/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. is neither necessary nor sufficient for the SLLN for any function $\phi(n) \uparrow \infty$, $\phi(n) = o(LLn)$. (This may be seen from Examples 2.7 and 2.8.)

Example 3.17. We showed in Example 2.7 that the rate in Theorem 2.1 could not be improved for any $q \geq 2$. We would like to show next that the rate in Theorem 3.14 is also the best possible, in the following sense: For each $p \in [1,2]$, and for each sequence $\phi(n) \uparrow \infty$, there exists a sequence $\{X_n\}$ of independent symmetric real random variables satisfying the condition $[(LLn)^{p-1}/n^p\phi(n)] \sum_{j=1}^{n} |X_j|^p \to 0$ a.s. but failing the SLLN. Such a sequence may be defined by

$$X_j = \varepsilon_j 2^n/LL2^n \qquad j = 2^n + 1, \ldots, 2^n + [LL2^n]$$

$$n = 1, 2, \ldots$$

$$X_j = 0 \qquad \text{Otherwise}$$

Then for any $K \in I(n)$, and $p \in [1,2]$,

$$[(LLK)^{p-1}/K^p\phi(K)]\sum_{j=1}^{K}|X_j|^p \leq [(LL2^{n+1})^{p-1}/2^{np}\phi(2^n)]\sum_{j=1}^{n+1}[LL2^j]2^{jp}/(LL2^j)^p$$

$$\leq C(LL2^{n+1})^{p-1}/2^{np} \cdot 2^{(n+1)p}/(LL2^{n+1})^{p-1} \cdot 1/\phi(2^n) \to 0 \text{ as } n \to \infty.$$

On the other hand, while (2.6) and (2.7) of Prokhorov's Lemma 2.6 are satisfied, we have $\Lambda(n) = 1/4^n \cdot [LL2^n] \cdot 4^n/(LL2^n)^2 \geq C/\log n$ so that $\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) = \infty$ for ε small enough. Hence $\{X_n\} \notin$ SLLN.

Remark 3.18. We have seen how the necessary and sufficient conditions for the SLLN "almost" coincide if B is a Hilbert space. If B is arbitrary, however, necessary and sufficient conditions that are close to one another cannot be formulated in terms of the validity of a real valued SLLN. Observe, however, that Remark 2.4 and Proposition 3.11(a) together imply that a B-valued SLLN holds if $LLn/n^2\, E_\varepsilon\|\sum_{j=1}^{n}\varepsilon_j X_j(\omega)\|^2 \to 0$ a.s. and only if $1/n^2\, E_\varepsilon\|\sum_{j=1}^{n}\varepsilon_j X_j(\omega)\|^2 \to 0$ a.s. These conditions are not, however, easily verifiable since there is no general technique of estimating $E\|S_n\|^2$ for an arbitrary sequence of B-valued random variables. (The two-sided estimates obtained by Giné and Zinn [14] may be expressed in terms of the individual summands only if B is a Hilbert space.)

We saw in Corollary 2.3 how the necessary condition of Theorem 2.1 could be expressed in terms of the individual moments if we assumed that $\|X_n\| \leq n$ a.s. $(n \geq 1)$. We show now that the sufficient condition of Theorem 3.14 may be similarly rephrased if $\|X_n\| \leq C\, n/LLn$ a.s. $(n \geq 1)$. The following Lemma of Loéve generalizes Prokhorov's result (Lemma 2.6).

Lemma 3.19. (Loéve [28]). Let $\{Y_n\}$ be a sequence of independent zero mean (real) random variables such that $|Y_n| \leq C\, b_n/LLb_n$ a.s. $(n \geq 1)$ where $b_n \uparrow \infty$ and $1 < C_1 \leq b_{2^{n+1}}/b_{2^n} \leq C_2 < \infty$ for some constants C_1 and C_2. Set $T_n = S_{2^{n+1}} - S_{2^n}/b_n$, $t_n^2 = ET_n^2 = 1/b_{2^n}^2 \sum_{j\in I(n)} EY_j^2$. Then $S_n/b_n \to 0$ a.s. iff the series $\sum_{n=1}^{\infty} \exp(-\varepsilon/t_n^2) < \infty$ for each $\varepsilon > 0$.

Proposition 3.20. Condition (3.27) may be expressed in terms of the moments of $\{X_n\}$ if $\|X_n\| \leq Cn/LLn$ a.s. $(n \geq 1)$. In fact, (3.27) holds iff

$$(3.32) \qquad [(LLn)^{p-1}/n^p] \sum_{j=1}^{n} \|X_j\|^p \to 0 \text{ in probability}$$

and

$$(3.33) \qquad \sum_{n=1}^{\infty} \exp[-\varepsilon\, 2^{2np}/\{(LL2^n)^{2p-2} \sum_{j\in I(n)} E(\|X_j\|^p - \|X_j'\|^p)^2\}]$$
$$< \infty, \text{ for each } \varepsilon > 0$$

where X_j' is an independent copy of $X_j (j \geq 1)$. (Here, (3.32) may be expressed in terms of the individual moments by the classical degenerate convergence criterion.)

Proof: Assume that $(LLn)^{p-1}/n^p \sum_{j=1}^{n} \|X_j\|^p \to 0$ a.s. for some $p \in [1,2]$.

By the result of Kuelbs and Zinn (see the Introduction), this is equivalent to (3.22) and

$$(3.34) \qquad (LLn)^{p-1}/n^p \sum_{j=1}^{n} (\|X_j\|^p - \|X_j'\|^p) \to 0 \text{ a.s.}$$

We have

$$\left|\|X_j\|^p - \|X_j'\|^p\right| \le 2Cj^p/(LLj)^p \le \frac{C'j^p/(LLj)^{p-1}}{LL(j^p/(LLj)^{p-1})},$$

so that Loéve's result aplies to (3.34).

It follows that (3.27) is equivalent to (3.32) and (3.33), as claimed. Notice that

$$E(\|X_j\|^p - \|X_j'\|^p)^2 = 2\,\mathrm{Var}(\|X_j\|^p) \le 2E\|X_j\|^{2p}$$

so that

$$(LL2^n)^{2p-2} \sum_{j\in I(n)} E(\|X_j\|^p - \|X_j'\|^p)^2/2^{2np} \le$$

$$2(LL2^n)^{2p-2} \sum_{j\in I(n)} E\|X_j\|^{2p}/2^{2np} \le$$

$$C(LL2^n)^{2p-2} \sum_{j\in I(n)} E\|X_j\|^2(2^n/LL2^n)^{2p-2}/2^{2np}$$

$$= C\Lambda(n),$$

so that (3.33) is a slight improvement of the Kuelbs-Zinn condition. One the other hand, (3.32) may do worse than their WLLN hypothesis.

Next, we proceed in a somewhat different direction to obtain a SLLN for Hilbert-space valued random variables. One may use the following example to motivate what follows: Define the sequence $\{X_n\}$ of ℓ^2-valued random variables by $X_n = \varepsilon_n e_n (n/LLn)^{1/2}$, where $\{e_n\}$ is the usual basis of ℓ^2. Does $\{X_n\} \in$ SLLN? Theorem 3.14 as well as the results of Hoffmann-Jorgensen and Piser, Kuelbs and Zinn, and Heinkel are all inconclusive. To see this, note that

$$\sum_{j=1}^{\infty} E\|X_j\|^2/j^2 = \sum_{j=1}^{\infty} 1/j\ LLj = \infty,$$

$$1/4^n \sum_{j\in I(n)} E\|X_j\|^2 = 1/4^n \sum_{j\in I(n)} \sup_{\|f\|\leq 1} E(f,X_j)^2$$

$$= 1/4^n \sum_{j\in I(n)} j/LLj \geq C/\log n$$

and

$$LLn/n^2 \sum_{j=1}^{n} \|X_j\|^2 \geq C > 0 \text{ a.s.}$$

On the other hand, $1/n^2 \sum_{j=1}^{n} \|X_j\|^2 \leq C/LLn \to 0$ a.s., so that no negative conclusion may reached using Theorem 2.1. The fact that $\{X_n\} \in$ SLLN may however be deduced from Theorem 3.22 below, which is a generalization to the Hilbert space setting of Teicher's [45] extension of the classical Kolmogorov SLLN. It extends the Hoffmann-Jorgensen and Pisier theorem in exactly the same way (for $B = H$). Teicher's result is stated next. It is really the first in a hierarchy of successfully stronger (and more complicated) results.

Theorem 3.21. (Teicher [45]). Let $\{X_n\}$ be a sequence of mean zero, independent (real) random variables satisfying

(3.35) $$\sum_{i=2}^{\infty} EX_i^2/i^4 \sum_{j=1}^{i-1} EX_j^2 < \infty$$

(3.36) $$\sum_{j=1}^{n} EX_j^2/n^2 \to 0$$

There exist constants C_j such that

(3.37) $$\sum_{j=1}^{\infty} P(|X_j| > C_j) < \infty \text{ and } \sum_{j=1}^{\infty} C_j^2\, EX_j^2/j^4 < \infty .$$

Then $\{X_n\} \varepsilon$ SLLN.

Egorov [10] showed that (3.35) and the condition $X_n = o(n)$ a.s. were sufficient for the SLLN to hold, but we shall see that such an extension will not be possible in the case of Hilbert space-valued random variables.

Theroem 3.22. Let $\{X_n\}$ be a sequence of independent symmetric random variables with values in the real separable Hilbert space H equipped with the inner product $(\cdot,\cdot)$. Assume that

(3.38) $$1/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0 \text{ a.s.}$$

and

(3.39) $$\sum_{k=2}^{\infty} k^{-4} \sum_{j=1}^{k-1} (X_j,X_k)^2 < \infty \text{ a.s.}$$

Then $\{X_n\} \varepsilon$ SLLN.

Before we prove the theorem, we shall need to state a basic lemma of Chow [7].

Lemma 3.23. (Chow [7]). Let $\{Y_n, \underline{F}_n; n \geq 1\}$ be a martingale difference sequence and suppose that $0 < a_j \uparrow \infty$, where a_j is $\underline{F}_{j-1}$ measurable for each $j \geq 1$. Then

$$(3.40) \qquad \sum_{j=1}^{\infty} E(|Y_j|^p | \underline{F}_{j-1})/a_j^p < \infty \text{ a.s. for some } 0 < p \leq 2$$

implies that $\sum_{j=1}^{n} Y_j/a_n \to 0$ a.s.

Proof of Theorem 3.22. We will show, as before, that $\{\varepsilon_n X_n(\omega)\} \in$ SLLN for almost all ω. Choose any ω satisfying (3.38) and (3.39). Denote $X_n(\omega)$ by $x_n (n \geq 1)$. We have

$$(3.41) \qquad \|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\|^2 = (\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n, \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n)$$

$$= \sum_{j=1}^{n} \|x_j\|^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} \varepsilon_i \varepsilon_j (x_i, x_j)$$

$$= \sum_{j=1}^{n} \|x_j\|^2 + 2 \sum_{k=2}^{n} (\varepsilon_k x_k, \varepsilon_1 x_1 + \ldots + \varepsilon_{k-1} x_{k-1})$$

so that we will have $\{\varepsilon_n x_n\} \in$ SLLN if we can show that

$$1/n^2 \sum_{k=2}^{n} (\varepsilon_k x_k, \varepsilon_1 x_1 + \ldots + \varepsilon_{k-1} x_{k-1}) \to 0 \text{ a.s.}$$

Set $Y_k = (\varepsilon_k x_k, \varepsilon_1 x_1 + \ldots + \varepsilon_{k-1} x_{k-1})$ and $\underline{F}_k = \sigma(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$.

Then $E(Y_k | \underline{F}_{k-1}) = \sum_{j=1}^{k-1} E(\varepsilon_j \varepsilon_k (x_j, x_k) | \underline{F}_{k-1}) = 0$ so that

$\{Y_k, \underline{F}_k; k \geq 2\}$ is a martingale difference sequence

We have,

$$E(E(Y_k^2|\underline{F}_{k-1}) = E(Y_k^2) = E(\sum_{j=1}^{k-1}(x_j,x_k)^2 + 2\sum_{\substack{i,j=1\\ i<j}}^{k-1}\varepsilon_i\varepsilon_j(x_i,x_k)(x_j,x_k))$$

$= \sum_{j=1}^{k-1}(x_j,x_k)^2$. It follows that $\sum_{k=2}^{\infty} k^{-4}E(E(Y_k^2|\underline{F}_{k-1})) < \infty$ so that

$\sum_{k=2}^{\infty} k^{-4}E(Y_k^2|\underline{F}_{k-1}) < \infty$ a.s. An application of Chow's Lemma with

$a_j = j^2$ and $p = 2$ shows that (3.42) holds.

<u>Corollary 3.24</u>. Suppose $\{X_n\}$ is a sequence of independent symmetric H-valued random variables satisfying (3.39). Then Theorems 2.1 and 3.22 show that $\{X_n\} \varepsilon$ SLLN iff $1/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. In particular, the ℓ^2-valued random variables $X_n = r_n e_n$ (where $\{r_n\}$ is a sequence of independent symmetric real random variables) satisfy the SLLN iff $1/n^2 \sum_{j=1}^{n} r_j^2 \to 0$ a.s.

<u>Remarks 3.25</u>.

(a) Theorem 3.22 is an improvement of Theorem 3.5 (Hoffmann-Jorgensen and Pisier) for B = H. If $\sum_{j=1}^{\infty} E\|X_j\|^2/j^2 < \infty$ then $\sum_{j=1}^{\infty}\|X_j\|^2/j^2 < \infty$ a.s. so that by Kronecker's Lemma, $1/n^2 \sum_{j=1}^{n} \|X_j\|^2 \to 0$ a.s. Thus (3.38) holds.

Also,

$$\sum_{k=2}^{\infty} k^{-4} \sum_{j=1}^{k-1} E(X_j,X_k)^2 \leq \sum_{k=2}^{\infty} k^{-4} \sum_{j=1}^{k-1} E\|X_j\|^2 E\|X_k\|^2$$

$$\leq \sum_{k=2}^{\infty} E\|X_k\|^2/k^2 \sum_{j=1}^{\infty} E\|X_j\|^2/j^2 < \infty$$

so that (3.39) holds. See Heinkel [15] for another improvement.

(b) Heinkel [15] has shown that (3.38) is implied by the condition $\sum_{j=1}^{n} E\|X_j\|^2/n^2 \to 0$ if $\|X_j\| \leq Cj/(LLj)^{1/2} (j \geq 1)$. The latter is clearly a more verifiable condition. This remark provides an analog to Lemma 2.1.

(c) One might ask whether (3.38) may be entirely dispensed with, as Egorov did for $B = \mathbb{R}$. Such a claim can easily be seen to be false by considering the ℓ^2-valued sequence $X_n = n\varepsilon_n e_n (n \geq 1)$. (3.39) is obviously satisfied and (3.38) obviously fails, so that the SLLN must fail by Theorem 2.1.

CHAPTER IV

THE BOUNDED LAW OF THE ITERATED LOGARITHM

In this chapter we will consider a sequence $\{X_n\}_{n=1}^{\infty}$ of independent random variables with values in an arbitrary Banach space B. We will not place any restrictions on the Banach space or on the moments of the $\{X_n\}$ sequence. In particular, we will not consider analogues of Theorems 2.1 and 3.14. We shall instead study specific kinds of sequences, especially Rademacher and independent Gaussian sequences.

One has to distinguish, in the Banach space setting, between the Bounded and Compact versions of the LIL (the definitions were provided in the Introduction). The formulation of both versions is due to Kuelbs [22,24] (the former reference treats the i.i.d. case while the latter deals with an extension of the classical Kolmogorov LIL (the non i.i.d. case)). We shall now state Kuelbs' basic bounded LIL in the non-i.i.d. case.

Theorem 4.1. (Kuelbs [24]). Let $\{X_n\}$ be a sequence of independent, zero mean, B valued random variables satisfying

(4.1) $\quad \|X_n\| \leq \Gamma_n \sigma_n/(LL\sigma_n^2)^{1/2}$ a.s. $(n \geq 1)$ where $\Gamma_n \to 0$ and

$$\sigma_n^2 = \sum_{j=1}^{n} E\|X_j\|^2 \to \infty.$$

(4.2) There exists $L > 0$ such that $\sup_n P(\|S_n\| > La_n) \leq 1/24$, where

$a_n = (2\sigma_n^2\, LL\sigma_n^2)^{1/2}$.

Then there exists a constant $\Lambda \in [0,\infty)$ such that

$$\limsup_{n \to \infty} \|S_n\|/a_n = \Lambda \text{ a.s.}$$

Notice that our definition of the BLIL (see Introduction) and that of Kuelbs are similar if B is a Hilbert space. We shall first prove a BLIL for Rademacher sequence $\{\varepsilon_n x_n\}$, $\{x_n\} \subset B$ under weaker hypotheses than (4.1) and (4.2). The method of proof relies on a result of Marcus and Zinn [32]. They generalize a Theorem and a construction of Volodin and Nagaev [46]. Let us describe this construction:

Given an increasing sequence $\{b_n\}$, fix $C > 1$ and consider the intervals $(0,C]$, $(C,C^2]$, ... From these, discard the ones for which $\{b_n\} \cap (C^k,C^{K+1}] = \phi$ and label the rest $(C^{t_r},C^{t_r+1}]$ $(r = 1,2,\ldots)$ in such a way that $t_r < t_{r+1}$. In other words, $(C^{t_r},C^{t_r+1}]$ is the rth interval having a non-empty intersection with the sequence $\{b_n\}$. Let $n_r = \sup\{n : b_n \in (C^{t_r},C^{t_r+1}]\}$ and consider the sequence $\{b_n\}$. It is clear that $t_r \geq r$. We shall, following Marcus and Zinn, call $\{n_r\}$ the Volodin-Nagaev (NV) subsequence determined by $\{b_n\}$ and C.

<u>Lemma 4.2</u>. (Marcus and Zinn [32]). Let $\{X_n\}$ be a sequence of independent B-valued random variables. Let $C > 1$ and $\{b_n\} \uparrow \infty$ be arbitrary. Assume that for some $a > 0$,

(4.3) $$\lim_{n\to\infty} P(\|S_n\| > a\, b_n) = 0 \, .$$

(4.4) For the NV subsequence $\{n_r\}$ determined by $\{b_n\}$ and C,

$$\sum_{r=1}^{\infty} P(\|S_{n_r}-S_{n_{r-1}}\| > 2ab_{n_r}) < \infty .$$

Then

(4.5) $$\limsup_{n\to\infty} \|S_n\|/b_n \leq \{(4C+[2/C-1])a\} \text{ a.s.}$$

We are now ready to state our BLIL for Rademacher sequences. Our relaxing of (4.1) in no way contradicts the classical examples of Marcinkiewicz and Zygmund [30] since we do not wish to insist that we obtain $\Lambda = 1$.

Theorem 4.3. Consider the Rademacher sequence $\{\varepsilon_n x_n\}$, where $\{x_n\} \subset B$. Assume that

$s_n^2 = E\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2 \to \infty$ as $n \to \infty$. Then there exists $\Lambda \in [0,\infty)$ such that

$$\limsup_{n\to\infty} \|\sum_{j=1}^{n} \varepsilon_j x_j\|/(2s_n^2\, LLs_n^2)^{1/2} = \Lambda \text{ a.s.}$$

Proof: Let $b_n = (2s_n^2\, LLs_n^2)^{1/2}$ $(n \geq 1)$. Theorem 2.6 in Hoffmann-Jorgensen [18] shows that $\{s_n^2\}$, and hence $\{b_n\}$, are increasing sequences. By Kolmogorov's zero-or-one law, we need to show that $\limsup_{n\to\infty} \|S_n\|/b_n < \infty$ a.s. Let $\{n_r\}$ be the NV subsequence based on $\{b_n\}$ and e. Let $\lambda > 0$ be arbitrary (we will choose a specified λ later). By Lemma 3.7,

$$P(\|S_n\| > \lambda\, b_n) = P(\|S_n\|/s_n > \lambda(2LLs_n^2)^{1/2}) \leq 3 \exp(-2\lambda^2 LLs_n^2/M^2) \to 0,$$

so that (4.3) holds for each $\lambda > 0$. Also,

$$(4.6)\quad P(\|S_{n_r}-S_{n_{r-1}}\| > 2\lambda b_{n_r}) \le$$

$$P(\|S_{n_r}\| > \lambda\, b_{n_r}) + P(\|S_{n_{r-1}}\| > \lambda\, b_{n_{r-1}}) =$$

$$P(\|S_{n_r}\|/s_{n_r} > \lambda(2LLs^2_{n_r})^{1/2}) + P(\|S_{n_{r-1}}\|/s_{n_{r-1}} > \lambda(2LLs^2_{n_{r-1}})^{1/2}) \le$$

$$6\exp(-2\lambda^2\, LLs^2_{n_{r-1}}/M^2)\ .$$

Recall that $n_r = \sup\{n : (2s_n^2 LLs_n^2)^{1/2} \in (e^{t_r}, e^{t_r+1})\}$. It is clear that $(2s^2_{n_r} LLs^2_{n_r})^{1/2} \ge e^{t_r}$ so that $\log 2 + \log s^2_{n_r} + LLLs^2_{n_r} \ge 2t_r$. It follows that $\log s^2_{n_r} \ge t_r \ge r$. (4.6) now yields

$$P(\|S_{n_r}-S_{n_{r-1}}\| > 2\lambda b_{n_r}) \le 6\exp(-2\lambda^2 \log(r-1)/M^2)$$

proving that (4.4) holds for $\lambda > M/\sqrt{2}$. It follows that

$$\limsup_{n\to\infty} \|S_n\|/b_n \le \frac{M}{\sqrt{2}}\ (4e+2/e-1) \text{ a.s.}$$

We turn next to the Gaussian case. Laws of the Iterated Logarithm for independent B-valued Gaussian random variables have been proved by Mangano [29] and Carmona and Kono [6]. We have, for example, the following.

Theorem 4.4. (Carmona and Kono [6]). Let $\{X_n\}$ be a sequence of independent Gaussian random variables with values in a Banach space B. Assume that $b_n \uparrow \infty$ and that $S_n/b_n^{1/2}$ converges in distribution to a (Gaussian) random variable X. We then have

(4.7) $$P(d(S_n/(2b_n\phi(b_n))^{1/2},D) \to 0) = 1$$

and

(4.8) $$P(C\{S_n/(2b_n\phi(b_n))^{1/2}\} = D) = 1$$

for each "admissible" function ϕ.

Remark: In the above theorem, D is the unit ball of the Reproducing Kernel Hilbert Space determined by L(X), and C(A) denotes the set of cluster points of the set A. An admissible function ϕ is defined as follows: Given a sequence $\{b_n\}$ as in Theorem 4.4 define the sequence $\{n_K\}$ by

$$n_1 = \inf\{n \geq 1 \mid b_n \neq 0\}$$

$$n_K = \inf\{n \geq 1 \mid b_n \geq e\, b_{n_{K-1}}\} \quad (K \geq 2) .$$

A strictly positive, increasing function ϕ on $(0,\infty)$ is said to be admissible if $\phi(b_{n_K}) = \log K$ $(K \geq 2)$.

If we take $b_n = s_n^2$, the function LLx clearly need not be admissible in general. Carmona and Kono show, however, that LLx is admissible if $C_1 n \leq s_n^2 \leq C_2 n$ for some $C_1, C_2 \in (0,\infty)$.

We shall prove a BLIL for independent B-valued Gaussian sequencs under less stringent conditions than those of Theorem 4.4. Also, we will prove the result for the (usually) inadmissible but more natural function LLx. The following basic inequality is due to Fernique.

Lemma 4.5. (Fernique [11,12]). Let X be a B-valued Gaussian random variable. Then there is a constant $N = N(B)$ such that for each $\varepsilon > 0$,

$$P(\|X\| > \varepsilon(E\|X\|^2)^{1/2}) \leq 3\exp(-\varepsilon^2/N^2). \tag{4.9}$$

Theorem 4.6. Let $\{X_n\}$ be a sequence of independent B-valued Gaussian random variables such that $s_n^2 = E\|\sum_{j=1}^{n} X_j\|^2 \to \infty$ as $n \to \infty$. Then there exists $\Lambda \in [0,\infty)$ such that

$$\limsup_{n \to \infty} \|S_n\|/(2s_n^2\, LLs_n^2)^{1/2} = \Lambda \text{ a.s.}$$

Proof: Exactly the same as that of Theorem 4.3.

The results of Kwapien and Fernique yielded BLIL's for Rademacher and Gaussian sequences respectively. One may, in the same way, use the following result of Kuelbs (see DeAcosta [1,2] for related results) to prove BLIL's for other classes of variables.

Lemma 4.7. (Kuelbs [23]). Let B be a cotype 2 Banach space with $\{x_n\} \subset B$. Let $\{Y_n\}$ be a sequence of i.i.d. real-valued random variables such that $EY_1 = 0$ and $E(\exp(\beta Y_1^2)) < \infty$ for some $\beta > 0$. If $S = \sum_{j=1}^{\infty} Y_j x_j$ converges a.s. then $E(\exp \beta\|S\|^2) < \infty$.

Proposition 4.8. Let $\{Y_n\}$, $\{x_n\}$ and B be as in Lemma 4.7. Assume Assume that $s_n^2 = E\|\sum_{j=1}^{n} Y_j x_j\|^2 \to \infty$ and let $S_n = \sum_{j=1}^{n} Y_j x_j$. Then there exists $\Lambda \in [0,\infty)$ such that

$$\limsup_{n \to \infty} \|S_n^2\|/(2s_n^2 LLs_n^2)^{1/2} = \Lambda \text{ a.s.}$$

Proof: Exactly the same as that of Theorems 4.3 and 4.6.

BIBLIOGRAPHY

BIBLIOGRAPHY

1) A. DeACOSTA, Exponential Moments of Vector Valued Random Series and Triangular Arrays, Ann. Probability, Vol. 8, 1980, pp. 381-389.

2) A. DeACOSTA, Strong Exponential Integrability of Sums of Independent B-Valued Random Vectors, Probability and Mathematical Statistics, Vol. 1, 1981, pp. 133-150.

3) A. BECK, A Convexity Condition in Banach Spaces and the Strong Law of Large Numbers, Proc. Amer. Math. Soc., Vol. 13, 1962, pp. 329-334.

4) A. BECK, On the Strong Law of Large Numbers, Ergodic Theory, F. B. Wright, ed., Academic Press, New York, 1963.

5) A. BECK, D. P. GIESY and P. WARREN, Recent Developments in the Theory of Strong Law of Large Numbers for Vector Valued Random Variables, Theor. Probability Appl. 20, 1975, pp. 127-134.

6) R. CARMONA and N. KONO, Convergence en loi et Lois du Logarithme Itéré pour les Vecteurs Gaussiens, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, Vol. 36, 1976, pp. 241-267.

7) Y. S. CHOW, A Martingale Inequality and the Law of Large Numbers, Proc. Amer. Math. Soc., Vol. 11, 1960, pp. 107-111.

8) K. L. CHUNG, The Strong Law of Large Numbers, Proceedings of the 2nd Berkeley Symposium on Statistics and Probability, 1951, pp. 341-352.

9) R. M. DUDLEY, The Sizes of Compact Subsets of Hilbert Space and Continuity of Gaussian Processes, J. Funct. Anal., Vol. 1, 1967, pp. 290-330.

10) V. A. EGOROV, On the Strong Law of Large Numbers and the Law of the Iterated Logarithm for Sequences of Independent Random Variables, Theor. Probability Appl., Vol. 15, 1970, pp. 509-513.

11) X. FERNIQUE, Intégrabilité des Vecteurs Gaussiens, C. R. Acad. Sc., Paris, Series A, Vol. 270, 1970, pp. 1698-1699.

12) X. FERNIQUE, Régularité des Trajectoires des Fonctions Aléatories Gaussiennes, Lecture Notes in Mathematics, Vol. 480, Springer Verlag, Berlin, 1975, pp. 1-96.

13) D. GILAT, Every Non-negative Submartingale is the Absolute Value of a Martingale, Ann. Probability, Vol. 5, 1977, pp. 475-481.

14) E. GINÉ and J. ZINN, Central Limit Theorems and Weak Laws of Large Numbers in Certain Banach Spaces, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, Vol. 62, 1983, pp. 323-354.

15) B. HEINKEL, On the Law of Large Numbers in 2-Uniformly Smooth Banach Spaces, 1983, to appear in Ann. Probability.

16) B. HEINKEL, Une Extension de la loi des Grands Nombres de Prokhorov, Preprint, 1983, University of Strasbourg.

17) B. HEINKEL, The non i.i.d. Strong Law of Large Numbers in 2-Uniformly Smooth Banach Spaces, Preprint, 1984, University of Strasbourg.

18) J. HOFFMANN-JORGENSEN, Sums of Independent Banach Space Valued Random Variables, Studia Math., Vol. 52, 1974, pp. 159-186.

19) J. HOFFMANN-JORGENSEN and G. PISIER, The Law of Large Numbers and the Central Limit Theorem in Banach Spaces, Ann. of Probability, Vol. 4, 1976, pp. 587-599.

20) K. ITO and M. NISIO, On the Convergence of Sums of Independent Banach Space Valued Random Variables, Osaka J. Math., Vol. 5, 1968, pp. 35-48.

21) J. P. KAHANE, Some Random Series of Functions, D. C. Heath and Company, Lexington, Massachusetts, 1968.

22) J. KUELBS, A Strong Convergence Theorem for Banach Space Valued Random Variables, Ann. Probability, Vol. 4, 1976, pp. 744-771.

23) J. KUELBS, Some Exponential Moments of Sums of Independent Random Variables, Trans. Amer. Math. Soc., Vol. 240, 1978, pp. 145-162.

24) J. KUELBS, Kolmogorov's Law of the Iterated Logarithm for Banach Space Valued Random Variables, Illinois J. Math., Vol. 21, 1977, pp. 784-800.

25) J. KUELBS and J. ZINN, Some Stability Results for Vector Valued Random Variables, Ann. Probability, Vol. 7, 1979, pp. 75-84.

26) S. KWAPIEN, Isomorphic Characterizations of Inner Product Spaces by Orthogonal Series with Vector Valued Coefficients, Studia Math., Vol. 44, 1972, pp. 583-595.

27) S. KWAPIEN, A Theorem on the Rademacher Series with Vector Coefficients, Probability in Banach Spaces, Lectures Notes in Math. 526, pp. 157-158, Springer-Verlag, Berlin.

28) M. LOEVE, Probability Theory, 3rd. Edition, D. Van Nostrand Co., Inc., Princeton, New Jersey, 1963.

29) G. C. MANGANO, On Strassen-type Laws of the Iterated Logarithm for Gaussian Elements in Abstract Spaces, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, Vol. 36, 1976, pp. 227-239.

30) J. MARCINKIEWICZ and A. ZYGMUND, Remarque sur la Loi du Logarithme Itere, Fund. Math., Vol. 29, 1937, pp. 215-222.

31) M. B. MARCUS and G. PISIER, Random Fourier Series with Applications to Harmonic Analysis, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1981.

32) M. B. MARCUS and J. ZINN, The Bounded Law of the Iterated Logarithm for the Weighted Empirical Distribution Process in the Non-I.I.D. Case. To appear in Ann. of Probability, May, 1984.

33) B. MAUREY, Espaces de Cotype p ($0 < p \leq 2$), Sem. Maurey-Schwartz, 1972-73, Ecole Polytechnique, Paris.

34) B. MAUREY and G. PISIER, Séries des Variables Aléatoires Vectorielles Independentes et Proprietés Géométriques des Espaces de Banach, Studia Math, Vol. 58, 1976, pp. 45-90.

35) E. MOURIER, Eléments aléatoires dans un espace de Banach, Ann. Inst. H. Poincaré, Vol. 13, 1952-53, pp. 159-244.

36) S. V. NAGAEV, On Necessary and Sufficient Conditions for the Strong Law of Large Numbers, Theor. Probability Appl., Vol. 17, 1972, pp. 573-581.

37) S. V. NAGAEV and N. A. VOLODIN, On the Strong Law of Large Numbers, Theor. Probability Appl., Vol. 20, 1975, pp. 626-631.

38) G. PISIER, Type des Espaces Normes, Sem. Maurey-Schwartz, 1973-74, Ecole Polytechnique, Paris, pp. III•1-III•11.

39) G. PISIER, Sur les espaces qui ne contiennent pas de l_n^1 uniformement, Sem. Maurey-Schwartz, 1973-74, Ecole Polytechnique, Paris, pp. VII•1-VII•19.

40) G. PISIER, Sur la Loi du Logarithme Itere dans les Espaces de Banach, Probability in Banach Spaces, Lecture Notes in Math 526, pp. 203-210, Springer-Verlag, Berlin.

41) G. PISIER, Le Theoreme de la Limite Centrale et la Loi du Logarithme Itere dans les Espaces de Banach, Sem. Maurey-Schwartz, 1975-76, Exposes III, IV and Annexe 1, Ecole Polytechnique, Paris.

42) YU. V. PROKHOROV, Some Remarks on the Strong Law of Large Numbers, Theor. Probability Appl., Vol. 4, 1959, pp. 204-208.

43) W. F. STOUT, Almost Sure Convergence, Academic Press, New York, 1974.

44) W. F. STOUT, A Martingale Analogue of Kolmogorov's Law of the Iterated Logarithm, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, Vol. 15, 1970, pp. 279-290.

45) H. TEICHER, Some new Conditions for the Strong Law, Proc. Nat. Acad. Sci. U.S.A., Vol. 59, 1968, pp. 705-707.

46) N. A. VOLODIN and S. V. NAGAEV, A Remark on the Strong Law of Large Numbers, Theor. Probability Appl., Vol. 23, 1978, pp. 810-813.

47) W. A. WOYCZYNSKI, Random Series and Law of Large Numbers in Some Banach Spaces, Theor. Probability Appl., Vol. 18, 1973, pp. 371-377.

48) V. V. YURINSKII, Exponential Bounds for Large Deviations, Theor. Probability Appl., Vol. 19, 1974, pp. 154-155.

49) J. ZINN, Private Communications.

50) N. DUNFORD and J. T. SCHWARTZ, Linear Operators, Vol. I, John Wiley and Sons, New York, 1957.