A DARBOUX PROPERTY FOR THE GRADIENT

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
RAYMOND PETER GOEDERT
1972

This is to certify that the

thesis entitled

A DARBOUX PROPERTY

FOR THE GRADIENT

presented by

Raymond Peter Goedert

has been accepted towards fulfillment
of the requirements for

Ph. D. degree in Mathematics

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Date July 28, 1972

O-7639

ABSTRACT

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By

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A real-valued function f of a real variable is said to have the Darboux property if $f(I)$ is connected for every closed interval I. A function is said to be Baire 1 if it is the pointwise limit of a sequence of continuous functions.

In Chapter I of this paper we discuss the following generalization of the Darboux property. (It is due to A.M. Bruckner and J.B. Bruckner.) If X and Y are topological spaces and $\mathfrak{B}$ is a basis for X, then a function $f : X \to Y$ is said to be Darboux $(\mathfrak{B})$ if $f(\bar{U})$ is connected for all $U \in \mathfrak{B}$. If $X = Y = E_1$ (the real line) and $\mathfrak{B}$ is the basis of open intervals for E_1, then this reduces to the usual notion of the Darboux property.

C.J. Neugebauer characterized (in terms of inverse images of closed sets) the class of Baire 1 functions having the ordinary Darboux property. In

Chapter II we consider the family $\mathfrak{Z}$ consisting of Baire 1 functions mapping E_n (Euclidean n-space) into a fixed separable metric space Y. A theorem is proved establishing a condition which distinguishes those functions in $\mathfrak{Z}$ which are Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis for E_n satisfying certain restrictions. The condition is stated in terms of inverse images of closed sets and is similar to Neugebauer's condition for Baire 1 functions having the ordinary Darboux property. Examples are given showing the various hypotheses in the theorem cannot be omitted. An example is also given showing the theorem is no longer valid if the domain E_n is replaced by an infinite-dimensional normed linear space.

A function $f : E_n \to E_1$ is said to be differentiable at x if

$$f(y) - f(x) = (y-x)\cdot \text{grad } f(x) + o(|y-x|)$$

as $y \to x$. A function is said to be differentiable if it is differentiable at x for all $x \in E_n$. In Chapter III we consider real-valued functions defined on E_n, each having a gradient everywhere in E_n. It is shown that the gradient of a differentiable function is Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis for E_n consisting of

connected sets satisfying a certain restriction. An example is given showing that this restriction cannot be omitted. An example is also presented showing that the gradient of a function which fails to be differentiable at even a single point need not be Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis whatever for E_n. In addition, a Darboux property for partial derivatives and an "intermediate-value property" for the norm of the gradient of a differentiable function are established.

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1972

ACKNOWLEDGEMENTS

I wish to express my gratitude to Professor Clifford E. Weil for suggesting the problems studied in this thesis and for his kind guidance throughout its preparation. I also wish to thank Professor Jan Mařík for many most valuable suggestions.

Finally I wish to thank my wife, Ann, for her patience and encouragement.

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CHAPTER I

INTRODUCTION, DEFINITIONS, AND NOTATION

Throughout this paper E_n will denote n-dimensional Euclidean space. The interior of a set S will be denoted by Int S, the boundary by Bd S, and the closure by $\bar{S}$.

A function $f : E_1 \to E_1$ is said to have the Darboux property if, given any a and b, $a < b$, and any c between $f(a)$ and $f(b)$, there is some $t \in (a,b)$ such that $f(t) = c$. (This defining property is usually termed the intermediate value property. A detailed exposition of functions with this property can be found in the survey article [3].) This definition gave rise to the following generalization due to Mišík [5].

Definition 1.1. Let X be a topological space, and let $\mathfrak{B}$ be a basis for X. A function $f : X \to E_1$ is said to have the strong Darboux property relative to $\mathfrak{B}$ if, given any $U \in \mathfrak{B}$ and any real number c for which there exist x_1 and x_2 in $\bar{U}$ with $f(x_1) < c < f(x_2)$, there is some $t \in U$ such that $f(t) = c$.

Notice that, if $X = E_1$ and $\mathfrak{B}$ is the basis for E_1 consisting of all open intervals, then this reduces to the ordinary Darboux property.

Now the main purpose of this paper is to establish a Darboux property of some sort for the gradient. Thus Mišík's generalization is not the appropriate one, for it requires the functions under consideration to be real-valued which the gradient is not.

The ordinary Darboux property for a real-valued function of a real variable can also be defined in the following fashion. (This definition is equivalent to the one given above.) A function $f : E_1 \to E_1$ is said to have the Darboux property if $f(I)$ is connected for each closed interval I. This definition gave rise to the following generalization due to Bruckner and Bruckner [2].

Definition 1.2. Let X and Y be topological spaces, and let $\mathfrak{B}$ be a basis for X. A function $f : X \to Y$ is said to have the Darboux property relative to $\mathfrak{B}$ (or, more briefly, is said to be Darboux $(\mathfrak{B})$) if $f(\bar{U})$ is connected for all $U \in \mathfrak{B}$.

It is this generalization of the Darboux property which will be used in this paper. Notice that, if $X = Y = E_1$ and $\mathfrak{B}$ is the basis for E_1 consisting of all open intervals, then this reduces to the ordinary Darboux property.

Suppose now that $\mathfrak{B}$ is a basis for a topological space X. Then, if a function $f : X \to E_1$ has Mišík's strong Darboux property relative to $\mathfrak{B}$, it is trivial to show that f is also Darboux ($\mathfrak{B}$). However, as the following example shows, the converse is not true.

Example 1.1. Let $\mathfrak{B}$ be the basis for E_2 consisting of the open rectangle R with corners at $(0,1)$, $(-1,1)$, $(-1,-1)$, and $(0,-1)$ and all open balls not tangent to the y-axis. Define $f : E_2 \to E_1$ as follows.

$$f(x,y) = \begin{cases} \sin\left(\frac{1}{x+1}\right), & x < -1 \\ 0, & -1 \leq x < 0 \\ 0, & x = 0 \text{ and } |y| > 1 \\ y, & x = 0 \text{ and } |y| \leq 1 \\ \sin\left(\frac{1}{x}\right), & x > 0 \end{cases}$$

It will be shown in Example 2.11 that f is Darboux ($\mathfrak{B}$). But f does not have the strong Darboux property relative to $\mathfrak{B}$, for $f(\bar{R}) = [-1,1]$ while $f(R) = \{0\}$.

There are certain terms and notational conventions which will be used throughout this paper. They are presented at this point.

Definition 1.3. Let (X,d) be a metric space. Then an open ball is denoted and defined as follows:

$$B(a,r) = \{x \in X : d(x,a) < r\}.$$

Definition 1.4. A topological space is said to be separable if it contains a countable dense subset.

Definition 1.5. A function is said to be Baire 1 if it is the pointwise limit of a sequence of continuous functions.

Definition 1.6. A function $f : E_n \to E_1$ is said to be differentiable at x if

$$f(y) - f(x) = (y-x)\cdot \text{grad } f(x) + o(|y-x|)$$

as $y \to x$, where $\cdot$ denotes the usual scalar product in E_n and grad $f(x)$ is the vector whose ith coordinate is the partial derivative of f with respect to the ith variable evaluated at x. The function f is said to be differentiable if it is differentiable at x for each $x \in E_n$.

Neugebauer [6] characterized (in terms of inverses images of closed sets) the class of Baire 1 functions having the ordinary Darboux property. (Also see Weil [7].) In Chapter II we consider the family $\mathfrak{F}$ consisting of Baire 1 functions mapping E_n into a fixed separable metric space Y. A theorem is proved establishing a condition which distinguishes those functions in $\mathfrak{F}$ which are Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis for E_n satisfying certain restrictions. The condition is stated in terms of inverse images of closed sets and is similar to Neugebauer's condition for Baire 1 functions having the ordinary Darboux property. Several examples are given demonstrating that the various hypotheses of the theorem cannot be omitted. In particular, a rather lengthy example is presented showing that the domain E_n cannot be replaced by a general normed linear space (in fact, not even by a separable Banach space) if the theorem is to remain valid.

There is a theorem due to Bruckner and Bruckner [2] establishing another condition which distinguishes those functions in $\mathfrak{F}$ which are Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis for E_n satisfying certain restrictions. The condition is stated in terms of inverse images of open sets and is similar to a condition due to Zahorski [8] for Baire 1 functions having the ordinary Darboux property. Bruckner and Bruckner's result follows quite easily from the theorem mentioned in the last paragraph and is presented as a simple consequence thereof.

In Chapter III the results of Chapter II are applied to the gradient. (When we consider grad f here, we tacitly assume f is such that grad f exists everywhere in E_n.) An example is presented showing that the gradient of a function which fails to be differentiable at even a single point need not be Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis whatever for E_n. Using a theorem from Chapter II, it is shown that the gradient of a differentiable function is Darboux ($\mathfrak{B}$) provided each element in $\mathfrak{B}$ is connected and none comes to a cusp on its boundary.

The major result in Chapter III has some rather interesting corollaries. In particular, a Darboux property is established for partial derivatives. In addition, it is shown that the norm of the gradient of a differentiable function has Mišík's strong Darboux property provided certain restrictions are placed on the basis for E_n under consideration.

CHAPTER II

A GENERALIZED DARBOUX PROPERTY

In [7] Weil defines a ball closed G_δ set as follows. A G_δ set $H \subset E_n$ is called a ball closed G_δ set if, whenever $B(x,r) \subset H$, one has $\mathrm{Bd}\, B(x,r) \subset H$. Neugebauer proved in [6] (without using Weil's terminology) that a real-valued function f of a real variable is Baire 1 and has the ordinary Darboux property if and only if the sets $\{x : f(x) \geq a\}$ and $\{x : f(x) \leq a\}$ are ball closed G_δ sets for each real number a.

In this chapter we shall consider the family $\mathfrak{F}$ consisting of all Baire 1 functions mapping E_n into a fixed separable metric space Y. As remarked earlier, the main objective of Chapter II is to establish a condition distinguishing those functions in $\mathfrak{F}$ which are Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis for E_n satisfying certain restrictions which will be introduced later. The condition will be stated in terms of inverse images of closed sets and will be similar to Neugebauer's condition for Baire 1 functions having the ordinary Darboux property. To establish this condition it is first necessary to isolate the "ball closed" part of the ball closed G_δ concept.

Definition 2.1. Let X be a topological space, and let $\mathfrak{B}$ be a basis for X. A set $S \subset X$ is said to be $\mathfrak{B}$ closed if, for each $U \in \mathfrak{B}$, $\text{Bd } U \subset S$ whenever $U \subset S$.

Example 2.1. Clearly any closed set in E_n is $\mathfrak{B}$ closed no matter what the basis $\mathfrak{B}$ is. However, a set can be $\mathfrak{B}$ closed without being closed. Let $\mathfrak{R}$ be the open-ball basis for E_2. Let

$$S = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ (x,y) \neq (1,1)\}.$$

S is not closed, but S is $\mathfrak{B}$ closed.

Example 2.2. Even an open set in E_n can be $\mathfrak{B}$ closed. Let $\mathfrak{B}$ be the basis for E_1 consisting of all open intervals except those having 0 as a right endpoint. Then $(-\infty,0)$ is both open and $\mathfrak{B}$ closed.

Example 2.3. A set which has no interior is vacuously $\mathfrak{B}$ closed no matter what $\mathfrak{B}$ is. For example, in E_1 both Q (the set of rationals) and $E_1 \sim Q$ are $\mathfrak{B}$ closed for every basis $\mathfrak{B}$.

Most of the results in this chapter require that certain restrictions be placed on the bases under consideration. The three restrictions that will be used are presented here with some discussion.

Definition 2.2. Let X be a metric space, and let $\mathfrak{B}$ be a basis for X. Then $\mathfrak{B}$ is said to satisfy (1) if, given any open ball B and any $x \in \mathrm{Bd}\, B$, there is some $V \in \mathfrak{B}$ such that $V \subset B$ and $x \in \mathrm{Bd}\, V$.

Definition 2.3. Let X be a metric space, and let $\mathfrak{B}$ be a basis for X. Then $\mathfrak{B}$ is said to satisfy (2), if, given any $U \in \mathfrak{B}$ and any $x \in \bar{U}$, there is some $V \in \mathfrak{B}$ such that $\bar{V} \sim \{x\} \subset U$ and $x \in \bar{V}$.

Definition 2.4. Let X be a metric space, and let $\mathfrak{B}$ be a basis for X. Then $\mathfrak{B}$ is said to satisfy (3) if, given any $U \in \mathfrak{B}$ and any $x \in U$, there is some $V \in \mathfrak{B}$ such that $V \subset U$ and $x \in \mathrm{Bd}\, V$.

Condition (1) says that, given any ball and any point on its boundary, you can touch that point with a basis element lying inside the ball. Condition (2) says that, given any basis element U and any point x in its closure, you can touch x with a basis element whose closure (except possibly for x) lies inside U. Condition (3) says that, given any basis element U and any point x in U, you can find another basis element contained in U having x on its boundary.

Conditions (1) and (2) are independent as Examples 2.4 and 2.5 show. Conditions (2) and (3) are independent as Examples 2.5 and 2.6 show. Example 2.7 shows that condition (3) does not imply condition (1). In general, condition (1) does not imply condition (3) either. For example, any basis for a discrete space satisfies (1) but not (3). However, if $\mathfrak{B}$ is a basis for E_n which satisfies (1), $\mathfrak{B}$ must also satisfy (3).

Example 2.4. Let $\mathfrak{B}$ be the basis for E_2 consisting of all open balls and the open rectangle R with corners at $(0,1)$, $(-1,1)$, $(-1,-1)$, and $(0,-1)$. $\mathfrak{B}$ satisfies (1), for clearly, given any ball and any point on its boundary, you can touch that point with another ball (which will be a basis element) lying inside the given ball. But $\mathfrak{B}$ does not satisfy (2), for you cannot touch any corner x of the rectangle with a basis element whose closure (except for x) lies inside the rectangle.

Example 2.5. Let $\mathfrak{B}$ be the basis for E_1 consisting of all open intervals not having 0 as an endpoint. $\mathfrak{B}$ clearly satisfies (2). However $\mathfrak{B}$ does not satisfy (3). For example, $(-1,1)$ is a basis element and 0 is in its interior. But there is no basis element lying in $(-1,1)$ having 0 as a boundary point. Notice that $\mathfrak{B}$ does not satisfy (1) either.

Example 2.6. Let $\mathfrak{B}$ be the basis for E_2 consisting of the open rectangle R with corners at $(0,1)$, $(-1,1)$, $(-1,-1)$, and $(0,-1)$ and all open balls not tangent to the y-axis. $\mathfrak{B}$ satisfies (3), for clearly, given any basis element and any point x inside it, there is an open ball which is not tangent to the y-axis (and, hence, is a basis element) lying inside the given basis element and having x as a boundary point. However $\mathfrak{B}$ does not satisfy (2), for you cannot touch any corner x of the rectangle with a basis element whose closure (except for x) lies inside the rectangle.

Example 2.7. Let $\mathfrak{B}$ be the basis for E_1 consisting of all open intervals except those having 0 as a right endpoint. Clearly $\mathfrak{B}$ satisfies (3) but not (1).

Remark. Most of the results in this chapter require that at least one of the three restrictions mentioned above be placed on the basis $\mathfrak{B}$ under consideration. It should be noted that, if $\mathfrak{B}$ is the basis for E_n consisting of all open balls, $\mathfrak{B}$ automatically satisfies (1), (2), and (3). However, as Bruckner and Bruckner mention in [2], this offers no help in overcoming the difficulties caused by other bases.

The following theorem is well known. It was originally proved (in an equivalent form) by R. Baire. (See, for example, [4, p.326].) It is presented here without proof.

Theorem 2.1. Let X be a complete metric space and Y a separable metric space. Let $f : X \to Y$ be Baire 1. Then, for any non-void G_δ set $S \subset X$, $f|S$ (f restricted to S) has a point of continuity.

The following theorem is the major result in this chapter.

Theorem 2.2. Let Y be a separable metric space. Let $f : E_n \to Y$ be Baire 1. Let $\mathfrak{B}$ be a basis of connected sets for E_n satisfying (1).

Then, if $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$, f is Darboux $(\mathfrak{R})$.

Proof. Let $f^{-1}(K)$ be $\mathfrak{B}$ closed for all closed $K \subset Y$. Suppose however that f is not Darboux $(\mathfrak{B})$. Then there is some $U \in \mathfrak{B}$ such that $f(\bar{U})$ is not connected. Hence there exist sets A^* and B^*, closed in Y, such that

$$f(\bar{U}) = (f(\bar{U}) \cap A^*) \cup (f(\bar{U}) \cap B^*)$$

is a decomposition of $f(\bar{U})$ into two disjoint, non-void, relatively closed sets.

Let $A = \{x \in \bar{U} : f(x) \in A^*\}$. Let $B = \{x \in \bar{U} : f(x) \in B^*\}$. Notice that $\bar{U} = A \cup B$. Let $P = U \cap \mathrm{Bd}\, A$. Notice that $P = U \cap \mathrm{Bd}\, B$ also since A and B are disjoint, $U \subset A \cup B$, and U is open.

It is claimed that $P \neq \emptyset$, for suppose P is void. Then each $x \in U$ is in either $\mathrm{Int}\, A$ or $\mathrm{Int}\, B$. Hence, since U is connected, either $U \subset A$ or $U \subset B$. If $U \subset A$, $\mathrm{Bd}\, U \subset A$ since $f^{-1}(A^*)$ is $\mathfrak{B}$ closed. Hence $B = \emptyset$ which is a contradiction. If $U \subset B$, $\mathrm{Bd}\, U \subset B$ since $f^{-1}(B^*)$ is $\mathfrak{B}$ closed. Hence $A = \emptyset$ which is a contradiction.

Notice that P is clearly a G_δ set.

It is claimed that $A \cap P$ is dense in P, for suppose not. Then there exists some $b_o \in P$ and some open ball $N = B(b_o,r) \subset U$ such that $N \cap (A \cap P) = \emptyset$. That is, there are no points of $A \cap \text{Bd } A$ in N. In particular, $b_o \notin A \cap \text{Bd } A$; so $b_o \in B$. Since $b_o \in B \cap \text{Bd } A$, there exists some $a_o \in A \cap B(b_o,\frac{r}{2})$. Let $d = \text{dist}(a_o,\bar{B})$. Then $0 < d < \frac{r}{2}$. The second inequality is clear. The first inequality follows from the fact that $a_o \notin \bar{B}$. (If $a_o \in \bar{B}$, then $a_o \in (A \cap \text{Bd } A) \cap N$ which is impossible.) There exists some $b_1 \in \bar{B}$ such that $|b_1 - a_o| = d$. It is claimed that $b_1 \in B$. (If $b_1 \in \bar{B} \sim B$, then $b_1 \in (A \cap \text{Bd } A) \cap N$ which is impossible.) Since $\mathfrak{B}$ satisfies (1) and since $b_1 \in \text{Bd } B(a_o,d)$, there is some $W \in \mathfrak{B}$ such that $W \subset B(a_o,d) \subset A \subset f^{-1}(A^*)$ and $b_1 \in \bar{W}$. But this is impossible since $f^{-1}(A^*)$ is $\mathfrak{B}$ closed.

The argument that $B \cap P$ is also dense in P is identical since $P = U \cap \text{Bd } B$.

Now P is a G_δ set, f is Baire 1, and Y is a separable metric space. Hence, by Theorem 2.1, $f|P$ has a point of continuity. But this is impossible, for suppose x is a point of continuity of $f|P$.

Either $x \in A$ or $x \in B$. Suppose $x \in A$. Since $B \cap P$ is dense in P, one can choose a sequence of points $b_n \in B \cap P$ such that b_n converges to x. Since x is a point of continuity of $f|P$, $f(b_n)$ converges to $f(x)$. Since B^* is closed and $f(b_n) \in B^*$ for all n, $f(x)$ is in B^*. Hence $x \in B$ which is a contradiction. (If one assumes that x is in B, an identical argument again leads to a contradiction.) Hence f must have been Darboux ($\mathfrak{B}$).

The hypothesis in Theorem 2.2 that f be Baire 1 cannot be omitted as the following example shows.

Example 2.8. Consider E_1 with basis $\mathfrak{B}$ consisting of all open intervals. Let f be the characteristic function of the rationals Q. (Then f is Baire 2 but not Baire 1.) The basis certainly satisfies (1). The inverse image of any closed set (in fact, of any set) is either Q, $E_1 \sim Q$, $\emptyset$, or E_1 and is therefore $\mathfrak{B}$ closed. However f is clearly not Darboux ($\mathfrak{B}$).

The hypothesis in Theorem 2.2 that the basis $\mathfrak{B}$ for E_n satisfies (1) cannot be omitted (or even replaced by the weaker condition (3)) as the following example shows.

Example 2.9. Consider E_1 with basis $\mathfrak{B}$ consisting of all open intervals not having 0 as a right endpoint. As was stated in Example 2.7, $\mathfrak{B}$ satisfies (3) but not (1). Let f be the characteristic function of $[0,\infty)$. Clearly f is Baire 1. The inverse image of any set is either $(-\infty,0)$, $[0,\infty)$, $\emptyset$, or E_1. Hence the inverse image of any closed set is $\mathfrak{B}$ closed. However, f is clearly not Darboux ($\mathfrak{B}$).

One might ask whether in Theorem 2.2 one could replace E_n by a normed linear space of some sort. It happens that, as the following example shows, replacing E_n even with a separable Banach space invalidates the theorem.

Example 2.10. Consider c_o, the space of all sequences of real numbers converging to 0. The norm on c_o is given by

$$\|x\| = \sup_k |x_k|, \quad \text{where} \quad x = \{x_k\} \in c_o.$$

The space c_o is a metric space with metric generated by the norm. Let $\mathfrak{B}$ be the open-ball basis for c_o. Clearly $\mathfrak{B}$ satisfies (1). It is well-known that c_o is a separable Banach space.

Let e^i be the sequence with a 1 in the ith place and 0's elsewhere. Let S be the collection of all finite rational combinations of the e^i's. It is easy to show that S is countable. To show that S is dense, let $x = \{x_k\} \in c_o$ and $\varepsilon > 0$ be given. Choose K sufficiently large that $|x_k| < \varepsilon$ for all $k \geq K$. Choose rationals $r_1,\ldots,r_{K-1}$ such that $|x_k - r_k| < \varepsilon$ for $k = 1,\ldots,K-1$. Let $s = (r_1,\ldots,r_{K-1},0,0,\ldots)$. Note that $s \in S$ and that $\|x-s\| < \varepsilon$. Hence S is indeed dense in c_o.

Now let

$$H = \{x = \{x_k\} \in c_o : \sum_{k=1}^{\infty} x_k 2^{-k} = 0\}.$$

It is claimed that H is closed. Let $\varepsilon > 0$ and $y = \{y_k\} \in \bar{H}$ be given. Then there is some $x = \{x_k\} \in H$ such that $\|y-x\| < \varepsilon$. Since $\sum_{k=1}^{\infty} x^k 2^{-k} = 0$, one has that

$$\left|\sum_{k=1}^{\infty} y_k 2^{-k}\right| = \left|\sum_{k=1}^{\infty} (y_k - x_k) 2^{-k}\right| \leq \sum_{k=1}^{\infty} |y_k - x_k| 2^{-k}$$
$$\leq \|y-x\| \sum_{k=1}^{\infty} 2^{-k} < \varepsilon.$$

But ε was arbitrary. Hence $\sum_{k=1}^{\infty} y_k 2^{-k} = 0$. Hence $y \in H$. Hence H is indeed closed.

Observe that, if $a \notin H$, then dist $(a,H) > 0$. This follows immediately from the fact that c_o is a metric space and H is closed. Observe also that, if $z = \{z_k\} \in c_o$ and $z \neq 0$, then

$$\left|\sum_{k=1}^{\infty} z_k 2^{-k}\right| < \|z\|.$$

To see this, note that $|z_k| < \|z\|$ for all k sufficiently large since $z_k \to 0$ as $k \to \infty$. Hence one has that

$$\left|\sum_{k=1}^{\infty} z_k 2^{-k}\right| \leq \sum_{k=1}^{\infty} |z_k| 2^{-k} < \|z\| \sum_{k=1}^{\infty} 2^{-k} = \|z\|.$$

It is claimed that $\operatorname{dist}(a,H) = \left|\sum_{k=1}^{\infty} a_k 2^{-k}\right|$, where $a = \{a_k\}$. Let $x = \{x_k\} \in H$. Then, since $\sum_{k=1}^{\infty} x_k 2^{-k} = 0$, one has that

$$\left|\sum_{k=1}^{\infty} a_k 2^{-k}\right| = \left|\sum_{k=1}^{\infty} (a_k - x_k) 2^{-k}\right| \leq \sum_{k=1}^{\infty} |a_k - x_k| 2^{-k}$$
$$\leq \|a-x\| \sum_{k=1}^{\infty} 2^{-k} = \|a-x\|.$$

Hence

$$\left|\sum_{k=1}^{\infty} a_k 2^{-k}\right| \leq \inf_{x \in H} \|a-x\| = \operatorname{dist}(a,H).$$

To show the reverse inequality, let $e^i = \{e_k^i\}$ be as defined above. Let

$$A_m = 2^m (2^m - 1)^{-1} \sum_{k=1}^{\infty} a_k 2^{-k}.$$

Let

$$x^m = \{x_k^m\} = a - A_m \sum_{i=1}^{m} e^i.$$

Note that $x^m \in c_o$. It is claimed that $x^m \in H$.

$$\sum_{k=1}^{\infty} x_k^m 2^{-k} = \sum_{k=1}^{m} (a_k - A_m) 2^{-k} + \sum_{k=m+1}^{\infty} a_k 2^{-k}$$

$$= \sum_{k=1}^{\infty} a_k 2^{-k} - A_m \sum_{k=1}^{m} 2^{-k}$$

$$= 2^{-m}(2^m-1)A_m - A_m 2^{-m}(2^m-1) = 0.$$

Hence $x^m \in H$. One has that

$$\|a-x^m\| = \|A_m \sum_{i=1}^{m} e^i\| = |A_m| \, \|\sum_{i=1}^{m} e^i\|$$

$$= |A_m| = 2^m(2^m-1)^{-1} |\sum_{k=1} a_k 2^{-k}|.$$

Therefore

$$\text{dist}(a,H) \leq \|a-x^m\| = 2^m(2^m-1)^{-1} |\sum_{k=1}^{\infty} a_k 2^{-k}|.$$

Letting $m \to \infty$, one gets $\text{dist}(a,H) \leq |\sum_{k=1}^{\infty} a_k 2^{-k}|$.

If $a \notin H$, there is no $x \in H$ such that $\text{dist}(a,x) = \text{dist}(a,H)$, for suppose there were such an x. Then

$$\|a-x\| = \text{dist}(a,x) = \text{dist}(a,H)$$

$$= |\sum_{k=1}^{\infty} a_k 2^{-k}| = |\sum_{k=1}^{\infty} (a_k - x_k) 2^{-k}|$$

which is a contradiction since $\|a-x\| \neq 0$.

Now let S and e^i be as defined above. $S \sim H$ is countably infinite since S is countably infinite and $e^i \in S \sim H$ for $i = 1,2,\cdots$. Hence one can write $S \sim H = \{s^1, s^2, \ldots\}$. Since $s^n \notin H$, $\mathrm{dist}(s^n, H) > 0$. Let $B_n = B(s^n, \mathrm{dist}(s^n, H))$. Note that $\bar{B}_n \subset c_o \sim H$, for, if not, there would be some $x \in H$ with $\mathrm{dist}(s^n, x) = \mathrm{dist}(s^n, H)$ which is impossible.

Define a continuous function f_n on a closed subset of c_o as follows.

$$f_n(x) = \begin{cases} 1, & x \in \bigcup_{k=1}^{n} \bar{B}_k \\ 0, & x \in H \end{cases}$$

Next use the Tietze Extension Theorem to extend f_n continuously to all of c_o. Denote the extended function by $\tilde{f}_n$. Do this for each $n = 1, 2, \cdots$.

Note that $\tilde{f}_n(x) = 0$ for each $x \in H$; so $f(x) = \lim_{n\to\infty} \tilde{f}_n(x)$ exists and equals 0 on H. It is claimed that $f(x) = \lim_{n\to\infty} \tilde{f}_n(x)$ exists and equals 1 on $c_o \sim H$. So suppose that $a \notin H$. Then $\mathrm{dist}(a, H) > 0$. Since S is dense in c_o and

$$B(a, \tfrac{1}{2}\,\mathrm{dist}(a, H)) \cap H = \emptyset,$$

there is some n such that

$$s^n \in (S \sim H) \cap B(a, \tfrac{1}{2}\,\mathrm{dist}(a, H)).$$

But then $\mathrm{dist}(s^n, H) > \frac{1}{2}\,\mathrm{dist}(a, H)$. Hence

$$\mathrm{dist}(a, s^n) < \tfrac{1}{2}\,\mathrm{dist}(a, H) < \mathrm{dist}(s^n, H).$$

Therefore $a \in B_n$, and, hence, $f(a) = 1$.

It is claimed that $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset E_1$. Clearly it suffices to show that $f^{-1}(1)$ and $f^{-1}(0)$ are each $\mathfrak{B}$ closed. First consider $f^{-1}(0) = H$. Since H is closed, $f^{-1}(0)$ is clearly $\mathfrak{B}$ closed. Next consider $f^{-1}(1) = c_0 \sim H$. Let $U = B(z,r) \subset f^{-1}(1)$. U is in $\mathfrak{B}$. Let $x \in \text{Bd } U$. Then $f(x) = 1$, for, if not, one has that $x \in H$ and $\text{dist}(z,H) = \text{dist}(z,x)$ which is impossible since $z \notin H$. Hence $\bar{U} \subset f^{-1}(1)$. Hence $f^{-1}(1)$ is $\mathfrak{B}$ closed.

Finally, it is clear that f is not Darboux ($\mathfrak{B}$). The unit ball, for example, is a basis element. But f maps its closure onto $\{0,1\}$, which is not connected.

If the restriction on the basis is changed, one can prove a converse to Theorem 2.2 even without the assumption that f is Baire 1.

Theorem 2.3. Let Y be a metric space. Let $\mathfrak{B}$ be a basis for E_n satisfying (2). Then, if $f : E_n \to Y$ is Darboux ($\mathfrak{B}$), $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$.

Proof. Let $K \subset Y$ be closed. Let U be any basis element contained in $f^{-1}(K)$. It must be shown that $\text{Bd } U \subset f^{-1}(K)$. Let $x \in \text{Bd } U$, but suppose that $x \notin f^{-1}(K)$. Since $\mathfrak{B}$ satisfies (2), there is some $V \in \mathfrak{B}$ such that $x \in \bar{V}$ and $\bar{V} \sim \{x\} \subset U \subset f^{-1}(K)$. (The set $\bar{V} \sim \{x\}$ cannot be void, for V is an open subset of E_n.) Hence $f(\bar{V} \sim \{x\}) \subset K$, and $f(x) \notin K$. But then $f(\bar{V}) = f(\bar{V} \sim \{x\}) \cup \{f(x)\}$ is a decomposition of $f(\bar{V})$ into two disjoint, non-void, relatively closed sets. Hence $f(\bar{V})$ is not connected which is a contradiction since $V \in \mathfrak{B}$ and f is Darboux ($\mathfrak{B}$).

The hypothesis in Theorem 2.3 that $\mathfrak{B}$ satisfies (2) cannot be omitted even if one assumes that f is Baire 1 as the following example shows.

Example 2.11. Let $E_n = E_2$, and let $Y = E_1$. Let $\mathfrak{B}$ be the basis for E_2 consisting of the open rectangle R with corners at $(0,1)$, $(-1,1)$, $(-1,-1)$, and $(0,-1)$ and all open balls not tangent to the y-axis. As was mentioned in Example 2.6, $\mathfrak{B}$ does not satisfy (2). Define $f : E_2 \to E_1$ as follows.

$$f(x,y) = \begin{cases} \sin\left(\frac{1}{x+1}\right), & x < -1 \\ 0, & -1 \leq x < 0 \\ 0, & x = 0 \text{ and } |y| > 1 \\ y, & x = 0 \text{ and } |y| \leq 1 \\ \sin\left(\frac{1}{x}\right), & x > 0 \end{cases}$$

It is easy to show that f is Baire 1. The function f is also Darboux ($\mathfrak{B}$). Any ball whose closure lies to the left of $x = -1$ or to the right of $x = 0$ automatically has its closure mapped onto a connected set since f is continuous in these regions. Any ball whose closure lies in the strip $-1 \leq x < 0$ has its closure mapped onto $\{0\}$. Any ball which lies to the left of and is tangent to $x = -1$ has its closure mapped onto $[-1,1]$. The same is true of any ball which overlaps $x = -1$ or $x = 0$. Finally $f(\bar{R}) = [-1,1]$. However, the inverse image of the closed set $(-\infty,0]$ is not $\mathfrak{B}$ closed since R is contained in $f^{-1}((-\infty,0])$ but $\bar{R}$ is not.

Remark. As before, let $\mathfrak{F}$ be the family of all Baire 1 functions mapping E_n into a fixed separable metric space Y. If one combines Theorem 2.2 and Theorem 2.3, one gets the desired condition distinguishing those functions in $\mathfrak{F}$ which are Darboux $(\mathfrak{B})$, where $\mathfrak{B}$ is any basis for E_n satisfying (1) and (2). As was remarked earlier, this condition is similar to a condition established by Neugebauer [6] and discussed by Weil [7] for Baire 1 functions having the ordinary Darboux property.

Let Y be a separable metric space. Let $f : E_n \to Y$ be Baire 1. Let $\mathfrak{B}$ be a basis of connected sets for E_n satisfying (1) and (2). Then $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$ if and only if f is Darboux $(\mathfrak{B})$.

The following definition is due to Bruckner and Bruckner [2].

Definition 2.5. Let X be a topological space, and let $\mathfrak{B}$ be a basis for X. A set $S \subset X$ is said to be dense-in-itself ($\mathfrak{B}$) if, given any $x \in S$ and any $U \in \mathfrak{B}$ with $x \in \bar{U}$, $S \cap U$ contains some point other than x.

Lemma 2.1. Let $f : X \to Y$, where X and Y are topological spaces. Let $\mathfrak{B}$ be a basis for X satisfying (3). Then $f^{-1}(V)$ is dense-in-itself ($\mathfrak{B}$) for all open $V \subset Y$ if and only if $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$.

Proof. Let $f^{-1}(V)$ be dense-in-itself ($\mathfrak{B}$) for all open $V \subset Y$, but suppose there is some closed $K \subset Y$ such that $f^{-1}(K)$ is not $\mathfrak{B}$ closed. Then there is some $U \in \mathfrak{B}$ and some $x \in \mathrm{Bd}\, U$ such that $U \subset f^{-1}(K)$ and $x \in f^{-1}(Y \sim K)$. But $f^{-1}(Y \sim K)$ is dense-in-itself ($\mathfrak{B}$). Hence $U \cap f^{-1}(Y \sim K) \neq \emptyset$ which is a contradiction.

For the reverse implication, let $f^{-1}(K)$ be $\mathfrak{B}$ closed for all closed $K \subset Y$, but suppose there is some open $V \subset Y$ such that $f^{-1}(V)$ is not dense-in-itself ($\mathfrak{B}$). Then there is some $U \in \mathfrak{B}$ and some $x \in f^{-1}(V)$

such that $x \in \bar{U}$ and $U \cap f^{-1}(V) \subset \{x\}$. If $x \in U$, then, by (3), there is a $U_1 \in \mathfrak{B}$ such that $U_1 \subset U$ and $x \in \text{Bd } U_1$. If $x \notin U$, then, obviously, $x \in \text{Bd } U$. We see that in either case there is a $W \in \mathfrak{B}$ such that $W \subset U$ and $x \in \text{Bd } W$. By the choice of U and x we have $W \subset f^{-1}(Y \sim V)$ which is a contradiction because $f^{-1}(Y \sim V)$ is $\mathfrak{B}$ closed.

Remark. Notice that, in Lemma 2.1, the hypothesis that $\mathfrak{B}$ satisfies (3) was used only to show that, if $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$, then $f^{-1}(V)$ is dense-in-itself ($\mathfrak{B}$) for all open $V \subset Y$. (The reverse implication was proved without using this assumption.) That this hypothesis cannot be omitted even if one assumes that f is Baire 1 is shown by the following example.

Example 2.12. Let $X = Y = E_1$. Let $\mathfrak{B}$ be the basis for E_1 consisting of all open intervals not having 0 as an endpoint. As was pointed out in Example 2.5, $\mathfrak{B}$ does not satisfy (3). Let f be the characteristic function of the singleton $\{0\}$. Clearly f is Baire 1. The inverse image of any set is either $\{0\}$, $E_1 \sim \{0\}$, $\emptyset$, or E_1. Hence the inverse image of any closed set is $\mathfrak{B}$ closed. However, the inverse image of the open set $(0, \infty)$, for example, is not dense-in-itself ($\mathfrak{B}$).

As before, let $\mathfrak{F}$ be the family of all Baire 1 functions mapping E_n into a fixed separable metric space Y. As was remarked earlier, Bruckner and Bruckner [2] established a condition (stated in terms of inverse images of open sets) distinguishing those functions in $\mathfrak{F}$ which are Darboux ($\mathfrak{B}$), where $\mathfrak{B}$ is any basis for E_n satisfying certain restrictions. This condition is similar to a condition established by Zahorski [8] for Baire 1 functions having the ordinary Darboux property. Bruckner and Bruckner's result is presented here as a simple consequence of the remark following Example 2.11.

Theorem 2.4. Let Y be a separable metric space. Let $f : E_n \to Y$ be Baire 1. Let $\mathfrak{B}$ be a basis of connected sets for E_n satisfying (1) and (2). Then $f^{-1}(V)$ is dense-in-itself ($\mathfrak{B}$) for all open $V \subset Y$ if and only if f is Darboux ($\mathfrak{B}$).

Proof. Since $\mathfrak{B}$ is a basis for E_n satisfying (1), $\mathfrak{B}$ satisfies (3) also. Hence, by Lemma 2.1, $f^{-1}(V)$ is dense-in-itself ($\mathfrak{B}$) for all open $V \subset Y$ if and only if $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$. But, by the remark following Example 2.11, $f^{-1}(K)$ is $\mathfrak{B}$ closed for all closed $K \subset Y$ if and only if f is Darboux ($\mathfrak{B}$).

Remark. In the proof of Theorem 2.4, the assumption that $\mathfrak{B}$ satisfies (2) was used only to show that, if f is Darboux ($\mathfrak{B}$), then $f^{-1}(V)$ is dense-in-itself ($\mathfrak{B}$) for all open $V \subset Y$. The assumption that $\mathfrak{B}$ satisfies (1), however, was used for both implications.

The following lemma gives an additional interesting property of Darboux ($\mathfrak{B}$) functions. This lemma will be used in the next chapter.

Lemma 2.2. Let X be a metric space and Y a topological space. Let $\mathfrak{B}$ be a basis for X. If $f : X \to Y$ is Darboux ($\mathfrak{B}$), then $f(V)$ is connected for each connected open set $V \subset X$.

Proof. Let $V \subset X$ be open and connected, but suppose that $f(V)$ is not connected. Then there exist sets A^* and B^*, open in Y, such that

$$f(V) = (f(V) \cap A^*) \cup (f(V) \cap B^*)$$

is a decomposition of $f(V)$ into two disjoint, non-void, relatively open sets.

Let $A = \{x \in V : f(x) \in A^*\}$. Let $B = \{x \in V : f(x) \in B^*\}$. Notice that $V = A \cup B$ and $A \cap B = \emptyset$. It is claimed that both A and B are open in X. To show that A is open, let $x \in A$.

Since V is open, one can find an open ball W such that $x \in W \subset \bar{W} \subset V$. Since W is open, one can find some $U \in \mathfrak{B}$ such that $x \in U \subset W$. Hence $x \in \bar{U} \subset \bar{W} \subset V$. It then follows that $\bar{U} \subset A$, for suppose not. One has $\bar{U} \cap A \neq \emptyset$ since $x \in \bar{U} \cap A$. If $\bar{U} \cap B \neq \emptyset$ also, then

$$f(\bar{U}) = (f(\bar{U}) \cap A^*) \cup (f(\bar{U}) \cap B^*)$$

is a decomposition of $f(\bar{U})$ into two disjoint, non-void, relatively open sets. But this is impossible since f is Darboux $(\mathfrak{B})$. Therefore one has $x \in U \subset \bar{U} \subset A$. Hence A is open. (The argument for B is identical.) But $V = A \cup B$, and V is connected. Therefore $A = \emptyset$ or $B = \emptyset$. Hence $f(V)$ is connected.

CHAPTER III

A DARBOUX PROPERTY FOR THE GRADIENT

This chapter deals with real-valued functions defined on E_n. It is tacitly assumed that, for each function f considered here, grad f exists everywhere.

Definition 3.1. Let (X,d) be a metric space. Let $S \subset X$ and let $x \in \bar{S}$. Then x is said to be accessible from S if there is some $\alpha > 0$ with the following property: there exists a sequence of points $c_n \in S \sim \{x\}$ such that $\lim c_n = x$ and $B(c_n, \alpha d(c_n, x)) \subset S$ for each n.

Definition 3.2. Let X be a metric space. A set $S \subset X$ is said to have an accessible boundary if each $x \in \mathrm{Bd}\, S$ is accessible from S.

Roughly speaking, a set S has an accessible boundary if it does not come to a cusp on its boundary. Consider the following example.

Example 3.1. Given $s > 0$, let x_s be the unique real root of $2x^3 + x - s = 0$. Then

$$r_s = [(s-x_s)^2 + x_s^4]^{1/2}$$

is the minimum distance from $(s,0)$ to the curve $y = x^2$. The minimum distance from $(s,0)$ to the curve $y = x$ is $\frac{s}{\sqrt{2}}$. Define U_s and V_s as follows.

$$U_s = \{(x,y) : |y| < x^2, 0 < x < x_s\} \cup B((s,0),r_s)$$

$$V_s = \{(x,y) : |y| < x,\ 0 < x < \tfrac{s}{2}\} \cup B((s,0),\tfrac{s}{\sqrt{2}})$$

V_s has an accessible boundary, but U_s does not. The point $(0,0)$ is on the boundary of V_s and is accessible from V_s. One can choose α to be any positive number not exceeding $\sin\frac{\pi}{4}$. The point $(0,0)$ is on the boundary of U_s also but is not accessible from U_s.

Remark. Notice that, if $x \in \operatorname{Int} S$, x is automatically accessible from S. In fact, one can choose α to be 1.

The following theorem is similar to a result established by Borwein and Meier [1] for the gradient of a function which is not necessarily differentiable.

Theorem 3.1. Let $\mathfrak{B}$ be a basis for E_n, each element of which has an accessible boundary. Let $f : E_n \to E_1$ be differentiable, and let $F = \operatorname{grad} f$.

Assume that $f(0) = 0$ and $F(0) = 0$. Let U be any element of $\mathfrak{B}$ such that $0 \in \bar{U}$. Then, given any $\epsilon > 0$, there is some $v \in U$ such that $v \neq 0$ and $|F(v)| \leq \epsilon$.

Proof. First we use the fact that U has an accessible boundary to select an α, $0 < \alpha < 1$, with the following property: given any $\xi > 0$, there is a point $c_\xi \in U$, such that $c_\xi \neq 0$ and

$$B(c_\xi, \alpha|c_\xi|) \subset U \cap B(0,\xi).$$

Next we use the fact that f is differentiable, that $f(0) = 0$, and that $F(0) = 0$ to find a $\delta > 0$ such that $|f(x)| < \frac{\alpha\epsilon}{2}|x|$ whenever $|x| \leq \delta$. Finally select a $c \in U$ such that $c \neq 0$ and

$$B = B(c,\alpha|c|) \subset U \cap B(0,\delta).$$

Now let $g(x) = \epsilon^2|x-c|^2 - f(x)^2$. Since g is continuous and $\bar{B}$ is compact, there is some $v \in \bar{B}$ such that

$$g(v) = \min\{g(x) : x \in \bar{B}\}.$$

For convenience let

$$\lambda = \min\{g(x) : x \in \bar{B}\}.$$

For $x \in \text{Bd } B$ one has

$$\begin{aligned} g(x) &\geq \epsilon^2\alpha^2|c|^2 - \tfrac{1}{4}\alpha^2\epsilon^2|x|^2 \\ &\geq \epsilon^2\alpha^2|c|^2 - \tfrac{1}{4}\alpha^2\epsilon^2(|c| + \alpha|c|)^2 \\ &> \epsilon^2\alpha^2|c|^2 - \tfrac{1}{4}\alpha^2\epsilon^2(2|c|)^2 = 0. \end{aligned}$$

But $\lambda \leq g(c) = -f(c)^2 \leq 0$. Hence $g(v) = \lambda$ for some $v \in B$. Therefore

$$\text{grad } g(v) = 2\varepsilon^2(v-c) - 2f(v) \text{ grad } f(v) = 0.$$

Hence $f(v) \text{ grad } f(v) = \varepsilon^2(v-c)$, which implies that

$$(*) \quad f(v)^2 |\text{grad } f(v)|^2 = \varepsilon^4 |v-c|^2 = \varepsilon^2(g(v)+f(v)^2)$$
$$= \varepsilon^2(\lambda+f(v)^2) \leq \varepsilon^2 f(v)^2.$$

If $\lambda < 0$, then

$$f(v)^2 = \varepsilon^2 |v-c|^2 - g(v) \geq -g(v) = -\lambda > 0.$$

Hence, dividing (*) by $f(v)^2$ yields the desired result.

Similarly, if $\lambda = 0$ and there is some $v \in B$ such that $v \neq c$ and $g(v) = \lambda$, then one has

$$f(v)^2 = \varepsilon^2 |v-c|^2 - g(v) = \varepsilon^2 |v-c|^2 > 0.$$

Again, dividing (*) by $f(v)^2$ yields the desired result.

It remains to consider the case where $\lambda = 0$ and $g(x) > 0$ for all $x \in B$ except $x = c$. (Notice that, in this case, $g(c) = f(c) = 0$.) Let $B_1 = B(c, \frac{\alpha}{2}|c|)$. For $x \in \text{Bd } B_1$ one has

$$f(x)^2 = \frac{1}{4}\varepsilon^2\alpha^2|c|^2 - g(x) < \frac{1}{4}\varepsilon^2\alpha^2|c|^2.$$

Hence $|f(x)| < \frac{1}{2}\varepsilon\alpha|c|$ for all $x \in \text{Bd } B_1$. Since f is continuous and $\text{Bd } B_1$ is compact, there is some $u \in \text{Bd } B_1$

such that

$$f(u) = \max\{f(x) : x \in \text{Bd } B_1\}.$$

Since $f(u) < \frac{1}{2}\epsilon\alpha|c|$, one can choose a $\nu > 0$ such that $f(u) + \nu < \frac{1}{2}\epsilon\alpha|c|$. Then, for all $x \in \text{Bd } B_1$, one has

$$-\frac{1}{2}\epsilon\alpha|c| < f(x) < f(x) + \nu \leq f(u) + \nu$$
$$< \frac{1}{2}\epsilon\alpha|c|.$$

Hence, for all $x \in \text{Bd } B_1$, $|f(x)+\nu| < \frac{1}{2}\epsilon\alpha|c|$. Now let

$$h(x) = \epsilon^2|x-c|^2 - (f(x)+\nu)^2.$$

Since h is continuous and $\bar{B}_1$ is compact, there is some $v \in \bar{B}_1$ such that

$$h(v) = \min\{h(x) : x \in \bar{B}_1\}.$$

For convenience let $\mu = \min\{h(x) : x \in \bar{B}_1\}$. For $x \in \text{Bd } B_1$ one has

$$h(x) > \frac{1}{4}\epsilon^2\alpha^2|c|^2 - \frac{1}{4}\alpha^2\epsilon^2|c|^2 = 0.$$

But

$$\mu \leq h(c) = -(f(c)+\nu)^2 = -\nu^2 < 0.$$

Hence $h(v) = \mu$ for some $v \in B_1$. Therefore

$$\text{grad } h(v) = 2\epsilon^2(v-c) - 2(f(v)+\nu)\text{ grad } f(v) = 0.$$

Hence $(f(v)+\nu)\text{ grad } f(v) = \epsilon^2(v-c)$, which implies that

$$(**) \qquad (f(v)+\nu)^2|\text{grad } f(v)|^2 = \epsilon^4|v-c|^2$$
$$= \epsilon^2(h(v)+(f(v)+\nu)^2)$$
$$= \epsilon^2(\mu + (f(v)+\nu)^2)$$
$$\leq \epsilon^2(f(v)+\nu)^2.$$

But

$$(f(v)+\nu)^2 = \varepsilon^2 |v-c|^2 - h(v)$$
$$= \varepsilon^2 |v-c|^2 - \mu$$
$$\geq -\mu > 0.$$

Hence, dividing (**) by $(f(v)+\nu)^2$ yields the desired result.

Corollary 3.1. Let $\mathfrak{B}$ be a basis for E_n, each element of which has an accessible boundary. Let $f : E_n \to E_1$ be differentiable, and let $F = \text{grad } f$. Let U be any element of $\mathfrak{B}$, and let x_o be any point in $\bar{U}$. Then, given any $\varepsilon > 0$, there is some $v \in U$ such that $v \neq x_o$ and $|F(v)-F(x_o)| \leq \varepsilon$.

Proof. Let $G(x) = F(x+x_o) - F(x_o)$, and let $g(x) = f(x+x_o) - f(x_o) - F(x_o) \circ x$, where $\circ$ denotes the usual scalar product in E_n. It is clear that g is differentiable, $G = \text{grad } g$, $g(0) = 0$, and $G(0) = 0$.

Let $\mathfrak{B}_1 = \{-x_o+W : W \in \mathfrak{B}\}$. $\mathfrak{B}_1$ is a basis for E_n, each element of which has an accessible boundary. Moreover, $V = -x_o+U$ is in $\mathfrak{B}_1$, and $0 \in \bar{V}$. Hence, by Theorem 3.1, there is some $v' \in V$ such that $v' \neq 0$ and $|G(v')| \leq \varepsilon$. Let $v = v' + x_o$. Then $v \in U$, $v \neq x_o$, and one has

$$|F(v)-F(x_o)| = |F(v'+x_o)-F(x_o)| = |G(v')| \leq \varepsilon.$$

The following theorem is the major result in this chapter.

Theorem 3.2. Let $\mathfrak{B}$ be a basis of connected sets for E_n, each element of which has an accessible boundary. Let $f : E_n \to E_1$ be differentiable, and let $F = \text{grad } f$. Then F is Darboux $(\mathfrak{B})$.

Proof. Let $\mathfrak{B}_o$ be the open-ball basis for E_n. Let $\mathfrak{B}_1 = \mathfrak{B} \cup \mathfrak{B}_o$. Then $\mathfrak{B}_1$ is a basis for E_n satisfying (1), and each element of $\mathfrak{B}_1$ is a connected set with an accessible boundary. Note that F, being the gradient of a continuous function, is Baire 1.

Suppose F is not Darboux $(\mathfrak{B}_1)$. Then, by Theorem 2.4, there is an open set $V \subset E_n$ such that $F^{-1}(V)$ is not dense-in-itself $(\mathfrak{B}_1)$. Hence there is some $U \in \mathfrak{B}_1$ and some $x_o \in \bar{U} \cap F^{-1}(V)$ such that U contains no point of $F^{-1}(V)$ with the possible exception of x_o.

We have $F(x_o) \in V$, and V is open. Hence there is some $r > 0$ such that $B(F(x_o),2r) \subset V$. Now use Corollary 3.1 to find a $v \in U$ such that $v \neq x_o$ and $|F(v)-F(x_o)| \leq r$. But then $F(v) \in B(F(x_o),2r) \subset V$ which is a contradiction. Thus F must have been Darboux $(\mathfrak{B}_1)$ and, hence, Darboux $(\mathfrak{B})$.

A Darboux property for partial derivatives follows easily from Theorem 3.2.

Corollary 3.2. Let $\mathfrak{B}$ be a basis of connected sets for E_n, each element of which has an accessible boundary. Let $f : E_n \to E_1$ be differentiable. Then $\partial_k f$ (the partial derivative with respect to the kth variable) is Darboux ($\mathfrak{B}$) for $k = 1,\ldots,n$.

Proof. Let $p_k : E_n \to E_1$ be the projection onto the kth component. Then

$$\partial_k f(x) = p_k(\text{grad } f(x)).$$

Since, as is well-known, p_k is continuous and, by Theorem 3.2, grad f is Darboux ($\mathfrak{B}$), it follows immediately that $\partial_k f$ is Darboux ($\mathfrak{B}$).

The following corollary establishes a sort of intermediate value property for the gradient of a differentiable function.

Corollary 3.3. Let $\mathfrak{B}$ be a basis of connected sets for E_n, each element of which has an accessible boundary. Let $F : E_n \to E_n$ be the gradient of a differentiable function. Suppose that $|F(x)|$ assumes the values a and b, $a < b$, at the points u and v respectively. Then, if U is any element of $\mathfrak{B}$ such that $u,v \in \bar{U}$, $|F(x)|$ assumes every value between a and b in U.

Proof. F is Darboux $(\mathfrak{B})$ by Theorem 3.2. Thus $F(U)$ is connected by Lemma 2.2. Hence $\{|F(w)| : w \in U\}$ is an interval. Now suppose there is some $c \in (a,b)$ with the following property: there exists no $w \in U$ such that $|F(w)| = c$. Let

$$\epsilon = \frac{1}{2}\min\{c-a, b-c\}.$$

By Corollary 3.1, one can find points $w_1, w_2 \in U$ such that $|F(w_1)-F(u)| \leq \epsilon$ and $|F(w_2)-F(v)| \leq \epsilon$. Hence

$$\begin{aligned} ||F(w_1)|-a| &= ||F(w_1)| - |F(u)|| \\ &\leq |F(w_1)-F(u)| \leq \epsilon, \end{aligned}$$

and

$$\begin{aligned} ||F(w_2)|-b| &= ||F(w_2)| - |F(v)|| \\ &\leq |F(w_2)-F(v)| \leq \epsilon. \end{aligned}$$

Therefore $|F(w_1)| < c < |F(w_2)|$. Hence $\{|F(w)| : w \in U\}$ is not connected which is a contradiction.

Remark. Let F and $\mathfrak{B}$ be as described in Corollary 3.3. Let $G(x) = |F(x)|$. Then what Corollary 3.3 says is that G has the "strong Darboux property relative to $\mathfrak{B}$" as defined by Mišík [5]. It should be noted that G must also then be Darboux $(\mathfrak{B})$.

The hypothesis in Theorem 3.2 that each element in $\mathfrak{B}$ have an accessible boundary cannot be omitted as the following example shows.

Example 3.2. Given $s > 0$, let U_s be as defined in Example 3.1. The collection of all such U_s together with translations of same is a basis for the usual topology on E_2. Call this basis $\mathfrak{B}$. Each $U \in \mathfrak{B}$ is connected, but each $U \in \mathfrak{B}$ has a point on its boundary which is not accessible from U. Let

$$h(x,y) = \frac{5}{27} x^{-4} y^3 - \frac{14}{9} x^{-2} y^2 + \frac{32}{9} y - \frac{32}{27} x^2$$

for $x > 0$. Define $f : E_2 \to E_1$ as follows.

$$f(x,y) = \begin{cases} 0 & , |y| \geq 4x^2 \text{ and } x > 0 \\ -h(x,-y) & , -4x^2 < y < -x^2 \text{ and } x > 0 \\ y & , |y| \leq x^2 \text{ and } x > 0 \\ h(x,y) & , x^2 < y < 4x^2 \text{ and } x > 0 \\ 0 & , x \leq 0 \end{cases}$$

(It is an elementary though rather tedious task to show that f is indeed everywhere differentiable.) Now let $F = \text{grad } f$. We have $F(0,0) = (0,0)$. However, F maps the set $\{(x,y) : |y| \leq x^2, x > 0\}$ onto the point $(0,1)$. Hence, for each $s > 0$, F maps $\bar{U}_s$ onto a set which is not connected. Therefore F is not Darboux $(\mathfrak{B})$.

The hypothesis in Theorem 3.2 that f be differentiable cannot be omitted as the following example shows.

Example 3.3. Let $\mathfrak{B}$ be any basis whatever for E_2. Let

$$f(x,y) = \begin{cases} 2xy(x^2+y^2)^{-1/2}, & (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0). \end{cases}$$

Then

$$\text{grad } f(x,y) = \begin{cases} (2y^3(x^2+y^2)^{-3/2},\ 2x^3(x^2+y^2)^{-3/2}), & (x,y) \neq (0,0) \\ (0,0) & , \quad (x,y) = (0,0). \end{cases}$$

For $(x,y) \neq (0,0)$,

$$|\text{grad } f(x,y)|^2 = 4(x^6+y^6)(x^2+y^2)^{-3}$$
$$= \frac{5}{2} + \frac{3}{2}\cos 4\theta \geq 1,$$

where $x = r\cos\theta$ and $y = r\sin\theta$. But $|\text{grad } f(0,0)| = 0$. Thus, for any open neighborhood U of $(0,0)$, grad $f(\bar{U})$ is not connected. The function f is differentiable everywhere save at the origin, for f, f_x, and f_y are continuous everywhere save at the origin. But grad f is certainly not Darboux ($\mathfrak{B}$).

Remark. Let $\mathfrak{B}$ be any basis of connected sets for E_n. Let $f : E_n \to E_1$ be continuously differentiable, and let $F = \text{grad } f$. Then F is clearly Darboux ($\mathfrak{B}$).

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