QUASISYMMETRIC FUNCTIONS AND PLANE QUASICONFORMAL MAPPINGS

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ABSTRACT

QUASISYMMETRIC FUNCTIONS AND PLANE QUASICONFORMAL MAPPINGS

By

Kenneth P. Goldberg

In 1928 H. Grötzsch gave a definition of quasiconformal mappings which sought to generalize the concept of conformal mappings. However, these mappings satisfy neither the reflection principle nor the normal family property, both of which are satisfied by conformal mappings. In 1954 L. V. Ahlfors gave a new definition for quasiconformal mappings which in fact extends the class of mappings that are quasi-conformal in the sense of Grötzsch. These new mappings do satisfy the above two properties, in addition to having many other properties of conformal mappings.

The dilatation $D \geq 1$ of a differentiable topological mapping $f: (x,y) \longrightarrow (u,v)$ of one plane domain onto another is determined by

$$D + D^{-1} = \frac{U_x{}^2 + U_y{}^2 + V_x{}^2 + V_y{}^2}{|U_x V_y - U_y V_x|}.$$

Geometrically, D represents the ratio between major and minor axes of the infinitesimal ellipse obtained by mapping

an infinitesimal circle of center (x,y). A mapping is said to be <u>quasiconformal</u> in the sense of Ahlfors if D is bounded. The least upper bound of D is called the <u>maximal dilatation</u>.

Beurling and Ahlfors showed (Acta Math. 1956) that there exists a quasiconformal mapping of the upper half plane onto itself with boundary correspondence $x \longrightarrow \varphi(x)$ if and only if

$$(1) \qquad \frac{1}{h} \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq h$$

for some constant h, $1 \leq h < \infty$, and for all real x and t. A function φ which satisfies (1) is said to be <u>quasisymmetric</u> and the least upper bound of h is called the <u>quasisymmetric dilatation</u> of φ and is denoted by $\rho(\varphi)$.

In Chapter I we show that (1) is really a generalized convexity-concavity condition, and that the assumptions in the definition of quasisymmetry can be significantly weakened without altering the class of such functions. We then use this fact to obtain sharp bounds for the dilatation of the sums, products, inverses and compositions of quasisymmetric functions on $(0,\infty)$.

In Chapter II we introduce, by means of a differentiability condition, a subclass of the quasisymmetric functions which we call <u>ratio-bounded</u>. We study closure properties of this class under sums, products, compositions and the taking of inverses and find sharp bounds for the quasisymmetric dilatation of such functions.

As already indicated, quasisymmetric functions of $(-\infty, \infty)$ onto itself can be extended to quasiconformal mappings of the upper half plane, but until now only one such extension had been given explicitly, namely that of Beurling and Ahlfors in 1956, and hardly anything was known about extremal extensions (that is, quasiconformal extensions with minimal maximal dilatation). In Chapters III and IV we use the results of Chapter II about ratio-bounded functions to obtain explicit extremal extensions for each quasisymmetric function in a certain class.

In Chapter V we continue our study of extremal quasiconformal extensions and generalize a result of Reich and Strebel (Comment. Math. Helv., 1970).

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TO
NEAL AND MYRNA,
JOAN AND SID,
AND ESPECIALLY TO MY MOTHER
DOLORES

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CHAPTER I

INTRODUCTION

A well-known theorem of Riemann [2, p.172] states that any simply-connected domain Ω whose boundary consists of more than one point can be mapped conformally onto the unit disk. Thus any two such domains Ω, Ω' can be mapped conformally onto each other.

In 1928 H. Grötzsch [8] posed the following problem: given a square D and a rectangle R which is not a square, can D be mapped conformally onto R so that the vertices correspond?

As was subsequently shown by Grötzsch [8], no such conformal mapping exists. He then asked for the most nearly conformal map of D onto R with vertices corresponding. In order to answer this new question one needs a method of measuring approximate (or quasi) conformality. It was in attempting to supply such a measure that Grötzsch laid the foundations for the modern theory of plane quasiconformal mappings [8].

Let Ω be a domain. A <u>quadrilateral</u> in Ω is a Jordan domain Q, $\bar{Q} \subset \Omega$, together with a pair of disjoint, closed arcs on the boundary of Q (called the b-arcs). If we map Q conformally onto a rectangle with side lengths a and b, with the b-arcs going onto the sides of length b (see Figure 1.1), then the <u>modulus</u> of Q is defined uniquely as

$$\text{mod } Q = \frac{a}{b}. \tag{1.1}$$

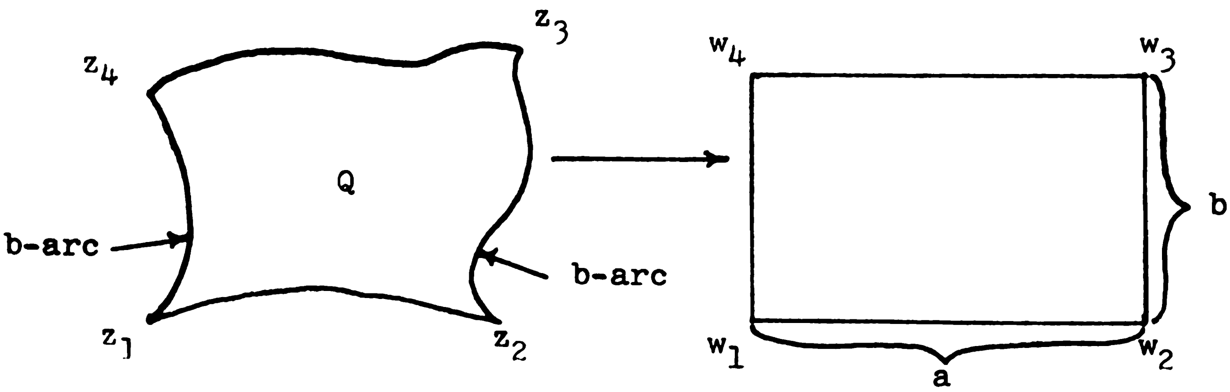

Figure 1.1

The modern definition of quasiconformality, as first given by L. V. Ahlfors in 1954 [4] is:

<u>Definition 1.1</u>. Let $w = f(z)$ be a sense-preserving homeomorphism from a domain Ω onto a domain Ω'. Then f is said to be <u>quasiconformal on</u> Ω, <u>or QC</u>, if there is some K, $1 \leq K < \infty$, satisfying

$$\frac{1}{K} \leq \frac{\text{mod } Q'}{\text{mod } Q} \leq K \tag{1.2}$$

for all quadrilaterals Q in Ω with $f(Q) = Q'$. We define the maximal, or QC, dilatation $K(f)$ of f on Ω to be the infimum of all numbers K satisfying (1.2). If $K(f) = K_0$ we say that f is K_0-quasiconformal (or K_0-QC) on Ω.

The above definition, as well as several equivalent ones, is given in [6].

As is easily shown [4, p.8], f is conformal if and only if $K(f) = 1$. Thus $K(f)$ can be used as a measure of approximate conformality.

We will also find the following definition quite useful later in this paper.

Definition 1.2. Let $f = u + iv$ be a sense-preserving homeomorphism of the domain Ω onto the domain Ω', and let (x_0, y_0) be a point of Ω at which both u and v have continuous partial derivatives. Let

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Then

$$\chi(x_0, y_0) = \frac{f_{\bar{z}}(x_0, y_0)}{f_z(x_0, y_0)}$$

is called the complex dilatation of f at the point (x_0, y_0) and

$$D(x_0, y_0) = \frac{1 + |\chi(x_0, y_0)|}{1 - |\chi(x_0, y_0)|}$$

is called the point dilatation of f at (x_0, y_0). Moreover,

$D(x_0, y_0)$ satisfies

$$(1.3)\quad D(x_0,y_0)+\frac{1}{D(x_0,y_0)}=\frac{u_x(x_0,y_0)^2+u_y(x_0,y_0)^2+v_x(x_0,y_0)^2+v_y(x_0,y_0)^2}{u_x(x_0,y_0)v_y(x_0,y_0)-u_y(x_0,y_0)v_x(x_0,y_0)}.$$

<u>Remark</u>: It is clear, from Definition B on page 24 of [3], that if f is a QC map of Ω onto Ω' then

$$K(f) = \operatorname*{ess\,sup}_{z \in \Omega} D(z).$$

A few of the more important properties of QC maps, whose proofs may be found in [3], are:

(i) f is conformal if and only if $K(f) = 1$.

(ii) f is QC if and only if f^{-1} is QC, and $K(f^{-1}) = K(f)$.

(iii) The QC dilatation $K(f)$ is invariant under composition with conformal mappings. (I.e., whenever f is QC, g is conformal, and $f \circ g$ $(g \circ f)$ is well-defined, then $K(f \circ g) = K(f)$ $(K(g \circ f) = K(f)))$.

By (iii) above and the theorem of Riemann mentioned at the beginning of this chapter, it can be assumed without loss of generality that both Ω and Ω' are the upper half plane $H = \{z = x+iy \mid y>0\}$.

A problem that aroused considerable interest during the early research in QC mappings was to determine necessary and sufficient conditions on a homeomorphism u of $(-\infty,\infty)$ onto itself which would allow u to be extended to a QC map of H onto itself.

This problem was completely solved by Ahlfors and Beurling in 1956 [5]. They proved that u can be extended to a QC map of H onto itself if and only if u satisfies a condition now referred to as quasisymmetry. The definition of quasisymmetry is as follows.

Definition 1.3. Let u be a continuous, strictly increasing function defined on an interval (a,b) with $-\infty \leq a < b \leq \infty$. Then u is said to be quasisymmetric on (a,b), or QS on (a,b), if there is some ρ, $1 \leq \rho < \infty$, satisfying

$$\frac{1}{\rho} \leq \frac{u(x+t)-u(x)}{u(x)-u(x-t)} \leq \rho \tag{1.4}$$

for all x and t with $a < x-t < x < x+t < b$. The QS dilatation $\rho(u)$ of u on (a,b) is defined as the infimum of all numbers ρ satisfying (1.4). If $\rho(u) = \rho_0$ we say that u is ρ_0-QS on (a,b).

In Chapter 2 we give an alternative definition of QS and prove it equivalent to Definition 1.3. Using the new definition the class of QS functions is then shown to be closed under the operations of addition, multiplication, composition and the taking of inverses. The remainder of Chapter 2 deals with the problem of finding explicit, sharp bounds for the QS dilatation of sums, products, compositions and inverses in terms of the QS dilatation of the original functions.

One of the difficulties frequently encountered when working with QS functions is that of determining whether or not a particular function u is QS and, if it is, finding $\rho(u)$. With this problem in mind we begin Chapter 3 by making the following definition.

Definition 1.4. Let u be a strictly increasing self-homeomorphism of $[0,\infty)$. Then u is said to be ratio bounded on $[0,\infty)$, or RB, if there are numbers L, M, $0 < L \leq M < \infty$, such that u satisfies

$$(1.5) \qquad L \leq \frac{xu'(x)}{u(x)} \leq M \quad \text{a.e. on } (0,\infty).$$

The lower (upper) ratio bound $L(u)$ $(M(u))$ of u on $[0,\infty)$ is defined as the supremum (infimum) of all numbers L (M) satisfying (1.5).

In the first part of Chapter 3 it is shown that the class of RB functions is closed under the operations of addition, multiplication, composition and the taking of inverses. Sharp bounds are found for the ratio bounds of these sums, products, compositions and inverses.

It is then shown that if a function u is RB on $[0,\infty)$ then u must also be QS on $(0,\infty)$ and sharp bounds are found for $\rho(u)$ on $(0,\infty)$ in terms of $L(u)$ and $M(u)$.

At the end of Chapter 3 we prove that if u is either convex or concave on $[0,\infty)$ then u is QS on $(0,\infty)$ if and only if it is RB on $[0,\infty)$.

In [3] Ahlfors and Beurling give an explicit extension for a QS self-homeomorphism u of $(-\infty,\infty)$ to a QC self-homeomorphism of H.

We begin Chapter 4 by defining a new extension for u to H. This extension is called the radial extension and is shown to be QC if and only if u is RB on $[0,\infty)$ and the function v given by $v(x) = -u(-x)$ is RB on $[0,\infty)$. Sharp bounds are found for the QC dilatation of the radial extension in terms of $L(u)$, $M(u)$, $L(v)$ and $M(v)$.

The remainder of Chapter 4 is concerned with the following generalization of the problem of Grötzsch: given a homeomorphism u between the boundaries of two domains, find the extension to the interiors of these domains which is most nearly conformal. Such an extension is said to be extremal for the given boundary homeomorphism u.

Conditions are given on u for which the radial extension is extremal. Questions of uniqueness and non-uniqueness are also investigated.

In Chapter 5 we continue our study of extremal mappings and generalize a result of Reich and Strebel [14].

CHAPTER II

QUASISYMMETRIC FUNCTIONS

1. A new definition.

As was pointed out in the introduction, a homeomorphism u of $(-\infty,\infty)$ onto itself can be extended to a QC map of H onto itself if and only if u is QS according to Definition 1.3. The following theorem gives an alternative formulation of quasisymmetry which is equivalent to Definition 1.3 but better suited to the estimates we want to make in this chapter.

Theorem 2.1. *Let* u *be a non-constant function defined on an interval* (a,b) *with* $-\infty \leq a < b \leq \infty$. *Then* u *is QS on* (a,b) *if and only if*

(i) u *is linear*

or (ii) *there is some* λ, $1/2 < \lambda < 1$, *such that*

$$\text{(2.1)}\quad \lambda u(x_1) + (1-\lambda)u(x_2) \leq u\left(\frac{x_1+x_2}{2}\right) \leq (1-\lambda)u(x_1) + \lambda u(x_2)$$

for all x_1, x_2 *with* $a < x_1 < x_2 < b$.

<u>Definition 2.1</u>. If u is a nonlinear QS function on (a,b) we define the <u>midpoint dilatation</u> $\lambda(u)$ of u to be the infimum of all numbers λ for which (2.1) holds. The relation (2.1) is called the <u>midpoint condition</u>. The relationship between $\lambda(u)$ and $\rho(u)$ is

$$\rho(u) = \frac{\lambda(u)}{1-\lambda(u)}, \qquad \lambda(u) = \frac{\rho(u)}{1+\rho(u)}. \tag{2.2}$$

<u>Proof of Theorem 2.1</u>.

(i) Let u be QS on (a,b). By Definition 1.2

$$\frac{1}{\rho_0} \leq \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq \rho_0 \tag{2.3}$$

for all x,t satisfying $a < x-t < x < x+t < b$, where $\rho_0 = \rho(u)$. Multiplying (2.3) by the positive expression $u(x) - u(x-t)$ and solving for $u(x)$ gives

$$\frac{\rho_0}{1+\rho_0}u(x-t) + \frac{1}{1+\rho_0}u(x+t) \leq u(x) \leq \frac{1}{1+\rho_0}u(x-t) + \frac{\rho_0}{1+\rho_0}u(x+t).$$

This double inequality becomes (2.1) if we set $x_1 = x-t$, $x_2 = x+t$, $\lambda(u) = \rho_0/(1+\rho_0)$.

(ii) Let u satisfy (2.1) with $\lambda_0 = \lambda(u)$. It must be shown that

a) u is continuous on (a,b).

b) u is strictly increasing on (a,b).

c) u satisfies (1.4) for some ρ, $1 \leq \rho < \infty$.

<u>Proof of a</u>: In [3, p.66] it is proved that (2.1) implies the continuity of u.

<u>Proof of b</u>: Let x_1, x_2 be given with $a < x_1 < x_2 < b$. Then by (2.1) it is clear that the inequality

$$\lambda_0 u(x_1) + (1-\lambda_0)u(x_2) \leq \lambda_0 u(x_2) + (1-\lambda_0)u(x_1)$$

implies $(2\lambda_0-1)u(x_1) < (2\lambda_0-1)u(x_2)$ or equivalently, since $1/2 < \lambda_0 < 1$, $u(x_1) \leq u(x_2)$. Hence u is non-decreasing.

To complete the proof of (b) assume there are points x_1, x_2 with $a < x_1 < x_2 < b$ and $u(x_1) = u(x_2) = M$. Since u is non-decreasing this would imply $u(x) = M$ identically on $[x_1,x_2]$. By assumption u is not constant. Hence there is some $x_3 \in (a,b)$ for which $u(x_3) \neq M$. Without loss of generality it can be assumed that $x_2 < x_3$. Then by the monotonicity of u, $u(x_2) = M < u(x_3)$.

Let S be defined as

$$S = \{x \mid x_2 \leq x \leq x_3 \text{ with } u(x) > u(x_2) = M\}. \tag{2.4}$$

Clearly $x_3 \in S$ so that S is not empty. In addition x_2 is a lower bound for S by (2.4). Thus S must have a greatest lower bound (g.l.b.) $\bar{x}$ with $x_2 \leq \bar{x}$. If $u(\bar{x}) > M$ then by continuity there is some $\epsilon > 0$ such that $u(x) > M$ on $(\bar{x}-\epsilon, \bar{x})$. But this contradicts the assumption that $\bar{x}$ is the g.l.b. of S. Hence $u(\bar{x}) = M$ and so $u(x) = M$ identically on $[x_1, \bar{x}]$. Pick $\epsilon > 0$ so small that $u(\bar{x}+\epsilon) > M$ and $u(\bar{x}-\epsilon) = M$. This is possible because $\bar{x}$ is the g.l.b. of S. Then by (2.1), using the points $\bar{x}-\epsilon$, $\bar{x}$, $\bar{x}+\epsilon$, we obtain

$$\lambda_0 u(\bar{x}-\epsilon) + (1-\lambda_0)u(\bar{x}+\epsilon) \leq u(\bar{x}) \leq (1-\lambda_0)u(\bar{x}-\epsilon) + \lambda_0 u(\bar{x}+\epsilon).$$

The left inequality implies that

$$u(\bar{x}) \geq \lambda_0 u(\bar{x}-\epsilon) + (1-\lambda_0)u(\bar{x}+\epsilon) > \lambda_0 M + (1-\lambda_0)M = M.$$

But this is obviously a contradiction of $u(\bar{x}) = M$. That is, there cannot be any $x_1 < x_2$ with $u(x_1) = u(x_2)$. Hence u must be strictly increasing on (a,b).

<u>Proof of c</u>: The proof is immediate since all the steps in the proof of (i) are reversible.

<u>Remark</u>: Looking at the statement of Theorem 2.1 it is reasonable to ask if condition (i) can be omitted by simply changing condition (ii) to allow $1/2 \leq \lambda_0 < 1$. It is obvious that if u is QS and linear then (2.1) does hold with $\lambda_0 = 1/2$.

The converse, however, is not true. It is possible to have a non-constant function which satisfies (2.1) with $\lambda_0 = 1/2$ but is not QS on (a,b). To show this we will exhibit an example.

In [7, p.150] a function f is constructed, using a Hamel basis $x_1, x_2, \ldots, x_\alpha, \ldots,$ $\alpha \in \Omega$, with the property $f(x+y) = f(x) + f(y)$ for all x, y. Taking $x = y$ we obtain $f(2x) = 2f(x)$, which leads to $f(x) = f(2x)/2$ or $f(x/2) = f(x)/2$. Thus for any x, y,

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{x}{2}+\frac{y}{2}\right) = f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

But this is just (2.1) with $\lambda_0 = 1/2$. As indicated in [7, p.150], f can be defined arbitrarily on the Hamel basis. Since a Hamel basis has an infinite number of elements, we can choose three elements x_1, x_2, x_3 from this particular basis. Assume without loss of generality that in the usual ordering of the reals we have $x_1 < x_2 < x_3$, and define $f(x_1) = 0$, $f(x_2) = 1$, $f(x_3) = 0$ and $f = 0$ for all the other elements in this basis. Let $u = f$. Then u satisfies (2.1) with $\lambda_0 = 1/2$ because f does. The function u is not constant since $u(x_1) \neq u(x_2)$. Yet u cannot be QS because it is not even monotonic since $x_1 < x_2 < x_3$ with $u(x_1) < u(x_2)$ and $u(x_2) > u(x_3)$. Hence condition (i) cannot be omitted in the statement of Theorem 2.1.

2. Upper and lower bounds.

In order to obtain upper and lower bounds for a QS function u on the interior of an interval when the function values are known at the endpoints, we define the following two functions $P(\lambda)$ and $p(\lambda)$ on $[0,1]$. If

$$\lambda = \frac{\theta_1}{2} + \frac{\theta_2}{2^2} + \ldots + \frac{\theta_n}{2^n} + \ldots \qquad (\theta_i = 0 \text{ or } 1)$$

is the binary expansion of $\lambda \in [0,1]$, then

(2.5)

$$P(\lambda) = P_{\lambda_0}(\lambda) = \lambda_0[\theta_1 + \lambda_{\theta_1}\theta_2 + \lambda_{\theta_1}\lambda_{\theta_2}\theta_3 + \ldots + \lambda_{\theta_1}\lambda_{\theta_2}\cdots\lambda_{\theta_{n-1}}\theta_n + \ldots],$$

$$p(\lambda) = p_{\lambda_0}(\lambda) = \lambda_1[\theta_1 + \lambda_{1-\theta_1}\theta_2 + \lambda_{1-\theta_1}\lambda_{1-\theta_2}\theta_3 + \ldots + \lambda_{1-\theta_1}\cdots\lambda_{1-\theta_{n-1}}\theta_n + \ldots],$$

where λ_0, $1/2 \leq \lambda_0 < 1$, is the midpoint dilatation of u (Cf. Definition 2.1) and $\lambda_1 = 1-\lambda_0$. When there is no chance of confusion we will use P, p in place of P_{λ_0}, p_{λ_0}, respectively.

Theorem 2.2. Let u be a QS function on (a,b) and let $x_1, x_2 \in (a,b)$ be given. Then for any $\lambda \in [0,1]$,

(2.6)
$$u[(1-\lambda)x_1 + \lambda x_2] \leq [1-P(\lambda)]u(x_1) + P(\lambda)u(x_2),$$

$$u[(1-\lambda)x_1 + \lambda x_2] \geq [1-p(\lambda)]u(x_1) + p(\lambda)u(x_2).$$

__Proof__: In a paper by R. Salem [15] P and p are shown to be continuous, strictly increasing functions mapping [0,1] onto itself. If v is QS with $v(0) = 0$, $v(1) = 1$, then by (2.1) and the method of construction of P and p in [15] we must have $p(\lambda) \leq v(\lambda) \leq P(\lambda)$ for all $\lambda \in [0,1]$ with a finite binary expansion. But the set of these numbers is dense in [0,1]. Hence, by the continuity of v,P and p, $p(\lambda) \leq v(\lambda) \leq P(\lambda)$ for all $\lambda \in [0,1]$.

If we now take any QS function u and set

$$v(\lambda) = \frac{u[(1-\lambda)x_1+\lambda x_2]-u(x_1)}{u(x_2)-u(x_1)}$$

then v is QS with $v(0) = 0$, $v(1) = 1$. Hence

$$p(\lambda) \leq v(\lambda) = \frac{u[(1-\lambda)x_1+\lambda x_2]-u(x_1)}{u(x_2)-u(x_1)} \leq P(\lambda).$$

Solving for $u[(1-\lambda)x_1+\lambda x_2]$ gives the desired bounds.

__Theorem 2.3__. $P(t) + p(1-t) = 1$ __identically on__ [0,1].

__Proof__: Define $f(t) = P(t) + p(1-t)$ on [0,1]. Since P and p are both continuous [15], so is f. It will be shown by induction on n that $f(t) = 1$ when t is of the form $t = m/2^n$, $n = 0,1,2,\ldots$ and $m = 0,1,\ldots,2^n$. Since the set of all such t is dense in [0,1] the continuity of f will then imply $f(x) = 1$ identically on [0,1].

(i) Let $n = 0$. Then m can be either 0 or 1 and $f(0) = P(0) + p(1-0) = 0 + 1 = 1$, $f(1) = P(1) + p(1-1) = 1$.

(ii) Let $n = 1$. Then m can be either 0, 1, or 2. But the cases $m = 0$ and $m = 2$ are covered by (i), while $m = 1$ gives $f(1/2) = P(1/2) + p(1-(1/2)) = \lambda_0 + \lambda_1 = 1$.

(iii) Assume $f(t) = 1$ for all t of the form $t = m/2^{N_0}$ and let $t_0 = m/2^{N_0+1}$, $0 \leq m \leq 2^{N_0+1}$. If $m = 0$ or $m = 2^{N_0+1}$ then $t = 0$ or 1, respectively. These cases have been treated in (i).

Let m be an even number. Then $m = 2k$, $1 \leq k < 2^{N_0}-1$, and so $t = m/2^{N_0+1} = 2k/2^{N_0+1} = k/2^{N_0}$. By the induction hypothesis this would imply $f(t) = f(k/2^{N_0}) = 1$.

Let m be an odd number. Then $m-1$ and $m+1$ are both even. Hence $m-1 = 2k_1$, $m+1 = 2k_2$ for some k_1, k_2 with $0 \leq k_1 < k_2 \leq 2^{N_0}$. Then

(2.7) $f(t) = f(m/2^{N_0+1})$

$$= f\left(\left(\frac{m-1}{2^{N_0+1}} + \frac{m+1}{2^{N_0+1}}\right)/2\right)$$

$$= P\left(\left(\frac{m-1}{2^{N_0+1}} + \frac{m+1}{2^{N_0+1}}\right)/2\right) + p\left(1-\left(\frac{m-1}{2^{N_0+1}} + \frac{m+1}{2^{N_0+1}}\right)/2\right)$$

$$= P([(k_1/2^{N_0}) + (k_2/2^{N_0})]/2) + p([(1-k_1/2^{N_0})+(1-k_2/2^{N_0})]/2)$$

$$= [\lambda_1 P(k_1/2^{N_0}) + \lambda_0 P(k_2/2^{N_0})]$$

$$+ [\lambda_1 p(1-k_1/2^{N_0}) + \lambda_0 p(1-k_2/2^{N_0})]$$

$$= \lambda_1 f(k_1/2^{N_0}) + \lambda_0 f(k_2/2^{N_0}).$$

Hence, by the induction hypothesis, (2.7) reduces to $\lambda_1 + \lambda_0 = 1$. The induction proof is completed.

Corollary 2.3. $P^{-1}(1/2) + p^{-1}(1/2) = 1$.

Proof: Since $P(0) = 0$, $P(1) = 1$ and P is continuous, there must be some $t_1 \in (0,1)$ with $P(t_1) = 1/2$ or, equivalently, $t_1 = P^{-1}(1/2)$. Similarly $p(0) = 0$, $p(1) = 1$ and p continuous imply the existence of a $t_2 \in (0,1)$ with $t_2 = p^{-1}(1/2)$. By Theorem 2.3, $1/2 = P(t_1) = 1-p(1-t_1)$. Hence $p(1-t_1) = 1/2 = p(t_2)$. Since p is strictly increasing, $p(1-t_1) = p(t_2)$ implies $1-t_1 = t_2$ or $t_1 + t_2 = 1$. Substituting $t_1 = P^{-1}(1/2)$, $t_2 = p^{-1}(1/2)$ gives $P^{-1}(1/2) + p^{-1}(1/2) = 1$.

3. <u>The power functions</u> $u(x) = x^\alpha$, $\alpha > 0$.

Corollary 2.3 will be found to be very useful later in this chapter. Also quite useful is

<u>Lemma 2.1</u>. Let $u(x) = x^\alpha$, $\alpha > 0$, on $[0,\infty)$. Then u is QS on $(0,\infty)$ with

$$\rho(u) = \max\left\{2^\alpha - 1, \frac{1}{2^\alpha - 1}\right\}. \tag{2.8}$$

<u>Proof</u>: It is clear that

$$\max\left\{2^\alpha-1, \frac{1}{2^\alpha-1}\right\} = \begin{cases} 2^\alpha-1 & \text{if } \alpha \geq 1 \\ \dfrac{1}{2^\alpha-1} & \text{if } 0 < \alpha < 1 \end{cases}.$$

The cases $\alpha \geq 1$ and $0 < \alpha < 1$ will be treated separately.

(i) Assume $\alpha \geq 1$ and let x,t be given arbitrarily with $x > 0$, $0 < t < x$. Then

$$\frac{u(x+t) - u(x)}{u(x) - u(x-t)} = \frac{(x+t)^\alpha - x^\alpha}{x^\alpha - (x-t)^\alpha} = \frac{\left(\frac{x+t}{x}\right)^\alpha - 1}{1 - \left(\frac{x-t}{x}\right)^\alpha} = \frac{(1+s)^\alpha - 1}{1 - (1-s)^\alpha} \tag{2.9}$$

$$= \frac{\dfrac{(1+s)^\alpha - 1}{s}}{\dfrac{1 - (1-s)^\alpha}{s}} \quad \text{with} \quad 0 < s = \frac{t}{x} < 1.$$

Clearly, $\left((1+s)^\alpha - 1\right)/s$ is just the slope of the secant line connecting the points $(1, 1^\alpha)$ and $(1+s, (1+s)^\alpha)$ on the graph of the convex function $y = x^\alpha$. Similarly $(1 - (1-s)^\alpha)/s$ is the slope of the secant connecting $(1-s, (1-s)^\alpha)$ and $(1, 1^\alpha)$. By [10, p.3], $((1+s)^\alpha - 1)/s$ is non-decreasing and $(1 - (1-s)^\alpha)/s$ non-increasing as s goes from 0 to 1. Hence

$$\frac{(1+s)^\alpha - 1}{1 - (1-s)^\alpha} = \frac{\dfrac{(1+s)^\alpha - 1}{s}}{\dfrac{1 - (1-s)^\alpha}{s}}$$

is non-decreasing for $s \in (0,1)$, and (2.9) together with

$$\lim_{s\to 0^+} \frac{(1+s)^\alpha - 1}{1 - (1-s)^\alpha} = 1, \qquad \lim_{s\to 1^-} \frac{(1+s)^\alpha - 1}{1 - (1-s)^\alpha} = 2^\alpha - 1$$

gives

$$\frac{1}{2^\alpha - 1} \le 1 \le \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \le 2^\alpha - 1. \tag{2.10}$$

Since the upper bound in (2.10) is actually approached as t approaches x, $2^\alpha - 1$ must be the best possible QS bound. That is, $\rho(u) = 2^\alpha - 1$.

(ii) Assume $0 < \alpha < 1$. Then the function $y = x^\alpha$ is concave instead of convex, as in (i). For concave functions, however, the double inequality in [10, p.3] is just reversed. The rest of the proof is the same as in (i) and we find that

$$2^{\alpha}-1 \le \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \le 1 \le \frac{1}{2^{\alpha}-1} .$$

Here the lower bound is approached as t approaches x so that $1/(2^{\alpha}-1)$ must be the best possible QS bound. That is, $\rho(u) = 1/(2^{\alpha}-1)$.

CLOSURE PROPERTIES

4. The sum of QS functions.

Theorem 2.4. Let $u_1, u_2, \ldots, u_n$ be QS functions on (a,b) with QS dilatations $\rho_1, \rho_2, \ldots, \rho_n$ and midpoint dilatations $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$, respectively. Then the function v defined as $v(x) = \sum_{i=1}^{n} u_i(x)$ is also QS on (a,b) and

$$\rho(v) \le \max_{1\le i\le n} \{\rho_i\} .$$

This result is sharp!

Proof: The proof is by induction on n.

(i) Assume $n = 2$ and let $c = \max\{\lambda^{(1)}, \lambda^{(2)}\}$. Then for $x_1, x_2 \in (a,b)$ (2.1) gives

$$v(\frac{x_1+x_2}{2}) = u_1(\frac{x_1+x_2}{2}) + u_2(\frac{x_1+x_2}{2})$$

$$\le [(1-\lambda^{(1)})u_1(x_1)+\lambda^{(1)}u_1(x_2)]+[(1-\lambda^{(2)})u_2(x_1)+\lambda^{(2)}u_2(x_2)]$$

$$\leq [(1-c)u_1(x_1)+cu_1(x_2)]+[(1-c)u_2(x_1)+cu_2(x_2)]$$

$$= (1-c)v(x_1)+cv(x_2).$$

Similarly,

$$v\left(\frac{x_1+x_2}{2}\right) = u_1\left(\frac{x_1+x_2}{2}\right) + u_2\left(\frac{x_1+x_2}{2}\right)$$

$$\geq [\lambda^{(1)}u_1(x_1)+(1-\lambda^{(1)})u_1(x_2)]+[\lambda^{(2)}u_2(x_1)+(1-\lambda^{(2)})u_2(x_2)]$$

$$\geq cv(x_1)+(1-c)v(x_2).$$

Hence

$$cv(x_1)+(1-c)v(x_2) \leq v\left(\frac{x_1+x_2}{2}\right) \leq (1-c)v(x_1)+cv(x_2).$$

By Theorem 2.1 v is QS on (a,b) while Definition 2.1 shows that $\lambda(v) \leq c = \max\{\lambda^{(1)}, \lambda^{(2)}\}$. Since $\rho(u) = \lambda(u)/(1-\lambda(u))$ is an increasing function of $\lambda(u)$, this is equivalent to $\rho(v) \leq \max\{\rho_1, \rho_2\}$.

(ii) Assume the theorem true for $n = N_0$ and let $\hat{v} = \sum_{i=1}^{N_0} u_i$. Then $v = \hat{v} + u_{N_0+1}$. But u_{N_0+1} is assumed QS and $\hat{v}$ is QS by the induction hypothesis, with $\rho(\hat{v}) \leq \max_{1\leq i\leq N_0} \{\rho_i\}$. Hence, by part (i), $v = \hat{v} + u_{N_0+1}$ is also QS on (a,b) with

$$\rho(v) \leq \max\{\rho(\hat{v}), \rho_{N_0+1}\} \leq \max_{1\leq i\leq N_0+1} \{\rho_i\}.$$

Equality holds when $u_1 = u_2 = \dots = u_n$.

An obvious analog of Theorem 2.4 would be to show that

$$\rho(v) \geq \min_{1\leq i\leq n} \{\rho_i\}.$$

This statement, however, is false even in the simple case $n = 2$, as the following Theorem shows.

<u>Theorem 2.5</u>. <u>There exist functions</u> u_1 <u>and</u> u_2, <u>QS on</u> $(-\infty, \infty)$, <u>such that</u> $\rho(u_1 + u_2) < \min(\rho_1, \rho_2)$.

<u>Proof</u>: For an arbitrary λ_0, $1/2 < \lambda_0 < 1$, and $\lambda_1 = 1-\lambda_0$, define $\hat{u}_1$ and $\hat{u}_2$ on $[0,1]$ as

$$\hat{u}_1(x) = \begin{cases} 2\lambda_0 x & \text{if } 0\leq x\leq 1/2 \\ (2-2\lambda_0)x+(2\lambda_0-1) & \text{if } 1/2<x\leq 1 \end{cases}$$

$$\hat{u}_2(x) = \begin{cases} 2\lambda_1 x & \text{if } 0\leq x\leq 1/2 \\ (2-2\lambda_1)x+(2\lambda_1-1) & \text{if } 1/2<x\leq 1 \end{cases}$$

It is easy to see that (1.4) is satisfied for both $\hat{u}_1$ and $\hat{u}_2$ on $[0,1]$ with $\rho = \lambda_0/\lambda_1$, and that $\hat{v}(x) = \hat{u}_1(x) + \hat{u}_2(x) = 2x$ for all $x \in [0,1]$. Next let us define functions u_1 and u_2 on $(-\infty,\infty)$ by

$$u_i(x) = \hat{u}_i(x) \quad \text{for } 0\leq x\leq 1, \quad u_i(x+2) = u_i(x)+2, \; i = 1,2.$$

Then by Theorem 5 of [9, p.239], u_1 and u_2 are both QS on $(-\infty,\infty)$. Also

$$\frac{u_i(\frac{3}{4}) - u_i(\frac{1}{2})}{u_i(\frac{1}{2}) - u_i(\frac{1}{4})} = \begin{cases} \frac{\lambda_0}{\lambda_1} & \text{if } i = 1, \\ \frac{\lambda_1}{\lambda_0} & \text{if } i = 2. \end{cases}$$

Therefore $\rho_1 = \rho(u_1) \geq \lambda_0/\lambda_1, \rho_2 = \rho(u_2) \geq \lambda_0/\lambda_1$. But by the construction of v, $v(x) = u_1(x) + u_2(x) = 2x$ identically on $(-\infty,\infty)$, so that $\rho(v) = 1$. Hence $\rho(v) = 1 < \lambda_0/\lambda_1 < \min\{\rho_1, \rho_2\}$.

5. <u>The product of QS functions</u>.

<u>Theorem 2.6</u>. <u>Let</u> $u_1, u_2, \ldots, u_n$ <u>be QS functions on</u> (a,b) <u>with</u> $u_i(a) = 0$ <u>for each</u> i. <u>Then the function</u> v <u>defined as</u> $v(x) = \prod_{i=1}^{n} u_i(x)$ <u>is also QS on</u> (a,b) <u>and</u>

$$\rho(v) \leq [\prod_{i=1}^{n} (1+\rho_i)] - 1.$$

<u>This result is sharp</u>!

<u>Proof</u>: The proof is by induction on n.

(i) Assume $n = 2$ and let $c = 1-(1-\lambda^{(1)})(1-\lambda^{(2)})$. Then for any $x_1, x_2 \in (a,b)$ we have by (2.1)

$$v(\frac{x_1+x_2}{2}) = u_1(\frac{x_1+x_2}{2})u_2(\frac{x_1+x_2}{2})$$

$$\leq [(1-\lambda^{(1)})u_1(x_1)+\lambda^{(1)}u_1(x_2)][(1-\lambda^{(2)})u_2(x_1)+\lambda^{(2)}u_2(x_2)]$$

$$= (1-\lambda^{(1)})(1-\lambda^{(2)})u_1(x_1)\ u_2(x_1)+(1-\lambda^{(1)})\lambda^{(2)}u_1(x_1)u_2(x_2)$$

$$+\lambda^{(1)}(1-\lambda^{(2)})u_1(x_2)u_2(x_1)+\lambda^{(1)}\lambda^{(2)}u_1(x_2)u_2(x_2)$$

$$\leq (1-\lambda^{(1)})(1-\lambda^{(2)})v(x_1)+(1-\lambda^{(1)})\lambda^{(2)}v(x_2)+\lambda^{(1)}(1-\lambda^{(2)})v(x_2)$$

$$+\lambda^{(1)}\lambda^{(2)}v(x_2)$$

$$= (1-\lambda^{(1)})(1-\lambda^{(2)})v(x_1)+(1-(1-\lambda^{(1)})(1-\lambda^{(2)}))v(x_2)$$

$$= (1-c)v(x_1)+cv(x_2).$$

Similarly

$$v(\frac{x_1+x_2}{2}) = u_1(\frac{x_1+x_2}{2})u_2(\frac{x_1+x_2}{2})$$

$$\geq cv(x_1)+(1-c)v(x_2).$$

Hence

$$cv(x_1)+(1-c)v(x_2)\leq v(\frac{x_1+x_2}{2})\leq(1-c)v(x_1)+cv(x_2).$$

By Theorem 2.1 v is QS on (a,b), while $\lambda(v) \leq c$. Changing to the QS dilatation, this becomes $\rho(v) \leq (\rho_1+1)(\rho_2+1) - 1$.

(ii) Assume the theorem true for $n = N_O$ and let $\hat{v} = \prod_{i=1}^{N_O} u_i$. Then $v = \hat{v}\cdot u_{N_O+1}$. But u_{N_O+1} is assumed QS and $\hat{v}$ is QS by the induction hypothesis, with $\rho(\hat{v}) \leq [\prod_{i=1}^{N_O} (\rho_i+1)]-1$. Hence, by part (i), $v = \hat{v}\cdot u_{N_O+1}$ is also QS on (a,b) with

$$\rho(v) \leq (\rho(\hat{v})+1)(\rho_{N_O+1}+1)-1 = \rho(\hat{v})(\rho_{N_O+1}+1)+\rho_{N_O+1} \leq [\prod_{i=1}^{N_O+1} (\rho_i+1)]-1.$$

Equality holds when $u_i(x) = x^{\alpha_i}$, $i = 1,2,\ldots,n$, on $[0,\infty)$ for any choice of $\alpha_1,\alpha_2,\ldots,\alpha_n$ all greater than or equal to 1. By Lemma 2.1 this choice of the u_i gives

$$[\prod_{i=1}^{n} (\rho_i+1)]-1 = (\prod_{i=1}^{n} 2^{\alpha_i})-1 = 2^{\Sigma\alpha_i}-1 = \rho(x^{\Sigma\alpha_i}) = \rho(v).$$

6. <u>The inverse of a QS function</u>.

<u>Theorem 2.7</u>. <u>Let</u> u <u>be a QS function from</u> (a,b) <u>onto</u> (c,d) <u>with midpoint dilatation</u> $\lambda_O = \lambda(u)$. <u>Then</u> u^{-1} <u>is a QS function of</u> (c,d) <u>onto</u> (a,b) <u>whose QS dilatation satisfies</u>

$$\rho(u^{-1}) \leq \frac{p_{\lambda_O}^{-1}(1/2)}{P_{\lambda_O}^{-1}(1/2)}$$

<u>where</u> p <u>and</u> P <u>denote the Salem functions in</u> (2.5). <u>This result is sharp</u>!

<u>Proof</u>: Let $y_1, y_3 \in (c,d)$ with $y_1 < y_3$ and let $y_2 = (y_1+y_3)/2$. Let $x_j = u^{-1}(y_j)$, $j = 1,2,3$, and let $\bar{\lambda} = P^{-1}(1/2) = 1-p^{-1}(1/2)$. Then

$$u[(1-\bar{\lambda})x_1+\bar{\lambda}x_3] \leq (1-P(\bar{\lambda}))u(x_1)+P(\bar{\lambda})u(x_3)$$

$$= (u(x_1)+u(x_3))/2 = y_2$$

by Theorem 2.2. Since u is monotone increasing this gives

$$x_2 \geq (1-\bar{\lambda})x_1 + \bar{\lambda}x_3.$$

Similarly,

$$u(\bar{\lambda}x_1+(1-\bar{\lambda})x_3) = u((1-(1-\bar{\lambda}))x_1 + (1-\bar{\lambda})x_3)$$

$$\geq (1-p(1-\bar{\lambda}))u(x_1) + p(1-\bar{\lambda})u(x_3)$$

$$= (u(x_1) + u(x_3))/2 = y_2.$$

This gives, by the monotonicity of u, $x_2 \leq \bar{\lambda}x_1 + (1-\bar{\lambda})x_3$.

Hence

$$(1-\bar{\lambda})u^{-1}(y_1)+(1-(1-\bar{\lambda}))u^{-1}(y_3)\leq u^{-1}(y_2)\leq(1-(1-\bar{\lambda}))u^{-1}(y_1)+(1-\bar{\lambda})u^{-1}(y_3)$$

and by Theorem 2.1 u^{-1} must be QS on (c,d) with $\lambda(u^{-1}) \leq 1-\bar{\lambda} = p^{-1}(1/2)$. By Definition 2.1 this becomes, in terms of the QS dilatation,

$$\rho(u^{-1}) \leq \frac{p^{-1}(1/2)}{P^{-1}(1/2)}.$$

Equality holds when $u(x) = x^{1/n}$ on $[0,\infty)$ for any integer $n \geq 1$. By Lemma 2.1, $\rho(u) = 1/(2^{1/n}-1)$. Hence, by Definition 2.1, $\lambda(u) = 1/2^{1/n}$. From (2.5) it is clear that

$$p(1-(1/2^n)) = 1-(\lambda(u))^n = 1-(1/2^{1/n})^n = 1-1/2 = 1/2,$$

or $p^{-1}(1/2) = 1-1/2^n$. For this choice of u, u^{-1} is given by $u^{-1}(y) = y^n$ and, as above, it is easy to see that $\rho(u^{-1}) = 2^n-1$. Hence

$$\rho(u^{-1}) = 2^n-1 = \frac{1-1/2^n}{1/2^n} = \frac{p^{-1}(1/2)}{1-p^{-1}(1/2)} = \frac{p^{-1}(1/2)}{P^{-1}(1/2)} .$$

For computations the following bound for $\rho(u^{-1})$, though no longer sharp, is probably easier to use than the bound in Theorem 2.7.

Corollary 2.7. Under the conditions of Theorem 2.7,

$$\rho(u^{-1}) \leq 2\cdot 2^{\frac{\log 2}{\log(1+1/\rho(u))}} - 1.$$

Proof: Let $\lambda_0 = \lambda(u)$ and $P = P_{\lambda_0}$. Since $P(1/2^n) = \lambda_0^n$ for each $n \geq 0$, and $P(1/2^0) = P(1) = 1 > 1/2$, there exists some integer k such that

$$(2.11) \qquad P(1/2^{k+1}) = \lambda_0^{k+1} \leq 1/2 < \lambda_0^k = P(1/2^k).$$

Since P is strictly increasing (2.11) is equivalent to $1/2^{k+1} \leq P^{-1}(1/2) < (1/2^k)$, which reduces easily to

$$(2.12) \qquad 1 - 1/2^k < 1-P^{-1}(1/2) = p^{-1}(1/2) \leq 1-1/2^{k+1}.$$

Solving (2.11) for k gives

$$k \leq \frac{\log 2}{\log (1/\lambda_0)} = \frac{\log 2}{\log (1+1/\rho(u))} < k+1,$$

and thus

$$2^k \leq 2^{\frac{\log 2}{\log(1+1/\rho(u))}}.$$

Hence, by Theorem 2.7 and (2.12) we find the desired bound

$$\rho(u^{-1}) \leq \frac{p^{-1}(1/2)}{P^{-1}(1/2)} \leq \frac{1-1/2^{k+1}}{1/2^{k+1}} = 2^{k+1}-1 = 2\cdot 2^k-1 \leq 2\cdot 2^{\frac{\log 2}{\log (1+1/\rho(u))}}-1.$$

7. <u>The composition of QS functions.</u>

<u>Theorem 2.8</u>. <u>Let</u> $u_1, u_2, \ldots, u_n$ <u>be QS functions such that the domain of each</u> u_{i+1} <u>is contained in the range of the preceding</u> u_i. <u>Then the composed function</u> u <u>defined as</u> $u(x) = u_n \circ u_{n-1} \circ \ldots \circ u_1(x)$ <u>is also QS and</u>

$$\rho(u) \leq \frac{P_n(1/2)}{1-P_n(1/2)}$$

<u>where</u> $P_n = P_{\lambda(u_n)} \circ P_{\lambda(u_{n-1})} \circ \ldots \circ P_{\lambda(u_1)}$. <u>This result is sharp!</u>

<u>Proof</u>: The proof is by induction on n.

(i) Assume $n = 2$ and let $x_1, x_2 \in (a,b)$ be given, where (a,b) is the domain of u_1. Let $c = P_{\lambda(u_2)}(\lambda(u_1))$. Then by Theorem 2.1 and Theorem 2.2

$$\begin{aligned} u(\tfrac{x_1+x_2}{2}) &= u_2(u_1(\tfrac{x_1+x_2}{2})) \\ &\leq u_2((1-\lambda(u_1))u_1(x_1)+\lambda(u_1)u_1(x_2)) \\ &\leq (1-P_{\lambda(u_2)}(\lambda(u_1)))u(x_1)+P_{\lambda(u_2)}(\lambda(u_1))u(x_2) \\ &= (1-c)u(x_1)+cu(x_2). \end{aligned}$$

Similarly

$$\begin{aligned} u(\tfrac{x_1+x_2}{2}) &= u_2(u_1(\tfrac{x_1+x_2}{2})) \\ &\geq (1-p_{\lambda(u_2)}(1-\lambda(u_1)))u(x_1)+p_{\lambda(u_2)}(1-\lambda(u_1))\ u(x_2) \\ &= P_{\lambda(u_2)}(\lambda(u_1))u(x_1)+(1-P_{\lambda(u_2)}(\lambda(u_1)))u(x_2) \\ &= cu(x_1)+(1-c)u(x_2). \end{aligned}$$

Hence

$$cu(x_1)+(1-c)u(x_2)\leq u(\tfrac{x_1+x_2}{2})\leq(1-c)u(x_1)+cu(x_2).$$

By Theorem 2.1 u is QS on (a,b) and $\lambda(u)\leq c = P_{\lambda(u_2)}(\lambda(u_1))$

$= P_{\lambda(u_2)}(P_{\lambda(u_1)}(1/2)) = P_2(1/2)$. Changing to the QS dilatation, this becomes

$$\rho(u) \leq \frac{P_2(1/2)}{1-P_2(1/2)} .$$

(ii) Assume the theorem true for $n = N_0$ and let $\hat{u} = u_{N_0} \circ u_{N_0-1} \circ \cdots \circ u_1$. Then $u = u_{N_0+1} \circ \hat{u}$. But u_{N_0+1} is assumed QS and $\hat{u}$ is QS by the induction hypothesis with $\lambda(\hat{u}) \leq P_{N_0}(1/2)$. Hence, by part (i), $u = u_{N_0+1} \circ \hat{u}$ is also QS and

$$\lambda(u) = \lambda(u_{N_0+1} \circ \hat{u}) \leq P_{\lambda(u_{N_0+1})}(P_{\lambda(\hat{u})}(1/2)) = P_{\lambda(u_{N_0+1})}(\lambda(\hat{u}))$$

$$= P_{N_0+1}(1/2).$$

By Definition 2.1 this is equivalent to

$$\rho(u) \leq \frac{P_{N_0+1}(1/2)}{1-P_{N_0+1}(1/2)} .$$

Equality holds when $u_i(x) = x^{\alpha_i}$, $i = 1,2,\ldots,n$, on $[0,\infty)$ for any choice of $\alpha_1, \alpha_2, \ldots, \alpha_n$ with all $\alpha_i \geq 1$. Obviously $u(x) = x^{\pi\alpha_i}$ on $[0,\infty)$, so that Lemma 2.1 gives

$$\lambda(u_i) = \frac{\rho(u_i)}{1+\rho(u_i)} = \frac{2^{\alpha_i}-1}{2^{\alpha_i}} = 1 - \frac{1}{2^{\alpha_i}} \quad \text{for each } i.$$

It is now trivial to show by induction that $P_n(1/2) = 1-1/2^{\pi\alpha_i}$.

Hence

$$\rho(u) = 2^{\pi\alpha_i}-1 = \frac{1-1/2^{\pi\alpha_i}}{1/2^{\pi\alpha_i}} = \frac{P_n(1/2)}{1-P_n(1/2)} .$$

<u>Remark</u>: It would be interesting, in Theorem 2.8, to see how $\rho(u)$ depends on the individual $\rho(u_i)$. For simplicity we will restrict the investigation to the case $n = 2$.

If $n = 2$ then by Theorem 2.8

$$(2.13) \quad \rho(u) = \rho(u_2 \circ u_1) \leq \frac{P_2(1/2)}{1-P_2(1/2)} = \frac{P_{\lambda(u_2)}(\lambda(u_1))}{1-P_{\lambda(u_2)}(\lambda(u_1))} .$$

Since $1/2 \leq \lambda(u_1) < 1$ there is some integer $k \geq 1$ with

$$1 - 1/2^k \leq \lambda(u_1) \leq 1 - 1/2^{k+1}.$$

Solving for k and using Definition 2.1 gives

$$k \leq \frac{\log(1+\rho(u_1))}{\log 2} \leq k+1.$$

Thus

$$P_{\lambda(u_2)}(\lambda(u_1)) \leq P_{\lambda(u_2)}(1-1/2^{k+1}) = 1-(1-\lambda(u_2))^{k+1}$$

and using this in (2.13) gives

$$(2.14) \quad \rho(u_2 \circ u_1) \leq (1+\rho(u_2))^{k+1} - 1 \leq (1+\rho(u_2))^{\frac{\log(2+2\rho(u_1))}{\log 2}} - 1.$$

Now suppose the function u_1 is fixed and let $\alpha = \log(2+2\rho(u_1))/\log 2$. Then $2 \leq \alpha < \infty$ and (2.14) shows that

$$(2.15) \quad \rho(u_2 \circ u_1) \leq (1+\rho(u_2))^{\alpha} - 1.$$

A simple expansion of the right-hand side in (2.15) shows that

$$\rho(u_2 \circ u_1) \leq \rho(u_2)^{\alpha} + O[\rho(u_2)^{\alpha-1}] \quad \text{as} \quad \rho(u_2) \quad \text{approaches} \quad \infty. \tag{2.16}$$

Suppose now that u_2 is fixed instead of u_1. Then using (2.14) and the identity $A^{\log B} = B^{\log A}$, with $A = (1+\rho(u_2))$, $B = (2+2\rho(u_1))$, we obtain

$$\rho(u_2 \circ u_1) \leq (1+\rho(u_2))^{\frac{\log (2+2\rho(u_1))}{\log 2}} - 1 \tag{2.17}$$

$$= (2+2\rho(u_1))^{\frac{\log(1+\rho(u_2))}{\log 2}} - 1.$$

Let $\beta = \log(1+\rho(u_2))/\log 2$. Then $1 \leq \beta < \infty$ and (2.17) gives

$$\rho(u_2 \circ u_1) \leq (2+2\rho(u_1))^{\beta} - 1. \tag{2.18}$$

A simple expansion of the right-hand side in (2.18) shows that

$$\rho(u_2 \circ u_1) \leq (2\rho(u_1))^{\beta} + O[\rho(u_1)^{\beta-1}] \quad \text{as} \quad \rho(u_1) \quad \text{approaches} \quad \infty.$$

The inequalities (2.15) and (2.18) show that if u_1 (u_2) is fixed, then $\rho(u_2 \circ u_1)$ is bounded by a power function of $\rho(u_2)$ $(\rho(u_1))$ as $\rho(u_2)$ $(\rho(u_1))$ tends to infinity, the power depending on the finite constant $\rho(u_1)$ $(\rho(u_2))$.

CHAPTER III

RATIO BOUNDED FUNCTIONS

1. Introduction.

Although the definition of quasisymmetry is quite simple, in practice it is often very difficult to determine whether or not a particular function u is QS, and to obtain bounds on its QS dilatation $\rho(u)$. Since all QS functions are by definition strictly increasing, however, and a monotone function must be differentiable almost everywhere, it would certainly be helpful to have a simple differentiability condition for quasisymmetry.

With this in mind we begin Chapter 3 by defining a new class of functions called ratio bounded, or RB, functions. This class is defined by a relatively simple differentiability condition. The major portion of this chapter is spent in an investigation of closure and other properties of this class, much as we spent Chapter 2 studying the properties of the class of QS functions.

The first application made of this class comes at the end of Chapter 3 when we show that the RB functions actually form a subclass of the QS functions and then use the differentiability condition to obtain sharp bounds for the QS dilatation $\rho(u)$ when u is RB.

A second, and more important, application of RB functions is their use in the problem of extending a QS function on $(-\infty,\infty)$ to a QC mapping of the upper half plane H. In particular we are interested in finding such an extension that is quasiconformally extremal. I.e., one that is as close to being conformal as possible. This second application will be treated in the next chapter, however.

2. Notation.

A function u will be said to be normalized if it satisfies

$$u(0) = 0, \quad u(1) = 1, \quad u(\infty) = \infty, \tag{3.1}$$

whenever the points $0, 1, \infty$ are in the domain of u. Throughout the remainder of this paper it will be assumed that all QS functions are normalized according to (3.1). Furthermore, if u is a QS function of $(-\infty,\infty)$ onto itself with QS dilatation $\rho(u)$ as given in Definition 1.3, then $\hat{\rho}(u)$ will mean the QS dilatation of the restriction of u to $(0,\infty)$. Obviously $\hat{\rho}(u) \leq \rho(u)$ always holds.

3. Alternative definitions.

For ease of reference we repeat here the definition of ratio boundedness (Cf. Definition 1.4).

Definition 3.1. Let u be a strictly increasing self-homeomorphism of $[0,\infty)$. Then u is said to be *ratio bounded on* $[0,\infty)$, *or RB, if there are numbers* L, M, $0 < L \leq M < \infty$, *such that*

$$(3.2) \qquad L \leq \frac{xu'(x)}{u(x)} \leq M \quad \text{a.e. on} \quad (0,\infty).$$

The lower (upper) ratio bound $L(u)$ $(M(u))$ *of* u *on* $[0,\infty)$ *is defined as the supremum (infimum) of all numbers* L (M) *satisfying* (3.2).

Our first goal is to give two equivalent, and very useful, alternative definitions for ratio boundedness.

Theorem 3.1. *A function* u *of* $[0,\infty)$ *onto itself is RB with ratio bounds* L_0, M_0 *if and only if*

$$(3.3) \quad u(x) = \exp\left(\int_1^x \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}\right) \quad \text{for all} \quad x > 0,\ u(0) = 0,$$

where χ *is some real valued, measurable function on* $(0,\infty)$ *with*

$$(3.4) \qquad \sup_{0<s<\infty} |\chi(s)| < 1, \quad \operatorname*{ess\,sup}_{0<s<\infty} |\chi(s)| = \frac{Q-1}{Q+1} < 1,$$

for some $Q \geq 1$. *Moreover, if* u *satisfies* (3.3), *then* $Q = \max\{M_0, 1/L_0\}$.

Proof:

(i) Assume that u is of the form (3.3). Then χ measurable implies that $(1+\chi(s))/(1-\chi(s))$ is also measurable since (3.4) shows that the denominator in the integral in (3.3) is never zero. Hence $\int_1^x \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}$, and thus also u in (3.3) must be continuous on $(0,\infty)$. It is also clear, taking a limit in (3.3) as x approaches 0, that u is continuous at 0 as well. By the second inequality in (3.4), $0 < 1/Q \leq (1+\chi(s))/(1-\chi(s))$ a.e. on $(0,\infty)$, so that

$$x_1 < x_2 \Rightarrow u(x_2) - u(x_1) = e^{\int_{x_1}^{x_2} \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}} > 0 \Rightarrow u(x_2) > u(x_1).$$

Thus u is strictly increasing. It is also trivial to show from the form of u (3.3) that u is normalized according to (3.1). Finally,

$$u'(x) = \frac{1+\chi(x)}{1-\chi(x)} \frac{1}{x} u(x) \qquad \text{a.e. on } (0,\infty),$$

whence

$$0 < \frac{1}{Q} \leq \frac{xu'(x)}{u(x)} = \frac{1+\chi(x)}{1-\chi(x)} \leq Q < \infty \qquad \text{a.e. on } (0,\infty).$$

By definition 3.1 u is RB with $1/Q \le L_0 \le M_0 \le Q$, and either $L_0 = 1/Q$ or $M_0 = Q$; otherwise there could not be equality in the second part of (3.4). Hence $Q = \max\{M_0, 1/L_0\}$.

(ii) Assume u is RB with ratio bounds L_0 and M_0. Then since u is monotonic, u' exists a.e. on $(0,\infty)$ and is measurable. Let $Q = \max\{M_0, 1/L_0\} \ge 1$ and define χ by

$$(3.5) \qquad \chi(s) = \begin{cases} \dfrac{su'(s)-u(s)}{su'(s)+u(s)} & \text{wherever } u' \text{ exists} \\ 0 & \text{elsewhere} \end{cases}$$

Then χ is measurable on $(0,\infty)$ and

$$\sup_{0<s<\infty} |\chi(s)| = \operatorname*{ess\,sup}_{0<s<\infty} |\chi(s)| = (Q-1)/(Q+1) < 1.$$

From (3.5), $\dfrac{u'(s)}{u(s)} = \dfrac{1+\chi(s)}{1-\chi(s)}\dfrac{1}{s}$ a.e. on $(0,\infty)$. Integrating and exponentiating gives

$$u(x) = e^{\int_1^x \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}},$$

and by continuity $u(0) = 0$.

<u>Theorem 3.2</u>. <u>Let</u> u <u>be a function mapping</u> $[0,\infty)$ <u>onto itself and normalized according to</u> (3.1). <u>Then</u> u <u>is RB on</u> $[0,\infty)$, <u>with ratio bounds</u> L_0 <u>and</u> M_0, <u>if and only if</u>

(3.6) $$\left(\frac{b}{a}\right)^{L_O} \leq \frac{u(b)}{u(a)} \leq \left(\frac{b}{a}\right)^{M_O}$$

<u>for all</u> $a, b \in (0,\infty)$ <u>with</u> $a < b$.

<u>Proof</u>:

(i) Assume u is RB with ratio bounds L_O, M_O. By Theorem 3.1

$$u(x) = e^{\int_1^x \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}} \quad \text{for all } x > 0, \quad u(0) = 0,$$

which implies that

$$L_O \leq \frac{1+\chi(s)}{1-\chi(s)} = \frac{su'(s)}{u(s)} \leq M_O \qquad \text{a.e. on } (0,\infty).$$

Hence

$$\left(\frac{b}{a}\right)^{L_O} = e^{\int_a^b \frac{L_O}{s} ds} \leq e^{\int_a^b \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}} = \frac{u(b)}{u(a)} \leq e^{\int_a^b \frac{M_O}{s} ds} = \left(\frac{b}{a}\right)^{M_O}.$$

(ii) Assume

(3.7) $$\left(\frac{b}{a}\right)^{L_O} \leq \frac{u(b)}{u(a)} \leq \left(\frac{b}{a}\right)^{M_O} \qquad \text{for all } 0<a<b<\infty.$$

Then $L_O > 0$ implies $u(b)/u(a) \geq (b/a)^{L_O} > 1$ for any $a < b$, which means u is strictly increasing. Hence u must also be differentiable a.e. on $(0,\infty)$. If $x \in (0,\infty)$ is a point of differentiability for u, then for any $h > 0$ (3.7) gives

$$\left(1+\frac{h}{x}\right)^{L_0} = \left(\frac{x+h}{x}\right)^{L_0} \leq \frac{u(x+h)}{u(x)} \leq \left(\frac{x+h}{x}\right)^{M_0} = \left(1+\frac{h}{x}\right)^{M_0} \quad \text{a.e.,}$$

or

$$(3.8) \qquad \frac{(1+\frac{h}{x})^{L_0}-1}{h}\, u(x) \leq \frac{u(x+h)-u(x)}{h} \leq \frac{(1+\frac{h}{x})^{M_0}-1}{h}\, u(x) \quad \text{a.e.}$$

Letting h approach 0, (3.8) reduces easily to

$$\frac{L_0}{x}\, u(x) \leq u'(x) \leq \frac{M_0}{x}\, u(x) \qquad \text{a.e. on} \quad (0,\infty),$$

which is equivalent to

$$L_0 \leq \frac{xu'(x)}{u(x)} \leq M_0 \qquad \text{a.e. on} \quad (0,\infty).$$

Therefore u is RB.

CLOSURE PROPERTIES

4. <u>The sum of RB functions</u>.

<u>Theorem 3.3</u>. <u>Let</u> $u_1, u_2, \ldots, u_n$ <u>be RB functions with ratio bounds</u> $L(u_i) = L_i$, $M(u_i) = M_i$, $i = 1,2,\ldots,n$. <u>Then the function</u> $v = (1/n)\sum_{i=1}^{n} u_i$ <u>is also RB, and</u>

$$L(v) \geq \min_{1\leq i\leq n} \{L_i\}, \quad M(v) \leq \max_{1\leq i\leq n} \{M_i\}.$$

<u>This result is sharp</u>!

<u>Proof</u>: Let $L_0 = \min_{1\leq i\leq n} \{L_i\}$, $M_0 = \max_{1\leq i\leq n} \{M_i\}$. Then

$$(3.9) \qquad L_0 u_i(x) \leq x u_i'(x) \leq M_0 u_i(x) \quad \text{a.e. on} \quad (0,\infty)$$

for each i. By the subadditivity of measure, however, the set of points at which <u>any</u> of the u_i does not satisfy (3.9) has measure 0. Hence

$$L_0 = \frac{\sum_1^n L_0 u_i}{\sum_1^n u_i} \leq \frac{xv'(x)}{v(x)} = \frac{x\sum_1^n u_i'(x)}{\sum_1^n u_i(x)} \leq \frac{\sum_1^n M_0 u_i}{\sum_1^n u_i} = M_0 \quad \text{a.e. on} \quad (0,\infty).$$

The proofs of continuity, monotonicity and normality follow trivially from the definition of v.

Equality holds when $u_i(x) = x^\alpha$ on $[0,\infty)$, $i = 1,2,\ldots,n$, for some $\alpha \geq 1$.

5. <u>The product of RB functions</u>.

<u>Theorem 3.4</u>. <u>Let</u> $u_1, u_2, \ldots, u_n$ <u>be RB functions</u>. <u>Then the function</u> $v = \prod_{i=1}^n u_i$ <u>is also RB, and</u>

$$L(v) \geq \sum_1^n L_i, \qquad M(v) \leq \sum_1^n M_i.$$

<u>This result is sharp</u>!

<u>Proof</u>: Let $L_0 = \sum_1^n L_i$, $M_0 = \sum_1^n M_i$. By the subadditivity of measure,

$$L_i u_i(x) \leq x u_i'(x) \leq M_i u_i(x) \qquad \text{a.e. on} \quad (0,\infty)$$

for each i implies

$$L_0 \leq \frac{xv'(x)}{v(x)} = \frac{x \sum_{i=1}^{n} \left(u_i'(x) \prod_{j\neq i} u_j(x)\right)}{\prod_{j=1}^{n} u_j(x)} = \sum_{i=1}^{n} \frac{x u_i'(x)}{u_i(x)} \leq M_0 \qquad \text{a.e. on} \quad (0,\infty).$$

The proofs of continuity, monotonicity and normality follow trivially from the definition of v.

Equality holds when $u_i(x) = u^{\sigma_i}$ on $[0,\infty)$, $i=1,2,\ldots,n$, for any choice of the α_i all greater than 0.

6. <u>The inverse of an RB function</u>.

<u>Theorem 3.5</u>. <u>If</u> u <u>is an RB function then so is</u> u^{-1}, <u>and</u>

$$L(u^{-1}) = 1/M(u), \quad M(u^{-1}) = 1/L(u).$$

<u>Proof</u>: Let $v = u^{-1}$. It is obvious that the continuity, monotonicity and normality of u imply the same properties for v. Since v is monotonic it is differentiable a.e. Let y_0 be a point at which v is differentiable. Then also u

must be differentiable at x_0, where $x_0 = v(y_0)$. Also $v'(y_0) = 1/u'(x_0)$ implies that

$$\frac{1}{M(u)} \leq \frac{y_0 v'(y_0)}{v(y_0)} = \frac{u(x_0)\frac{1}{u'(x_0)}}{x_0} = \frac{u(x_0)}{x_0 u'(x_0)} \leq \frac{1}{L(u)} \quad \text{a.e. on} \quad (0,\infty)$$

7. <u>The composition of RB functions</u>.

<u>Theorem 3.6</u>. <u>Let</u> $u_1, u_2, \ldots, u_n$ <u>be RB functions</u>. <u>Then the composition function</u> $v = u_n \circ u_{n-1} \circ \cdots \circ u_1$ <u>is also RB, and</u>

$$L(v) \geq \prod_{i=1}^{n} L_i, \quad M(v) \leq \prod_{i=1}^{n} M_i.$$

<u>This result is sharp</u>!

<u>Proof</u>: The proof is by induction on n.

(i) Assume $n = 2$. Then $v = u_2 \circ u_1$. The proofs of continuity, monotonicity and normality follow trivially from the definition of v. Hence v is differentiable a.e. on $(0,\infty)$. Let x_0 be a point of differentiability for v and let $y_0 = u_1(x_0)$. Then $v'(x_0) = u_2'(y_0)u_1'(x_0)$. Since $v(x_0) = u_2(y_0)$ and $y_0 = u_1(x_0)$ we thus have

$$L_1 L_2 \leq \frac{x_0 v'(x_0)}{v(x_0)} = \frac{x_0 u_2'(y_0) u_1'(x_0)}{u_2(y_0)} \frac{y_0}{u_1(x_0)} = \frac{y_0 u_2'(y_0)}{u_2(y_0)} \cdot \frac{x_0 u_1'(x_0)}{u_1(x_0)}$$

$$\leq M_1 M_2 \quad \text{a.e. on} \quad (0,\infty).$$

(ii) Assume the theorem true for $n = N_0$ and let $\hat{v} = u_{N_0} \circ u_{N_0-1} \circ \cdots \circ u_1$. Then $v = u_{N_0+1} \circ \hat{v}$. But u_{N_0+1} is RB by assumption and $\hat{v}$ is RB by the induction hypothesis with

$$L(\hat{v}) \geq \prod_{1}^{N_0} L_i, \quad M(\hat{v}) \leq \prod_{1}^{N_0} M_i.$$

Hence, by part (i), $v = u_{N_0+1} \circ \hat{v}$ is also RB and

$$L(v) = L(u_{N_0+1} \circ \hat{v}) \geq L_{N_0+1} \circ L(\hat{v}) \geq \prod_{1}^{N_0+1} L_i, \quad M(v) = M(u_{N_0+1} \circ \hat{v})$$

$$\leq M_{N_0+1} \cdot M(\hat{v}) \leq \prod_{1}^{N_0+1} M_i.$$

Equality holds when $u_i(x) = x^{\alpha_i}$ on $[0, \infty)$, $i = 1, 2, \ldots, n$, for any choice of the α_i all greater than 0.

8. <u>The class</u> $T_{L,M}$ <u>of RB functions</u>.

<u>Definition 3.2</u>. For given $L, M \in (0, \infty)$ with $L < M$, let $T_{L,M}$ denote the class of RB functions u with $L(u) \geq L$, and $M(u) \leq M$.

<u>Lemma 3.1</u>. <u>Let</u> u <u>be continuous on</u> $[0, \infty)$ <u>and suppose</u> $\{u_i\}$ <u>converges to</u> u <u>pointwise on</u> $[0, \infty)$ <u>and uniformly on compact subsets, where</u> $u_i \in T_{L,M}$ <u>for each</u> i. <u>Then</u> $u \in T_{L,M}$ <u>also</u>.

Proof: Since $u_i(0) = 0$ and $u_i(1) = 1$ for each i, we obviously have $u(0) = 0$, $u(1) = 1$ as well, by the pointwise convergence of the u_i. Also, $u_i(x) > 0$ for each i and any $x > 0$, so that $u(x) \geq 0$. Suppose there exist $x_1, x_2 \in (0,\infty)$ with $u(x_1) > u(x_2)$. Let $d = u(x_1) - u(x_2) > 0$. Since $\{u_i\}$ converges uniformly to u on the compact set $[x_1/2, 2x_2]$, there exists an N_0 such that

$$|u_{N_0}(x_1)-u(x_1)| < d/4, \; | u_{N_0}(x_2)-u(x_2)| < d/4.$$

Then

$$u_{N_0}(x_1) \geq u(x_1)-d/4 = u(x_2)+3d/4 > u(x_2)+d/4 > u_{N_0}(x_2).$$

Therefore $u_{N_0}(x_1) > u_{N_0}(x_2)$, which is impossible since u_{N_0} is strictly increasing. Therefore u is non-decreasing.

The proof will be completed by showing that for any $x_1, x_2 \in (0,\infty)$ with $x_1 < x_2$,

$$(3.10) \qquad \left(\frac{x_2}{x_1}\right)^L \leq \frac{u(x_2)}{u(x_1)} \leq \left(\frac{x_2}{x_1}\right)^M .$$

Given any integer N_0, let $\epsilon = u(x_1)/2(2N_0+1)$. By the uniform convergence of $\{u_i\}$ to u on $[x_1/2, 2x_2]$, there exists an n such that

$$0 < u_n(x_1)-\epsilon \leq u(x_1) \leq u_n(x_1)+\epsilon, \quad u_n(x_2)-\epsilon \leq u(x_2) \leq u_n(x_2)+\epsilon.$$

Thus

$$\frac{u(x_2)}{u(x_1)} \leq \frac{u_n(x_2)+\varepsilon}{u_n(x_1)-\varepsilon} = \frac{u_n(x_2)}{u_n(x_1)} \frac{1+\frac{\varepsilon}{u_n(x_2)}}{1-\frac{\varepsilon}{u_n(x_1)}}$$

$$\leq \left(\frac{x_2}{x_1}\right)^M \frac{1+\frac{\varepsilon}{u_n(x_1)}}{1-\frac{\varepsilon}{u_n(x_1)}} \leq \left(\frac{x_2}{x_1}\right)^M \frac{1+\frac{\varepsilon}{u(x_1)-\varepsilon}}{1-\frac{\varepsilon}{u(x_1)-\varepsilon}}$$

$$= \left(\frac{x_2}{x_1}\right)^M \frac{u(x_1)}{u(x_1)-2\varepsilon} = \left(\frac{x_2}{x_1}\right)^M \left[1+\frac{1}{2N_0}\right].$$

Since N_0 can be chosen arbitrarily large, this means that $u(x_2)/u(x_1) \leq (x_2/x_1)^M$. Similarly, we find that $u(x_2)/u(x_1) \geq (x_2/x_1)^L$. Thus (3.10) holds. By Theorem 3.2 u must be RB with $L(u) \geq L$, $M(u) \leq M$. Hence $u \in T_{L,M}$.

Theorem 3.7. $T_{L,M}$ is compact under uniform convergence on compact subsets of $(0,\infty)$.

Proof: We begin by showing that $T_{L,M}$ is equicontinuous on compact subsets of $(0,\infty)$.

Let S be a compact subset of $(0,\infty)$. Then there exist positive numbers a,b with $0<a<1<b<\infty$ such that $a \leq x \leq b$ for all $x \in S$. Given any $\varepsilon > 0$, let $\delta = a((1+\varepsilon/2b^M)^{1/2}-1)$. Let $u \in T_{L,M}$ and $x_1, x_2 \in S$ with $x_1 < x_2$. Then by Theorem 3.2, $u(b) = u(b)/u(1) \leq b^{M(u)} \leq b^M$. Hence

$|x_2 - x_1| < \delta$ implies that $x_2/x_1 < 1 + (\delta/x_1) \leq 1 + (\delta/a)$, which implies further that

$$|u(x_2)-u(x_1)| = u(x_2)-u(x_1) = u(x_1)\left\lfloor\frac{u(x_2)}{u(x_1)} - 1\right\rfloor$$

$$\leq u(b)\left\lfloor\left(\frac{x_2}{x_1}\right)^M - 1\right\rfloor \leq b^M\left\lfloor\left(1 + \frac{\delta}{a}\right)^M - 1\right\rfloor$$

$$= b^M\left\lfloor\frac{\varepsilon}{2b^M}\right\rfloor = \frac{\varepsilon}{2} < \varepsilon.$$

Thus $T_{L,M}$ is equicontinuous on compact subsets of $(0,\infty)$.

Now let $x \in (0,\infty)$. It must be shown that $S_x = \{u(x) | u \in T_{L,M}\}$ is bounded. If $x = 1$ then $S = \{1\}$ by the normality condition, and $\{1\}$ is obviously bounded. Suppose $0 < x < 1$. Then by Theorem 3.2, using $a = x$ and $b = 1$, we obtain $x^M \leq u(x) \leq x^L$, so $S_x \subset [x^M, x^L]$. Hence S_x is bounded. Finally, if $x > 1$ then by Theorem 3.2 with $a = 1$ and $b = x$ we find $x^L \leq u(x) \leq x^M$, so $S_x \subset [x^L, x^M]$ is bounded.

Hence $T_{L,M}$ satisfies all the conditions of Ascoli's Theorem, so any sequence $\{u_i\}$ of elements from $T_{L,M}$ has a subsequence which converges to a continuous function u pointwise on $(0,\infty)$ and uniformly on compact subsets of $(0,\infty)$. By Lemma 3.1 $u \in T_{L,M}$. Therefore $T_{L,M}$ is compact.

<u>Corollary 3.7</u>. <u>For any</u> $\alpha > 0$, <u>let</u> $u_\alpha(x) = x^\alpha$ <u>on</u> $[0,\infty)$. <u>Then for any</u> $L, M \in (0,\infty)$ <u>with</u> $L \leq M$, $u_L \in T_{L,M}$ <u>and</u> $u_M \in T_{L,M}$. <u>Also, for any</u> $u \in T_{L,M}$ <u>we have</u>

$$u_M(x) \leq u(x) \leq u_L(x) \qquad \text{for all } x \in [0,1],$$

$$u_L(x) \leq u(x) \leq u_M(x) \qquad \text{for all } x \in [1,\infty).$$

<u>Proof</u>: The proof is immediate from Theorem 3.2. If $x \in [0,1]$ take $a = x$, $b = 1$. If $x \in [1,\infty)$ take $a = 1$, $b = x$.

FIRST APPLICATION

9. <u>The relationship between RB and QS</u>.

It will be shown in this section that any function u which is RB on $[0,\infty)$ must also be QS on $(0,\infty)$. Sharp bounds on $\rho(u)$ will be found in terms of $L(u)$ and $M(u)$. The following lemmas will be found quite useful.

<u>Lemma 3.2</u>. <u>Let</u> $T > 0$. <u>Then</u> $T > 2^T - 1$ <u>if</u> $0<T<1$, $T = 2^T - 1$ <u>if</u> $T = 1$, <u>and</u> $T < 2^T - 1$ <u>if</u> $T > 1$.

<u>Proof</u>: Let $f(T) = 2^T - 1 - T$ for $T \geq 0$. Clearly $f(0) = 0$ and $f(1) = 0$. Furthermore, $f'(T) = 2^T \log 2 - 1$ has only one zero T_0 with $0<T_0<1$, and $f''(T) = 2^T(\log 2)^2 > 0$ everywhere. Therefore the critical point T_0 must be a

minimum and so f is decreasing on $(0,1)$ and increasing on $(1,\infty)$. This completes the proof.

Lemma 3.3. *If* $T > 0$ *and* $0<s\leq 1$ *then*

$$\min\{T,\ 2^T-1\} \leq \frac{(1+s)^T-1}{s} \leq \max\{T,\ 2^T-1\}.$$

Proof: Let $f(s) = ((1+s)^T-1)/s$ for $0<s\leq 1$. Then

$$f'(s) = \frac{Ts(1+s)^{T-1} + 1 - (1+s)^T}{s^2},$$

$$f''(s) = \frac{2(1+s)^T - 2Ts(1+s)^{T-1} + T(T-1)s^2(1+s)^{T-2} - 2}{s^3}.$$

If there is some $s_1 \in (0,1)$ with $f'(s_1) = 0$ then

$$(1+s_1)^T - 1 = Ts_1(1+s_1)^{T-1}.$$

Therefore

$$f''(s_1) = \frac{T(T-1)(1+s_1)^{T-2}}{s_1},$$

so

(3.11) $$f''(s) \text{ is } \begin{cases} < 0 & \text{if } 0<T<1 \\ = 0 & \text{if } T = 1 \\ > 0 & \text{if } T > 1. \end{cases}$$

Case 1: If $0<T<1$ then by (3.11) s_1 is a maximum point of f and

$$(3.12) \qquad f(s_1) = \frac{(1+s_1)^T - 1}{s_1} = \frac{Ts_1(1+s_1)^{T-1}}{s_1}$$

$$= T(1+s_1)^{T-1} = \frac{T}{(1+s_1)^{1-T}}$$

$$\leq T.$$

Since any maximum or minimum of f on $[0,1]$ must occur either at one of the endpoints or at an interior point s_1 with $f'(s_1) = 0$, (3.12) shows that for any $s \in (0,1]$,

$$f(s) \leq \max\{f(0), f(1), T\} = \max\{T, 2^T - 1\},$$

$$f(s) \geq \min\{f(0), f(1)\} = \min\{T, 2^T - 1\}.$$

<u>Case 2</u>: If $T = 1$ then

$$f(s) = \frac{(1+s) - 1}{s} = 1 = T = 2^T - 1.$$

Therefore, once again,

$$\min\{T, 2^T - 1\} \leq f(s) \leq \max\{T, 2^T - 1\}.$$

<u>Case 3</u>: If $T > 1$ then by (3.11) s_1 is a minimum point of f and

$$f(s_1) = \frac{(1+s_1)^T - 1}{s_1} = \frac{Ts_1(1+s_1)^{T-1}}{s_1} = T(1+s_1)^{T-1} \geq T.$$

Hence

$$f(s) \geq \min\{f(0),\ f(1),\ T\} = \min\{T,\ 2^T - 1\},$$

$$f(s) \leq \max\{f(0),\ f(1)\} = \max\{T, 2^T - 1\}.$$

<u>Lemma 3.4</u>. <u>If</u> $T > 0$ <u>and</u> $0<s\leq 1$ <u>then</u>

$$\min\{1,T\} \leq \frac{1-(1-s)^T}{s} \leq \max\{1,T\}.$$

<u>Proof</u>: Let

$$g(s) = \frac{1-(1-s)^T}{s}.$$

Then

$$g'(s) = \frac{Ts(1-s)^{T-1} + (1-s)^T - 1}{s^2} \tag{3.13}$$

and

$$g''(s) = \frac{-2(1-s)^T - 2Ts(1-s)^{T-1} - T(T-1)s^2(1-s)^{T-2} + 2}{s^3}. \tag{3.14}$$

If there is some $s_1 \in (0,1)$ with $g'(s_1) = 0$ then by (3.13)

$$1-(1-s_1)^T = Ts_1(1-s_1)^{T-1}.$$

Therefore, by (3.14),

$$g''(s_1) = \frac{-T(T-1)s_1^2(1-s_1)^{T-2}}{s_1^2}.$$

Hence

$$g''(s_1) \text{ is } \begin{cases} > 0 & \text{if } 0<T<1 \\ = 0 & \text{if } T = 1 \\ < 0 & \text{if } T > 1. \end{cases} \tag{3.15}$$

<u>Case 1</u>: If $0<T<1$ then by (3.15) s_1 is a minimum point of g on $(0,1)$ and

$$g(s_1) = \frac{1-(1-s_1)^T}{s_1} = T(1-s_1)^{T-1} = \frac{T}{(1-s_1)^{1-T}} \geq T.$$

Thus

$$g(s) \geq \min\{g(0),\ g(1),\ T\} = \min\{1,T\},$$

$$g(s) \leq \max\{g(0),\ g(1)\} = \max\{1,T\}.$$

<u>Case 2</u>: If $T = 1$ then $g(s) = 1 = T$, so again

$$\min\{1,\ T\} \leq g(s) \leq \max\{1,\ T\}.$$

<u>Case 3</u>: If $T > 1$ then by (3.15) s_1 is a maximum point for g and

$$g(s_1) = \frac{1-(1-s_1)^T}{s_1} = T(1-s_1)^{T-1} \leq T.$$

Hence

$$g(s) \leq \max\{g(0),\ g(1),\ T\} = \max\{1,T\},$$

$$g(s) \geq \min\{g(0),\ g(1)\} = \min\{1,\ T\}.$$

<u>Theorem 3.8</u>. <u>Let</u> u <u>be</u> RB <u>on</u> $[0,\infty)$ <u>with ratio bounds</u> $L = L(u)$, $M = M(u)$. <u>Then</u> u <u>must also be QS on</u> $(0,\infty)$ <u>with</u>

(i) $2^L-1 \leq \hat{\rho}(u) \leq 2^M-1$ if $1 \leq L \leq M$,

(ii) $\dfrac{1}{2^M-1} \leq \hat{\rho}(u) \leq \dfrac{1}{2^L-1}$ if $L \leq M \leq 1$,

(iii) $\hat{\rho}(u) \leq \max \{\dfrac{1}{2^L-1}, 2^M-1, \dfrac{M}{L}, \dfrac{M}{L}\left(\dfrac{2}{M-L}\right)^{M-L}(M-1)^{M-1}(1-L)^{1-L}\}$

if $L<1<M$.

<u>These results are sharp</u>!

Proof: The proof will be in two parts. In (a) we will treat cases (i) and (ii). In (b) we will treat case (iii).

(a) To find lower bounds for $\hat{\rho}(u)$, we notice that by the definition of $\hat{\rho}(u)$, if it exists,

$$\frac{1}{\hat{\rho}(u)} \leq \frac{u(x+t)-u(x)}{u(x)-u(x-t)} = \frac{\frac{u(x+t)}{u(x)}-1}{1-\frac{u(x-t)}{u(x)}} \leq \hat{\rho}(u) \quad \text{for all} \quad x>0,\ 0<t<x.$$

Letting $t \longrightarrow x$, and remembering that u is continuous on $[0,\infty)$ with $u(0) = 0$, we find that

$$\frac{1}{\hat{\rho}(u)} \leq \frac{u(2x)}{u(x)} - 1 \leq \hat{\rho}(u).$$

Thus

$$\hat{\rho}(u) \geq \frac{u(2x)}{u(x)} - 1 \geq \left(\frac{2x}{x}\right)^L - 1 = 2^L - 1,$$

$$\frac{1}{\hat{\rho}(u)} \leq \frac{u(2x)}{u(x)} - 1 \leq \left(\frac{2x}{x}\right)^M - 1 = 2^M - 1.$$

Hence $\hat{\rho}(u) \geq \max\{2^L-1, 1/(2^M-1)\}$. The lower bounds in cases (i) and (ii) follow simply from this inequality.

To find upper bounds for $\hat{\rho}(u)$ let $x \in (0,\infty)$, $0<t<x$, and define s as $0 < s = t/x < 1$. Then by Theorem 3.2

$$\frac{u(x+t)-u(x)}{u(x)-u(x-t)} = \frac{\frac{u(x+t)}{u(x)}-1}{1-\frac{u(x-t)}{u(x)}} \geq \frac{\left(\frac{x+t}{x}\right)^L-1}{1-\left(\frac{x-t}{x}\right)^M} = \frac{(1+s)^L-1}{1-(1-s)^M},$$

(3.16)

$$\frac{u(x+t)-u(x)}{u(x)-u(x-t)} = \frac{\frac{u(x+t)}{u(x)}-1}{1-\frac{u(x-t)}{u(x)}} \leq \frac{\left(\frac{x+t}{x}\right)^M-1}{1-\left(\frac{x-t}{x}\right)^L} = \frac{(1+s)^M-1}{1-(1-s)^L}.$$

But by Lemma 3.3 and Lemma 3.4, if $0<s<1$ then

$$\frac{(1+s)^M-1}{1-(1-s)^L} = \frac{\frac{(1+s)^M-1}{s}}{\frac{1-(1-s)^L}{s}} \leq \frac{\max\{M, 2^M-1\}}{\min\{1, L\}} = \alpha,$$

(3.17)

$$\frac{(1+s)^L-1}{1-(1-s)^M} = \frac{\frac{(1+s)^L-1}{s}}{\frac{1-(1-s)^M}{s}} \geq \frac{\min\{L, 2^L-1\}}{\max\{1, M\}} = \beta.$$

Thus, if $c = \max\{\alpha, 1/\beta\}$, then by (3.16) and (3.17)

$$\frac{1}{c} \leq \frac{u(x+t)-u(x)}{u(x)-u(x-t)} \leq c,$$

so $\hat{\rho}(u) \leq c$. Now by Lemma 3.2, $1 \leq L \leq M$ implies $\alpha = 2^M-1$, $\beta = L/M \geq 1/M$. Hence $c = \max\{2^M-1, M\} = 2^M-1$. This is just the bound we are looking for in case (i). Similarly in case (ii), $L \leq M \leq 1$ implies by Lemma 3.2 that $\alpha = M/L \leq 1/L$, $\beta = 2^L - 1$. Hence $c = \max\{1/L, 1/(2^L-1)\} = 1/(2^L-1)$. Again, this is the bound we want.

(b) For any $x \in (0,\infty)$, $0<t<x$, let s be defined as $0 < s = t/x < 1$,

$$f(s) = \frac{(1+s)^M - 1}{1-(1-s)^L}, \quad g(s) = \frac{(1+s)^L - 1}{1-(1-s)^M}, \quad \text{for } s \in (0,1),$$

and $\alpha = \sup_{0<s<1} f(s)$, $\beta = \inf_{0<s<1} g(s)$. Then as we showed in (a),

$$\beta \leq g(s) \leq \frac{u(x+t)-u(x)}{u(x)-u(x-t)} \leq f(s) \leq \alpha.$$

If, as before, we set $c = \max\{\alpha, 1/\beta\}$, then we have $\hat{\rho}(u) \leq c$ in case (iii) as well. Our problem is to estimate α and β in this case.

It is clear that both f and g are continuous on $(0,1)$ and that by l'Hôpital's rule

$$\lim_{s\to 0^+} f(s) = \frac{M}{L}, \quad \lim_{s\to 1^-} f(s) = 2^M - 1, \quad \lim_{s\to 0^+} g(s) = \frac{L}{M}, \quad \lim_{s\to 1^-} g(s) = 2^L - 1.$$

Hence, if we define f and g at 0 and 1 by their limit values, then f and g will be continuous on $[0,1]$. Since $[0,1]$ is compact f and g must actually attain their maximum and minimum values on $[0,1]$. That is,

$$\alpha = \max_{0\le s\le 1} f(s), \quad \beta = \min_{0\le s\le 1} g(s).$$

Since we already know the values of f and g at the end-points 0 and 1 we need only investigate possible interior maxima for f and interior minima for g to obtain bounds on α and β.

Let s_0, $0<s_0<1$, be any critical point for f. Then

$$0 = f'(s_0) = \frac{(1-(1-s_0)^L)M(1+s_0)^{M-1}-((1+s_0)^M-1)L(1-s_0)^{L-1}}{(1-(1-s_0)^L)^2}$$

implies

$$(1-(1-s_0)^L)M(1+s_0)^{M-1} = ((1+s_0)^M-1)L(1-s_0)^{L-1},$$

which in turn implies

$$f(s_0) = \frac{(1+s_0)^M-1}{1-(1-s_0)^L} = \frac{M(1+s_0)^{M-1}}{L(1-s_0)^{L-1}} = \frac{M}{L}(1+s_0)^{M-1}\cdot(1-s_0)^{1-L}.$$

If we define $h(s_0) = (M/L)(1+s_0)^{M-1}(1-s_0)^{1-L}$ for $0<s_0<1$, then we find that $f(s_0) \le \sup_{0<s_0<1} h(s_0)$. But now

$$\frac{h'(s_0)}{h(s_0)} = \frac{M-1}{1+s_0} - \frac{1-L}{1-s_0} = \frac{1}{1-s_0^{\,2}}\,(M+L-2-s_0(M-L)).$$

If $M+L-2\le 0$ then h is a decreasing function of s_0, so that $h(s_0) \le M/L$ on $0<s_0<1$. But if $M+L-2>0$ then $h'(s_0) = 0$ for $s_0 = (M+L-2)/(M-L) \in (0,1]$ since $L\le 1$ implies $M+L-2\le M-L$.

And in this case h increases from $s_0 = 0$ to $s_0 = (M+L-2)/(M-L)$ and then decreases until $s_0 = 1$. So if $M+L-2>0$ then the supremum of h on $(0,1)$ is

$$h\left(\frac{M+L-2}{M-L}\right) = \frac{M}{L}\left(\frac{2}{M-L}\right)^{M-L}(M-1)^{M-1}(1-L)^{1-L}.$$

Thus, in any case, if s_0 is any critical point for f on $(0,1)$ then

$$f(s_0) \leq \max\left\{\frac{M}{L},\ \frac{M}{L}\left(\frac{2}{M-L}\right)^{M-L}(M-1)^{M-1}(1-L)^{1-L}\right\}.$$

This means, finally, that

$$\alpha = \max\{f(0),\ f(1),\ f(s_0)\} \leq \max\left\{\frac{M}{L},\ 2^M-1,\ \frac{M}{L}\left(\frac{2}{M-L}\right)^{M-L}(M-1)^{M-1}(1-L)^{1-L}\right\}$$

Now suppose s_0, $0<s_0<1$, is a critical point for g. Then in a manner similar to our treatment of f, we must have

$$g(s_0) = (L/M)(1+s_0)^{L-1}(1-s_0)^{1-M}.$$

Setting $j(s_0) = (L/M)(1+s_0)^{L-1}(1-s_0)^{1-M}$ for $0<s_0<1$ and differentiating j logarithmically as we did with h, we find that $M+L-2\geq 0$ implies j increasing so that $j(s_0)\geq j(0) = L/M$ for $s_0 \in (0,1)$. And, as before, if $M+L-2<0$ then j decreases from $s_0 = 0$ to $s_0 = (M+L-2)/(L-M)$, then increases until $s_0 = 1$. Hence we must have

$$j(s_0) \geq j\left(\frac{M+L-2}{L-M}\right) = \left(\frac{L}{M}\right)\left(\frac{2}{M-L}\right)^{L-M}(1-L)^{L-1}(M-1)^{1-M}.$$

Thus, in any case, if s_O is any critical point of g with $0<s_O<1$ then

$$g(s_O) \geq \min\{\frac{L}{M}, \frac{L}{M}(\frac{2}{M-L})^{L-M}(1-L)^{L-1}(M-1)^{1-M}\}$$

and so, finally,

$$\beta = \min\{g(0), g(1), g(s_O)\} \geq \min\{\frac{L}{M}, 2^L-1, \frac{L}{M}(\frac{2}{M-L})^{L-M}(1-L)^{L-1}(M-1)^{1-M}\}.$$

And, at long last,

$$\hat{\rho}(u) \leq c = \max\{\alpha, \frac{1}{\beta}\} = \max\{2^M-1, \frac{1}{2^L-1}, \frac{M}{L}, \frac{M}{L}(\frac{2}{M-L})^{M-L}(M-1)^{M-1}(1-L)^{1-L}\}.$$

Equality will hold in cases (i), (ii) and (iii) by the following examples 3.1, 3.2 and 3.3, respectively.

Example 3.1. Let $u(x) = x^\alpha$ on $[0,\infty)$ with $\alpha \geq 1$. Then u is differentiable on $(0,\infty)$ and

$$\frac{xu'(x)}{u(x)} = \frac{x\alpha x^{\alpha-1}}{x^\alpha} = \alpha \geq 1 \quad \text{for all} \quad x \in (0,\infty).$$

Therefore u is RB with $L = M = \alpha$. Hence the lower and upper bounds in case (i) of Theorem 3.8 are the same and so both inequalities must actually be equalities. Thus $\hat{\rho}(u) = 2^\alpha-1$ as was previously shown in Lemma 2.1.

Example 3.2. Let $u(x) = x^\alpha$ on $[0,\infty)$, with $0<\alpha\leq 1$. As in the previous example we find that u is RB on $(0,\infty)$ with $L = M = \alpha$. But since $\alpha \leq 1$ we must use case (ii) of Theorem 3.8. Once again both the lower and upper bounds are the same, so both inequalities must be equalities and $\hat{\rho}(u) = 1/(2^{\alpha}-1)$, again as was shown in Lemma 2.1.

<u>Example 3.3</u>. Define u on $[0,\infty)$ as

$$u(x) = \begin{cases} x^{1/2} & \text{if } 0 \le x \le 1 \\ x^{3/2} & \text{if } 1 < x. \end{cases}$$

Then

$$\frac{xu'(x)}{u(x)} = \begin{cases} 1/2 & \text{if } 0<x<1 \\ 3/2 & \text{if } 1<x. \end{cases}$$

Hence u is RB with $L = 1/2 < 1$, $M = 3/2 > 1$ and so we are in case (iii). Looking at the upper bound in case (iii) of Theorem 3.8 we see that

$$2^{M}-1 = 2^{3/2}-1 = 2\sqrt{2}-1 < 2(3/2)-1 = 3-1 = 2,$$

$$1/(2^{L}-1) = 1/(2^{1/2}-1) < 1/(1.4-1) = 1/.4 = 5/2,$$

$$M/L = (3/2)/(1/2) = 3,$$

$$(M/L)(2/(M-L))^{M-L}(M-1)^{M-1}(1-L)^{1-L} = (3)(2)(1/2)^{1/2}(1/2)^{1/2} = 3.$$

Hence the upper bound says $\hat{\rho}(u) \le 3$. But if we take $x = 1$ in the definition of QS then we find

$$\frac{u(1+t)-u(1)}{u(1)-u(1-t)} = \frac{(1+t)^{3/2}-1}{1-(1-t)^{1/2}}.$$

Thus $\lim_{t\to 0^+} \frac{u(1+t)-u(1)}{u(1)-u(1-t)} = 3$ so that $\hat{\rho}(u) \geq 3$. Hence we must have $\hat{\rho}(u) = 3$, which shows that equality holds in case (iii).

10. Convex and concave functions.

If we look back at Theorem 2.1 we can think of (2.1) as a generalized convexity-concavity condition. For if $\lambda_0 = 1/2$ then the left inequality becomes the condition for the concavity of u and the right inequality becomes the condition for the convexity of u. Then it is natural to ask: If u is concave or convex, what else is needed to show that u is QS? The answer is contained in the next two theorems.

Theorem 3.9. *Let u be a continuous, monotone function on $[0,\infty)$, and normalized according to (3.1). Furthermore, suppose u is convex on $(0,\infty)$. Then u is QS if and only if it is RB. Moreover, if u is QS, then*

(3.18) $$1\leq L(u), \quad 2^L-1\leq\hat{\rho}(u)\leq 2^M-1.$$

This result is sharp!

Proof:

(i) Assume u RB and let x_0 be any point of differentiability for u. (Since u is monotonic it must be differentiable a.e.). Since u is assumed to be convex we can use statement (1.4) of [10, p.3] if we simply reverse both of the inequality symbols. Hence

$$\frac{u(x_0)-u(x_0-h)}{h} \geq \frac{u(x_0)-u(0)}{x_0-0} = \frac{u(x_0)}{x_0} .$$

Letting $h \to 0^+$, we find that

$$\lim_{h\to 0^+} \frac{u(x_0)-u(x_0-h)}{h} = u'(x_0) \geq \frac{u(x_0)}{x_0} ,$$

or

$$\frac{x_0 u'(x_0)}{u(x_0)} \geq 1.$$

Therefore $L(u) \geq 1$. Case (i) of Theorem 3.8 now shows that u is QS, and gives the desired bounds.

(ii) Now assume that u is QS on $(0,\infty)$ with QS dilatation $\hat{\rho}(u)$. Then again by (1.4) of [10, p.3] with both inequalities reversed,

$$\begin{aligned} \frac{u(x_0)-u(x_0-h)}{h} &\leq \frac{u(2x_0)-u(x_0)}{x_0} \\ &= \frac{u(2x_0)-u(x_0)}{u(x_0)-u(0)} \cdot \frac{u(x_0)-u(0)}{x_0} \\ &\leq \hat{\rho}(u)\frac{u(x_0)}{x_0} . \end{aligned}$$

Letting $h \to 0^+$, we find that $u'(x_0) \leq \hat{\rho}(u)\ u(x_0)/x_0$ or

$$\frac{x_0 u'(x_0)}{u(x_0)} \leq \hat{\rho}(u).$$

And, as we showed in part (i) of this proof,

$$\frac{x_0 u'(x_0)}{u(x_0)} \geq 1.$$

Therefore u is RB. Again, since $L(u) \geq 1$ we may apply case (i) of Theorem 3.8, and the bounds follow. Equality holds in (3.18) if we take $u(x) = x$ for $0 \leq x < \infty$. In this case we have $L(u) = M(u) = 1$ and all the inequalities in Theorem 3.9 become equalities.

<u>Theorem 3.10</u>. <u>Let</u> u <u>be a continuous, monotone function on</u> $[0,\infty)$, <u>normalized according to</u> (3.1). <u>Furthermore, suppose</u> u <u>is concave on</u> $(0,\infty)$. <u>Then</u> u <u>is QS if and only if it is RB. Moreover, if</u> u <u>is QS, then</u>

$$M(u) \leq 1, \; \frac{1}{2^M-1} \leq \hat{\rho}(u) \leq \frac{1}{2^L-1} \; .$$

<u>This result is sharp</u>!

<u>Proof</u>: The proof is the same as for Theorem 3.9 except that here, since u is assumed to be concave, we use (1.4) of [10, p.3] exactly as it is given. Equality again holds when $u(x) = x$, $0 \leq x < \infty$.

CHAPTER IV

RADIAL EXTENSIONS

1. Notation.

For a given QS function u mapping $(-\infty,\infty)$ onto itself let

(i) $u_1(x) = u(x)$ on $[0,\infty)$,

(ii) $u_2(x) = -u(-x)$ on $[0,\infty)$,

(iii) $U(x) = \log(u_1(e^x))$ for $-\infty<x<\infty$,

(iv) $V(x) = \log(u_2(e^x))$ for $-\infty<x<\infty$,

(v) $h(x) = (V(x)-U(x))/\pi$.

Furthermore we say that

(vi) u is RB on $[0,\infty)$ if u_1 if RB on $[0,\infty)$, and its ratio bounds are $L_1 = L(u_1)$, $M_1 = M(u_1)$,

(vii) u is RB on $(-\infty,0]$ if u_2 is RB on $[0,\infty)$, and its ratio bounds are $L_2 = L(u_2)$, $M_2 = M(u_2)$.

Lemma 4.1. If u is RB on $(-\infty,0]$ and on $[0,\infty)$ then

$0<L_1\leq U'\leq M_1<\infty$ for almost all x, $-\infty<x<\infty$,

$0<L_2\leq V'\leq M_2<\infty$ for almost all x, $-\infty<x<\infty$.

Proof: The proof follows immediately from the definition of RB and the fact (cf. (iii) and (iv)) that

$$U'(x) = \frac{e^x u_1'(e^x)}{u_1(e^x)}, \quad V'(x) = \frac{e^x u_2'(e^x)}{u_2(e^x)}$$

whenever the right-hand sides exist.

If we let R be the subset of $[0,\infty)$ where either u_1' or u_2' fails to exist, then since u_1 and u_2 are monotone functions we know that $\ell(R) = 0$, where ℓ is one-dimensional Lebesgue measure. Finally, if we let R' be the set of points in $(-\infty,\infty)$ where either U or V is not differentiable, we must have also $\ell(R') = 0$ since U and V are strictly increasing.

2. The radial extension.

Definition 4.1. Let u be a QS function on $(-\infty,\infty)$. Then the function f_u defined as

$$(4.1) \qquad f_u(z) = f_u(re^{i\theta}) = [u_1(r)^{(\pi-y)/\pi} \cdot u_2(r)^{y/\pi}]e^{i\theta}$$

for $r > 0$, $0<\theta<\pi$, is called the radial extension of u.

Lemma 4.2. The function f_u is an extension of u to a homeomorphism of the upper half plane $H = \{z = x+iy \mid y > 0\}$ onto itself.

Proof: It is obvious from the definition of f_u that $\arg f_u(z) = \arg z$ for $\text{Im } z > 0$; hence f_u maps each ray with argument between 0 and π onto itself. But $|f_u(re^{i\theta})| = u_1(r)^{(\pi-y)/\pi} \cdot u_2(r)^{y/\pi}$ increases monotonically from 0 to ∞ as r increases from 0 to ∞. Hence f_u is a one-to-one map of H onto itself. Since $\arg f_u$ and $|f_u|$ are both continuous functions of $\arg z$ and $|z|$, f_u must be a bi-continuous mapping. Finally, taking $\theta = 0$, $\theta = \pi$ shows that $f_u = u$ on $(-\infty,\infty)$, so that f_u is an extension of u to H.

Theorem 4.1. Let u be a QS map of $(-\infty,\infty)$ onto itself. Then the radial extension f_u is QC if and only if u is RB on both $(-\infty,0]$ and $[0,\infty)$. Furthermore, if f_u is QC, then

$$Q = K(f_u) = \frac{\hat{Q} + \sqrt{\hat{Q}^2-4}}{2}$$

where

$$\hat{Q} = \operatorname*{ess\,sup}_{-\infty<x<\infty} \hat{Q}(x),$$

and

$$\hat{Q}(x) = \max\{U'(x) + \frac{1+h^2(x)}{U'(x)},\ V'(x) + \frac{1+h^2(x)}{V'(x)}\}$$

for all x with $-\infty<x<\infty$.

Proof: Since, by Lemma 4.2, f_u is a homeomorphism, the proof will depend primarily on a study of the point dilatation (cf. Definition 1.2). In order to simplify the proof we begin by forming the composed mapping $g = f_2 \circ f_u \circ f_1$, where $f_1(z) = e^z$ and $f_2(z) = \log z$. Letting $\Omega = \{x+iy \mid 0<y<\pi\}$ and $\Omega' = \{S+iT \mid 0<T<\pi\}$, g is clearly a homeomorphism of Ω onto Ω' given by

$$(4.2) \qquad g: x+iy \longrightarrow \left[\frac{\pi-y}{\pi} U(x) + \frac{y}{\pi} V(x)\right] + iy$$

for $-\infty<x<\infty$, $0<y<\pi$. Therefore

$$(4.3) \qquad S(x,y) = \frac{\pi-y}{\pi}U(x) + \frac{y}{\pi}V(x), \quad T(x,y) = y,$$

and by the invariance of the QC dilatation under composition with conformal maps we can study the simpler function g instead of f_u.

In order to prove that g, and hence also f_u, is QC, we will use the analytic definition as given in [3, p.24]. According to this definition we must show that

(i) g is ACL (absolutely continuous on lines) in Ω,

(ii) $D \leq K$ a.e. in Ω, where D is the point dilatation of g and $K \geq 1$ is a finite constant.

(i) To show that g is absolutely continuous on every horizontal line in Ω, fix y_0, $0<y_0<\pi$, and let $x_1 + iy_0$, $x_2 + iy_0$ be any two points on this horizontal line with $x_1 < x_2$. Then

$$(4.4)\qquad |g(x_2+iy_0) - g(x_1+iy_0)| = \frac{\pi-y}{\pi}[U(x_2) - U(x_1)] + \frac{y}{\pi}[V(x_2) - V(x_1)].$$

But by Theorem 3.2,

$$(4.5)\qquad L_1(x_2-x_1) = \log (e^{x_2}/e^{x_1})^{L_1} \leq U(x_2) - U(x_1)$$

$$= \log u_1(e^{x_2}) - \log u_1(e^{x_1})$$

$$= \log u_1(e^{x_2})/u_1(e^{x_1})$$

$$\leq \log (e^{x_2}/e^{x_1})^{M_1} = M_1(x_2-x_1).$$

Similarly,

$$(4.6)\qquad L_2(x_2-x_1) = \log (e^{x_2}/e^{x_1})^{L_2} \leq V(x_2) - V(x_1)$$

$$\leq \log (e^{x_2}/e^{x_1})^{M_2} = M_2(x_2-x_1).$$

Using (4.5) and (4.6) in (4.4) gives

$$|g(x_2+iy_0) - g(x_1+iy_0)| = \frac{\pi-y}{\pi}[U(x_2) - U(x_1)] + \frac{y}{\pi}[V(x_2) - V(x_1)]$$

$$\leq \frac{\pi-y}{\pi}M_1(x_2-x_1) + \frac{y}{\pi}M_2(x_2-x_1)$$

$$\leq \max\{M_1,M_2\}\cdot(x_2-x_1).$$

Thus

$$\frac{|g(x_2+iy_0) - g(x_1+iy_0)|}{|(x_2+iy_0) - (x_1+iy_0)|} \leq \max\{M_1,M_2\} < \infty.$$

Therefore g is Lipschitz continuous, hence absolutely continuous, on each horizontal line in Ω.

To show that g is absolutely continuous on every vertical line in Ω, fix x_0, $-\infty<x_0<\infty$, and let x_0+iy_1, x_0+iy_2 be any two points on this vertical line with $0<y_1\leq y_2<\pi$. Then

$$|g(x_0+iy_2) - g(x_0+iy_1)| = \left|\frac{y_2}{\pi}[V(x_0)-U(x_0)]+\frac{y_1}{\pi}[U(x_0)-V(x_0)]+i(y_2-y_1)\right|$$

$$\leq \left|\frac{y_2-y_1}{\pi}[V(x_0)-U(x_0)]\right| + |y_2-y_1|$$

$$= \frac{y_2-y_1}{\pi}[V(x_0) - U(x_0) + \pi]$$

$$\leq (y_2-y_1)[1 + \frac{\log \rho}{\pi}].$$

This last statement is true because, from the quasisymmetry of u,

$$U(x_0) - V(x_0) = \log \frac{u_1(e^{x_0})}{u_2(e^{x_0})} \leq \log \rho(u),$$

$$V(x_0) - U(x_0) = \log \frac{u_2(e^{x_0})}{u_1(e^{x_0})} \leq \log \rho(u).$$

Therefore,

$$\frac{|g(x_0+iy_2) - g(x_0+iy_1)|}{|(x_0+iy_2) - (x_0+iy_1)|} \leq 1 + \frac{\log \rho}{\pi} < \infty .$$

Thus g is Lipschitz continuous, and hence absolutely continuous, on each vertical line in Ω.

(ii) Finally we show that $D \leq K$ a.e. in Ω for some finite constant $K \geq 1$. As was shown in the proof of Lemma 4.1, the set R' of points in $(-\infty,\infty)$ at which either U' or V' does not exist has one-dimensional Lebesgue measure 0. Therefore the set of points obtained by the cartesian product $R'\times(0,\pi)$ must have two-dimensional Lebesgue measure 0. Thus, for investigating the QC dilatation of g we need only look at the points in Ω of the form x_0+iy_0 where $U'(x_0)$ and $V'(x_0)$ exist and $0<y_0<\pi$.

Let x_0 be any fixed point for which U' and V' exist, and let y satisfy $0<y<\pi$. Then, recalling that $h(x_0) = (V(x_0)-U(x_0))/\pi$, we have

$$S_x(x_0,y_0) = U'(x_0)+yh'(x_0), \quad S_y(x_0,y_0) = h(x_0), \quad T_x = 0, \quad T_y = 1.$$

Hence

$$(4.7) \qquad D + \frac{1}{D} = \frac{S_x^2+S_y^2+T_x^2+T_y^2}{|S_xT_y-S_yT_x|} = (U'(x_0)+yh'(x_0)) + \frac{1+h^2(x_0)}{U'(x_0)+yh'(x_0)} .$$

We wish to consider $D + 1/D$ as a function of y, $0<y<\pi$. Clearly (4.7) is a continuous function of y on $[0,\pi]$, and the values at the end-points are

$$(D + \frac{1}{D})(0) = U'(x_0) + \frac{1+h^2(x_0)}{U'(x_0)} ,$$

$$(D + \frac{1}{D})(\pi) = U'(x_0)+\pi h'(x_0) + \frac{1+h^2(x_0)}{U'(x_0)+yh'(x_0)} = V'(x_0)+\frac{1+h^2(x_0)}{V'(x_0)}.$$

To check on possible interior extrema, we observe that

$$\frac{d}{dy}(D + \frac{1}{D}) = h'(x_0)\left(1 - \frac{1+h^2(x_0)}{U'(x_0)+yh'(x_0)}\right),$$

$$\frac{d^2}{dy^2}(D + \frac{1}{D}) = 2(h'(x_0))^2\left(\frac{1+h^2(x_0)}{(U'(x_0)+yh'(x_0))^3}\right) > 0.$$

Therefore, any interior extremum will be a minimum, not a maximum. So

$$\sup_{0<y<\pi} \left(D(x_0,y) + \frac{1}{D(x_0,y)}\right) = \max\left\{U'(x_0) + \frac{1+h^2(x_0)}{U'(x_0)},\ V'(x_0) + \frac{1+h^2(x_0)}{V'(x_0)}\right\}$$

$$= \hat{Q}(x_0).$$

We know, from the analytic definition of quasiconformality [3, p.24], that

$$K(g) + \frac{1}{K(g)} = \hat{Q} = \operatorname*{ess\ sup}_{-\infty<x<\infty} \hat{Q}(x),$$

and solving this for $K(g)$, or equivalently for $K(f_u)$, we get the desired result.

Finally, if u is not RB on both $(-\infty,0]$ and $[0,\infty)$ it is because one of the following is true: $L_1 = 0$, $M_1 = \infty$, $L_2 = 0$, or $M_2 = \infty$. Since all the cases are similar we will treat only $M_1 = \infty$. Since

$$M_1 = \operatorname*{ess\ sup}_{0<x<\infty} \frac{xu'(x)}{u(x)} = \operatorname*{ess\ sup}_{-\infty<x<\infty} U'(x),$$

we see immediately that if $M_1 = \infty$ then for each $A > 0$ there exists a set $W \subset (-\infty,\infty)$ with positive one-dimensional measure on which $U'(x) > A$. But then the set $Wx(0,\pi/2)$ has positive two-dimensional measure and for any (x_0,y_0) in $Wx(0,\pi/2)$ we have

$$D(x_0,y_0) + \frac{1}{D(x_0,y_0)} = (U'(x_0)+yh'(x_0)) + \frac{1+h^2(x_0)}{U'(x_0)+yh'(x_0)}$$

$$= (\frac{\pi-y}{\pi} U'(x_0) + \frac{y}{\pi} V'(x_0)) + \frac{1+h^2(x_0)}{(\frac{\pi-y}{\pi} U'(x_0) + \frac{y}{\pi} V'(x_0))}$$

$$\geq (\frac{\pi-\frac{\pi}{2}}{\pi} U'(x_0)+0) + 0$$

$$= U'(x_0)/2 > A/2.$$

Therefore, for any $A > 0$ there exists a set of positive measure in Ω on which $D + 1/D > A/2$. Thus $D + 1/D$, and hence also D, cannot be bounded a.e. in Ω. This means that g, and hence also f_u, is not QC.

<u>Corollary 4.1</u>. <u>Under the hypotheses of Theorem</u> 4.1 <u>let</u> L_1,M_1 <u>and</u> L_2,M_2 <u>be the ratio bounds of</u> u <u>on</u> $[0,\infty)$ <u>and</u> $(-\infty,0]$, <u>respectively. Also let</u> $\rho_0 = \rho_0(u)$ <u>be the smallest constant greater than or equal to</u> 1 <u>for which</u>

$$(4.8) \qquad \frac{1}{\rho_0} \leq \frac{u(t)-u(0)}{u(0)-u(-t)} = \frac{u_1(t)}{u_2(t)} \leq \rho_0 \qquad \text{for all } t > 0.$$

<u>Then</u>

$$K(f_u) + \frac{1}{K(f_u)} \leq \max\{L + \frac{1}{L}(1+(\frac{\log \rho_0}{\pi})^2),\ M + \frac{1}{M}(1+(\frac{\log \rho_0}{\pi})^2)\}$$

<u>where</u> $L = \min\{L_1,L_2\}$, $M = \max\{M_1,M_2\}$.

<u>Proof</u>: Consider the function $x + a/x$, where a is a positive constant. Then

$$\frac{d}{dx}(x+\frac{a}{x}) = 1-\frac{a}{x^2}, \quad \frac{d^2}{dx^2}(x+\frac{a}{x}) = \frac{2a}{x^3} > 0 \qquad \text{for} \quad x > 0.$$

So if this function is defined on an interval, any maximum must occur at an endpoint. Now apply this reasoning to the expression

$$U'(x_0) + \frac{1+h^2(x_0)}{U'(x_0)} .$$

Since $L_1 \leq U' \leq M_1$ and $h(x) = \frac{V(x)-U(x)}{\pi} = \frac{1}{\pi} \log \frac{u_2(e^x)}{u_1(e^x)} \leq \frac{\log \rho_0}{\pi}$,

we have

$$(4.9) \quad U'(x_0)+\frac{1+h^2(x_0)}{U'(x_0)} \leq \max\{L_1+\frac{1}{L_1}(1+(\frac{\log \rho_0}{\pi})^2),\ M_1+\frac{1}{M_1}(1+(\frac{\log \rho_0}{\pi})^2)\}.$$

Similarly

$$(4.10) \quad V'(x_0)+\frac{1+h^2(x_0)}{V'(x_0)} \leq \max\{L_2+\frac{1}{L_2}(1+(\frac{\log \rho_0}{\pi})^2),\ M_2+\frac{1}{M_2}(1+(\frac{\log \rho_0}{\pi})^2)\}.$$

By Theorem 4.1 we are looking for the maximum of the four right-hand terms in (4.9) and (4.10). But these are all of the form $x + a/x$ with $a = 1 + (\frac{\log \rho_0}{\pi})^2$ and, as we have already shown, any maximum must occur at an endpoint. Since the values L_1, M_1, L_2, M_2 are contained in the interval $[L,M]$, where $L = \min\{L_1, L_2\}$ and $M = \max\{M_1, M_2\}$, all four terms can be replaced by the maximum of the two terms using L and M. The desired bound follows easily.

Theorem 4.2. *Let* u *be QS on* $(-\infty,\infty)$ *and RB on both* $[0,\infty)$ *and* $(-\infty,0]$. *Let* $\rho_1 = \rho(u_1)$ *and* $\rho_2 = \rho(u_2)$. *If*

(i) u_1 *and* u_2 *are convex, then*

$$K(f_u)+1/K(f_u) \leq \max\{2+(\frac{\log \rho_0}{\pi})^2,\ \max\{\rho_1,\rho_2\}+\frac{1+((\log \rho_0)/\pi)^2}{\max\{\rho_1,\rho_2\}}\},$$

(ii) u_1 *and* u_2 *are concave, then*

$$K(f_u)+1/K(f_u) \leq \max\{2+(\frac{\log \rho_0}{\pi})^2,\ \frac{1}{\max\{\rho_1,\rho_2\}}+(\max\{\rho_1,\rho_2\})(1+(\frac{\log \rho_0}{\pi})^2)\}$$

(iii) u_1 *convex and* u_2 *concave, then*

$$K(f_u)+1/K(f_u) \leq \max\{\rho_1+\frac{1+((\log \rho_0)/\pi)^2}{\rho_1},\ \frac{1}{\rho_2}+\rho_2(1+(\frac{\log \rho_0}{\pi})^2)\},$$

(iv) u_1 *concave and* u_2 *convex, then*

$$K(f_u)+1/K(f_u) \leq \max\{\rho_2+\frac{1+((\log \rho_0)/\pi)^2}{\rho_2},\ \frac{1}{\rho_1}+\rho_1(1+(\frac{\log \rho_0}{\pi})^2)\}.$$

Proof: The proof for each of the four cases listed above depends on Theorem 4.1 and Theorems 3.9 and 3.10. Since all the cases are similar, we will prove only (iii).

By Theorem 3.9, since u_1 is convex on $(0,\infty)$, we have $1 \leq L_1$ and $\max\{M_1, 2^{L_1}-1\} \leq \rho(u_1) \leq 2^{M_1}-1$. Hence $L_1 \leq U' \leq M_1$ implies $1 \leq U' \leq M_1 \leq \rho_1$. Similarly by Theorem 3.10, since u_2 is concave on $(0,\infty)$, we have $M_2 \leq 1$ and $\max\{1/L_2, 1/(2^{M_2}-1)\} \leq \rho(u_2) \leq 1/(2^{L_2}-1)$. Hence, $1/\rho_2 \leq L_2 \leq V' \leq 1$.

Therefore

$$1/\rho_2 \leq U' \leq \rho_1 \quad \text{and} \quad 1/\rho_2 \leq V' \leq \rho_1,$$

and we already know that $h \leq (\log \rho_0)/\pi$.

As in the proof of Corollary 4.1, we can get an upper bound for both

$$U' + \frac{1+h'}{U'} \quad \text{and} \quad V' + \frac{1+h'}{V'}$$

by taking the maximum of $x + a/x$ with $x = 1/\rho_2$ and $a = ((\log \rho_0)/\pi)^2$, or $x = \rho_1$ and $a = ((\log \rho_0)/\pi)^2$. Hence, by Theorem 4.1,

$$K(f_u)+1/K(f_u) \leq \max\{\frac{1}{\rho_2}+\rho_2(\frac{\log \rho_0}{\pi})^2, \quad \rho_1 + \frac{1+((\log \rho_0)/\pi)^2}{\rho_1}\}.$$

A special case of the radial extension and of the results we have just obtained occurs when the QS function u is odd, i.e., when $u(-x) = -u(x)$ for all $x \geq 0$. In this case $\rho_0 = 1$ so that $\log \rho_0 = 0$, whence $L_1 = L_2$ and $M_1 = M_2$. Thus all the calculations and numerical results become much simpler. In this case we will call f_u <u>the odd radial extension</u>. We now investigate this special case.

3. <u>The odd radial extension</u>.

<u>Theorem 4.3</u>. <u>Let</u> u <u>be an odd QS mapping of</u> $(-\infty,\infty)$ <u>onto itself. Then the odd radial extension</u>

$$f_u(z) = f_u(re^{i\theta}) = e^{i\theta}u(r) = e^{i\arg z}u(|z|) \quad \text{for} \quad z \in H,$$

is QC in H if and only if u is RB on $[0,\infty)$. Moreover, if f_u is QC, then

$$K(f_u) = Q = \max\{M, 1/L\}.$$

Proof: Since u is odd we have $u_1 = u_2$, which implies that $L_1 = L_2 = L$, $M_1 = M_2 = M$, $U = V$ and $h = 0$. Hence, by Theorem 4.1,

$$\hat{Q}(x) = U'(x) + \frac{1}{U'(x)}$$

whenever $U'(x)$ exists and so

$$\hat{Q} = \operatorname*{ess\,sup}_{-\infty<x<\infty} \hat{Q}(x) = \max\{M, 1/L\} + \frac{1}{\max\{M, 1/L\}} .$$

Thus

$$Q = K(f_u) = \max\{M, 1/L\}$$

and the result then follows easily from Theorem 4.1.

Corollary 4.3. If u is RB on $[0,\infty)$ then u must also be QS on $(0,\infty)$.

Remark: In Theorem 3.8 we proved this corollary and in addition found bounds on $\rho(u)$ in terms of $L(u)$ and $M(u)$. We now give a much simpler proof of the corollary making use of Theorem 4.3.

Proof of Corollary 4.3: Define a function v on $(-\infty,\infty)$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \geq 0 \\ -u(-x) & \text{if } x < 0. \end{cases}$$

Then v is clearly odd on $(-\infty,\infty)$ and RB on $[0,\infty)$ with ratio bounds $L(v) = L(u)$ and $M(v) = M(u)$. By Theorem 4.3 f_v is a QC extension of v to H. But by Theorem 1 of Ahlfors and Beurling [5, p.126], the existence of the QC extension f_v implies that v is QS on $(-\infty,\infty)$. Since $v = u$ on $(0,\infty)$ u must be QS on $(0,\infty)$.

Theorem 4.4. Let u be odd on $(-\infty,\infty)$ and RB on $[0,\infty)$. If u is either convex or concave on $(0,\infty)$ then

$$K(f_u) \leq \hat{\rho}(u) \leq \rho(u),$$

where $\hat{\rho}(u)$ is the QS dilatation of the restriction of u to $(0,\infty)$.

Proof: As we saw in the proof of Theorem 4.2, if u is odd on $(-\infty,\infty)$ and either convex or concave on $(0,\infty)$ then $\max\{M,1/L\} \leq \hat{\rho}(u)$. Furthermore it is clear that we always have $\hat{\rho}(u) \leq \rho(u)$. Therefore, from Theorem 4.3, $K(f_u) = \max\{M,1/L\}$ and the result follows trivially.

From [12] we get the following very useful theorem dealing with odd radial extensions.

Theorem 4.5. Let f be a Q-quasiconformal mapping of the upper half plane H onto itself with $f(0) = 0$ and $f(1) = 1$. Then the following statements are equivalent:

(i) f is an odd radial mapping, i.e., there is some odd QS function u defined on $(-\infty,\infty)$ for which $f = f_u$.

(ii) The complex dilatation χ of f in H satisfies $\chi(z) = e^{2i \arg z}\chi(|z|)$ a.e. in H.

(iii) f is given by the formula

$$f(z) = \exp\left(\int_1^{|z|} \frac{1+\chi(r)}{1-\chi(r)} \frac{dr}{r} + i \arg z\right) \quad \text{for} \quad z \in H,$$

with boundary values $f(0) = 0$ and $f(\infty) = \infty$, where χ, the complex dilatation, is measurable with

$$\sup_{0<r<\infty} |\chi(r)| < 1, \quad \operatorname{ess\,sup}_{0<r<\infty} |\chi(r)| \leq \frac{Q-1}{Q+1} .$$

4. Extremal radial extensions.

Having found necessary and sufficient conditions for a radial extension to be QC in the upper half plane, as well as a means for determining the QC dilatation when the extension is QC, we now seek conditions under which the radial extension is extremal. That is, we wish to know when the radial extension is as close to being conformal as possible. For this purpose the following definition is helpful.

Definition 4.2. Let u be a QS function on $(-\infty,\infty)$. Then we define

$$K_u = \inf K(f_u)$$

where the infimum is taken over all QC extensions f_u of u to the upper half plane. If f^* is any extension of u satisfying $K(f^*) = K_u$, then f^* is called an <u>extremal extension</u> of u.

On the problem of finding extremal extensions, we need a lower bound on K_u for an arbitrary u. The following theorem will give such a bound.

<u>Theorem 4.6</u>. <u>Let</u> u <u>be a QS function defined on</u> $(-\infty,\infty)$. <u>Then</u>

$$\frac{1}{K_u} \leq \min\{\overline{\lim_{x\to\infty}} \frac{\log u(x)}{\log x}, \overline{\lim_{x\to-\infty}} \frac{\log u(x)}{\log x}, \overline{\lim_{x\to 0^+}} \frac{\log u(x)}{\log x}, \overline{\lim_{x\to 0^-}} \frac{\log u(x)}{\log x}\},$$

$$K_u \geq \max\{\underline{\lim_{x\to\infty}} \frac{\log u(x)}{\log x}, \underline{\lim_{x\to-\infty}} \frac{\log u(x)}{\log x}, \underline{\lim_{x\to 0^+}} \frac{\log u(x)}{\log x}, \underline{\lim_{x\to 0^-}} \frac{\log u(x)}{\log x}\},$$

<u>where</u> $\overline{\lim}$ <u>denotes the limit superior and</u> $\underline{\lim}$ <u>denotes the limit inferior</u>.
<u>Both bounds are sharp</u>!

<u>Proof</u>: For any $b < 0$ we can consider the upper half plane H, with the points $b, 0, x, \infty$ as vertices, as a quadrilateral (cf. page 1). Thus, if f^* is any extension of u to H whatsoever, it must map the quadrilateral H

onto the quadrilateral H' consisting of the upper half plane with the vertices $u(b)$, $u(0) = 0$, $u(x)$ and $u(\infty) = \infty$. Thus [3, p.21] we must have

$$1/K(f^*) \leq \text{mod } H'/\text{mod } H \leq K(f^*).$$

Now by [5, p.130] we see that

$$\text{mod } H = P(x/-b) \quad \text{and} \quad \text{mod } H' = P(u(x)/-u(b)),$$

where P is a function such that $P(0) = 0$, $P(1) = 1$, $P(1/\lambda) = 1/P(\lambda)$, $P(\infty) = \infty$ and $P(\lambda) = 1 + \theta(\lambda)\log\lambda$ for $\lambda \geq 1$, where $\theta(\lambda)$ increases from $\theta(1) = .2284...$ to $\theta(\infty) = 1/\pi = .3183...$. Now, as we let x approach ∞, we see that

$$\overline{\lim_{x\to\infty}} \frac{\text{mod } H'}{\text{mod } H} = \overline{\lim_{x\to\infty}} \frac{P(u(x)/-u(b))}{P(x/-b)}$$

$$= \overline{\lim_{x\to\infty}} \frac{1+\theta(u(x)/-u(b))\log(u(x)/-u(b))}{1+\theta(x/-b)\log(x/-b)}$$

$$= \overline{\lim_{x\to\infty}} \frac{\log(u(x)/-u(b))}{\log(x/-b)}$$

$$= \overline{\lim_{x\to\infty}} \frac{\log u(x)}{\log x} .$$

Similarly,

$$\underline{\lim_{x\to\infty}} \frac{\text{mod } H'}{\text{mod } H} = \underline{\lim_{x\to\infty}} \frac{\log u(x)}{\log x} .$$

Thus

$$\frac{1}{K(f^*)} \le \overline{\lim_{x\to\infty}} \frac{\log u(x)}{\log x} \le K(f^*),$$

(4.11)

$$\frac{1}{K(f^*)} \le \underline{\lim_{x\to\infty}} \frac{\log u(x)}{\log x} \le K(f^*).$$

On the other hand, since $P(1/\lambda) = 1/P(\lambda)$, we have as x approaches 0^+

$$\overline{\lim_{x\to 0^+}} \frac{\text{mod } H}{\text{mod } H'} = \overline{\lim_{x\to 0^+}} \frac{P(x/-b)}{P(u(x)/-u(b))}$$

$$= \overline{\lim_{x\to 0^+}} \frac{P(-u(b)/u(x))}{P(-b/x)}$$

$$= \overline{\lim_{x\to 0^+}} \frac{1+\theta(-u(b)/u(x))\log(-u(b)/u(x))}{1+\theta(-b/x)\log(-b/x)}$$

$$= \overline{\lim_{x\to 0^+}} \frac{\log u(x)}{\log x}.$$

Similarly,

$$\underline{\lim_{x\to 0^+}} \frac{\text{mod } H}{\text{mod } H'} = \underline{\lim_{x\to 0^+}} \frac{\log u(x)}{\log x}.$$

Thus

$$\frac{1}{K(f^*)} \le \overline{\lim_{x\to 0^+}} \frac{\log u(x)}{\log x} \le K(f^*),$$

(4.12)

$$\frac{1}{K(f^*)} \le \underline{\lim_{x\to 0^+}} \frac{\log u(x)}{\log x} \le K(f^*).$$

Of course we could just as easily have used $b > 0$ and $x < 0$. In this case we would have ended up with $x \to -\infty$ and $x \to 0^-$ instead of $x \to \infty$ and $x \to 0^+$ in (4.11) and (4.12), respectively. This completes the proof. As we shall soon see, equality holds when $u(x) = (\text{sign } x)|x|^\alpha$, $\alpha > 0$. Equality holds in the lower bound for $0<\alpha\leq 1$; in the upper bound for $\alpha \geq 1$.

Corollary 4.6. Under the same hypotheses as for Theorem 4.6,

$$\frac{1}{K_u} \leq \min\{ \lim_{x\to 0^+} \frac{xu'(x)}{u(X)}, \lim_{x\to 0^-} \frac{xu'(x)}{u(x)}, \lim_{x\to\infty} \frac{xu'(x)}{u(x)}, \lim_{x\to-\infty} \frac{xu'(x)}{u(x)}\},$$

$$K_u \geq \max\{ \lim_{x\to 0^+} \frac{xu'(x)}{u(x)}, \lim_{x\to 0^-} \frac{xu'(x)}{u(x)}, \lim_{x\to\infty} \frac{xu'(x)}{u(x)}, \lim_{x\to-\infty} \frac{xu'(x)}{u(x)}\},$$

whenever any of these limits exists. Both bounds are sharp!

Proof: Since all the cases are similar we will treat only $x \to \infty$. Assume

$$\lim_{x\to\infty} xu'(x)/u(x) = A$$

exists, where A can be finite or infinite. This implies that $xu'(x)/u(x)$ is actually defined on some interval (a,∞). But we know that x and $u(x)$ exist on (a,∞). Thus u' must also exist on (a,∞). That is, u is differentiable on (a,∞). Thus we can apply l'Hôpital's rule to the ratio

$\log u(x)/\log x$ to obtain

$$\lim_{x\to\infty} \frac{\log u(x)}{\log x} = \lim_{x\to\infty} \frac{u'(x)/u(x)}{1/x} = \lim_{x\to\infty} \frac{xu'(x)}{u(x)} .$$

Equality again holds for $u(x) = (\text{sign } x)|x|^{\alpha}$, $\alpha > 0$.

Theorem 4.7. Let u be QS on $(-\infty,\infty)$ and RB on both $(-\infty,0]$ and $[0,\infty)$. Let

$$B = \lim_{x\to 0^+} \frac{xu'(x)}{u(x)}, \qquad C = \lim_{x\to 0^-} \frac{xu'(x)}{u(x)},$$

$$D = \lim_{x\to\infty} \frac{xu'(x)}{u(x)}, \qquad E = \lim_{x\to -\infty} \frac{xu'(x)}{u(x)},$$

and put

$$A = \max\{B, \frac{1}{B}, C, \frac{1}{C}, D, \frac{1}{D}, E, \frac{1}{E}\}.$$

If $A \geq Q$, where Q is defined as in Theorem 4.1, then the radial extension f_u is extremal for u, and hence $K_u = K(f_u) = Q$.

Proof: By Corollary 4.6 we must have $K_u \geq A$. Thus $K(f_u) \geq A$. If $K(f_u) = Q \leq A$, however, then by the definition of K_u we must have $K(f_u) = K_u$. Hence the radial extension f_u is extremal.

Theorem 4.8. Let u be QS on $(-\infty,\infty)$, odd, and RB on $[0,\infty)$ with ratio bounds L and M. Let $Q = \max\{M, 1/L\}$. If any one of the four conditions

(i) $\lim_{x\to 0^+} \frac{xu'(x)}{u(x)} = Q,$

(ii) $\lim_{x\to 0^+} \frac{xu'(x)}{u(x)} = \frac{1}{Q},$

(iii) $\lim_{x\to\infty} \frac{xu'(x)}{u(x)} = Q,$

(iv) $\lim_{x\to\infty} \frac{xu'(x)}{u(x)} = \frac{1}{Q},$

holds, then the radial extension f_u is extremal.

Proof: The proof follows immediately from Theorem 4.7 since any one of the four conditions leads to $A \geq Q$.

Corollary 4.8. Let u be QS on $(-\infty,\infty)$ and odd. Then u satisfies the hypotheses of Theorem 4.8 if and only if u is of the form

(4.13) $$u(x) = \exp\left(\int_1^{|x|} \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s} + i \arg x\right), \quad x \neq 0, \; u(0) = 0,$$

where χ is real-valued and measurable on $(0,\infty)$,

$$\sup_{0<s<\infty} |\chi(s)| < 1, \quad \operatorname{ess\,sup}_{0<s<\infty} |\chi(s)| = \frac{Q-1}{Q+1} < 1,$$

and either

(i) $\lim_{s\to 0^+} |\chi(s)| = \frac{Q-1}{Q+1}$ or (ii) $\lim_{s\to\infty} |\chi(s)| = \frac{Q-1}{Q+1}$.

Proof: We know from Theorem 3.1 that u is RB on $[0,\infty)$ if and only if it has the form (3.3). But (3.3) and (4.13) are identical on $[0,\infty)$. The term $i \arg x$ in (4.13) just extends the function u symmetrically about the origin. Since

$$\frac{1+\chi(s)}{1-\chi(s)} = \frac{su'(s)}{u(s)}$$

for such an RB function we see that (i) of Corollary 4.8 corresponds to either (i) or (ii) of Theorem 4.8, while (ii) of Corollary 4.8 corresponds to either (iii) or (iv) of Theorem 4.8.

<u>Example 4.1</u>: Let $u(x) = (\text{sign } x) \sum_{i=1}^{n} a_i x^{c_i}$ where $a_i \geq 0$, $c_i > 0$, the a_i and c_i not necessarily integers, and $\sum_{i=1}^{n} a_i = 1$. We obviously may assume for simplicity that $c_1 < c_2 < \ldots < c_n$. Then it is easy to see that u is odd, $u(0) = 0$, $u(1) = 1$, and u is continuous and monotonic. Now for any $x > 0$,

$$\frac{xu'(x)}{u(x)} = \frac{\sum_1^n a_i c_i x^{c_i}}{\sum_1^n a_i x^{c_i}} .$$

Thus, since

$$c_1 \sum_1^n a_i x^{c_i} \leq \sum_1^n c_i a_i x^{c_i} \leq c_n \sum_1^n a_i x^{c_i} ,$$

we easily see that

$$L = c_1 \leq \frac{xu'(x)}{u(x)} \leq c_n = M.$$

Thus u is RB on $[0,\infty)$. Also

$$\lim_{x\to\infty} \frac{xu'(x)}{u(x)} = c_n, \quad \lim_{x\to 0^+} \frac{xu'(x)}{u(x)} = c_1,$$

so that one of the conditions of Theorem 4.8 must be satisfied -- condition (iii) if $c_n \geq 1/c_1$, condition (ii) if $c_n \leq 1/c_1$. Hence the radial extension is extremal.

Theorem 4.9. Let u satisfy the hypotheses of Theorem 4.7 so that the radial extension is extremal, and let $\hat{Q}(x)$, $\hat{Q}$ and Q be defined as in Theorem 4.1. If there is some $\epsilon > 0$ and some interval $(c,d) \subset (-\infty,\infty)$ on which $\hat{Q}(x) \leq \hat{Q} - \epsilon$ then the radial extension f_u is not unique extremal.

Proof: As in the proof of Theorem 4.1, if we use the auxiliary conformal mappings e^z and $\log z$, then we need only consider functions of Ω onto Ω' where

$$\Omega = \{x+iy \mid 0<y<\pi\} \quad \text{and} \quad \Omega' = \{S+iT \mid 0<T<\pi\} .$$

The boundary conditions then become

$$(4.14) \quad x + 0i \longrightarrow U(x) + 0i, \quad x + \pi i \longrightarrow V(x) + \pi i,$$

for $-\infty<x<\infty$ and $0<y<\pi$. Furthermore the radial extension f_u then corresponds to the function $\hat{f}_u : \Omega \longrightarrow \Omega'$ given by

$$\hat{f}_u : x + iy \longrightarrow \left(\frac{\pi - y}{\pi} U(x) + \frac{y}{\pi} V(x)\right) + iy.$$

Let $W(x) = (U(x) + V(x))/2 + \sigma(x)$ where $\sigma \geq 0$ is a function to be defined later in the proof, and let $g:\Omega \longrightarrow \Omega'$ be defined as

$$g(x+iy) = \begin{cases} \left[\frac{\pi-2y}{\pi} U(x) + \frac{2y}{\pi} W(x)\right] + iy & \text{if } 0<y<\pi/2 \\ \left[\frac{2\pi-2y}{\pi} W(x) + \frac{2y-\pi}{\pi} V(x)\right] + iy & \text{if } \pi/2\leq y<\pi. \end{cases}$$

Obviously g satisfies (4.14). Furthermore it is clear that the proof of Theorem 4.1 can be used here if the cases $0<y\leq\pi/2$ and $\pi/2\leq y<\pi$ are treated separately. Hence g is QC.

In order to calculate $K(g)$ we will use the fact that $K(g) = \operatorname{ess\,sup}_{z \in \Omega} D(z)$, where D is the point dilatation of g on Ω [3, p.24]. Moreover, since the cases $0<y\leq\pi/2$ and $\pi/2\leq y<\pi$ are similar, we will treat only $0<y\leq\pi/2$. Let $x_0 \in (c,d)$ be a point at which U and V are differentiable, and consider $D(x_0,y) + 1/D(x_0,y)$ for $0<y<\pi/2$. As was shown in the proof of Theorem 4.1, $D + 1/D$ must attain its maximum at one of the endpoints $y = 0$, $y = \pi/2$. But

$$D(x_0,0) + \frac{1}{D(x_0,0)} = U'(x_0) + \frac{1+(h(x_0)+2\sigma(x_0))^2}{U'(x_0)}$$

$$= U'(x_0) + \frac{1+h(x_0)^2}{U'(x_0)} + \frac{4h(x_0)\sigma(x_0)+4\sigma(x_0)^2}{U'(x_0)}$$

$$\leq \hat{Q} - \epsilon + \frac{4h(x_0)\sigma(x_0)+4\sigma(x_0)^2}{U'(x_0)}$$

and

$$D(x_0,\pi/2) + \frac{1}{D(x_0,\pi/2)} = \left[\frac{U'(x_0)+V'(x_0)}{2}+\pi\sigma'(x_0)\right]$$

$$+ \frac{1+(h(x_0)+2\sigma(x_0))^2}{(U'(x_0)+V'(x_0))+\pi\sigma'(x_0)} .$$

Hence, if we choose σ to be positive but sufficiently small on (c,d), and with a sufficiently small derivative a.e. on (c,d), then

$$D(x_0,y) + \frac{1}{D(x_0,y)} \leq \max\{D(x_0,0)+\frac{1}{D(x_0,0)},\ D(x_0,\pi/2)+\frac{1}{D(x_0,\pi/2)}\} \leq \hat{Q}$$

for all y with $0<y\leq\pi/2$. A similar result holds for $\pi/2\leq y<\pi$. Choose σ in this way on (c,d) and let $\sigma = 0$ everywhere else. Then $g(x,y) = \hat{f}_u(x,y)$ for all (x,y) with $x \notin (c,d)$. Hence

$$D + \frac{1}{D} \leq \hat{Q} \quad \text{a.e. in } \Omega,$$

and so

$$K(g) \leq Q = K(\hat{f}_u) = K_u.$$

Hence g is also extremal. But $\sigma > 0$ on (c,d) implies that $g \neq f_u$ on $(c,d)\times(0,\pi)$. That is, $\hat{f}_u$ is not unique extremal.

The statement of this theorem is much simpler in the case where u is an odd function. In fact the result becomes:

Theorem 4.10. Let u satisfy the hypotheses of Theorem 4.8, so that the odd radial extension f_u is extremal and $Q = \max\{M, 1/L\}$. If there is some $\varepsilon > 0$ and some interval $(a,b) \subset (0,\infty)$ on which

$$\frac{1}{Q-\varepsilon} \leq \frac{xu'(x)}{u(x)} \leq Q-\varepsilon, \tag{4.15}$$

then the odd radial extension is not unique extremal.

Proof: Suppose $Q = 1$. Then $M = L = 1$ as well and thus $Q = 1 = xu'(x)/u(x)$ a.e. on $(0,\infty)$. That is, (4.15) could not hold on the interval (a,b). Hence we can assume that $Q > 1$. Pick ε in (4.15) small enough that $Q - \varepsilon > 1$. Since the expression $x + 1/x$ is increasing for $x > 1$, $1<Q-\varepsilon<Q$ implies

$$\left(Q + \frac{1}{Q}\right) - \left(Q-\varepsilon + \frac{1}{Q-\varepsilon}\right) = \hat{\varepsilon} > 0.$$

Since u is odd we must have $h = 0$ (Cf. (v), page 62) and hence $\hat{Q}(x) = U'(x) + 1/U'(x)$. Furthermore, by (4.15),

$$\frac{1}{Q-\varepsilon} \leq \frac{xu'(x)}{u(x)} = U'(x) \leq Q-\varepsilon,$$

so by the monotonicity of $x + 1/x$ we must also have

$$U'(x) + \frac{1}{U'(x)} \leq Q-\varepsilon + \frac{1}{Q-\varepsilon} .$$

Finally, by definition of $\hat{Q}$ and Q, $\hat{Q} = Q + 1/Q$. Putting all this together we find

$$\hat{Q}(x) = U'(x) + \frac{1}{U'(x)} \leq Q-\varepsilon + \frac{1}{Q-\varepsilon} < Q + \frac{1}{Q} - \hat{\varepsilon} = \hat{Q} - \hat{\varepsilon}$$

for all $x \in (a,b)$. The result follows immediately from Theorem 4.9.

Corollary 4.10. The condition for non-uniqueness in Theorem 4.10 is satisfied if there is any point $x_0 \in (0,\infty)$ at which u' is continuous and

$$\frac{x_0 u'(x_0)}{u(x_0)} \neq Q \quad \text{or} \quad \frac{1}{Q}\,.$$

Proof: Since u' is continuous at x_0, and $x_0 > 0$, also $xu'(x)/u(x)$ is continuous at x_0. If $xu'(x)/u(x)$ is either $> Q$ or $< 1/Q$ for $x = x_0$, then the same must be true in some small interval containing x_0. But this is impossible, since $Q = \max\{M, 1/L\}$, where L and M are the ratio bounds of u. Thus we must have

$$\frac{1}{Q} < \frac{x_0 u'(x_0)}{u(x_0)} < Q.$$

We can therefore choose $t > 0$ so small that

$$\frac{1}{Q} < \frac{xu'(x)}{u(x)} < Q$$

on the entire interval $[x_0-t, x_0+t]$. By continuity the function $xu'(x)/u(x)$ must attain its maximum and minimum on this compact set. Hence there is some $\varepsilon > 0$ for which

$$\frac{1}{Q-\varepsilon} \leq \frac{xu'(x)}{u(x)} \leq Q - \varepsilon$$

on $(x_0-t/2, x_0+t/2) \subset [x_0-t, x_0+t]$. We can now appeal to Theorem 4.10.

Example 4.2: Consider again the function

$$u(x) = (\text{sign } x) \sum_1^n a_i x^{c_i}$$

introduced in Example 4.1 on page 83. If $n > 1$ then the radial extension is not unique extremal for this u. The function u is obviously odd, and if $n > 1$ then

$$L = c_1 = \frac{c_1 \sum_1^n a_i}{\sum_1^n a_i} < \frac{1 \cdot u'(1)}{u(1)} = \frac{\sum_1^n a_i c_i}{\sum_1^n a_i} < \frac{c_n \sum_1^n a_i}{\sum_1^n a_i} = c_n = M,$$

so that

$$\frac{1}{Q} < \frac{1\, u'(1)}{u(1)} < Q,$$

where $Q = \max\{c_n, 1/c_1\}$. Also u' is clearly continuous at $x = 1$, since $u(x)$ is a linear combination of powers of x. Therefore the hypotheses of Corollary 4.10 are fulfilled by u, and we conclude that the radial extension is not unique extremal.

Theorem 4.11. Let u be a QS function of $[-1,1]$ onto itself and suppose there are numbers L_1, M_1, L_2, M_2

such that

$$0 < L_1 \leq xu'(x)/u(x) \leq M_1 < \infty \quad \text{a.e. on} \quad (0,1),$$

$$0 < L_2 \leq xu'(x)/u(x) \leq M_2 < \infty \quad \text{a.e. on} \quad (-1,0).$$

Let $C = \lim_{x \to 0^-} \frac{xu'(x)}{u(x)}$ *and* $D = \lim_{x \to 0^+} \frac{xu'(x)}{u(x)}$, *and put*

$$A = \max\{C, \frac{1}{C}, D, \frac{1}{D}\}.$$

If $A + 1/A \geq \operatorname{ess\,sup}_{-\infty<x<0} \hat{Q}(x)$, *where* $\hat{Q}(x)$ *is defined as in Theorem* 4.1, *then the radial extension of* u *to* $G = \{z \mid |z| < 1, \text{ Im } z > 0\}$, (i.e. the restriction of f_u in (4.1) to $0<r<1$), *is extremal for the boundary values*

$$f(z) = \begin{cases} u(z), & \text{for } z = x \in [-1,1] \\ z, & \text{for } |z| = 1, \text{ Im } z \geq 0. \end{cases}$$

Proof: Let g be any other QC map of G onto itself with boundary values f. Let f_u denote the radial extension of u to G, and let $K(f_u)$ and $K(g)$ denote the QC dilatations of f_u and g, respectively. By extending both f_u and g by reflection in the semicircle $|z| = 1$, $\text{Im } z \geq 0$ we obtain QC mappings $\hat{f}_u$ and $\hat{g}$, respectively, of the upper half plane onto itself, with dilatations $K(\hat{f}_u) = K(f_u)$, $K(\hat{g}) = K(g)$. But it is easy to see that if f_u satisfies the hypotheses of this theorem on $[-1,1]$, then the reflection $\hat{f}_u$ satisfies the hypotheses of Theorem 4.7 on $(-\infty,\infty)$. Therefore $K(\hat{f}_u)$ is minimal for the boundary values of $\hat{f}_u$ on $(-\infty,\infty)$. But $g = f_u$ on $[-1,1]$ implies $\hat{g} = \hat{f}_u$ on $(-\infty,\infty)$; so we must have $K(g) = K(\hat{g}) \geq K(\hat{f}_u) = K(f_u)$. Hence f_u is extremal as a map of G onto itself.

Corollary 4.11. Let u satisfy the hypotheses of Theorem 4.11, so that the radial extension is extremal. If there is some $\varepsilon > 0$ and some interval $(c,d) \subset (-\infty,0)$ on which $\hat{Q}(x) < Q_0-\varepsilon$, where $Q_0 = \operatorname{ess\,sup}_{-\infty<x<0} \hat{Q}(x)$, then the radial extension is not unique extremal.

Proof: The proof is basically the same as the proof of Theorem 4.9.

Theorem 4.12. Let u be an odd QS function of $[-1,1]$ onto itself and suppose there are numbers L and M such that

$$0 < L \leq \frac{xu'(x)}{u(x)} \leq M < \infty \quad \text{a.e. on} \quad (0,1).$$

Let $Q_0 = \max\{M, 1/L\}$. If either of the two conditions

$$\text{(i)} \quad \lim_{x\to 0^+} \frac{xu'(x)}{u(x)} = Q_0,$$

$$\text{(ii)} \quad \lim_{x\to 0^+} \frac{xu'(x)}{u(x)} = \frac{1}{Q_0}$$

holds, then the radial extension of u to a map of G, the upper half of the unit disk, onto itself, is extremal for the boundary values it assumes.

Proof: The proof follows immediately from Theorem 4.11.

Corollary 4.12. Let u satisfy the hypotheses of Theorem 4.12, so that the radial extension is extremal. If there is some $\epsilon > 0$, and some interval $(a,b) \subset (0,1)$ on which

$$\frac{1}{Q_0-\epsilon} \leq \frac{xu'(x)}{u(x)} \leq Q_0-\epsilon,$$

then the radial extension is not unique extremal.

Proof: This theorem follows immediately from Corollary 4.11.

Theorem 4.13. Let u be an odd QS function of $[-1,1]$ onto itself. Then u satisfies the hypotheses of Theorem 4.12 if and only if u is of the form

$$u(x) = \exp\left(\int_1^{|x|} \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s} + i \arg x\right),$$

$$0 \neq x \in [-1,1], \quad u(0) = 0,$$

where χ is real-valued and measurable on $(0,1)$,

$$\sup_{0<s<1} |\chi(s)| < 1, \quad \operatorname{ess\,sup}_{0<s<1} |\chi(s)| = \frac{Q_0-1}{Q_0+1} < 1,$$

and

$$\lim_{s\to 0^+} |\chi(s)| = \frac{Q_0-1}{Q_0+1}.$$

Proof: Except for slight modifications, the proof is the same as that for Corollary 4.8.

5. An application of the extremal radial extension.

It is well known [11, p.411] that in the class of all QC mappings of the upper half plane onto itself satisfying a finite number of boundary conditions $w(z_i) = w_i$, $i = 1,\dots,n$, the extremal map has constant point dilatation $D(z) = K$ in the upper half plane. It is also well known, and quite evident from Theorem 4.7 and Theorem 4.8, that when the entire boundary correspondence u is specified, then an extremal mapping need no longer have constant point dilatation. As a side result we will use Theorem 4.8 to show that the extremal mapping need not have constant point dilatation even when a countably infinite number of boundary conditions is specified.

Theorem 4.14. *There exists a mapping w defined on the non-negative integers, and an extremal QC extension f^* of w to the upper half plane, such that f^* has non-constant point dilatation.*

Proof: Let $w(n) = n+n^2$ for $n = 0,1,2,\dots$ and let $u(x) = (\text{sign } x)(|x| + |x|^2)$ for all $x \in (-\infty,\infty)$. Clearly

$$1 \leq \frac{xu'(x)}{u(x)} = \frac{x+2x^2}{x+x^2} = \frac{1+2x}{1+x} \leq 2 \quad \text{for} \quad x \in (0,\infty),$$

and

$$\lim_{x\to\infty} \frac{xu'(x)}{u(x)} = 2.$$

Hence condition (iii) of Theorem 4.8 is satisfied and so the radial extension f_u of u is extremal with $K(f_u) = 2$. But $f_u(n) = u(n) = w(n)$ for $n = 1,2,\ldots,$ so that f_u is also an extension of w. Moreover, if g is any QC extension of w then

$$\frac{\log g(n)}{\log n} = \frac{\log (n+n^2)}{\log n} = \frac{\log n^2 + \log (1+1/n)}{\log n} = 2 + \frac{\log (1+1/n)}{\log n}$$

shows that $\lim_{n\to\infty} \frac{\log g(n)}{\log n} = 2$. Hence, by Theorem 4.6, $K(g) \geq 2$. That is, $K(f_u) \leq K(g)$ for all QC extensions g of w. Thus letting $f^* = f_u$ we see that f^* is an extremal extension of w and f^* has non-constant point dilatation since $D(re^{i\theta}) = (1+2r)/(1+r)$.

6. General radial mappings.

Looking back at Theorem 4.5, we see that parts (ii) and (iii) give a characterization of a general odd radial extension--part (ii) by the explicit formula for the complex dilatation and part (iii) by the explicit form of the function itself. Notice, however, that nowhere in Theorem 4.5 is the function χ assumed to be real-valued.

In most of our applications of the radial mapping we have taken χ to be real-valued so that the half-lines $(-\infty,0]$ and $[0,\infty)$ would be mapped by the radial mapping f onto themselves. This was necessary if we wanted a map of either the upper half plane onto itself or the upper half of the unit disk onto itself. Theorem 4.5 continues to hold, however, as we see in [12], even if χ is complex-valued. We can therefore still talk about the radial mapping in this case, although it is easy to see from (iii) of Theorem 4.5 that if χ is not real-valued in $[0,\infty)$ then f will not map $[0,\infty)$ onto itself. The most important property of the radial mapping continues to hold even if χ is complex-valued, however. This is property (iii) of Theorem 4.5,

$$\chi(z) = e^{2\, i\arg z}\chi(|z|),$$

so that

$$|\chi(z)| = \chi(|z|).$$

Property (iii) shows that we can find the QC dilatation $K(f)$ of the radial mapping just by investigating the point dilatation $D(z)$ on the ray $\arg z = 0$, and taking the essential supremum of these values. This certainly saves a lot of work and is what enables us to find conditions under which the radial mapping is extremal for the boundary homeomorphism it induces.

Since we are no longer considering the radial map as the extension of a given u on $(-\infty,\infty)$ to the upper half plane, we can no longer denote it by f_u. Henceforth we will consider a map to be radial in the general sense if it satisfies

parts (ii) or (iii) of Theorem 4.5, in some domain Ω with $(0,0) \in \Omega$, but not necessarily satisfies part (i). We will denote such a general radial mapping by $f^*(z)$ or f^*.
We are now ready to continue our study of extremal radial maps.

Up to now, as can be seen in part (ii) of Theorem 4.5, we have been restricting our investigation of extremal radial maps to those having complex dilatation $X(z) = \chi(re^{i\theta})$ of the form

$$(4.16) \qquad X(re^{i\theta}) = e^{2i\theta}X(r),$$

where X is real-valued and measurable on its domain of definition. We now generalize this somewhat by considering instead complex dilatations of the form

$$(4.17) \qquad X(re^{i\theta}) = e^{2i\theta}\frac{\alpha(r)-1+i\beta(r)}{\alpha(r)+1+i\beta(r)}, \quad 0<r\leq 1, \quad 0\leq\theta<2\pi,$$

with α and β real-valued and measurable on their domains of definition. If $\beta(r) = 0$ identically then (4.17) reduces to (4.16) as a special case with $\chi = (\alpha-1)/(\alpha+1)$. The following lemma will prove quite useful.

<u>Lemma 4.3</u>. <u>Let</u> f <u>be a QC mapping of the open unit disk</u> Δ <u>onto itself with</u> $f(0) = 0$. <u>Then</u>

$$K(f) \geq \max\{\overline{\lim_{r\to 0^+}}\frac{\log|f(r)|}{\log(r/16)}, \quad \overline{\lim_{r\to 0^+}}\frac{\log r}{\log(|f(r)|/16)}\}.$$

<u>Proof</u>: It is well known [13] that for any function f satisfying the hypotheses of Lemma 4.3,

$$|f(z_1)-f(z_2)| \leq 16|z_1-z_2| \tag{4.18}$$

for all $z_1, z_2 \in \Delta$, where $K = K(f)$ is the QC dilatation. Taking $z_1 = r \in (0,1]$ and $z_2 = 0$ in (4.18) gives $|f(r)| \leq 16r^{1/K}$ and solving for K we find that $K \geq \log r/\log(|f(r)|/16)$, which implies $K \geq \overline{\lim_{r\to 0^+}} \log r/\log(|f(r)|/16)$. Now f^{-1} also satisfies the hypotheses of Lemma 4.3. Hence we can use f^{-1} in (4.18) with $z_1 = r \in (0,1]$ and $z_2 = 0$. Again solving for K gives $K \geq \log|f(r)|/\log(r/16)$, which implies $K \geq \overline{\lim_{r\to 0^+}} \log|f(r)|/\log(r/16)$. This proves the lemma.

<u>Theorem 4.15</u>. <u>Let</u>

$$\chi(re^{i\theta}) = e^{2i\theta}\,\frac{\alpha(r)-1+i\beta(r)}{\alpha(r)+1+i\beta(r)} \quad \underline{\text{for}} \quad 0<r\leq 1, \quad 0\leq\theta<2\pi,$$

<u>where</u> $\alpha(r)$ <u>and</u> $\beta(r)$ <u>are real-valued and measurable on</u> $(0,1]$, <u>and satisfy the following conditions</u>:

(i) $0<A\leq\alpha(r)\leq B<\infty$, $|\beta(r)|\leq C<\infty$ <u>for some</u> A, B, C.

(ii) α <u>is continuous in some right-hand neighborhood</u> $(0,\lambda)$ <u>of the origin</u>.

(iii) $\lim_{r\to 0^+} \alpha(r) = \alpha_0 \geq A>0$ <u>exists</u>.

Let f^* be the radial mapping of Δ onto itself that leaves 0 and 1 fixed and has χ as its complex dilatation. If

(4.19) $$\operatorname*{ess\,sup}_{0<r\leq 1} |\chi(r)| \leq \left|\frac{\alpha_0 - 1}{\alpha_0 + 1}\right|,$$

then f^* is extremal in the class of all QC maps of Δ' onto $f^*(\Delta')$ that agree with f^* on the boundary. Here Δ' denotes the open unit disk minus the closed interval $[0,1]$ so that $\partial\Delta' = \{z \mid |z| = 1\} \cup \{z = x+iy \mid 0\leq x\leq 1,\ y = 0\}$.

Proof: Let g be any QC map of Δ' onto $f^*(\Delta')$ that agrees with f^* on the boundary of Δ'. Then, by (iii) of Theorem 4.5,

$$g(r) = f^*(r) = \exp\left(\int_1^r \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}\right) = \exp\left(-\int_r^1 (\alpha(s)+i\beta(s))\frac{ds}{s}\right)$$

for $r \in (0,1]$.

Hence

$$|g(r)| = \exp\left(-\int_r^1 \alpha(s) \frac{ds}{s}\right).$$

But by (ii) and (iii) of Theorem 4.15 we can use l'Hôpital's rule to find

$$\lim_{r\to 0^+} \frac{\log |g(r)|}{\log (r/16)} = \lim_{r\to 0^+} \frac{-\int_r^1 \alpha(s) \frac{ds}{s}}{\log (r/16)}$$

$$= \lim_{r\to 0^+} \frac{\alpha(r)/r}{1/r} = \lim_{r\to 0^+} \alpha(r)$$

$$= \alpha_0 ,$$

and

$$\lim_{r\to 0^+} \frac{\log r}{\log (|g(r)|/16)} = \lim_{r\to 0^+} \frac{\log r}{-\int_r^1 \alpha(s)\,\frac{ds}{s} - \log 16}$$

$$= \lim_{r\to 0^+} \frac{1/r}{\alpha(r)/r} = \lim_{r\to 0^+} \frac{1}{\alpha(r)}$$

$$= \frac{1}{\alpha_0}\,.$$

Hence, by Lemma 4.3 and (4.19),

$$K(g) \ge \max\{\alpha_0, 1/\alpha_0\} = \frac{1+|(\alpha_0-1)/(\alpha_0+1)|}{1-|(\alpha_0-1)/(\alpha_0+1)|} \ge \frac{1+\operatorname{ess\,sup}_{0<r\le 1} |\chi(r)|}{1-\operatorname{ess\,sup}_{0<r\le 1} |\chi(r)|} = K(f^*).$$

That is, f^* is extremal.

<u>Corollary 4.15</u>. <u>Let</u>

$\chi(re^{i\theta}) = \chi(r)e^{2i\theta}$ <u>for</u> $0<r\le 1,\quad 0\le\theta<2\pi,$

<u>where</u>

(i) $\chi(r)$ <u>is real-valued and measurable on</u> $(0,1]$.

(ii) $\sup_{0<r\le 1} |\chi(r)| < 1,\quad \operatorname{ess\,sup}_{0<r\le 1} |\chi(r)| = M < 1,$

(iii) $\chi(r)$ <u>is continuous in some right-hand neighborhood</u> $(0,\lambda)$ <u>of the origin</u>,

(iv) $\lim_{r\to 0^+} \chi(r)$ <u>exists</u>.

<u>Let</u> f^* <u>be the radial map of</u> Δ <u>onto itself with</u> 0 <u>and</u> 1 <u>fixed and with</u> χ <u>as its complex dilatation. If</u>

$\lim_{r\to 0^+} |\chi(r)| = M$ *then* f^* *is extremal in the class of all QC maps of* Δ' *onto* $f^*(\Delta')$ *that agree with* f^* *on the boundary*.

Proof: The proof follows immediately from Theorem 4.15 if we take $\beta(r) = 0$ and $\alpha(r) = (1+\chi(r))/(1-\chi(r))$.

We shall need the following result of T. Sasaki [16].

Theorem of Sasaki. Let f be a QC map of the open unit disk onto itself with $f(0) = 0$ and $f(1) = 1$. Then

$$\overline{\lim_{r\to 0^+}} \left|\frac{\arg f(r)}{\log r}\right| \leq \frac{1}{2}\left[K(f) - \frac{1}{K(f)}\right].$$

Theorem 4.16. *Let*

$$\chi(re^{i\theta}) = e^{2i\theta}\,\frac{\alpha(r)-1+i\beta(r)}{\alpha(r)+1+i\beta(r)} \quad \text{for} \quad 0<r\leq 1, \quad 0\leq\theta<2\pi,$$

where $\alpha(r)$ *and* $\beta(r)$ *are real-valued and measurable on* $(0,1]$ *and satisfy the following conditions*:

(i) $0<A\leq\alpha(r)\leq B<\infty$, $0<D\leq\beta(r)\leq C<\infty$ *for some* A, B, C, D,

(ii) $\beta(r)$ *is continuous in some right-hand neighborhood* $(0,\lambda)$ *of the origin*,

(iii) $\lim_{r\to 0^+} \beta(r) = \beta_0 \geq D > 0$ *exists*.

Let f^* *be the radial mapping of* Δ *onto itself that leaves* 0 *and* 1 *fixed and has* χ *as its complex dilatation. If*

(4.20) $$\operatorname*{ess\,sup}_{0<r\leq 1} \frac{\sqrt{(\alpha(r)-1)^2+\beta(r)^2}\ \sqrt{(\alpha(r)+1)^2+\beta(r)^2}}{2\alpha(r)} \leq \beta_0$$

<u>then</u> f^* <u>is extremal in the class of all QC maps of</u> Δ' <u>to</u> $f^*(\Delta')$ <u>that agree with</u> f^* <u>on the boundary</u>.

<u>Proof</u>: Let g be any QC map of Δ' onto $f^*(\Delta')$ that agrees with f^* on the boundary of Δ'. Then, by (iii) of Theorem 4.5,

$$g(r) = f^*(r) = \exp\left(\int_1^r \frac{1+\chi(s)}{1-\chi(s)}\frac{ds}{s}\right) = \exp\left(\int_1^r \frac{\alpha(s)}{s}ds + i\int_1^r \frac{\beta(s)}{s}ds\right)$$

for $r \in (0,1]$,

so that $\arg g(r) = \int_1^r \frac{\beta(s)}{s}ds$. But by (ii) and (iii) of Theorem 4.16 we can use l'Hôpital's rule to evaluate

$$\lim_{r\to 0^+} \left|\frac{\arg g(r)}{\log r}\right| = \lim_{r\to 0^+} \frac{\int_1^r \frac{\beta(s)}{s}ds}{\log r}$$

$$= \lim_{r\to 0^+} \frac{\beta(r)/r}{1/r} = \lim_{r\to 0^+} \beta(r)$$

$$= \beta_0 .$$

Hence, using the Theorem of Sasaki and (4.20), we find that

$$(4.21)\quad \frac{1}{2}[K(g)-\frac{1}{K(g)}] \geq \beta_0 \geq \operatorname*{ess\,sup}_{0<r\leq 1} \frac{\sqrt{(\alpha(r)-1)^2+\beta(r)^2}\sqrt{(\alpha(r)+1)^2+\beta(r)^2}}{2\alpha(r)}$$

$$= \frac{1}{2}\left[\frac{1+\operatorname*{ess\,sup}_{0<r\leq 1}|\chi(r)|}{1-\operatorname*{ess\,sup}_{0<r\leq 1}|\chi(r)|} - \frac{1-\operatorname*{ess\,sup}_{0<r\leq 1}|\chi(r)|}{1+\operatorname*{ess\,sup}_{0<r\leq 1}|\chi(r)|}\right]$$

$$= \frac{1}{2}\left[K(f^*) - \frac{1}{K(f^*)}\right].$$

Since $x - 1/x$ is a non-decreasing function of x if $x > 0$, (4.21) implies that $K(g) \geq K(f^*)$. Hence f^* is extremal.

<u>Corollary 4.16</u>. <u>Let</u>

$$\chi(re^{i\theta}) = e^{2i\theta}\frac{\alpha(r)-1+i\beta(r)}{\alpha(r)+1+i\beta(r)} \quad \text{for } 0<r\leq 1, \quad 0\leq\theta<2\pi,$$

<u>with</u> $\alpha(r) = \sqrt{\beta(r)^2+1}$, and <u>such that</u>

(i) $\beta(r)$ <u>is real-valued and measurable on</u> $(0,1]$,

(ii) $\beta(r)$ <u>is continuous in some right-hand neighborhood</u> $(0,\lambda)$ <u>of the origin</u>,

(iii) $0<D\leq\beta(r)\leq C<\infty$ <u>for</u> $r \in (0,1]$.

<u>Let</u> f^* <u>be the radial map of</u> Δ <u>onto itself leaving</u> 0 <u>and</u> 1 <u>fixed and having</u> χ <u>as its complex dilatation. If</u>

$$(4.22)\quad \lim_{r\to 0^+} \beta(r) \geq M = \operatorname*{ess\,sup}_{0<r\leq 1} \beta(r),$$

<u>then</u> f^* <u>is extremal in the class of all QC maps of</u> Δ' <u>onto</u> $f^*(\Delta')$ <u>that agree with</u> f^* <u>on the boundary of</u> Δ'. <u>Moreover, if</u> (4.22) <u>holds, then</u> $K(f^*) = M + \sqrt{M^2+1}$.

<u>Proof</u>: The relation $\alpha(r) = \sqrt{\beta(r)^2+1}$ in the hypothesis of Corollary 4.16 implies that

$$\operatorname*{ess\,sup}_{0<r\leq 1} \frac{\sqrt{(\alpha(r)-1)^2+\beta(r)^2}\,\sqrt{(\alpha(r)+1)^2+\beta(r)^2}}{2\alpha(r)} = \operatorname*{ess\,sup}_{0<r\leq 1} \beta(r) = M.$$

Hence (4.22) of Corollary 4.16 implies (4.20) of Theorem 4.16. All the other hypotheses of Theorem 4.16 are obviously satisfied, and it follows that f^* is extremal. To find $K(f^*)$ we notice that with the given relation between α and β, we easily get

$$|\chi(re^{i\theta})| = \left|\frac{\alpha(r)-1+i\beta(r)}{\alpha(r)+1+i\beta(r)}\right| = \frac{\sqrt{(\alpha(r)-1)^2+\beta(r)^2}}{\sqrt{(\alpha(r)+1)^2+\beta(r)^2}}$$

$$= \frac{\sqrt{(\alpha(r)-1)^2+\alpha(r)^2-1}}{\sqrt{(\alpha(r)+1)^2+\alpha(r)^2-1}} = \frac{\sqrt{\alpha(r)-1}}{\sqrt{\alpha(r)+1}}$$

$$= \frac{\sqrt{\alpha(r)^2-1}}{\sqrt{(\alpha(r)+1)^2}} = \frac{\beta(r)}{\alpha(r)+1}.$$

Hence

$$D(re^{i\theta}) = \frac{1+|\chi(re^{i\theta})|}{1-|\chi(re^{i\theta})|} = \frac{1+\frac{\beta(r)}{\alpha(r)+1}}{1-\frac{\beta(r)}{\alpha(r)+1}} = \frac{\alpha(r)+1+\beta(r)}{\alpha(r)+1-\beta(r)} = \alpha(r) + \beta(r)$$

and so

$$K(f^*) = \operatorname*{ess\ sup}_{0<r\leq 1} D(re^{i\theta}) = \operatorname*{ess\ sup}_{0<r\leq 1} (\alpha(r)+\beta(r)) = \sqrt{M^2+1} + M.$$

CHAPTER V

ADDITIONAL EXTREMAL EXTENSIONS; A GENERALIZATION OF A RESULT OF REICH AND STREBEL

Although we now leave the study of radial maps, we are far from finished with the study of extremal QC mappings with given boundary homeomorphisms. We continue with the following definition.

Definition 5.1. A function U mapping $[0,\infty)$ onto itself will be called linear radial, or LR, if

$$U(x) = \log u(e^x) \qquad \text{for all} \quad x\in[0,\infty)$$

where u is any function satisfying the conditions

(i) u RB on $[0,\infty)$ with $u(1) = 1$,

(ii) $0<L\leq xu'(x)/u(x)\leq M<\infty$ a.e. on $(1,\infty)$,

(iii) $\lim\limits_{x\to\infty} \dfrac{xu'(x)}{u(x)} = Q = \max\{M,1/L\}$.

Remark: Since an explicit formula is given in (iii) of Theorem 4.5 for a function u satisfying the above conditions, we know that U is LR if and only if

$$U(x) = \log u(e^x) = \int_1^{e^x} \frac{1+\chi(s)}{1-\chi(s)} \frac{ds}{s}, \quad x\in[0,\infty)$$

where χ is a real-valued, measurable function on $(0,\infty)$ with

$$\sup_{0<s<\infty} |\chi(s)|<1, \quad \operatorname*{ess\,sup}_{0<s<\infty} |\chi(s)| \le \frac{Q-1}{Q+1},$$

and

$$\lim_{s\to\infty} |\chi(s)| = \frac{Q-1}{Q+1}$$

by condition (iii) of Definition 5.1.

By its definition U is a continuous, strictly increasing map of $[0,\infty)$ onto itself with

$$0<L\le U'\le M<\infty \quad \text{a.e. on } (0,\infty)$$

and

$$\lim_{x\to\infty} U'(x) = Q = \max\{M, 1/L\}.$$

<u>Lemma 5.1</u>. Let U be LR on $[0,\infty)$. Then for each $\epsilon>0$ there exists an $N(\epsilon)>0$ such that

$$(1-\epsilon)Qx\le U(x) \quad \text{for all } x>N(\epsilon).$$

Proof: Consider

$$\frac{U(x)}{Qx} = \frac{\int_0^x U'(s)ds}{Qx} .$$

Since $\lim_{x\to\infty} U'(x) = Q$, there is some $\hat{x}>0$ such that $x\geq\hat{x}$ implies that $U'(x)\geq Q-\hat{\epsilon}$, where $\hat{\epsilon} = Q\epsilon/2$. Therefore

$$\frac{U(x)}{Qx} = \frac{\int_0^{\hat{x}} U'(s)ds}{Qx} + \frac{\int_{\hat{x}}^{x} U'(s)ds}{Qx}$$

$$\geq \frac{\int_{\hat{x}}^{x} U'(s)ds}{Qx} \geq \frac{\int_{\hat{x}}^{x} (Q-\hat{\epsilon})ds}{Qx}$$

$$= (1-\frac{\hat{\epsilon}}{Q})(\frac{x-\hat{x}}{x}) .$$

Now since $\lim_{x\to\infty} \frac{x-\hat{x}}{x} = 1$, there exists an $N(\epsilon)$ for which

$$(1-\frac{\hat{\epsilon}}{Q})(\frac{x-\hat{x}}{x})\geq(1-\frac{\hat{\epsilon}}{Q}) - \frac{\hat{\epsilon}}{Q} = 1-\epsilon \quad \text{for} \quad x>N(\epsilon).$$

Hence

$$\frac{U(x)}{Qx}\geq 1-\epsilon \quad \text{for} \quad x>N(\epsilon),$$

and the lemma is proved.

Theorem 5.1. Let h be a normalized RB function on $[0,\infty)$ with ratio bounds L, M, where $1<L\leq M$. Let G be the domain

$$G = \{z = x+iy \mid -\infty<x<\infty,\ y>h(|x|)\}.$$

Finally, let

$$F(z) = F(x+iy) = (\text{sign } x)U(|x|) + iy$$

in G, where U is LR. If $G' = F(G)$ then F is extremal in the class of all QC maps from G to G' that agree with F on the boundary of G. Moreover,

$$K(F) = Q = \max \{M, 1/L\}.$$

(See Figure 5.1).

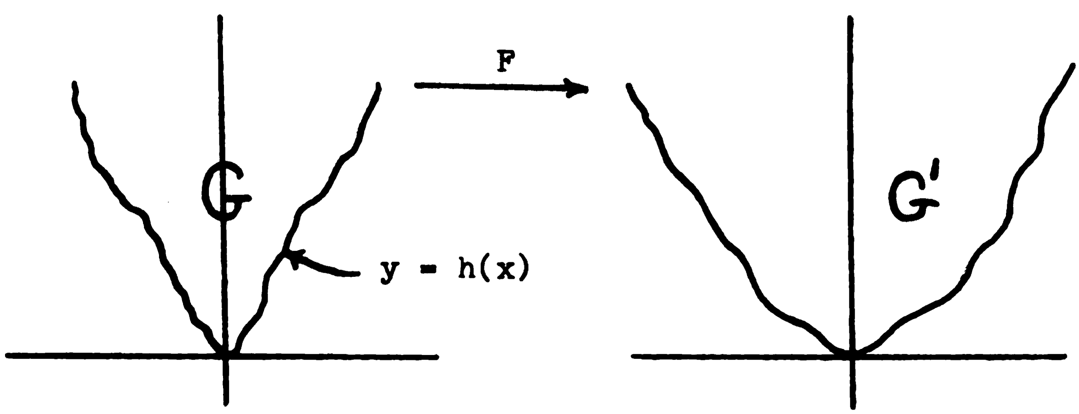

Figure 5.1

Proof: Let $f:G \to G'$ be any K-QC map of G to G' that agrees with F on the boundary of G. Now choose $\epsilon > 0$. Then by Lemma 5.1 there exists an $N(\epsilon) > 0$ such that $U(x) \geq (1-\epsilon)Qx$ whenever $x > N(\epsilon)$. Let $y_0 = h(N(\epsilon))$. Then for any $\eta > y_0$, we have $h^{-1}(\eta) \geq h^{-1}(y_0) = N(\epsilon)$, so that

$$(5.1) \qquad 2(1-\varepsilon)Qh^{-1}(\eta) \le 2U(h^{-1}(\eta)) \le L(\eta) = \int_{-h^{-1}(\eta)}^{h^{-1}(\eta)} |f_z+f_{\bar{z}}|d\xi ,$$

for any $\eta > y_0$, where $L(\eta)$ is the length of the f-image of the segment

$$\gamma_\eta = \{z \mid \text{Im } z = \eta, \ -h^{-1}(\eta) \le \text{Re } z \le h^{-1}(\eta)\}.$$

(See figure 5.2)

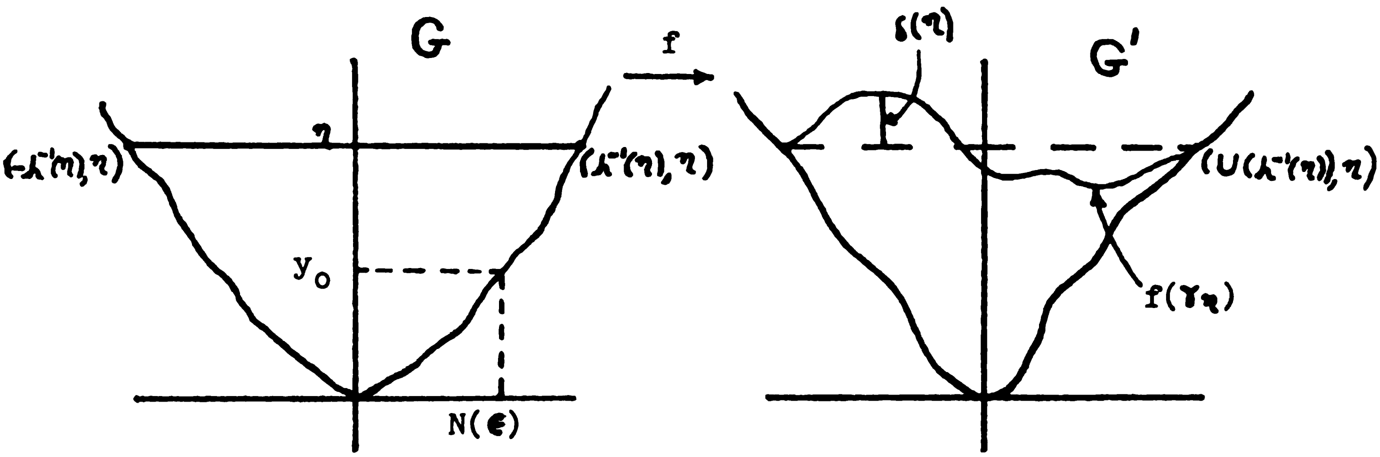

Figure 5.2

Integrating (5.1) with respect to η from 0 to y for any $y > y_0$ gives

$$2(1-\varepsilon)Q\int_{y_0}^{y} h^{-1}(\eta)\,d\eta \le \int_{y_0}^{y} L(\eta)\,d\eta \le \int_0^y\int_{-h^{-1}(\eta)}^{h^{-1}(\eta)} |f_z+f_{\bar{z}}|d\xi.$$

Squaring and applying the Schwarz inequality gives

$$(2(1-\varepsilon)Q\int_{y_0}^{y} h^{-1}(\eta)\,d\eta)^2 \leq \left(\int_0^y L(\eta)\,d\eta\right)^2 \leq \iint_{G_y} J\,d\xi\,d\eta \iint_{G_y} \frac{|1+\chi|^2}{1-|\chi|^2}\,d\xi\,d\eta ,$$

where $J(z) = |f_z|^2 - |f_{\bar{z}}|^2$ and $\chi(z) = \dfrac{f_{\bar{z}}}{f_z}$ are the Jacobian and complex dilatation, respectively, of f, and $G_y = G \cap \{z | \mathrm{Im}\, z < y\}$.

Clearly

$$U(x) = \int_0^x U'(s)\,ds \leq \int_0^x Qs\,ds = Qx \quad \text{for any} \quad x \geq 0.$$

Hence, if $\delta(y) \geq 0$ is the maximal upper deviation of $f(\gamma_y)$ above the horizontal line $\mathrm{Im}\, z = y$, i.e. $\delta(y) = \sup_{z \in \gamma_y} \{\mathrm{Im}\, f(x) - y\}$, then it is easy to see by considering the relevant areas (cf. Figure 5.2) that

$$\iint_{G_y} J\,d\xi\,d\eta \leq 2\int_0^{y+\delta(y)} U(h^{-1}(\eta))\,d\eta \leq 2Q\int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta$$

and, using the same reasoning as on page 354 of [14],

$$\iint_{G_y} \frac{|1+\chi|^2}{1-|\chi|^2}\,d\xi\,d\eta \leq 2K\int_0^y h^{-1}(\eta)\,d\eta,$$

where $(K-1)/(K+1) = \operatorname{ess\,sup}_{z \in G_y} |\chi(z)|$. Therefore

$$(2(1-\varepsilon)Q\int_{y_0}^{y} h^{-1}(\eta)\,d\eta)^2 \leq 4KQ\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta$$

or

$$(5.2) \qquad Q \le \frac{K}{(1-\varepsilon)^2} \; \frac{\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\left(\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\right)^2} \; .$$

If we can show that the term in brackets in (5.2) approaches 1 as y tends to ∞, then we will have $Q \le K/(1-\varepsilon)$. After this we can let ε tend to 0 and achieve $Q \le K$, from which it will follow that F is extremal for its boundary values. For convenience we write

$$(5.3) \qquad \frac{\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\left(\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\right)^2} = \left(\frac{\int_0^y h^{-1}(\eta)\,d\eta}{\int_{y_0}^y h^{-1}(\eta)\,d\eta}\right) + \left(\frac{\int_0^y h^{-1}(\eta)\,d\eta}{\int_{y_0}^y h^{-1}(\eta)\,d\eta}\right)\left(\frac{\int_y^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\int_{y_0}^y h^{-1}(\eta)\,d\eta}\right) .$$

In order to simplify calculations, we let $h^{-1}(\eta) = g(\eta)$ in the rest of the proof. Now by Theorem 3.5, since h is an RB function with

$$(5.4) \qquad 1 < L \le \frac{\eta h'(\eta)}{h(\eta)} \le M,$$

it follows that g is also RB with

$$\frac{1}{M} \le \frac{\eta g'(\eta)}{g(\eta)} \le \frac{1}{L} < 1.$$

Clearly

$$\lim_{y\to\infty} \frac{\int_0^y g(\eta)\,d\eta}{\int_{y_0}^y g(\eta)\,d\eta} = 1 + \lim_{y\to\infty} \frac{\int_0^{y_0} g(\eta)\,d\eta}{\int_{y_0}^y g(\eta)\,d\eta} = 1$$

and, using (5.4) and integration by parts,

$$\int_y^{y+\delta(y)} g(\eta)\,d\eta \le M\int_y^{y+\delta(y)} \eta g'(\eta)\,d\eta = M\eta g(\eta)\Big]_y^{y+\delta(y)} - M\int_y^{y+\delta(y)} g(\eta)\,d\eta \,.$$

Thus

$$(M+1)\int_y^{y+\delta(y)} g(\eta)\,d\eta \le M\eta g(\eta)\Big]_y^{y+\delta(y)}$$

or

$$\int_y^{y+\delta(y)} g(\eta)\,d\eta \le \frac{M}{M+1}[(y+\delta(y))g(y+\delta(y))-yg(y)].$$

Similarly,

$$\int_{y_0}^y g(\eta)\,d\eta \ge \frac{L}{L+1}[yg(y)-y_0 g(y_0)].$$

Therefore, if we let $C = M(L+1)/L(M+1)$, then

$$(5.5)\qquad 0 \le \frac{\int_y^{y+\delta(y)} g(\eta)\,d\eta}{\int_{y_0}^y g(\eta)\,d\eta} \le C\,\frac{(y+\delta(y))g(y+\delta(y))-yg(y)}{yg(y)-y_0 g(y_0)}$$

$$= \frac{C}{1-\frac{y_0 g(y_0)}{yg(y)}}\left[\left(1+\frac{\delta(y)}{y}\right)\frac{g(y+\delta(y))}{g(y)}-1\right].$$

But by Theorem 3.2

$$\frac{g(y+\delta(y))}{g(y)} \le \left(\frac{y+\delta(y)}{y}\right)^{\frac{1}{L}} = \left(1+\frac{\delta(y)}{y}\right)^{\frac{1}{L}}.$$

Hence (5.5) gives

$$(5.6) \qquad 0 \le \frac{\int_y^{y+\delta(y)} g(\eta)\,d\eta}{\int_{y_0}^{y} g(\eta)\,d\eta} \le \frac{C}{1-(y_0 g(y_0)/yg(y))}\left[(1+\frac{\delta(y)}{y})^{\frac{L+1}{L}} - 1\right].$$

Now for any $\eta > 1$, Theorem 3.2 shows that $g(\eta) \le \eta^{1/L}$, $1/L < 1$. Hence, the proof on the top of page 355 in [14] that

$\lim_{y\to\infty} \frac{\delta(y)}{y} = 0$ still goes through and so

$$\lim (1 + \frac{\delta(y)}{y})^{1+1/L} - 1 = 0.$$

Moreover, since

$$\lim_{y\to\infty} \frac{C}{1-(y_0 g(y_0)/yg(y))} = C,$$

we conclude from (5.6) that

$$\lim_{y\to\infty} \frac{\int_y^{y+\delta(y)} g(\eta)\,d\eta}{\int_{y_0}^{y} g(\eta)\,d\eta} = 0.$$

Hence, by (5.3),

$$\lim_{y\to\infty} \frac{\int_0^y h^{-1}(\eta)\,d\eta \int_0^{y+\delta(y)} h^{-1}(\eta)\,d\eta}{\left(\int_{y_0}^{y} h^{-1}(\eta)\,d\eta\right)^2} = 1,$$

so that $Q \le K$.

Therefore any QC mapping $f: G \longrightarrow G'$ agreeing with F on the boundary of G must have QC dilatation $K(f) \ge Q$.

But for the original function

$$F(x+iy) = \begin{cases} U(x) + iy & \text{if } x \geq 0 \\ -U(-x) + iy & \text{if } x < 0, \end{cases}$$

it is clear from (4.7) in the proof of Theorem 4.1 that the point dilatation of F is

$$D(x+iy) + \frac{1}{D(x+iy)} = U'(|x|) + \frac{1}{U'(|x|)}$$

wherever U' exists (i.e., almost everywhere). Thus, since $0<L\leq U'\leq M<\infty$, $D(x+iy) \leq \max\{M, 1/L\} = Q$. But in fact, by taking $\operatorname{Im} z$ large enough, we can find points in G with arbitrarily large real part x, and hence $\lim_{x\to\infty} U'(x) = Q$ implies that

$$K(F) = \operatorname*{ess\,sup}_{z \in G} D(z) = Q \leq K(f).$$

That is, F is extremal.

<u>Remark</u>: If, in Theorem 5.1 we take $h(\eta) = \eta^L$, $L > 1$, and $U(x) = Qx$, $Q \geq 1$, then we get the extremality results of Reich and Strebel [14] as a special case.

As before, the question of uniqueness in Theorem 5.1 is partially resolved by the following corollary.

<u>Corollary 5.1</u>. <u>Let</u> U <u>and</u> h <u>satisfy the hypotheses of Theorem</u> 5.1, <u>so that the map</u> F <u>is extremal for the boundary</u>

values it assumes. If there is some $\epsilon > 0$ *and some interval* $(a,b) \subset (0,\infty)$ *on which*

$$\frac{1}{Q-\epsilon} \leq U'(x) \leq Q-\epsilon,$$

then F *is not unique extremal.*

Proof: Let y_0 be so large that $h^{-1}(y_0) > b$. Define a map $g: G \longrightarrow G'$ as

$$g(x+iy) = \begin{cases} \left(\frac{2y_0-y}{y_0} U(x) + \frac{y-y_0}{y_0} W(x)\right) + iy & \text{if } y_0 \leq y \leq 2y_0 \\ \left(\frac{3y_0-y}{y_0} W(x) + \frac{y-2y_0}{y_0} U(x)\right) + iy & \text{if } 2y_0 \leq y \leq 3y_0 \\ F(x+iy) & \text{if } y \notin [y_0, 3y_0], \end{cases}$$

where $W(x) = U(x) + \sigma(x)$ for $x \in [0,\infty)$ and some function $\sigma \geq 0$ to be picked. It is clear that between $y = y_0$ and $y = 2y_0$, g is a linear combination of U and W, and the same is true between $y = 2y_0$ and $y = 3y_0$. Hence we have the same kind of mapping as in the proof of Theorem 4.9. The proof here is almost identical to the proof of Theorem 4.9 and it is easy to see that if we choose σ so that $\sigma(x) = 0$ for $x \notin [a,b]$, $\sigma(x) > 0$ for $x \in (a,b)$, and both σ and $|\sigma'|$ are sufficiently small on $[a,b]$, then $D(x+iy) \leq Q$ for all $x+iy$ with $a \leq x \leq b$ and $y_0 \leq y \leq 3y_0$. Since $g = F$ everywhere else, this implies $K(g) = Q = K(F)$. Hence g is also extremal. It is also clear that $g(z) = F(z)$ on the boundary of G but that g is not the same as F everywhere on G since $\sigma > 0$ on (a,b). Thus F is not unique extremal.

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