INVESTIGATION OF A DYNAMIC SAINT-VENANT REGION IN A SEMI-INFINITE STRIP

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ABSTRACT

INVESTIGATION OF A DYNAMIC SAINT-VENANT
REGION IN A SEMI-INFINITE STRIP

By

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The investigation examines the steady state response of a semi-infinite strip with stress-free edges to time-harmonic self-equilibrated shear and normal stresses on the finite edge. The mathematical analysis is based on the equations of linear elasticity for generalized plane stress and involves a biorthogonal eigenfunction expansion of a four component stress vector.

Solutions for three different boundary stress distributions at one frequency are examined in detail and reveal significant non-decaying stress modes. The shapes of these modes are shown graphically.

The eigenvalues are tabulated for seven different frequencies between 100 and 100,000 cycles per second.

INVESTIGATION OF A DYNAMIC SAINT-VENANT REGION IN A SEMI-INFINITE STRIP

By

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TABLE OF CONTENTS

LIST OF TABLES

Table		Page

LIST OF FIGURES

LIST OF PRINCIPAL SYMBOLS

$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	stress components in Cartesian coordinates.
Q	auxiliary "stress like" variable.
u_x, u_y	displacement components in Cartesian coordinates.
$\sigma_{xxb}, \sigma_{yyb}, \sigma_{xyb}$, Q_b, u_{xb}, u_{yb}	boundary stress and displacement functions on the finite edge $x = 0$.
ω	angular frequency.
t	time.
ρ	density.
E	Young's modulus.
ν	Poisson's ratio.
α	propagation constant in the x direction equal to 2π divided by the wavelength.
m	$= \frac{1}{\alpha}\sqrt{\alpha^2 - \frac{\rho\omega^2}{E}(1 - \nu^2)}$
n	$= \frac{1}{\alpha}\sqrt{\alpha^2 - \frac{2\rho\omega^2}{E}(1 + \nu)}$
K	$= (1 + n^2)/(\nu n^2 - \nu - 2)$
R	$= (\nu n^2 + 2n^2 - \nu)/(1 + n^2)$
V	$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, eigenvector of the first mixed boundary case.
Z	$= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, adjoint problem eigenvector for the first mixed boundary case.

W $= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, eigenvector of the second mixed boundary case.

U $= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, adjoint problem eigenvector for the second mixed boundary case.

η_i eigenfunctions of the four-vector formulation.

$N^{(j)}$ normalization constants.

$k^{(j)}$ arbitrary constants of the adjoint functions.

c $= \omega/\alpha$, phase velocity.

c_o $= \sqrt{E/\rho}$, phase velocity of infinitely long waves in a cylinder.

a $= \frac{\rho\omega^2}{E} (1 - \nu^2)$.

$(\ )^+$ represents complex conjugate transpose.

$(\bar{\ })$ represents complex conjugate.

CHAPTER I

INTRODUCTION

1.1 Historical Background

In 1853, Barre de Saint-Venant presented his solution for torsion in long prismatic bars of various cross sectional shapes (1). In this solution, he was able to satisfy the end conditions producing the twist in the bar only up to a resultant force and couple, the distribution of which, he assumed, could differ from the required distribution in the body of the bar. He postulated that variations in the distribution of statically equivalent end loadings must have little effect on the twisted bar except near the ends. This declaration became the basis for what is now known as the Saint-Venant principle. In a footnote of his memoir, Saint-Venant went on to say that the influence of forces in equilibrium acting on a small portion of a body extends very little beyond the parts upon which they act (2).

The importance of this principle in the subsequent developments of the theory of elasticity cannot be over-stated, and the problem of providing mathematical clarification of the principle and justification of its use has been the subject of serious consideration since its enunciation. As a practical matter, analytical stress analysis of equilibrium systems using classical theory constantly resorts to the principle by replacing one force system with a statically equivalent one with the assumption that the resulting errors exist in a region extending very little beyond the surface of application. Indeed, without the use of this principle, many problems would be too complicated to solve.

The earliest solution for a system of forces in equilibrium on the edge of a plate is attributed to Thomson and Tait (1867), followed by a more complete solution by Maurice Levy (1877), (3). M.J. Boussinesq (1885), (4), one of the most distinguished pupils of Saint-Venant, was able to define the region of local perturbation resulting from the application of statically equivalent systems of loads normal to the infinite half space. This work has been a standard reference in textbooks as proof of the Saint-Venant principle. As recently as 1945, von Mises (5) argued that Boussinesq's solution was for a particular loading, and that Saint-Venant's principle in its traditional form does not hold true if equilibrant force systems are introduced tangent to the plane surface. He proposed to modify the principle with the introduction of the concept of astatic equilibrium which requires forces to remain in equilibrium when rotated through any angle. E. Sternberg (1953), (6) supplied a precise formulation and proof of the Saint-Venant principle as modified by von Mises.

The von Mises-Sternberg papers define the Saint-Venant boundary region for bodies having very general geometries. Recent engineering requirements have necessitated the determination of the magnitude of the stresses in this boundary region for specific geometries. Techniques now exist to investigate the stress distribution in the Saint-Venant boundary region for particular time-independent boundary conditions on the semi-infinite strip (7), cylinder (8), wedge and cone (9).

The investigation of the possible existence of a dynamic Saint-Venant region is limited. B.A. Boley (1954), (10), using a simple model, found that a region similar to the static boundary region exists for slowly applied loads; but as the rate of loading is increased, the region extends to longer portions. L.W. Kennedy and O.E. Jones (1969), (11) examined the effect of altering the radial distribution of statically equivalent pressure step loads on a semi-infinite cylinder having zero lateral displacement on the end. They concluded that any differences in time-average dynamic stresses and strains are negligible at distances greater than five times the cylinder diameter, and differences in peak values are small at distances greater than twenty diameters. The effect of a self-equilibrated load representing the difference between the two statically equivalent applied loads remains undetermined.

The Kennedy and Jones investigation, as well as other of the most recent studies in wave propagation, employed transform techniques which permit asymptotic solutions to the equations of motion at large propagation distances only for the mixed end boundary conditions of one stress and one displacement (12). The important Saint-Venant boundary region with pure end stress conditions specified still requires investigation.

1.2 Problem Statement

A semi-infinite strip with stress free edges is end loaded with time-harmonic self-equilibrated shear and normal stresses. The decay characteristics of the resulting stress distribution and propagation modes are investigated through the analytic solution of the equations of linear elasticity.

1.3 Assumptions and General Solution Outline

The homogeneous, isotropic, linearly-elastic, semi-infinite strip occupies the region $-1 \leq y \leq 1$ and $x \leq \infty$. The solution is based on the linearized equations of elasticity for generalized plane stress and is obtained by use of an extension of a technique of biorthogonal eigenfunction expansion developed by M.W. Johnson and R.W. Little (1965), (7). The boundary conditions for this problem are:

a bounded solution as x approaches infinity

stress free infinite boundaries, $y = \pm 1$ (1.3-1)

$$\sigma_{yy}(x, \pm 1) = \sigma_{xy}(x, \pm 1) = 0 \tag{1.3-2}$$

and one of the following conditions on the finite boundary $x = 0$:

the first mixed boundary condition

$$\begin{aligned} \sigma_{xx} &= \sigma_{xxb}(y) \cos \omega t \\ u_y &= u_{yb}(y) \cos \omega t \end{aligned} \tag{1.3-3a}$$

the second mixed boundary condition

$$\begin{aligned} \sigma_{xy} &= \sigma_{xyb}(y) \cos \omega t \\ u_x &= u_{xb}(y) \cos \omega t \end{aligned} \tag{1.3-3b}$$

the pure stress boundary condition

$$\begin{aligned} \sigma_{xx} &= \sigma_{xxb}(y) \cos \omega t \\ \sigma_{xy} &= \sigma_{xyb}(y) \cos \omega t \end{aligned} \tag{1.3-3c}$$

the pure displacement boundary condition

$$u_x = u_{xb}(y) \cos \omega t$$
$$u_y = u_{yb}(y) \cos \omega t \ . \qquad (1.3\text{-}3d)$$

This problem is formulated to investigate the Saint-Venant boundary region with pure stress end conditions. In the course of the development, formulations are obtained which accept the specification of each of the mixed boundary conditions. The mixed boundary cases are not examined separately because these boundary conditions are not of immediate interest to this problem. Their development, however, is a vital step in the formulation of the pure stress problem.

The formulation of the pure stress case involves three major steps, each of which is discussed separately in Chapters II, III, and IV.

In Chapter II, a second-order partial differential vector equation is developed in terms of the normal stresses σ_{xx} and σ_{yy}. This development is called the first mixed boundary case because specification of σ_{xx} and σ_{yy} on the boundary $x = 0$ is equivalent to specification of the normal stress σ_{xx} and the displacement u_y in the y direction. The stress σ_{yy} by itself is not a boundary stress on the surface $x = 0$, but in combination with σ_{xx} allows specification of u_y. The homogeneous boundary conditions on the infinite edges leads to an eigenvalue problem which is solved for the eigenvalues and eigenfunctions. A biorthogonality operator is found which permits the direct calculation of the eigenfunction constants from the prescribed mixed boundary conditions. As was indicated earlier, the investigation of the Saint-Venant boundary region is to be made for pure stress end conditions, not conditions involving one boundary stress and one boundary displacement. Thus, for this study, the solutions developed in this chapter will not be used to solve a particular mixed boundary value problem, but rather, they constitute an intermediate step in the pure stress boundary formulation continued in Chapters III and IV.

In Chapter III, a second-order partial differential vector equation is developed in terms of the shear stress σ_{xy} and a new function Q. This development is called the second mixed case because specification of σ_{xy} and Q on the surface $x = 0$ is equivalent to specification of the shear stress σ_{xy} and a displacement u_x in the x direction. The solution proceeds in the same manner described for the first mixed case. Again, the solutions obtained for the second mixed case are treated as elements of the final formulation.

In Chapter IV, the results of Chapters II and III are combined to write a series expansion of the vector $[\sigma_{xx},\sigma_{yy},\sigma_{xy},Q]$. The two biorthogonality operators are combined to produce a biorthogonality operator which can be applied to the four component vector.

In Chapter V, boundary functions σ_{xxb} and σ_{xyb} are specified, and σ_{yy} and Q are written as a series expansion on the surface $x = 0$. The eigenfunction constants are obtained by applying the four-vector biorthogonal operator. Chapter VII includes a discussion of three examples.

CHAPTER II

FORMULATION OF PROBLEM FOR FIRST MIXED BOUNDARY CASE

2.1 Equations of Elasticity

The governing equations for the first mixed case are derived from the following linearized equations of elasticity using the generalized plane stress assumption.

Equations of Motion

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho \frac{\partial^2 u_x}{\partial t^2} \tag{2.1-1}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho \frac{\partial^2 u_y}{\partial t^2} \tag{2.1-2}$$

Stress-Displacement Relations

$$\frac{\partial u_x}{\partial x} = \frac{1}{E} (\sigma_{xx} - \nu\sigma_{yy}) \tag{2.1-3}$$

$$\frac{\partial u_y}{\partial y} = \frac{1}{E} (\sigma_{yy} - \nu\sigma_{xx}) \tag{2.1-4}$$

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{2(1+\nu)}{E} \sigma_{xy} \tag{2.1-5}$$

2.2 Boundary Conditions

For the first mixed case, the boundary conditions specified on the finite boundary $x = 0$ are

$$\sigma_{xx} = \sigma_{xxb}(y) \cos \omega t \tag{2.2-1}$$

$$u_y = u_{yb}(y) \cos \omega t \ . \tag{2.2-2}$$

As has been indicated earlier in Section 1.3, equation (2.1-4) shows that the condition (2.2-2) is equivalent to the specification of (2.2-1) and σ_{yy} on the boundary $x = 0$.

$$\sigma_{yy} = \sigma_{yyb}(y) \cos \omega t \tag{2.2-3}$$

The infinite boundaries, $y = \pm 1$, are taken to be stress free,

$$\sigma_{yy}(x, \pm 1) = \sigma_{xy}(x, \pm 1) = 0, \tag{2.2-4}$$

and the solution is required to be bounded as x approaches infinity.

2.3 Equations for the First Mixed Case

The general equations given in Section 2.1 can be reduced to two equations for σ_{xx} and σ_{yy}. Differentiating equation (2.1-5) once with respect to x and once with respect to y yields the following expression

$$\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial y \partial x^2} = \frac{2(1+\nu)}{E} \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \ . \tag{2.3-1}$$

Substituting equations (2.1-3) and (2.1-4) gives

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = 2(1+\nu) \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \ . \tag{2.3-2}$$

Differentiating equation (2.1-1) with respect to x, (2.1-2) with respect to y and adding gives

$$2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \rho \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) - \frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} \ . \tag{2.3-3}$$

Substituting equation (2.3-3) into (2.3-2) and using equations (2.1-3) and (2.1-4) yields the first equation.

$$\frac{\partial^2}{\partial x^2}(\sigma_{xx}+\sigma_{yy}) + \frac{\partial^2}{\partial y^2}(\sigma_{xx}+\sigma_{yy}) = \frac{\rho(1-\nu^2)}{E}\frac{\partial^2}{\partial t^2}(\sigma_{xx}+\sigma_{yy}) \qquad (2.3\text{-}4)$$

The second equation is obtained by differentiating equation (2.1-1) with respect to x, (2.1-2) with respect to y, subtracting and substituting equations (2.1-3) and (2.1-4).

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} = \frac{\rho(1+\nu)}{E}\frac{\partial^2}{\partial t^2}(\sigma_{xx}-\sigma_{yy}) \qquad (2.3\text{-}5)$$

Equations (2.3-4) and (2.3-5) can be written in matrix form

$$A\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = N\frac{\partial^2 S}{\partial t^2}, \qquad (2.3\text{-}6)$$

where

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix}$$

$$N = \frac{\rho}{E}\begin{bmatrix} 2+\nu-\nu^2 & -\nu(1+\nu) \\ -(1+\nu) & 1+\nu \end{bmatrix}.$$

This matrix equation is the desired first mixed boundary case formulation.

2.4 Solution of the Equations for First Mixed Case

The boundary stresses (2.2-1) and (2.2-3) are time-harmonic of frequency ω. Thus, a time-harmonic solution form is assumed using separation of variables

$$S = V(y)X(x)e^{i\omega t} \quad , \tag{2.4-1}$$

where

$$V(y) = \begin{bmatrix} v_1(y) \\ v_2(y) \end{bmatrix} \quad .$$

Substituting this assumed form of solution into equation (2.3-6) yields

$$AV \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 V}{\partial y^2} + \omega^2 \, NVX = 0 \quad . \tag{2.4-2}$$

This suggests the possible form $X(x) = e^{i\alpha x}$ which when substituted into equation (2.4-2) yields

$$\frac{d^2 V}{dy^2} + (\omega^2 N - \alpha^2 A)V = 0 \quad , \tag{2.4-3}$$

or

$$\frac{d^2 V}{dy^2} + HV = 0 \quad , \tag{2.4-4}$$

where

$$H = \begin{bmatrix} \frac{\rho\omega^2}{E}(2+\nu-\nu^2) - 2\alpha^2 & -\frac{\rho\omega^2}{E}(\nu+\nu^2) - \alpha^2 \\ -\frac{\rho\omega^2}{E}(1+\nu) + \alpha^2 & \frac{\rho\omega^2}{E}(1+\nu) \end{bmatrix} \quad .$$

Equation (2.4-4) is an ordinary second-order matrix differential equation with constant coefficients. Expanding equation (2.4-4) yields

$$\frac{d^2v_1}{dy^2} + h_{11}v_1 + h_{12}v_2 = 0 \tag{2.4-5}$$

$$\frac{d^2v_2}{dy^2} + h_{21}v_1 + h_{22}v_2 = 0 \ . \tag{2.4-6}$$

These two equations may be written in the form

$$\left(\frac{d^4}{dy^4} + (h_{11} + h_{22})\frac{d^2}{dy^2} + h_{11}h_{22} - h_{12}h_{21}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \ . \tag{2.4-7}$$

The roots of the characteristic equation obtained from (2.4-7) are:

$$r_{1,2} = \pm\sqrt{\alpha^2 - \frac{\rho\omega^2}{E}(1-\nu^2)} = \pm\,\alpha m \tag{2.4-8a}$$

$$r_{3,4} = \pm\sqrt{\alpha^2 - \frac{2\rho\omega^2}{E}(1+\nu)} = \pm\,\alpha n \ , \tag{2.4-8b}$$

where

$$m = \frac{1}{\alpha}\sqrt{\alpha^2 - \frac{\rho\omega^2}{E}(1-\nu^2)} \tag{2.4-9}$$

$$n = \frac{1}{\alpha}\sqrt{\alpha^2 - \frac{2\rho\omega^2}{E}(1+\nu)} \ . \tag{2.4-10}$$

The solution of equation (2.4-4) has the following form:

$$v_1 = C_1 \sinh \alpha my + C_2 \cosh \alpha my + C_3 \sinh \alpha ny + C_4 \cosh \alpha ny$$

$$v_2 = C_5 \sinh \alpha my + C_6 \cosh \alpha my + C_7 \sinh \alpha ny + C_8 \cosh \alpha ny \ .$$

Substituting these expressions for v_1 and v_2 into either of the equations (2.4-5) or (2.4-6) defines four of the constants.

$$v_1 = C_1 \sinh \alpha my + C_2 \cosh \alpha my + C_3 \sinh \alpha ny + C_4 \cosh \alpha ny \tag{2.4-11}$$

$$v_2 = KC_1 \sinh \alpha my + KC_2 \cosh \alpha my - C_3 \sinh \alpha ny - C_4 \cosh \alpha ny \ , \quad (2.4\text{-}12)$$

where

$$K = \frac{1 + n^2}{\nu n^2 - \nu - 2} \ .$$

The remaining constants are determined from the boundary conditions.

Satisfaction of the boundary conditions on $y = \pm 1$.

(a) On the surfaces $y = \pm 1$, the normal stress σ_{yy} is zero for all x and time t. This implies, from equation (2.4-1), that $v_2(\pm 1) = 0$. Thus from equation (2.4-12):

$$KC_1 \sinh \alpha m - C_3 \sinh \alpha n = 0 \quad (2.4\text{-}13)$$

$$KC_2 \cosh \alpha m - C_4 \cosh \alpha n = 0 \ . \quad (2.4\text{-}14)$$

Two equations appear as a result of separating the solution into even and odd functions of y.

(b) On the surfaces $y = \pm 1$, the shear stress σ_{xy} is zero for all x and t. The governing equations do not explicitly contain σ_{xy}, so that a relationship between σ_{xx}, σ_{yy}, and σ_{xy} must be constructed. Differentiating equation (2.1-2) with respect to x and substituting equation (2.1-5) yields

$$\frac{\partial^2 \sigma_{xy}}{\partial x^2} = \rho \frac{\partial^2}{\partial t^2}\left[\frac{2(1+\nu)}{E}\sigma_{xy} - \frac{\partial u_x}{\partial y}\right] - \frac{\partial^2 \sigma_{yy}}{\partial x \partial y} \ .$$

Differentiating this equation with respect to x and substituting equation (2.1-3) yields

$$\left[\frac{\partial^2}{\partial x^2} - \frac{2\rho(1+\nu)}{E}\frac{\partial^2}{\partial t^2}\right]\frac{\partial\sigma_{xy}}{\partial x} = -\left[\frac{\partial^2}{\partial x^2} - \frac{\rho\nu}{E}\frac{\partial^2}{\partial t^2}\right]\frac{\partial\sigma_{yy}}{\partial y}$$

$$-\frac{\rho}{E}\frac{\partial^2}{\partial t^2}\frac{\partial\sigma_{xx}}{\partial y} . \qquad (2.4\text{-}15)$$

If σ_{xy} is constant at $y = \pm 1$, then $\frac{\partial\sigma_{xy}}{\partial x} = 0$ and equation (2.4-15) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\rho\nu}{E}\frac{\partial^2}{\partial t^2}\right]\frac{\partial\sigma_{yy}}{\partial y} + \frac{\rho}{E}\frac{\partial^2}{\partial t^2}\frac{\partial\sigma_{xx}}{\partial y} = 0 \qquad \text{at} \quad y = \pm 1 .$$

Using the assumed solution form from equation (2.4-1) gives

$$\left[-\alpha^2 + \frac{\rho\nu}{E}\omega^2\right]\frac{dv_2(\pm 1)}{dy} = \frac{\rho\omega^2}{E}\frac{dv_1(\pm 1)}{dy} . \qquad (2.4\text{-}16)$$

Substituting equations (2.4-11) and (2.4-12) into equation (2.4-16) and separating into even and odd functions of y yields

$$C_1\alpha m\left(\alpha^2 K + \frac{\rho\omega^2}{E} - \frac{K\nu\rho\omega^2}{E}\right)\cosh \alpha m$$

$$+ C_3\alpha n\left(-\alpha^2 + \frac{\rho\omega^2}{E} + \frac{\nu\rho\omega^2}{E}\right)\cosh \alpha n = 0 \qquad (2.4\text{-}17)$$

and

$$C_2\alpha m\left(\alpha^2 K + \frac{\rho\omega^2}{E} - \frac{K\nu\rho\omega^2}{E}\right)\sinh \alpha m$$

$$+ C_4\alpha n\left(-\alpha^2 + \frac{\rho\omega^2}{E} + \frac{\nu\rho\omega^2}{E}\right)\sinh \alpha n = 0 . \qquad (2.4\text{-}18)$$

The solution of equations (2.4-13) and (2.4-17) yields the transcendental equation for the odd eigenvalues.

$$\frac{\tanh \alpha m}{\tanh \alpha n} = \frac{4mn}{(1+n^2)^2} . \qquad (2.4\text{-}19)$$

In a similar manner the equation for the even eigenvalues is obtained by solution of equations (2.4-14) and (2.4-18).

$$\frac{\tanh \alpha n}{\tanh \alpha m} = \frac{4mn}{(1+n^2)^2} \tag{2.4-20}$$

The transcendental equation, (frequency equation), (2.4-19) is used to define the eigenvalue α when the eigenfunctions v_1 and v_2 are odd in y, and equation (2.4-20) is used when these eigenfunctions are even.

The eigenfunctions divide into even and odd functions of the form:

$$\begin{aligned}
{}_o v_1 &= C_1 \left(\sinh \alpha my + \frac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right) \\
{}_o v_2 &= C_1 \left(K \sinh \alpha my - \frac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right) \\
{}_e v_1 &= C_2 \left(\cosh \alpha my + \frac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny\right) \\
{}_e v_2 &= C_2 \left(K \cosh \alpha my - \frac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny\right) \quad .
\end{aligned} \tag{2.4-21}$$

The constants C_1 and C_2 are determined from the boundary conditions on the remaining surface $x = 0$.

2.5 Definition of the Biorthogonal Operator and the Adjoint Equation

Consider V_r as the solution vector of the differential equation (2.4-3) associated with the r^{th} eigenvalue α_r.

$$\frac{d^2V_r}{dy^2} = (\alpha_r^2 A - \omega^2 N)V_r \tag{2.5-1}$$

Premultiplying equation (2.5-1) by the complex conjugate transpose of some arbitrary vector function of y, Z_s, and integrating from

$y = -1$ to $y = 1$ yields[1]

$$\int_{-1}^{1} Z_s^+ \frac{d^2V_r}{dy^2}\,dy = \int_{-1}^{1} Z_s^+(\alpha_r^2 A - \omega^2 N)V_r\,dy \quad , \quad Z_s = \begin{bmatrix} z_{s1} \\ z_{s2} \end{bmatrix} .$$

Integrating the left side by parts gives

$$\left(Z_s^+ \frac{dV_r}{dy} - \frac{dZ_s^+}{dy} V_r\right)\Bigg|_{-1}^{1} + \int_{-1}^{1} \frac{d^2Z_s^+}{dy^2} V_r\,dy = \int_{-1}^{1} Z_s^+(\alpha_r^2 A - \omega^2 N)V_r\,dy. \qquad (2.5\text{-}2)$$

Consider the arbitrary vector function Z_s as being the s^{th} solution vector of the following differential equation, termed the adjoint equation[2]

$$\frac{d^2Z_s}{dy^2} = (\bar{\alpha}_s^2 A^+ - \omega^2 N^+)Z_s \ , \qquad (2.5\text{-}3)$$

where the eigenvalue α_s is determined from the same transcendental equation obtained for the equation (2.5-1). It remains to be shown that this condition is satisfied.

Premultiplying equation (2.5-3) by V_r^+ and integrating from $y = -1$ to $y = 1$ produces

$$\int_{-1}^{1} V_r^+ \frac{d^2Z_s}{dy^2}\,dy = \int_{-1}^{1} V_r^+(\bar{\alpha}_s^2 A^+ - \omega^2 N^+)Z_s\,dy \ .$$

Taking the complex conjugate transpose of this equation,

$$\int_{-1}^{1} \frac{d^2Z_s^+}{dy^2} V_r\,dy = \int_{-1}^{1} Z_s^+(\alpha_s^2 A - \omega^2 N)V_r\,dy \ ,$$

---

(1) $()^+$ represents complex conjugate transpose

(2) $(\bar{\ })$ represents complex conjugate

and subtracting from equation (2.5-2) yields

$$\left(Z_s^+ \frac{dV_r}{dy} - \frac{dZ_s^+}{dy} V_r \right) \Bigg|_{-1}^{1} = (\alpha_r^2 - \alpha_s^2) \int_{-1}^{1} Z_s^+ A V_r \, dy . \tag{2.5-4}$$

Expanding the left side of equation (2.5-4) gives

$$\left(\bar{z}_{s1} \frac{dv_{r1}}{dy} + \bar{z}_{s2} \frac{dv_{r2}}{dy} - \frac{d\bar{z}_{s1}}{dy} v_{r1} - \frac{d\bar{z}_{s2}}{dy} v_{r2} \right) \Bigg|_{-1}^{1} =$$

$$(\alpha_r^2 - \alpha_s^2) \int_{-1}^{1} Z_s^+ A V_r \, dy . \tag{2.5-5}$$

The boundary conditions on the surface $y = \pm 1$ in Section 2.2 for the original differential equation give:

(a) $v_{r2}(\pm 1) = 0$

(b) From equation (2.4-16)

$$\frac{dv_{r1}(\pm 1)}{dy} = -\alpha_r^2 \frac{E}{\rho\omega^2} \frac{dv_{r2}(\pm 1)}{dy} + \nu \frac{dv_{r2}(\pm 1)}{dy} ,$$

where it is noted that the eigenvalue α_r appears in the second boundary condition. Substituting these values into equation (2.5-5) yields

$$\left[\bar{z}_{s1} \left(-\alpha_r^2 \frac{E}{\rho\omega^2} \frac{dv_{r2}}{dy} + \nu \frac{dv_{r2}}{dy} \right) + \bar{z}_{s2} \frac{dv_{r2}}{dy} - \frac{d\bar{z}_{s1}}{dy} v_{r1} \right]_{-1}^{1} =$$

$$(\alpha_r^2 - \alpha_s^2) \int_{-1}^{1} Z_s^+ A V_r \, dy . \tag{2.5-6}$$

Imposing the following boundary conditions on the adjoint problem:

(a) $\dfrac{d\bar{z}_{s1}(\pm 1)}{dy} = 0$

(b) $\bar{z}_{s2}(\pm 1) = \alpha_s^2 \dfrac{E}{\rho\omega^2} \bar{z}_{s1}(\pm 1) - \nu \bar{z}_{s1}(\pm 1)$, and substituting these into equation (2.5-6) yields

$$(\alpha_s^2 - \alpha_r^2)\bar{z}_{s1} \frac{E}{\rho\omega^2} \frac{dv_{r2}}{dy}\Bigg|_{-1}^{1} = (\alpha_r^2 - \alpha_s^2)\int_{-1}^{1} Z_s^+ AV_r dy \ ,$$

or

$$(\alpha_r^2 - \alpha_s^2)\left[\int_{-1}^{1} Z_s^+ AV_r dy + \bar{z}_{s1} \frac{E}{\rho\omega^2} \frac{dv_{r2}}{dy}\Bigg|_{-1}^{1}\right] = 0 \ . \qquad (2.5\text{-}7)$$

Thus, biorthogonality is defined by equation (2.5-7) for $\alpha_r^2 \neq \alpha_s^2$.

2.6 Solution of the Adjoint Differential Equation

The adjoint differential equation (2.5-3) can be written as

$$\frac{d^2Z}{dy^2} + H^+Z = 0 \ , \qquad (2.6\text{-}1)$$

where the matrix H is defined with equation (2.4-4). The solution of equation (2.6-1) is obtained by the same procedure used in the solution of equation (2.4-4). Expanding equation (2.6-1) gives

$$\frac{d^2z_1}{dy^2} + \bar{h}_{11}z_1 + \bar{h}_{21}z_2 = 0 \qquad (2.6\text{-}2)$$

$$\frac{d^2z_2}{dy^2} + \bar{h}_{12}z_1 + \bar{h}_{22}z_2 = 0 \ . \qquad (2.6\text{-}3)$$

These equations can be written in the following form

$$\left(\frac{d^4}{dy^4} + (\bar{h}_{11} + \bar{h}_{22})\frac{d^2}{dy^2} + \bar{h}_{11}\bar{h}_{22} - \bar{h}_{12}\bar{h}_{21}\right)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0 \ . \qquad (2.6\text{-}4)$$

The roots of the characteristic equation obtained from equation (2.6-4) are:

$$r_{1,2} = \pm\sqrt{\bar{\alpha}^2 - \frac{\rho\omega^2}{E}(1-\nu^2)} = \pm\,\overline{\alpha m} \qquad (2.6\text{-}5a)$$

$$r_{3,4} = \pm\sqrt{\bar{\alpha}^2 - \frac{2\rho\omega^2}{E}(1+\nu)} = \pm\,\overline{\alpha n}\ , \tag{2.6-5b}$$

where

$$\bar{m} = \frac{1}{\bar{\alpha}}\sqrt{\bar{\alpha}^2 - \frac{\rho\omega^2}{E}(1-\nu^2)} \tag{2.6-6}$$

$$\bar{n} = \frac{1}{\bar{\alpha}}\sqrt{\bar{\alpha}^2 - \frac{2\rho\omega^2}{E}(1+\nu)}\ . \tag{2.6-7}$$

The complex conjugate of the adjoint functions representing the solution of the adjoint equation are:

$$\bar{z}_1 = B_1\sinh\alpha my + B_2\cosh\alpha my + B_3\sinh\alpha ny + B_4\cosh\alpha ny$$

$$\bar{z}_2 = B_5\sinh\alpha my + B_6\cosh\alpha my + B_7\sinh\alpha ny + B_8\cosh\alpha ny\ .$$

Substitution of these functions into either equation (2.6-2) or (2.6-3) defines four of the constants.

$$\bar{z}_1 = B_1\sinh\alpha my + B_2\cosh\alpha my - KB_3\sinh\alpha ny - KB_4\cosh\alpha ny \tag{2.6-8}$$

$$\bar{z}_2 = B_1\sinh\alpha my + B_2\cosh\alpha my + B_3\sinh\alpha ny + B_4\cosh\alpha ny \tag{2.6-9}$$

<u>Satisfaction of the Adjoint Problem Boundary Conditions</u>.

In defining orthogonality in Section 2.5, the following two boundary conditions were imposed on the adjoint problem:

(a) $$\frac{d\bar{z}_1(\pm 1)}{dy} = 0$$

(b) $$\bar{z}_2(\pm 1) = \alpha^2\frac{E}{\rho\omega^2}\bar{z}_1(\pm 1) - \nu\bar{z}_1(\pm 1)\ .$$

Substitution of equations (2.6-8) and (2.6-9) into (a) and (b) and separating into even and odd functions of y yields the following two transcendental equations which are identical to those defining the eigenvalue α for the original governing equation (2.4-4). For the adjoint functions odd in y,

$$\frac{\tanh \alpha m}{\tanh \alpha n} = \frac{4mn}{(1+n^2)^2} .$$

For the adjoint functions even in y,

$$\frac{\tanh \alpha n}{\tanh \alpha m} = \frac{4mn}{(1+n^2)^2} .$$

Similar to the eigenfunctions, the adjoint functions divide into even and odd functions of the form:

$$\begin{aligned}
{}_o\bar{z}_1 &= B_1 \left(\sinh \alpha my - \frac{m}{n}\frac{\cosh \alpha m}{\cosh \alpha n} \sinh \alpha ny\right) \\
{}_o\bar{z}_2 &= B_1 \left(\sinh \alpha my + \frac{m}{Kn}\frac{\cosh \alpha m}{\cosh \alpha n} \sinh \alpha ny\right) \\
{}_e\bar{z}_1 &= B_2 \left(\cosh \alpha my - \frac{m}{n}\frac{\sinh \alpha m}{\sinh \alpha n} \cosh \alpha ny\right) \\
{}_e\bar{z}_2 &= B_2 \left(\cosh \alpha my + \frac{m}{Kn}\frac{\sinh \alpha m}{\sinh \alpha n} \cosh \alpha ny\right)
\end{aligned} \qquad (2.6\text{-}10)$$

2.7 Summary

The y-dependence of the solution is carried by the eigenfunctions (2.4-21) of which there are an infinite number corresponding to the roots, α, of the transcendental equations. The correct solution of the governing differential equations requires the summation of the distinct solutions. Separating the solution into even and odd functions of y, this sum may be expressed as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} = \sum_{r=1}^{\infty} C_{1r} \begin{bmatrix} \sinh \alpha my + \dfrac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny \\ K \sinh \alpha my - \dfrac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny \end{bmatrix}_{(r)} e^{i({}_o\alpha_r x + \omega t)}$$

$$+ \sum_{r=1}^{\infty} C_{2r} \begin{bmatrix} \cosh \alpha my + \dfrac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny \\ K \cosh \alpha my - \dfrac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny \end{bmatrix}_{(r)} e^{i({}_e\alpha_r x + \omega t)} . \qquad (2.7\text{-}1)$$

The constants C_{1r} and C_{2r} are obtained formally by the specification of σ_{xx} and σ_{yy} on the surface $x = 0$ and the application of the biorthogonality operator (2.5-7). This solution and the solution developed in Chapter III will be combined in Chapter IV in a formulation which allows specification of pure stress boundary conditions, σ_{xx} and σ_{xy}, on the surface $x = 0$.

CHAPTER III

FORMULATION OF PROBLEM FOR SECOND MIXED BOUNDARY CASE

3.1 General Remarks

In this chapter, the equations given in Section 2.1 are used to derive governing equations for specifications of σ_{xy} and the displacement u_x on the finite surface $x = 0$. To permit formulation of the problem in this manner, a new variable Q is introduced, and the governing equations involving only the stress σ_{xy} and the "stress-like" variable Q are obtained. It will be shown in Section 3.3 that the specification of σ_{xy} and Q on the surface $x = 0$ is equivalent to specification of σ_{xy} and u_x up to a rigid body displacement.

As in Chapter II, the solutions of the second mixed case developed in this chapter will not be used to solve a particular mixed value problem, but will be combined with those obtained in Chapter II for specification of the stresses σ_{xx} and σ_{xy} on the surface $x = 0$.

3.2 Boundary Conditions

For the second mixed case, the boundary conditions specified on the finite boundary $x = 0$ are

$$\sigma_{xy} = \sigma_{xyb}(y) \cos \omega t \tag{3.2-1}$$

$$u_x = u_{xb}(y) \cos \omega t \tag{3.2-2}$$

It will be shown in Section 3.3 that condition (3.2-2) is equivalent to specification of (3.2-1) and Q on the surface $x = 0$.

$$Q = Q_b(y) \cos \omega t \tag{3.2-3}$$

The infinite boundaries, $y = \pm 1$, are stress free,

$$\sigma_{yy}(x, \pm 1) = \sigma_{xy}(x, \pm 1) = 0 , \tag{3.2-4}$$

and the solution is required to be bounded as x approaches infinity.

3.3 The Auxiliary Variable, Q

To construct equations which do not involve σ_{xx} and σ_{yy}, the variable Q is introduced and is defined

$$\frac{\partial Q}{\partial x} = \frac{\partial \sigma_{xx}}{\partial y} - \nu\rho \frac{\partial^2 u_y}{\partial t^2} \tag{3.3-1}$$

with the requirement that it be bounded as dimension x approaches infinity.

Differentiating equation (2.1-3) with respect to y and substituting equation (2.1-2) yields

$$\frac{\partial^2 u_x}{\partial x \partial y} = \frac{1}{E}\left[\frac{\partial \sigma_{xx}}{\partial y} - \nu\left(\rho \frac{\partial^2 u_y}{\partial t^2} - \frac{\partial \sigma_{xy}}{\partial x}\right)\right] . \tag{3.3-2}$$

Substituting equation (3.3-1) into equation (3.3-2) gives

$$\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial y}\right) = \frac{1}{E}\left(\frac{\partial Q}{\partial x} + \nu \frac{\partial \sigma_{xy}}{\partial x}\right) .$$

Integrating with respect to x gives

$$\frac{\partial u_x}{\partial y} = \frac{1}{E} (Q + \nu\sigma_{xy}) + f(y) ,$$

and substituting equation (2.1-5) yields

$$\frac{\partial u_y}{\partial x} = \frac{2(1+\nu)}{E} \sigma_{xy} - \frac{1}{E} (Q + \nu\sigma_{xy}) - f(y) . \qquad (3.3\text{-}3)$$

If u_y is bounded as x approaches infinity, $f(y)$ may be taken to be zero. Equation (3.3-3) may be written

$$\frac{\partial u_x}{\partial y} = \frac{1}{E} (Q + \nu\sigma_{xy}) . \qquad (3.3\text{-}4)$$

Specification of Q and σ_{xy} on the surface $x = 0$ is equivalent to specification of σ_{xy} and u_x up to a rigid body displacement.

3.4 Equations for the Second Mixed Case

Differentiating equation (3.3-3) with respect to y and substituting equation (2.1-4) into the result gives

$$\frac{\partial Q}{\partial y} + \frac{\partial \sigma_{yy}}{\partial x} - \nu \frac{\partial \sigma_{xx}}{\partial x} - (2+\nu) \frac{\partial \sigma_{xy}}{\partial y} = 0 .$$

Differentiating again with respect to y and using equations (2.1-1) and (2.1-2) gives

$$\frac{\partial^2 Q}{\partial y^2} - \frac{\partial^2 \sigma_{xy}}{\partial x^2} - 2 \frac{\partial^2 \sigma_{xy}}{\partial y^2} + \rho \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_y}{\partial x} - \nu \frac{\partial u_x}{\partial y}\right) = 0 .$$

Substituting equations (2.1-5)

$$\frac{\partial^2 Q}{\partial y^2} - \frac{\partial^2 \sigma_{xy}}{\partial x^2} - 2 \frac{\partial^2 \sigma_{xy}}{\partial y^2} + \rho \frac{\partial^2}{\partial t^2} \frac{\partial u_y}{\partial x} - \nu\rho \frac{\partial^2}{\partial t^2} \frac{2(1+\nu)}{E} \sigma_{xy} + \nu\rho \frac{\partial^2}{\partial t^2} \frac{\partial u_y}{\partial x} = 0$$

and (3.3-1) yields

$$\frac{\partial^2 Q}{\partial y^2} - \frac{\partial^2 \sigma_{xy}}{\partial x^2} - 2\,\frac{\partial^2 \sigma_{xy}}{\partial y^2} + \rho\,\frac{\partial^2}{\partial t^2}\,\frac{\partial u_y}{\partial x} - \nu\rho\,\frac{\partial^2}{\partial t^2}\,\frac{2(1+\nu)}{E}\,\sigma_{xy}$$
$$+ \frac{\partial^2 \sigma_{xx}}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x^2} = 0 \ .$$

Using equations (2.1-1) and (2.1-5) yields the first equation for σ_{xy} and Q.

$$\frac{\partial^2 \sigma_{xy}}{\partial x^2} + 3\,\frac{\partial^2 \sigma_{xy}}{\partial y^2} + \frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial y^2} = \frac{2(1-\nu^2)}{E}\,\rho\,\frac{\partial^2 \sigma_{xy}}{\partial t^2} \qquad (3.4\text{-}1)$$

The second equation for σ_{xy} and Q is obtained by adding the derivative of equation (3.3-1) with respect to x to the derivative of equation (2.1-1) with respect to y and substituting equations (2.1-5) and (3.3-4) into the resulting equation.

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial y^2} = \frac{\rho(1+\nu)}{E}\,\frac{\partial^2}{\partial t^2}\,(Q - \nu\sigma_{xy}) \qquad (3.4\text{-}2)$$

Equations (3.4-1) and (3.4-2) may be written in matrix form

$$D\,\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = M\,\frac{\partial^2 T}{\partial t^2} \ , \qquad (3.4\text{-}3)$$

where

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} , \qquad T = \begin{bmatrix} \sigma_{xy} \\ Q \end{bmatrix}$$

$$M = \frac{(1+\nu)\rho}{E} \begin{bmatrix} -\nu & 1 \\ -(2+\nu) & 3 \end{bmatrix} .$$

This matrix equation is the desired second mixed boundary case formulation.

3.5 Solution of the Equations for Second Mixed Case

The boundary conditions (3.2-1) and (3.2-3) are time-harmonic of frequency ω. A time-harmonic solution form is assumed

$$T = W(y)e^{i\alpha x}e^{i\omega t} \quad , \tag{3.5-1}$$

where

$$W(y) = \begin{bmatrix} w_1(y) \\ w_2(y) \end{bmatrix} \quad .$$

Substituting this assumed form of solution into equation (3.4-3) gives

$$\frac{d^2W}{dy^2} + (\omega^2 M - \alpha^2 D)W = 0 \quad , \tag{3.5-2}$$

or

$$\frac{d^2W}{dy^2} + GW = 0 \quad , \tag{3.5-3}$$

where

$$G = \begin{bmatrix} -\frac{\omega^2\rho}{E}(\nu + \nu^2) & \frac{\omega^2\rho}{E}(1+\nu) - \alpha^2 \\ -\frac{\omega^2\rho}{E}(2 + 3\nu + \nu^2) + \alpha^2 & \frac{\omega^2\rho}{E}(3 + 3\nu) - 2\alpha^2 \end{bmatrix} \quad .$$

Expanding equation (3.5-3) yields

$$\frac{d^2w_1}{dy^2} + g_{11}w_1 + g_{12}w_2 = 0 \tag{3.5-4}$$

$$\frac{d^2w_2}{dy^2} + g_{21}w_1 + g_{22}w_2 = 0 \quad . \tag{3.5-5}$$

These two equations may be written

$$\left(\frac{d^4}{dy^4} + (g_{11} + g_{22})\frac{d^2}{dy^2} + g_{11}g_{22} - g_{12}g_{21}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \ . \qquad (3.5\text{-}6)$$

The roots of the characteristic equation obtained from equation (3.5-6) are identical to the roots of the first mixed problem of Chapter II:

$$r_{1,2} = \pm\sqrt{\alpha^2 - \frac{\rho\omega^2}{E}(1 - \nu^2)} = \pm\,\alpha m$$

$$r_{3,4} = \pm\sqrt{\alpha^2 - \frac{2\rho\omega^2}{E}(1 + \nu)} = \pm\,\alpha n$$

The solution of equation (3.5-3) has the following form:

$$w_1 = C_1 \sinh \alpha m y + C_2 \cosh \alpha m y + C_3 \sinh \alpha n y + C_4 \cosh \alpha n y$$

$$w_2 = C_5 \sinh \alpha m y + C_6 \cosh \alpha m y + C_7 \sinh \alpha n y + C_8 \cosh \alpha n y \quad .$$

Substituting these expressions for w_1 and w_2 into either of the equations (3.5-4) or (3.5-5) defines four of the constants yielding:

$$w_1 = C_1 \sinh \alpha m y + C_2 \cosh \alpha m y + C_3 \sinh \alpha n y + C_4 \cosh \alpha n y \qquad (3.5\text{-}7)$$

$$w_2 = C_1 \sinh \alpha m y + C_2 \cosh \alpha m y + RC_3 \sinh \alpha n y + RC_4 \cosh \alpha n y \ , \qquad (3.5\text{-}8)$$

where

$$R = \frac{\nu n^2 + 2n^2 - \nu}{1 + n^2} \quad .$$

The remaining constants are determined by specification of the boundary conditions on the surfaces $y = \pm 1$ and $x = 0$.

<u>Satisfaction of the boundary conditions on</u> $y = \pm 1$.

(a) On the surfaces $y = \pm 1$, the shear stress σ_{xy} is zero for all x and t. This implies, from equation (3.5-1), that $w_1(\pm 1) = 0$. Thus, from equation (3.5-7):

$$C_1 \sinh \alpha m + C_3 \sinh \alpha n = 0 \tag{3.5-9}$$

$$C_2 \cosh \alpha m + C_4 \cosh \alpha n = 0 \; . \tag{3.5-10}$$

(b) On the surfaces $y = \pm 1$, the normal stress σ_{yy} is zero for all x and t. Because the equations do not explicitly contain σ_{yy}, a relation between σ_{yy}, Q, and σ_{xy} must be constructed. Differentiating equation (3.3-3) with respect to y and substituting into equation (2.1-4) gives

$$\frac{\partial Q}{\partial y} + \frac{\partial \sigma_{yy}}{\partial x} - \nu \frac{\partial \sigma_{xx}}{\partial x} - (2+\nu)\frac{\partial \sigma_{xy}}{\partial y} = 0 \quad . \tag{3.5-11}$$

Differentiating equation (2.1-1) with respect to x and substituting from equation (2.1-3) yields

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \frac{\rho}{E} \frac{\partial^2}{\partial t^2} (\sigma_{xx} - \nu \sigma_{yy}) \; . \tag{3.5-12}$$

Multiplying the derivative of equation (3.5-12) with respect to x by Poisson's ratio, ν, and substituting equation (3.5-11) gives

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\rho}{E}(1-\nu^2)\frac{\partial^2}{\partial t^2}\right]\frac{\partial \sigma_{yy}}{\partial x} + \left[\frac{\partial^2}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2}{\partial t^2}\right]\frac{\partial Q}{\partial y}$$
$$+ \left[-2\frac{\partial^2}{\partial x^2} + \frac{\rho}{E}(2+\nu)\frac{\partial^2}{\partial t^2}\right]\frac{\partial \sigma_{xy}}{\partial y} = 0 \; . \tag{3.5-13}$$

If σ_{yy} is constant on the surfaces $y = \pm 1$, then $\frac{\partial \sigma_{yy}}{\partial x} = 0$ and equation (3.5-13) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2}{\partial t^2}\right]\frac{\partial Q}{\partial y} + \left[-2\frac{\partial^2}{\partial x^2} + \frac{\rho}{E}(2+\nu)\frac{\partial^2}{\partial t^2}\right]\frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \text{for} \quad y = \pm 1.$$

Using the assumed solution form from equation (3.5-1) gives

$$\left[-\alpha^2 + \frac{\rho\omega^2}{E}\right] \frac{dw_2(\pm 1)}{dy} + \left[2\alpha^2 - \frac{\rho}{E}(2+\nu)\omega^2\right] \frac{dw_1(\pm 1)}{dy} = 0 \; . \qquad (3.5\text{-}14)$$

Substituting equations (3.5-7) and (3.5-8) into equation (3.5-14) and separating into even and odd functions of y yields

$$C_3\alpha n\left[\left(-\alpha^2 + \frac{\rho\omega^2}{E}\right)R + 2\alpha^2 - (2+\nu)\frac{\rho\omega^2}{E}\right]\cosh \alpha n$$
$$+ C_1\alpha m\left[\alpha^2 - (1+\nu)\frac{\rho\omega^2}{E}\right]\cosh \alpha m = 0 \qquad (3.5\text{-}15)$$

and

$$C_4\alpha n\left[\left(-\alpha^2 + \frac{\rho\omega^2}{E}\right)R + 2\alpha^2 - (2+\nu)\frac{\rho\omega^2}{E}\right]\sinh \alpha n$$
$$+ C_2\alpha m\left[\alpha^2 - (1+\nu)\frac{\rho\omega^2}{E}\right]\sinh \alpha m = 0 \; . \qquad (3.5\text{-}16)$$

Solution of equations (3.5-9) and (3.5-15) yields

$$\frac{\tanh \alpha n}{\tanh \alpha m} = \frac{4mn}{(1+n^2)^2} \; . \qquad (3.5\text{-}17)$$

Solution of equations (3.5-10) and (3.5-16) yields

$$\frac{\tanh \alpha m}{\tanh \alpha n} = \frac{4mn}{(1+n^2)^2} \; . \qquad (3.5\text{-}18)$$

The transcendental equation (3.5-17) is used to define the eigenvalue α when the eigenfunctions w_1 and w_2 are odd in y. Note that this equation is identical to equation (2.4-20) which defines the eigenvalue α for the first mixed problem when the eigenfunctions v_1 and v_2 are even in y. The transcendental equation (3.5-18) defines α when the eigenfunctions w_1 and w_2 are even in y. Likewise, this equation defines the eigenvalue for odd functions of v_1 and v_2. This relation is equivalent to the fact that if σ_{xx} and σ_{yy} are even functions of a given variable, σ_{xy} is odd in that variable.

It is desirable at this point to define as the even problem the one which has σ_{xx} and σ_{yy} in terms of functions even in the y - variable and σ_{xy} in terms of functions odd in the y - variable. The odd problem is defined as the one with σ_{xx} and σ_{yy} odd in the y - variable and σ_{xy} even in the y - variable. The presubscripts "o" and "e" used to designate odd and even will now be used to identify quantities related to the odd and even problems respectively. Thus, for example, for the odd problem the presubscript "o" is applied to the eigenfunctions of the first mixed case which are odd in the y - variable, and to the eigenfunctions of the second mixed case which are even in the y - variable.

The eigenfunctions of the second mixed case divide into sets for the even and odd problems of the form:

$$
\begin{aligned}
{}_e w_1 &= C_1 \left(\sinh \alpha m y - \frac{\sinh \alpha m}{\sinh \alpha n} \sinh \alpha n y\right) \\
{}_e w_2 &= C_1 \left(\sinh \alpha m y - \frac{R \sinh \alpha m}{\sinh \alpha n} \sinh \alpha n y\right) \\
{}_o w_1 &= C_2 \left(\cosh \alpha m y - \frac{\cosh \alpha m}{\cosh \alpha n} \cosh \alpha n y\right) \\
{}_o w_2 &= C_2 \left(\cosh \alpha m y - \frac{R \cosh \alpha m}{\cosh \alpha n} \cosh \alpha n y\right)
\end{aligned}
\tag{3.5-19}
$$

The constants C_1 and C_2 are determined from the boundary conditions on the surface $x = 0$.

3.6 Definition of the Biorthogonal Operator and the Adjoint Equation

Consider W_r as the solution vector of the differential equation (3.5-2) associated with the r^{th} eigenvalue α_r.

$$\frac{d^2W_r}{dy^2} + (\omega^2 M - \alpha_r^2 D)W_r = 0 \tag{3.6-1}$$

Premultiplying equation (3.6-1) by the complex conjugate transpose of some arbitrary vector function of y, U_s, and integrating from $y = -1$ to $y = 1$ yields

$$\int_{-1}^{1} U_s^+ \frac{d^2W_r}{dy^2}\, dy = \int_{-1}^{1} U_s^+(\alpha_r^2 D - \omega^2 M)W_r dy \ , \qquad U_s = \begin{bmatrix} u_{s1} \\ u_{s2} \end{bmatrix} \ .$$

Integrating the left side by parts gives

$$\left. \left(U_s^+ \frac{dW_r}{dy} - \frac{dU_s^+}{dy} W_r \right) \right|_{-1}^{1} + \int_{-1}^{1} \frac{d^2U_s^+}{dy^2} W_r dy = \int_{-1}^{1} U_s^+(\alpha_r^2 D - \omega^2 M)W_r dy \ . \tag{3.6-2}$$

Consider the vector function U_s as being the s^{th} solution vector of the following differential equation, termed the adjoint equation

$$\frac{d^2U_s}{dy^2} = (\bar{\alpha}_s^2 D^+ - \omega^2 M^+)U_s \ , \tag{3.6-3}$$

where, as will be shown later, the eigenvalue α_s is determined from the same transcendental equation obtained for the equation (3.6-1).

Premultiplying equation (3.6-3) by W_r^+ and integrating from $y = -1$ to $y = 1$ gives

$$\int_{-1}^{1} W_r^+ \frac{d^2U_s}{dy^2}\, dy = \int_{-1}^{1} W_r^+(\bar{\alpha}_s^2 D^+ - \omega^2 M^+)U_s dy \ .$$

Taking the complex conjugate transpose of this equation,

$$\int_{-1}^{1} \frac{d^2U_s^+}{dy^2} W_r dy = \int_{-1}^{1} U_s^+(\alpha_s^2 D - \omega^2 M)W_r dy \ ,$$

and subtracting it from equation (3.6-2) yields

$$\left(U_s^+ \frac{dW_r}{dy} - \frac{dU_s^+}{dy} W_r \right) \Bigg|_{-1}^{1} = (\alpha_r^2 - \alpha_s^2) \int_{-1}^{1} U_s^+ DW_r \, dy \; . \qquad (3.6\text{-}4)$$

Expanding the left side of equation (3.6-4) yields

$$\left(\bar{u}_{s1} \frac{dw_{r1}}{dy} + \bar{u}_{s2} \frac{dw_{r2}}{dy} - \frac{d\bar{u}_{s1}}{dy} w_{r1} - \frac{d\bar{u}_{s2}}{dy} w_{r2} \right) \Bigg|_{-1}^{1} =$$

$$(\alpha_r^2 - \alpha_s^2) \int_{-1}^{1} U_s^+ DW_r \, dy \; . \qquad (3.6\text{-}5)$$

The boundary conditions on the surface $y = \pm 1$ may be written in the form:

(a) $w_{r1}(\pm 1) = 0$

(b) $\left[-\alpha_r^2 + \frac{\rho\omega^2}{E} \right] \frac{dw_{r2}(\pm 1)}{dy} + \left[2\alpha_r^2 - \frac{\rho}{E} (2+\nu)\omega^2 \right] \frac{dw_{r1}(\pm 1)}{dy} = 0 \; ,$

where it is noted that eigenvalue α_r appears in the second boundary condition. Substituting these values into equation (3.6-5) gives

$$\left[\bar{u}_{s1} \frac{dw_{r1}}{dy} - \bar{u}_{s2} \left(\frac{2\alpha_r^2 - \frac{\rho}{E}(2+\nu)\omega^2}{-\alpha_r^2 + \frac{\rho\omega^2}{E}} \right) \frac{dw_{r1}}{dy} - \frac{d\bar{u}_{s2}}{dy} w_{r2} \right]_{-1}^{1} =$$

$$(\alpha_r^2 - \alpha_s^2) \int_{-1}^{1} U_s^+ D \, W_r \, dy \; . \qquad (3.6\text{-}6)$$

The following boundary conditions are imposed on the adjoint functions:

(a) $\frac{d\bar{u}_{s2}(\pm 1)}{dy} = 0$

(b) $\bar{u}_{s1}(\pm 1) = \frac{2\alpha_s^2 - \frac{\rho}{E}(2+\nu)\omega^2}{-\alpha_s^2 + \frac{\rho\omega^2}{E}} \; \bar{u}_{s2}(\pm 1) \; .$

Substituting these boundary conditions into equation (3.6-6) gives

$$\left[\frac{2\alpha_s^2 - \frac{\rho}{E}(2+\nu)\omega^2}{-\alpha_s^2 + \frac{\rho\omega^2}{E}} \bar{u}_{s2} \frac{dw_{r1}}{dy} - \frac{2\alpha_r^2 - \frac{\rho}{E}(2+\nu)\omega^2}{-\alpha_r^2 + \frac{\rho}{E}\omega^2} \bar{u}_{s2} \frac{dw_{r1}}{dy} \right]_{-1}^{1} =$$

$$(\alpha_r^2 - \alpha_s^2)\int_{-1}^{1} U_s^+ DW_r dy \ ,$$

or

$$(\alpha_r^2 - \alpha_s^2)\left[\int_{-1}^{1} U_s^+ DW_r dy - \frac{\frac{\nu\rho\omega^2}{E}}{(-\alpha_s^2 + \frac{\rho\omega^2}{E})(-\alpha_r^2 + \frac{\rho\omega^2}{E})} \bar{u}_{s2} \frac{dw_{r1}}{dy} \Big|_{-1}^{1} \right] = 0 \ . \tag{3.6-7}$$

Biorthogonality is defined by (3.6-7), but it is not in a usable form because the eigenvalue α_r appears in the operator. Multiplying equation (3.6-7) by $(-\alpha_r^2 + \frac{\rho\omega^2}{E})$

$$(\alpha_r^2 - \alpha_s^2)\left[\int_{-1}^{1} U_s^+ (-\alpha_r^2 + \frac{\rho\omega^2}{E}) DW_r dy - \frac{\frac{\nu\rho\omega^2}{E}}{-\alpha_s^2 + \frac{\rho\omega^2}{E}} \bar{u}_{s2} \frac{dw_{r1}}{dy} \Big|_{-1}^{1} \right] = 0 \ ,$$

and recalling from equation (3.6-1)

$$\alpha_r^2 DW_r = \frac{d^2 W_r}{dy^2} + \omega^2 MW_r \ ,$$

the term under the integral sign can be written in the following form

$$U_s^+(-\alpha_r^2 + \frac{\rho\omega^2}{E})DW_r = \frac{\rho\omega^2}{E} U_s^+ DW_r - U_s^+ \frac{d^2 W_r}{dy^2} - \omega^2 U_s^+ MW_r \ .$$

After these substitutions, the biorthogonality can be defined as

$$\int_{-1}^{1} \left(U_s^+ DW_r - \frac{E}{\rho\omega^2} U_s^+ \frac{d^2 W_r}{dy^2} - \frac{E}{\rho} U_s^+ MW_r \right) dy$$

$$- \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2} \bar{u}_{s2} \frac{dw_{r1}}{dy} \Big|_{-1}^{1} = 0, \ \alpha_r^2 \neq \alpha_s^2 \ . \tag{3.6-9}$$

3.7 Solution of the Adjoint Differential Equation

The adjoint differential equation (3.6-3) can be written as

$$\frac{d^2U}{dy^2} + G^+U = 0 \ , \tag{3.7-1}$$

where the matrix G is defined with equation (3.5-3). The solution of equation (3.7-1) is obtained by the same procedure used in the solution of equation (3.5-3). Expanding equation (3.7-1) gives

$$\frac{d^2u_1}{dy^2} + \bar{g}_{11}u_1 + \bar{g}_{21}u_2 = 0 \tag{3.7-2}$$

$$\frac{d^2u_2}{dy^2} + \bar{g}_{12}u_1 + \bar{g}_{22}u_2 = 0 \ . \tag{3.7-3}$$

These equations can be written in the following form

$$\left(\frac{d^4}{dy^4} + (\bar{g}_{11} + \bar{g}_{22})\frac{d^2}{dy^2} + \bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}\bar{g}_{21}\right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \ .$$

The roots of the characteristic equation are identical to those obtained for the adjoint equation of the first mixed problem:

$$r_{1,2} = \pm\sqrt{\bar{\alpha}^2 - \frac{\rho\omega^2}{E}(1-\nu^2)} = \pm\,\overline{\alpha m}$$

$$r_{3,4} = \pm\sqrt{\bar{\alpha}^2 - \frac{2\rho\omega^2}{E}(1+\nu)} = \pm\,\overline{\alpha n} \ .$$

The complex conjugate of the adjoint functions representing the solution of the adjoint equation are:

$$\bar{u}_1 = B_1\sinh\alpha my + B_2\cosh\alpha my + B_3\sinh\alpha ny + B_4\cosh\alpha ny$$

$$\bar{u}_2 = B_5\sinh\alpha my + B_6\cosh\alpha my + B_7\sinh\alpha ny + B_8\cosh\alpha ny \ .$$

Substitution of these functions into either equation (3.7-2) or (3.7-3) defines four of the constants, thus:

$$\bar{u}_1 = -RB_1 \sinh \alpha my - RB_2 \cosh \alpha my - B_3 \sinh \alpha ny - B_4 \cosh \alpha ny \qquad (3.7\text{-}4)$$

$$\bar{u}_2 = B_1 \sinh \alpha my + B_2 \cosh \alpha my + B_3 \sinh \alpha ny + B_4 \cosh \alpha ny \quad . \qquad (3.7\text{-}5)$$

Satisfaction of the Adjoint Problem Boundary Conditions.

In developing the biorthogonality operator in Section 3.6, the following two boundary conditions were imposed on the adjoint functions:

(a) $$\frac{d\bar{u}_2(\pm 1)}{dy} = 0$$

(b) $$(-\alpha^2 + \frac{\rho\omega^2}{E})\bar{u}_1(\pm 1) = [2\alpha^2 - \frac{\rho}{E}(2+\nu)\omega^2]\bar{u}_2(\pm 1) \quad .$$

Substitution of equations (3.7-4) and (3.7-5) into (a) and (b) and separating into even and odd functions yields the following two transcendental equations which are identical to those defining the eigenvalue α for the original governing equation (3.5-3). For the even problem, the adjoint functions are odd in the y-variable, and the corresponding eigenvalues are the roots of

$$\frac{\tanh \alpha n}{\tanh \alpha m} = \frac{4mn}{(1+n^2)^2} \quad .$$

For the odd problem, the adjoint functions are even in the y-variable, and the eigenvalues are the roots of

$$\frac{\tanh \alpha m}{\tanh \alpha n} = \frac{4mn}{(1+n^2)^2} \quad .$$

The adjoint functions divide into sets corresponding to the even and odd problem in a similar manner as the eigenfunctions. They are of the form:

$$
\begin{aligned}
{}_e\bar{u}_1 &= B_1 \left(-R \sinh \alpha my + \frac{m}{n} \frac{\cosh \alpha m}{\cosh \alpha n} \sinh \alpha ny\right) \\
{}_e\bar{u}_2 &= B_1 \left(\sinh \alpha my - \frac{m}{n} \frac{\cosh \alpha m}{\cosh \alpha n} \sinh \alpha ny\right) \\
{}_o\bar{u}_1 &= B_2 \left(-R \cosh \alpha my + \frac{m}{n} \frac{\sinh \alpha m}{\sinh \alpha n} \cosh \alpha ny\right) \\
{}_o\bar{u}_2 &= B_2 \left(\cosh \alpha my - \frac{m}{n} \frac{\sinh \alpha m}{\sinh \alpha n} \cosh \alpha ny\right) .
\end{aligned}
\tag{3.7-6}
$$

3.8 Summary

The y-dependence of the solution is carried by the eigenfunctions (3.5-19) of which there are an infinite number corresponding to the roots, α, of the transcendental equations (3.5-17) and (3.5-18). The solution of the differential equation requires the summation of the distinct solutions. Separating the solution into even and odd functions of y, this sum may be expressed as

$$
\begin{bmatrix} \sigma_{xy} \\ Q \end{bmatrix} = \sum_{r=1}^{\infty} C_{1r} \begin{bmatrix} \cosh \alpha my - \dfrac{\cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny \\ \cosh \alpha my - \dfrac{R \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny \end{bmatrix}_{(r)} e^{i({}_o\alpha_r x + \omega t)}
$$

$$
+ \sum_{r=1}^{\infty} C_{2r} \begin{bmatrix} \sinh \alpha my - \dfrac{\sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny \\ \sinh \alpha my - \dfrac{R \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny \end{bmatrix}_{(r)} e^{i({}_e\alpha_r x + \omega t)} . \tag{3.8-1}
$$

The arbitrary constants C_{1r} and C_{2r} are obtained formally by the specification of σ_{xy} and Q on the finite edge $x = 0$ and the application of the biorthogonality operator (3.6-9). This solution will now be combined with the first mixed case to form a general solution for any acceptable boundary conditions. As indicated earlier, the stress boundary problem is the one of most importance.

CHAPTER IV

FOUR-VECTOR FORMULATION FOR THE PURE STRESS CASE

4.1 The Odd Problem Four-Vector Formulation

In Chapters II and III, eigenfunction expansions were developed for σ_{xx} and σ_{yy}, and for σ_{xy} and Q, respectively. These solutions are given by equations (2.7-1) and (3.8-1), and the functions in the odd problem have the following form:

$$
\begin{aligned}
\sigma_{xx} &= \sum_{r=1}^{\infty} C_{1r}\left(\sinh \alpha my + \frac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right)_{(r)} e^{i(\alpha_{o}\alpha_{r}x + \omega t)} \\
\sigma_{yy} &= \sum_{r=1}^{\infty} C_{1r}\left(K \sinh \alpha my - \frac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right)_{(r)} e^{i(\alpha_{o}\alpha_{r}x + \omega t)} \\
\sigma_{xy} &= \sum_{r=1}^{\infty} C_{2r}\left(\cosh \alpha my - \frac{\cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny\right)_{(r)} e^{i(\alpha_{o}\alpha_{r}x + \omega t)} \\
Q &= \sum_{r=1}^{\infty} C_{2r}\left(\cosh \alpha my - \frac{R \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny\right)_{(r)} e^{i(\alpha_{o}\alpha_{r}x + \omega t)} \quad .
\end{aligned}
\qquad (4.1\text{-}1)
$$

In this chapter, the four solutions are combined to form a four-vector $[\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, Q]$. Substituting the r^{th} term of the solutions (4.1-1) into the coupled differential equation (3.5-11)

$$
\frac{\partial Q}{\partial y} + \frac{\partial \sigma_{yy}}{\partial x} - \nu \frac{\partial \sigma_{xx}}{\partial x} - (2 + \nu) \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad (3.5\text{-}11) \text{ repeated}
$$

yields the following relation between the constants C_{1r} and C_{2r}

$$C_{2r} = \left(\frac{i\,2m}{\nu n^2 - \nu - 2}\right)_{(r)} C_{1r} \quad .$$

The desired odd eigenfunction expansion of the four-vector becomes

$$_o\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ Q \end{bmatrix} = \sum_{r=1}^{\infty} {_oC_r} \begin{bmatrix} _o\eta_1 \\ _o\eta_2 \\ _o\eta_3 \\ _o\eta_4 \end{bmatrix}_{(r)} e^{i({_o\alpha_r}x + \omega t)} \quad , \qquad (4.1\text{-}2)$$

where

$$_o\eta_1 = \sinh \alpha m y + \frac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha n y$$

$$_o\eta_2 = K \sinh \alpha m y - \frac{K \sinh \alpha m}{\sinh \alpha n} \sinh \alpha n y$$

$$_o\eta_3 = \frac{i\,2m}{\nu n^2 - \nu - 2} \left(\cosh \alpha m y - \frac{\cosh \alpha m}{\cosh \alpha n} \cosh \alpha n y\right)$$

$$_o\eta_4 = \frac{i\,2m}{\nu n^2 - \nu - 2} \left(\cosh \alpha m y - \frac{R \cosh \alpha m}{\cosh \alpha n} \cosh \alpha n y\right)$$

$$K = \frac{1 + n^2}{\nu n^2 - \nu - 2} \quad , \quad R = \frac{\nu n^2 + 2n^2 - \nu}{1 + n^2}$$

α are the roots of $\dfrac{\tanh \alpha m}{\tanh \alpha n} = \dfrac{4mn}{(1+n^2)^2}$.

4.2 The Even Problem Four-Vector Formulation

The even problem formulation involves the following variable forms:

$$\sigma_{xx} = \sum_{r=1}^{\infty} C_{3r} \left(\cosh \alpha m y + \frac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha n y\right)_{(r)} e^{i({_e\alpha_r}x + \omega t)}$$

$$\sigma_{yy} = \sum_{r=1}^{\infty} C_{3r} \left(K \cosh \alpha m y - \frac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha n y\right)_{(r)} e^{i({_e\alpha_r}x + \omega t)}$$

$$(4.2\text{-}1)$$

$$\sigma_{xy} = \sum_{r=1}^{\infty} C_{4r}\left(\sinh \alpha my - \frac{\sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right)_{(r)} e^{i({}_e\alpha_r x + \omega t)}$$

$$Q = \sum_{r=1}^{\infty} C_{4r}\left(\sinh \alpha my - \frac{R \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right)_{(r)} e^{i({}_e\alpha_r x + \omega t)} .$$

Substituting the r^{th} term of the solutions into equation (3.5-11) yields

$$C_{4r} = \left(\frac{i\ 2m}{\nu n^2 - \nu - 2}\right)_{(r)} C_{3r} .$$

The desired even problem eigenfunction expansion becomes

$$_e\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ Q \end{bmatrix} = \sum_{r=1}^{\infty} {}_eC_r \begin{bmatrix} {}_e\eta_1 \\ {}_e\eta_2 \\ {}_e\eta_3 \\ {}_e\eta_4 \end{bmatrix}_{(r)} e^{i({}_e\alpha_r x + \omega t)} , \qquad (4.2\text{-}2)$$

where

$${}_e\eta_1 = \cosh \alpha my + \frac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny$$

$${}_e\eta_2 = K \cosh \alpha my - \frac{K \cosh \alpha m}{\cosh \alpha n} \cosh \alpha ny$$

$${}_e\eta_3 = \frac{i\ 2m}{\nu n^2 - \nu - 2}\left(\sinh \alpha my - \frac{\sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right)$$

$${}_e\eta_4 = \frac{i\ 2m}{\nu n^2 - \nu - 2}\left(\sinh \alpha my - \frac{R \sinh \alpha m}{\sinh \alpha n} \sinh \alpha ny\right)$$

K and R are defined with equation (4.1-2), and α are the roots of

$$\frac{\tanh \alpha n}{\tanh \alpha m} = \frac{4mn}{(1+n^2)^2} .$$

The general solution is the sum of the expansions (4.1-2) and (4.2-2). However, every function can be replaced with the sum of an even and odd function, and the series expansions will be used separately for convenience.

4.3 Four-Vector Form of the Biorthogonality Operator

The biorthogonality for the first mixed case is given by (2.5-7).

$$\int_{-1}^{1} Z_s^+ A V_r \, dy + \frac{E}{\rho\omega^2} \bar{z}_{s1} \frac{dv_{r2}}{dy} \Bigg|_{-1}^{1} = 0 \;, \quad \alpha_r^2 \neq \alpha_s^2$$

The eigenfunctions η_{ri} of the four-vector form equal the eigenfunctions v_{ri} of the two-vector form divided by a constant, so that the biorthogonality is not violated.

$$\int_{-1}^{1} [\bar{z}_{s1}, \bar{z}_{s2}] \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_{r1} \\ \eta_{r2} \end{bmatrix} dy + \frac{E}{\rho\omega^2} \bar{z}_{s1} \frac{d\eta_{r2}}{dy} \Bigg|_{-1}^{1} = 0, \; \alpha_r^2 \neq \alpha_s^2 \tag{4.3-1}$$

The biorthogonality for the second mixed case is given by (3.6-9).

$$\int_{-1}^{1} \left(U_s^+ D W_r - \frac{E}{\rho\omega^2} U_s^+ \frac{d^2 W_r}{dy} - \frac{E}{\rho} U_s^+ M W_r \right) dy$$

$$- \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2} \bar{u}_{s2} \frac{dw_{r1}}{dy} \Bigg|_{-1}^{1} = 0, \; \alpha_s^2 \neq \alpha_r^2$$

The functions η_{ri} equal the two-vector form eigenfunctions w_{ri} multiplied by a constant, and the biorthogonality is not violated.

$$\int_{-1}^{1}\left\{[\bar{u}_{s1},\bar{u}_{s2}]\;D\begin{bmatrix}\eta_{r3}\\ \eta_{r4}\end{bmatrix} - \frac{E}{\rho\omega^2}[\bar{u}_{s1},\bar{u}_{s2}]\frac{d^2}{dy^2}\begin{bmatrix}\eta_{r3}\\ \eta_{r4}\end{bmatrix}\right.$$

$$\left. - \left(\frac{E}{\rho}\right)[\bar{u}_{s1},\bar{u}_{s2}]\;M\begin{bmatrix}\eta_{r3}\\ \eta_{r4}\end{bmatrix}\right\}dy - \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2}\,\bar{u}_{s2}\,\frac{d\eta_{r3}}{dy}\Bigg|_{-1}^{1} = 0,$$

$$\alpha_s^2 = \alpha_r^2 \tag{4.3-2}$$

Combining the biorthogonality relations (4.3-1) and (4.3-2) yields

$$\int_{-1}^{1}\left\{[\bar{z}_{s1},\bar{z}_{s2},\bar{u}_{s1},\bar{u}_{s2}]\begin{bmatrix}2 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 2\end{bmatrix}\begin{bmatrix}\eta_{r1}\\ \eta_{r2}\\ \eta_{r3}\\ \eta_{r4}\end{bmatrix}\right.$$

$$- \frac{E}{\rho\omega^2}[\bar{z}_{s1},\bar{z}_{s2},\bar{u}_{s1},\bar{u}_{s2}]\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\frac{d^2}{dy^2}\begin{bmatrix}\eta_{r1}\\ \eta_{r2}\\ \eta_{r3}\\ \eta_{r4}\end{bmatrix}$$

$$\left. - [\bar{z}_{s1},\bar{z}_{s2},\bar{u}_{s1},\bar{u}_{s2}]\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -(\nu+\nu^2) & (1+\nu)\\ 0 & 0 & -(\nu^2+3\nu+2) & (3+3\nu)\end{bmatrix}\begin{bmatrix}\eta_{r1}\\ \eta_{r2}\\ \eta_{r3}\\ \eta_{r4}\end{bmatrix}\right\}dy$$

$$+ \left[\frac{E}{\rho\omega^2}\,\bar{z}_{s1}\,\frac{d\eta_{r2}}{dy} - \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2}\,\bar{u}_{s2}\,\frac{d\eta_{r3}}{dy}\right]_{-1}^{1} = 0, \quad \alpha_s^2 \neq \alpha_r^2 . \tag{4.3-3}$$

The adjoint functions z_{si}, defined by equations (2.6-10), and u_{si}, (3.7-6), are determined up to an arbitrary constant for each set. The determination of these constants is explained in Section 5.2.

The four-vector form of the biorthogonality operator Φ can be written as

$$\Phi_s = [Z_s^+, U_s^+] \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} - [Z_s^+, U_s^+] \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix} - (\frac{E}{\rho})[Z_s^+, U_s^+] \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix}$$

$$+ [Z_s^+, U_s^+] \begin{bmatrix} J & 0 \\ 0 & P \end{bmatrix} \delta(y-1) - [Z_s^+, U_s^+] \begin{bmatrix} J & 0 \\ 0 & P \end{bmatrix} \delta(y+1) \quad , \qquad (4.3\text{-}4)$$

where matrices A, D, F, J, M, and P are defined as

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad F = \frac{E}{\rho\omega^2} \begin{bmatrix} \frac{d^2}{dy^2} & 0 \\ 0 & \frac{d^2}{dy^2} \end{bmatrix}$$

$$J = \frac{E}{\rho\omega^2} \begin{bmatrix} 0 & \frac{d}{dy} \\ 0 & 0 \end{bmatrix}, \quad M = \frac{(1+\nu)\rho}{E} \begin{bmatrix} -\nu & 1 \\ -(2+\nu) & 3 \end{bmatrix}, \quad P = \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2} \begin{bmatrix} 0 & 0 \\ \frac{d}{dy} & 0 \end{bmatrix}.$$

CHAPTER V

SATISFACTION OF BOUNDARY CONDITIONS ON THE SURFACE X = 0

5.1 Procedure Outline

The eigenfunction constants are determined in terms of the time harmonic stresses σ_{xx} and σ_{xy} on the surface $x = 0$. On this boundary, the general eigenfunction expansion has the form

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ Q \end{bmatrix}_{x=0} = \begin{bmatrix} \sigma_{xxb} \\ \sum_{j=1}^{\infty} C_j \eta_{j2} \\ \sigma_{xyb} \\ \sum_{j=1}^{\infty} C_j \eta_{j4} \end{bmatrix} \cos \omega t = \sum_{r=1}^{\infty} C_r \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}_{(r)} \cos \omega t \ ,$$

or

$$\begin{bmatrix} \sigma_{xxb} \\ \sum_{j=1}^{\infty} C_j \eta_{j2} \\ \sigma_{xyb} \\ \sum_{j=1}^{\infty} C_j \eta_{j4} \end{bmatrix} = \sum_{r=1}^{\infty} C_r \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}_{(r)} . \qquad (5.1\text{-}1)$$

Multiplying both sides of equation (5.1-1) by the biorthogonality operator Φ_s, (4.3-4), and integrating from $y = -1$ to $y = 1$ yields the following equations

$$F_s + \sum_{i=1}^{\infty} B_{si} C_i = N_s C_s \ , \quad s = 1,2,\ldots,\infty \ . \tag{5.1-2}$$

N_s is a normalization factor, F_s involves the boundary stresses σ_{xxb} and σ_{xyb}, and the summed term arises from the series expansion of σ_{yy} and Q. The system of equations is solved for the s constants by truncating the series.

5.2 The Adjoint Function Arbitrary Constants

The arbitrary constants of each set of adjoint function vectors are chosen such that the adjoint function vectors and the eigenfunction vectors of each mixed case are a biorthonormal set. The necessity of this requirement becomes evident in the actual development of equation (5.1-2).

Consider the solution of either of the mixed boundary cases. For the first mixed case, the expansion (2.7-1) can be used, and the biorthogonality operator of (2.5-7) applied in the manner described in Section 5.1, or the four-vector expansion can be used by substituting the series expansion for σ_{xy} and Q. In the latter procedure, an inner biorthogonality condition yields the same relation developed using the two-vector expansion. The series term of equation (5.1-2) does not appear in this mixed boundary case, thereby yielding an explicit solution for the eigenfunction constants. Also, each term of the resulting equation for the eigenfunction constants is multiplied by a single adjoint function arbitrary constant which can be taken equal to unity. These same characteristics are true of the second mixed case. In the four-vector formulation for the pure stress end condition, the equations

(5.1-2) involve a sum of terms, some of which are multiplied by the adjoint function arbitrary constant of the first mixed case and others multiplied by the constant of the second mixed case. This condition requires the evaluation of each constant or at least a determination of the ratio of the constants. The second alternative is dismissed because equations coupling the two adjoint problems are not evident.

The normalization constant N_s of equation (5.1-2) for the four-vector form is the sum of the normalization constants for the first and second mixed cases

$$N_s = k_s^{(1)} N_s^{(1)} + k_s^{(2)} N_s^{(2)} ,$$

where $k_s^{(1)}$ and $k_s^{(2)}$ are the adjoint function arbitrary constants of the first and second mixed cases respectively. In constructing a biorthonormal set for each case, these constants are chosen as

$$k_s^{(1)} = 1/N_s^{(1)} \quad , \quad k_s^{(2)} = 1/N_s^{(2)}$$

with the result that $N_s = 2$.

5.3 Satisfaction of Boundary Conditions for the Odd Problem

The eigenfunction expansion for the odd problem is given by (4.1-2). Substituting the stress boundary conditions at $x = 0$, premultiplying both sides by the odd problem biorthogonality operator, ${}_o\Phi_s$, and integrating from $y = -1$ to $y = 1$

$$\int_{-1}^{1} {}_o\Phi_s \begin{bmatrix} \sigma_{xxb} \\ \sum_{j=1}^{\infty} C_j \eta_{j2} \\ \sigma_{xyb} \\ \sum_{j=1}^{\infty} C_j \eta_{j4} \end{bmatrix} dy = C_s \int_{-1}^{1} {}_o\Phi_s \; {}_o\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}_{(s)} dy \tag{5.3-1}$$

yields

$$F_s + \sum_{i=1}^{\infty} B_{si} C_i = (k_s^{(1)} N_s^{(1)} + k_s^{(2)} N_s^{(2)}) C_s \ , \tag{5.3-2}$$

where

$$F_s = \int_{-1}^{1} \left\{ (2\bar{z}_{s1} - \bar{z}_{s2}) \sigma_{xxb} + \left[(\nu+\nu^2) \bar{u}_{s1} + (\nu^2+3\nu+1) \bar{u}_{s2} - \frac{E}{\rho\omega^2} \bar{u}_{s1} \frac{d^2}{dy^2} \right] \sigma_{xyb} \right\} dy - \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2} \bar{u}_{s2} \frac{d\sigma_{xyb}}{dy} \Bigg|_{-1}^{1} \tag{5.3-3}$$

$$B_{si} = \int_{-1}^{1} \left\{ \bar{z}_{s1} \eta_{i2} - \left[\nu \bar{u}_{s1} + (1+3\nu) \bar{u}_{s2} + \frac{E}{\rho\omega^2} \bar{u}_{s2} \frac{d^2}{dy^2} \right] \eta_{i4} \right\} dy + \frac{E}{\rho\omega^2} \bar{z}_{s1} \frac{d\eta_{i2}}{dy} \Bigg|_{-1}^{1} \tag{5.3-4}$$

$$k_s^{(1)} N_s^{(1)} = \int_{-1}^{1} [(2\bar{z}_{s1} - \bar{z}_{s2}) \eta_{s1} + \bar{z}_{s1} \eta_{s2}] dy + \frac{E}{\rho\omega^2} \bar{z}_{s1} \frac{d\eta_{s2}}{dy} \Bigg|_{-1}^{1} \tag{5.3-5}$$

$$k_s^{(2)} N_s^{(2)} = \int_{-1}^{1} \left\{ \left[(\nu+\nu^2) \bar{u}_{s1} + (\nu^2+3\nu+1) \bar{u}_{s2} - \frac{E}{\rho\omega^2} \bar{u}_{s1} \frac{d^2}{dy^2} \right] \eta_{s3} - \left[\nu \bar{u}_{s1} + (1+3\nu) \bar{u}_{s2} + \frac{E}{\rho\omega^2} \bar{u}_{s2} \frac{d^2}{dy^2} \right] \eta_{s4} \right\} dy - \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2} \bar{u}_{s2} \frac{d\eta_{s3}}{dy} \Bigg|_{-1}^{1} \ . \tag{5.3-6}$$

The odd problem adjoint functions z_i are given by equations (2.6-10), the functions u_i by (3.7-6), and the eigenfunctions η_i by (4.1-2).

The evaluation of the coefficients B_{si} requires care. The arguments of the hyperbolic functions are related to the eigenvalue α through the following expressions

$$\alpha m = \sqrt{\alpha^2 - \frac{\rho\omega^2}{E}(1-\nu^2)} \quad , \quad \alpha n = \sqrt{\alpha^2 - \frac{2\rho\omega^2}{E}(1+\nu)} \quad .$$

The eigenvalues occur in pairs, one of the pair being the negative of the complex conjugate of the other, and some of the eigenvalues are real. B_{si} must be determined independently for each of the following relations between the roots α_i:

(1) $\alpha_i^2 \neq \alpha_s^2$

(2) $\alpha_i = \alpha_s$

(3) $\alpha_i = -\alpha_s$ and:

(a) $\alpha_i m_i = \alpha_s m_s$, $\alpha_i n_i = \alpha_s n_s$, $m_i = -m_s$, $n_i = -n_s$

(b) $\alpha_i m_i = \alpha_s m_s$, $\alpha_i n_i = -\alpha_s n_s$, $m_i = -m_s$, $n_i = n_s$

(c) $\alpha_i m_i = -\alpha_s m_s$, $\alpha_i n_i = -\alpha_s n_s$, $m_i = m_s$, $n_i = n_s$.

B_{si} for Case (1)

Substitution of the eigenfunctions and adjoint functions into (5.3-4) and performing the required operations yields

$$B_{si} = 2k_s^{(1)} \cosh \alpha_s m_s \sinh \alpha_i m_i \left\{ \alpha_s m_s K_i \left[\frac{2}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} - \frac{1}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} - \frac{1}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \right] \right.$$
$$\left. + \frac{\tanh \alpha_s m_s}{\tanh \alpha_i m_i} \frac{\alpha_i m_i K_i}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2) - 1 \right] \right.$$

$$- \frac{\tanh \alpha_s n_s}{\tanh \alpha_i m_i} \frac{\alpha_i m_i K_i}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \frac{m_s}{n_s} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2) - 1 \right]$$

$$- \frac{\tanh \alpha_s m_s}{\tanh \alpha_i n_i} \frac{\alpha_i n_i K_i}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2) - 1 \right]$$

$$+ \frac{\tanh \alpha_s n_s}{\tanh \alpha_i n_i} \frac{\alpha_i n_i K_i}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \frac{m_s}{n_s} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2) - 1 \right] \Bigg\}$$

$$+ i\, 2k_s^{(2)} \cosh \alpha_s m_s \sinh \alpha_i m_i \Bigg\{ \frac{2\alpha_i m_i^2 K_i}{1 + n_i^2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \right.$$

$$\left. - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \frac{4m_s^2}{(1+n_s^2)^2} \right]$$

$$- \frac{\alpha_i (1+n_i^2) R_i K_i}{2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 m_i^2} \frac{4m_s^2}{(1+n_s^2)^2} \right]$$

$$- \frac{2\alpha_s m_s m_i K_i}{1+n_i^2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 m_i^2} - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \right] \frac{\tanh \alpha_s m_s}{\tanh \alpha_i m_i}$$

$$+ \frac{2\alpha_s m_s m_i K_i R_i}{1 + n_i^2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} \right.$$

$$\left. - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \right] \frac{\tanh \alpha_s m_s}{\tanh \alpha_i m_i} \Bigg\} \; . \qquad (5.3\text{-}7)$$

<u>B_{si} for Case (2)</u>

$$B_{ss} = k_s^{(1)} K \Bigg\{ -1 - \frac{m}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} + \alpha m \sinh \alpha m \cosh \alpha m \left[\frac{1}{\alpha^2 m^2} + \frac{1}{\alpha^2 n^2} \right.$$

$$+ \frac{2E}{\rho\omega^2} \left(2 - \frac{n}{m} \frac{\tanh \alpha m}{\tanh \alpha n} - \frac{m}{n} \frac{\tanh \alpha n}{\tanh \alpha m} \right)$$

$$+ \frac{2}{\alpha^2 m^2 - \alpha^2 n^2} \left(\frac{n}{m} \frac{\tanh \alpha m}{\tanh \alpha n} - \frac{m}{n} \frac{\tanh \alpha n}{\tanh \alpha m} \right) \Bigg] \Bigg\}$$

$$- \frac{i\ 2mk_s^{(2)} K}{1+n^2} \Bigg\{ (1+3\nu-\nu R) \Bigg[1 + \frac{\sinh \alpha m \cosh \alpha m}{\alpha m}$$

$$- \frac{2\alpha m R \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{n}{m} \frac{\tanh \alpha n}{\tanh \alpha m} \right) \Bigg]$$

$$+ (1+2\nu) \Bigg[\frac{m}{n} \frac{R \sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} \left(1 + \frac{\sinh \alpha n \cosh \alpha n}{\alpha n} \right)$$

$$+ \frac{2\alpha m \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{m}{n} \frac{\tanh \alpha m}{\tanh \alpha n} \right) \Bigg]$$

$$+ \frac{E}{\rho\omega^2} \alpha^2 m^2 \Bigg[1 + \frac{\sinh \alpha m \cosh \alpha m}{\alpha m} + \frac{2\alpha m \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{m}{n} \frac{\tanh \alpha m}{\tanh \alpha n} \right) \Bigg]$$

$$+ \frac{E}{\rho\omega^2} \alpha^2 n^2 \Bigg[\frac{mR}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} \left(1 + \frac{\sinh \alpha n \cosh \alpha n}{\alpha n} \right)$$

$$- \frac{2\alpha m R \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{n}{m} \frac{\tanh \alpha n}{\tanh \alpha m} \right) \Bigg] \Bigg\} . \qquad (5.3\text{-}8)$$

B_{si} for Cases (3a) and (3b)

Eliminating the i subscript from the functions in (5.3-4) through the substitution of the relations given in (3a) gives the same expression for B_{si} as with the substitution of the relations given in (3b). The resulting expression is almost identical to (5.3-8) except for one sign change.

Consider (5.3-8) written as

$$B_{ss} = k_s^{(1)} K \left\{ \qquad \right\} - \frac{i\ 2mk_s^{(2)} K}{1 + n^2} \left\{ \qquad \right\} . \qquad (5.3\text{-}8)$$

B_{si} for the cases (3a) and (3b) are

$$B_{si} = k_s^{(1)} K \left\{ \qquad \right\} + \frac{i\ 2mk_s^{(2)} K}{1+n^2} \left\{ \qquad \right\} . \tag{5.3-9}$$

B_{si} for Case (3c)

Substituting the relations listed in (3c) again yields a relation similar to (5.3-8) except for a sign change on the first term

$$B_{si} = -k_s^{(1)} K \left\{ \qquad \right\} - \frac{i\ 2mk_s^{(2)} K}{1+n^2} \left\{ \qquad \right\} . \tag{5.3-10}$$

Substituting the adjoint functions into (5.3-5) and (5.3-6), and factoring out the arbitrary constants $k_s^{(1)}$ and $k_s^{(2)}$, the following is obtained for the normalization constants $N_s^{(1)}$ and $N_s^{(2)}$:

$$N_s^{(1)} = \frac{(1+\nu)(n^2-1)}{\nu n^2 - \nu - 2} \left\{ \frac{m}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} - 1 \right.$$
$$\left. + \frac{(n^2-1)\sinh \alpha m \cosh \alpha m}{2\alpha m n^2 (1+n^2)} \left[(2-2\nu)n^4 + (1+5\nu)n^2 - (1+\nu) \right] \right\} \tag{5.3-11}$$

$$N_s^{(2)} = \frac{i\ 2m(1+\nu)(1+2\nu+n^2)}{(\nu n^2-\nu-2)(1+n^2)} \left\{ \frac{m}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} - 1 \right.$$
$$\left. + \frac{(n^2-1)\sinh \alpha m \cosh \alpha m}{2\alpha m n^2 (1+n^2)} \left[(2-2\nu)n^4 + (1+5\nu)n^2 - (1+\nu) \right] \right\} . \tag{5.3-12}$$

In the derivation, biorthogonality was defined for $\alpha_i^2 \neq \alpha_s^2$ but examination of the operator reveals biorthogonality holds for $\alpha_i \neq \alpha_s$ also.

5.4 Satisfaction of Boundary Conditions for the Even Problem

The even problem is handled in exactly the same way using the eigenfunction expansion (4.2-2) and the even problem biorthogonality operator. Substituting the even problem eigenfunctions and adjoint functions into (5.3-3) for F_s, (5.3-4) for B_{si}, (5.3-5) and (5.3-6) for the normalization constants yields the following:

$$
\begin{aligned}
B_{si} = {} & 2k_s^{(1)} \sinh \alpha_s m_s \cosh \alpha_i m_i \Bigg\{ \alpha_s m_s K_i \left[\frac{2}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} - \frac{1}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} - \frac{1}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \right] \\
& + \frac{\tanh \alpha_i m_i}{\tanh \alpha_s m_s} \, \frac{\alpha_i m_i K_i}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2) - 1 \right] \\
& - \frac{\tanh \alpha_i m_i}{\tanh \alpha_s n_s} \, \frac{\alpha_i m_i K_i}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \, \frac{m_s}{n_s} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2) - 1 \right] \\
& - \frac{\tanh \alpha_i n_i}{\tanh \alpha_s m_s} \, \frac{\alpha_i n_i K_i}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2) - 1 \right] \\
& + \frac{\tanh \alpha_i n_i}{\tanh \alpha_s n_s} \, \frac{\alpha_i n_i K_i}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \, \frac{m_s}{n_s} \left[\frac{E}{\rho\omega^2} (\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2) - 1 \right] \Bigg\} \\
& + i\, 2k_s^{(2)} \sinh \alpha_s m_s \cosh \alpha_i m_i \Bigg\{ \frac{2\alpha_i m_i^2 K_i}{1+n_i^2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \right. \\
& \left. - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \, \frac{4m_s^2}{(1+n_s^2)^2} \right] \\
& - \frac{\alpha_i (1+n_i^2) R_i K_i}{2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 m_i^2} \, \frac{4m_s^2}{(1+n_s^2)^2} \right] \\
& - \frac{2\alpha_s m_s m_i K_i}{1+n_i^2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 m_i^2} - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 m_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 m_i^2} \right] \frac{\tanh \alpha_i m_i}{\tanh \alpha_s m_s} \\
& + \frac{2\alpha_s m_s m_i R_i K_i}{1+n_i^2} \left[\frac{1+3\nu-\nu R_s + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 m_s^2 - \alpha_i^2 n_i^2} - \frac{1+2\nu + \frac{E}{\rho\omega^2} \alpha_i^2 n_i^2}{\alpha_s^2 n_s^2 - \alpha_i^2 n_i^2} \right] \frac{\tanh \alpha_i m_i}{\tanh \alpha_s m_s} \Bigg\}
\end{aligned}
\tag{5.4-1}
$$

B_{si} for Case (2)

$$B_{ss} = k_s^{(1)} K \left\{ 1 + \frac{m}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} + \alpha m \sinh \alpha m \cosh \alpha m \left[\frac{1}{\alpha^2 m^2} + \frac{1}{\alpha^2 n^2} \right.\right.$$

$$+ \frac{2E}{\rho\omega^2} \left(2 - \frac{n}{m} \frac{\tanh \alpha n}{\tanh \alpha m} - \frac{m}{n} \frac{\tanh \alpha m}{\tanh \alpha n} \right)$$

$$\left.\left. + \frac{2}{\alpha^2 m^2 - \alpha^2 n^2} \left(\frac{n}{m} \frac{\tanh \alpha n}{\tanh \alpha m} - \frac{m}{n} \frac{\tanh \alpha m}{\tanh \alpha n} \right) \right] \right\}$$

$$+ \frac{i\ 2mk_s^{(2)} K}{1+n^2} \left\{ (1+3\nu-\nu R) \left[1 - \frac{\sinh \alpha m \cosh \alpha m}{\alpha m} \right.\right.$$

$$\left. + \frac{2\alpha m\ R \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{n}{m} \frac{\tanh \alpha m}{\tanh \alpha n} \right) \right]$$

$$+ (1+2\nu) \left[\frac{m}{n} \frac{R \sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} \left(1 - \frac{\sinh \alpha n \cosh \alpha n}{\alpha n} \right) \right.$$

$$\left. - \frac{2\alpha m \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{m}{n} \frac{\tanh \alpha n}{\tanh \alpha m} \right) \right]$$

$$+ \frac{E}{\rho\omega^2} \alpha^2 m^2 \left[1 - \frac{\sinh \alpha m \cosh \alpha m}{\alpha m} - \frac{2\alpha m \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \right.$$

$$\left. \left(1 - \frac{m}{n} \frac{\tanh \alpha n}{\tanh \alpha m} \right) \right]$$

$$+ \frac{E}{\rho\omega^2} \alpha^2 n^2 \left[\frac{m}{n} \frac{R \sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} \left(1 - \frac{\sinh \alpha n \cosh \alpha n}{\alpha n} \right) \right.$$

$$\left.\left. + \frac{2\alpha m\ R \sinh \alpha m \cosh \alpha m}{\alpha^2 m^2 - \alpha^2 n^2} \left(1 - \frac{n}{m} \frac{\tanh \alpha m}{\tanh \alpha n} \right) \right] \right\} . \qquad (5.4\text{-}2)$$

B_{si} for Cases (3a), (3b), (3c)

For the three cases of condition 3, the expression for B_{si} differs from (5.4-2) by only one sign change. Consider (5.4-2) written as

$$B_{ss} = k_s^{(1)} K \left\{ \qquad \right\} + \frac{i\, 2m\, k_s^{(2)} K}{1+n^2} \left\{ \qquad \right\} .$$

B_{si} for the case (3) are

$$B_{si} = k_s^{(1)} K \left\{ \qquad \right\} - \frac{i\, 2m k_s^{(2)} K}{1+n^2} \left\{ \qquad \right\} . \qquad (5.4\text{-}3)$$

The normalization constants for the even problem are:

$$N_s^{(1)} = \frac{(1+\nu)(n^2-1)}{\nu n^2 - \nu - 2} \left\{ 1 - \frac{m}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} \right.$$

$$\left. + \frac{(n^2-1) \sinh \alpha m \cosh \alpha m}{(1+n^2)\, 2\alpha m n^2} \left[(2-2\nu) n^4 + (1+5\nu) n^2 - (1+\nu) \right] \right\} \qquad (5.4\text{-}5)$$

$$N_s^{(2)} = \frac{i\, 2m(1+\nu)(1+2\nu+n^2)}{(\nu n^2 - \nu - 2)(1+n^2)} \left\{ 1 - \frac{m}{n} \frac{\sinh \alpha m \cosh \alpha m}{\sinh \alpha n \cosh \alpha n} \right.$$

$$\left. + \frac{(n^2-1) \sinh \alpha m \cosh \alpha m}{2\alpha m n^2 (1+n^2)} \left[(2-2\nu) n^4 + (1+5\nu) n^2 - (1+\nu) \right] \right\} . \qquad (5.4\text{-}6)$$

CHAPTER VI

DETERMINATION OF THE EIGENVALUES

6.1 General Remarks

The eigenvalues α are the roots of the transcendental equations

$$\frac{\tanh \alpha m}{\tanh \alpha n} = \frac{4mn}{(1+n^2)^2} \tag{6.1-1}$$

$$\frac{\tanh \alpha n}{\tanh \alpha m} = \frac{4mn}{(1+n^2)^2} \quad . \tag{6.1-2}$$

Equation (6.1-1) is used for the odd problem, and equation (6.1-2) is used for the even problem. Similar equations occur in the investigation of wave propagation in an infinite plate. This problem has received much attention, and the solutions of these equations have been made, but, as far as this writer has been able to determine, not for the complex values of α associated with the decaying wave modes close to the finite edge of the plate. This study is concerned with the investigation of all bounded modes for each frequency of propagation. Equations (6.1-1) and (6.1-2) define the eigenvalues on the real axis and the upper half of the complex plane.

6.2 Determination of the Real Eigenvalues for the Odd Problem

The transcendental equations are solved with the frequency ω as a parameter and the Poisson ratio ν taken to be equal to

1/3. Recalling that

$$\alpha^2 m^2 = \alpha^2 - \frac{\rho\omega^2}{E}(1-\nu^2)$$

$$\alpha^2 n^2 = \alpha^2 - \frac{2\rho\omega^2}{E}(1+\nu) \quad ,$$

it is convenient to substitute constants "a" and "b" such that

$$\alpha^2 m^2 = \alpha^2 - a$$

$$\alpha^2 n^2 = \alpha^2 - b \quad .$$

For $\nu = 1/3$, the constant b equals 3a. Equation (6.1-1) can now be written as

$$\frac{\tanh\sqrt{\alpha^2 - a}}{\tanh\sqrt{\alpha^2 - 3a}} = \frac{4\alpha^2\sqrt{\alpha^2 - a}\sqrt{\alpha^2 - 3a}}{(2\alpha^2 - 3a)^2} \tag{6.2-1}$$

Consider the solution of equation (6.2-1) when α is very large compared to "a" and the hyperbolic terms approach unity.

$$\alpha^6 - 5.25\alpha^4 a + 6.75\alpha^2 a^2 - 2.53a^3 = 0 \tag{6.2-2}$$

Only one root of equation (6.2-2) yields a real argument for the hyperbolic functions. This root is

$$\alpha = 1.884\sqrt{a} \quad . \tag{6.2-3}$$

Consider the solution of equation (6.2-1) when α and "a" are very small and the hyperbolic functions can be replaced by their arguments.

$$4\alpha^4 - 12\alpha^2 a + 9a = 4\alpha^4 - 12\alpha^2 a \tag{6.2-4}$$

For this equation to be satisfied, "a" (or ω) must equal zero.

The phase velocity, c, of a given mode equals ω/α, and the wave length Λ is defined in terms of α by $\alpha = 2\pi/\Lambda$. When α is very large (small wave length), the limiting phase velocity is

$$c = \omega/\alpha = \omega/1.884\sqrt{a} \quad , \; c = 0.563\sqrt{E/\rho} \quad .$$

When α approaches zero the phase velocity approaches zero.

Intermediate values of α have been obtained numerically with the computer for seven values of frequency. For each frequency, the root yielding the lowest phase velocity is identified as the first mode root, and the mode with the next highest phase velocity is mode two, and so on.

Table VI-1 lists the odd problem real eigenvalues of the first four modes for the frequencies examined and the corresponding phase velocity ratio c/c_o where $c_o = \sqrt{\frac{E}{\rho}}$, the velocity of propagation of plane longitudinal waves along a rod. Tabulation of all the eigenvalues for these frequencies is found in the appendix.

Figure VI-1 is presented to graphically illustrate the phase velocity-wave length dependence. Phase velocity variation with changes in wave length is termed dispersion stemming from the fact that this characteristic causes distortion of the shape of a propagating pulse. The first mode curve is in close agreement with the dispersion curve for flexural waves in cylindrical bars, and the trend of the higher mode curves indicated by the few data points plotted is in agreement with published phase velocity curves for flat plates and cylindrical bars. The intent here is not to present a complete solution of the transcendental equations, but to attach some physical significance to the real eigenvalues and make some comparisons of the values used to solve the problem with published results (13)-(15).

6.3 Determination of the Real Eigenvalues for the Even Problem

Following the same procedure as in article 6.2, equation (6.1-2) can be written as

$$\frac{\tanh\sqrt{\alpha^2 - 3a}}{\tanh\sqrt{\alpha^2 - a}} = \frac{4\alpha^2\sqrt{\alpha^2 - a}\,\sqrt{\alpha^2 - 3a}}{(2\alpha^2 - 3a)^2} \tag{6.3-1}$$

Considering the solution of (6.3-1) when α is very large relative to "a" yields the result (6.2-3). Considering the solution when α and "a" are very small yields

$$-8\alpha^2 + 9a = 0 \ ,$$

or

$$c = \frac{\omega}{\alpha} = \sqrt{\frac{E}{\rho}} \ .$$

Thus, when α is large the phase velocity approaches 0.563 c_o. When α approaches zero, the phase velocity approaches c_o.

Intermediate values of α were obtained numerically, and Table VI-2 lists the even problem real eigenvalues of the first four modes and the corresponding phase velocities for the frequencies examined. Figure VI-2 illustrates the dispersive nature of the first three modes. The implied shape and values of the curves are close to those found in the literature for flat plates and cylindrical bars. For the first mode, the velocity ratio c/c_o approaches 1.062 for the flat plate and approaches one for the cylindrical bar as α approaches zero.

6.4 Determination of the Complex Eigenvalues for the Odd and Even Problem

Writing equation (6.2-1) in the following form

$$(2\alpha^2 - 3a)^2 \tanh\sqrt{\alpha^2 - a} - 4\alpha^2\sqrt{\alpha^2 - a}\sqrt{\alpha^2 - 3a}\,\tanh\sqrt{\alpha^2 - 3a} = 0, \tag{6.4-1}$$

and (6.2-2) as

$$(2\alpha^2 - 3a)^2 \tanh\sqrt{\alpha^2 - 3a} - 4\alpha^2\sqrt{\alpha^2 - a}\sqrt{\alpha^2 - 3a}\,\tanh\sqrt{\alpha^2 - a} = 0, \tag{6.4-2}$$

and letting $\alpha = x + iy$, solutions were made numerically by use of a digital computer. An asymptotic solution of (6.4-1) for α very large compared to "a" aided the search for the roots by isolating the roots in the complex plane and giving very good approximations of the roots at low frequencies.

It is of interest to note the comparison of the complex roots of (6.4-1) and (6.4-2) with the roots obtained by Johnson and Little in their solution of the static problem. This comparison is given in Tables VI-3 and VI-4.

The first quadrant complex roots for two frequencies, 1000 and 31600 cycles per second, are plotted in Figure VI-3. The roots for all frequencies examined are tabulated in the appendix. The calculations were made for a material density of 15.1 slugs per cubic foot and a Young's modulus of $30(10^6)$ pounds per square inch. All dimensions are in units of feet.

Table VI-1 -- Real Eigenvalues and Phase Velocities for the First Four Modes of the Odd Problem

Frequency ω cps	Constant "a"	Eigen-value $_o\alpha$	Velocity Ratio c/c_o	$\frac{\text{Wavelength}}{\text{Strip Width}}$
100	0.00123	0.259	0.144	12.130
1000	0.123	0.969	0.384	3.242
6380	5.0	4.30	0.552	0.731
		2.42	0.980	1.300
10000	12.3	6.63	0.561	0.474
		5.38	0.691	0.584
		3.29	1.131	0.955
		1.44	2.583	2.182
31600	123.0	20.89	0.563	0.150
		19.12	0.615	0.164
		18.38	0.640	0.171
		16.90	0.696	0.186
		-----	-----	-----
63800	500.0	42.12	0.563	0.074
		38.69	0.613	0.081
		38.39	0.618	0.082
		37.78	0.628	0.083
		-----	-----	-----
100000	1230.0	66.07	0.563	0.048
		60.18	0.618	0.052
		59.63	0.624	0.053
		58.89	0.632	0.053
		-----	-----	-----

Table VI-2 -- Real Eigenvalues and Phase Velocities for the First Four Modes of the Even Problem

Frequency ω cps	Constant "a"	Eigen-value e^{α}	Velocity Ratio c/c_o	$\frac{\text{Wavelength}}{\text{Strip Width}}$
100	0.00123	0.0372	1.000	84.451
1000	0.123	0.373	0.997	8.422
6380	5.0	4.056	0.584	0.774
		2.238	1.060	1.404
		1.131	2.097	2.778
10000	12.3	6.580	0.565	0.477
		3.926	0.947	0.800
		3.316	1.120	0.947
31600	123	20.89	0.563	0.150
		18.84	0.624	0.167
		17.74	0.663	0.177
		15.85	0.742	0.198
63800	500	42.12	0.563	0.074
		38.58	0.615	0.081
		38.12	0.622	0.082
		37.36	0.635	0.084
100000	1230	66.07	0.563	0.048
		60.65	0.619	0.052
		60.38	0.616	0.052
		59.93	0.621	0.052

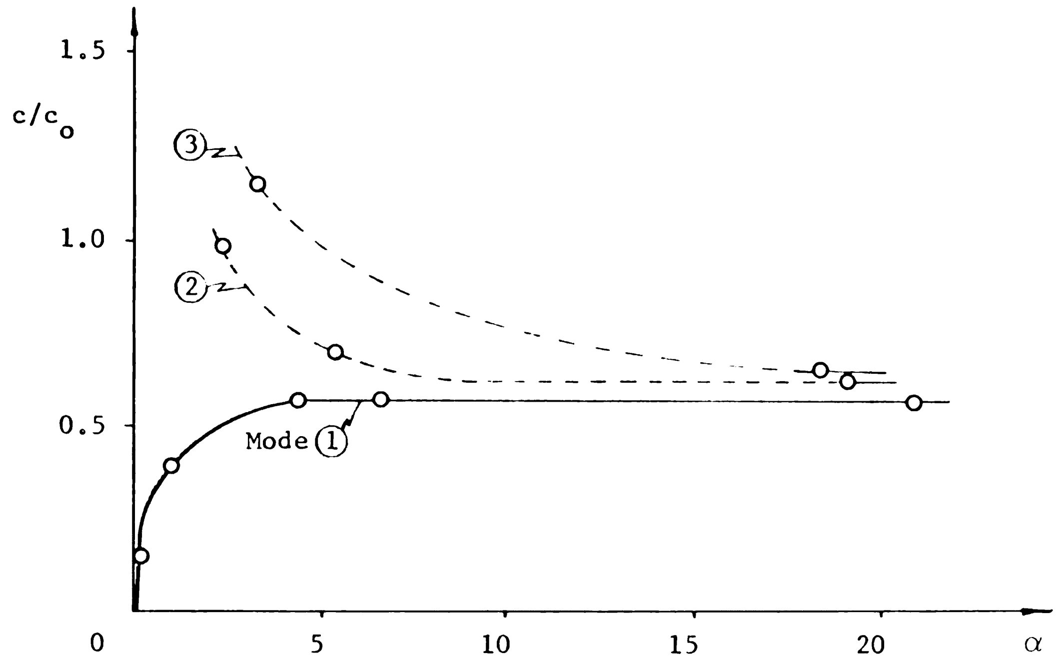

Figure VI-1 -- Phase Velocity Variation for the First Three Modes of the Odd Problem

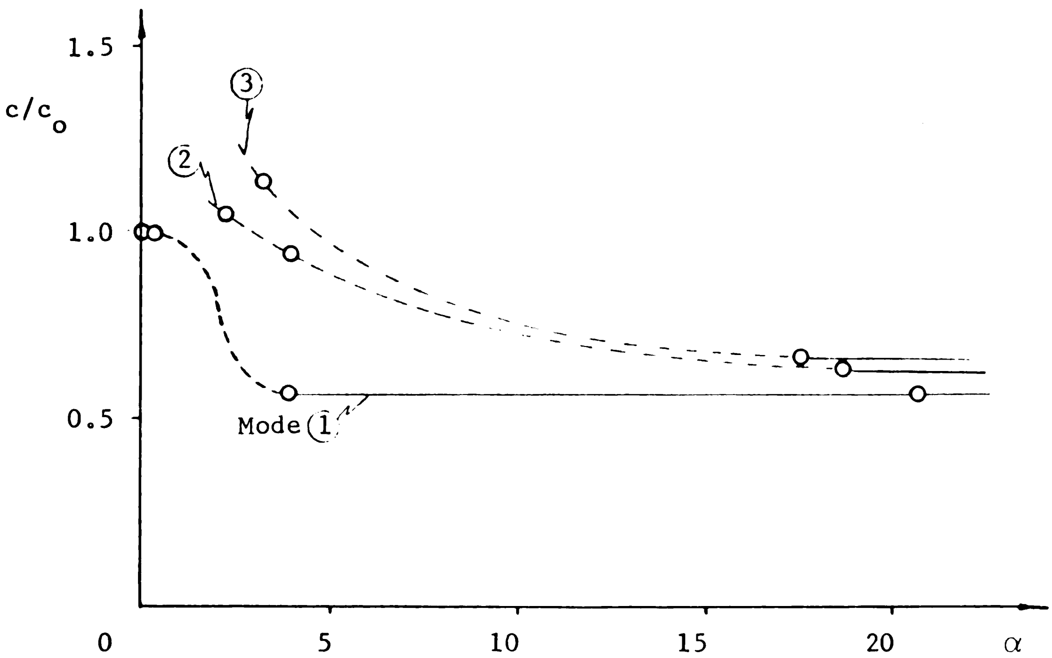

Figure VI-2 -- Phase Velocity Variation for the First Three Modes of the Even Problem

Table VI-3 -- Comparison of Low Frequency, Odd Problem, Complex Eigenvalues with the Static Problem Eigenvalues*

Static, Odd Problem		100 cps, Odd Problem	
± 1.384339 +	i3.748838	± 1.384371 +	i3.748499
1.676105	6.949980	1.676120	6.949802
1.858384	10.119259	1.858393	10.119137
1.991571	13.277274	1.991577	13.277181
2.096626	16.429872	2.096630	16.429796
2.183398	19.579409	2.183401	19.579346
2.257320	22.727036	2.257323	22.726982
2.321714	25.873384	2.321716	25.873337
2.378758	29.018831	2.378759	29.018789
2.429959	32.163617	2.429960	32.163579

Table VI-4 -- Comparison of Low Frequency, Even Problem, Complex Eigenvalues with the Static Problem Eigenvalues*

Static, Odd Problem		100 cps, Odd Problem	
± 1.125365 +	i2.106196	± 1.125419 +	i2.105550
1.551575	5.356269	1.551595	5.356036
1.775544	8.536683	1.775555	8.536538
1.929405	11.699178	1.929412	11.699072
2.046853	14.854060	2.046858	14.853977
2.141891	18.004933	2.141895	18.004865
2.221723	21.153414	2.221726	21.153355
2.290553	24.300342	2.290555	24.300292
2.351048	27.446203	2.351050	27.446158
2.405013	30.591295	2.405014	30.591255

* From Johnson and Little (7)

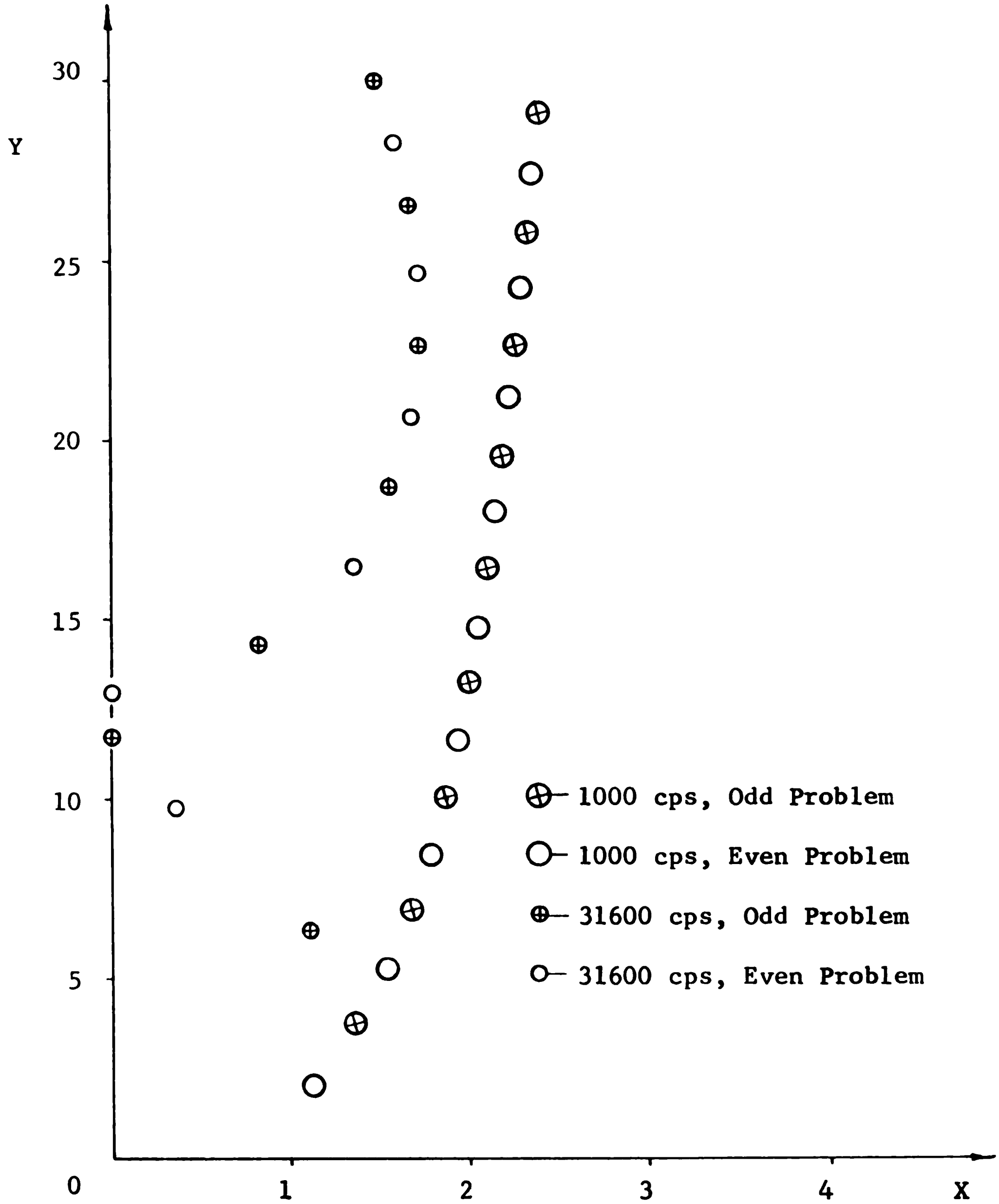

Figure VI-3 -- Eigenvalues in First Quadrant of Complex Plane for Frequencies of 1000 and 31600 cycles per second; $\alpha = X + iY$

CHAPTER VII

SPECIFIC BOUNDARY VALUE PROBLEMS

7.1 General Remarks

The general solution of the problem is in the form of an eigenfunction expansion of the four-vector

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ Q \end{bmatrix} = \sum_{r=1}^{\infty} C_r \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}_{(r)} e^{i(\alpha_r x + \omega t)} , \qquad (7.1\text{-}1)$$

where the constants C_r are determined from the particular boundary stresses σ_{xxb} and σ_{xyb} on the surface $x = 0$ through the application of the biorthogonality operator Φ_s, (4.3-4). This procedure was detailed in Chapter V, and it was seen that the evaluation of the constants requires the solution of the infinite set of equations (5.3-2)

$$F_s + \sum_{i=1}^{\infty} B_{si} C_i = [k_s^{(1)} N_s^{(1)} + k_s^{(2)} N_s^{(2)}] C_s \; , \; s = 1,2,\ldots,\infty. \qquad (5.3\text{-}2)$$

repeated

The F_s term is the only term containing the boundary functions.

$$F_s = \int_{-1}^{1} \left\{ (2\bar{z}_{s1} - \bar{z}_{s2})\sigma_{xxb} + \left[(\nu^2 + 3\nu + 1)\bar{u}_{s2} + (\nu + \nu^2)\bar{u}_{s1} - \frac{E}{\rho\omega^2} \bar{u}_{s1} \frac{d^2}{dy^2} \right] \sigma_{xyb} \right\} dy - \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha_s^2} \bar{u}_{s2} \frac{d\sigma_{xyb}}{dy} \Bigg|_{-1}^{1} . \qquad (5.3\text{-}3)$$

repeated

In this chapter, solutions are made for the following self-equilibrated end loadings:

Problem	σ_{xxb}	σ_{xyb}
1	$-3y + 10y^3 - 7y^5$	0
2	$\cos \frac{\pi}{2} y - \frac{2}{\pi}$	0
3	0	$-\sin \pi y$

7.2 The Boundary Term F_s

Problem 1 $\quad \sigma_{xxb} = -3y + 10y^3 - 7y^5$

$\sigma_{xyb} = 0$

Equation (5.3-3) gives

$$F_s = \int_{-1}^{1} (2\,{}_o\bar{z}_{s1} - {}_o\bar{z}_{s2})\sigma_{xxb}\, dy ,$$

where

$${}_o\bar{z}_{s1} = k_s^{(1)} \left(\sinh \alpha_s m_s y - \frac{m_s}{n_s} \frac{\cosh \alpha_s m_s}{\cosh \alpha_s n_s} \sinh \alpha_s n_s y\right)$$

$${}_o\bar{z}_{s2} = k_s^{(1)} \left(\sinh \alpha_s m_s y + \frac{m_s}{K_s n_s} \frac{\cosh \alpha_s m_s}{\cosh \alpha_s n_s} \sinh \alpha_s n_s y\right) .$$

Performing the required integration yields

$$F_s = k_s^{(1)} \Bigg\{ R_s \frac{m_s}{n_s} \cosh \alpha_s m_s \left[\frac{\tanh \alpha_s n_s}{\alpha_s^6 n_s^6} (-16\alpha_s^4 n_s^4 - 720\alpha_s^2 n_s^2 - 1680) + \frac{1}{\alpha_s^5 n_s^5} (160\alpha_s^2 n_s^2 + 1680) \right]$$

$$+ \cosh \alpha_s m_s \left[\frac{\tanh \alpha_s m_s}{\alpha_s^6 m_s^6} (16\alpha_s^4 m_s^4 + 720\alpha_s^2 m_s^2 + 1680) + \frac{1}{\alpha_s^5 m_s^5} (-160\alpha_s^2 m_s^2 - 1680) \right] \Bigg\} .$$

<u>Problem 2</u>

$$\sigma_{xxb} = \cos \frac{\pi}{2} y - \frac{2}{\pi}$$

$$\sigma_{xyb} = 0$$

Following the same procedure using the even adjoint functions gives

$$F_s = k_s^{(1)} \left\{ R_s \frac{m_s}{n_s} \cosh \alpha_s m_s \left[\frac{4}{\alpha_s n_s \pi} \tanh \alpha_s m_s - \frac{4\pi}{\pi^2 + 4\alpha_s^2 n_s^2} \frac{\tanh \alpha_s m_s}{\tanh \alpha_s n_s} \right] \right.$$

$$\left. + \cosh \alpha_s m_s \left[\frac{4\pi}{\pi^2 + 4\alpha_s^2 m_s^2} - \frac{4 \tanh \alpha_s m_s}{\alpha_s m_s \pi} \right] \right\} .$$

<u>Problem 3</u>

$$\sigma_{xxb} = 0$$

$$\sigma_{xyb} = -\sin \pi y$$

Substitution of these functions into (5.3-3) and integrating gives

$$F_s = k_s^{(2)} 2\pi \sinh \alpha_s m_s \left\{ \frac{1}{\alpha_s^2 m_s^2 + \pi^2} \left[R_s \left(\nu + \nu^2 + \frac{E}{\rho\omega^2} \pi^2\right) \right. \right.$$

$$\left. - (\nu^2 + 3\nu + 1) - \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha^2} (\alpha_s^2 m_s^2 + \pi^2) \right]$$

$$+ \frac{1}{\alpha_s^2 n_s^2 + \pi^2} \frac{m_s}{n_s} \frac{\tanh \alpha_s n_s}{\tanh \alpha_s m_s} \left[(\nu^2 + 3\nu + 1) - \left(\nu + \nu^2 + \frac{E}{\rho\omega^2} \pi^2\right) \right.$$

$$\left. \left. + \frac{\nu}{\frac{\rho\omega^2}{E} - \alpha^2} (\alpha_s^2 n_s^2 + \pi^2) \right] \right\} .$$

7.3 <u>Results and Conclusions</u>

A detailed study of the steady state solution was made for the intermediate frequency, 6380 cycles per second, for each of the time-harmonic boundary loadings defined by problems 1, 2, and 3. The eigenvalues used for this frequency are tabulated in Table A-3 in the appendix. Tables VII-1 and VII-2 illustrate the convergence of the eigenfunction constants for problems 1 and 2; Table VII-3 gives the comparison of the prescribed boundary functions with the truncated eigenfunction expansion approximation for problem 1. These examples are representative of the solutions for the three problems at this frequency. The boundary functions and the non-decaying stress wave modes of problems 1, 2, and 3 have been plotted in Figures VII-1 through VII-7. The decaying modes at $x = 0$, the sum of which is the difference between the boundary functions and the sum of the non-decaying modes at $x = 0$, are not shown as the major concern is the stress magnitude and distribution beyond the decay region. The phase velocities and wave lengths of each mode are given in Tables VI-1 and VI-2.

Examination of the results shown in Figures VII-1 through VII-7 leads to the following conclusions:

1. A Saint-Venant boundary region does not exist for the dynamic problem. The self equilibrated, time-harmonic boundary stresses of this problem produce stress waves which do not decay. The existence of non-decaying stress waves at this frequency insures the existence of non-decaying stress waves for any time-dependent boundary loading which includes this frequency component.
2. Application of the time-harmonic, self-equilibrated stress σ_{xxb} with zero shear stress σ_{xyb} in problems 1 and 2 produces significant non-decaying σ_{xx}, σ_{yy}, and σ_{xy} stress modes. These modes are the only significant modes remaining beyond a distance of 2 strip widths for the odd problem and 1 strip width for

the even problem from the loading surface $x = 0$.

3. The time-harmonic, self-equilibrated shear stress σ_{xyb} with zero normal stress σ_{xxb} in problem 3 yields a non-decaying σ_{xy} stress mode very similar in magnitude and shape to the applied stress without producing σ_{xx} and σ_{yy} modes.
4. The shape and magnitude of the non-decaying stress modes are very sensitive to the boundary stress distribution. Problems 1 and 2 each involve self-equilibrated normal boundary stress σ_{xxb} distributions of approximately the same peak magnitude as shown in Figures VII-1 and VII-4. Figure VII-2 of problem 1 shows a resulting mode 1 σ_{xx} stress wave having a peak magnitude 5.7 times greater than the peak magnitude of the applied stress. Figure VII-5 of problem 2 shows three non-decaying modes of σ_{xx} stress waves, all of which have peak values less than the peak value of the boundary stress.

Solutions of these problems were attempted for other frequencies, but the results were not conclusive because of unsatisfactory convergence of the eigenfunction constants. At the lower frequencies, round-off error is suspected as the main source of difficulty. At the higher frequencies, the equations (5.3-2) become highly ill-conditioned. Examining problem 1 at the lower frequency of 1000 cycles per second, the trend of the convergence of the single non-decaying eigenmode constant appeared to be toward zero. The higher frequency solutions indicated the existence of more non-decaying modes, but the convergence of the eigenfunction constants was too unstable to permit conclusive statements.

Table VII-1 -- Convergence of Eigenfunction Constants for Problem 1

Boundary Functions: $\sigma_{xxb} = -3y + 10y^3 - 7y^5$

$\sigma_{xyb} = 0$

Constant	8 Pair of Roots	10 Pair of Roots	12 Pair of Roots	16 Pair of Roots
C_1	0.015 - 0.904 i	0.025 - 1.187 i	0.025 - 1.208 i	0.026 - 1.208 i
$C_2 = \overline{C}_1$				
C_3	0.288 + 0.296 i	0.206 + 0.194 i	0.198 + 0.184 i	0.201 + 0.188 i
$C_4 = \overline{C}_3$				
C_5	4.045 + 3.297 i	3.317 + 2.589 i	3.236 + 2.513 i	3.272 + 2.545 i
$C_6 = \overline{C}_5$				
C_7	-0.559 + 1.856 i	-0.511 + 1.432 i	-0.505 + 1.380 i	-0.508 + 1.406 i
$C_8 = \overline{C}_7$				
C_9	-0.251 - 0.348 i	-0.143 - 0.315 i	-0.131 - 0.308 i	-0.137 - 0.314 i
$C_{10} = \overline{C}_9$				

Table VII-2 -- Convergence of Eigenfunction Constants for Problem 2

Boundary Functions: $\sigma_{xxb} = \cos \frac{\pi}{2} y - \frac{2}{\pi}$

$\sigma_{xyb} = 0$

Constant	8 Pair of Roots	9 Pair of Roots	10 Pair of Roots	11 Pair of Roots
C_1	-0.012 + 0.004 i	-0.013 + 0.002 i	-0.013 + 0.002 i	-0.013 + 0.003 i
$C_2 = \bar{C}_1$				
C_3	-0.052 - 1.882 i	-0.052 - 2.106 i	-0.052 - 2.180 i	-0.052 - 2.000 i
$C_4 = \bar{C}_3$				
C_5	0.120 - 0.166 i	0.119 - 0.182 i	0.118 - 0.187 i	0.119 - 0.175 i
$C_6 = \bar{C}_5$				
C_7	0.045 + 0.008 i	0.040 + 0.010 i	0.038 + 0.010 i	0.042 + 0.009 i
$C_8 = \bar{C}_7$				
C_9	-0.004 + 0.005 i	-0.005 + 0.004 i	-0.005 + 0.003 i	-0.004 + 0.004 i
$C_{10} = \bar{C}_9$				

Table VII-3 -- Approximation of Boundary Function, Problem 1

Boundary Function: $\sigma_{xxb} = -3y + 10y^3 - 7y^5$

$\sigma_{xyb} = 0$

Series Solution with 15 pairs of Eigenvalues

Y	Prescribed Stresses		Series Solution	
	σ_{xxb}	σ_{xyb}	σ_{xx}	σ_{xy}
0.0	0.00000	0.00000	0.00000	0.00000
0.1	-0.29007	0.00000	-0.29013	0.00002
0.2	-0.52224	0.00000	-0.52222	0.00006
0.3	-0.64701	0.00000	-0.64696	-0.00005
0.4	-0.63168	0.00000	-0.63174	-0.00004
0.5	-0.46875	0.00000	-0.46878	0.00009
0.6	-0.18432	0.00000	-0.18422	-0.00001
0.7	0.15351	0.00000	0.15348	-0.00011
0.8	0.42624	0.00000	0.42612	0.00008
0.9	0.45657	0.00000	0.45662	0.00010
1.0	0.00000	0.00000	-0.00051	0.00000

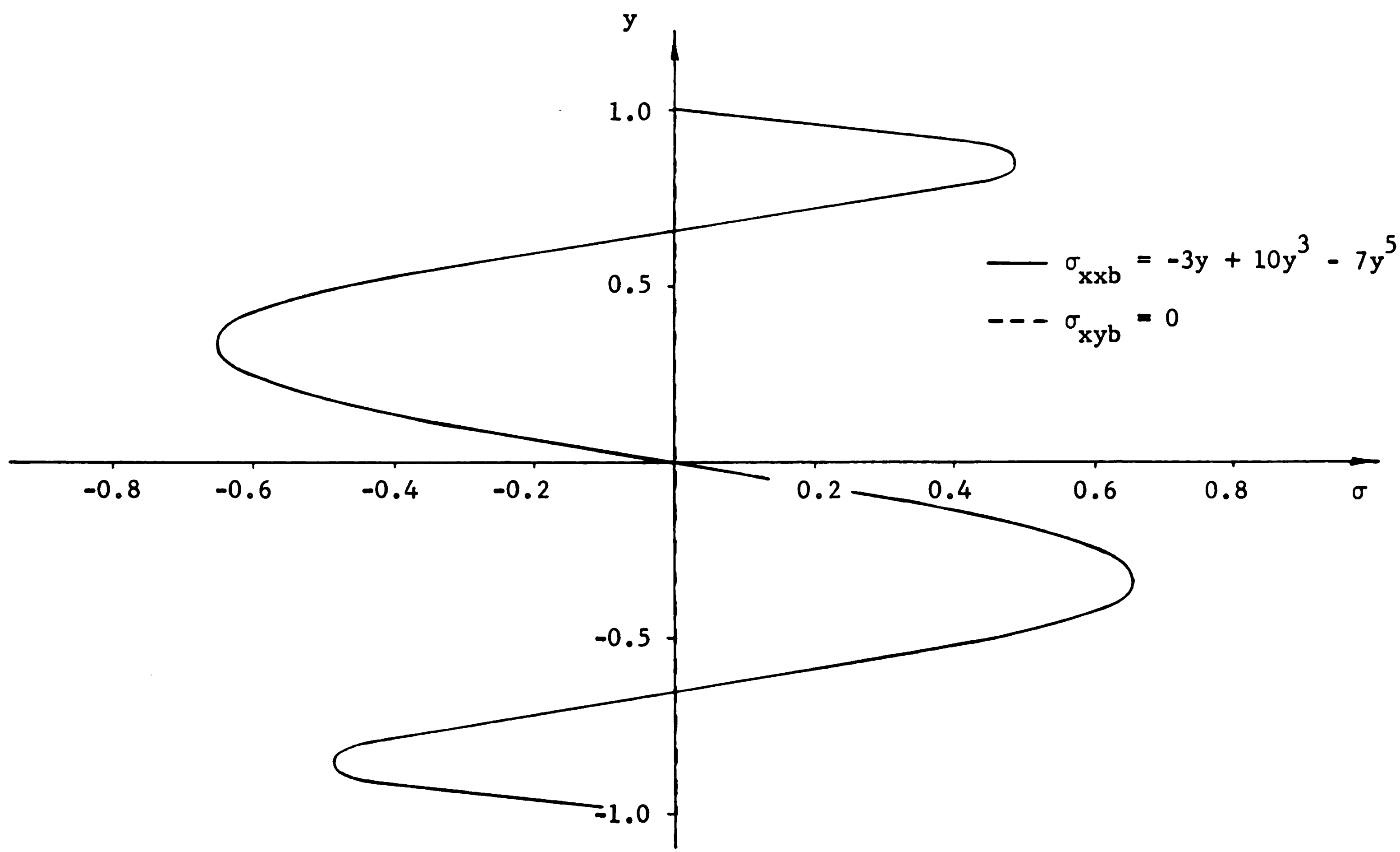

Figure VII-1 -- Boundary Stresses on Surface x = 0, Problem 1

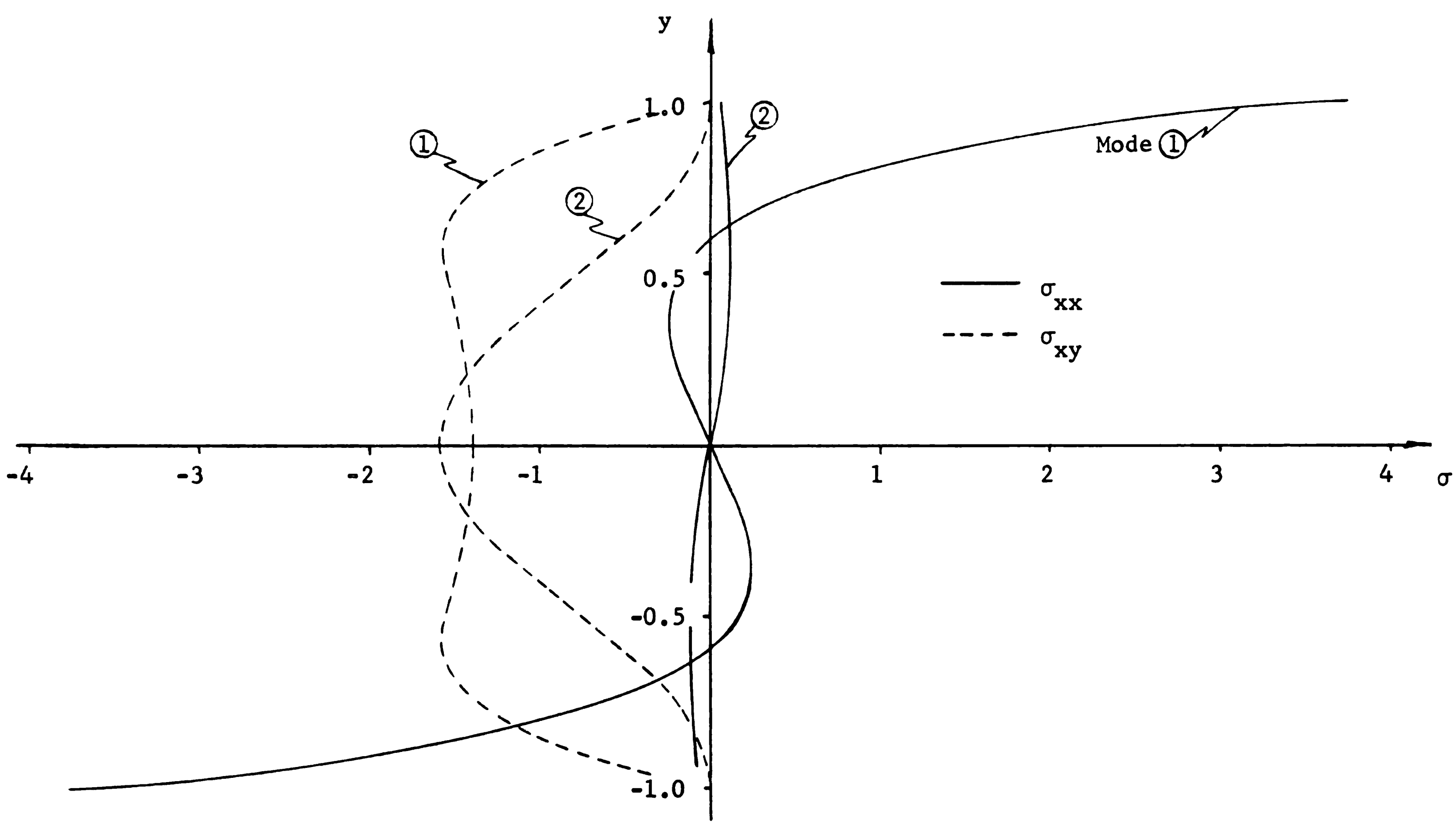

Figure VII-2 -- The Non-Decaying Wave Modes at x = 0, Problem 1

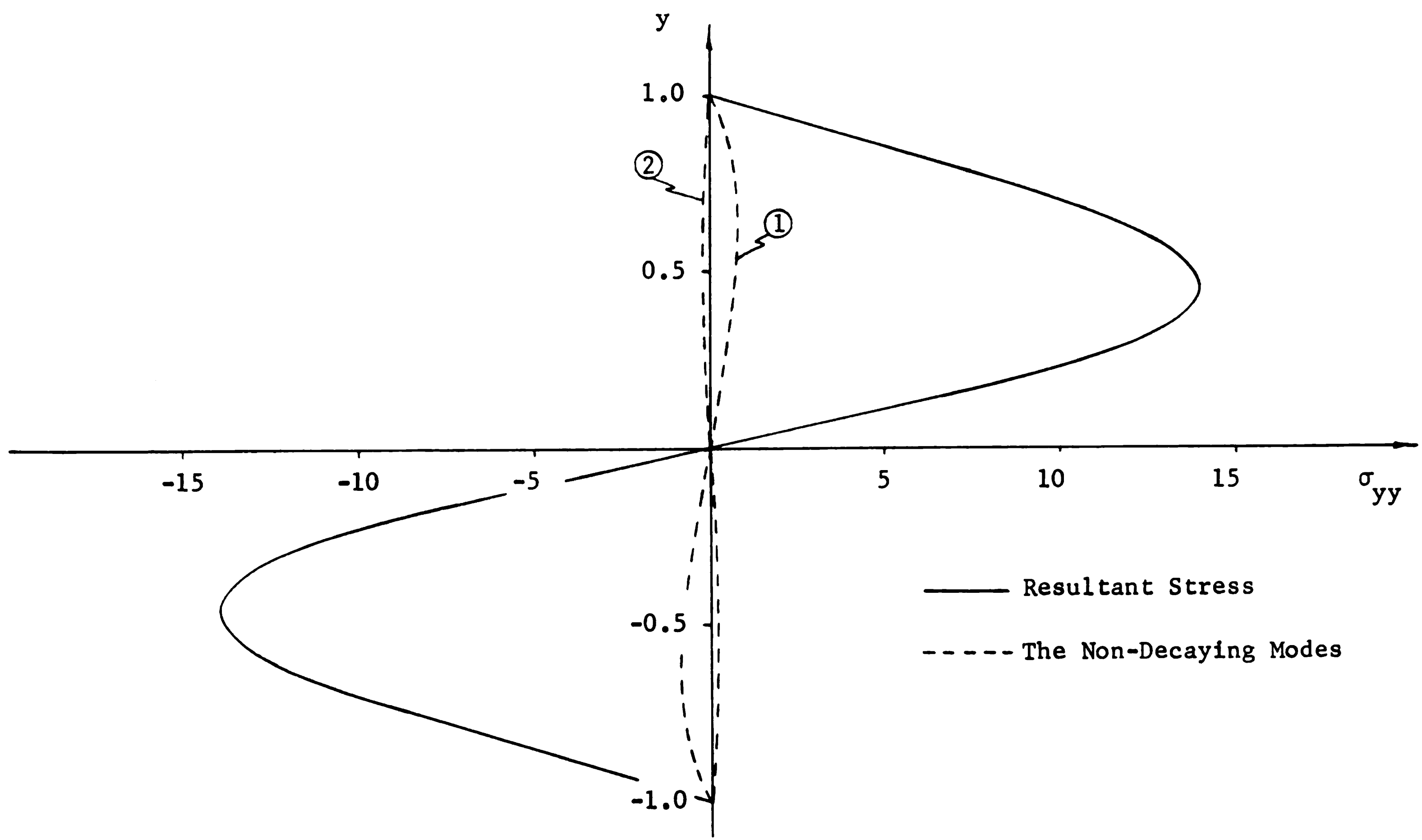

Figure VII-3 -- Stress σ_{yy} at x = 0, Problem 1

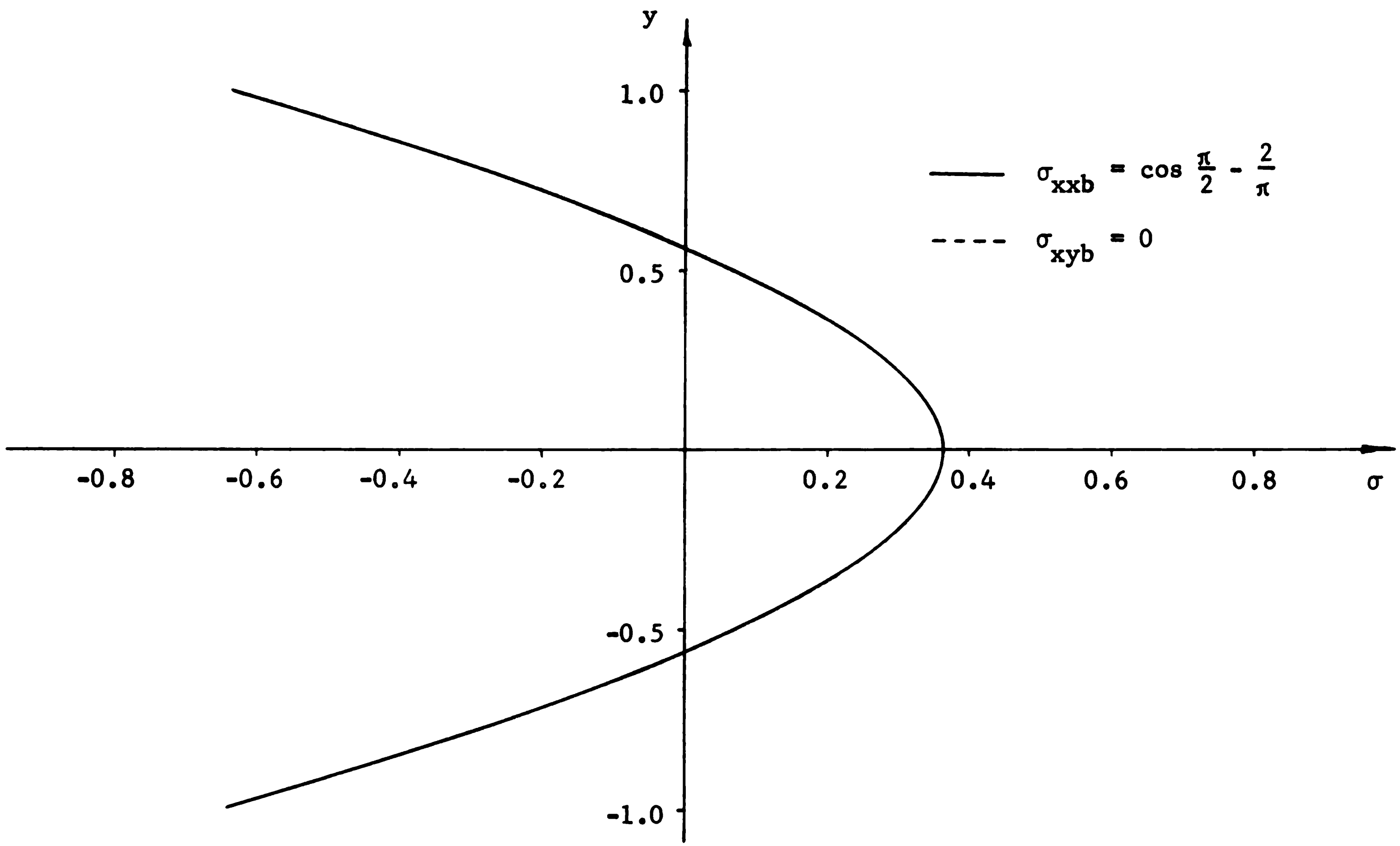

Figure VII-4 -- Boundary Stresses on Surface x = 0, Problem 2

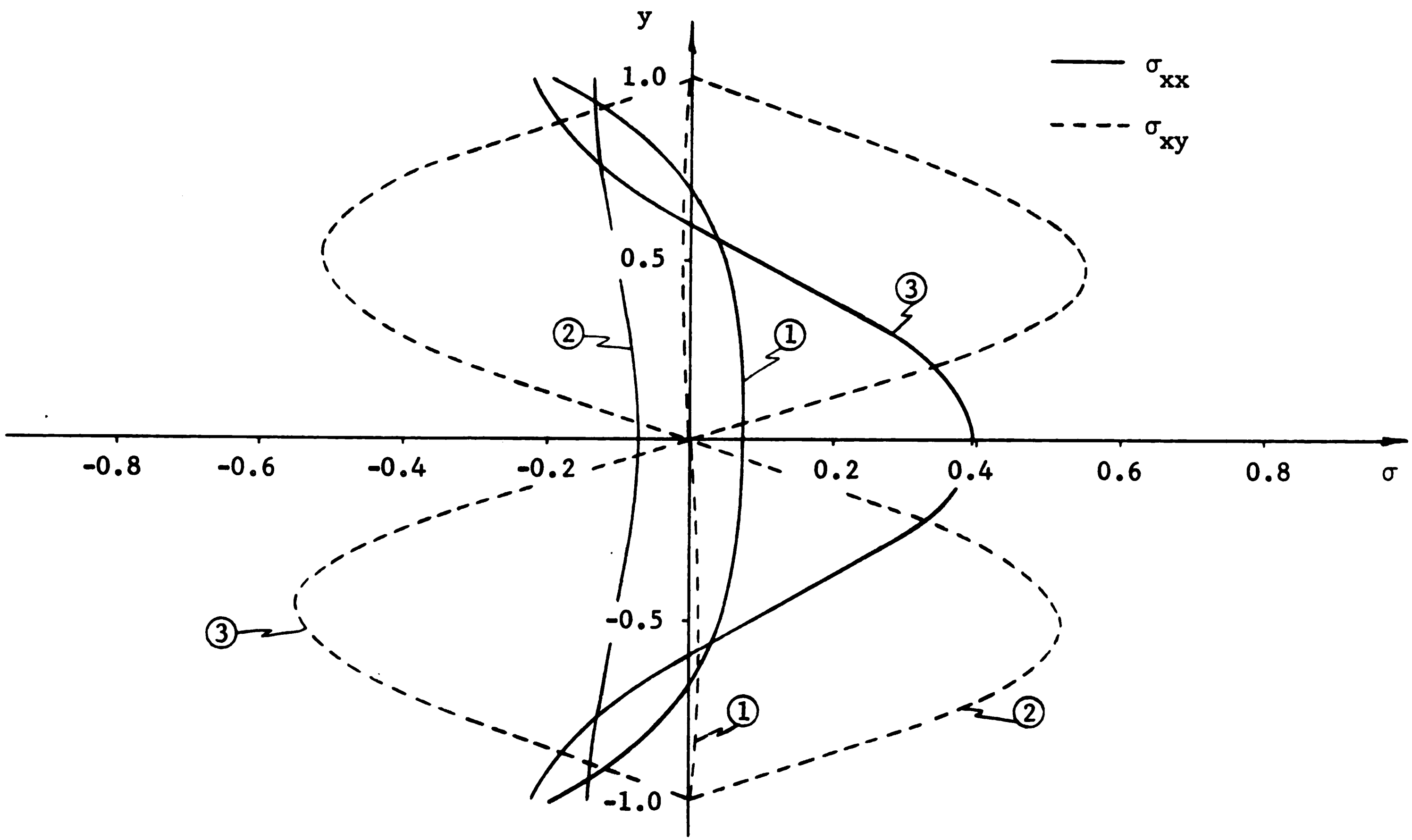

Figure VII-5 -- The Non-Decaying Wave Modes at x = 0, Problem 2

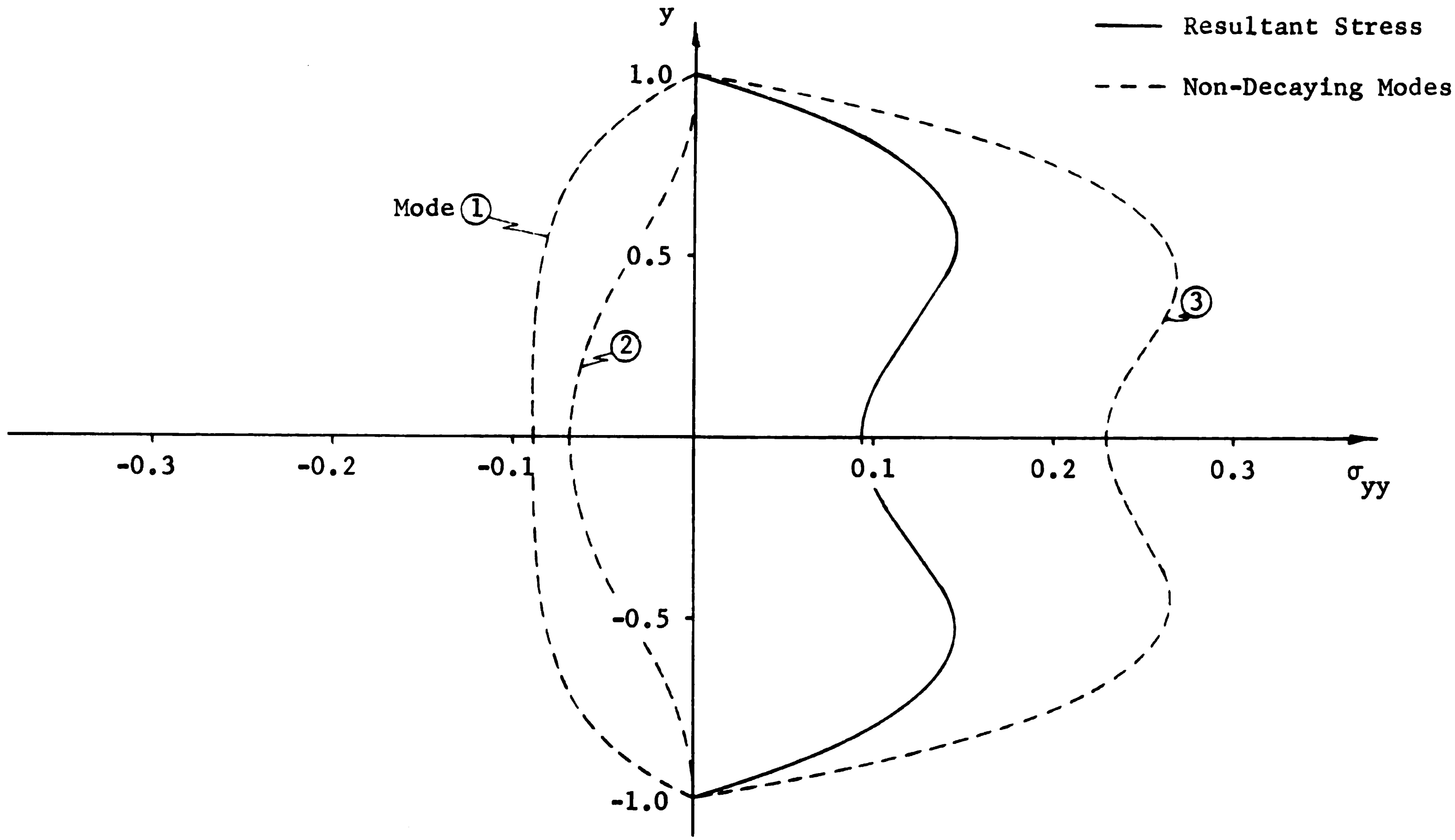

Figure VII-6 -- Stress σ_{yy} at x = 0, Problem 2

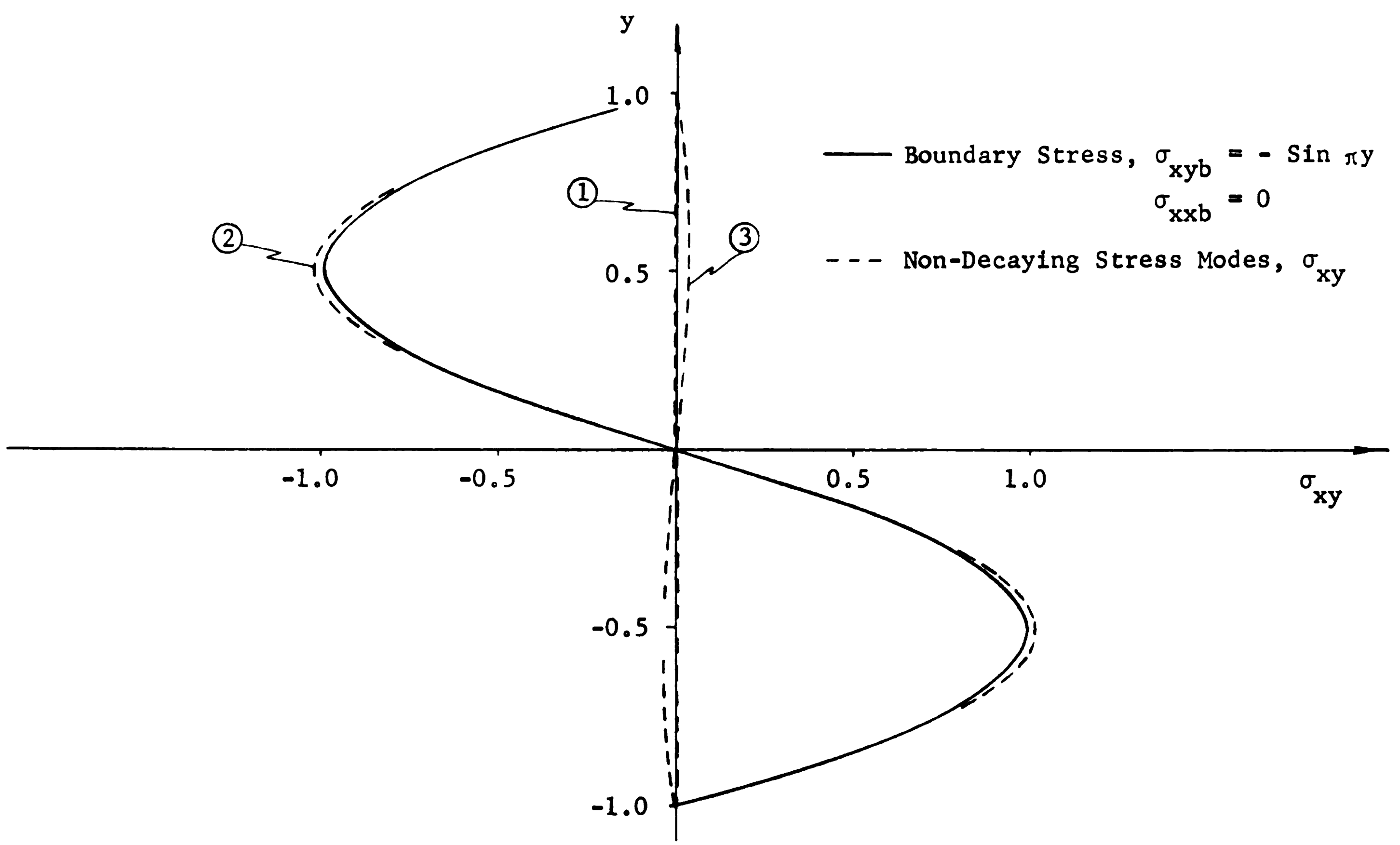

Figure VII-7 -- Boundary Stress σ_{xyb} and the Non-Decaying σ_{xy} Modes at x = 0, Problem 3

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APPENDICES

APPENDIX A

TABULATED EIGENVALUES

TABLE A-1 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 100 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	± .259341	+ i 0.000000	± .0371997	+ i 0.000000
2	1.384371	3.748499	1.125419	2.105550
3	1.676120	6.949802	1.551595	5.356036
4	1.858393	10.119137	1.775555	8.536538
5	1.991577	13.277181	1.929412	11.699072
6	2.096630	16.429796	2.046858	14.853977
7	2.183401	19.579346	2.141895	18.004865
8	2.257323	22.726982	2.221726	21.153355
9	2.321716	25.873337	2.290555	24.300292
10	2.378759	29.018789	2.351050	27.446158
11	2.429960	32.163579	2.405014	30.591255

TABLE A-2 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 1000 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	± .969392 + i	0.000000	± .373002 + i	0.000000
2	1.387431	3.714782	1.130700	2.040549
3	1.677591	6.932145	1.553632	5.332933
4	1.859280	10.107082	1.776676	8.522219
5	1.992178	13.268010	1.930134	11.688658
6	2.097068	16.422389	2.047367	14.845783
7	2.183736	19.573132	2.142276	17.998107
8	2.257588	22.721629	2.222023	21.147604
9	2.321933	25.868635	2.290794	24.295285
10	2.378940	29.014596	2.351247	27.441726
11	2.430112	32.159796	2.405180	30.587278
12	2.476535	35.304422	2.453862	33.732171
13	2.519015	38.448604	2.498226	36.876562
14	2.558169	41.592435	2.538975	40.020558
15	2.594481	44.735984	2.576654	43.164241
16	2.628334	47.879302	2.611694	46.307669
17	2.660042	51.022428	2.644439	49.450887
18	2.689860	54.165395	2.675173	52.593930
19	2.717999	57.308225	2.704127	55.736826

TABLE A-3 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 6380 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	± 4.295876 + i	0.000000	± 4.055999 + i	0.000000
2	2.421033	0.000000	2.237781	0.000000
3	1.355817	1.992252	1.130842	0.000000
4	1.702163	6.179568	1.577832	4.304994
5	1.877602	9.609329	1.798075	7.924157
6	2.005841	12.893890	1.945888	11.261712
7	2.107627	16.122062	2.059326	14.512666
8	2.192156	19.322040	2.151671	17.724623
9	2.264475	22.505794	2.229616	20.915484
10	2.327682	25.679321	2.297072	24.093585
11	2.383821	28.845969	2.356535	27.263356
12	2.434316	32.007763	2.409702	30.427379
13	2.480197	35.165999	2.457780	33.587264
14	2.522237	38.321548	2.501657	36.744066
15	2.561030	41.475016	2.542008	39.898511
16	2.597039	44.626840	2.579357	43.051111
17	2.630638	47.777341	2.614120	46.202239
18	2.662129	50.926763	2.646631	49.352174

TABLE A-4 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 10000 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	± 6.630079	+ i 0.000000	± 6.579686	+ i 0.000000
2	5.378575	0.000000	3.926267	0.000000
3	3.286863	0.000000	3.315675	0.000000
4	1.436153	0.000000	.805028	1.973710
5	1.517139	4.818572	1.710236	6.917090
6	1.826559	8.800626	1.913085	10.582463
7	1.983445	12.306705	2.043406	13.994701
8	2.095982	15.658226	2.142976	17.304375
9	2.185563	18.937701	2.224562	20.561275
10	2.260571	22.177246	2.294043	23.787170
11	2.325330	25.392200	2.354712	26.993212
12	2.382416	28.590880	2.408631	30.185738
13	2.433512	31.778207	2.457193	33.368631
14	2.479787	34.957288	2.501392	36.544408
15	2.522091	38.130183	2.541960	39.714775
16	2.561062	41.298318	2.579457	42.880930
17	2.597194	44.462708	2.614321	46.043740
18	2.630877	47.624099	2.646901	49.203849
19	2.662425	50.783048	2.677480	52.361744

TABLE A-5 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 31600 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	± 20.893342	+ i 0.000000	± 20.893334	+ i 0.000000
2	19.116382	0.000000	18.839587	0.000000
3	18.381749	0.000000	17.740090	0.000000
4	16.903148	0.000000	15.850131	0.000000
5	14.554569	0.000000	11.088598	0.000000
6	11.561352	0.000000	10.700487	0.000000
7	10.709123	0.000000	9.681430	0.000000
8	9.690573	0.000000	7.950855	0.000000
9	8.071672	0.000000	4.312302	0.000000
10	5.638132	0.000000	1.216463	0.000000
11	1.109372	6.327763	.382669	9.825665
12	0.000000	11.332493	0.000000	12.971777
13	.821718	14.202922	1.365751	16.518870
14	1.574184	18.660664	1.669688	20.694499
15	1.705364	22.653411	1.701246	24.557542
16	1.665112	26.420677	1.597806	28.253380
17	1.493388	30.065269	1.333585	31.868434
18	1.058952	33.689863	0.000000	35.246044
19	0.000000	35.977305	.363617	36.386089
20	1.174472	38.321627	1.461126	40.076479

TABLE A-6 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 63800 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	±42.125041 + i	0.000000	± 42.125041 + i	0.000000
2	38.691841	0.000000	38.577849	0.000000
3	38.387766	0.000000	38.121319	0.000000
4	37.777907	0.000000	37.356459	0.000000
5	36.855312	0.000000	36.272106	0.000000
6	35.603678	0.000000	34.845962	0.000000
7	33.993858	0.000000	33.041099	0.000000
8	31.980117	0.000000	30.802011	0.000000
9	29.496923	0.000000	24.795354	0.000000
10	26.477242	0.000000	22.382882	0.000000
11	23.205347	0.000000	22.202007	0.000000
12	22.171072	0.000000	21.644669	0.000000
13	21.657992	0.000000	20.797816	0.000000
14	20.824644	0.000000	19.682077	0.000000
15	19.670625	0.000000	18.195303	0.000000
16	18.256867	0.000000	16.543159	0.000000
17	16.281783	0.000000	14.153948	0.000000
18	14.064993	0.000000	10.156298	0.000000
19	11.648308	0.000000	7.960167	0.000000
20	4.201674	0.000000	.802532	13.652395
21	0.000000	11.271708	1.161230	20.331783
22	1.311928	17.237081	0.000000	24.405508
23	.292534	23.194187	.904466	27.419767
24	0.000000	25.992610	1.521996	32.102359

TABLE A-7 --ROOTS OF TRANSCENDENTAL EQUATIONS (6.1-1) AND (6.1-2)

FREQUENCY = 100000 CYCLES PER SECOND
POISSON RATIO = 1/3

N	EQUATION (6.1-1) ODD PROBLEM $_o\alpha_N$		EQUATION (6.1-2) EVEN PROBLEM $_e\alpha_N$	
1	± 66.070545 + i	0.000000	± 66.070545 + i	0.000000
2	60.177115	0.000000	60.654611	0.000000
3	59.629017	0.000000	60.381975	0.000000
4	58.893544	0.000000	59.926245	0.000000
5	57.965518	0.000000	59.284975	0.000000
6	56.837344	0.000000	58.454013	0.000000
7	55.498506	0.000000	57.427032	0.000000
8	53.934958	0.000000	56.195045	0.000000
9	52.128313	0.000000	54.745858	0.000000
10	50.054807	0.000000	53.063368	0.000000
11	47.684164	0.000000	51.126645	0.000000
12	44.979451	0.000000	48.908797	0.000000
13	41.904074	0.000000	40.226163	0.000000
14	38.478920	0.000000	36.751606	0.000000
15	35.378636	0.000000	34.853644	0.000000
16	34.900756	0.000000	34.455069	0.000000
17	34.463530	0.000000	33.829152	0.000000
18	33.836205	0.000000	32.991782	0.000000
19	33.033031	0.000000	31.995858	0.000000
20	32.012504	0.000000	30.813558	0.000000
21	30.800500	0.000000	29.399656	0.000000
22	29.420589	0.000000	27.860995	0.000000
23	27.730862	0.000000	23.511384	0.000000
24	24.000258	0.000000	21.533973	0.000000
25	20.984984	0.000000	18.332334	0.000000
26	17.915348	0.000000	12.807290	0.000000
27	15.191106	0.000000	10.498085	0.000000
28	6.107614	0.000000	.825374	16.772158
29	0.000000	13.641388	.553100	24.873496
30	1.230817	21.138086	0.000000	28.918273

APPENDIX B

ASYMPTOTIC SOLUTION OF THE ODD PROBLEM TRANSCENDENTAL EQUATION

The transcendental equation for the odd problem is

$$\frac{\tanh \alpha m}{\tanh \alpha n} = \frac{4mn}{(1+n^2)^2} \qquad (B.1)$$

where

$$\alpha^2 m^2 = \alpha^2 - \frac{\rho\omega^2}{E}(1 - \nu^2) \qquad (B.2)$$

$$\alpha^2 n^2 = \alpha^2 - \frac{2\rho\omega^2}{E}(1 + \nu) \quad . \qquad (B.3)$$

The eigenvalue α is found for a given frequency ω and Poisson ratio equal to 1/3. Thus, (B.2) and (B.3) can be written in terms of the constant "a"

$$a = \frac{\rho\omega^2}{E}(1 - \nu^2)$$

$$\alpha^2 m^2 = \alpha^2 - a$$

$$\alpha^2 n^2 = \alpha^2 - 3a \quad .$$

Substituting these two expressions into equation (B.1) yields

$$\frac{\tanh \sqrt{\alpha^2 - a}}{\tanh \sqrt{\alpha^2 - 3a}} = \frac{4\alpha^2 \sqrt{\alpha^2 - a}\ \sqrt{\alpha^2 - 3a}}{(2\alpha^2 - 3a)^2} \qquad (B.4)$$

The asymptotic solution yields values of α in the complex plane for α very large relative to the constant "a". Substituting $\alpha = x + iy$ gives

$$\sqrt{\alpha^2 - a} = \sqrt{x^2 - y^2 - a + 2ixy} = x + \epsilon_1 + i(y + \epsilon_2), \qquad (B.5)$$

where $\epsilon_1, \epsilon_2 \ll 1$.

Squaring both sides of (B.5), cancelling like terms, and neglecting the higher order terms of ϵ_i yields

$$x\epsilon_1 - y\epsilon_2 = -\frac{a}{2}$$

$$y\epsilon_1 + x\epsilon_2 = 0 \quad .$$

Solving these two equations for ϵ_1 and ϵ_2 gives

$$\epsilon_1 = -\frac{ax}{2y^2} \qquad (B.6)$$

$$\epsilon_2 = \frac{a}{2y} \quad . \qquad (B.7)$$

The right hand side of equation (B.4) becomes

$$\frac{4\alpha^2\sqrt{\alpha^2 - a}\,\sqrt{\alpha^2 - 3a}}{(2\alpha^2 - 3a)^2} = \frac{4\alpha^2[\alpha + \epsilon_1 + i\epsilon_2][\alpha + 3(\epsilon_1 + i\epsilon_2)]}{(2\alpha^2 - 3a)^2}$$

Substituting $\epsilon_c = \epsilon_1 + i\epsilon_2$, the right side is

$$\frac{4\alpha^2(\alpha + \epsilon_c)(\alpha + 3\epsilon_c)}{(2\alpha^2 - 3a)^2} = \frac{4\alpha^4 + 16\alpha^2\epsilon_c + 12\alpha^2\epsilon_c^2}{4\alpha^4 - 12\alpha^2 a + 9a^2} \quad .$$

As α becomes very large relative to "a", this ratio approaches 1.

The left side of (B.4) can be written as

$$\frac{\tanh\sqrt{\alpha^2 - a}}{\tanh\sqrt{\alpha^2 - 3a}} = \frac{\tanh[x + \epsilon_1 + i(y + \epsilon_2)]}{\tanh[x + 3\epsilon_1 + i(y + 3\epsilon_2)]} = \frac{\tanh(p + iq)}{\tanh(r + is)}$$

$$= \frac{\tanh p + i\tan q}{1 + i\tanh p\tan q} \cdot \frac{1 + i\tanh r\tan s}{\tanh r + i\tan s} \quad , \qquad (B.8)$$

where

$$p = x(1 - \frac{a}{2y^2}) \quad , \quad r = x(1 - \frac{3a}{2y^2})$$

$$q = y(1 + \frac{a}{2y^2}) \quad , \quad s = y(1 + \frac{3a}{2y^2}) \quad .$$

As X and Y become very large,

$$\tanh p \to \tanh x \to 1$$

$$\tanh r \to \tanh x \to 1$$

and (B.8) may be written as

$$\frac{\tanh x[1 - \tan q \tan s] + i[\tan q + \tanh^2 x \tan s]}{\tanh x[1 - \tan q \tan s] + i[\tan s + \tanh^2 x \tan q]} \quad . \tag{B.9}$$

The ratio (B.9) must approach unity for large α. This requires that $\tan q$ equal $\tan s$.

$$\tan q = \tan s$$

$$\tan (y + \frac{a}{2y}) = \tan(y + \frac{3a}{2y})$$

Using the identity for the tangent of the sum, and replacing the tangent of the small argument with the argument itself yields

$$\frac{\tan y + \frac{a}{2y}}{1 - \frac{a}{2y} \tan y} = \frac{\tan y + \frac{3a}{2y}}{1 - \frac{3a}{2y} \tan y} \quad .$$

As y increases without limit, $\tan y$ should remain of the order of magnitude 1. Thus, $\tan q$ and $\tan s$ approach unity. Assume for y:

$$y = (\frac{4n+1}{4})\pi + \delta \ , \quad \delta << 1 \quad . \tag{B.10}$$

$$\tan q = \tan(y + \frac{a}{2y}) = \frac{\tan y + \frac{a}{2y}}{1 - \frac{a}{2y} \tan y} = 1 \tag{B.11}$$

For y defined by (B.10),

$$\tan y = \tan[(\frac{4n+1}{4})\pi + \delta] = \frac{1+\delta}{1-\delta} \quad .$$

Substituting into (B.11) yields

$$\frac{1+\delta}{1-\delta} + \frac{a}{2y} = 1 - \frac{a}{2y}(\frac{1+\delta}{1-\delta}) \quad ,$$

or $\delta = -\frac{a}{2y}$ which is negligible for large values of y. Likewise, requiring $\tan s$ to approach unity and solving for δ gives $\delta = -\frac{3a}{2y}$ which is negligible for large values of y. Thus, for large values of y, the imaginary part of α is approximated by

$$y = (\frac{4n+1}{4})\pi \quad .$$

Having isolated the imaginary part of α, the real part, x, is determined from equation (B.4) numerically with the computer.