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DISCRETE AND CONTINUOUS MODELS OF CRITICAL DAMPING FOR A SIMPLY SUPPORTED BEAM

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ABSTRACT

DISCRETE AND CONTINUOUS MODELS OF CRITICAL DAMPING FOR A SIMPLY SUPPORTED BEAM

By

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The concept of "critical damping" for both multidegree of freedom, discrete linear systems and distributed parameter systems, has been investigated by a number of researchers. Conditions under which a system is critically damped, which depend on certain combinations of the coefficient operators, have been obtained. Conditions under which a damping mechanism provides underdamping, overdamping, or mixed damping have also been described.

In this thesis, specific results are applied to a particular continuous system, a simply supported beam. Results are obtained by considering both the analytic model of the beam and an approximate, discrete version of this model obtained by the finite element method. A variety of damping mechanisms are included in a number of example problems.

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I. INTRODUCTION

The damped, free vibrations of both multidegree of freedom discrete linear systems and continuous systems have received considerable attention. The first matter to receive attention concerned the definitions of the various possible types of damping. For the simplest systems, the various types of damping, which result in systems that are underdamped (oscillatory), overdamped (nonoscillatory), or critically damped (also nonoscillatory), are easily defined in terms of the parameters of the system. Defining similar criteria for more complex systems has been an area of current interest.

Another area of current interest concerns the qualitative nature of the motion of a system, under a given type of damping. It is desired to determine, without resorting to lengthy calculations, the type of damping a particular damping mechanism provides. Ideally, this problem is to be attacked by examining the parameters of the system, not by solving the equations of motion of the system, or, alternatively, resorting to experiment.

The inverse of this problem has also been addressed. Given a system with a predetermined desired motion, it is necessary to identify a damping mechanism, or mechanisms, which will insure that the desired motion is attained. Again, this problem is to be attacked without resorting to lengthy calculations or experiment.

In this paper, recent results on each of these topics will be applied to a particular continuous system, a simply supported beam. The last of the above topics will be addressed by considering an analytic model of the beam. The first two of the above problems will be attacked by considering an approximate, discrete version of this model obtained by the finite element method. First, however, a brief summary of other results on these topics will be presented.

It is well known that the qualitative nature of the solution of one of the simplest vibratory systems, a single degree of freedom spring-mass-damper system (spring and damper in parallel) can be determined by examining the coefficients of the ordinary differential equation which describes its motion. This equation is

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.1}$$

where m, c, and k are the positive values of the mass, damper, and spring, respectively, and x(t) is the "displacement" from static equilibrium of the system. Equation (1.1) can be rewritten as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0 \tag{1.2}$$

where the natural frequently of the undamped system is $\omega = \sqrt{\frac{k}{m}}$, and the damping ratio is $\zeta = c/2\sqrt{km}$. The system is said to be overdamped if $\zeta > 1$, critically damped if $\zeta = 1$, and underdamped if $\zeta < 1$. For more complicated single degree of freedom systems, the analysis is still straightforward and simple.

The equation of motion for the more complex, multidegree of freedom, discrete, damped linear system is an analogous matrix differential equation

$$M\ddot{\vec{x}}(t) + C\dot{\vec{x}}(t) + K\vec{x}(t) = 0 \tag{1.3}$$

where M and K are the positive definite, symmetric, mass and stiffness matrices respectively, and C is the positive semi-definite symmetric damping matrix. For this problem, the concepts of over, under, and critical damping are more subtle and require closer scrutiny than the single degree of freedom case. In fact, a number of results addressing the qualitative nature of damping of solutions of this equation have been published, each shedding some light in different areas.

In previous work Duffin [1], in considering (1.3) defined an over-damped system in terms of a function of the quadratic forms of the M, C, and K matrices. More recently, Nicholson [2] defined an underdamped system in terms of the eigenvalues of the mass, damping, and stiffness matrices. Müller [3], upon considering Nicholson's work, defined an underdamped system similar to that of Duffin and derived a sufficient condition for underdamping in terms of the definiteness of the coefficient matrices.

Recently, additional results have been obtained by Beskos and Boley [4]. Techniques used in [4] are different from those found in matrix theory which were used in [1,2,3]. Essentially Beskos and Boley address the following question: Given $\{c_k\}_{k=1}^{m}$ different damping coefficients, find the locus of combinations of the $\{c_k\}$ leading to critically damped motion. Their approach requires calculating critical damping surfaces in a parameter space (this surface corresponds to the concept of finding the "minimum value of damping required for non-oscillation"). By calculating critical damping surfaces in parameter space, one is led to the following observations: A combination of damping coefficient which correspond to a point in the parameter space "above" the critical damping surface yields a solution to (1.3) which is overdamped (nonoscillatory); a combination of damping coefficients yields an underdamped (oscillatory) solution of (1.3) if the point in parameter space corresponding to the combination lies "below" the critical surface.

The conditions derived by Duffin, Nicholson, and Beskos and Boley require substantial calculations. In the cases of Duffin and Nicholson, the calculations are used for verification of their respective conditions; there are different reasons for the computational load found in [4]. It appears that the technique of Beskos and Boley works well when the damping is Rayleigh ($C = \alpha K + \beta M$) and the order of the system is small. This stems from the fact that the technique requires differentiations for which there are presently no efficient numerical schemes available.

More complete results, which do not require substantial calculations, have recently been obtained by Inman and Andry [5]. In this paper, the definition of critical damping is stated in terms of the coefficient matrices in a manner analogous to the single degree of freedom case. A critical damping matrix is defined in terms of the unique positive definite square roots of the matrices M and K. Having obtained C_{cr}, a system is defined as overdamped, critically damped, or underdamped if the matrix $(C - C_{cr})$ is positive definite, zero, or negative definite, respectively. A fourth possibility, mixed damping, is defined if $(C - C_{cr})$ is indefinite.

The computational load for these results is nominal and does not encounter any special problems due to high order. In the special case that the damping matrix is diagonalized by the undamped modal matrix, all the conditious presented become necessary and sufficient; for the general case (C not diagonalized by the undamped modal matrix), sufficient conditions are presented which are easily verified. For these reasons, the results in [5] are more general than Müller's and reduce to Müller's in the special case.

II. A DISCRETE MODEL OF CRITICAL DAMPING

2.0 Introduction

In this chapter the first two problems outlined in the general introduction are considered for the special case of a simply supported beam. The beam is discretized using a finite element model and the definitions of the various types of damping in [5], for multidegree of freedom discrete systems, are directly applied to the discretized continuous system.

The definitions of the various types of damping, including critical damping, are presented in the first section. A discussion of the physical implications of these definitions is also included. The consistent finite element model of the simply supported beam is presented in the second section. The coefficient matrices necessary to apply the definitions from section one are obtained from this model.

The next two sections present derivations which make the definition of section one easier to apply. In the first, a simple method for calculating the critical damping matrix with respect to the original coordinate system is derived. In the second, it is shown that the test for the type of damping a given damping matrix provides can be carried out in the original coordinate system. Working in the original coordinate system eliminates the need to calculate square roots and inverses of matrices. Numerical accuracy is increased and computing costs are decreased.

In the last section, the critical damping matrix is calculated and introduced into the equations of motion of the beam. These equations are solved directly using MSC/NASTRAN finite element code, and the resulting motion is shown to be indeed critically damped. Other example damping matrices are compared to the critical damping matrix and the results are presented.

2.1 Definitions

Consider a multidegree of freedom, discrete, damped linear system described by the matrix differential equation

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = 0 \qquad (2.1.1)$$

where M and K are the positive definite, symmetric mass and stiffness matrices, respectively, and C is the positive semi-definite damping matrix. In previous work, Inman and Andry [5] rewrite (2.1.1) as

$$\ddot{\vec{y}} + \tilde{C}\dot{\vec{y}} + \tilde{K}\vec{y} = 0 \qquad (2.1.2)$$

where $\tilde{C} = M^{-\frac{1}{2}} CM^{-\frac{1}{2}}$, $\tilde{K} = M^{-\frac{1}{2}} KM^{-\frac{1}{2}}$, and $M^{\frac{1}{2}}$ denotes the unique positive definite square root of the positive definite matrix M.

The critical damping matrix for this system is defined to be

$$C_{cr} = 2(M^{-\frac{1}{2}} KM^{-\frac{1}{2}})^{\frac{1}{2}} = 2\tilde{K}^{\frac{1}{2}} \qquad (2.1.3)$$

in direct analogy to the single degree of freedom case. The motion of the system is defined to be critically damped if $\tilde{C} = C_{cr}$, to be over-damped if $(\tilde{C} - C_{cr})$ is positive definite, and to be underdamped if $(\tilde{C} - C_{cr})$ is negative definite. A fourth possibility, mixed damping, is defined if $(\tilde{C} - C_{cr})$ is indefinite.

Defining the various types of damping in this way results in physically reasonable eigenvalues. As usual, the eigenvalues of (2.1.2) are taken to be the 2n roots of the polynomial equation

$$|\lambda^2 I + \lambda\tilde{C} + \tilde{K}| = 0 \tag{2.1.4}$$

where $|\cdot|$ denotes the determinant. In particular, Inman and Andry show that (i) if (2.1.1) is critically damped then there are at most n distinct negative real eigenvalues and no complex eigenvalues, (ii) if (2.1.1) is underdamped the eigenvalues are all complex and appear in complex conjugate pairs with negative real parts, which corresponds to all modes of the system oscillating in damped harmonic motion, and (iii) if (2.1.1) is overdamped, then the eigenvalues are all negative real numbers, and none of the modes oscillates. For the special case that $\tilde{C}$ is diagonalized by the undamped modal matrix (2.1.1) exhibits mixed damping if and only if there is at least one real eigenvalue and at least one complex conjugate pair of eigenvalues. One of the modes of the damped system will oscillate and at least one will not.

2.2 The Finite Element Model

The partial differential equation of motion for the damped, transverse, free vibrations of a simply supported beam is

$$\frac{\partial^4 v(x,t)}{\partial x^4} + c\,\frac{\partial}{\partial t}\, v(x,t) + \frac{\rho}{EI}\,\frac{\partial^2 v(x,t)}{\partial t^2} = 0 \tag{2.2.1}$$

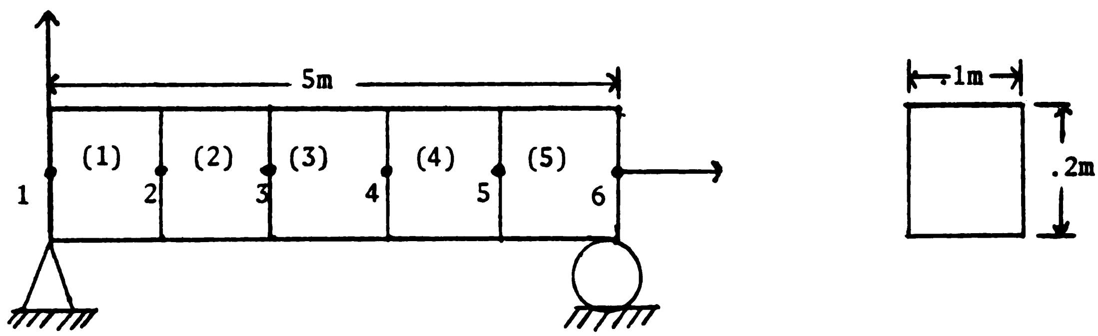
with boundary conditions

$$v(0,t) = v(L,t) = 0 \tag{2.2.2}$$

$$v''(0,t) = v''(L,t) = 0 \tag{2.2.3}$$

where $v(x,t)$ is the transverse deflection of the beam. This equation is valid for the case in which damping of constant magnitude c, is applied over the entire length of the beam, and the mass per unit length ρ, modulus of elasticity E, and moment of inertia I, are all assumed to be constant.

This equation will be approximated by a five element, consistent finite element model [6]. The finite element model for the simply supported steel beam being considered is shown in Figure 1.



The simply supported beam with finite element grid.

The beam cross section.

Figure 1

The physical quantities assumed are

$$E = 2.0 \times 10^{11} \text{ PA} \qquad m = 7.852 \times 10^{3} \frac{\text{kg}}{\text{m}^3}$$

$$I = 6.67 \times 10^{-5} \text{ m}^4 \qquad A = .02\text{m}^2 \text{ (cross sectional area)}$$

The node number for each node is located to the left and slightly below each grid point, and the element number is enclosed by parentheses on each element.

Consider a typical element of the beam, (e), shown in Figure 2.

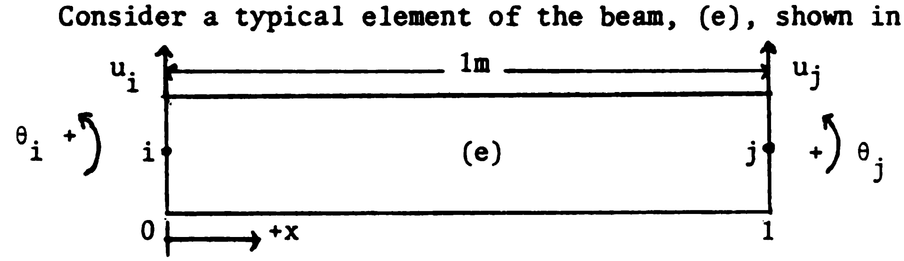


Figure 2. A Typical Element of the Beam

Each node has two degrees of freedom associated with it; one vertical translation u and one rotation θ. Four unknowns must be determined for each element. A cubic displacement function, whose four coefficients will be determined in terms of the four unknown displacements, will be assumed of the form

$$v^{(e)}(x,t) = a_0(t) + a_1(t)\ x + a_2(t)\ x^2 + a_3(t)\ x^3 \qquad (2.2.4)$$

Substituting the known boundary conditions for each element, and making the small angle assumption, $\tan\theta = \theta$, gives

$$v^{(e)}(0,t) = a_0(t) = u_i(t)$$
$$v^{(e)}(1,t) = a_0(t) + a_1(t) + a_2(t) + a_3(t) = u_j(t)$$
$$v^{(e)'}(0,t) = a_1(t) = -\theta_i(t)$$
$$v^{(e)'}(1,t) = a_1(t) + 2a_2(t) + 3a_3(t) = \theta_j(t)$$

where the superscript (e) denotes a typical element and ( )' denotes differentiation of ( ) with respect to the spatial coordinate x. Solving for the unknown coefficients $a_i(t)$, collecting terms, and rearranging gives

$$v^{(e)}(x,t) = N_1^{(e)}(x)u_i(t) + N_2^{(e)}(x)\theta_i(t) + N_3^{(e)}(x)u_j(t) + N_4^{(e)}(x)\theta_j(t)$$

or in vector notation

$$v^{(e)}(x,t) = \vec{N}^{(e)}(x)\ \vec{w}^{(e)}(t)$$

where

$$N_1(x) = 1 - 3x^2 + 2x^3 \qquad \vec{w}^{(e)}(t) = \begin{bmatrix} u_i(t) \\ \theta_i(t) \\ u_j(t) \\ \theta_j(t) \end{bmatrix}$$
$$N_2(x) = -x(1 - 2x + x^2)$$
$$N_3(x) = 3x^2 - 2x^3$$
$$N_4(x) = x(x - x^2)$$

The interpolating polynomials chosen are by definition Hermitian polynomials since both the function and its first derivative are specified at nodes i and j. The vector $N^{(e)}(x)$ is composed of the shape functions for the element. As an approximation to equation (2.2.1), the kinetic and potential energy of the beam and the Rayleigh dissipation function will be calculated from the finite element model. These quantities will then be introduced into Lagrange's equations of motion and the approximate equations of motion will be obtained.

The kinetic energy of a typical element, dropping the superscript (e) for convenience, is

$$
\begin{aligned}
T &= \frac{1}{2} \int_{vol} m \, \dot{v}^T \, \dot{v} \, d \, vol \\
&= \frac{mA}{2} \int_0^1 \dot{\vec{w}}^T \, \vec{N}^T \, \vec{N} \, \dot{\vec{w}} \, dx \\
&= \frac{mA}{2} \dot{\vec{w}}^T \left[\int_0^1 \vec{N}^T \, \vec{N} \, dx \right] \dot{\vec{w}} \\
&= \frac{1}{2} \dot{\vec{w}}^T \, [m]^{(e)} \, \dot{\vec{w}}
\end{aligned}
$$

where $[m]^{(e)} = mA \int_0^1 \vec{N}^T(x) \, \vec{N}(x) \, dx$ is the element mass matrix

$$
[m]^{(e)} = \frac{mA}{420} \begin{bmatrix} 156 & -22 & 54 & 13 \\ -22 & 4 & -13 & -3 \\ 54 & -13 & 156 & 22 \\ 13 & -3 & 22 & 4 \end{bmatrix}
$$

The Rayleigh dissipation function for a typical element is

$$
\begin{aligned}
F &= \frac{1}{2} \int_0^1 c(x) \, \dot{\vec{w}}^T \, \vec{N}^T(x) \, \vec{N}(x) \, \dot{\vec{w}} \, dx \\
&= \frac{1}{2} c \, \dot{\vec{w}}^T \int_0^1 \vec{N}^T(x) \, \vec{N}(x) \, dx \, \dot{\vec{w}}
\end{aligned}
$$

$$= \frac{1}{2} \dot{\vec{w}}^T [c]^{(e)} \dot{\vec{w}}$$

where $[c]^{(e)}$ is the element damping matrix

$$[c]^{(e)} = c \int_0^1 \vec{N}^T(x) \vec{N}(x) \, dx$$

$$= \frac{c}{420} \begin{bmatrix} 156 & -22 & 54 & 13 \\ -22 & 4 & -13 & -3 \\ 54 & -13 & 156 & 22 \\ 13 & -3 & 22 & 4 \end{bmatrix}$$

The potential energy of a typical element is

$$V = \frac{EI}{2} \int_0^1 \left(\frac{\partial^2 v}{\partial x^2}\right) dx$$

But $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2}{\partial x^2} [\vec{N}^T(x) \vec{w}(t)] = \vec{B}^T(x) \vec{w}(t)$ where $\vec{B}(x) = \frac{\partial^2}{\partial x^2} (\vec{N}(x))$

Substituting gives

$$V = \frac{EI}{2} \vec{w}^T \int_0^1 \vec{B}^T \vec{B} \, dx \, \vec{w}$$

$$= \frac{1}{2} \vec{w}^T [k]^{(e)} \vec{w}$$

where $[k]^{(e)} = EI \int_0^1 \vec{B}^T \vec{B} \, dx$ is the element stiffness matrix

$$[k]^{(e)} = EI \begin{bmatrix} 6 & -3 & -6 & -3 \\ -3 & 2 & 3 & 1 \\ -6 & 3 & 6 & 3 \\ -3 & 1 & 3 & 2 \end{bmatrix}$$

Using the usual direct stiffness method of assembly [7] one obtains the mass matrix M_0, the viscous damping matrix C_0, and the stiffness matrix K_0, for the entire beam. Hence, the kinetic and potential energy and the dissipation function for the entire beam is

$$T = \frac{1}{2} \dot{\vec{w}}^T M_0 \dot{\vec{w}}, \quad V = \frac{1}{2} \vec{w}^T K_0 \vec{w}, \quad F = \frac{1}{2} \dot{\vec{w}}^T C_0 \dot{\vec{w}}$$

where $\vec{w}^T = [u_1 \; \theta_1 \; u_2 \; \ldots \; u_6 \; \theta_6]$

Substituting into Lagrange's equations of motion.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{w}_i}\right) - \frac{\partial T}{\partial w_i} + \frac{\partial f}{\partial \dot{w}_i} + \frac{\partial V}{\partial w_i} = 0, \quad i = 1, 2, \ldots, 12$$

gives

$$M_0 \ddot{\vec{w}} + C_0 \dot{\vec{w}} + K_0 \vec{w} = 0$$

This is a system of 12 equations in 12 unknowns. A usual occurrence in constructing a finite element model is that the system does not satisfy the vertical translation boundary conditions. It also contains spurious equations which arise by differentiating with respect to known constant quantities. The overall analysis is not affected by eliminating the spurious equations and satisfying the boundary conditions. In particular, notice $u_1 = u_{11} = 0$. Substituting this fact into the system will eliminate the first and eleventh columns of M_0, C_0, and K_0. Also notice the first and eleventh equations were obtained by differentiating with respect to the known quantities u_1 and u_{11}, and hence must be eliminated from the final system.

The final equations of motion for the simply supported, steel beam, with the boundary conditions satisfied, are

$$M\ddot{\vec{w}} + C\dot{\vec{w}} + K\vec{w} = 0$$

For the final mass, stiffness, and damping matrices see Appendix A.

2.3 A Simple Method for Calculating C_{cr}

Consider again the matrix differential equation of motion for a continuous system obtained from a finite element model of the system

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = 0 \tag{2.3.1}$$

Proceeding in the same manner as Inman and Andry in [5], let $\vec{x} = M^{-\frac{1}{2}}\vec{y}$ and premultiply (2.3.1) by $M^{-\frac{1}{2}}$ to give

$$M^{-\frac{1}{2}} MM^{-\frac{1}{2}} \ddot{\vec{y}} + M^{-\frac{1}{2}} CM^{-\frac{1}{2}} \dot{\vec{y}} + M^{-\frac{1}{2}} KM^{-\frac{1}{2}} \vec{y} = 0 \tag{2.3.2}$$

Let S be the orthogonal, undamped modal matrix ($S^TS = I$), let $\vec{y} = S\vec{z}$, and premultiply (2.3.2) by S^T to obtain

$$S^TM^{-\frac{1}{2}} MM^{-\frac{1}{2}} S\ddot{\vec{z}} + S^TM^{-\frac{1}{2}}CM^{-\frac{1}{2}} S\dot{\vec{z}} + S^TM^{-\frac{1}{2}} KM^{-\frac{1}{2}} S\vec{z} = 0 \tag{2.3.3}$$

Notice $S^TM^{-\frac{1}{2}} MM^{-\frac{1}{2}} S = I$ and $S^TM^{-\frac{1}{2}} KM^{-\frac{1}{2}} S = \Lambda$, a diagonal matrix with the eigenvalues of the undamped system along the diagonal. Let $\Phi = M^{-\frac{1}{2}} S$. Then $\Phi^T = S^T M^{-\frac{1}{2}T}$. But $(M^{-\frac{1}{2}})^T = M^{-\frac{1}{2}}$, since M is symmetric. Substituting into (2.3.3) gives

$$\Phi^TM\Phi\ddot{\vec{z}} + \Phi^TC\Phi\dot{\vec{z}} + \Phi^TK\Phi\vec{z} = 0 \tag{2.3.4}$$

Notice Φ is orthogonal with weight M ($\Phi^TM\Phi = I$) and $\Phi^TK\Phi = \Lambda$. Hence, from vibration theory [8], Φ is the undamped modal matrix normalized with respect to the mass matrix M.

For critical damping, $M^{-\frac{1}{2}} CM^{-\frac{1}{2}} = C_{cr}$ in (2.3.3). But, from matrix theory [9], $S^T C_{cr} S = 2\Lambda^{\frac{1}{2}}$. Thus (2.3.3) becomes

$$\ddot{\vec{z}} + 2\Lambda^{\frac{1}{2}} \dot{\vec{z}} + \Lambda\vec{z} = 0 \tag{2.3.5}$$

Comparing the coefficient of $\dot{\vec{z}}$ in (2.3.4) and (2.3.5) one concludes that for critical damping

$$\Phi^T C_{cr} \Phi = 2\Lambda^{\frac{1}{2}}$$

or

$$C_{cr} = 2(\Phi^T)^{-1} \Lambda^{\frac{1}{2}} \Phi^{-1} \tag{2.3.6}$$

Note, however, that $\Phi^{-1} = \Phi^T M$, and $(\Phi^T)^{-1} = M\Phi$. Substituting into (2.3.6) gives

$$C_{cr} = 2M\Phi\Lambda^{\frac{1}{2}}\Phi^T M \tag{2.3.7}$$

Thus, the critical damping matrix in the original coordinates is obtained by simply performing the indicated matrix multiplications.

2.4 A Note On The Comparison Test

Given an arbitrary damping matrix C, with respect to the original coordinates, the test for the type of damping provided by C can be carried out in the original coordinates. By definition, $\tilde{C} - C_{cr}$ is positive definite if and only if

$$\vec{x}^T(\tilde{C} - C_{cr})\, \vec{x} > 0 \tag{2.4.1}$$

for $\vec{x}$ an arbitrary nonzero vector. Denoting the critical damping matrix in original coordinates as $\hat{C}_{cr}$, we have $C_{cr} = M^{-\frac{1}{2}}\, \hat{C}_{cr}\, M^{-\frac{1}{2}}$. Substituting into (2.4.1) with $\tilde{C} = M^{-\frac{1}{2}}\, CM^{-\frac{1}{2}}$ gives

$$\vec{x}^T(M^{-\frac{1}{2}}\, CM^{-\frac{1}{2}} - M^{-\frac{1}{2}}\, \hat{C}_{cr}\, M^{-\frac{1}{2}})\vec{x} > 0$$

or

$$\vec{x}^T M^{-\frac{1}{2}}(C - \hat{C}_{cr})\, M^{-\frac{1}{2}}\, \vec{x} > 0 \tag{2.4.2}$$

Letting $\vec{y} = M^{-\frac{1}{2}}\, \vec{x}$ in (2.4.2) gives

$$\vec{y}^T(C - \hat{C}_{cr})\vec{y} > 0 \qquad (2.4.3)$$

But, since $\vec{x}$ is an arbitrary nonzero vector, so is $\vec{y}$. Hence $\tilde{C} = C_{cr}$ is positive definite if and only if $C - \hat{C}_{cr}$ is positive definite. A similar argument also works for the negative definite and indefinite cases.

2.5 Results

The critical damping matrix for the simply supported steel beam under consideration is calculated using (2.3.7). Both the eigenvalues and the eigenvectors, normalized with respect to the mass matrix, of the undamped system are needed to calculate the matrices in (2.3.7). They are obtained by using MSC/NASTRAN finite element code. The NASTRAN model (see Appendix A) of the beam consists of 5 CBAR elements. As a model check, the consistent mass and stiffness matrices generated by NASTRAN are printed out. These matrices are exactly the same as the mass and stiffness matrices calculated by hand.

As a further model check, the eigenvalues generated by NASTRAN, using the Given's method, are compared to the eigenvalues obtained by solving the separated partial differential equation of motion of the beam, see Table 1. Good agreement in the first four modes indicates that the NASTRAN model is working properly. Table 1 also shows that better numerical accuracy in the higher modes can be obtained by modeling the beam with a larger number of elements. An examination of the eigenvectors generated by NASTRAN shows the expected sinusoidal behavior. As a final check, all calculations were performed using IBM double precision scientific subroutines, with nearly identical results.

The matrix $\Lambda^{\frac{1}{2}}$ is formed by replacing the diagonal of the 10×10 identity matrix with the square root of the eigenvalues obtained from

TABLE I

COMPARISON OF THREE NASTRAN MODELS WITH THEORETICAL MODEL

MODE NO.	MSC/NASTRAN FREQUENCIES (HZ) 5 ELEMENT MODEL	10 ELEMENT MODEL	20 ELEMENT MODEL	DIFFERENTIAL EQUATION FREQUENCIES (HZ)	PERCENT DIFFERENCES *	**	***
1	0.183147E+02	0.183128E+02	0.183128E+02	0.183128E+02	0.01	0.0	-0.00
2	0.733722E+02	0.732587E+02	0.732517E+02	0.732514E+02	0.16	0.01	0.00
3	0.166123E+03	0.164903E+03	0.164821E+03	0.164816E+03	0.79	0.05	0.00
4	0.299753E+03	0.293489E+03	0.293036E+03	0.293005E+03	2.30	0.17	0.01
5	0.508140E+03	0.459625E+03	0.457939E+03	0.457821E+03	10.99	0.39	0.03
6	0.733498E+03	0.664494E+03	0.695613E+03	0.659262E+03	11.26	0.79	5.51
7	0.106796E+04	0.910045E+03	0.898207E+03	0.897329E+03	19.01	1.42	0.10
8	0.151681E+04	0.119901E+04	0.117396E+04	0.117202E+04	29.42	2.30	0.17
9	0.204149E+04	0.153351E+04	0.148723E+04	0.148334E+04	37.63	3.38	0.26
10	0.232859E+04	0.203256E+04	0.183851E+04	0.183128E+04	27.16	10.99	0.39

* ((MODEL 1-THEORETICAL)/THEORETICAL)*100%

** ((MODEL 2-THEORETICAL)/THEORETICAL)*100%

*** ((MODEL 3-THEORETICAL)/THEORETICAL)*100%

NASTRAN, ordered from smallest to largest. The modal matrix of the undamped system, ϕ, is formed by placing the i^{th} eigenvector (which NASTRAN has already normalized with respect to the mass matrix) in the i^{th} column of ϕ. The critical damping matrix, with respect to the original coordinates, is calculated using (2.3.7), see Appendix B.

The accuracy of this procedure is checked by performing the matrix multiplication in (2.3.4). As expected, the result is the decoupled system (2.3.5).

The matrix differential equation of motion (2.3.1) is solved directly on the computer using NASTRAN with $C = \hat{C}_{cr}$. A solution of (2.3.1) of the form $\vec{x} = \vec{\psi}e^{\lambda t}$ is assumed, where $\vec{\psi}$ is a complex vector and λ is a complex number. The resulting complex eigenvalue problem is solved by NASTRAN using the upper Hessenberg method. Very little alteration of the original NASTRAN model is necessary to perform these calculations. The real eigenvalue extraction technique is replaced with a complex eigenvalue extraction technique and the critical damping matrix is entered using direct matrix input bulk data cards.

Ten complex conjugate pairs of eigenvalues and eigenvectors are calculated. An examination of the eigenvalues (see Appendix B) shows that the motion of the system is indeed critically damped. The frequency of oscillation of any mode, $f_j = |I_m(\lambda_j)|/2\pi$, where λ_j is the eigenvalue of the j^{th} mode, is less than .38 Hertz. These frequencies are nonzero due to round off error in the above calculations. The eigenvalues appear in complex conjugate pairs, with imaginary parts that approach zero. Thus, there are only ten distinct negative, real eigenvalues, as expected for the critically damped case.

As an example of a damping matrix that theory predicts will overdamp the system, let $C = 2\hat{C}_{cr}$. Then $C - \hat{C}_{cr} = \hat{C}_{cr}$. From matrix theory [10], a matrix A is positive definite if and only if there exists a nonsingular matrix W such that $A = W^T W$. Letting $A = \hat{C}_{cr}$ and $W = \sqrt{2}\, \Lambda^{\frac{1}{4}}\, \Phi^{-1}$, it is easily seen that $\hat{C}_{cr}$ is positive definite. Note that W is nonsingular since both $\sqrt{2}\, \Lambda^{\frac{1}{4}}$ and Φ^{-1} are nonsingular. The frequency of oscillation of all modes, as calculated directly by NASTRAN (see Appendix B) using $C = 2\hat{C}_{cr}$ in (2.3.1), is zero, as expected for an overdamped system.

As an example of a damping matrix that theory predicts will underdamp the system, let $C = .1\, C_{cr}$. Then $C - \hat{C}_{cr} = -.9C_{cr}$, and hence is negative definite. Ten complex conjugate pairs of eigenvalues, all with nonzero imaginary parts (see Appendix B) are calculated using NASTRAN. Thus the eigenvalues are of the expected type for an underdamped system and all modes oscillate.

As a final example, let C in (2.3.1) represent damping of constant magnitude, $c = 903468.04\ \frac{NT/m}{m/sec}$ distributed over the entire beam. The theory of the next chapter predicts that the resulting system will display mixed damping, with the first five modes being nonoscillatory and the last five modes being oscillatory. See example 8, Appendix C for the actual finite element damping matrix associated with this type of damping.

The theory of Inman and Andry also predicts that the resulting system will display mixed damping. The matrix $C - \hat{C}_{cr}$, (see Appendix B) is clearly indefinite. Calculating the eigenvalues of the system with this damping matrix in place, it is clear that the first four modes are nonoscillatory with zero frequency and the other modes are oscillatory with frequencies between 220 and 2300 Hertz, (see Appendix B).

III. AN ANALYTIC MODEL OF CRITICAL DAMPING

3.0 Introduction

In this chapter, the last of the problems outlined in the general introduction is considered for the special case of a simply supported beam. This problem is approached by considering an analytic model of the beam, consisting of a partial differential equation with boundary conditions. Both discrete damping, centered at a finite number of points on the beam, and damping distributed over the entire beam are considered.

The analytic model of both the undamped and damped simply supported beam is presented in the first two sections of this chapter. Although this information is well known and readily available (for instance, see [11]), it is presented here to aid in the understanding of the definitions and derivations presented in the remainder of this chapter.

In the next section, an expression for the amount of damping necessary to insure a particular mode to be critically damped or overdamped is derived, for a particular configuration of discrete damping. This expression is obtained by applying a direct extension to continuous systems of the definitions presented in Chapter 2. In particular, it is shown that, for a system possessing classical normal modes, the qualitative nature of the motion of the system can be determined by examining the definiteness of certain combinations of the coefficient operators.

In the final section, two sets of examples, which illustrate these results, are presented. A finite element model is obtained for each example and the resulting damping matrices are introduced into the equations of motion for the beam. The equations are solved directly using MSC/NASTRAN finite element code and the results are discussed.

3.1 The Undamped Simply Supported Beam

The equation of motion for the undamped, transverse free vibrations of a uniform beam, with constant mass per unit length ρ, modulus of elasticity E, and moment of inertia I is

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\rho}{EI} \frac{\partial^2 v(x,t)}{\partial t^2} = 0 \tag{3.1.1}$$

where $v(x,t)$ is the transverse deflection of the beam. For the simply supported beam being considered, the deflection and the bending moment are zero at each end. This leads to the two boundary conditions

$$v(0,t) = v(L,t) = 0 \tag{3.1.2}$$

$$v''(0,t) = v''(L,t) = 0 \tag{3.1.3}$$

where ( )' denotes differentiation of ( ) with respect to the spatial coordinate x.

Assume that the solution to the above system is separable in time and space, and is of the form

$$v(x,t) = V(x)\ f(t) \tag{3.1.4}$$

Introducing the assumed solution into (3.1.1) and simplifying yields

$$\frac{EI}{\rho} \frac{1}{V(x)} \frac{d^2 V(t)}{dx^4} = \frac{-1}{f(t)} \frac{d^2 f(t)}{dt^2} \tag{3.1.5}$$

The left side of (3.1.5) depends only on x, and the right side depends only on t. Both x and t are independent variables, so (3.1.5) has a solution only if both sides are constant. Choosing a positive value ω^2, for this constant, called a characteristic value or eigenvalue, leads to two ordinary differential equations

$$\frac{d^4V(x)}{dx^4} - \frac{\rho}{EI}\omega^2 v(x) = 0 \tag{3.1.6}$$

$$\frac{d^2f(t)}{dt^2} + \omega^2 f(t) = 0 \tag{3.1.7}$$

The choice of a positive value for the eigenvalues of the problem leads to the harmonic, rather than exponential, solution of (3.1.7)

$$f(t) = C_1 \cos \omega t + C_2 \sin \omega t \tag{3.1.8}$$

which is consistent with the fact that a conservative system has constant total energy.

Equation (3.1.6) is a fourth order, homogeneous, ordinary differential equation. Four boundary conditions are required for its solution. Substituting the assumed solution (3.1.4) into the boundary conditions (3.1.2) and (3.1.3) yields the four required boundary conditions

$$V(0) = V(L) = 0 \tag{3.1.9}$$

$$V''(0) = V''(L) = 0. \tag{3.1.10}$$

The general solution of (3.1.6), with $\beta^4 = \frac{\rho}{EI}\omega^2$, is

$$V(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \tag{3.1.11}$$

Using boundary condition (3.1.9) and (3.1.10), we obtain the frequency equation

$$\sin \beta L = 0 \tag{3.1.12}$$

with solution $\beta L = n\pi$, $\quad n = 1, 2, 3, \ldots$ or

$$\beta^4 = \left(\frac{n\pi}{L}\right)^4 = \frac{\rho}{EI}\,\omega^2 \qquad n = 1, 2, 3, \ldots$$

Thus, this system possesses a countably infinite sequence of eigenvalues

$$\omega_n^2 = \frac{EI}{\rho}\left(\frac{n\pi}{L}\right)^4 \qquad n = 1, 2, 3, \ldots \tag{3.1.13}$$

Corresponding to the eigenvalues ω_n, one obtains the eigenfunctions or natural modes of the system

$$V_n(x) = a_n \sin\frac{n\pi x}{L} \tag{3.1.14}$$

which is the solution of the ordinary differential equation (3.1.6) with boundary conditions (3.1.9) and (3.1.10).

The final solution of (3.1.1) is obtained from the expansion theorem of modal analysis and is of the form

$$v(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\,\eta_n(t) \tag{3.1.15}$$

where $\eta_n(t)$ are time dependent generalized coordinates. Introducting (3.1.15) into (3.1.1), multiplying by $\sin\frac{r\pi x}{L}$ and integrating from zero to L gives

$$\sum_{n=1}^{\infty}\int_0^L \eta_n(t)\,\sin\frac{r\pi x}{L}\,\frac{\partial^4}{\partial x^4}\left(\sin\frac{n\pi x}{L}\right)dx$$

$$+ \sum_{n=1}^{\infty} \ddot{\eta}_n(t)\frac{\rho}{EI}\,\sin\frac{n\pi x}{L}\;\sin\frac{r\pi x}{L}\,dx = 0$$

Simplifying yields

$$\ddot{\eta}_n(t) + \omega_n^2\,\eta_n(t) = 0 \qquad n = 1, 2, 3, \ldots \tag{3.1.16}$$

This is a set of <u>uncoupled</u> ordinary differential equations with solution

$$\eta_n(t) = \eta_n(0) \cos \omega_n t + \frac{\dot{\eta}_n(0)}{\omega_n} \sin \omega_n t$$

where $$\eta_n(0) = \int_0^L \rho \sin \frac{n\pi x}{L} \; v(x,0) \; dx \qquad (3.1.17)$$

$$\dot{\eta}_n(0) = \int_0^L \rho \sin \frac{n\pi x}{L} \; \dot{v}(x,0) \; dx \qquad (3.1.18)$$

Thus knowledge of the initial deflection and velocity is required to proceed further.

3.2 The Damped Simply Supported Beam

The equation of motion for the damped, transverse free vibrations of a uniform beam, in operator notation, is

$$L[v(x,t)] + \frac{\partial}{\partial t} C[v(x,t)] + M(x) \frac{\partial^2 v(x,t)}{\partial t^2} = 0 \qquad (3.2.1)$$

Where $L = EI \dfrac{\partial^4}{\partial x^4}$ and $M(x) = \rho$. C is a linear homogeneous differential operator consisting of derivatives with respect to the spatial coordinate, x, but not with respect to time. C represents the effects of continuous or discrete viscous damping. The boundary conditions for this system are (3.1.2) and (3.1.3), unchanged from the undamped system.

Proceeding in the same manner as in the undamped case, assume the solution of (3.2.1) to be a superposition of the product of the eigenfunctions of the undamped system times time dependent generalized coordinates, $\eta_n(t)$

$$v(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \eta_n(t) \qquad (3.2.2)$$

Substituting into (3.2.1), multiplying by $\sin \frac{r\pi x}{L}$, integrating from zero to L, and simplifying gives

$$\ddot{\eta}_n(t) + \sum_{r=1}^{\infty} C_{nr}\,\dot{\eta}_n(t) + \omega_n^2\,\eta_n(t) = 0 \qquad n = 1,2,3,\ldots \tag{3.2.3}$$

where

$$C_{nr} = \int_0^L \sin\frac{n\pi x}{L}\;C[\sin\frac{r\pi x}{L}]\,dx \tag{3.2.4}$$

Equation (3.2.3) represents an infinite set of <u>coupled</u> ordinary differential equations. In the special case where the operator C represents constant damping distributed over the entire beam

$$C_{nr} = \begin{cases} C\displaystyle\int_0^{L_2} \sin^2\frac{n\pi x}{L}\,dx = \frac{CL}{2} & n = r \\ 0 & n \neq r \end{cases} \tag{3.2.5}$$

where C is a viscous damping constant with units of distributed force per unit velocity. This type of damping leads to an infinite <u>uncoupled</u> set of ordinary differential equations.

$$\ddot{\eta}_n(t) + \frac{c}{\rho}\,\dot{\eta}_n(t) + \omega_n^2\,\eta(t) = 0 \qquad n = 1,\ 2,\ 3,\ \ldots \tag{3.2.6}$$

or

$$\ddot{\eta}_n(t) + 2\zeta_n\,\omega_n\,\dot{\eta}_n(t) + \omega_n^2\,\eta_n(t) = 0 \quad n = 1,2,3,\ldots \tag{3.2.7}$$

The notation in (3.2.7) was chosen to render it similar in structure to the equation of a single degree-of-freedom system. For critical damping of the n^{th} mode, $\zeta_n = 1$.

Hence $\quad \zeta_n = \dfrac{c}{2\rho\omega_n} = 1 \quad \text{or} \quad c = 2\rho\omega_n = 2\rho\left(\dfrac{n\pi}{L}\right)^2\sqrt{\dfrac{EI}{\rho}}$

$$= 2\left(\frac{n\pi}{L}\right)^2\sqrt{EI\rho}\;. \tag{3.2.8}$$

We note from equation (3.2.7) that an infinite amount of damping would be required in order to overdamp every mode of the system; the best one could obtain would be the overdamping of a finite number of modes.

3.3 A Result on Discrete Damping

Consider a simply supported beam with discrete damping centered at a finite number of points, m, on the beam. For this case, the operator C in equation (3.2.1) is

$$C = \sum_{i=1}^{m} c_i(x)\ \chi_{E_i}(x) \tag{3.3.1}$$

where $\chi_{E_i}(x) = \begin{cases} 0 & x \notin E_i \\ 1 & x \in E_i \end{cases}$ is the characteristic function for the set E_i. The set E_i is an open interval (a_i, b_i) where $0 \leq a_i < b_i \leq L$, and, for any pair of open intervals, $E_i \cap E_j = \emptyset$, for $i \neq j$. Defining C in this way reflects the fact that the damping is not applied to a set of discrete points, but instead, is distributed over a collection of discrete open intervals.

Define the linear operator L_1, to be

$$L_1[\Phi] = \sum_{i=1}^{m} \frac{c_i(x)}{\rho}\ \chi_{E_i}(x)\ \Phi(x) \tag{3.3.2}$$

and the linear differential operator L_2 to be

$$L_2[\Phi] = \frac{1}{\rho} L[\Phi] \tag{3.3.3}$$

where $L = EI \frac{\partial^4}{\partial x^4}$ as in equation (3.2.1). Also define the operator L_3 to be

$$L_3[\Phi] = (4L_2 - L_1^2)[\Phi] \tag{3.3.4}$$

Inman and Andry have shown in [12] that the type of damping provided by the damping mechanism described by (3.3.1) depends on the definiteness of the operator L_3. Recall, an operator L is said to be positive definite if and only if, for any comparison function u, the following holds,

$$\int_D uL[u]dD \geq 0 \tag{3.3.5}$$

and equality holds only for $u = 0$. A function is a comparison function if it is sufficiently differentiable over the domain D and it satisfies all the boundary conditions of the eigenvalue problem. One set of comparison functions for the simply supported beam being considered is the eigenfunctions of the undamped system, $\left\{\sin \frac{n\pi x}{L}\right\}_{n=1}^{\infty}$.

Utilizing the above definitons, the definiteness of the operator L_3 is seen to depend on the sign of the quantity d_n, defined by

$$d_n = 4\int_0^L v_n(x)\ L_2[v_n(x)]dx - \int_0^L \sum_{i=1}^{m}\sum_{j=1}^{m} \chi_{E_i}(x)\chi_{E_j}(x)C_i(x)C_j(x)v_n^2(x)$$

$$= 4\int_0^L v_n(x)\ L_2[v_n(x)\ dx - \sum_{i=1}^{m}\int_0^L C_i^2\ (x)\ \chi_{E_i}(x)\ v_n^2\ (x)\ dx \tag{3.3.6}$$

where $v_n(x)$ is the n^{th} eigenfunction of the undamped system. It can be shown that the n^{th} mode is

underdamped if $d_n > 0$,.

critically damped if $d_n = 0$ and

overdamped if $d_n < 0$.

Notice the similarity of these results to the results obtained for the discrete, multidegree of freedom systems discussed in section 2.1. These results can be used to determine either the placement of damping or the magnitude of damping necessary for the system to be critically damped when one or the other of the parameters is in some way constrained.

Consider the special case of damping of constant magnitude c_i placed on the beam such that the intervals over which the damping is distributed are centered about the maximum and minimum values of the eigenfunctions. For this case, the open intervals E_i are of the form

$$E_i = \left(\left[\frac{2i-1}{2n} - \delta\right]L, \left[\frac{2i-1}{2n} + \delta\right]L\right) \quad i = 1, \ldots, n$$

with $2n\delta \leq 1$. Evaluating the first integral in (3.3.6) with $v_n(x) = \sin\frac{n\pi x}{L}$ and L_2 as defined in (3.3.3) gives

$$4\int_0^L \sin\frac{n\pi x}{L}\left(\frac{EI}{\rho}\right)\frac{\partial^4}{\partial x^4}\left(\sin\frac{n\pi x}{L}\right)dx = 2\,\frac{EIL}{\rho}\left(\frac{\pi n}{L}\right)^4 . \quad (3.3.7)$$

Evaluating the second integral gives

$$\sum_{i=1}^{m}\int_0^L \frac{C_i^2(x)}{\rho^2}\chi_{E_i}(x)\sin^2\frac{n\pi x}{L}\,dx = \sum_{i=1}^{m}\frac{C_i^2}{\rho^2}\int_{(\frac{2i-1}{2n}-\delta)L}^{(\frac{2i-1}{2n}+\delta)L}\sin^2\frac{n\pi x}{L}\,dx$$

$$= \sum_{i=1}^{m} L\,\frac{C_i^2}{\rho^2}\left[\delta - \frac{1}{4n\pi}\left(\sin 2n\pi\left(\frac{2i-1}{2n}+\delta\right) - \sin 2n\pi\left(\frac{2i-1}{2n}-\delta\right)\right)\right]$$

$$= \sum_{i=1}^{m}\frac{L\,C_i^2}{2n\pi\rho^2}\,[2n\pi\delta + \sin 2n\pi\delta\,] . \quad (3.3.8)$$

For the n^{th} mode of the beam to be critically damped or over-damped $d_n \leq 0$. Substituting (3.3.7,8) into (3.3.6) gives

$$\frac{2EI}{\rho L^3}(\pi n)^4 - \frac{L}{\rho^2 2n\pi}\sum_{i=1}^{m}C_i^2\,[2n\pi\delta + \sin 2n\pi\delta] \leq 0$$

or

$$4\rho EI\left(\frac{\pi n}{L}\right)^4\frac{n\pi}{[2n\pi\delta + \sin 2n\pi\delta]} \leq \sum_{i=1}^{m}C_i^2 \quad (3.3.9)$$

If the magnitude of damping is the same over each interval E_i, then for critical damping we have

$$(c_{cr})_n = 2(\frac{\pi n}{L})^2 \sqrt{\frac{\rho EI(\pi)}{2n\pi\delta + \sin 2n\pi\delta}} \tag{3.3.10}$$

Notice, if $2n\delta = 1$, that is, the damping is of constant magnitude, continuously distributed along the entire beam, there results

$$(c_{cr})_n = 2\ (\frac{\pi n}{L})^2\ \sqrt{\rho EI} \tag{3.3.11}$$

This is the same value as was obtained by considering the partial differential equation for this type of damping.

3.4 Examples

Two sets of example problems, consisting of four related examples each, are considered. In the first set, the amount of damping necessary to critically damp the first mode of vibration is calculated. In each example, the damping is distributed over a single interval whose center coincides with the geometric center of the beam. The width of this interval is varied from .04m to 5m; a width of .04m being a physical model of damping applied at a single point, and a width of 5m representing damping applied over the entire beam. The intermediate cases considered include damping distributed over the entire center element and center three elements. See Table 2 for the amounts of damping necessary to critically damp the first mode using these configurations of dampers.

In the second set of example problems, the amount of damping necessary to critically damp the fifth mode of vibration is calculated. In each example, the damping is distributed over five intervals, each of whose centers coincides with the geometric center of an element. The widths of each of these intervals is varied from .04m to 1m; the first case representing damping applied at five discrete points, and the last case representing damping applied over the entire beam. The intermediate cases considered include damping applied over widths of .2m and .6m. See Table 2 for the amounts of damping necessary to critically damp the fifth mode using these configurations of dampers.

Table 2

Summary of Data for Examples 1-8

Example Number	Damping Magnitude $\left(\frac{NT/m}{m/sec}\right)$	Complex Eigenvalues (real)	(imag)	Frequency (Hertz)
1	127853.760	-6.30172	±114.9086	18.28827
2	58084.188	-72.70735	± 90.34698	14.37917
3	38035.899	-109.5956	± 35.55132	5.658167
4	36138.722	-115.0625	± 1.682878	.2678384
5	3196344.000			
6	1452119.700	Examples 5 through 8		
7	950897.460	see Appendix		
8	903468.040	for this information		

The original finite element damping model (see Chapter 2) can be modified to represent the various cases of discrete damping described above by recognizing a characteristic each of these cases has in common. In each case, the damping is applied over one or more regions, each of whose center coincides with the center of an element. Hence an element damping matrix for this general type of damping is calculated, and the final finite element damping matrix is obtained using the same procedures employed in the original model.

The general element damping matrix for discrete damping is obtained in a manner analogous to the original model. Assume the damping is distributed over the intervals $E_i = (a_i, b_i)$, where $0 \le a_i < b_i \le 1$, and i = number of elements which experience damping. Then the Rayleigh dissipation function for a typical element experiencing damping is

$$
\begin{aligned}
F &= \frac{1}{2} \int_0^1 c(x)\ \chi_{E_i}(x)\ \dot{\vec{w}}^T\ \vec{N}^T\ \vec{N}\ \dot{\vec{w}}\ dx \\
&= \frac{1}{2}\ \ddot{w}^T \left[c \int_{a_i}^{b_i} \vec{N}^T(x)\ \vec{N}(x)\ dx \right] \dot{\vec{w}} \\
&= \frac{1}{2}\ \dot{\vec{w}}^T\ [c]^{(e)}\ \dot{\vec{w}}
\end{aligned}
$$

where $[c]^{(e)}$ is the element damping matrix

$$
[c]^{(e)} = c \int_{a_i}^{b_i} \vec{N}^T(x)\ \vec{N}(x)\ dx
$$

$$
= \frac{c}{420} \begin{bmatrix} 156(b_i-a_i) & -22(b_i^2-a_i^2) & 54(b_i-a_i) & 13(b_i^2-a_i^2) \\ & 4(b_i^3-a_i^3) & -13(b_i^2-a_i^2) & -3(b_i^3-a_i^3) \\ & & 156(b_i-a_i) & 22(b_i^2-a_i^2) \\ \text{SYM} & & & 4(b_i^3-a_i^3) \end{bmatrix}
$$

The Rayleigh dissipation function for elements which do not experience damping is identically zero, and hence the element damping matrix for this type of damping is the 4×4 zero matrix. The final finite element damping matrix for each case is obtained by proceeding as in the original model, using the appropriate elements damping matrices. See Appendix C for the damping matrices for each example described above.

The discretized matrix differential equations of motion (2.1.1) is solved directly on the computer using NASTRAN with the various damping matrices in place. The eigenvalues and frequency of the first mode of the first four examples are listed in Table 2. Notice, as the width of the region over which the damping is distributed is increased, the magnitude of the real part of the eigenvalue also increases. This indicates that the motion of the first mode decays faster as the width of the region over which the damping is applied is increased. Also notice, as the width of the region over which the damping is applied is increased, the imaginary part of the eigenvalues and the frequency decreases. Critical damping is attained only in the last example, while the first three examples exhibit underdamped motion. This could be due to the finite element model itself, which consistently overestimates these quantities. Indeed, the imaginary part of the eigenvalue for the last example can be shown to be zero, by examining the decoupled ordinary differential equations which govern the motion of this example.

The eigenvalues and frequencies of all ten modes of the last four examples can be found in Appendix D. The first four modes should be critically damped. Again, as the width of the regions over which the damping is applied is increased, the theoretically predicted behavior is more closely achieved. Indeed, the fifth example shows only the first mode overdamped, the sixth example shows only the first two modes overdamped, the seventh shows the first three modes overdamped, and the last example shows the first four modes overdamped, with the remainder underdamped. Again, this behavior could be due to the finite element model. This model results in accurate eigenvalues for only the first four modes of the last example. This can be seen by examining Table 3 where the theoretically predicted eigenvalues are compared to the eigenvalues obtained from the finite element model.

Table 3

Comparison of NASTRAN Model with Analytical Model

Mode No.	THEORETICAL Eigenvalues (Real)	(Imag.)	NASTRAN Eigenvalues (Real)	(Imag.)
1	-2.3021644		-2.302657	
1	-5750.9155		-5750.801	
2	-37.058599		-37.18228	
2	-5716.1591		-5715.922	
3	-192.86618		-196.0541	
3	-5560.3515		-5557.051	
4	-666.28015		-702.3069	
4	-5086.9375		-5050.801	
5	-2864.0435		-2876.552	-1385.286
5	-2889.1742		-2876.554	+1385.286
6	-2876.6088	±5961.0677	-2876.554	±3600.778
7	-2876.6088	±9698.1044	-2876.554	±6062.379
8	-2876.6088	±13557.925	-2876.554	±9085.953
9	-2876.6088	±17730.181	-2876.556	±12500.37
10	-2876.6088	±22281.889	-2876.558	±14345.40

IV. CONCLUSION

The damped transverse free vibrations of a simply supported beam have been considered. The question of critical damping for this beam has been addressed in two different ways. Both a continuous model and a discretized approximation to this model have provided information.

The continuous model of the beam was discretized using a five element, consistent finite element model. A critical damping matrix for this model was calculated, and subsequently shown to critically damp the first ten modes of vibration simultaneously. Various other damping matrices were also shown to provide the predicted type of damping.

It should be emphasized that the critical damping matrix which has been calculated is very heavily model dependent. A different finite element model will result in a different, although possibly similar (in structure), critical damping matrix. Alternatives to the finite element model presented in this paper include a model consisting of a larger number of elements and a lumped parameter finite element model. Note that the latter of these alternatives would not be appropriate for this analysis. The mass matrix in a lumped parameter model is singular, not positive definite. Thus the theory on which these calculations are based would not apply.

It has already been demonstrated that a model consisting of a larger number of elements provides more accurate results. A critical damping matrix developed for such a model would critically damp a larger number of modes. Naturally, the physical effort involved in computing such a critical damping matrix would also increase. However, the theory upon which these calculations are based does not increase in complexity as the order of the system is increased. Thus, the number of modes which are critically damped is limited only by the amount of physical effort one is willing to expend in the computation of the critical damping matrix.

The critical damping matrix which has been developed was used primarily for comparison. A given damping matrix was compared to the critical damping matrix to determine the type of damping it would provide. Although this is a very immediate and important use for this matrix, a more satisfying use would be the direct application of this matrix to the system. This, however, requires that the critical damping matrix be interpreted physically, i.e., what magnitude of damping, placed where on the beam would result in this damping matrix.

This question is difficult to answer because the critical damping matrix is fully populated. A typical finite element damping matrix possesses a very orderly "block diagonal" structure. A closer look at the critical damping matrix reveals an approximate block diagonal structure. One possible cause of the nonzero "off-block" terms could be that the columns of the orthogonal modal matrix used to calculate the critical damping matrix are not quite orthogonal. This is a matter which requires further investigation.

The question of critical damping for the beam was addressed in a second way, utilizing a continuous model of the beam. An expression for critical damping of a given mode was developed for a particular configuration of discrete regions on the beam over which the damping was applied. The model was discretized using the same finite element model as in the first approach, with a damping matrix modified to describe discrete damping. Several examples were considered.

The results from these examples were mixed. The model worked well for the cases in which the damping was distributed over the entire beam, or a large fraction of the entire beam. However, the model worked less well for the cases in which the damping was more truly discrete. This is not surprising, since the original, consistent finite element model was designed to model continously distributed parameter system. Perhaps a different finite element model, such as a lumped model, would give improved results in the discrete cases.

The above calculations were carried out for a particular configuration of discrete damping. This configuration was chosen for a number of reasons. It was symmetric about the center of the beam, it simplified the analysis, and it simplified the finite element model. However, at this time, it is impossible to say if this particular configuration was physically the best choice for this problem. Determining the best possible configuration of discrete damping for this problem is another matter which requires further study.

APPENDICES

APPENDIX A - FINITE ELEMENT MATRICES AND THE UNDAMPED NASTRAN MODEL

THE MASS MATRIX IS

$$M=\begin{bmatrix}
1.4956200 & -4.8607600 & -1.1217100 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-4.8607600 & 116.6580000 & 0.0 & 20.1908000 & 4.8607600 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-1.1217100 & 0.0 & 2.9912400 & -4.8607600 & -1.1217100 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 20.1908000 & -4.8607600 & 116.6580000 & 0.0 & 20.1908000 & 4.8607600 & 0.0 & 0.0 & 0.0 \\
0.0 & 4.8607600 & -1.1217100 & 0.0 & 2.9912400 & -4.8607600 & -1.1217100 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 20.1908000 & -4.8607600 & 116.6580000 & 0.0 & 20.1908000 & 4.8607600 & 0.0 \\
0.0 & 0.0 & 0.0 & 4.8607600 & -1.1217100 & 0.0 & 2.9912400 & -4.8607600 & -1.1217100 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 20.1908000 & -4.8607600 & 116.6580000 & 0.0 & 4.8607600 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 4.8607600 & -1.1217100 & 0.0 & 2.9912400 & -1.1217100 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 4.8607600 & -1.1217100 & 1.4956200
\end{bmatrix}$$

THE DAMPING MATRIX IS

$$\frac{C}{420}\begin{bmatrix}
4.0000 & -13.0000 & -3.0000 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-13.0000 & 312.0000 & 0.0 & 54.0000 & 13.0000 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-3.0000 & 0.0 & 8.0000 & -13.0000 & -3.0000 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 54.0000 & -13.0000 & 312.0000 & 0.0 & 54.0000 & 13.0000 & 0.0 & 0.0 & 0.0 \\
0.0 & 13.0000 & -3.0000 & 0.0 & 8.0000 & -13.0000 & -3.0000 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 54.0000 & -13.0000 & 312.0000 & 0.0 & 54.0000 & 13.0000 & 0.0 \\
0.0 & 0.0 & 0.0 & 13.0000 & -3.0000 & 0.0 & 8.0000 & -13.0000 & -3.0000 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 54.0000 & -13.0000 & 312.0000 & 0.0 & 13.0000 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 13.0000 & -3.0000 & 0.0 & 8.0000 & -3.0000 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 13.0000 & -3.0000 & 4.0000
\end{bmatrix}$$

THE STIFFNESS MATRIX IS

0.5336D+08	0.8004D+08	0.2668D+08	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.8004D+08	0.3202D+09	0.0	-0.1601D+09	-0.8004D+08	0.0	0.0	0.0	0.0	0.0
0.2668D+08	0.0	0.1067D+09	0.8004D+08	0.2668D+08	0.0	0.0	0.0	0.0	0.0
0.0	-0.1601D+09	0.8004D+08	0.3202D+09	0.0	-0.1601D+09	-0.8004D+08	0.0	0.0	0.0
0.0	-0.8004D+08	0.2668D+08	0.0	0.1067D+09	0.8004D+08	0.2668D+08	0.0	0.0	0.0
0.0	0.0	0.0	-0.1601D+09	0.8004D+08	0.3202D+09	0.0	-0.1601D+09	-0.8004D+08	0.0
0.0	0.0	0.0	-0.8004D+08	0.2668D+08	0.0	0.1067D+09	0.8004D+08	0.2668D+08	0.0
0.0	0.0	0.0	0.0	0.0	-0.1601D+09	0.8004D+08	0.3202D+09	0.0	-0.8004D+08
0.0	0.0	0.0	0.0	0.0	-0.8004D+08	0.2668D+08	0.0	0.1067D+09	0.2668D+08
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.8004D+08	0.2668D+08	0.5336D+08

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N A S T R A N   E X E C U T I V E   C O N T R O L   D E C K   E C H O

```
ID GRAY,VIBBEAMC!
APP DISP
TIME 5
SOL 25
DIAG 9, 13, 14, 17, 21, 22
$THE NEXT STATEMENTS WILL CAUSE THE FINAL MASS AND STIFFNESS MATRICES
$TO BE PRINTED.
ALTER 142
MATGPR GPLD,USETD,SILD,MAA//C,N,A/C,N,AS
MATGPR GPLD,USETD,SILD,KAA//C,N,A/C,N,AS
ENDALTER
CEND
```

VIBRATION ANALYSIS OF AN
UNDAMPED SIMPLY SUPPORTED BEAM

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C A S E   C O N T R O L   D E C K   E C H O

```
CARD
COUNT
  1      TITLE = VIBRATION ANALYSIS OF AN
  2      SUBTITLE = UNDAMPED SIMPLY SUPPORTED BEAM
  3      ECHO = BOTH
  4      SPC = 74
  5      METHOD = 60
  6      OUTPUT
  7      DISP = ALL
  8      SUBCASE 1
  9      LABEL=OUTPUT FOR MODES 1 THRU 10
 10      MODES=10
 11      SUBCASE 11
 12      DISP=NONE
 13      VELO=NONE
 14      ACCEL=NONE
 15      BEGIN BULK
```

```
                     I N P U T   B U L K   D A T A   D E C K   E C H O
.   1  ..   2  ..   3  ..   4  ..   5  ..   6  ..   7  ..   8  ..   9  ..  10  .
$
$      ****   GRID  POINTS    ****
$
GRID,1,0,0.0,0.0,0.0,0,,,
GRID,2,0,1.0,C.0,0.0,C,,,
GRID,3,0,2.0,C.0,0.0,C,,,
GRID,4,0,3.0,0.0,0.0,C,,,
GRID,5,0,4.0,0.0,0.0,C,,,
GRID,6,0,5.0,C.0,0.0,0,,,
$
$      ****   AUXILIARY  GRID  POINT    ****
$
GRID,200,0,100.0,100.0,0.C,0,,,
$
$      ****   ELEMENT  DATA    ****
$
CBAR,1,10,1,2,200,,,2,,
CBAR,2,10,2,3,200,,,2,,
CBAR,3,10,3,4,200,,,2,,
CBAR,4,10,4,5,200,,,2,,
CBAR,5,10,5,6,200,,,2,,
$
$ SINCE ALL ELEMENTS HAVE THE SAME LENGTH, GEOMETRIC PROPERTIES,
$ MATERIAL PROPERTIES,AND THE SAME DIRECTION COSINES, THEY WILL
$ ALL POSSESS THE SAME MASS AND STIFFNESS MATRICES RELATIVE TO
$ THE LOCAL AXES.  IT IS POSSIBLE TO CALCULATE THE MASS AND STIFFNESS
$ MATRICES FOR ONE ELEMENT AND USE THE INFORMATION FOR THE REMAINING
$ ELEMENTS.
$
$ THIS IS ACCOMPLISHED BY USING THE CNGRNT STATEMENT.
$
CNGRNT,1,2,3,4,5
$
$      **** CHOOSE REAL EIGENVALUE EXTRACTION TECHNIQUE ****
$
EIGR,50,GIV,,,,10,,0,+E1
+E1,MASS
EIGR,60,MGIV,0,2.+03,,10,,0,+E2
+E2,MASS
$
$      ****     STATIC CONDENSATION     ****
$
$
$      ****   GEOMETRIC  PROPERTIES    ****
$
PBAR,1C,20,.02,,6.67-05,,,,,
$
$      ****   MATERIAL  DATA    ****
$
MAT1,2C,2.0+11,,,7.852+C3,,,,,
```

```
                         I N P U T   B U L K   D A T A   D E C K   E C H O

              .   1  ..  2  ..  3  ..  4  ..  5  ..  6  ..  7  ..  8  ..  9  ..  10  .
              $
              $       ****    PURGES    ****
              $
              $ PURGE NONEXISTING DEGREES OF FREEDOM
              $
              SPC1,74,1246,1,THRU,6
              $
              $ PURGE DEGREE OF FREEDOM FOR AUXILLIARY GRID POINT.
              $
              SPC1,74,123456,200
              $
              $       ****    BOUNDARY  CONDITIONS    ****
              $
              $  THIS STATEMENT SETS THE Z-TRANSLATION AT NODES 1 AND 6 EGUAL TO ZERO.
              $
              SPC1,74,3,1,6
              $
              $       ****    PARAMETER STATEMENTS    ****
              $
              $ THIS STATEMENT REQUESTS COUPLED MASS FORMULATION.
              $
              PARAM,COUPMASS,1
              $
              ENDDATA
         TOTAL COUNT=   74
```

R E A L　E I G E N V A L U E S

MODE NO.	EXTRACTION ORDER	EIGENVALUE	RADIANS	CYCLES	GENERALIZED MASS	GENERALIZED STIFFNESS
1	1	1.324213E+04	1.150744E+02	1.831467E+01	9.999998E-01	1.324212E+04
2	2	2.125312E+05	4.610110E+02	7.337218E+01	9.999999E-01	2.125311E+05
3	3	1.089483E+06	1.043783E+03	1.661232E+02	9.999998E-01	1.089482E+06
4	4	3.547213E+06	1.883405E+03	2.997529E+02	9.999998E-01	3.547212E+06
5	5	1.019358E+07	3.192738E+03	5.081399E+02	9.999995E-01	1.019357E+07
6	6	2.124016E+07	4.608703E+03	7.334978E+02	9.999996E-01	2.124014E+07
7	7	4.502699E+07	6.710215E+03	1.067964E+03	9.999998E-01	4.502698E+07
8	8	9.082912E+07	9.530430E+03	1.516815E+03	9.999999E-01	9.082910E+07
9	9	1.645340E+08	1.282708E+04	2.041493E+03	9.999998E-01	1.645340E+08
10	10	2.140651E+08	1.463096E+04	2.328590E+03	9.999998E-01	2.140651E+08

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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 1.324213E+04

SUBCASE 1

R E A L E I G E N V E C T O R N O. 1

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	3.171740E-02	0.0
2	G	0.0	0.0	-2.967129E-02	0.0	2.565992E-02	0.0
3	G	0.0	0.0	-4.800916E-02	0.0	9.801216E-03	0.0
4	G	0.0	0.0	-4.800916E-02	0.0	-9.801216E-03	0.0
5	G	0.0	0.0	-2.967129E-02	0.0	-2.565992E-02	0.0
6	G	0.0	0.0	0.0	0.0	-3.171740E-02	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

VIBRATION ANALYSIS OF AN
UNDAMPED SIMPLY SUPPORTED BEAM

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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 2.125312E+05

SUBCASE 1

R E A L E I G E N V E C T O R N O. 2

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	6.362975E-02	0.0
2	G	0.0	0.0	-4.815835E-02	0.0	1.966267E-02	0.0
3	G	0.0	0.0	-2.976350E-02	0.0	-5.147754E-02	0.0
4	G	0.0	0.0	2.976350E-02	0.0	-5.147754E-02	0.0
5	G	0.0	0.0	4.815835E-02	0.0	1.966267E-02	0.0
6	G	0.0	0.0	0.0	0.0	6.362975E-02	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

VIBRATION ANALYSIS OF AN
UNDAMPED SIMPLY SUPPORTED BEAM

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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 1.089493E+06

SUBCASE 1

R E A L E I G E N V E C T O R N O. 3

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	9.656256E-02	0.0
2	G	0.0	0.0	-4.877972E-02	0.0	-2.983949E-02	0.0
3	G	0.0	0.0	3.014752E-02	0.0	-7.812077E-02	0.0
4	G	0.0	0.0	3.014752E-02	0.0	7.812077E-02	0.0
5	G	0.0	0.0	-4.877972E-02	0.0	2.983949E-02	0.0
6	G	0.0	0.0	0.0	0.0	-9.656256E-02	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

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OUTPUT FOR MODES 1 THRU 10 — SUBCASE 1

EIGENVALUE = 3.547213E+06

REAL EIGENVECTOR NO. 4

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	1.309117E-01	0.0
2	G	0.0	0.0	-3.128890E-02	0.0	-1.059098E-01	0.0
3	G	0.0	0.0	5.062633E-02	0.0	4.045396E-02	0.0
4	G	0.0	0.0	-5.062633E-02	0.0	4.045396E-02	0.0
5	G	0.0	0.0	3.128890E-02	0.0	-1.059098E-01	0.0
6	G	0.0	0.0	0.0	0.0	1.309117E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

VIBRATION ANALYSIS OF AN UNDAMPED SIMPLY SUPPORTED BEAM — APRIL 22, 1981 — NASTRAN 5/ 1/79 — PAGE 23

OUTPUT FOR MODES 1 THRU 10 — SUBCASE 1

EIGENVALUE = 1.019359E+07

REAL EIGENVECTOR NO. 5

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	1.954656E-01	0.0
2	G	0.0	0.0	-7.733812E-17	0.0	-1.954656E-01	0.0
3	G	0.0	0.0	2.214112E-16	0.0	1.954656E-01	0.0
4	G	0.0	0.0	-2.214679E-16	0.0	-1.954656E-01	0.0
5	G	0.0	0.0	-2.618252E-17	0.0	1.954656E-01	0.0
6	G	0.0	0.0	0.0	0.0	-1.954656E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

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OUTPUT FOR MODES 1 THRU 10 — SUBCASE 1

EIGENVALUE = 2.124016E+07

REAL EIGENVECTOR NO. 6

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	2.709467E-01	0.0
2	G	0.0	0.0	2.848219E-02	0.0	-2.192005E-01	0.0
3	G	0.0	0.0	-4.608516E-02	0.0	8.372712E-02	0.0
4	G	0.0	0.0	4.608516E-02	0.0	8.372712E-02	0.0
5	G	0.0	0.0	-2.848219E-02	0.0	-2.192005E-01	0.0
6	G	0.0	0.0	0.0	0.0	2.709467E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

VIBRATION ANALYSIS OF AN
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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 4.502699E+07

SUBCASE 1

R E A L E I G E N V E C T O R N O . 7

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	3.610297E-01	0.0
2	G	0.0	0.0	4.569918E-02	0.0	-1.115643E-01	0.0
3	G	0.0	0.0	-2.824365E-02	0.0	-2.920792E-01	0.0
4	G	0.0	0.0	-2.824365E-02	0.0	2.920792E-01	0.0
5	G	0.0	0.0	4.569918E-02	0.0	1.115643E-01	0.0
6	G	0.0	0.0	0.0	0.0	-3.610297E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

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UNDAMPED SIMPLY SUPPORTED BEAM

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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 9.092912E+07

SUBCASE 1

R E A L E I G E N V E C T O R N O . 8

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	-4.903537E-01	0.0
2	G	0.0	0.0	-4.020059E-02	0.0	-1.515276E-01	0.0
3	G	0.0	0.0	-2.484533E-02	0.0	3.967045E-01	0.0
4	G	0.0	0.0	2.484533E-02	0.0	3.967045E-01	0.0
5	G	0.0	0.0	4.020059E-02	0.0	-1.515276E-01	0.0
6	G	0.0	0.0	0.0	0.0	-4.903537E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 1.645340E+09

SUBCASE 1

R E A L E I G E N V E C T O R N O . 9

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	-6.455241E-01	0.0
2	G	0.0	0.0	-1.601216E-02	0.0	-5.222399E-01	0.0
3	G	0.0	0.0	-2.590822E-02	0.0	-1.994779E-01	0.0
4	G	0.0	0.0	-2.590822E-02	0.0	1.994779E-01	0.0
5	G	0.0	0.0	-1.601216E-02	0.0	5.222399E-01	0.0
6	G	0.0	0.0	0.0	0.0	6.455241E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

VIBRATION ANALYSIS OF AN
UNDAMPED SIMPLY SUPPORTED BEAM

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OUTPUT FOR MODES 1 THRU 10
EIGENVALUE = 2.140651E+08

SUBCASE 1

R E A L   E I G E N V E C T O R   N O .   10

POINT ID.	TYPE	T1	T2	T3	R1	R2	R3
1	G	0.0	0.0	0.0	0.0	5.171535E-01	0.0
2	G	0.0	0.0	1.843113E-15	0.0	5.171535E-01	0.0
3	G	0.0	0.0	3.989081E-15	0.0	5.171535E-01	0.0
4	G	0.0	0.0	4.190360E-15	0.0	5.171535E-01	0.0
5	G	0.0	0.0	1.576049E-15	0.0	5.171535E-01	0.0
6	G	0.0	0.0	0.0	0.0	5.171535E-01	0.0
200	G	0.0	0.0	0.0	0.0	0.0	0.0

APPENDIX B - EIGENVALUES, AND MATRICES FROM CHAPTER II

THE CRITICAL DAMPING MATRIX IS

14012.9450	11709.6011	-2810.7737	-115.5000	-176.9951	-154.3940	-73.0420	-12.9561	-8.7790	-1.8799
11709.6011	226023.0224	-115.4941	-111344.5052	-11864.0087	-1707.2609	102.5410	408.9595	154.4024	12.9561
-2810.7737	-115.4941	27848.8867	11555.2157	-2883.8155	-128.4550	-185.7704	-154.4024	-76.7959	-8.7790
-115.5000	-111344.5052	11555.2157	224315.7866	-12.9489	-110935.5371	-11709.6153	-1707.2609	128.4550	154.3940
-176.9951	-11864.0087	-2883.8155	-12.9489	28017.1112	11709.6153	-2814.5353	-102.5410	-185.7704	-73.0420
-154.3940	-1707.2609	-128.4550	-110935.5371	11709.6153	224315.7866	12.9489	-111344.5052	-11555.2157	115.5000
-73.0420	102.5410	-185.7704	-11709.6153	-2814.5353	12.9489	28017.1112	11864.0087	-2883.8155	-176.9951
-12.9561	408.9595	-154.4024	-1707.2609	-102.5410	-111344.5052	11864.0087	226023.0224	115.4941	-11709.6011
-8.7790	154.4024	-76.7959	128.4550	-185.7704	-11555.2157	-2883.8155	115.4941	27848.8867	-2810.7737
-1.8799	12.9561	-8.7790	154.3940	-73.0420	115.5000	-176.9951	-11709.6011	-2810.7737	14012.9450

N A S T R A N   E X E C U T I V E   C O N T R O L   D E C K   E C H O

CRITICALLY DAMPED EXAMPLE

```
ID GRAY,DAMPED,BEAM02
APP DISP
TIME 5
SOL 7
DIAG 9, 13, 14, 17, 21, 22
$THE NEXT STATEMENTS WILL CAUSE THE FINAL MASS AND STIFFNESS MATRICES
$TO BE PRINTED.
$ALTER 142
$MATGPR GPLD,USETD,SILD,MAA//C,N,A/C,N,A$
$MATGPR GPLD,USETD,SILD,KAA//C,N,A/C,N,A$
$ENDALTER
CEND
```


C A S E   C O N T R O L   D E C K   E C H O

CARD COUNT	
1	TITLE = VIBRATION ANALYSIS OF A
2	SUBTITLE = DAMPED SIMPLY SUPPORTED BEAM
3	ECHO = BOTH
4	SPC = 74
5	CMETHOD = 5
6	B2PP=CMATR
7	OUTPUT
8	DISP = ALL
9	SUBCASE 1
10	LABEL=OUTPUT FOR MODES 1 THRU 20
11	MODES=20
12	SUBCASE 21
13	DISP=NONE
14	VELO=NONE
15	ACCEL=NONE
16	BEGIN BULK

```
                         I N P U T   B U L K   D A T A   D E C K   E C H O

          .  1  ..  2  ..  3  ..  4  ..  5  ..  6  ..  7  ..  8  ..  9  ..  10  .
          $
          $     ****    GRID  POINTS  ****
          $
          GRID,1,0,0.0,0.0,0.0,0,,,,
          GRID,2,0,1.0,0.0,0.0,0,,,,
          GRID,3,0,2.0,0.0,0.0,0,,,,
          GRID,4,0,3.0,0.0,0.0,0,,,,
          GRID,5,0,4.0,0.0,0.0,0,,,,
          GRID,6,0,5.0,0.0,0.0,0,,,,
          $
          $     ****    AUXILIARY  GRID  POINT  ****
          $
          GRID,200,0,100.0,100.0,0.0,0,,,,
          $
          $     ****    ELEMENT  DATA  ****
          $
          CBAR,1,10,1,2,200,,,2,,
          CBAR,2,10,2,3,200,,,2,,
          CBAR,3,10,3,4,200,,,2,,
          CBAR,4,10,4,5,200,,,2,,
          CBAR,5,10,5,6,200,,,2,,
          $
          $ SINCE ALL ELEMENTS HAVE THE SAME LENGTH, GEOMETRIC PROPERTIES,
          $ MATERIAL PROPERTIES,AND THE SAME DIRECTION COSINES, THEY WILL
          $ ALL POSSESS THE SAME MASS AND STIFFNESS MATRICES RELATIVE TO
          $ THE LOCAL AXES.  IT IS POSSIBLE TO CALCULATE THE MASS AND STIFFNESS
          $ MATRICES FOR ONE ELEMENT AND USE THE INFORMATION FOR THE REMAINING
          $ ELEMENTS.
          $
          $ THIS IS ACCOMPLISHED BY USING THE CNGRNT STATEMENT.
          $
          $CNGRNT,1,2,3,4,5
          $
          $     **** CHOOSE COMPLEX EIGENVALUE EXTRACTION TECHNIQUE ****
          $
          EIGC,5,HESS,MAX,,,1.-8,,,+EIG1
          +EIG1,,,,,,,,20,,,
          $
          $     ****    GEOMETRIC  PROPERTIES  ****
          $
          PBAR,10,20,.02,,6.67-05,,,,,
          $
          $     ****    MATERIAL  DATA  ****
          $
          MAT1,20,2.0+11,,,7.852+03,,,,,
          $
          $     ****    PURGES  ****
          $
          $ PURGE NONEXISTING DEGREES OF FREEDOM
```

```
                    I N P U T   B U L K   D A T A   D E C K   E C H O

.   1  ..   2  ..   3  ..   4  ..   5  ..   6  ..   7  ..   8  ..   9  ..  10  .
SPC1,74,1246,1,THRU,6
$
$ PURGE DEGREE OF FREEDOM FOR AUXILLIARY GRID POINT.
$
SPC1,74,123456,200
$
$     ****    BOUNDARY  CONDITIONS  ****
$
$  THIS STATEMENT SETS THE Z-TRANSLATION AT NODES 1 AND 6 EQUAL TO ZERO.
$
SPC1,74,3,1,6
$
$     ****    DIRECT INPUT DAMPING MATRIX ****
$
DMIG,CMATR,0,6,1,0,,,,,
DMIG*   CMATR           1               5                               *C1
*C1      1              5               1.40128736+4
DMIG*   CMATR           2               3                               *C2
*C2     1               5               1.17097543+4                    *C3
*C3     2               3               2.260247338+5                   *C35
*C35
DMIG*   CMATR           2               5                               *C4
*C4     1               5               -2.8109489+3                    *C5
*C5     2               3               -1.154856+2                     *C6
*C6     2               5               2.78487051+4
DMIG*   CMATR           3               3                               *C65
*C65    1               5               -1.154786+2                     *C7
*C7     2               3               -1.113449144+5                  *C75
*C75    2               5                1.15553603+4                   *C8
*C8     3               3               2.243172513+5
DMIG*   CMATR           3               5                               *C9
*C9     1               5               -1.770424+2                     *C10
*C10    2               3               -1.16641471+4                   *C11
*C11    2               5               -2.8839969+3                    *C12
*C12    3               3               -1.29620+1                      *C13
*C13    3               5                2.80169658+4
DMIG*   CMATR           4                3                              *C145
*C145   1               5               -1.543982+2                     *C146
*C146   2               3               -1.7074734+03                   *C147
*C147   2               5               -1.284357+02                    *C14
*C14    3               3                -1.109360160+5                 *C15
*C15    3               5                1.17097569+4                   *C16
*C16    4               3                2.243172513+5
DMIG*   CMATR           4                5                              *C165
*C165   1               5               -7.30479+01                     *C166
*C166   2               3                1.025275+02                    *C167
*C167   2               5               -1.858220+02                    *C17
*C17    3               3                -1.17097569+4                  *C18
*C18    3               5                -2.8147116+3                   *C19
*C19    4               3                1.29620+1                      *C20
```

```
                         I N P U T   B U L K   D A T A   D E C K   E C H O

      .  1  ..  2  ..  3  ..  4  ..  5  ..  6  ..  7  ..  8  ..  9  ..  10  .
      *C20     4              5               2.80169658+4
      DMIG*    CMATR          5               3                          *C215
      *C215    1              5              -1.29599+01                 *C216
      *C216    2              3              4.089009+02                 *C217
      *C217    2              5              -1.543978+02                *C218
      *C218    3              3               -1.7074734+3               *C219
      *C219    3              5               -1.025275+02               *C21
      *C21     4              3               -1.113449144+5             *C22
      *C22     4              5                1.18641471+4              *C23
      *C23     5              3                2.260247338+5
      DMIG*    CMATR          5                5                         *C235
      *C235    1              5              -8.7787+00                  *C236
      *C236    2              3                1.543978+02               *C237
      *C237    2              5               -7.68130+01                *C238
      *C238    3              3                1.284357+02               *C239
      *C239    3              5               -1.858220+02               *C24
      *C24     4              3               -1.15553603+4              *C25
      *C25     4              5                -2.8839969+3              *C26
      *C26     5              3                1.154856+2                *C27
      *C27     5              5                2.78487051+4
      DMIG*    CMATR          6                5                         *C281
      *C281    1              5               -1.8816+00                 *C282
      *C282    2              3               1.29599+01                 *C283
      *C283    2              5               -8.7787+00                 *C284
      *C284    3              3               1.543982+2                 *C285
      *C285    3              5               -7.30479+01                *C286
      *C286    4              3               1.154786+2                 *C287
      *C287    4              5               -1.770424+2                *C28
      *C28     5              3               -1.17097543+4              *C29
      *C29     5              5               -2.8109489+3               *C30
      *C30     6              5                1.40128736+4
      $
      $   ****     PARAMETER STATEMENTS****
      $
      $ THIS STATEMENT REQUESTS COUPLED MASS FORMULATION.
      $
      PARAM,COUPMASS,1
      $
      ENDDATA

TOTAL COUNT=  139
```

CRITICALLY DAMPED EXAMPLE

COMPLEX EIGENVALUE SUMMARY

ROOT NO.	EXTRACTION ORDER	EIGENVALUE = λ_j (REAL)	(IMAG)	FREQUENCY = f_j (CYCLES)	DAMPING COEFFICIENT = g_j
1	2	-1.175290E+02	-2.563473E-03	0.0	0.0
2	16	-1.280185E+04	-2.924336E-02	0.0	0.0
3	12	-6.694738E+03	-3.937056E-02	0.0	0.0
4	19	-1.444282E+04	-7.617366E-02	0.0	0.0
5	8	-3.184565E+03	-9.545189E-02	0.0	0.0
6	15	-9.416734E+03	-1.266748E-01	0.0	0.0
7	11	-4.541984E+03	-2.223572E-01	0.0	0.0
8	7	-1.829238E+03	-1.193933E+00	0.0	0.0
9	5	-1.028147E+03	-1.205450E+00	1.918533E-01	1.705831E+03
10	1	-4.697312E+02	-2.420364E+00	3.852130E-01	3.881489E+02
11	3	-1.126191E+02	2.364078E-03	0.0	0.0
12	18	-1.285231E+04	2.824545E-02	0.0	0.0
13	14	-6.725691E+03	3.936117E-02	0.0	0.0
14	20	-1.481910E+04	7.617211E-02	0.0	0.0
15	10	-3.200915E+03	9.544837E-02	0.0	0.0
16	17	-9.644121E+03	1.266742E-01	0.0	0.0
17	13	-4.675422E+03	2.223681E-01	0.0	0.0
18	9	-1.937573E+03	1.193943E+00	0.0	0.0
19	6	-1.059419E+03	1.205379E+00	1.918420E-01	1.757819E+03
20	4	-4.522915E+02	2.420626E+00	3.852546E-01	3.736980E+02

OVERDAMPED EXAMPLE

COMPLEX EIGENVALUE SUMMARY

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	20	-4.787227E+04	-1.554272E-09	0.0	0.0
2	19	-5.460524E+04	-2.121145E-09	0.0	0.0
3	2	-5.046587E+02	-4.011145E-09	0.0	0.0
4	5	-3.083429E+01	-5.943509E-09	0.0	0.0
5	9	-1.720508E+03	-2.103234E-08	0.0	0.0
6	14	-7.028934E+03	-4.645550E-08	0.0	0.0
7	6	-8.554949E+02	-6.697741E-08	0.0	0.0
8	1	-4.294609E+02	-1.808701E-07	0.0	0.0
9	10	-2.553663E+03	-8.829416E-07	0.0	0.0
10	7	-1.234906E+03	-2.144658E-06	0.0	0.0
11	4	-1.235282E+02	-1.846305E-05	0.0	0.0
12	17	-2.504277E+04	6.750829E-12	0.0	0.0
13	16	-1.719983E+04	6.304955E-11	0.0	0.0
14	15	-1.191541E+04	6.875427E-10	0.0	0.0
15	18	-3.556818E+04	3.111451E-09	0.0	0.0
16	13	-3.895298E+03	5.030375E-08	0.0	0.0
17	11	-3.436939E+03	1.557702E-07	0.0	0.0
18	12	-3.920375E+03	7.246537E-07	0.0	0.0
19	8	-1.798004E+03	2.147575E-06	0.0	0.0
20	3	-2.796819E+02	1.873687E-05	0.0	0.0

UNDERDAMPED EXAMPLE

C O M P L E X   E I G E N V A L U E   S U M M A R Y

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	1	-1.149687E+01	-1.144987E+02	1.822302E+01	2.008210E-01
2	3	-4.611426E+01	-4.586937E+02	7.300417E+01	2.010655E-01
3	5	-1.043985E+02	-1.038549E+03	1.652902E+02	2.010469E-01
4	6	-1.882945E+02	-1.873969E+03	2.982512E+02	2.009579E-01
5	9	-3.192739E+02	-3.176734E+03	5.055928E+02	2.010076E-01
6	10	-4.609570E+02	-4.585594E+03	7.298198E+02	2.010457E-01
7	12	-6.709507E+02	-6.676586E+03	1.062612E+03	2.009861E-01
8	14	-9.529583E+02	-9.482668E+03	1.509213E+03	2.009894E-01
9	16	-1.282824E+03	-1.276277E+04	2.031259E+03	2.010260E-01
10	17	-1.463096E+03	-1.455762E+04	2.316918E+03	2.010075E-01
11	2	-1.149686E+01	1.144987E+02	1.822302E+01	2.008207E-01
12	4	-4.611426E+01	4.586987E+02	7.300417E+01	2.010655E-01
13	7	-1.043985E+02	1.038549E+03	1.652902E+02	2.010469E-01
14	8	-1.882945E+02	1.873969E+03	2.982512E+02	2.009579E-01
15	11	-3.192739E+02	3.176734E+03	5.055928E+02	2.010076E-01
16	13	-4.609570E+02	4.585594E+03	7.298198E+02	2.010457E-01
17	15	-6.709507E+02	6.676586E+03	1.062612E+03	2.009861E-01
18	18	-9.529583E+02	9.482668E+03	1.509213E+03	2.009894E-01
19	20	-1.282824E+03	1.276277E+04	2.031259E+03	2.010260E-01
20	19	-1.463096E+03	1.455762E+04	2.316918E+03	2.010075E-01

C MINUS C CRITICAL IS

-5408.4161	-39674.2413	-3642.3943	115.4786	177.0424	154.3982	73.0479	12.9599	8.7787	1.8815
-39674.2413	445122.9531	115.4856	227505.0910	39828.6341	1707.4734	-102.5275	-408.9009	-154.3978	-12.9599
-3642.3943	115.4856	-10639.7900	-39519.8473	-3569.3462	128.4357	185.8220	154.3978	76.8130	8.7787
115.4786	227505.0910	-39519.8473	446830.4355	12.9620	227096.1925	39674.2438	1707.4734	-128.4357	-154.3982
177.0424	39828.6341	-3569.3462	12.9620	-10808.0507	-39674.2433	-3638.6315	102.5275	185.8220	73.0477
154.3982	1707.4734	128.4357	227096.1926	-39674.2438	446830.4355	-12.9620	227505.0910	39519.8473	-115.4786
73.0479	-102.5275	185.8220	39674.2438	-3638.6315	-12.9620	-10808.0507	-39828.6341	-3569.3462	177.0424
12.9599	-408.9009	154.3978	1707.4734	102.5275	227505.0910	-39828.6341	445122.9531	-115.4856	39674.2413
8.7787	-154.3978	76.8130	-128.4357	185.8220	39519.8473	-3569.3462	-115.4856	-10639.7900	-3642.3943
1.8816	-12.9599	8.7787	-154.3982	73.0479	-115.4785	177.0424	39674.2413	-3642.3943	-5408.4161

MIXED DAMPING EXAMPLE

C O M P L E X   E I G E N V A L U E   S U M M A R Y

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	11	-2.876552E+03	-1.385286E+03	2.204751E+02	4.153006E+00
2	13	-2.876554E+03	-3.600778E+03	5.730815E+02	1.597739E+00
3	15	-2.876554E+03	-6.062379E+03	9.648574E+02	9.489848E-01
4	18	-2.876554E+03	-9.085953E+03	1.446074E+03	6.331868E-01
5	20	-2.876556E+03	-1.250037E+04	1.989496E+03	4.602349E-01
6	19	-2.876558E+03	-1.434540E+04	2.283142E+03	4.010423E-01
7	8	-7.023069E+02	4.998132E-09	0.0	0.0
8	4	-5.050901E+03	5.064507E-09	0.0	0.0
9	3	-5.557051E+03	5.103693E-09	0.0	0.0
10	1	-5.750801E+03	5.103693E-09	0.0	0.0
11	2	-5.715922E+03	5.103693E-09	0.0	0.0
12	6	-3.718228E+01	8.601909E-09	0.0	0.0
13	7	-1.960541E+02	1.504080E-08	0.0	0.0
14	5	-2.302657E+00	6.969027E-08	0.0	0.0
15	9	-2.876554E+03	1.385286E+03	2.204751E+02	4.153008E+00
16	10	-2.876554E+03	3.600778E+03	5.730815E+02	1.597739E+00
17	12	-2.876554E+03	6.062379E+03	9.648574E+02	9.489848E-01
18	14	-2.876554E+03	9.085953E+03	1.446074E+03	6.331868E-01
19	16	-2.876556E+03	1.250037E+04	1.989496E+03	4.602349E-01
20	17	-2.876558E+03	1.434540E+04	2.283142E+03	4.010423E-01

APPENDIX C - FINITE ELEMENT DAMPING MATRICES FOR DISCRETE AND CONTINUOUS DAMPING

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 1 IS

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	1899.5416	-267.8841	657.5336	158.2951	0.0	0.0	0.0
0.0	0.0	0.0	-267.8841	36.5491	-158.2951	-27.4118	0.0	0.0	0.0
0.0	0.0	0.0	657.5336	-158.2951	1899.5416	267.8841	0.0	0.0	0.0
0.0	0.0	0.0	158.2951	-27.4118	267.8841	36.5491	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 2 IS

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	21574.1270	-3042.5051	1467.9670	1797.8439	0.0	0.0	0.0
0.0	0.0	0.0	-3042.5051	553.1827	-1797.8439	-414.8871	0.0	0.0	0.0
0.0	0.0	0.0	1467.9670	-1797.8439	21574.1270	3042.5051	0.0	0.0	0.0
0.0	0.0	0.0	1797.8439	-414.8871	3042.5051	553.1827	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 3 IS

0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	14127.0196	1992.3566	4890.3299	1177.3016	0.0	0.0	0.0	0.0	0.0
0.0	1992.3566	362.2467	-1177.3016	-271.0850	0.0	0.0	0.0	0.0	0.0
0.0	4890.3299	-1177.3016	28255.2393	0.0	4890.3299	1177.3016	0.0	0.0	0.0
0.0	1177.3016	-271.0850	0.0	724.4933	-1177.3016	-271.0850	0.0	0.0	0.0
0.0	0.0	0.0	4890.3299	-1177.3016	28255.2393	0.0	4890.3299	1177.3016	0.0
0.0	0.0	0.0	1177.3016	-271.0850	0.0	724.4933	-1177.3016	-271.0850	0.0
0.0	0.0	0.0	0.0	0.0	4890.3299	-1177.3016	14127.0196	1992.3566	0.0
0.0	0.0	0.0	0.0	0.0	1177.3016	-271.0850	1992.3566	362.2467	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 4 IS

344.1783	-1118.5795	-258.1337	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-1118.5795	26845.9078	0.0	4646.4071	1118.5795	0.0	0.0	0.0	0.0	0.0
-258.1337	0.0	688.3566	-1118.5795	-258.1337	0.0	0.0	0.0	0.0	0.0
0.0	4646.4071	-1118.5795	26845.9078	0.0	4646.4071	1118.5795	0.0	0.0	0.0
0.0	1118.5795	-258.1337	0.0	688.3566	-1118.5795	-258.1337	0.0	0.0	0.0
0.0	0.0	0.0	4646.4071	-1118.5795	26845.9078	0.0	4646.4071	1118.5795	0.0
0.0	0.0	0.0	1118.5795	-258.1337	0.0	688.3566	-1118.5795	-258.1337	0.0
0.0	0.0	0.0	0.0	0.0	4646.4071	-1118.5795	26845.9078	0.0	1118.5795
0.0	0.0	0.0	0.0	0.0	1118.5795	-258.1337	0.0	688.3566	-258.1337
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1118.5795	-258.1337	344.1783

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 5 IS

913.7282	-3957.3783	-685.2962	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-3957.3783	94977.0789	0.0	16438.3406	3957.3783	0.0	0.0	0.0	0.0	0.0
-685.2962	0.0	1827.4564	-3957.3783	-685.2962	0.0	0.0	0.0	0.0	0.0
0.0	16438.3406	-3957.3783	94977.0789	0.0	16438.3405	3957.3783	0.0	0.0	0.0
0.0	3957.3783	-685.2962	0.0	1827.4564	-3957.3783	-685.2962	0.0	0.0	0.0
0.0	0.0	0.0	16438.3406	-3957.3783	94977.0787	0.0	16438.3406	3957.3783	0.0
0.0	0.0	0.0	3957.3783	-685.2962	0.0	1827.4564	-3957.3783	-685.2962	0.0
0.0	0.0	0.0	0.0	0.0	16438.3405	-3957.3783	94977.0789	0.0	3957.3783
0.0	0.0	0.0	0.0	0.0	3957.3783	-685.2962	0.0	1827.4564	-685.2962
0.0	0.0	0.0	0.0	0.0	0.0	0.0	3957.3783	-685.2962	913.7282

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 6 IS

2102.1161	-8989.3124	-1576.5871	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-8989.3124	215743.4983	0.0	37340.2209	8989.3124	0.0	0.0	0.0	0.0	0.0
-1576.5871	0.0	4204.2323	-8989.3124	-1576.5871	0.0	0.0	0.0	0.0	0.0
0.0	37340.2209	-8989.3124	215743.4983	0.0	37340.2209	8989.3124	0.0	0.0	0.0
0.0	8989.3124	-1576.5871	0.0	4204.2323	-8989.3121	-1576.5871	0.0	0.0	0.0
0.0	0.0	0.0	37340.2209	-8989.3124	215743.4983	0.0	37340.2209	8989.3124	0.0
0.0	0.0	0.0	8989.3124	-1576.5871	0.0	4204.2323	-8989.3124	-1576.5871	0.0
0.0	0.0	0.0	0.0	0.0	37340.2209	-8989.3124	215743.4983	0.0	8989.3124
0.0	0.0	0.0	0.0	0.0	8989.3121	-1576.5871	0.0	4204.2323	-1576.5871
0.0	0.0	0.0	0.0	0.0	0.0	0.0	8989.3124	-1576.5871	2102.1161

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 7 IS

4564.3078	-17659.5243	-3423.2309	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-17659.5243	423828.5822	0.0	73354.9469	17659.5243	0.0	0.0	0.0	0.0	0.0
-3423.2309	0.0	9128.6156	-17659.5243	-3423.2309	0.0	0.0	0.0	0.0	0.0
0.0	73354.9469	-17659.5243	423828.5822	0.0	73354.9469	17659.5243	0.0	0.0	0.0
0.0	17659.5243	-3423.2309	0.0	9128.6156	-17659.5243	-3423.2309	0.0	0.0	0.0
0.0	0.0	0.0	73354.9469	-17659.5243	423828.5822	0.0	73354.9469	17659.5243	0.0
0.0	0.0	0.0	17659.5243	-3423.2309	0.0	9128.6156	-17659.5243	-3423.2309	0.0
0.0	0.0	0.0	0.0	0.0	73354.9469	-17659.5243	423828.5822	0.0	17659.5243
0.0	0.0	0.0	0.0	0.0	17659.5243	-3423.2309	0.0	9128.6156	-3423.2309
0.0	0.0	0.0	0.0	0.0	0.0	0.0	17659.5243	-3423.2309	4564.3078

ASSEMBLED DAMPING MATRIX FOR EXAMPLE NO. 8 IS

8004.4575	-27964.4870	-6453.3431	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-27964.4870	671147.0869	0.0	116160.1766	27964.4870	0.0	0.0	0.0	0.0	0.0
-6453.3431	0.0	17208.9150	-27964.4870	-6453.3431	0.0	0.0	0.0	0.0	0.0
0.0	116160.1766	-27964.4870	671147.0869	0.0	116160.1766	27964.4870	0.0	0.0	0.0
0.0	27964.4870	-6453.3431	0.0	17208.9150	-27964.4870	-6453.3431	0.0	0.0	0.0
0.0	0.0	0.0	116160.1766	-27964.4870	671147.0869	0.0	116160.1766	27964.4870	0.0
0.0	0.0	0.0	27964.4870	-6453.3431	0.0	17208.9150	-27964.4870	-6453.3431	0.0
0.0	0.0	0.0	0.0	0.0	116160.1766	-27964.4870	671147.0869	0.0	27964.4870
0.0	0.0	0.0	0.0	0.0	27964.4870	-6453.3431	0.0	17208.9150	-6453.3431
0.0	0.0	0.0	0.0	0.0	0.0	0.0	27964.4870	-6453.3431	8004.4575

APPENDIX D - EIGENVALUES FOR EXAMPLES 5-8

EXAMPLE 5

C O M P L E X E I G E N V A L U E S U M M A R Y

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	1	-7.969307E+02	-4.686704E-09	0.0	0.0
2	3	-1.661642E+01	-1.433971E-07	0.0	0.0
3	4	-4.047163E+02	-2.207690E+02	3.513646E+01	3.666424E+00
4	6	-3.983857E+02	-9.648198E+02	1.535558E+02	8.258241E-01
5	8	-3.863196E+02	-1.843764E+03	2.934441E+02	4.190553E-01
6	10	-3.054680E+02	-3.178092E+03	5.058088E+02	1.922336E-01
7	12	-3.172803E+02	-4.596797E+03	7.316028E+02	1.380441E-01
8	14	-2.850457E+02	-6.703832E+03	1.066948E+03	8.503962E-02
9	18	-2.667144E+02	-9.526629E+03	1.516210E+03	5.599344E-02
10	20	-2.825679E+02	-1.282396E+04	2.040997E+03	4.406874E-02
11	19	-3.054680E+02	-1.462777E+04	2.328083E+03	4.176548E-02
12	2	-4.047163E+02	2.207690E+02	3.513646E+01	3.666424E+00
13	5	-3.983857E+02	9.648198E+02	1.535558E+02	8.258241E-01
14	7	-3.863196E+02	1.843764E+03	2.934441E+02	4.190553E-01
15	9	-3.054680E+02	3.178092E+03	5.058088E+02	1.922336E-01
16	11	-3.172803E+02	4.596797E+03	7.316028E+02	1.380441E-01
17	13	-2.850457E+02	6.703832E+03	1.066948E+03	8.503962E-02
18	15	-2.667144E+02	9.526629E+03	1.516210E+03	5.599344E-02
19	16	-2.825679E+02	1.282396E+04	2.040997E+03	4.406874E-02
20	17	-3.054680E+02	1.462777E+04	2.328083E+03	4.176548E-02

EXAMPLE 6

COMPLEX EIGENVALUE SUMMARY

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	5	-1.239154E+02	-5.663580E-09	0.0	0.0
2	2	-1.840864E+03	-1.521861E-08	0.0	0.0
3	4	-7.193463E+00	-7.312730E-08	0.0	0.0
4	3	-9.060334E+02	-5.187292E+02	8.255832E+01	3.493280E+00
5	7	-8.807703E+02	-1.666862E+03	2.652893E+02	1.056800E+00
6	9	-7.027576E+02	-3.114436E+03	4.956777E+02	4.512904E-01
7	11	-7.271377E+02	-4.546398E+03	7.235818E+02	3.198741E-01
8	13	-6.578230E+02	-6.676363E+03	1.062576E+03	1.970602E-01
9	15	-6.180625E+02	-9.510039E+03	1.513570E+03	1.299810E-01
10	17	-6.527366E+02	-1.281043E+04	2.038844E+03	1.019070E-01
11	18	-7.027571E+02	-1.461408E+04	2.325903E+03	9.617531E-02
12	1	-1.715250E+03	0.0	0.0	0.0
13	6	-9.060334E+02	5.187292E+02	8.255832E+01	3.493280E+00
14	8	-8.807703E+02	1.666862E+03	2.652893E+02	1.056800E+00
15	10	-7.027576E+02	3.114436E+03	4.956777E+02	4.512904E-01
16	12	-7.271377E+02	4.546398E+03	7.235818E+02	3.198741E-01
17	14	-6.578230E+02	6.676363E+03	1.062576E+03	1.970602E-01
18	16	-6.180625E+02	9.510039E+03	1.513570E+03	1.299810E-01
19	20	-6.527366E+02	1.281043E+04	2.038844E+03	1.019070E-01
20	19	-7.027571E+02	1.461408E+04	2.325903E+03	9.617531E-02

EXAMPLE 7

C O M P L E X E I G E N V A L U E S U M M A R Y

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	1	-3.650274E+00	-7.163077E-09	0.0	0.0
2	3	-3.354338E+02	-9.115193E-07	0.0	0.0
3	5	-1.762470E+03	-6.725488E+02	1.070395E+02	5.241167E+00
4	10	-1.525892E+03	-2.804502E+03	4.463503E+02	1.088173E+00
5	12	-1.554381E+03	-4.330812E+03	6.892700E+02	7.178243E-01
6	14	-1.466228E+03	-6.545477E+03	1.041745E+03	4.480124E-01
7	16	-1.414842E+03	-9.424266E+03	1.499918E+03	3.002551E-01
8	20	-1.460371E+03	-1.274363E+04	2.028211E+03	2.291924E-01
9	19	-1.525891E+03	-1.455118E+04	2.315892E+03	2.097275E-01
10	8	-3.627734E+03	7.517507E-10	0.0	0.0
11	7	-3.560275E+03	7.930140E-08	0.0	0.0
12	2	-5.970209E+01	1.100511E-07	0.0	0.0
13	6	-3.250429E+03	3.635762E-07	0.0	0.0
14	4	-1.762470E+03	6.725488E+02	1.070395E+02	5.241167E+00
15	9	-1.525892E+03	2.804502E+03	4.463503E+02	1.088173E+00
16	11	-1.554381E+03	4.330812E+03	6.892700E+02	7.178243E-01
17	13	-1.466228E+03	6.545477E+03	1.041745E+03	4.480124E-01
18	15	-1.414842E+03	9.424266E+03	1.499918E+03	3.002550E-01
19	17	-1.460372E+03	1.274363E+04	2.028211E+03	2.291924E-01
20	18	-1.525890E+03	1.455118E+04	2.315892E+03	2.097274E-01

EXAMPLE 8

C O M P L E X   E I G E N V A L U E   S U M M A R Y

ROOT NO.	EXTRACTION ORDER	EIGENVALUE (REAL)	EIGENVALUE (IMAG)	FREQUENCY (CYCLES)	DAMPING COEFFICIENT
1	11	-2.876552E+03	-1.385286E+03	2.204751E+02	4.153006E+00
2	13	-2.876554E+03	-3.600778E+03	5.730815E+02	1.597739E+00
3	15	-2.876554E+03	-6.062379E+03	9.648574E+02	9.489948E-01
4	18	-2.876554E+03	-9.085953E+03	1.446074E+03	6.331868E-01
5	20	-2.876556E+03	-1.250037E+04	1.989496E+03	4.602349E-01
6	19	-2.876558E+03	-1.434540E+04	2.283142E+03	4.010423E-01
7	8	-7.023069E+02	4.998132E-09	0.0	0.0
8	4	-5.050801E+03	5.364507E-09	0.0	0.0
9	3	-5.557051E+03	5.103693E-09	0.0	0.0
10	1	-5.750801E+03	5.103693E-09	0.0	0.0
11	2	-5.715922E+03	5.103693E-09	0.0	0.0
12	6	-3.718228E+01	8.601909E-09	0.0	0.0
13	7	-1.960541E+02	1.504680E-08	0.0	0.0
14	5	-2.302657E+00	6.369027E-08	0.0	0.0
15	9	-2.876554E+03	1.385286E+03	2.204751E+02	4.153008E+00
16	10	-2.876554E+03	3.600778E+03	5.730815E+02	1.597739E+00
17	12	-2.876554E+03	6.062379E+03	9.648574E+02	9.489848E-01
18	14	-2.876554E+03	9.085953E+03	1.446074E+03	6.331868E-01
19	16	-2.876556E+03	1.250037E+04	1.989496E+03	4.602349E-01
20	17	-2.876558E+03	1.434540E+04	2.283142E+03	4.010423E-01

LIST OF REFERENCES

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1. Duffin, R. J., "A Minimax Theory for Overdamped Networks," Journal of Rational Mechanics and Analysis, Vol. 4, 1955, pp 221-233.

2. Nicholson, D. W., "Eigenvalue Bounds for Damped Linear Systems," Mechanics Research Communications, Vol. 5, No. 3, 1978, pp. 147-152.

3. Müller, P. C., "Oscillatory Damped Linear Systems," Mechanics Research Communications, Vol. 6, No. 2, 1979, pp. 81-85.

4. Beskos, D. E. and Boley, B. A., "Critical Damping in Linear Discrete Dynamic Systems," Technical Report, Grant / Contract #N00014-75-C-1042.

5. Inman, D. J., and Andry, A. N., "Some Results on the Nature of Eigenvalues of Discrete Damped Linear Systems," Journal of Applied Mechanics, Dec. 1980.

6. Cook, R. D., "Concepts and Applications of Finite Element Analysis." John Wiley and Sons, Inc., New York, 1974, pp 237-239.

7. Segerlind, L. J., "Applied Finite Element Analysis," John Wiley and Sons, Inc., New York, 1976, pp 105-109.

8. Meirovitch, L., "Analytic Methods in Vibrations," the Macmillan Co., N.Y., 1967, pg. 80

9. Bellman, R., "Introduction to Matrix Analysis," 2nd Ed., McGraw-Hill, New York, 1970, p. 95.

10. Strang, G., "Linear Algebra and Its Applications," Academic Press, New York, 1976, p.241.

11. Meirovitch, L., "Analytic Methods in Vibrations," The Macmillan Co., N.Y., 1967, Chs. 5., 9.

12. Inman, D. J., and Andry, A. N., "The Nature of Temporal Solutions of Damped Distributed Systems with Classical Normal Modes," 18th Annual Meeting of the Society of Engineering Science, Providence, Rhode Island, Sept. 1981.