PLANAR SETS HAVING PROPERTY P^n

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This is to certify that the

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ABSTRACT

PLANAR SETS HAVING PROPERTY P^n

by Merle D. Guay

As a natural generalization of convexity, a subset X of a set S in a linear space L is said to have property P^n relative to S if for every n distinct points of X at least two of the points are joined by a line segment which lies entirely in S; if X = S, then S is said to have property P^n. Property P^2 is the usual definition of convexity.

It is first shown that a set having property P^n may be expressed as the union of (n-1) or fewer starlike sets. Several results which depend primarily upon the linearity of the containing space are then obtained for sets having property P^n.

In an attempt to determine the number of closed convex subsets which are required to express a closed, connected P^n set as the union of convex sets several results are obtained. For n = 4, the maximum number is shown to be 5 if S bounds a bounded domain of its complement; and to be 4 if S has a cut point, a one-dimensional kernel, contains a point at which S is both locally convex and one-dimensional, or has at most one point of local non-convexity which is not in the kernel of S. If S has exactly one point of local non-convexity q, then S is shown to be starlike from q, without assuming that S has property P^n; if in addition, S has property P^n, then it

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is shown that S may be expressed as the union of $(n-1)$ or fewer closed convex sets. Finally, if S has two or more points of local non-convexity each of which is contained in the kernel of S, then S is shown to be expressible as the union of 3 or fewer closed convex sets, independent of property P^n.

Finally, the higher dimensional case, the topological properties of P^n sets, and the problem of obtaining an upper bound on the number of convex sets required to express a set having property P^n as the union of convex sets are briefly considered.

THESIS

PLANAR SETS HAVING PROPERTY P^n

By

Merle D. Guay

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CONTENTS

CHAPTER I

INTRODUCTION

As a natural generalization of convexity, a set S in a linear topological space L is said to have property P^n if for every n distinct points of S at least two of the points are joined by a line segment which lies entirely in S. Property P^2 is equivalent to convexity. For $n = 3$, Valentine [17] found this concept to be useful in the study of sets each of which is the union of two convex sets. He was able to show that a closed connected set in E^2 having property P^3 can always be expressed as the union of three or fewer closed convex sets having a non-empty intersection, and that the number three is best. He later found this same concept useful in proving that the boundaries of two compact, convex bodies S_1 and S_2 in a Minkowski space L_n intersect in a finite number of $(n - 2)$ - dimensional manifolds, provided that the intersection of the interiors of S_1 and S_2 be contained in the interior of the convex hull of the union of S_1 and S_2 [18]. The definition of property P^3 given by Valentine suggested to me the definition given above as a natural generalization. It was later discovered that Allen [1] and in a joint paper, Danzer, Grunbaum and Klee [6] had given generalizations of convexity which encompass the definition above as a special case. However, no relevant publications have appeared to date.

The results of Valentine [17] suggest the possiblity that a closed connected set in E^2 having property P^n should be expressible as the union of n or fewer closed convex sets. However, this conjecture is false. For example, if the set S is closed, connected, has property P^4, and bounds a bounded residual domain of S, then S may be expressed as the union of 5 or fewer closed convex sets, and the number 5 is best. This example and the fact that a set with property P^3 is starlike suggests that the condition of starlikeness be added to property P^n in the hypothesis of the conjecture. If, in addition, certain restrictions are placed upon the nature of the set of points of local non-convexity, the result is forthcoming. In general, however, the result is still a conjecture for the case $n = 4$, while for values of n greater than four starlikeness does not restrict the number of convex sets to be n.

The results contained herein were obtained in an attempt to determine the properties of sets having property P^n $(n \geq 3)$, and to determine how such sets may be expressed as the union of their convex subsets.

CHAPTER II

SETS HAVING PROPERTY P^n

The results of this chapter are of an intrinsic nature, depending primarily upon the properties attributed to the set itself. The linearity of the containing space is indispensable, of course. While it is assumed that the sets being considered are embedded in Euclidean m-dimensional space, E^m, many of the results could have been stated with a general linear topological space as the containing space. Since the results are also of a heterogeneous nature, and are not required, for the most part, in the proofs of later results, they are numbered as propositions rather than being called lemmas.

With rare exception familiarity with the common terminology of convexity and topology is assumed. Notation used is explained as its introduction becomes necessary. The following less familiar definitions are essential to the understanding of most of that which follows. Each is a natural generalization of convexity.

<u>Definition 2.1</u> A set S contained in a linear space L, is said to be starlike if there exists a point x in S such that for each y in S, it is true that the line segment xy lies entirely in S.

<u>Remark:</u> A non-empty convex set is starlike from each of its points.

<u>Definition 2.2</u> A subset X of a set S in a linear space L is said to have property P^n, $(n \geq 2)$, relative to S if for every n distinct points of X at least two of the points are joined by a line segment which lies

entirely in S. If X = S, the set X is said to have property P^n.

Remark: Property P^2 is the usual definition of convexity.

The next two results help to explain the intimate relationship between the two concepts.

Proposition 2.1 Let $S \subset E^m$ have property P^n. Then S may be expressed as the union of (n - 1) or fewer starlike sets.

Proof. For n = 2, the set is convex. Assume then that the result is true for n = k - 1, and consider the case n = k. There must exist (k - 1) points of S no two of which are joined by a line segment lying entirely in S, since otherwise S has property P^j for $j < k$ and the induction hypotheses applies. Hence, let $p_1,\dots,p_{k-1}$, be the (k - 1) points no two of which are joined by a line segment lying entirely in S, and let x be an element of S different from p_i, $i = 1,\dots,k-1$. Then the line segment xp_i is contained in S for some i, since otherwise $x,p_1,\dots,p_{k-1}$ would violate property P^k. Thus S is a union of sets X_i starlike from p_i and the result follows.

As trivial examples of sets in E^2 having property P^n one might consider the boundary of a regular n - sided polygon as a set having property P^{n+1} for $n \geq 3$. A set consisting of n distinct line segments which intersect in the origin is an example of a starlike set having property P^{n+1}.

Remarks: It is clear that any set which is the union of exactly (n - 1) convex sets has property P^n.

It is also clear that property P^n implies property P^m for $m > n$.

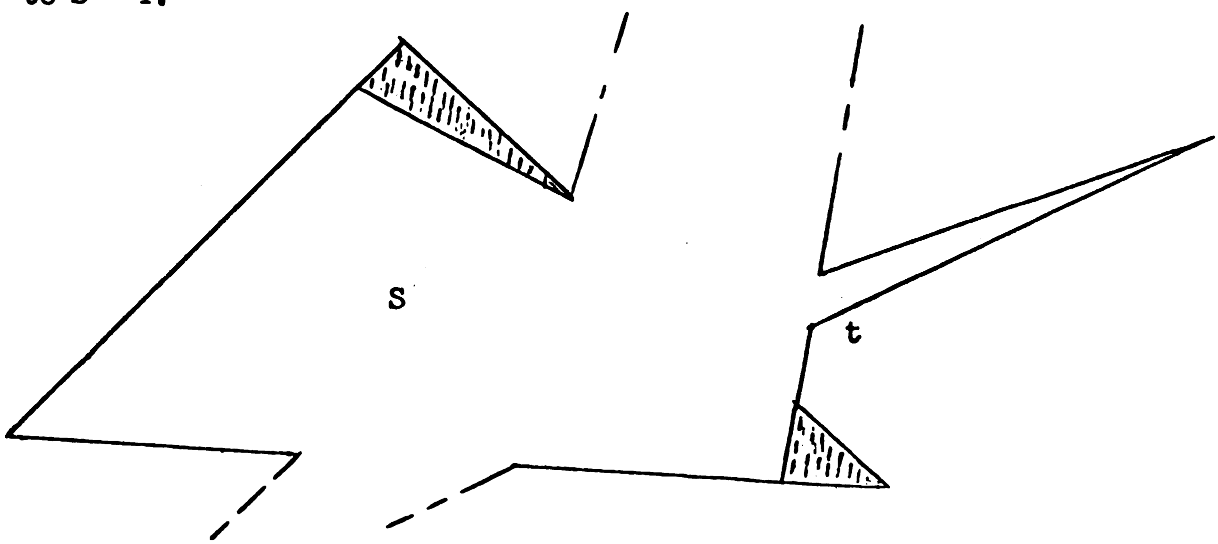
The well-known definition of local convexity proves to be extremely useful and so is included.

Definition 2.3 A set S is said to be locally convex at a point p in S if there exists an open spherical neighborhood N of p such that $S \cap N$ is convex. If a set is locally convex at each of its points, it is said to be locally convex. A point p of S is a point of local non-convexity if S is not locally convex at p.

Proposition 2.2 If $S \subset E^m$ is a closed connected set having property P^n, $n > 3$, then S is the union of a starlike set and a set having property P^{n-2} relative to S.

Proof. Tietze [14] has shown that a closed connected set in E^m which is locally convex is in fact convex. Hence, if S has no points of local non-convexity we are done. Let t be a point of local non-convexity and $T = \{x \in S \mid xt \subset S\}$ let $x_1, \ldots, x_{n-2}$ be points of $S - T$ which are not joined in S, and N_i be a spherical neighborhood of t of radius $\frac{1}{i}$. By the closure of S and the definition of t, for i sufficiently large, there exist y_i and z_i in $N_i \cap S$ such that $y_i, z_i, x_1, \ldots, x_{n-2}$ are not joined in S, contradicting the fact that S has property P^n. Hence, $S - T$ is contained in a subset of S which has property P^{n-2} relative to S, and the result follows.

Remark: Instinctively one considers the above result as an invitation to attempt an induction on n when seeking to prove a given result. While this is sometimes effective, the set T will, in general, have the same property P^n as did S, and S - T is only contained in a subset of S having property P^{n-2} relative to S. For n = 4, consider the following example which illustrates the difficulty: the shaded area corresponds to S - T.



As the union of three convex sets S quite obviously has property P^4, and also quite obviously S - T is contained in a set having property P^2, a convex set, while T again has property P^4.

While almost all examples given are polygonal, no proof given depends upon this property. It is simply easier to construct such examples, and, having constructed them, to determine whether or not they do indeed have property P^n for some predetermined value of n.

Since a set having property P^n is the union of a finite number of starlike sets, it is not surprising that the following concept and result are of interest when considering such sets.

<u>Definition 2.4</u> Let S be contained in a linear topological space L. The kernel K of S is the set of all points of S with respect to which S is starlike. That is, $K = \{z \in S \mid zx \subset S \text{ for all } x \in S\}$.

Brunn showed that K is a closed convex set, provided that S is a closed subset of E^2. The following result generalizes the Brunn theorem [4] and provides a useful characterization of the kernel of S.

Proposition 2.3 Let S be a set in a linear topological space L. Then the kernel K of S is the intersection of all maximal convex subsets of S.

Proof. First of all, every point x of S is contained in a maximal convex subset, M_x of S. Let x be in S. Then $\{x\}$ is a convex set. Partially order by inclusion the collection C_α^x of all convex subsets of S containing x. Using the maximal principal extract a maximal simply - ordered subcollection $\{C_{\alpha'}^x\}$, and let $M_x = \bigcup_{\alpha'} \{C_{\alpha'}^x\}$.

Now, let v be in K. Then v is contained in every maximal convex set in S for otherwise vM_x, the join of v with M_x would be a convex set containing M_x properly. Next, assume that $\bar{v}$ is in the intersection of all maximal convex sets of S. Then, since every point x of X is in some maximal convex set, $x\bar{v}$ is in S. It follows that $\bar{v} \in K$.

Corollary. The kernel K of $S \subset L$ is a convex set which is closed if S is closed.

Proof. The set K is convex since the intersection of any number of convex sets is known to be convex. If S is closed, M_x is closed for x in S since the closure of a convex set is a convex set. Finally, the intersection of an arbitrary number of closed sets is a closed set.

Remark: Helly [8] proved the following interesting result: If F is a family of compact convex sets in an N-dimensional Minkowski space L_N, then a necessary and sufficient condition that all members of F have a point in common is that every N + 1 members of F have a point in common.

Using the third result and the fact that K, as a closed subset of a compact set, is compact, the theorem could be stated: If F is a family of compact starlike sets in an N-dimensional Minkowski space L_N, then a necessary and sufficient condition that all members of F be starlike from a point common to all of their kernels, is that the intersection of each N + 1 members of the family contain a point common to the kernels of the N + 1 members.

The Helly number of a family F of sets is defined to be the smallest cardinal k such that whenever **G** is a finite subfamily of F and $\cap$ G $\neq \emptyset$ for all G $\subset$ **G** with card G < k + 1, then $\cap$ **G** $\neq \emptyset$. Helly's theorem asserts that the Helly number of the family F of compact convex sets in E^m is m + 1. An intriguing but extremely difficult question is: Does the family of compact (connected) sets in E^m having property P^n have a finite Helly number?

It is of possible interest to mention, in passing, that the analogue of the separation theorem for convex sets and the Krein-Millman theorem for convex sets are obtainable for starlike sets using the concepts of a homeomorphism [2] and relative extreme points [12], respectively.

Although there are a number of elementary results which one may prove for starlike sets which are the natural analogues of those usually encountered for convex sets, our interest here is in sets having property P^n, and so only a few elementary results which do not hold for sets having property P^n will be included.

The following results are an indication of the fact that property P^n is preserved under many of the usual operations which are in some sense "linear" operations.

Proposition 2.4 If $S \subset E^m$ has property P^n and L is a linear transformation of S, then L(S) has property P^n.

Proof. Let $y_1, \ldots, y_n$ be n distinct points of L(S). Then there exist distinct points $x_1, \ldots, x_n$ in S such that $y_i = L(x_i)$, $i = 1, 2, \ldots, n$. Since S has property P^n, $x_i x_j \subset S$ for some i and j. But $L(\alpha x_i + (1 - \alpha)x_j) = \alpha y_i + (1 - \alpha)y_j$ for $0 \leq \alpha \leq 1$. Hence $y_i y_j \subset L(S)$ for some i and j, as was claimed.

Proposition 2.5 Let $S \subset E^m$ have property P^n and λ be any real number. Then $\lambda S = \{\lambda s \mid s \in S\}$ has property P^n.

Proof. $\lambda S = \{\lambda s \mid s \in S\}$ defines a linear transformation.

Proposition 2.6 Let $S \subset E^m$ be a set having property P^n which is contained in a linear variety T of dimension m - 1. Let v be some point of $E^m - T$. Then vS, the cone over S with vertex v, has property P^n.

Proof. Let x_1, x_2 be points of vS - v such that $x_1 x_2 \not\subset$ vS (which clearly do not lie along the same generator of the cone), and let π be the projection map which carries x_1 and x_2 into S along the generators of the cone through x_1 and x_2, respectively. Then the segment $\pi(x_1)\pi(x_2) \not\subset S$, since $\pi(x_1)$, $\pi(x_2)$ and v determine a plane containing x_1 and x_2: by the definition of vS, $x_1 x_2$ would then be in vS. Thus, if $x_1, x_2, \ldots, x_n$ were distinct points of vS which were not joined in vS, then $\pi(x_1), \pi(x_2), \ldots, \pi(x_n)$ would be distinct points of S which would violate property P^n.

Corollary. Let $S \subset E^m$ be a closed connected starlike set having property P^n. Let $\bar{S}$ be a suspension of S constructed by choosing the suspension points v_1 and v_2 to lie on a line orthogonal to E^m in E^{m+1} such that v_1v_2 intersects the kernel of S. Then $\bar{S}$ has property P^n.

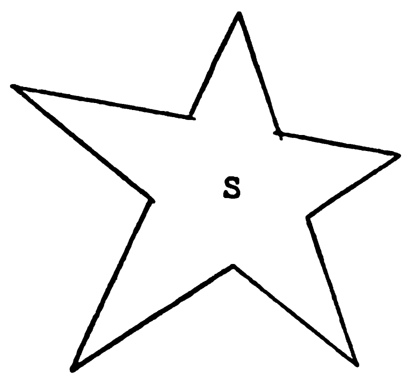
Proof. Let $\bar{K}$ be the kernel of $\bar{S}$. Clearly, $\bar{K}$ is nonempty since $K \subset \bar{K}$. Let $v \in v_1v_2$. Then $v \in \bar{K}$. This follows from the fact that the suspension of a convex set M in E^m, having as suspension points two points such as v_1 and v_2 which lie on a line orthogonal to E^m in E^{m+1} and intersecting M, is quite evidently a convex set. Since K is known to be the intersection af all maximal convex sets in S, and $\bar{S}$ is contained in the union of the suspensions of all maximal convex sets in S, it follows easily that $v_1v_2$is contained in every maximal convex set in $\bar{S}$ and hence in $\bar{K}$.

Now let $x_1, x_2, \ldots, x_n$ be n distinct points of $\bar{S}$ which are not joined in $\bar{S}$. Since by Proposition 2.6, v_1S and v_2S have property P^n, not all of the points $x_1, x_2, \ldots, x_n$ can lie in one of these two sets. Let $x_1 \in v_1S$ and $x_2 \in v_2S$. Then $x_1 = \alpha x + (1 - \alpha)v_1$, $0 \leq \alpha \leq 1$, for some $x \in S$ and $x_2 = \beta y + (1 - \beta)v_2$, $0 \leq \beta \leq 1$, for some $y \in S$. Since $v_1v_2 \subset \bar{K}$, if $xy \subset S$, then the join of v_1v_2 and xy would be a 3-simplex or a 2-simplex, (in either case a convex set), which contains x_1x_2 and lies entirely in S. This implies that $x_1x_2 \subset \bar{S}$, contrary to assumption. Thus if $x_1x_2 \not\subset \bar{S}$, $xy \not\subset S$. By the proof of Proposition 2.6, the same conclusion may be drawn if x_1 and x_2 are both in v_1S or in v_2S. Thus if $x_1, x_2, \ldots, x_n$ are not joined in $\bar{S}$, there exist n distinct points which are not joined in S. This contradiction proves the result.

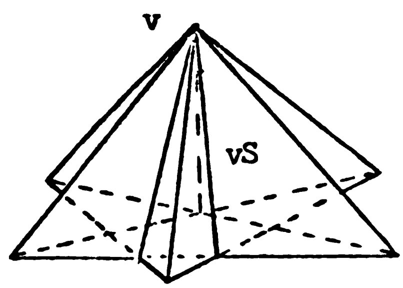
<u>Remarks</u>: From Proposition 2.6 it is clear that one may obtain a closed, connected set having property P^n which is starlike by simply constructing the cone over a closed, connected set having property P^n.

It also follows from Proposition 2.6 that the suspension of the set S between two points v_1 and v_2 must have property P^{2n-1}. It may, of course, have property P^m for $n \leq m \leq 2n-1$, as illustrated by the Corollary to Proposition 2.6

To illustrate how the situation changes, (and consequently the methods of proof), as the dimension of the set S increases, and to provide an example for the preceeding result, we consider the conventional five-pointed star in E^2.



This set obviously has property P^3, may be written as the union of three (and no fewer) convex sets, and has exactly five isolated points of local nonconvexity. The cone over S is a three dimensional set having property P^3, but has no <u>isolated</u> points of local non-convexity.



The following result of Valentine [17] is a clear indication of how the situation changes: Let S be a closed set in a linear topological space L where the dimension of L is greater than two. Assume that S has property P^3, and that S is not contained in any two-dimensional variety of L. Then if S has one isolated point of local non-convexity, S has at most two points of local non-convexity.

<u>Proposition 2.7</u> Let $S \subset E^m$ have property P^n and C be a convex set in E^m. Then $C + S = \{x + y \mid x \in C, y \in S\}$ has property P^n.

<u>Proof.</u> Let $c_i + x_i$, $i = 1.,,,.n$, be n distinct elements of $C + S$. Since $x_i x_j \subset S$ for some i, j, and $c_i c_j \subset C$ for all i, j, we have $0 \leq \alpha \leq 1$, $\alpha(c_i + x_i) + (1 - \alpha)(c_j + x_j) = \alpha c_i + (1 - \alpha)c_j + \alpha x_i + (1 - \alpha)x_j$ is in $C + S$. Hence, $C + S$ has property P^n.

<u>Corollary:</u> If under the hypotheses of Proposition 2.7, $C = \{x\}$, then $x + S$ has property P^n.

<u>Proposition 2.8</u> The Cartesian product of two starlike sets is starlike.

<u>Proof.</u> Let $A \subset E^k$ and $B \subset E^m$ be starlike sets, and let a_o and b_o be elements of the kernels of A and B, respectively. In E^{k+m} consider the vector expression

$$\alpha(a_o,b_o) + (1 - \alpha)(a,b) = (\alpha a_o, b_o) + ((1 - \alpha)a,(1 - \alpha)b)$$
$$(\alpha a_o + (1 - \alpha)a, \alpha b_o + (1 - \alpha)b)$$

which is an element of AXB for all $0 \leq \alpha \leq 1$.

since

$$\alpha a_o + (1 - \alpha)a \in A, \ 0 \leq \alpha \leq 1$$

and

$$\alpha b_o + (1 - \alpha)b \in B, \ 0 \leq \alpha \leq 1.$$

Corollary (to the Proof) The product of the kernels of A and B is the kernel of the product A X B.

Proposition 2.9 The Cartesian product of a convex set, C, with a set S having property P^n, has property P^n.

Proof. Let (c_i, x_i), $i = 1,2,\ldots,n$, be n distinct points of C X S. Then $c_ic_j \subset C$ for all i and j and $x_ix_j \subset S$ for some i and j. Thus $\alpha c_i + (1 - \alpha)c_j$ is an element of C and $\alpha x_i + (1 - \alpha)x_j$ is an element of S for some i and j and $0 \leq \alpha \leq 1$. But this implies that $(\alpha c_i + (1 - \alpha)c_j, \alpha x_i + (1 - \alpha)x_j)$ is an element of C X S for some i and j, and for all $0 \leq \alpha \leq 1$ as was to be shown.

Proposition 2.10 Let $S \subset E^m$ have property P^n and $\varepsilon > 0$. Then $U(S,\varepsilon)$, the parallel body of S, also has property P^n.

Proof. Let $x_1, x_2, \ldots, x_n$ be distinct elements of $U(S,\varepsilon)$. There exist elements $y_1, \ldots, y_n$ of S such that $||x_i - y_i|| < \varepsilon$ for $i = 1,\ldots,n$, by the definition of $U(S,\varepsilon)$. Assume first that all of the y_i are distinct. Then there exist i and j such that $y_iy_j \subset S$ by property P^n. Fixing i and j, consider x_ix_j. Let $x = \alpha x_i + (1 - \alpha)x_j$ for some α between 0 and 1. Then for the same α, $y = \alpha y_i + (1 - \alpha)y_j$ belongs to S. Moreover,

$$\begin{aligned} ||x - y|| &= ||\alpha x_i + (1 - \alpha)x_j - \alpha y_i + (1 - \alpha)y_j|| \\ &= ||\alpha(x_i - y_i) + (1 - \alpha)(x_j - y_j)|| \\ &< \alpha\varepsilon + (1 - \alpha)\,\varepsilon = \varepsilon \end{aligned}$$

which implies that x is an element of $U(S,\varepsilon)$ which in turn implies $U(S,\varepsilon)$ has property P^n.

Suppose now that the $y_1,\ldots,y_n$ are not all distinct. Let $y_i = y_j$, $i \neq j$. If it is possible to select an element $\bar{y}_j$ in S such that $y_i \neq \bar{y}_j$ and $d(x_j,\bar{y}_j) < \varepsilon$, and this is indeed possible for each pair which are equal, then the above argument applies. If not, then $U(y_i,\varepsilon) \cap S = y_i$. Since x_i,x_j are in $U(y_i,\varepsilon)$, which is convex, $x_ix_j \subset U(y_i,\varepsilon) \subset U(S,\varepsilon)$ which again implies $U(S,\varepsilon)$ has property P^n.

Proposition 2.11 If S is a set in E^m, and $X \subset S$ has property P^n relative to S, then cl(X) has property P^n relative to cl(S).

Proof. Let $x_1,x_2,\ldots,x_n$ be distinct elements in cl(X), then for any $\varepsilon > 0$ there exist $u_1,\ldots,u_n$, such that $||u_i|| < \varepsilon$ and $x_i + u_i$, $i = 1,2,\ldots,n$ are distinct elements of X. Since X has property P^n relative to S, the line segment $(x_i + u_i)(x_j + u_j)$, say, is in S. Thus for any α such that $0 \leq \alpha \leq 1$, $\alpha(x_i + u_i) + (1 - \alpha)(x_j + u_j)$ is in S. Now

$$||[\alpha(x_i + u_i) + (1 - \alpha)(x_j + u_j)] - [\alpha\, x_i + (1 - \alpha)x_j]||$$
$$= ||\, \alpha\, u_i + (1 - \alpha)\, u_j\, || < \alpha\, \varepsilon + (1 - \alpha)\, \varepsilon = \varepsilon$$

Since ε is arbitrary, $\alpha\, x_i + (1 - \alpha)\, x_j$ is in cl(S).

Proposition 2.12 Let $n \geq 2$ be a positive integer. The limit S of a sequence $\{S^k\}$ of compact sets having property P^n is a compact set having property P^n.

Proof. It is well known that S is compact [10]. Hence, let $x^1,x^2,\ldots,x^n$, be n distinct points of S, and let $\rho(S,S^k) = \varepsilon_k$ in the Hausdorff metric. Since $S^k \subset U(S,\varepsilon_k)$ and $\varepsilon_k \to 0$, we may find a sequence

$\{x_k^i\}$ $(i = 1,2,\ldots,n)$ such that x_k^i is in S^k and $\lim x_k^i = x^i$ $(i = 1,2,\ldots,n)$. For each value of k, at least two points of $x_k^1, x_k^2, \ldots, x_k^n$ are joined by a line segment which lies entirely in S^k. Since the number of possible pairs which may be joined in S^k for each k is finite, we may choose a subsequence $\{S^j\}$ of $\{S^k\}$ for which the sequences $\{x_j^1\}$, $\{x_j^2\}, \ldots, \{x_j^n\}$ converge to $x^1, x^2, \ldots, x^n$, and such that for some pair of points x^1 and x^2, say, $x_j^1 x_j^2 \subset S^j$ for all values of j. Since S is closed and $\epsilon_j \to 0$ we have $x^1 x^2 \subset S$; that is, S has property P^n.

Horn and Valentine [10] have generalized the notion of a convex set in the following manner: A set X in E_2 is called an L_n set if for every pair of points x and y in X, there is a polygonal path, consisting, of at most n segments, lying entirely in X, which joins x to y.

Perhaps the most striking result obtained for this class of sets is the following result which was proved by Bruchner [3]. Theorem: A necessary and sufficient condition that the set X in E_2 be compact and connected is that X be the limit of a sequence of compact L_n sets for some natural number n. This result has been generalized by J. W. McCoy to a set X contained in a complete, convex, locally compact metric space [13]. Kay [1[illegible]] has shown that a closed, connected P^n set S in a Minkowski space is an L_{n-1} set.

Thus, the class of all closed, connected sets having property P^n is a subclass of the class of L_{n-1} sets. The following example shows that they form a <u>proper</u> subclass. For $n \geq 2$, one needs only to take the cone at $(\frac{1}{2},1)$ over the points $(0,0)$, $(0,\frac{1}{n}),\ldots,(0,\frac{n-1}{n})$, $(0,1)$ to obtain an L_2 set with property P^{n+2}.

As a generalization of Definition 1.2 of property P^n, the following is given.

Definition 2.5 A set S in E^m is said to have property P^n_r if for each n distinct points of S, at least r + 1 of the points, $1 \leq r \leq n - 1$, are joined by line segments which lie entirely in S.

Proposition 2.13 Let S be a closed connected set in E^m $(m \geq 1)$. Then S has property P^n_2 if and only if S has property $P^{n-1}_1 \equiv P^{n-1}$.

It is immediate that S has property P^n_2 whenever S has property P^{n-1}. Hence, let S have property P^n_2, and assume that $\{x_k\}$, $k = 1,2,\ldots,n-1$ is a collection of $n - 1$ distinct points no two of which are joined in S. Let $x \in S$, $x \neq x_k$, $k = 1,2,\ldots,n-1$. Then by property P^n_2, $x\,x_1$ and $x\,x_2$, say, are in S. Let $\{z_i\}$ be a sequence of points in $(x\,x_1)$ converging to x_1. Then since $x_1\,x_k \not\subset S$, $k = 2,\ldots,n-1$, there must exist a neighborhood of N of x_1 such that for all z_i in N, $z_i\,x_k \not\subset S$, $k = 2,3,\ldots,n-1$. Then z_i and x_k, $k = 1,2,\ldots,n-1$, violate property P^{n-1}_2, a contradiction.

Corollary: Under the conditions of the Theorem, S is convex if and only if S has property P^3_2.

CHAPTER III

PLANAR SETS HAVING PROPERTY P^n

The results of this chapter were obtained in an attempt to determine the number of convex sets which are required if a closed, connected set S in E^2 having property P^n is to be expressed as the union of closed, convex sets. Two results of Valentine are extended and several new results obtained.

Unlike the results of Chapter II, many of the proofs of this Chapter depend upon the properties of the containing space, E^2.

The following notation and terminology will be standard throughout the remaining chapters.

The letter S denotes a closed connected set in E^2 unless otherwise stated. K is the convex kernel of the set S. The letter Q always denotes the set of points of local non-convexity (lnc) of S. (Q is evidently closed if S is closed).

The closed line segment joining x to y is denoted by xy; the corresponding open line segment is denoted by (xy). The line determined by the points x and y is denoted by L(x,y). By R(x,y) is meant the ray emanating from x and passing through y. By W(x,y) we shall mean the open half-plane determined by L(x,y) and lying to the left of the line L(x,y) if L(x,y) is considered as directed from x to y. While the meaning of the notation W(x,y) as given above is not standard, the economy of words which it allows in that which follows justifies its usage.

The interior, closure, boundary, and convex hull of a set A are denoted by intA, clA, bdA, and convA, respectively.

Definition 3.1 Let S be a connected space. A point q of S is called a cut point of S provided that $S - \{q\} = A \cup B$, where A and B are disjoint, nonempty, open subsets of S.

Theorem 3.1 If q is a cut point of the P^n set S, and all the components of $S - \{q\}$ are convex, then S is the union of $n - 1$ or fewer convex sets.

Proof. If all the components of $S - \{q\}$ are one dimensional the proof is immediate so assume that at least one of the components, S^*, is two dimensional. We proceed by induction on the number of components.

If there is but one additional component the conclusion is clear. If not, q is a cut point of $(S - S^*) \cup \{q\}$ and we claim $(S - S^*) \cup \{q\}$ is a P^{n-1} set. For if $(S - S^*) \cup \{q\}$ contains points $x_1, x_2, \ldots, x_{n-1}$ no two of which are joined in that set, then since S^* is not a subset of the union of the lines $L(q, x_i)$, there is a point x in S^* not joined in S to any x_i. Thus S is not P^n. Hence, by induction, $(S - S^*) \cup \{q\}$ is the union of $n - 2$ or fewer convex sets and S itself the union of $n - 1$ or fewer.

Theorem 3.2 If a closed, connected set S in E^m has exactly one point, q, of local non-convexity, then S is starlike from q.

Proof. By Proposition 2.3, it suffices to show that q is contained in every maximal convex subset of S. Let M be such a set and suppose $q \notin M$. Since M is closed, there exists a hyperplane, L, such that $L \cap M = \emptyset$ with q in one open half-space and M in the other. Let W be the closed half-space containing M.

If for each $y \in M$ there exists a sphere, $\sigma(y,\rho) \subset M$ then M is both open and closed relative to S and S is not connected. Thus for some z, $z \in M$, each sphere $\sigma(z,\rho)$ intersects $S - M$.

Since S is locally convex there exists at z, a sphere $\sigma(z,\rho_1)$ with $\sigma(z,\rho_1) \cap S$ convex and furthermore for some $\rho_2 < \rho_1$, $\sigma(z,\rho_2) \subset W$. Thus, $\sigma(z,\rho_2) \cap S \subset W \cap S$, is non empty, convex and is not a subset of M. It follows that M is a proper subset of the component, K, of $W \cap S$ which contains M.

Suppose, now, that x is any point of K. There exists $\sigma(x,\rho_3)$ such that $\sigma(x,\rho_3) \cap S \cap W$ is convex, and since $x \in K$, this set is a subset of K. That is to say, K is locally convex at each point and, being closed and connected is by Tietze's theorem, convex. This contradicts the maximality of M and shows that q must be an element of M.

Definition 3.2 If the rays R(q,p) and R(q,r) are not on a line they bound a convex and a non convex sector of the plane. The closed convex sector will be denoted by T(pqr) and the closed non convex sector by $T^*(pqr)$.

A sector of a circle which is non convex is a major circular sector. The center of the circle is also called the center of the sector.

<u>Lemma 3.1</u> If q is the only point of local non-convexity of the closed, connected set S in E^2, and q is not a cut point of S, then corresponding to points p and r of S such that $pr \not\subset S$ there exists a circular disk D(pqr) such that $D(pqr) \cap T^*(pqr) \subset S$.

<u>Proof</u>. Since S is starlike from q, the points q, p and r are not collinear. Since S - {q} is connected, locally compact, and locally connected, S - {q} is arcwise connected [20], and there exists an arc C in S - {q} containing p and r. Let the distance from q to the compact set C be α, and consider the circular disk D(pqr) with center q and radius $\alpha/2$.

If every ray R(q,x) in T(pqr) intersects C, then q is a point of local convexity of $T(pqr) \cap S$ and the component of $T(pqr) \cap S$ containing $qr \cup qp$ is convex. Then $pr \subset S$, a contradiction. Thus there exists a ray R(q,x) in T(pqr) which fails to intersect C. Now if any ray from q in $T^*(pqr)$ fails to intersect C, then C would lie in two separated subsets of the plane. Hence every such ray intersects C and $D(pqr) \cap T^*(pqr) \subset S$ as required.

<u>Lemma 3.2</u> If S and q are as in Lemma 3.1, then S has property P^3.

<u>Proof</u>. Suppose x,y,z are points of S no two of which are joined in S. If $q \in conv\{x,y,z\}$, then the smallest of the three disks D(xqy), D(yqz), and D(xqz), guaranteed by Lemma 3.1, is a subset of S, and S is locally convex at q, a contradiction.

Suppose then that $q \notin \text{conv}\{x,y,z\}$. One of the rays $R(q,x)$, $R(q,y)$, and $R(q,z)$ is in the interior of the convex sector defined by the other two. Assume that $R(q,y) \subset \text{int}(T(xqz))$. Then $D(xqy) \subset T(yqz) \cap S$, and q is a point of local convexity of $S \cap T(yqz)$. The component of this set containing y, q and z is thus a convex set and yz is in S. This contradiction establishes the theorem.

<u>Theorem 3.3</u> If q is the only point of local non-convexity of the closed, connected P^n set S in E^2, then S is the union of $n - 1$ or fewer convex sets.

<u>Proof.</u> If q is not a cut point of S then S is P^3 and it follows from [17] that it is the union of two convex sets.

Suppose then that q is a cut point of S. If all the components of $S - \{q\}$ are convex, the conclusion follows from Theorem 3.1. We consider now the remaining possibility that one of the components, S^*, is non-convex. Now q is clearly the only point of non-convexity of $S^* \cup \{q\}$, and q is not a cut point of this set. The set $S^* \cup \{q\}$ then satisfies the hypotheses of Lemma 3.1 and q is the center of a major circular sector, D, lying wholly in $S^* \cup \{q\}$. This means that the remaining components of $S - \{q\}$ lie in the convex sector of the plane defined by the rays which intersect D only in q. If M is such a component, it is clearly convex, since its only possible point of non-convexity is q and q is hardly the center of a major circular sector lying wholly in $M \cup \{q\}$. So $M \cup \{q\}$ is convex.

Since $(S - S^*) \cup \{q\}$ is now clearly P^{n-2} and satisfies the hypotheses of Theorem 3.1, it is the union of $n - 3$ or fewer convex sets while S^* is P^3 and is the union of two convex sets. Thus S is the union of $n - 1$ or fewer convex sets.

In the proof of Theorem 2 of [17] it may be observed that the use of property P^3 is unnecessary in the case where each point of local non-convexity of the set S is in the convex kernel of S if one introduces Lemma 3.4 below which itself is independent of property P^3 and requires only that S be closed and connected. That is, the following theorem, a generalization of the theorem cited above, can be proved.

<u>Theorem 3.4</u> Let S have at least two points of local non-convexity. If every point of local non-convexity is in the kernel, K, of S, then S may be written as the union of three closed convex sets. The number three is best.

The proof of this theorem is a modification of that given by Valentine [17] which avoids the use of property P^3. Four definitions and five lemmas are needed.

<u>Definition 3.3</u> A cross-cut of a set Y contained in E^2 is a closed segment xy such that $(xy) \subset \mathrm{int}Y$ and such that x and y are in bdY.

<u>Lemma 3.3</u> Each open segment (uv) of the convex kernel, K, of S contains no points of local non-convexity.

Proof. Let w be an element of $Q \cap (uv)$. Clearly $S \not\subset L(u,v)$. Let z be in $S - L(u,v)$. Since $uv \subset K$, $\Delta uzv \subset S$. Hence, each sufficiently small neighborhood of w contains no crosscuts of $E^2 - S$, since such a crosscut xy would have to have its interior (xy) in one of the open half-planes bounded by $L(u,v)$.

Definition 3.4 A component of the complement of a closed connected set S is called a residual domain of S.

Lemma 3.4 Let D be a bounded residual domain of S. Then, the bdD contains at least three points of local non-convexity of S.

Proof. Consider the set $E^2 - D$ which contains S. $E^2 - D$ is closed since D is by definition an open subset of an open set in E^2. Moreover, $bdD = clD \cap (E^2 - D)$ is closed and bounded, and hence compact. Let p be a fixed element of bdD and x be an arbitrary element of bdD. As x varies over bdD the distance from p to x defines a continuous function d from the compact set $p \times bdD$ into the reals. Hence, d must attain its maximum at some point q_1 of bdD. Consider the sphere $\sigma(p,r)$ having center at p and radius $r = d(p,q_1)$. Evidently $D \subset C$, where C is the open disk bounded by $\sigma(p,r)$. Thus, each point of $\sigma(p,r)$ is contained in $E^2 - D$. If q_1 is a point of local convexity of $E^2 - D$, then there exists an open spherical neighborhood N of q_1 such that $N \cap (E_2 - D)$ is convex. In particular, if x and y in $\sigma(p,r) \cap N$ are such that $pq_1 \cap xy \neq \emptyset$, we have $xy \subset E^2 - D$ which contradicts the assumption that q_1 is in the bdD. It follows that q_1 is a point of local non-convexity for $E^2 - D$. Moreover, q_1 is a point of local non-convexity of S. Let x and y be elements of $E^2 - D$ such that $xy \not\subset E^2 - D$. Then $xy \cap D \neq \emptyset$. Since D is an open set $xy \cap D$ may be expressed as a countable union of disjoint open intervals. Let (uv) be one such interval. Then $u, v \in \mathrm{bd}D \subset S$, and $uv \not\subset S$. More specifically, if in every spherical neighborhood N' of q_1 there exist points x and y of $E^2 - D$ such that $xy \not\subset E^2 - D$, then there exist points u and v in $S \cap xy \cap N'$ such that $uv \not\subset S$. That is, if q_1 is a point of local non-convexity of $E^2 - D$, then q_1 is a point of local non-convexity of S. In the same way we may next locate a second point of local non-convexity q_2 in bdD at a maximal distance from q_1. (Which will not be p, in general). The third point q_3 is obtained in like manner by maximizing the sum of the distances $d(q_1,q_3)$ and $d(q_2,q_3)$ to obtain an ellipse with foci at q_1 and q_2 passing through q_3. Because the ellipse, like the circle, is a convex curve, the very same argument gives the desired result.

Remark: The boundary of the triangle indicates that the number 3 is best.

Lemma 3.5 Under the hypothesis of Theorem 3.4, S has at least one isolated point of local non-convexity.

Proof. Let xy be a crosscut of a residual domain of S. The set D - (xy) is the union of two disjoint open sets, denoted by D_1 and D_2 [14]. Since S is starlike, D_1, say, is bounded while D_2 is not. Then $\mathrm{bd}D_1$ is a continuum [14]. By Lemma 3.4, $\mathrm{bd}D_1$ contains a point q different from x or y which is a point of local non-convexity of S. Since S is starlike from q, xq and yq are in S. This implies that $D_1 \subset \Delta xqy$ since D was a residual domain and q is in bdD_1.

Consider the lines $L(x,q)$ and $L(y,q)$ or more specifically, the v - shaped domains V_i, $i = 1,2,3,4$, that they determine. Order the V_i in a clockwise direction about q so that $V_1 \supset D_1$. Suppose q_1 is an element of $(clV_1 - q) \cap Q$. Then since $(q_1x \cup q_1y) \subset S$ we have $D_1 \subset \Delta xyq_1$. But this contradicts the fact that q is in bdD_1. Suppose next that there exists an element q_1 in $V_2 \cap Q$. Then Δxqq_1 and Δyqq_1 are contained in S and once more we would contradict the fact that q is an element of bdD_1. Similarly for V_4.

Now if q_1 is in $V_3 \cap Q$, $(\Delta yqq_1 \cup \Delta xqq_1) \subset S$ which implies q is isolated since V_1 contains no points of Q. Finally, since no open segment of K contains a point of Q, there does not exist a sequence of points of Q along $L(x,q) \cap clV_3$ or $L(y,q) \cap clV_3$ with q as a limit point. Thus, q is an isolated point of local non-convexity of S.

<u>Corollary</u>. Let Q' be the set of isolated points of local non-convexity. Then $Q = cl(Q')$.

<u>Proof</u>. Let $q \in Q$. If $q \in Q'$, then $q \in cl(Q')$. If $q \notin Q'$, let N be an open spherical neighborhood of q. Since $q \in Q$, there exist x, $y \in N \cap S$ such that $xy \not\subset S$ and xy defines a crosscut x'y' of a residual domain D of S. As in the proof of Lemma 3.5, the boundary of the bounded component D_1 of $D-(x'y')$ contains an element q' of Q'. Since $q \in K$,

$clD_1 \subset \Delta x'qy' \subset N$. Thus every neighborhood N of q contains an element q' of Q'. That is, $Q \subset cl(Q')$. Since Q is closed, $cl(Q') \subset Q$, and we have $Q = cl(Q')$.

<u>Lemma 3.6</u> The boundary of convQ is connected, and contains at most one ray.

<u>Proof.</u> Since $H \equiv$ convQ is convex, if bdH were not connected, it is well known that it would consist of two parallel lines. Then Lemma 3.3 would imply that each of these parallel lines contains at most two points of local non-convexity. But then Q would be bounded and bdH would be connected. If bdH contains two rays, then Lemma 3.3 would again imply that Q is bounded, a contradiction.

<u>Definition 3.5</u> An edge of bd(convQ) is a closed segment xy or a closed ray $x\infty$ whose endpoints are elements of Q.

<u>Lemma 3.7</u> Let x and y be successive points of local non-convexity in bd(convQ), and W be an open half-plane of support to convQ which abuts on the edge xy (or $x\infty$). Then $(\text{conv}Q) \cup (W \cap S)$ is a convex subset of S.

<u>Proof.</u> If u is in convQ and v is an element of $S \cap W$, then $uv \subset S$, since $\text{conv}Q \subset K$, K being convex. Now, to show that $uv \subset (\text{conv}Q \cup (S \cap W))$ we show that $xy \cap uv \neq \emptyset$ (or $uv \cap x\infty \neq \emptyset$).

Recall that x and y are in bdS. Suppose $uv \cap xy = \emptyset$. Then xv and yv are in S which implies that x is in $\operatorname{int}(\Delta uyv) \cap S$ or y is in $\operatorname{int}(\Delta uxv) \cap S$ which implies x or y is not in Q, a contradiction.

Suppose next that u and v are elements of $S \cap W$. Let z be an element of (xy) (or $(x\,\infty)$). Suppose $uv \not\subset S \cap W$. Since $uz, zv \subset S$, Δuvz would contain a point of Q, by Lemma 3.4. This is a contradiction since $W \cap Q = \emptyset$ and by Lemma 3.3 (xy) (or $x\,\infty$) contain no points of Q. Hence $(\operatorname{conv}Q) \cup (W \cap S)$ is convex. (If $\operatorname{conv}Q = xy$, then $(\operatorname{conv}Q \cup (W \cap S))$ may or may not be closed).

<u>Lemma 3.8</u> Let $\{x_i y_i\}$ be a countable number of pairwise disjoint edges in bd(convQ). Assume that bd(convQ) contains at least three edges, and let W be the open half-plane of support to convQ whose boundary contains $(x_i y_i)$, ($x_i y_i$ may be $x_i\infty$). Then the set $X \equiv (\operatorname{conv}Q) \cup (S \cap (\bigcup_i W_i))$ is a closed convex set.

<u>**Proof.**</u> Choose an order on the boundary bd(convQ), and assume that in this ordering x_i is the beginning of the edge $x_i y_i$ and that y_i is the end point of $x_i y_i$.

Let $x_i y_i$ and $x_j y_j$ be two disjoint edges, and consider the convex region V bounded by the lines $L(x_i, y_j)$ and $L(x_j, y_i)$ and containing the quadrilateral $x_i y_i x_j y_j$. Let V_i and V_j be the portions of V adjacent to $x_i y_i$ and $x_j y_j$, respectively. (These two sets may not be bounded). If $x_j y_j = x_j\infty$, say, then $L(x_i, \infty)$ is a line parallel to the ray $x_j\infty$. Now, $S \cap W_j \subset V_j$ since otherwise x_j or y_j are not in Q, a contradiction.

Let u and v be elements of S. If u and v are in $(\text{conv}Q) \cup (S \cap W_i)$, then by Lemma 3.7, $uv \subset X$. If u is in $S \cap W_i$ and v is in $S \cap W_j$, $i \neq j$, then u is in V_i and v is in V_j. Since V is convex, $V_i \cap x_iy_ix_jy_j = x_iy_i$, and we have $uv \cap x_iy_i \neq \emptyset$ which implies $uv \subset X$.

Finally, X is closed. The finite case is immediate since $\text{cl}W_i \cap S \subset V_i$ implies $\text{cl}(W_i \cap S) \subset (W_i \cap S) \cup \text{bd}(\text{conv}Q)$. If there are an infinite number of disjoint edges, let r be a limit point of the sequence of sets $W_{i_n} \cap S$. Since $W_{i_n} \cap S \subset V_{i_n}$, by fixing (x_jy_j) of the preceeding paragraph it follows that $(x_{i_n}y_{i_n}) \to q$, a fixed point of bd(convQ), as $i_n \to \infty$. Since then $V_{i_n} \to q$ as $i_n \to \infty$ we have $r = q$, an element of convQ. Hence X is closed since convQ is closed.

<u>Proof of Theorem 3.4</u> First assume that $Q = \{q_1 \cup q_2\}$. The line $L(q_1,q_2)$ divides the plane into the two open half-planes W_i $(i = 1,2)$. By Lemma 3.7, $W_i \cap S$ is convex $(i = 1,2)$. Hence, $S = \text{cl}(W_1 \cap S) \cup \text{cl}(W_2 \cap S) \cup L(q_1,q_2) \cap S$ is the desired decomposition.

Next assume that $Q = \{q_1 \cup q_2 \cup \ldots \cup q_{2m}\}$ where $m > 1$. Order the edges of bd(convQ) in a counterclockwise manner so that $q_1 \equiv q_{2m+1}$. Let W_i denote the open half-plane of support to convQ adjacent to q_iq_{i+1}. By Lemma 3.8 each of the sets

$$S_1 = (\text{conv}Q) \cup S \cap (\bigcup_{i=1}^{m} W_{2i-1})$$

$$S_2 = (\text{conv}Q) \cup S \cap (\bigcup_{i=1}^{m} W_{2i})$$

is a closed convex set. Since $S \subset (\text{conv}Q) \cup S \cap (\bigcup_{i=1}^{2m} W_i)$ we have $S = S_1 \cup S_2$.

Next, if $Q = \{q_1 \cup q_2 \cup \ldots \cup q_{2m+1}\}$ $m \geq 1$, we add $S_3 = (\mathrm{conv}Q) \cup (S \cap W_{2m+1})$ to the sets S_1 and S_2. Finally, if Q is infinite, we need the following definition:

Definition 3.6 A closed, connected subset I of bd(convQ) is called a polygonal element if the following conditions hold:

1. It is the closure of the union of edges of bd(convQ). (An edge of bd(convQ) is a closed segment xy or a closed ray $x\infty$ whose endpoints are contained in clQ.)
2. Its endpoints are limit points of elements in Q.
3. If I = bd(convQ), then I contains at most one limit point of elements in Q. If I $\neq$ bd(convQ), then only its endpoints are limit points of elements in Q.

Note that a polygonal element is maximal in the sense that it is not a proper subset of a larger polygonal element. The number of polygonal elements of bd(convQ) is countable. This follows from the fact that they are convex subarcs of the boundary of a convex set, convQ, which do not overlap. By definition, each I contains at least one segment. Hence, relative to bd(convQ), each polygonal element has a non-empty relative interior, and the non-overlapping of the polygonal elements implies countability. If convQ is bounded, it is clear that there can be at most finitely many polygonal elements of length at least 1/n times the perimeter of bd(convQ). In the unbounded case, we may simply consider a (countable) monotone increasing sequence $\{\sigma_i\}$ of closed disks concentric about the origin. Then $\mathrm{bd}(\sigma_i \cap \mathrm{conv}Q)$ contains at most a countable number of polygonal elements for each value of i. Since $\{\sigma_i\}$ is countable, we have bd(conv$Q$) contains at most a countable number of polygonal elements.

Let $I_1, I_2, \ldots, I_k, \ldots$ be a well ordering of these elements. For each polygonal element I_k, divide the edges it contains into two classes M_k^1 and M_k^2 such that no two edges of M_k^i $(i = 1,2)$ are adjacent, that is, have an endpoint in common. It may happen that one of the M_k^i may be empty. For each edge $e \in M_k^i$ we let W_e^i denote the open half-plane of support to bd(conv$Q$) whose boundary contains e. Define

$$F_k^i = \bigcup_{e \in M_k^i} (W_e^i \cap S) \qquad (i = 1,2) \quad \text{and let}$$

$$S_i = \text{conv}Q \cup \left(\bigcup_k F_k^i \right) \quad (i = 1,2)$$

Since each edge in M_k^i is separated from each edge in M_k^i $(k \neq m)$, Lemma 3.8 implies that S_1 and S_2 are closed convex subsets of S. Moreover, since for each point $x \in S$, either $x \in$ conv$Q$, or x is in some $W_e^i \cap S$, we have $S = S_1 \cup S_2$ and $S_1 \cap S_2$ is non-empty.

Corollary. If S has property P^3 and two or more points of local non-convexity, then S may be expressed as the union of three or fewer closed convex sets.

Proof. If q is a point of local non-convexity of S, then S is starlike from q, [17]. The result now follows, since the hypothesis of the theorem is satisfied.

If the set S bounds a bounded residual domain of itself, the following two results may be obtained.

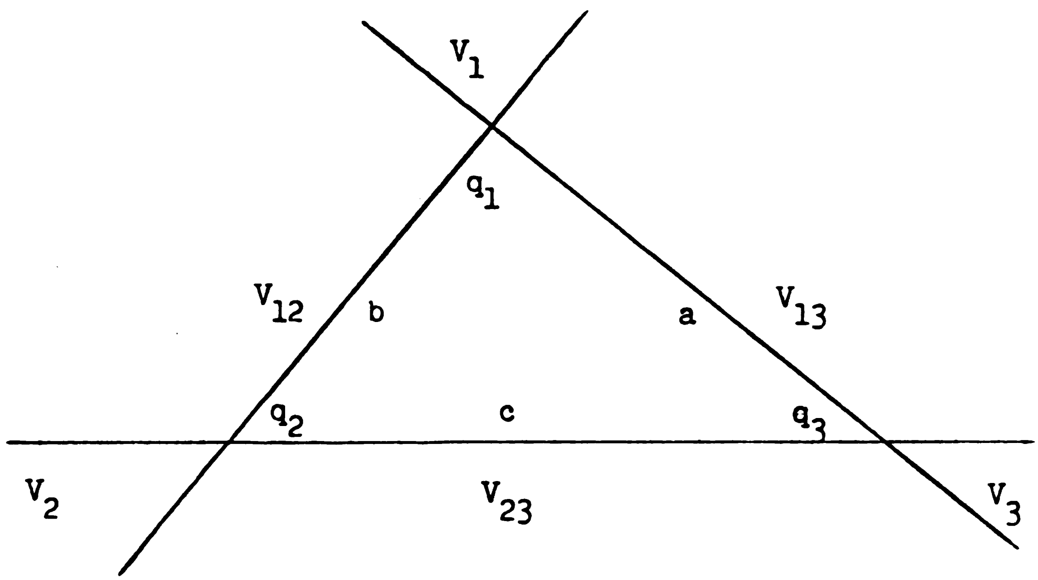
<u>Lemma 3.9</u> Let S have property P^4. Then S can bound at most one bounded residual domain of S.

<u>Proof.</u> Let K_1 be such a domain. By Lemma 3.4, bdK_1 contains three or more points of local non-convexity of S. Let q_1, q_2, q_3 be three such points. Then, by property P^4 and the closure of S, these points are joined by line segments lying entirely in S. Thus, by the definition of K_1, $K_1 \subset \Delta q_1q_2q_3$. Moreover, there can not exist a fourth point q_4 of local non-convexity in $\Delta q_1q_2q_3$ since then $q_4q_i \subset S$, $i = 1,2,3$, which would either contradict the definition of K_1 or the fact that q_i is an element of bdK_1. For the same reasons if x_1, y_1, and z_1 are elements of (q_1q_2), (q_2q_3) and (q_1q_3), respectively, then the line joining any two of these points does not lie entirely in S. Hence, by property P^4, every point of S is joined to at least one of the points x_1, y_1 and z_1 by a line segment lying entirely in S. If a second residual domain, K_2 of S exists, then there exist points of bdK_2 corresponding to q_1, q_2 and q_3 of bdK_1 at least one of which is distinct from q_1, q_2, or q_3 and points x_2, y_2, and z_2 corresponding to x_1, y_1 and z_1, respectively of $\Delta q_1q_2q_3$ such that the three line segments joining x_1, y_1, and z_1 to x_2, y_2 or z_2 are not in S or the three line segments joing x_2, y_2 and z_2 to x_1, y_1, or z_1 are not in S. In either case property P^4 is violated. Thus S can bound at most one bounded residual domain of S.

<u>Theorem 3.5</u> Let S have property P^4. If S bounds a bounded residual domain of S, then S may be expressed as the union of five or fewer closed convex sets. The number five is best.

Proof. Let B be the bounded residual domain which is bounded by S. Then B is entirely contained in some triangle whose vertices q_1, q_2 and q_3 are points of local non-convexity of S, as was shown in the proof of Lemma 3.4. We denote the six unbounded domains into which the plane is divided by extensions of the sides of $\Delta q_1q_2q_3$ as shown in Figure 3.2. By V_0 we shall mean the set $\mathrm{cl}(\Delta q_1q_2q_3) - B$. The component of $\mathrm{int}(S \cap V_0)$ whose closure contains q_iq_j will be denoted by J_{ij}. Denote $(S \cap V_{ij}) \cup J_{ij} \cup (q_iq_j)$ by D_{ij}.

Figure 3.2



Observe first of all that $S \cap D_{13}$, $S \cap D_{23}$ and $S \cap D_{12}$ are each convex sets since if x and y are elements of $S \cap D_{23}$, say, such that $xy \not\subset S \cap D_{23}$, then, a and b (as shown in Figure 3.2) together with x and y are four points which violate property P^4. The same argument applies to $S \cap D_{13}$ and $S \cap D_{12}$. Observe next that the points c, a, and b are not joined to any point in $S_1 = (S \cap W(q_2,q_3)) - J_{23}$, $S_2 = (S \cap W(q_3,q_1)) - J_{13}$, and $S_3 = (S \cap W(q_1,q_2)) - J_{12}$, respectively, which implies that these sets have property P^3. Hence, by Proposition 2.11

and the Corollary to Theorem 3.4, the closure of each of these sets may be written as the union of three or fewer closed convex sets. Let Q_i denote the set of points of local non-convexity of S_i. Since S_i has property P^3, Q_i is contained in the kernel of S_i. Clearly, Q_i is contained in $cl(V_i \cap S)$, and is non-empty, since q_i is contained in Q_i. We denote the cardinality of Q_i by N_i. Since $S = S_1 \cup S_2 \cup S_3$, the remainder of the proof consists of showing that under all possible circumstances these three starlike sets may always be written as the union of five or fewer closed convex sets.

Order the points of Q_1 and Q_2 in a counterclockwise direction starting with $q_1 \equiv q_{1,1}$ and $q_2 \equiv q_{2,1}$, respectively. Let $Q_1 = \bigcup_{j=1}^{N_1} \{q_{1,j}\}$ and $Q_2 = \bigcup_{k=1}^{N_2} \{q_{2,k}\}$.

CASE I Assume first that $N_1 = N_2 = 1$. We consider a closed disk C_i with center at q_i, $i = 1,2$, and with radius sufficiently small so that $C_i \cap S \subset S_i$. In the proof of Theorem 3[16] it is shown that the number of components of $S \cap bd(conv(C_i \cap S))$ is 1,2, or 4. If the number of components is 4, then it is shown that clS_i is the union of two distinct rays or line segments intersecting at q_i. Here this implies that S_i is contained in $L(q_1,q_2) \cup L(q_i,q_3)$, $i = 1,2$. For future reference, we denote the convex subset $S_i \cap L(q_1,q_2)$ of S_i by X^i_{12} and denote the convex subset $S_i \cap L(q_i,q_3)$ by X^i_{13}.

If the number of components of $S \cap bd(conv(C_i \cap S))$ is 2, then S_i is known to be expressible as the union of two convex sets each of which is determined by one of the two distinct components of

$S \cap bd(conv(C_i \cap S))$. (By the proof of Theorem 3[17]). The intersection of these two convex subsets of S_i is $\{q_i\}$. Let B_i be the component of $S \cap bd(conv(C_i \cap S))$ which intersects $S \cap D_{12}$. Let X^i_{12} be the unique convex subset of S_i determined by B_i. By the above discussion of the proof of Theorem 3[17], q_i is a cut point of S_i and $X^i_{12} - \{q_i\}$ is a component of $S_i - \{q_i\}$. Since $X^i_{12} - \{q_i\}$ intersects the convex (connected) set $S \cap D_{12}$, the latter must be contained in $X^i_{12} - \{q_i\}$, and consequently in X^i_{12}, $i = 1,2$. By the same argument $(S_i - X^i_{12}) \cup \{q_i\} \equiv X^i_{13}$ is a convex subset of S_i containing $S \cap D_{13}$. If $S \cap bd(conv(C_i \cap S))$ has exactly one component, $i = 1,2$, then the support line to the convex set J_{12} through q_i, distinct from $L(q_1,q_2)$ if $J_{12} \neq \emptyset$ and $L(q_1,q_2)$ if $J_{12} = \emptyset$, determines two convex subsets X^i_{12} and X^i_{13} of S_i such that $q_i \in (clX^i_{12} \cap clX^i_{13})$, $S_i = clX^i_{12} \cup clX^i_{13}$, $X^i_{12} \supset S \cap D_{12}$ and $X^i_{13} \supset S \cap D_{13}$.

<u>CASE II</u> Assume now that $N_i = 2m_i$, $i = 1,2$. Consider the edges $q_{1,j}q_{1,j+1}$ and $q_{2,k}q_{2,k+1}$ $j = 1,\ldots,2m_1$, $k = 1,\ldots,2m_2$ where $q_{1,2m_1+1} \equiv q_{1,1}$ and $q_{2,2m_2+1} \equiv q_{2,1}$. Let the open half-plane of support to $convQ_1$ and $convQ_2$ adjacent to $q_{1,j}q_{1,j+1}$ and $q_{2,k}q_{2,k+1}$, respectively, be denoted by W^1_j and W^2_k, respectively. By Lemma 3.8, the sets $(convQ_1) \cup cl(S_1 \cap (\bigcup_{j=1}^{m_1} W_{2j})) \equiv X^1_{12}$ and $(convQ_2) \cup cl(S_2 \cap (\bigcup_{k=1}^{m_2} W_{2k-1})) \equiv X^2_{12}$ are convex sets, having $S \cap D_{12}$ in common. Moreover, by the proof of Lemma 3.8, X^i_{12} is known to be one of the two convex sets whose union is S_i, $i = 1,2$. The closed convex sets

$$(\mathrm{conv}Q_1) \cup \mathrm{cl}(S_1 \cap (\bigcup_{j=1}^{m_1} W_{2j-1})) \equiv X_{13}$$

and

$$(\mathrm{conv}Q_2) \cup \mathrm{cl}(S_2 \cap (\bigcup_{k=1}^{m_2} W_{2k})) \equiv X^2_{23}$$

are such that $S_i = X^i_{12} \cup X^i_{13}$ and $S \cap D_{13} \subset X^i_{13}$ for $i = 1,2$, as in the proof of Lemma 3.8.

<u>CASE III</u> If one (or both) of the N_i, $i = 1,2$, is infinite, then by considering polygonal elements of $\mathrm{bd}(\mathrm{conv}Q_i)$, as in the proof of Theorem 3.4, we may express the set S_i as the union of two distinct convex subsets X^i_{12} and X^i_{13}, such that X^i_{12} contains $S \cap D_{12}$ and X^i_{13} contains $S \cap D_{13}$, $i = 1,2$.

<u>CASE IV</u> We next assume that $N_1 = 2m_1+1$ for $m_1 > 1$. Again, by Lemma 3.8, the sets

$$(\mathrm{conv}Q_1) \cup \mathrm{cl}(S_1 \cap (\bigcup_{j=1}^{m_1-1} W^1_{2j} \cup W^1_{2m_1+1})) \equiv Y^1_{12}$$

$$(\mathrm{conv}Q_1) \cup \mathrm{cl}(S_1 \cap (\bigcup_{j=1}^{m_1} W^1_{2j+1})) \equiv Z^1_{12}$$

are closed convex sets. Likewise,

$$(\mathrm{conv}Q_2) \cup \mathrm{cl}(S_2 \cap (\bigcup_{k=1}^{m_2} W^2_{2k-1})) \equiv Z^2_{12}$$

and

$$(\mathrm{conv}Q_2) \cup \mathrm{cl}(S_2 \cap (\bigcup_{k=2}^{m_2} W^2_{2k} \cup W^2_1)) \equiv Y^2_{12}$$

are closed convex sets. The same Lemma 3.8 shows that

$$(\mathrm{conv}Q_1) \cup \mathrm{cl}(S_1 \cap (W^1_1 \cup W^1_{2m_1})) \equiv X^1_{13}$$

and

$$(\mathrm{conv}Q_2) \cup \mathrm{cl}(S_2 \cap (W^2_2 \cup W^2_{2m_1+1})) \equiv X^2_{23}$$

are closed convex sets containing $S \cap D_{13}$ and $S \cap D_{23}$, respectively. Moreover, $S_i = X^i_{13} \cup Y^i_{12} \cup Z^i_{12}$, and the sets Y^i_{12} and Z^i_{12} each contain $S \cap D_{12}$, $i = 1,2$.

In the four cases considered we have selected convex subsets of S_1 and S_2 denoted by X^1_{12}, X^2_{12}, Y^1_{12}, Y^2_{12}, Z^1_{12}, and Z^2_{12}, each of which contains the convex set $S \cap D_{12}$. At the same time we have selected the convex subsets X^1_{13} and X^2_{23} of S_1 and S_2, respectively, in such a way that X^1_{13} contains $S \cap D_{13}$ while X^2_{23} contains $S \cap D_{23}$. Moreover, in each case these subsets have been chosen so that $S_i = X^i_{13} \cup Y^i_{12} \cup Z^i_{12}$ or $S_i = X^i_{13} \cup X^i_{12}$. It is clear that several other cases arise as combinations of the four cases considered; for example, N_1 might be an even positive integer while N_2 is an odd positive integer. However, in each of these cases, the decomposition of S_1 and S_2 may be taken to be exactly the same as in the four cases considered.

To reduce the number of convex sets from 9 to 5 we first show that $cl(S_1 \cup S_2)$ may be expressed as the union of 4 or fewer closed convex sets. This is accomplished by showing that $cl(G^1_{12} \cup H^2_{12})$ is a closed convex subset of $cl(S_1 \cup S_2)$ where X, Y, and Z may be substituted for either G or H. Assume then that $G^1_{12} \equiv G$ is a convex subset of S_1 containing $S \cap D_{12}$ and that $H^2_{12} \equiv H$ is a convex subset of S_2 also containing $S \cap D_{12}$. Consider the closed subset $cl(G \cup H)$ of the closed set S. Since G and H are convex sets, each is connected. Since $G \cap H = S \cap D_{12}$, a non-empty, convex (connected) set, $G \cup H$ is a connected set.

Hence, $cl(G \cup H)$ is a connected set. By Tietze's Theorem, if $cl(G \cup H)$ is locally convex, then $cl(G \cup H)$ is convex. Assume, to the contrary, that $cl(G \cup H)$ has a point of local non-convexity p. Since clG and clH are convex subsets of clS_1 and clS_2, respectively, p is not a point of local non-convexity of clG or clH. Thus, if σ is a spherical neighborhood of p of radius r, and $x,y \in \sigma \cap cl(G \cup H)$ such that $xy \not\subset cl(G \cup H)$, then $x \in (clG - clH)$ while $y \in (clH - clG)$. This implies $x \in cl(S_1 \cap V_1)$ while $y \in cl(S_2 \cap V_2)$. If we take r to be less than one-half the distance from q_1 to q_2, then no such x and y can exist. Since this contradicts the definition of p as a point of local non-convexity of $cl(G \cup H)$, it must be true that $cl(G \cup H)$ is locally convex, and hence convex.

For the cases discussed, it has thus far been shown that the 6 possible closed convex sets which could arise from $cl(S_1 \cup S_2)$ may always be reduced to 4 or fewer closed convex sets. The arguments given for the four cases above when applied to clS_3 show that clS_3 may be expressed as the union of 3 or fewer closed convex sets in such a way that one of these sets X^3_{13} contains $S \cap D_{13}$ while another X^3_{23} contains $S \cap D_{23}$. Then the same argument given above applies to show that $cl(X^1_{13} \cup X^3_{13})$ and $cl(X^2_{23} \cup X^3_{23})$ are closed convex subsets of S. Recalling the arguments given in the four cases above, it is clear that this can be done in such a way that there will remain in clS_3 at most one closed convex set. It has now been shown that the seven possible closed convex sets which remained may be expressed as five or fewer closed convex sets, the desired result.

<u>CASE V.</u> There is one case which remains to be settled; namely, the case which arises when $N_i = 3$ for one or more values of i.

If $N_i = 3$ for exactly one value of i, then without loss of generality we may assume that $N_3 = 3$. As in Cases II, III, and IV we consider the edges q_{31},q_{32}, $q_{32}q_{33}$, and $q_{33}q_{31}$ of $\operatorname{conv}Q_3$. We again denote the open half-plane of support to $\operatorname{conv}Q_3$ adjacent to $q_{3,j}q_{3,j+1}$ by W_j^3, $j = 1,2,3$ letting $q_{34} \equiv q_{31}$. By Lemma 3.8 the set $(\operatorname{conv}Q_3) \cup \operatorname{cl}(S_3 \cap W_j^3)$ $j = 1,2,3$, is a convex set. Moreover, $(\operatorname{conv}Q_3) \cup \operatorname{cl}(S_3 \cap W_1^3) \equiv X_{23}^3$ contains D_{23} while $(\operatorname{conv}Q_3) \cup \operatorname{cl}(S_3 \cap W_3^3)$ $\equiv X_{13}^3$ contains D_{13}. Finally, $S_3 = X_{23}^3 \cup X_{13}^3 \cup ((\operatorname{conv}Q_3) \cup \operatorname{cl}(S_3 \cap W_2^3))$

By the same arguments given for the preceeding three cases $\operatorname{cl}(X_{13}^1 \cup X_{13}^3)$ and $\operatorname{cl}(X_{23}^2 \cup X_{23}^3)$ are convex sets. The only distinction to be made now between this case and previous cases is that the remaining convex set in $\operatorname{cl}S_3$ which is the fifth convex set in that decomposition is the set $(\operatorname{conv}Q_3) \cup \operatorname{cl}(S_3 \cap W_2^3)$.

We next assume that $N_i = 3$ for $i = 1,2$ and that N_3 is one, even or infinite. In this case, as in Cases I, II, and III, the set $\operatorname{cl}S_3$ may be expressed as the union of two closed convex sets $\operatorname{cl}(X_{23}^3)$ and $\operatorname{cl}(X_{13}^3)$, containing D_{23} and D_{13}, respectively.

Since $N_i = 3$ for $i = 1,2$, S_i may be expressed as the union of the three convex sets, $(\operatorname{conv}Q_i) \cup \operatorname{cl}(S_i \cap W_j^i)$, $j = 1,2,3$, $i = 1,2$, with the property that

$$X_{12}^1 \equiv (\mathrm{conv}Q_1) \cup \mathrm{cl}(S_1 \cap W_3^1) \supset D_{12}$$

$$X_{12}^2 \equiv (\mathrm{conv}Q_2) \cup \mathrm{cl}(S_2 \cap W_1^2) \supset D_{12}$$

and

$$X_{13}^1 \equiv (\mathrm{conv}Q_1) \cup \mathrm{cl}(S_1 \cap W_1^1) \supset D_{13}$$

$$X_{23}^2 \equiv (\mathrm{conv}Q_2) \cup \mathrm{cl}(S_2 \cap W_3^2) \supset D_{23}$$

By the argument given after Case IV, $\mathrm{cl}(X_{12}^1 \cup X_{12}^2)$, $\mathrm{cl}(X_{13}^1 \cup X_{13}^3)$, and $\mathrm{cl}(X_{23}^2 \cup X_{23}^3)$ are convex sets. Since S is evidently the union of these three sets together with the convex sets $(\mathrm{conv}Q_i) \cup \mathrm{cl}(S_i \cap W_2^i)$ $i = 1,2$, we see that S is once again expressible as the union of five or fewer closed convex sets. Finally, we consider the case where $N_i = 3$, $i = 1,2$, and $N_3 = 2m + 1$ for $m \geq 1$. It is convenient to consider the case $m = 1$ first.

For $m = 1$, we define the following sets

$$A_1 = \mathrm{cl}[(S_2 \cup S_3) \cap (W(q_{22},q_{23}) \cap W(q_{32},q_{21}) \cap W(q_{32},q_{33}))]$$

$$A_2 = \mathrm{cl}[(S_1 \cup S_3) \cap (W(q_{12},q_{13}) \cap W(q_{11},q_{33}) \cap W(q_{32},q_{33}))]$$

$$A_3 = \mathrm{cl}[(S_2 \cup S_3) \cap (W(q_{21},q_{22}) \cap W(q_{23},q_{32}) \cap W(q_{33},q_{31}))]$$

$$A_4 = \mathrm{cl}[(S_1 \cup S_3) \cap (W(q_{13},q_{11}) \cap W(q_{31},q_{32}) \cap W(q_{33},q_{12}))]$$

Finally, let

$$B_1 = ((S_1 \cap W_2^1) \cap W(q_{12},q_{33}))$$

$$B_2 = ((S_2 \cap W_2^2) \cap W(q_{32},q_{23}))$$

and

$$B_{12} = D_{12} \cap (W(q_{33},q_{11}) \cap W(q_{21},q_{32}))$$

Since each is the closure of the intersection of convex sets, each is convex. We define A_5 to be the set $\mathrm{cl}(\mathrm{conv}(B_1 \cup B_2 \cup B_{12}))$. Consider the set A_1. Since S_2 and S_3 have property P^3, each point of local non-convexity of S_2 and S_3 is in the kernel of S_2 and S_3, respectively. By Lemma 3.7, $cl(convQ_3 \cup (S_3 \cap W_1^3))$ and $cl(convQ_2 \cup (S_2 \cap W_3^2))$ are convex sets. The argument following Case IV may be applied to these two sets to show that their union is a convex set. Thus, the intersection X of the closed half-plane $cl(W(q_{32},q_{21}))$ with this set is convex. From Lemma 3.7, it is also known that $cl(S_2 \cap W_1^2) \cup convQ_2$ is a convex set. Thus, $Y \equiv (cl(S_2 \cap W_1^2) \cup convQ_2) \cap cl(W(q_{32},q_{21}))$ is a convex set. From its definition, $A_1 = X \cup Y$. Let $x, y \in A_1$. If $x, y \in X$ or $x, y \in Y$, then $xy \subset A_1$ since X and Y are convex. If $x \in X$ and $y \in Y$, then $xy \cap convQ_2 \neq \emptyset$ which implies $xy \subset A_1$, since $convQ_2 \subset X \cap Y$. In exactly the same manner it can be shown that A_2 is a convex set.

We next consider A_3. By Lemma 3.7, $\overline{X} \equiv convQ_3 \cup cl(S_3 \cap W_2^3)$, $\overline{Y} \equiv convQ_2 \cup cl(S_2 \cap W_2^3)$, $Z_1 \equiv convQ_2 \cup cl(S_2 \cap W_3^2)$ and $Z_2 \equiv convQ_3 \cup cl(S_3 \cap W_1^3)$ are convex sets. By the argument following Case IV, $\overline{Z} \equiv Z_1 \cup Z_2$ is a convex set. By definition, A_3 is the intersection of $\overline{X} \cup \overline{Y} \cup \overline{Z}$ with the closed convex region bounded by $L(q_{33},q_{31})$, $L(q_{21},q_{22})$ and $L(q_{23},q_{32})$. Thus, if $x, y \in A_3 \cap \overline{X}$, $A_3 \cap \overline{Y}$, or $A_3 \cap \overline{Z}$, then $xy \subset A_3$. If $x \in A_3 \cap \overline{X}$, $y \in A_3 \cap \overline{Y}$, then $xy \cap convQ_2$ and $xy \cap convQ_3$ and $xy \cap D_{23}$ are non-empty. Since $convQ_2$ and $convQ_3$ are in the kernels of S_2 and S_3, respectively, $xy \subset A_3$. If $x \in \overline{X}$ and $z \in \overline{Z}$, then $xz \cap convQ_3$ is non-empty which implies $xz \subset A_3$, since $convQ_3 \subset \overline{X} \cap \overline{Z}$. Similarly, if $y \in \overline{Y}$ and $z \in \overline{Z}$, then $yz \subset A_3$, since $yz \cap convQ_2$ is nonempty and $convQ_2 \subset \overline{Y} \cap \overline{Z}$. The set A_4 is defined in precisely the same way as A_3, and may be shown to be convex in exactly the same manner. Lastly, we consider the set A_5. By construction, each element of $B_1 \cup B_{12}$ is not joined to the point of local non-convexity q_{33}; otherwise q_{12}, q_{11} or q_{21} would be a point of local convexity. Similarly each element of $B_2 \cup B_{12}$ is not joined to q_{32} by a line segment lying entirely in S, since then q_{11}, q_{21} or q_{23} would not be in Q. Therefore, by Proposition 2.2, $B_1 \cup B_{12}$ and $B_2 \cup B_{12}$ are convex relative to S. By Bunt's refinement of Fenchel's Theorem [5](see Theorem 3.6 for statement) the convex hull of the two components B_1 and B_{12}, and the convex hull of the two components B_2 and B_{12} are contained in S.

Next consider the set $\mathrm{conv}(B_1 \cup B_2)$. By Bunt's refinement of Fenchel's Theorem $\mathrm{conv}(B_1 \cup B_2) \subset S$ if $B_1 \cup B_2$ is convex relative to S. Since by definition B_1 and B_2 are convex subsets of S, it suffices to show that for $x \in B_1$ and $y \in B_2$, $xy \subset S$. By the definition of B_1 and B_2, $x \in W_2^1 \cap S_1$ and $y \in W_2^2 \cap S_2$. Since $q_{11} \in K_1$ and $q_{21} \in K_2$, $x, y \in W(q_{11}, q_{12}) \cap W(q_{23}, q_{21})$. Evidently, if $xy \not\subset S$, then either x or y is contained in $W(q_{22}, q_{13})$. Assume that $x \in W(q_{22}, q_{13})$. By property P^4 and the closure of S, we know that $q_{22}q_{13} \subset S$. Since $q_{33} \in \mathrm{cl}V_3$, there exists an element z in $W_2^1 \cap q_{22}q_{13} \cap W(q_{33}, q_{11}) \cap W(q_{21}, q_{33})$ such that $z\, q_{33} \not\subset S$. Since $x \in W(q_{22}, q_{13})$ if $xz \subset S$, then $q_{13} \notin Q_3$, a contradiction. But if $xz \not\subset S$, then property P^4 is violated, since $xq_{33} \not\subset S$ and $q_{33} \in Q$ implies that each point which is not joined to q_{33} by a line segment lying entirely in S, is joined to x by such a line segment. (See the proof of Proposition 2.2). This contradiction implies $xy \subset S$. That is, $B_1 \cup B_2$ is convex relative to S. The same argument applies if $y \in W(q_{22}, q_{13})$, or if $x, y \in W(q_{22}, q_{13})$. It follows that $\mathrm{conv}(B_1 \cup B_2) \subset S$.

We now have three convex components B_1, B_2, and B_{12} with the property that $\text{conv}(B_1 \cup B_{12})$, $\text{conv}(B_2 \cup B_{12})$ and $\text{conv}(B_1 \cup B_2)$ are contained in S. We wish to show that $\text{conv}(B_1 \cup B_2 \cup B_{12}) \subset S$. Assume that this is not the case and let $w \in \text{conv}(B_1 \cup B_2 \cup B_{12}) - S$. It is well known that this would imply the existence of three or fewer points x_1, x_2, and x_3 of $B_1 \cup B_2 \cup B_{12}$ such that $w \in \text{conv}\{x_1, x_2, x_3\} - S$. Since $\text{conv}(B_1 \cup B_{12})$, $\text{conv}(B_2 \cup B_{12})$ and $\text{conv}(B_1 \cup B_2)$ are contained in S, the existence of such a w is impossible unless there exists exactly three non-collinear points x_1, x_2, and x_3 such that $x_1 \in B_1$, $x_2 \in B_2$, and $x_3 \in B_{12}$, say, and $w \in \text{conv}\{x_1, x_2, x_3\} - S = \Delta x_1x_2x_3 - S$. We now have $\text{bd}(\Delta x_1x_2x_3) \subset S$ while $\text{int}(\Delta x_1x_2x_3)$ contains $w \notin S$. Since S is closed, there exists an open spherical neighborhood σ of w such that $\sigma \cap S = \emptyset$ and $\sigma \subset \text{int}(\Delta x_1x_2x_3)$. Thus, $\text{bd}(\Delta x_1x_2x_3)$ bounds a residual domain of S. From Lemma 3.9, it follows that $\Delta q_{11}q_{21}q_{31} \subset \Delta x_1x_2x_3$. But, since $x_i \in \text{cl}V_i$, $i = 1,2$, and $x_3 \in D_{12}$ while $q_{11} \in \text{cl}V_3$, $q_{11} \notin \Delta x_1x_2x_3$, a contradiction. Thus, $\text{conv}(B_1 \cup B_2 \cup B_{12}) \subset S$, and since S is closed, $\text{cl}(\text{conv}(B_1 \cup B_2 \cup B_{12}) \subset S$. That is A_5 is a closed convex subset of S.

It must now be shown that $S = \bigcup_{k=1}^{5} A_k$. Consider the set $S_i = (\text{conv}Q_i) \cup (\bigcup_{j=1}^{3} (S_i \cap W_j^i))$ $i = 1,2,3$, and recall that $Q_i \subset \text{cl}V_i$, $i = 1,2,3$. Evidently, $(S_1 - W_2^1) \subset A_2 \cup A_4 \cup B_{12} \cup A_1$, $(S_2 - W_2^2) \subset A_1 \cup A_3 \cup B_{12} \cup A_2$, and $(S_3 - W_2^3) \subset A_1 \cup A_2 \cup A_3 \cup A_4$. The claim is that $(S_1 \cap W_2^1) \subset A_4 \cup A_5$, $(S_2 \cap W_2^2) \subset A_3 \cup A_5$ and $(S_3 \cap W_2^3) \subset A_3 \cup A_4$. It is important to recall that $(S_1 \cap W_2^1)$ is a convex set whose closure intersects $(S_1 - W_2^1)$ in the line segment $q_{12}q_{13}$, and that $(S_1 \cap W_2^1) \subset$ $cl(W(q_{11},q_{12}) \cap W(q_{13},q_{11}))$ since $q_{11} \in K_1$, and $q_{12},q_{13} \in Q_1$, $i = 1,2,3$. With this in mind, we direct our attention first to $(S_3 \cap W_2^3)$. Since by property P^4 and the closure of S, Q is convex relative to S, the line segments $q_{32}q_{23}$ and $q_{33}q_{12}$ lie in S. If $(S_3 \cap W_2^3) \not\subset A_3 \cup A_4$, then there would exist three points $x \in (S_3 \cap W_2^3) \cap W(q_{33},q_{12}) \cap W(q_{23},q_{32})$, $y \in S_3 \cap (q_{32},q_{23})$ and $z \in S_3 \cap (q_{33},q_{12})$ such that x, y and z violate property P^3 of S_3. Thus, $(S_3 \cap W_2^3) \subset A_3 \cup A_4$.

It is clear that the convex sets $(S_1 \cap W_2^1)$ and $(S_2 \cap W_2^2)$ are each separated into two convex sets by the lines $L(q_{33},q_{12})$ and $L(q_{32},q_{23})$, respectively. One of the convex subsets of S_i is $B_i \subset A_5$, $i = 1,2$. Thus, it suffices to show that $(S_1 \cap W_2^1) - B_1$ is contained in $A_3 \cup A_4$. This in turn reduces to showing that $(S_1 \cap W_2^1) - B_1 \subset W(q_{33},q_{12}) \cap$ $W(q_{13},q_{11})$ and that $(S_2 \cap W_2^2) - B_2 \subset W(q_{21},q_{22}) \cap W(q_{23},q_{32})$. This, however, is evident from the fact that $(S_1 \cap W_2^1) \subset cl(W(q_{11},q_{12}) \cap$ $W(q_{13},q_{11}))$.

Consider now the case $N_i = 3$, $i = 1,2$, and $N_3 = 2m + 1$ for $m > 1$. We first define the sets

$$A_1' = (S_3 \cap (\bigcup_{j=1}^{m-1} W_{2j+1}^3))$$

and

$$A_4' = (S_3 \cap (\bigcup_{j=1}^{m-1} W_{2j}^3))$$

The set S is expressible as the union of the following convex sets.

$$\bar{A}_1 = \mathrm{cl}[(S_2 \cup S_3) \cap (W(q_{3,2m},q_{21}) \cap W(q_{22},q_{23}) \cap W(q_{32},q_{33}))] - A_4'$$

$$\bar{A}_2 = \mathrm{cl}[(S_1 \cup S_3) \cap (W(q_{12},q_{13}) \cap W(q_{11},q_{3,2m+1}) \cap W(q_{32},q_{3,2m+1}))]$$

$$\bar{A}_3 = \mathrm{cl}[(S_2 \cup S_3) \cap (W(q_{3,2m+1},q_{31}) \cap W(q_{21},q_{22}) \cap W(q_{23},q_{3,2m}))]$$

$$\bar{A}_4 = \mathrm{cl}[(S_1 \cup S_3) \cap (W(q_{3,2m+1},q_{12}) \cap W(q_{13},q_{11}) \cap W(q_{31},q_{32}) \cap$$
$$W(q_{3,2m},q_{3,2m+1})] - A_1'$$

and

$$\bar{A}_5 = \mathrm{cl}(\mathrm{conv}(\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_{12})).$$

where $\bar{B}_1 = (S_1 \cap W_2^1) \cap W(q_{12},q_{3,2m+1})$

$$\bar{B}_2 = (S_2 \cap W_2^2) \cap W(q_{3,2m},q_{23}),$$

and $\bar{B}_{12} = D_{12} \cap W(q_{21},q_{3,2m}) \cap W(q_{3,2m+1},q_{11})$.

The sets $\bar{A}_2, \bar{A}_3$ and $\bar{A}_5$ are defined in the same way that A_2, A_3, and A_5 were defined, respectively. The proof that each is convex requires only the obvious change of subscript; that is, the replacement of q_{32} by $q_{3,2m}$ and of q_{33} by $q_{3,2m+1}$ in the above argument.

Consider the set $\bar{A}_1$. By Lemma 3.8, $\mathrm{conv}Q_3 \cup (S_3 \cap (\bigcup_{j=0}^{m-1} W_{2j+1}^3))$ is a convex set. Thus, $W(q_{3,2m},q_{21}) \cap W(q_{32},q_{33})$ intersects this set in a convex set. Moreover, A_1' is contained in this intersection since $q_{21}q_{3,2m}$ and $q_{32}q_{33}$ intersect $\mathrm{conv}Q_3$ in line segments and if A_1' were not so contained, either $q_{3,2m}$ or q_{33} would be a point of local convexity, contrary to assumption.

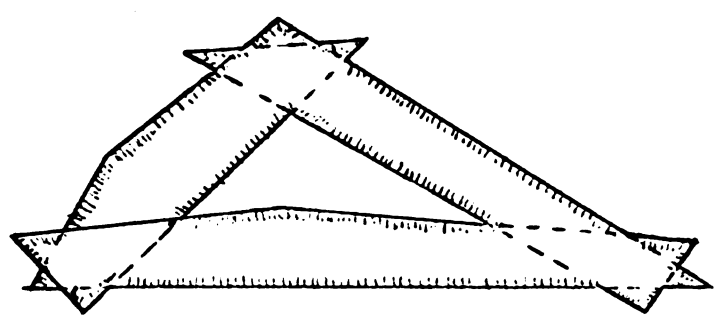
The remainder of the proof that $\bar{A}_1$ is convex is the same as that for A_1.

The set $\overline{A}_4$ is defined in the same way that A_4 was defined except for its intersection with $W(q_{3,2m+1},q_{32})$. By Lemma 3.8, $\mathrm{conv}Q_3 \cup (S_3 \cap W^3_{2m+1}) \cup A'_4$ is a convex set. Thus, the intersection of this set with $W(q_{3,2m},q_{3,2m+1}) \cap W(q_{3,2m+1},q_{12}) \cap W(q_{31},q_{32})$ is a convex set. Again, A'_4 is contained in this set since $q_{31}q_{32}$ and $q_{3,2m}q_{3,2m+1}$ are contained in the kernel of S_3. If A'_4 were not so contained then q_{32} or $q_{3,2m}$ would not be in Q_3. The remainder of the proof that $\overline{A}_4$ is convex is the same as that for A_4.

The proof that $S = \bigcup_{i=1}^{5} \overline{A}_i$ is identical to that of $N_3 = 3$ with the exception of S_3. This case differs for the set $(S_3 \cap W(q_{3,2m+1},q_{32}))$, which is seen to be the union of A_1', A_4', and $(S_3 \cap W^3_{2m})$ by checking the indices of summation. It is known from Theorem 3.4 that $S_3 = \mathrm{conv}Q_3 \cup (S_3 \cap (\bigcup_{j=1}^{2m+1} W^3_j))$. Since $(S_3 \cap W^3_1) \subset \overline{A}_1 \cup \overline{A}_3$, $(S_3 \cap W^3_{2m+1}) \subset \overline{A}_2 \cup \overline{A}_4$ and $\mathrm{conv}Q_3 \subset \overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3 \cup \overline{A}_4$, we see that S_3 is contained in $\bigcup_{i=1}^{5} \overline{A}_i$.

Thus, in every case, S is expressible as the union of five or fewer closed convex sets.

To see that the number five is best consider the following example which satisfies the hypothesis of the theorem but which can not be expressed as the union of fewer than five convex sets.



Theorem 3.6 If S has at most one point of local non-convexity which is not in the kernel K of S, and S has property P^4, then S may be expressed as the union of four or fewer closed convex sets.

Proof. If Q, the set of points of local non convexity of S, is empty, then S is a closed, connected, locally convex set, which, by Tietze's theorem is convex. If Q has exactly one element, then S is expressible as the union of three or fewer closed convex sets, by Theorem 3.3. If Q has two or more elements, each of which is in the kernel, K, of S, then S may be expressed as the union of three or fewer closed convex sets, by Theorem 3.4. Thus, we may assume that Q has two or more elements, exactly one of which is not in K.

Let $q \in Q-K$. It was shown in the proof of Proposition 2.2 that S is expressible as the union of a starlike set whose kernel contains q, and the set X of all points of S which are not joined to q by a line segment lying entirely in S. By Proposition 2.2, since S has property P^4, the set X has property P^2 relative to S. That is, X is convex relative to S. We show that conv$X \subset S$, and that S-X may be expressed as the union of three or fewer closed convex sets.

It is first shown that X has at most two components. Assume that X has three components C_1, C_2, and C_3 in S. Let $c_i \in C_i$, $i = 1,2,3$. Since X is convex relative to S, we have that c_1c_2, c_2c_3, and c_1c_3 are entirely contained in S. Since, however, $C_1 \cap C_2$, $C_2 \cap C_3$, and $C_1 \cap C_3$ are empty, we have that c_1c_2, c_2c_3, and c_1c_3 each intersect $S - X$. Let a_{12}, a_{23}, and a_{13} be elements of $c_1c_2 \cap (S-X)$, $c_2c_3 \cap (S-X)$, and

$c_1c_3 \cap (S-X)$, respectively. Then qa_{12}, qa_{23}, and qa_{13} lie entirely in $S-X \subset S$. The elements c_1, c_2, c_3, and q are not collinear since the fact that qa_{12}, qa_{23}, and qa_{13} lie in S would imply $qc_i \subset S$ for some i. If c_1, c_2, and c_3 are collinear in that order, say, then qa_{12}, qa_{23}, and $a_{12}a_{23}$ are contained in S while $qc_2 \not\subset S$. Since $S-X$ is closed, there exists an open spherical neighborhood N of c_2 such that $x \in N \cap S$ implies $xq \not\subset S$. Thus $\mathrm{bd}(\Delta a_{12}qa_{23}) \subset S$ bounds one or more residual domains of S. Since S is starlike, this is impossible. If c_1, c_2, and c_3 are not collinear, then they determine the triangle $\Delta c_1c_2c_3 \equiv T$, whose vertices we may assume to be in that counterclockwise order. Then $q \in \mathrm{int}T$, or $S-(S \cap T)$. (For the same reasons as given above, q can not lie on an extended side of T.) If $q \in \mathrm{int}T$, $\mathrm{bd}(qa_{12}c_2a_{23})$ bounds one or more residual domains of S, a contradiction. If $q \in S-(S \cap T)$, then without loss of generality, we may assume that q is in the v-shaped region determined by $L(c_2,c_3)$ and $L(c_2,c_1)$ and containing (c_1c_3), or that q is in the v-shaped region determined by these lines which intersects T in the point c_2. In the former case $\mathrm{bd}(qa_{23}c_2a_{12})$ bounds a residual domain of S, and in the latter case $\mathrm{bd}(qa_{12}c_1c_3a_{23})$ bounds a residual domain of S. Since in each case we are led to a contradiction of the starlikeness of S, we have that X has at most two components C_1 and C_2, say, in S.

W. Fenchel [7] showed that a necessary and sufficient condition for a point x to be contained in the convex hull of a compact, connected set X in E^m is that there exist m (or fewer) points $p_1, p_2, \ldots, p_m$ in X such that $x \in \mathrm{conv}(p_1 \cup p_2 \cup \ldots \cup p_m)$. Later L.N.H. Bunt [5] showed that compactness was unnecessary and that "connected" may be replaced by "having at most m components". For $m = 2$, this theorem may be stated: A necessary and sufficient condition for a point x to be contained in the convex hull of a set having at most 2 components is that there exist 2 (or fewer) points p_1, p_2 in X, such that $x \in \mathrm{conv}(p_1 \cup p_2) \equiv p_1p_2$. Since in this case $X = \{z \mid zq \not\subset S\}$ has the property that for all z_1 and z_2 in X, $z_1z_2 \subset S$, if we considered the set Y of all points of S which are in X or are contained in a line segment, the end points of which are in X, Bunt's refinement of Fenchel's theorem applies to give $Y \subset S$ is the convX. That is, X is contained in a convex subset of S.

We now direct our attention to the closed subset S-X of S. Let $\overline{Q}$ and $\overline{K}$ denote the set of points of local non-convexity and kernel of S-X, respectively. Let $p \in \overline{Q}$ and assume that $p \notin \overline{K}$. Then $p \neq q$ since $q \in \overline{K}$. Let $x \in S\text{-}X$ be such that $px \not\subset S\text{-}X$. If $p \in Q$ of S, then $px \subset S$, which then implies that $px \cap X \neq \emptyset$. If $x \in L(p,q)$, $px \subset S\text{-}X$ since qx and qp are in S-X, a contradiction. If $x \notin L(p,q)$, then we may consider Δpxq. Since $p \in Q$, $px \subset S$. Since $q \in \overline{K}$, and $x, p \in S\text{-}X$, qx and $qp \subset S\text{-}X \subset S$. Also, since $px \cap X \neq \emptyset$, and S-X is closed, there exists an open subinterval (uv) of (px) such that $(uv) \subset X$ which implies that $bd(\Delta xpq) \subset S$ bounds a bounded residual domain of S, a contradiction, since S is starlike. Finally, we show that the case where $p \notin Q$ is impossible by showing $p \in \overline{Q}$ implies $p \in Q$. Assume, to the contrary, that $p \in \overline{Q}$, and $p \notin Q$. Then there exists a spherical neighborhood N of p of radius r such that $N \cap S$ is convex. Since $p \in \overline{Q}$, there exist elements y and z of $N \cap (S-X)$ such that $yz \not\subset S-X$. Since $N \cap S$ is convex, $yz \subset S$ which implies that $yz \cap X \neq \emptyset$. Consider the points y, z, and q. If $q \in L(y,z)$, then $yz \subset S-X$ since qy and qz are contained in $S-X$. If $q \notin L(y,z)$, we may then consider Δyqz. As before, since $yz \cap X \neq \emptyset$ and $S-X$ is closed, there eixsts an open interval $(uv) \subset (yz)$ such that $(uv) \subset X$. This implies that $\mathrm{bd}(\Delta yqz)$, which is contained in S, bounds a bounded residual domain of S, which again contradicts the starlikeness of S. Hence, each point of local non-convexity of $S-X$ is in the kernel of $S-X$. Since $S-X$ is a closed, connected set, if $S-X$ has two or more points of local non-convexity, then $S-X$ may be expressed as the union of three or fewer closed convex sets, by Theorem 3.4. If $S-X$ has exactly one point of local non-convexity, then $S-X$ is expressible as the union of three or fewer closed, convex sets, by Theorem 3.3. If $S-X$ has no points of local non-convexity, then $S-X$ is convex, by Tietze's theorem. Thus, in each case, $S-X$ may be expressed as the union of three or fewer closed, convex sets, M_i, $i = 1,2,3$. Then $S = \mathrm{cl}Y \cup (\bigcup_1^3 M_i)$ is the desired decomposition of S into closed convex sets.

<u>Lemma 3.10</u> Let S have property P^n, and z be a point of local convexity of S. If S is one-dimensional at z, then there exists a line X through z such that $\mathrm{cl}(S-X)$ has property P^{n-1}.

<u>Proof</u>. Since z is a point of local convexity, there exists an open spherical neighborhood, $\sigma(z,\varepsilon)$ of z such that $\sigma(z,\varepsilon) \cap S$ is convex. Since S is one-dimensional at z, there exists an $\bar{\varepsilon} \leq \varepsilon$ such that

$bd(\sigma(z,\overline{\varepsilon}) \cap S)$ is zero-dimensional. That is, $cl(\sigma(z,\overline{\varepsilon}) \cap S)$ is a line segment, Y. Hence, $xz \subset S$ implies $x \in S \cap X$, where X is the line determined by Y. Thus if $x_1, x_2, \ldots, x_{n-1}$ are n-1 points of S-X which are not joined in S, the points $x_1, x_2, \ldots, x_{n-1}$, z would violate property P^n. Hence S-X, and consequently $cl(S - X) \equiv A$, have property P^{n-1} relative to S. Let W_1 and W_2 be the open half-planes determined by X. Then $A = cl[(W_1 \cap S) \cup (W_2 \cap S)]$. Assume now that A does not have property P^{n-1} (relative to A). That is, let $x_1, x_2, \ldots, x_{n-1}$ be n-1 distinct points of A which are not joined in A. Not all of these points can be in $cl(W_1 \cap S)$, or in $cl(W_2 \cap S)$, since these sets have property P^{n-1}. Consider first the case where $x_i \notin X \cap S$ for $i = 1,2,\ldots,n-1$. Since $zx_i \not\subset S$ for $i = 1,\ldots,n-1$, $x_i x_j \subset S$ for some i and j because A has property P^{n-1} relative to S. Since $x_i x_j \not\subset A$, $x_i \in W_1 \cap S$ while $x_j \in W_2 \cap S$, say. Thus $x_i x_j \cap (X \cap S)$ is non-empty. Let $u = x_i x_j \cap (X \cap S)$. Since A is closed and $x_i x_j - \{u\} \subset A$, $u \in A$. That is, A has property P^{n-1}. Next consider the case where $x_i \in (X \cap S)$ for some i. Without loss of generality we may let $i = 1$. Since $x_1 \in cl(W_1 \cap S)$ or $x_1 \in cl(W_2 \cap S)$, for each neighborhood N of x_1, there exists an element $\overline{x}_1$ in $W_1 \cap S$ or $W_2 \cap S$ such that $\overline{x}_1, x_2, \ldots, x_{n-1}$ are not joined in A, since A is closed. Then $\overline{x}_1, x_2, \ldots, x_{n-1}$, z violate property P^n. Thus in this case too, A has property P^{n-1}. That is, $cl(S - X)$ has property P^{n-1}.

<u>Corollary</u> If S is as in the lemma and $n = 4$, then S is expressible as the union of four or fewer closed convex sets.

Proof. Since S satisfies the hypothesis of the lemma, $cl(S - X) = A$ has property P^3 and is expressible as the union of three or fewer closed convex sets. If $X \cap S$ has exactly one (convex) component, the result follows.

If there exist two components X_1 and X_2 of $X \cap S$ with the property that neither is a subset of A, then there exist points $z_1 \in X_1$ and $z_2 \in X_2$ with the property that $z_1z_2 \not\subset S$. Moreover, the conditions of the lemma are satisfied at z_1 and z_2. It is clear that A must be a convex set and the result follows.

Finally, the set $X \cap S$ can have at most two such components. For then $cl(S - X) = \emptyset$ and S is not a connected set.

Theorem 3.7 Let S have property P^4. If S has a cut-point q, then S may be expressed as the union of 4 or fewer closed convex sets.

Proof. Since q is a cut-point of S, $S - \{q\}$ has two or more components.

If one of the components, C, of $S - \{q\}$ is convex, we consider the two cases, C is one-dimensional and C is two-dimensional. If C is one-dimensional, then clC is evidently a line segment qy or a ray $R(q,y)$ emanating from q and containing qy. In either case the hypothesis of Lemma 3.10 is clearly satisfied at $y \in C$, and the result follows from the Corollary to Lemma 3.10.

Assume next that the convex component C of $S - \{q\}$ is two-dimensional. Since $clC \cap cl(S - C) = \{q\}$, if $c \in clC$ and $b \in cl(S - C)$,

$c \neq q \neq b$, then $cb \subset S$ implies $q \in (cb)$. Moreover, if $a, b \in cl(S - C)$ and $ab \subset S$, then $ab \subset cl(S - C)$. Assume, to the contrary, that $ab \not\subset cl(S - C)$. Since $q \in cl(S - C)$ and $ab \subset S$, $ab \cap (clC - \{q\}) \neq \emptyset$. That is, $ab \cap C \neq \emptyset$ which implies C is not a component of $S - \{q\}$, a contradiction. Now, let x_1, x_2, x_3 be three distinct points of $cl(S - C)$ which are not pairwise joined in $cl(S - C)$, then by the above remarks, if $c \in clC$, $c \neq q$, then $cx_i \subset S$ implies $q \in (cx_i)$ and the points x_1, x_2, x_3 are not pairwise joined in S. Since clC is two-dimensional at c, and $\{x_1, x_2, x_3\}$ is a finite set, there exists in some neighborhood N of c, a point c' of clC such that $c' x_i \not\subset S$, $i = 1,2,3$. The points c', x_1, x_2, x_3 now violate property P^3. Thus, S is once again expressible as the union of three or fewer closed convex subsets of $cl(S - C)$ and the closed convex subset clC.

Let us now assume that the closure of each component of $S - \{q\}$ is not convex. Let C and C' be two such components. Let $x, y \in clC$ and $a, b \in clC'$ such that $xy \not\subset clC$ and $ab \not\subset clC'$. By a previous argument this implies $xy \not\subset S$ and $ab \not\subset S$. We first assume that q is not one of the four points x, y, a, b. By property P^4, two of the four points x, y, a, b must be joined by a line segment lying in S. Without loss of generality we may assume that $xa \subset S$. Once again, this implies $q \in (xa)$. Since C is a component of $S - \{q\}$ and $x \neq q$, there exists a spherical neighborhood σ of x such that $\sigma \cap S \subset C$. If there exists a spherical neighborhood $\sigma' \subset \sigma$ of x such that for all $x' \in \sigma' \cap S$, $x' a \subset S$, then $q \in (x'a)$ for each x' and we have $cl(\sigma' \cap S)$ is a line segment containing x. ($(\sigma' \cap S) \neq \{x\}$ since S is connected) Clearly x satisfies the hypothesis of Lemma 3.10 and the result follows from the arguments used in the one-dimensional convex case above. Assume then that for all spherical neighborhood σ' of x there exists $x' \neq x$ in $\sigma' \cap S$ such that $x'a \not\subset S$. Since S is a closed set and $xy \not\subset S$, there exists an open spherical neighborhood $\sigma' \subset \sigma$ such that for all $x' \in \sigma' \cap S$, $x'y \not\subset S$. Let $x' \in \sigma' \cap S$ such that $x'a \not\subset S$. Then, since $x'a$, $x'y$, and ab are not contained in S, by property P^4, $x'b$, yb, or ya is contained in S. If $ya \subset S$, then q, y and a are collinear since $\mathrm{cl}C \cap \mathrm{cl}C' = \{q\}$. Since $xa \subset S$, q, x, and a are collinear for the same reason. Thus, if $ya \subset S$, $xy \subset xa \cup ya \subset S$, a contradiction. Assume then that $x'b \subset S$. Since $x' \in \sigma \cap S \subset C$, $x' \neq q$ and $x'b \subset S$ implies $q \in (x'b)$.

If there exists a spherical neighborhood τ of x' such that for all $z \in \tau \cap S$, $zb \subset S$, then $q \in (zb)$ for all $z \in \tau$ which implies $\mathrm{cl}(\tau \cap S)$ is a line segment containing x'. Then x' satisfies the hypothesis of Lemma 3.10 and the arguments of the one-dimensional convex case apply to give the desired result. Therefore, we may assume that there exists in every spherical neighborhood τ of x' an element z such that $zb \not\subset S$. Let $\tau \subset \sigma'$ be a spherical neighborhood of x' which does not intersect the component of $L(x,a) \cap S$ containing xa. Let $z \in \tau$ such that $zb \not\subset S$. Since $z \in \sigma'$, $zy \not\subset S$. Since $z \in \tau$ and τ does not intersect the component of $L(x,a) \cap S$ which contains xa, $za \not\subset S$. We now have a point $z \in C$ such that za, zb, zy are not contained in S. Since ab and ya are not contained in S, it must happen that yb is contained in S, by property P^4. If, however, $yb \subset S$, then q, y, b are collinear, since $\mathrm{cl}C \cap \mathrm{cl}C' = \{q\}$. Since $x'b \subset S$,

q, x', b are collinear for the same reason. Thus, if $yb \subset S$, $x'y \subset x'b \cup yb \subset S$, a contradiction.

Thus, if xa and $x'b$ are contained in S, x, y, a, b are different from q, and neither x nor x', as defined above, has a spherical neighborhood whose closure intersects S in a line segment, we are able to exhibit four points z, y, a, and b which violate property P^4.

We now consider the case where xa and yb are contained in S, and x, y, a, and b are each difference from q. Without loss of generality we may assume that none of the elements x, y, a, b has a neighborhood whose closure intersects S in a line segment. If x' is defined as above, then $x'b \not\subset S$, for if $x'b \subset S$, then $\mathrm{cl}C \cap \mathrm{cl}C' = \{q\}$ implies q, x', b are collinear. Since $yb \subset S$, q, y, b are collinear for the same reason, and we have $x'y \subset x'b \cup yb \subset S$, a contradiction. Let Γ be a spherical neighborhood of b such that $\Gamma \cap S \subset C'$. Since S is closed, and neither $x'b$ nor ab is in S, there exists a spherical neighborhood Γ' of b, $\Gamma' \subset \Gamma$, such that $b' \varepsilon \Gamma'$ implies $b'x'$ and $b'a$ are not contained in S. By our original assumption each spherical neighborhood of b contains an element which is not joined to y by a line segment in S. Let $b' \varepsilon \Gamma' \cap S$ have this property. Again, since $yb \subset S$, $xa \subset S$ and $xy \not\subset S$, $ya \not\subset S$. Thus x', y, a, b' violate property P^4. In the case where $xa, yb \subset S$, it has been shown that $ya \not\subset S$. The same reasoning shows that $xa, yb, xb \subset S$ can not occur. Finally, assume that one of the points x, y, a, or b is q. Without loss of generality we may assume that $a = q$. Then since S is a closed set and $qb \not\subset S$, there exists a spherical neighborhood λ of q such that for all $a' \in \lambda \cap S$, $a'b \not\subset S$. Since $q \in clC'$, $\lambda \cap C' \neq \emptyset$. Let $a' \in \lambda \cap C'$. The arguments given above are now applicable to the four points x, y, a', b.

Hence, in each case, we are led to a contradiction unless the closure of some component of $S - \{q\}$ is convex or unless there exists a point having a neighborhood the intersection of whose closure is a line segment.

Since the result has been established in each of these cases, the proof is complete.

<u>Theorem 3.8</u> Let S be a closed starlike set and have property P^4. Let the kernel K of S be a one-dimensional set. Then S may be expressed as the union of four or fewer closed convex sets.

<u>Proof</u>. Since S is closed, by the corollary to Proposition 2.3, the kernel K of S is closed and convex. Since K is one-dimensional, K is a line, ray, or line segment. Clearly, K is not a line unless $S \subset K$. Since $S \subset K$ implies that S is convex, we may assume that K is a line segment xy, or a ray R(x,y) containing xy. Let W(x,y) and W(y,x) be the open half-planes determined by L(x,y). Since $S \cap L(x,y)$ contains K, $S \cap L(x,y)$ is a convex set. Thus, if $S \subset S \cap L(x,y)$, S is convex, and we are done. Moreover, if $S \cap L(x,y) \not\subset cl[(W(x,y) \cup W(y,x)) \cap S]$, then there exists a point of local convexity, z, in $S \cap L(x,y)$ such that S is one-dimensional at z. By Lemma 3.10, $cl[S-(S \cap L(x,y))]$ has property P^3. Thus, the set S may be expressed as the union of the 3

or fewer closed convex sets which arise from $cl[S-(S \cap L(x,y))]$ and the closed convex set $S \cap L(x,y)$. Therefore, we may assume that $S \cap L(x,y) \subset cl[(W(x,y) \cup W(y,x)) \cap S]$, and that $W(x,y) \cap S$ and $W(y,x) \cap S$ are not both empty. If $W(x,y) \cap S$ and $W(y,x) \cap S$ are convex, we are again done, since then S is seen to be the union of the two closed convex sets $cl(W(x,y) \cap S)$, and $cl(W(y,x) \cap S)$. Assume then that $W(x,y) \cap S$ is non-empty, and that $cl(W(x,y) \cap S)$ is not convex. Then $W(x,y) \cap S$ must contain a point of local non-convexity of S. Assume, to the contrary, that $(W(x,y) \cap S)$ contains no points of local non-convexity of S. We show that $(W(x,y) \cap S)$ is a convex set. For if $u, v \in (W(x,y) \cap S)$ such that $uv \not\subset (W(x,y) \cap S)$. Then evidently $uv \not\subset S$. Since $x, y \in K$, and $uv \not\subset S$, $x, y \notin L(u,v)$. Thus, we may consider the quadrilateral xyvu whose vertices may be considered to be ordered in a counterclockwise manner x, y, v, u, as indicated. Let $p = xv \cap yu$, and consider the component C of the closed set $S \cap \Delta upv$ containing the connected subset of S, $up \cup pv$. Since C is a closed, connected subset of S, if C is also locally convex, then, by Tietze's result, C is convex. Since $C \subset W(x,y) \cap S$, if $W(x,y) \cap S$ contains no points of local non-convexity then C is locally convex in which case $u, v \in C$ implies $uv \subset C \subset S$, a contradiction.

If $y = \infty$, the quadrilateral xyvu becomes an infinite convex strip. The same arguments apply, however. Thus, in the case being considered, if $W(x,y) \cap S$ contains no points of local non-convexity, then $W(x,y) \cap S$ and, consequently $cl(W(x,y) \cap S)$ is a convex set. We see that if neither $W(x,y) \cap S$, nor $W(y,x) \cap S$ contains a point of local non-convexity of S, then S is expressible as the union of the two closed convex sets $cl(W(x,y) \cap S)$ and $cl(W(y,x) \cap S)$.

We next show that each point of local non-convexity of $W(x,y)$ S is in the kernel of $W(x,y) \cap S$. Let q be a point of local non-convexity of S in $W(x,y) \cap S$. Let $z \neq q$ be a point of $W(x,y) \cap S$ such that $zq \not\subset S$. By the proof of Proposition 2.2, the set S is starlike from $q \cup z$; that is, for each element u of S, either $qu \subset S$ or $zu \subset S$. We first assume that $K = xy$, $y \neq \infty$. Evidently, $L(z,q) \cap xy = \emptyset$ since otherwise $zq \subset S$. Since $\Delta xqy \subset S$ and $\Delta xzy \subset S$, we have $\Delta xqy \cap \Delta xzy \equiv T \subset X$. Let $u \in intT$. Then since $u \notin K$, there exists an element v of S such that $uv \not\subset S$. Order the four v-shaped regions V_i, $i = 1,2,3,4$ determined by $L(x,u)$ and $L(y,u)$ in a counterclockwise manner starting with that region which contains (xy). Evidently $v \notin clV_1 \cup clV_3$ since $xy = K$ and there would then exist an element w in xy such that w, u, and v are collinear which would imply $uv \subset S$, a contradiction. Let $v \in V_2$. Since S is starlike from $q \cup z$, vq or vz is contained in S. We have, $uq \subset \Delta xqy \cap S$ and $uz \subset \Delta xzy \cap S$. The segments vy and yu are also contained in S since $y \in K$. Thus uv is contained in the quadrilateral $uqvy$ or $uzvy$, each of which lies entirely in S, and we have $uv \subset S$, a contradiction. If $v \in V_4$, we have $uv \subset uqvx \subset S$ or $uv \subset uzvx \subset S$, again a contradiction. This contradiction implies that every point of $W(x,y) \cap S$ and hence of $cl(W(x,y) \cap S)$ is joined to q by a line segment which lies entirely in S. Thus each point of local non-convexity q of S in $W(x,y) \cap S$ is in the kernel of $cl(W(x,y) \cap S)$.

If $y = \infty$, the same argument applies to the regions clV_1, clV_3, and V_4. If, however, $v \in V_2$, we have uv is contained in the convex strip uqv ∞ if $qv \subset S$ or in uzv ∞ if $zy \subset S$, which again leads to the same conclusion.

Let $\overline{Q}$ be the set of points of local non-convexity of $cl(W(x,y) \cap S) \equiv A$. The set $\overline{Q}$ is non-empty, by assumption. If $q \in \overline{Q} \cap L(x,y)$, we show that q is the limit of a sequence of points of $\overline{Q} \cap W(x,y)$. Since each element of $\overline{Q} \cap W(x,y)$ is in $\overline{K}$, the kernel of A, it will follow from the closure of S, that $q \in \overline{K}$. Assume that $q \in \overline{Q} \cap L(x,y)$. If q is an isolated point of $\overline{Q}$, then there exists an open spherical neighborhood σ of q such that $\sigma \cap \overline{Q} = \{q\}$. Let $u, v \in \sigma \cap A$ such that $uv \not\subset A$. Evidently $uv \not\subset S$. If $u, v \in L(x,y) \cap S$, then $uv \subset S$, since $L(x,y) \cap S$ contains xy, the kernel of S. Thus, at most one of the points u and v is in $L(x,y) \cap S$. Assume that $u \in L(x,y) \cap S$. Then since $u \in cl(W(x,y) \cap S)$, every spherical neighborhood λ of u intersects $W(x,y) \cap S$. Moreover, since S is closed, there exists a spherical neighborhood λ' of u such that $u' \in \lambda' \cap A$ implies $u'v \not\subset S$. We may assume that $\lambda' \subset \sigma$. The above argument indicates that we may assume u and v to be in $W(x,y) \cap S$, whenever it is desirable to do so. If $q \in \overline{K}$, the kernel of A, then qu and qv are in S. If $q \notin \overline{K}$, then there exists a point $z \in W(x,y) \cap S$, such that $qz \not\subset S$. Since S is closed, there will then exist a spherical neighborhood τ of q, $\tau \subset \sigma$ such that $w \in \tau \cap A$ implies $wz \not\subset S$. Let $u, v \in \tau \cap (W(x,y) \cap S)$ such that $uv \not\subset S$. Then, by Proposition 2.2, each point of S is either joined to q or to z by a line segment lying entirely in S. Since uz and vz are not contained in S, uq and vq are contained in S. Thus, in either case ($q \in K$ and $q \notin K$) we are able to find points u and v in $\sigma \cap (W(x,y) \cap S)$ such that $uv \not\subset S$ while $qu, qv \subset S$. By Lemma 3.3, $q \notin (x,y)$. Let us assume that q, x, y is the ordering of these three points along the line $L(x,y)$. (We do not exclude the possibility that $x = q$). Since $y \in K$, $y \in \overline{K}$. Since $uv \not\subset S$, u, v and y are not collinear. Thus, we may consider Δyqu and Δyqv. Since $qu, qv \subset S$ and $y \in \overline{K}$, each of these triangles lies entirely in A. Since qu and qv are in S and $uv \not\subset S$, u, v, and q are not collinear. Hence either $yu \cap qv$ or $yv \cap qu$ defines a unique point p in $W(x,y) \cap S$ with the property that pu and pv are contained in $\sigma \cap (W(x,y) \cap S)$. Thus, Δpuv lies entirely in $\sigma \cap (W(x,y))$. Let C be the component of $\Delta puv \cap (W(x,y) \cap S)$ which contains the connected set $up \cup pv$. Since $\Delta puv \subset \sigma \cap (W(x,y)$, C is a closed, connected, locally convex set which must then be convex, by Tietze's theorem. Since $u, v \in C$, $uv \subset C \subset S$, a contradiction, unless C contains a point of local non-convexity of S in $\sigma \cap W(x,y)$ distinct from q.

If $y = \infty$, then the same argument holds for a suitable choice of y' along the ray $x\infty$. It follows that every open spherical neighborhood of q contains a point of local non-convexity of S lying in $W(x,y) \cap S$. That is, q is a limit point of a sequence $\{p_n\}$ of points of local non-convexity of S in $W(x,y) \cap S$. Since $p_n \in \overline{K}$ for each n, $p_n a \subset S$ for all $a \in A$. For each $a \in A$, the set $\{p_n a\}$ converges to the set qa which is in A, since A is a closed subset of the closed set S. Thus, $q \in \overline{K}$, the kernel of A. We now have that every point of $\overline{Q}$, the set of points of local non-convexity of A is in $\overline{K}$, the kernel of A. Moreover, if $\overline{Q} \cap L(x,y)$ is non-empty, then the cardinality $\overline{N}$ of $\overline{Q}$ is infinite.

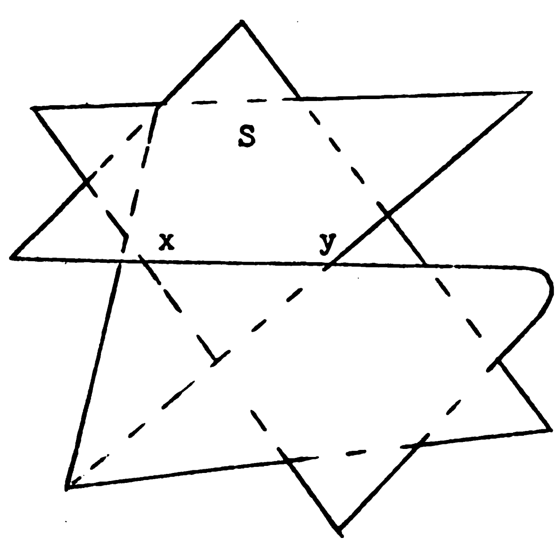
Evidently the set A is a closed, connected set each of whose points of local non-convexity is contained in its kernel $\overline{K}$. That is, the set A satisfies the hypothesis of Theorem 3.4. It follows that if $\overline{N}$ is one, even, or infinite, that A is expressible as the union of 2 convex sets, as was shown in the proof of Theorem 3.4.

If $\overline{N} = 2m + 1$ is an odd integer greater than one then $\overline{Q} \cap L(x,y) = \emptyset$ and there exists an edge q_0q_1 of conv$\overline{Q}$ and a support plane W_0 of conv$\overline{Q}$ which abuts conv$\overline{Q}$ along q_0q_1 such that $xy \subset \mathrm{cl}(W_0 \cap (W(x,y) \cap S))$. As in the proof of Theorem 3.4, $W_0 \cap (W(x,y) \cap S)$, and, consequently, $\mathrm{cl}(W_0 \cap (W(x,y) \cap S))$, are convex. Starting with q_0q_1 we order the remaining edges of conv$\overline{Q}$ in a counterclockwise manner. If we let W_i be the open half-plane of support to conv$\overline{Q}$ whose closure contains q_iq_{i+1}, we have the convex sets

$$S_1 = \mathrm{cl}\,\{(\mathrm{conv}\overline{Q}) \cup [(S \cap W(x,y)) \cap \bigcup_{i=1}^{m} W_{2i-1}]\}$$

$$S_2 = \mathrm{cl}\,\{(\mathrm{conv}\overline{Q}) \cup [(S \cap W(x,y)) \cap \bigcup_{i=1}^{m} W_{2i}]\}$$

and $S_3 = \mathrm{cl}(W_0 \cap (W(x,y) \cap S))$ as three convex sets containing $\mathrm{cl}(S \cap W(x,y))$, as in the proof of Theorem 3.4. Let M_1 and M_2 be maximal convex sets of $\mathrm{cl}(W(x,y) \cap S)$ containing S_1 and S_2, respectively. We show that $S_3 \subset M_1 \cup M_2$. Let $w \in S_3$ such that $w \notin M_1 \cup M_2$. Then there exists an element s in M_1 and t in M_2 such that $sq \not\subset S$ and $tw \not\subset S$. By Lemma 3.8, $s \in cl(W_1 \cap (W(x,y) \cap S))$ and $t \in cl(W_{2m} \cap (W(x,y) \cap S))$. Let x,y,q_1,q_0 be the natural ordering of the vertices of the quadrilateral xyq_1q_0. Then S_3 must be contained in the union of the two convex regions C_2 and C_1 bounded by $L(x,q_0)$ and $L(q_1,q_{2m})$, and $L(y,q_1)$ and $L(q_0,q_1)$ respectively, and containing the quadrilateral xyq_1q_0. Moreover, $cl(W_1 \cap W(x,y) \cap S)) \subset C_1$ and $cl(W_{2m} \cap (W(x,y) \cap S)) \subset C_2$. If this were not the case either q_0 or q_1 would not be in Q. Since the quadrilateral $xyq_1q_0 \subset \bar{K}$ and the above mentioned sets are contained in $C_1 \cup C_2$, we have $sw \cap \bar{K}$ or $tw \cap \bar{K}$ is non-empty. That is, sw or tw is contained in S. This contradiction shows that $S_3 \subset M_1 \cup M_2$. As in the proof of Theorem 3.4, the same arguments hold for $y = \infty$, with only the obvious changes. Since the above arguments may also be applied to $cl(W(y,x) \cap S)$, we see that S is the union of 4 or fewer closed convex sets. The following example is representative:



An examination of the proof of Theorem 3.6 discloses the following:

1. If S has property P^4, and q is a point of local non-convexity of S, then the convex hull of the set X of points of S which are not joined to q by a line segment lying entirely in S is contained in S.
2. If w is a point in the kernel of S, then w is a point in the kernel of S - X.
3. If w is a point of local non-convexity of S - X, then w is a point of local non-convexity of S.

If S, X, and q are defined as above, and S does not bound a bounded domain of its complement, then S-X has property P^4. Assume to the contrary, that x_1, x_2, x_3, and x_4 are four points of S-X which are not joined in S-X. Then $X \cap x_1x_2$, say, is non-empty since S has property P^4. Clearly, q, x_1, and x_2 are not collinear since qx_1 and qx_2 are contained in S-X. Thus, we may consider Δqx_1x_2. Since S-X is closed, there exists an open subinterval (uv) in $(x_1x_2) \cap X$ which implies $bd(\Delta qx_1x_2) \subset S$ bounds a residual domain of S, contrary to assumption.

Since by Theorem 3.5, a closed, connected set in E^2 having property P^4 which bounds a bounded domain of its complement may be expressed as the union of 5 or fewer closed convex sets, the problem of determining the number of closed convex sets which are required if a closed, connected set in E^2 having property P^4 is to be expressed as the union of such sets, may be reduced to the case where the set

S does not bound such a domain. Since S is either convex or contains a point of local non-convexity q, (by Tietze's Theorem), we may assume the existence of such a point. By the above remarks, S may be considered to be the union of a starlike set having property P^4 with q in its kernel, and a closed convex set. Thus, if the case where S is starlike with one or more points of local non-convexity in the kernel of S were settled, the general case would then follow. Several of the cases settled by the theorems of this chapter allow one to place even greater restrictions upon the nature of the set S.

CHAPTER IV

FURTHER CONSIDERATIONS AND EXAMPLES

While it may be difficult to determine how many convex sets are required to express a closed, connected set S having property P^n as the union of closed convex sets it may not be as difficult to establish an upper bound on the number required. The following theorem is a step in that direction.

<u>Theorem 4.1</u> If S is a closed, connected P^n set in E^2 and the set Q of points of local non-convexity of S is a finite set, then S is expressible as the union of a finite number of convex sets.

<u>Proof.</u> Let N be the cardinality of Q. If N = 1, the result follows from Theorem 3.3. Hence, assume the result to be true for $N \leq k - 1$ and consider the case $N = k$.

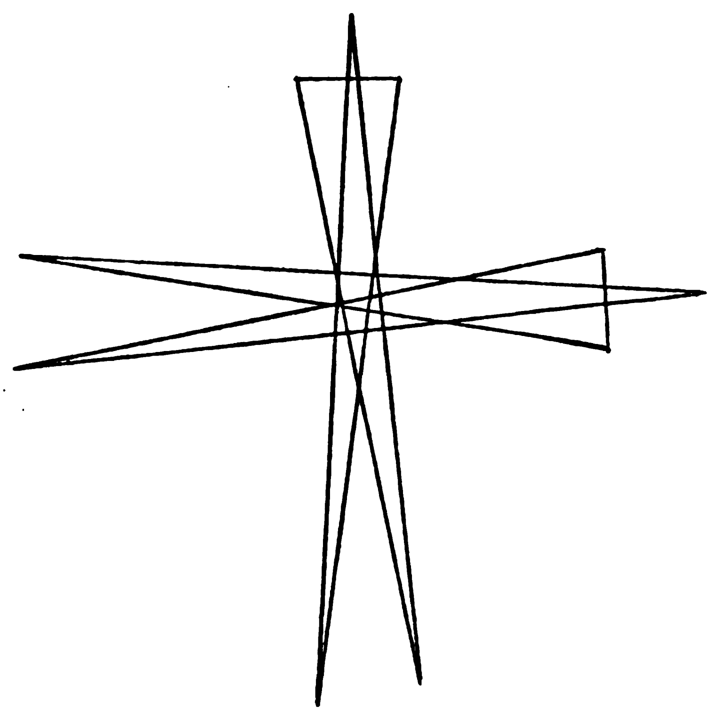
Since convQ is a compact, convex set in E^m, there exists an element q of Q such that $q \notin \text{conv}(Q - \{q\})$. Thus, there exists a line L which strictly separates q and the compact convex set $\text{conv}(Q - \{q\})$.

Let U be the closed half-space determined by L and containing $\text{conv}(Q - \{q\})$ in its interior, and V be the closed half-space containing q in its interior. It is clear that $U \cap S$ and $V \cap S$ are closed sets having property P^n. Thus each has at most $n - 1$ components, and each component is a P^n set containing at most $k - 1$ points of Q. Thus, by our induction hypothesis, each component is expressible as the union of a finite number of closed convex sets. The result now follows.

Remark: An example given by Kay [11] shows that a finite number of convex sets may not suffice if S is not closed. The case where S is closed remains unsettled.

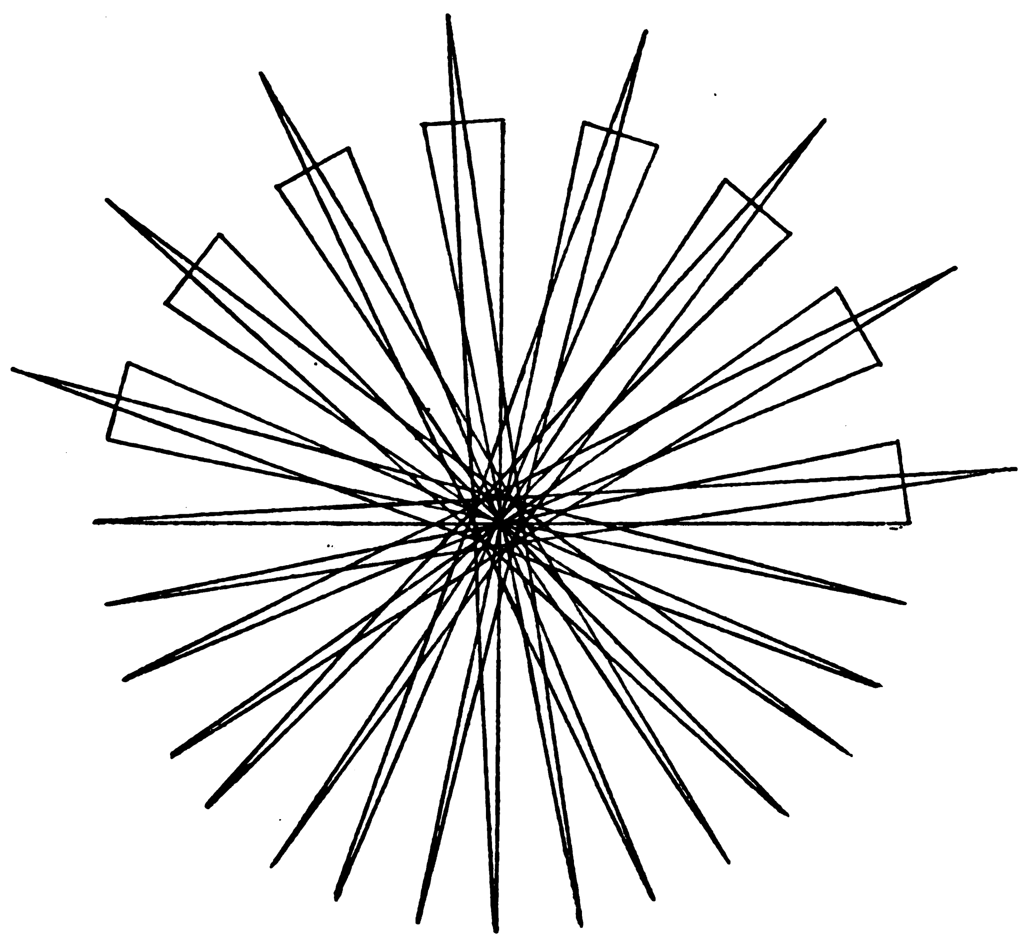
Considering the results obtained in Chapter III for closed, connected sets in the plane having property P^4, one might conjecture that a closed, starlike set in E^2 with property P^n is expressible as the union of n or fewer closed convex sets. The following examples show that this is not the case for $n > 4$.

The first example given has property P^5, but can not be expressed as the union of fewer than 6 closed convex sets. (Each star is expressible as the union of no fewer than 3 convex sets, and no point of one may be entirely included in convex subsets of the other).



The second example given indicates how one might subdivide the circumference of a circle to obtain a starlike set having property P^{2k+1} which is expressible as the union of no fewer than 3k convex sets, where k is the number of stars.

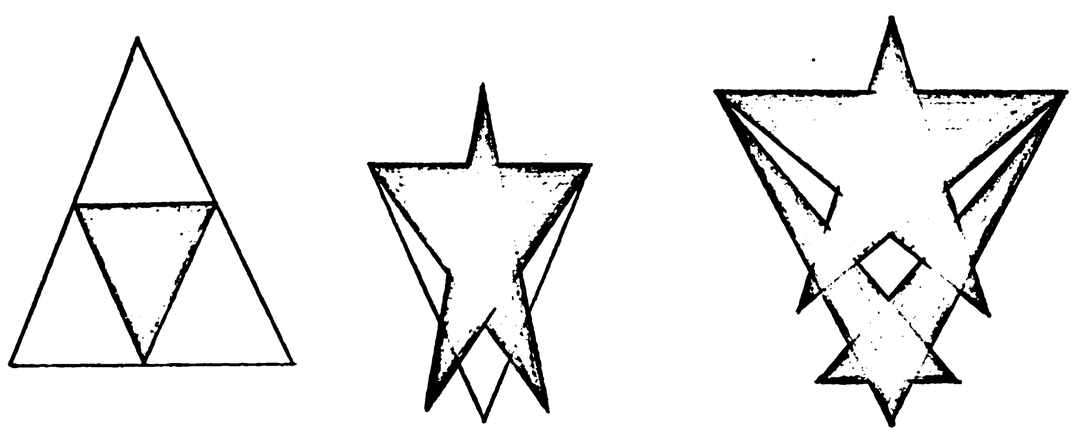
This particular example has property P^{17} and is expressible as the union of 24 closed convex sets. Its kernel is the center of the circle.



Some of the topological properties of a set having property P^n have been established. For example, if S is closed and connected, then S is arcwise connected by Proposition 2.2; by Theorem 7 of Kay [11] which is stated without proof, S is locally starlike, and hence locally connected.

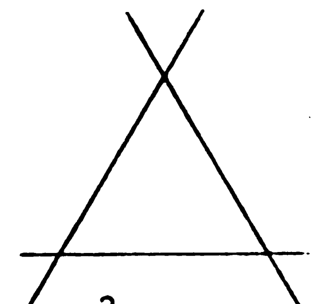
For $n \geq 4$, a closed connected set need not be simply connected since the boundary of a triangle in E^2 has property P^4. The question of the connectivity of a closed, connected set in E^2 is of some interest.

For $n = 4$ it was shown in Lemma 3.9 that a closed, connected set S in E^2 with property P^4 can bound at most one bounded domain of its complement. For $n = 5$, each of the following examples having property P^5 bounds 3 such domains. (In these examples, 4, 5, and 6 convex sets have been used to bound the three domains).

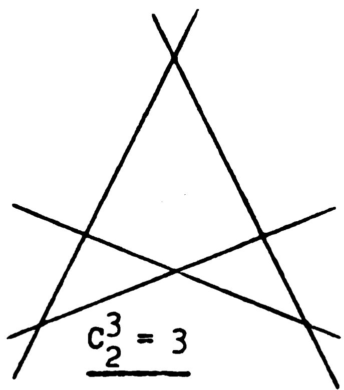


For $n > 4$, the number of bounded domains which a closed, connected set in E^2 having property P^n can bound is an open question.

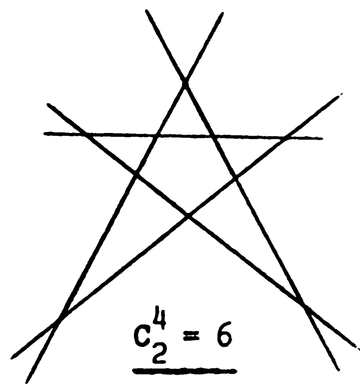
The following class of examples of sets having property P^{k+1} which bound C_2^{k-1} domains indicates that the number is greater than or equal to C_2^{k-1}. Let the k vertices of a convex polygon be chosen so that the extension of each side intersects the extension of every other side. The configuration consisting of the k extended sides will bound C_2^{k-1} domains and will have property P^{k+1}. Given below are examples for k = 3,4,5,6,7,8. Several other configurations have been considered, but each has resulted in C_2^{k-1} or fewer bounded domains



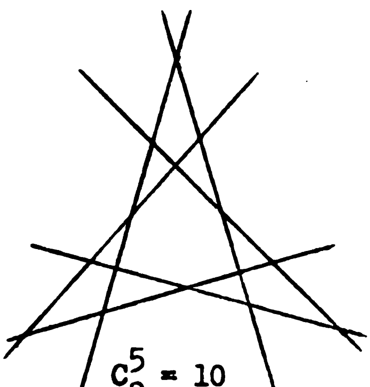
$C_2^2 = 1$



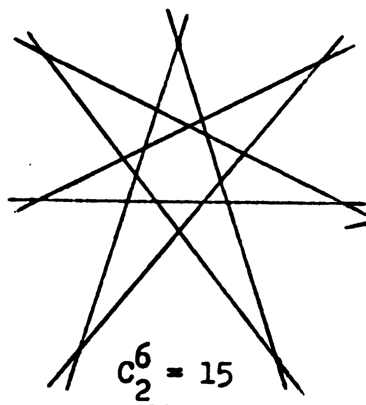
$C_2^3 = 3$



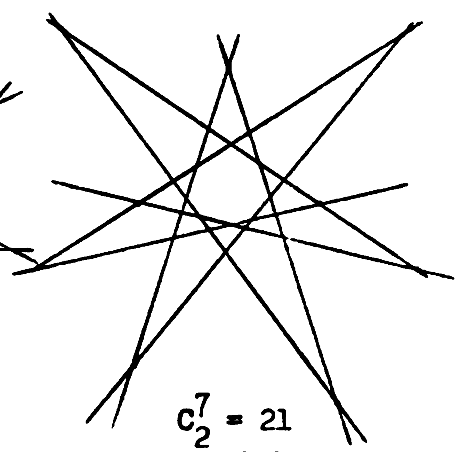
$C_2^4 = 6$



$C_2^5 = 10$



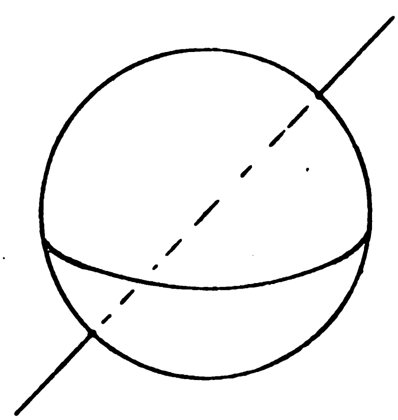
$C_2^6 = 15$



$C_2^7 = 21$

The nature of P^n sets in higher dimensions has received very little attention. If the proofs given in E^2 are to be generalized, it would seem that the generalization of Valentine's results, Theorem 3.3 and Theorem 3.4 should first be considered, and the nature of the set of points of local non-convexity understood.

For $n = 3$, Valentine [18] has given the following result: Let S be a closed set in a linear topological space L where the dimension of L is greater than two. If S has property P^3, is not contained in any two-dimensional variety of L, and has one isolated point of local nonconvexity, then S has at most two points of local non-convexity. The proof uses strongly the fact that for $n = 3$ the set of points of local non-convexity of S are in the kernel of S, which is not true for $n > 3$. The following example is a P^3 set in E^3 having exactly two points of local non-convexity.



By adding lines which pierce the sphere one may obtain a P^n set having $2(n - 2)$ isolated points of local non-convexity.

It is clear from the example given after Proposition 2.6, and the boundary of a 3-simplex which has property P^5 that S need not have any isolated points of local non-convexity.

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