CONTINUUM THEORY FOR GAS - SOLID - LIQUID MEDIA

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ABSTRACT

CONTINUUM THEORY FOR GAS-SOLID-LIQUID MEDIA

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The objective of this work was to develop a continuum model which adequately describes the mechanical behavior of a biological product. The medium was assumed constructed of two sets of interconnected pores separated by a solid. One set of pores contained a gas; the other a liquid. Constitutive and stress-strain relations were derived through energy considerations for the gas-solid-liquid medium.

A series of loading conditions was outlined for determination of the material constants. Experimental attempts were made to determine the compressibility of apple parenchyma using three loading conditions which involved the internal gas pressure and a hydrostatic pressure. Apparatus limitations prevented successful determination of the compressibilities. Suggestions for improvement of the apparatus were made.

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The finite element method was used to obtain numerical solutions to the axisymmetric boundary value problem for the gas-solid-liquid medium. The stress distribution in a spherical body without a skin (unrestrained), with a skin (restrained), and a restrained body subjected to flat plate compression were studied. Internal liquid pressure was varied between 689.5 kPa (100 psi) and 2758 kPa (400 psi) using 689.5 kPa (100 psi) increments while a zero gas pressure was used. Material properties of an apple were used in the contitutive equations.

An unrestrained homogeneous body with the liquid under pressure expanded without developing stress. The restrained body was found to have a hydrostatic (compressive) stress in the parenchyma and tension stress in the skin. Flat plate compression combined with the liquid pressure produced shear stresses and a hydrostatic stress in the parenchyma.

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By

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ABBREVIATIONS AND SYMBOLS

A	area
a	ratio of gas pressure to applied hydrostatic pressure
a_1, a_2, a_3	coefficients for shape functions
b_1, b_2, b_3	
c_1, c_2, c_3	
[B]	matrix defined by $\{\varepsilon\} = [B]\ [U]$
C	material property
C_f	compressibility of the liquid
C_g	compressibility of the gas
^oC	degree Centigrade
cm	centimeter
[D]	material properties matrix
D	material property
E	material property - modulus of elasticity
e	dilatation of bulk material
e_f	dilatation of the liquid
e_g	dilatation of the gas
e_s	dilatation of the solid
e_{ij}	strain tensor
e_x, e_y, e_z	strains in the coordinate directions

F_i	traction on the liquid
f	liquid porosity
f_i	tractions on the solid
$\{f_{(e)}\}$	element force matrix
G_i	traction on the gas
g	gas porosity
H	material property
I_1, I_2, I_3	strain invariants
in	inch
K	bulk modulus of elasticity - "open system"
K_c	bulk modulus of elasticity - "closed system"
k_f	liquid permeability
k_g	gas permeability
$\{k_{(e)}\}$	element stiffness matrix
kPa	kilopascal = $(N/m^2) \times 10^3$
$[M_i]$	matrix of shape functions
M	material property
mm	milimeter
N	material property
N_i	shape function
n_i	unit normal vector
$\{P\}$	force matrix
P	material property
P'	hydrostatic pressure
p_f	pressure of liquid
p_g	pressure of gas

psi	pounds per square inch
Q_x, Q_y, Q_z	displacement of liquid
q_r	stress component in r direction
q_z	stress component in z direction
S	surface
SE	strain energy
$\{U\}$	displacement matrix
U_x, U_y, U_z	displacement of gas
u	displacement in the radial direction
u_i	displacements
u_x, u_y, u_z	displacements of solid matrix
V	volume
V_B	volume of bulk material
V_f	volume of liquid
V_g	volume of gas
v	displacement in z direction
v_i or $\{v\}$	gas flow relative to solid
W	strain energy per unit volume
W_c	work by concentrated forces
W_q	work by applied stresses
WL	work by internal and applied loads
w_i or $\{w\}$	liquid flow relative to solid
α	material property
α_1, α_2, α_3	coefficients for approximating polynomials of the displacement components
β_1, β_2, β_3	
δ_1, δ_2, δ_3	
ϕ_1, ϕ_2, ϕ_3	

β	material property
$\gamma_x, \gamma_y, \gamma_z$	shear strain components
$\dot{\gamma}_1$	gas flow parameter
$\dot{\gamma}_2$	liquid flow parameter
Δ	jacketed compressibility - no gas flow
δ	jacketed compressibility - no liquid flow
δ_{ij}	Kronecker delta, 0 for i $\neq$ j, and 1 for i = j
$\{\varepsilon\}$	strain matrix
ζ	relative gas flow
η	unjacketed compressibility
θ	jacketed compressibility - no gas flow, no liquid pressure
κ	jacketed compressibility - no liquid flow, no gas pressure
λ	material property - Lamé elastic constant for "open system"
λ_c	material property - Lamé elastic constant for "closed system"
μ	material property - Poisson's ratio
ν_f	viscosity of liquid
ν_g	viscosity of gas
Π	total potential energy
ρ	density
σ_f	stress component on liquid
σ_g	stress component on gas
σ_{ij}	stress components on the solid
$\{\tau\}$	stress matrix
τ_{ij}	total stress components - bulk material

τ_{xx}, τ_{yy}, τ_{zz}	normal stress components - bulk material
τ_{xy}, τ_{xz}, τ_{yz}	shear stress components - bulk material
τ'_{ij}	effective stress
ϕ	volume of gas
Ψ	relative liquid flow
ω	jacketed compressibility - no liquid flow, no gas flow

I. INTRODUCTION

Extensive loss is often incurred in the production of various fruits; notably sweet cherries, tomatoes, and certain varieties of apples; as a result of cracking of the skin and fleshy tissues some time prior to harvest. Cracks affect the appearance of the fruit, encourage mold and insect contamination, cause trimming losses for the canner, and result in consumer dissatisfaction. Rapid deterioration usually follows the exposure of the ruptured tissue to air, and the injured fruit becomes worthless or of inferior grade. Cracked fruit, therefore, represent a considerable loss in income to both processing and fresh market industries.

Most of the previous research on fruit cracking has been concerned with the environmental conditions conducive to fruit splitting and breeding crack-resistant varieties. It was the author's contention that present mechanics models of materials were inadequate for representation of the behavior of many biological products. A model which incorporates the effects of the liquid and gas elements was needed. The objective of this study was the development of such a model.

The work reported in this thesis may be divided into three parts:

1. The development of a model containing parameters and variables necessary to more adequately represent the mechanical behavior of fruit tissue.
2. The experimental determination of the parameters necessary for use of the model developed.
3. Use of the finite element method for solution of the continuum equations yielding stress distributions for a body approximating the shape of a fruit.

II. REVIEW OF LITERATURE

Two areas must be considered when developing a continuum mechanics model for a biological product such as a fruit: the plant physiology and horticultural area and the material science and mechanics area. This review of literature is divided into two sections: i) that work related to plant materials and ii) the mechanics theory applicable to the present approach.

2.1 Mechanical Properties of Plant Materials

Several authors have recently reviewed the available literature related to fruit anatomy pertinent to mechanical properties and splitting of fruits. Tennes (1973) reviewed literature available on fruit structure of tomatoes and cherries as related to fruit splitting. He also reviewed the cultural practices which have been used in an attempt to alleviate the problem. Brusewitz (1969) reviewed in detail the anatomy of plant material which might be pertinent to the mechanical properties of the material. Akyurt (1969) also discussed available literature on the cellular structure of plant material and modeling attempts which have been recorded. Because of these studies, the present review of literature was restricted to works which have direct bearing on the proposed method of modeling the elastic action of a system of cells. Literature on the growth of plants was not reviewed unless it is pertinent to the elastic action of the material.

Many researchers pursuing the phenomenological approach to the mechanics of biological materials have assumed the tissues are continuous, homogeneous, and isotropic. Only a few researchers have attempted to describe the effect of such variables as cell dimensions and turgor pressure. Of those attempting to include such variables, the mechanical properties of plant materials have been studied on three levels: the cell wall, the cell, and tissue.

The structure of the cell wall has been studied in great detail. A number of references, such as Frey-Wyssling (1952), describe the make-up of the cell wall. Probine and Prestone (1962) concluded that the anisotropic nature of the cell wall affects its mechanical properties and cell growth.

Cleland (1971) concentrated on certain aspects of cell wall extension including the mechanical properties of primary walls and their relation to cell enlargement. He presented two conclusions from his rheological studies. First, the mechanical properties of all primary cell walls are probably qualitatively similar. Second, the difference that exists between cell elongation and mechanical properties of the walls are sufficient to indicate that cell wall extension does not involve simple physical stretching of the wall.

Frey-Wyssling (1952) pointed out that mechanical stresses on tissues not only involve elastic or plastic alteration of the cell wall, but they also bring about morphological deformations of cell shape, which may be even more important than the rheological behavior of the cell wall.

Most attempts at describing cell action have been based on the assumption of some regular geometric pattern and shape for the cells. Matzke and Duffy (1955), however, stated that rigid conformity to pattern does not exist. They found the average number of faces per cell close to fourteen and that the faces vary in shape from triangular to nonagonal.

Haines (1950) found that the relation between cell extension and turgor pressure for spherical isotropic cells obeying Hooke's Law is not linear but hyperbolic. He further states, "There can be no approximate linear relationship to any cell dimension satisfactory for purposes of calculating turgor or osmotic pressure."

Broyer (1952) defined a coefficient of distention of a boundary as $\alpha = \frac{\Delta P}{\Delta A/A}$ and a coefficient of enlargement as $e = \frac{\Delta P}{\Delta V/V}$. He calculated relations between and for several simple geometric shapes and found that relative volume and changes in volume can be used to determine e as defined for actual pressure changes. Work required for change in relative volume was calculated assuming constant pressure.

Philip (1958b) developed a dynamic theory of the classical osmotic plant cell in quantitative form and extended it to the case where diffusible solute is present. He assumed change in cell volume to be linearly proportional to change in turgor pressure, and developed a first-order expression for the change in relative volume with respect to time for several initial and boundary conditions. In a second paper, Philip (1958a) developed relations for propagation of turgor pressure and other properties through cell aggregates describing them mathematically as diffusion phenomena and assuming the elastic modulus of the material defined by

$$1/E = 1/T(V/V_o - 1)$$

where T is the turgor pressure. He stated further study of the cell-wall stress distribution was needed to refine the propagation equations.

Building on the work of Broyer and Philip, Slayter (1967) concluded that for given cell dimensions and permeabilities, both the elastic properties of the cell and the internal osmotic pressure influence the rate of swelling and shrinking.

Mela (1967a), using microscopic photography, developed a method for studying Young's modulus by extension of mitochondria cells measured for different concentrations of salt water. He assumed membranes of uniform thickness which do not have pores large enough to affect stress-strain calculations. In a second article, Mela (1967b) reported Young's modulus to be a non-linear function of temperature with a minimum at 12-13°C.

Studies must be done on the level of a multi-cell structure, or tissue, to include the effect of interaction between cells. These studies have been performed by both removing a segment of tissue for analysis and by studying the action of the whole body such as a fruit.

Falk _et al_. (1958) studied the relations between turgor pressure and Young's modulus using the resonant frequency of potato tuber parenchyma. They concluded that the cell wall material follows Hooke's law and that there are changes in elasticity of the whole parenchyma due to turgor pressure which are in turn reversible thanks to the ideal cell wall material.

Nilsson _et al_. (1958) studied the dependence of Young's modulus of potato tuber parenchyma on turgor pressure using a simple theoretical model. The cells of the parenchyma were approximated by regular geometric cell-forms (spheres or polyhedra), each cell being bounded by an elastic membrane and filled with an incompressible fluid. It was shown that this model yields the correct dependence of cell diameter on turgor pressure and that certain cell-wall constants can be determined using the relation.

Burstrom et al. (1967) used the resonance frequency method for determining Young's modulus of internodes of etiolated pea seedlings. They found the modulus to increase nearly proportional to turgor pressure and that at water saturation the modulus is more than fifty times higher than at plasmolysis.

Studies by Meynhardt (1964) indicated it may be possible to predict the susceptibility to splitting of different grape cultivars by an anatomical investigation of the berry tissue. He concluded that it seems possible that the subepidermal cell dimension ratio (longitudinal to radial) and the number of subepidermal cell layers may contribute to the resistance or susceptibility of grape berry tissue to splitting. Working with tomatoes, Cotner et al. (1969) found that fruit with flattened epidermal cells were less susceptible to concentric cracks than those with rounded cells, but no such correlation existed for radial cracks.

Clevenger and Hamann (1968) studied the mechanical properties of apple skin. They determined material properties, including elastic modulus and Poisson's ratio, for three varieties of apples. All skins were found to be anisotropic with the greatest strength in the longitudinal direction. Relaxation and creep experiments showed that apple skin tends to be viscoelastic in behavior. Four element (Burger) models were found to describe the action of the material very well.

Akyurt (1969) and Akyurt et al. (1972) attempted to develop methods for studying the stress-strain relations in plant materials. With the cell wall idealized as a shell, the finite element method was proposed for the solution of the corresponding linear equilibrium problem. Akyurt showed that macrodisplacements as well as stresses and couple stresses acting on cellular bodies emerge as solutions of the field equations of the micropolar theory of Eringen (1962). The linear theory of viscoelasticity was also employed.

Considine and Kriedeman (1972) devised a laboratory technique to measure the internal turgor pressure required for fruit rupture in order to assess resistance to splitting of grapes. Fruit of uniform maturity and known osmotic potential were immersed in a range of osmotics to create a known turgor pressure at equilibrium. "Critical turgor," the pressure which resulted in 50 percent of the grapes splitting, was approximately 15 atm in grape cultivars prone to splitting and 40 atm in resistant cultivars. They found splitting was not necessarily related to berry size or to the presence of seeds. No dominant relationship was found to exist between the berry shape and the susceptibility to splitting. The authors concluded that it is the epidermal and subepidermal layers which limit berry enlargement. Two uniform groups of berries were immersed in distilled water to emphasize this point. One group was intact, the other peeled. Within 30 minutes, the intact fruit were ruptured. Peeled fruit, on the other hand, absorbed twice as much water without suggestion of splitting.

Raschke (1970) studied the transmission of changes in water potential in leaves. He found the epidermis of the leaf Zea mays transmits changes in water potential in the water supply of the leaf to the stomata within 0.1 second. Also, reduction in water supply can cause the subsidiary cells surrounding the stomata to collapse within 1.5 minutes, and the epidermis to shrink to one-third of its original thickness within 20 minutes.

2.2 Biot Theory of Elastic Porous Media

Theories of deformation of a porous material containing a viscous compressible fluid and the theory of flow of the fluid through the material have been developed and discussed in a series of papers by Biot and his co-workers (Biot, 1941, 1955, 1956b, 1962, 1963; and Biot and Clinger, 1941, 1942). The theories were first applied to consolidation and settlement of foundations for both isotropic and anisotropic media. Later developments have been in dynamic problems (Biot, 1956a) and finite deformation (Biot, 1972).

Paria (1957-58, 1958a, 1958b, 1966) applied Biot's theories to axisymmetric consolidation of isotropic material under static as well as impulsive loads, transverse isotropic semi-infinite mass under normal loads, deformation of viscoelastic body under pressure, spherical isotropic body under pressure, and flow of fluids through deformable bodies.

Freudenthal and Spillers (1962) developed theoretical solutions, using Biot's theory, for the infinite layer and the half-space assuming a quasi-static consolidating elastic media.

Only a limited number of experimental determinations of coefficients of the equations for Biot's theory have been reported. Biot and Willis (1957) measured the elastic coefficients for sandstone. Fatt (1957, 1959) reported compressibilities of petroleum-bearing sandstones in the range of 0 to 15,000 psi. He also noted a useful model for sandstone can be developed by a sphere pack composed of a mixture of very hard and very soft spheres.

Considerable interest in the use of Biot's theory combined with the finite element has been recently shown in the area of soil consolidation. Most researchers have used variational principles equivalent to the governing equations in Biot's consolidation theory (Sanhu and Wilson, 1969; Yokoo et al., 1971; Hwang et al., 1971) to solve for pore pressure and settlement under various loading conditions. Hwang et al. (1972) used a formulation by the method of weighted residuals.

III. DEVELOPMENT OF QUASI-STATIC THEORY OF A GAS-SOLID-LIQUID SYSTEM

3.1 Model Description

Consider a medium which is the combination of a deformable solid material, a gas, and a liquid, compressible or incompressible. The solid forms the skeleton or the framework of the body and forms a division between two sets of small pores. One set of interconnected pores is filled with a liquid, the the other set contains a gas.

The formulation of a mathematical theory of such a three-phase medium starts with the definition of certain relevant variables. Using the differential element pictured in Figure 3.1, we shall define:

u_x, u_y, u_z as the displacements of the solid matrix parallel to the coordinate axes,

U_x, U_y, U_z as the displacements of the gas,

Q_x, Q_y, Q_z as the displacements of the liquid,

f as the liquid porosity, defined by $f = V_f/V_B$ where V_f is the volume of the liquid and V_B the volume of the bulk material within the element,

g as the gaseous porosity, defined by $g = V_g/V_B$,

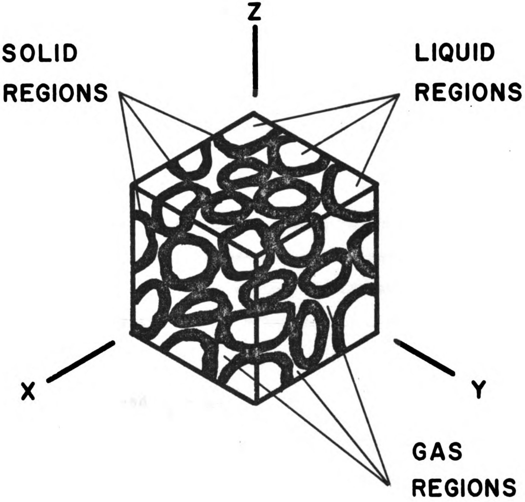

Figure 3.1 Differential Element of Medium

where V_g is the volume of the gas within the element.

The volumes of liquid displaced through unit areas normal to the coordinate directions, X, Y, Z, would be fQ_x, fQ_y, and fQ_z. Similar values for the gas are gU_x, gU_y, and gU_z.

3.2 Theoretical Development

The total stress components of the bulk material, τ_{ij}[1], can be expressed using components as

$$\tau_{ij} = \sigma_{ij} + \delta_{ij} (\sigma_f + \sigma_g) \tag{3.1}$$

where σ_{ij} results from forces applied to the solid part of the body, δ_{ij} is the Kronecker delta, and σ_f and σ_g result from forces applied to the liquid and gas, respectively. Denoting the liquid pressure, p_f, and the gas pressure, p_g,

$$\sigma_f = -fp_f \text{ and } \sigma_g = gp_g. \tag{3.2}$$

---

[1] The standard indicial system for a rectangular Cartesian reference frame is employed for the following section: Repeating the subscripts i, j, or k implies summation, Kronecker's delta is denoted by δ_{ij}, differentiation with respect to space is indicated by subscripts preceded by comma. The subscripts f and g are not to be confused with the summation indices; they indicate liquid and gas, respectively.

Since the system is considered in equilibrium, p_f and p_g are assumed constant throughout their respective regions of the body.

The strain energy of a porous elastic medium can be defined as the isothermal free energy of the gas-solid-liquid system. Let W denote the strain energy per unit volume. The variation of the strain energy for a volume V bounded by surface S is equal to the virtual work of the surface forces, i.e.,

$$\begin{aligned}\iiint_V \delta W dV = \iint_S (&f_x \delta u_x + f_y \delta u_y + f_z \delta u_z + G_x \delta U_x \\ &+ G_y \delta U_y + G_z \delta U_z + F_x \delta Q_x + F_y \delta Q_y + F_z \delta Q_z)\, dS\end{aligned} \tag{3.3}$$

where f_i, G_i, and F_i are the tractions acting on the solid, gas, and liquid regions of dS respectively. They can be expressed

$$\begin{aligned} f_i &= \sigma_{ij}\, n_j \\ F_i &= \sigma_f\, \delta_{ij}\, n_j \\ G_i &= \sigma_g\, \delta_{ij}\, n_j \end{aligned} \tag{3.4}$$

where n_j is an outward normal to the surface. Forces can be expressed in terms of τ_{ij}, p_f, and p_g by using (3.1) and (3.2) to obtain

$$f_i = (\tau_{ij} + \delta_{ij}\, (gp_g + fp_f))n_j$$

$$F_i = \delta_{ij}\, (-fp_f)\, n_j$$

$$G_i = \delta_{ij} (-gp_g) n_j.$$

Introducing the above expressions into (3.3) yields

$$\iiint_V \delta W dV = \iint_S (f_i \delta u_i + G_i \delta U_i + F_i \delta Q_i) dS$$

$$= \iint_S ((\tau_{ij} + \delta_{ij}(gp_g + fp_f)n_j)\delta u_i + \delta_{ij}(-gp_g)\delta U_i n_j$$

$$+ \delta_{ij}(-fp_f)\delta Q_i n_j) dS.$$

By defining

$$\begin{aligned} w_i &= g(U_i - u_i) \\ v_i &= f(Q_i - u_i) \end{aligned} \qquad (3.5)$$

we can obtain

$$\begin{aligned} \iiint_V \delta W dV &= \iint_S \tau_{ij}\, \delta u_i\, n_j\, dS \\ &- p_g \iint_S \delta w_i n_i\, dS \qquad (3.6) \\ &- p_f \iint_S \delta v_i n_i\, dS. \end{aligned}$$

The vectors v_i and w_i represent gas and liquid flows relative to the solid and are measured in terms of volume per unit surface area of the bulk material.

The first of the surface integrals of (3.6) can be transformed to a volume integral by means of Gauss' theorem to produce

$$\iint_S \tau_{ij}\, n_j\, \delta u_i dS = \iint_V (\tau_{ij} \delta u_i),_j\, dV.$$

Expanding the integrand

$$(\tau_{ij}\ \delta u_i)_{,j} = \delta u_i\ \tau_{ij,j} + \tau_{ij}\ \delta u_{i,j}.$$

The total stress field being in equilibrium implies

$$\delta u_i\ \tau_{ij,j} = 0.$$

Therefore, the integrand can be expressed as

$$\begin{aligned}(\tau_{ij}\ \delta u_i)_{,j} &= \tau_{xx}\ \delta e_x + \tau_{yy}\ \delta e_y + \tau_{zz}\ \delta e_z \\ &+ \tau_{yz}\ \delta\gamma_x + \tau_{zx}\ \delta\gamma_y + \tau_{xy}\ \delta\gamma_z\end{aligned} \tag{3.7}$$

where

$$\begin{aligned} e_x &= \frac{\partial u_x}{\partial x} & \gamma_x &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ e_y &= \frac{\partial u_y}{\partial y} & \gamma_y &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ e_z &= \frac{\partial u_z}{\partial z} & \gamma_z &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\ . \end{aligned} \tag{3.8}$$

Similarly, the second and third surface integrals of (3.6) can be transformed to volume integrals. Defining

$$\begin{aligned}\zeta &= -\ w_{i,i} = \mathrm{div}[\ g(u_i - U_i)] \\ \Psi &= -\ v_{i,i} = \mathrm{div}[\ f(u_i - Q_i)]\end{aligned} \tag{3.9}$$

the second and third integrals become

$$- \iint_S \delta w_i n_i dS = \iiint_V \delta\zeta dV$$

and

$$- \iint_S \delta v_i n_i dS = \iiint_V \delta\Psi dV.$$

Substituting the results of (3.7), (3.8), and (3.9) into (3.6) yields

$$\begin{aligned} \iiint_V \delta W dV = \iiint_V (&\tau_{xx}\,\delta e_x + \tau_{yy}\,\delta e_y + \tau_{zz}\,\delta e_z \\ &+ \tau_{yz}\,\delta\gamma_x + \tau_{zx}\,\delta\gamma_y + \tau_{xy}\,\delta\gamma_z \\ &+ p_f\,\delta\Psi + p_g\,\delta\zeta)\,dV. \end{aligned} \tag{3.10}$$

Hence, for an arbitrary volume

$$\begin{aligned} \delta W = \;&\tau_{xx}\,\delta e_x + \tau_{yy}\,\delta e_y + \tau_{zz}\,\delta e_z \\ &+ \tau_{yz}\,\delta\gamma_x + \tau_{xy}\,\delta\gamma_z + \tau_{zx}\,\delta\gamma_y \\ &+ p_f\,\delta\Psi + p_g\,\delta\zeta. \end{aligned} \tag{3.11}$$

ζ and Ψ are defined for nonhomogeneous porosity. If uniform porosity is assumed throughout the body, (3.8) and (3.9) become

$$\begin{aligned} \zeta &= g \operatorname{div} (u_i - U_i) \\ \Psi &= f \operatorname{div} (u_i - Q_i) \end{aligned} \tag{3.12}$$

These variables are now measures of the amount of each substance which has moved in or out of a given element attached to the solid frame. They also represent the increments of fluid and gas contents.

The strain energy W must be a function of ζ, Ψ, and the six strain components,

$$W = W(e_x, e_y, e_z, \gamma_x, \gamma_y, \gamma_z, \zeta, \Psi). \tag{3.13}$$

δW must be an exact differential; therefore,

$$\begin{aligned}
\tau_{xx} &= \frac{\partial W}{\partial e_x} & \tau_{yz} &= \frac{\partial W}{\partial \gamma_x} \\
\tau_{yx} &= \frac{\partial W}{\partial e_y} & \tau_{zx} &= \frac{\partial W}{\partial \gamma_y} \\
\tau_{zz} &= \frac{\partial W}{\partial e_z} & \tau_{xy} &= \frac{\partial W}{\partial \gamma_z} \\
p_f &= \frac{\partial W}{\partial \Psi} & p_g &= \frac{\partial W}{\partial \zeta}.
\end{aligned} \tag{3.14}$$

These relations lead directly to the formulation of the general stress-strain relations for a gas-solid-liquid medium. Biot (1962) points out several major aspects of this type of derivation where W is the isothermal free energy. The stress-strain relations include phenomena which may depend on the physical chemistry of the gas-solid-liquid system; as well as, phenomena which are expressible by means of thermodynamic variables such as interfacial and surface tension effects.

3.3 Linear Stress-Strain Relations

Considering the case of an isotropic medium, the strain energy is a function of five variables, the three strain invariants, I_1, I_2, I_3, and the fluid components ζ and Ψ, i.e.

$$W = W(I_1, I_2, I_3, \zeta, \Psi). \tag{3.15}$$

The strain energy is quadratic in form for a linear material, Love (1944). Only the first- and second-order variables are included in the energy expression. The invariant terms remaining are

$$I_1 = e_x + e_y + e_z$$

$$I_2 = e_y e_z + e_z e_y + e_x e_y - \tfrac{1}{4}(\gamma_x{}^2 + \gamma_y{}^2 + \gamma_x{}^2).$$

It is more convenient to use the invariant

$$I_2' = -4I_2 = \gamma_x{}^2 + \gamma_y{}^2 + \gamma_z{}^2 - 4e_z e_y - 4e_z e_x - 4e_x e_y.$$

The quadratic form for W now becomes

$$2W = He^2 + \mu I_2' - 2Ce\zeta + M\zeta^2 - 2De\Psi$$
$$+ N\Psi^2 + p\zeta\Psi.$$

Sokolnikoff (1956) shows that the coefficients on any linear terms in W must be zero. Therefore

$$\begin{aligned} 2W &= H(e_x{}^2 + e_y{}^2 + e_z{}^2 + 2e_xe_y + 2e_xe_z + 2e_ye_z) \\ &+ \mu(\gamma_x{}^2 + \gamma_y{}^2 + \gamma_z{}^2 - 4e_ye_z - 4e_ze_x - 4e_xe_y) \\ &-2C(e_x + e_y + e_z)\zeta - 2D(e_x + e_y + e_z)\Psi \\ &+ M\zeta^2 + N\Psi^2 + P\zeta\Psi, \end{aligned} \tag{3.16}$$

where H, μ, C, D, M, N, P are coefficients which depend on the material properties.

Substituting expression (3.16) in the general equations (3.14), we obtain

$$\begin{aligned} \tau_{xx} &= He - 2\mu(e_y + e_z) - C\zeta - D\Psi \\ \tau_{yy} &= He - 2\mu(e_x + e_z) - C\zeta - D\Psi \\ \tau_{zz} &= He - 2\mu(e_y + e_x) - C\zeta - D\Psi \\ \tau_{yz} &= \mu\gamma_x \qquad \tau_{zx} = \mu\gamma_y \qquad \tau_{xy} = \mu\gamma_z \\ p_f &= -De + N\Psi + P\zeta/2 \\ p_g &= -Ce + M\zeta + P\Psi/2. \end{aligned} \tag{3.17}$$

Letting $H = \lambda_c + 2\mu$, $C = \alpha M$, and $D = \beta N$

(3.17) can be given in the form

$$\tau_{xx} = \lambda_c e + 2\mu e_x - \alpha M\zeta - \beta N\Psi$$

$$\tau_{yy} = \lambda_c e + 2\mu e_y - \alpha M\zeta - \beta N\Psi$$

$$\tau_{zz} = \lambda_c e + 2\mu e_z - \alpha M\zeta - \beta N\Psi \qquad (3.18)$$

$$\tau_{yz} = \mu\gamma_x \qquad \tau_{zx} = \mu\gamma_y \qquad \tau_{xy} = \mu\gamma_z$$

$$p_f = -\beta N e + N\Psi + P\zeta/2$$

$$p_g = -\alpha M e + M\zeta + P\Psi/2.$$

Written in matrix form

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \\ p_g \\ p_f \end{bmatrix} = \begin{bmatrix} \lambda_c+2\mu & \lambda_c & \lambda_c & 0 & 0 & 0 & -\alpha M & -\beta N \\ \lambda_c & \lambda_c+2\mu & \lambda_c & 0 & 0 & 0 & -\alpha M & -\beta N \\ \lambda_c & \lambda_c & \lambda_c+2\mu & 0 & 0 & 0 & -\alpha M & -\beta N \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ -\alpha M & -\alpha M & -\alpha M & 0 & 0 & 0 & M & P \\ -\beta N & -\beta N & -\beta N & 0 & 0 & 0 & P & N \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ \gamma_x \\ \gamma_y \\ \gamma_z \\ \zeta \\ \Psi \end{bmatrix}$$

We can define, using (3.18), the bulk modulus of jacketed compressibility as

$$K_c = \lambda_c + 2\mu/3 \qquad (3.19)$$

for this "closed system" in which the fluid pores are sealed.

Several relationships between coefficients in (3.16) can be determined by considering the non-negative nature of the strain energy W. Using (3.19), strain energy can be expressed as

$$
\begin{aligned}
2W = {} & K_c\, e^2 - Ce\zeta - 2De\Psi + M\zeta^2 \\
& + N\Psi^2 + D\zeta\Psi + 2\mu/3[(e_y-e_z)^2 \\
& + (e_z - e_x)^2 + (e_x - e_y)^2] \\
& + \mu(\gamma_x^2 + \gamma_y^2 + \gamma_z^2).
\end{aligned}
\tag{3.20}
$$

Letting $e = \zeta = \Psi = 0$ directly implies $\mu \geq 0$.

By putting $\gamma_x = \gamma_y = \gamma_z$ and $e_x = e_y = e_z$ the strain energy expression reduces to

$$
\begin{aligned}
2W = {} & K_c e^2 - 2Ce\zeta - 2De\Psi \\
& + M\zeta^2 + N\Psi^2 + P\zeta\Psi.
\end{aligned}
\tag{3.21}
$$

By letting $\Psi = 0$ in (3.19), we obtain the expression

$$2W = K_c e^2 - 2C\zeta e + M\zeta^2. \tag{3.22}$$

Since W must be equal to or greater than zero, it can be shown that

$$K_c \geq 0 \text{ and } M \geq 0. \tag{3.23}$$

Using the quadratic form for solution with respect to e, we find the discriminate

$$4C^2\,\zeta^2 - 4K_c(M\zeta^2 - 2W) \geq 0$$

for real solutions. We find, for $\zeta \neq 0$

$$K_cM - C^2 \geq \frac{2K_cW}{\zeta^2} \geq 0. \qquad (3.24)$$

Similarly, by letting $\zeta = 0$ in (3.19), we obtain

$$2W = K_c\,e^2 - 2D\Psi + N\Psi^2. \qquad (3.25)$$

This expression is never negative if

$$K_c \geq 0, \quad N \geq 0, \quad \text{and } K_cN - D^2 \geq 0. \qquad (3.26)$$

Again considering (3.20) when neither ζ or Ψ is zero, the discriminate yields

$$(2C\zeta + 2D\Psi)^2 - 4K_c(M\zeta^2 + N\Psi^2 + P\zeta\Psi - 2W) \geq 0$$

when solving for e. This relationship can be modified to

$$(K_cM - C^2)\zeta^2 + (K_cN - D^2)\Psi^2 + (K_cP - 2CD)\Psi\zeta - 2K_cW \geq 0. \qquad (3.27)$$

For real solutions for ζ, the discriminate of the solution must be non-negative, that is

$$(K_cP - 2CD)^2\Psi^2 - 4(K_cM - C^2)[(K_cN - D^2)\Psi^2 - 2K_cW]$$

$$\geq 0.$$

Therefore, considering $\Psi \neq 0$,

$$(K_cM - C^2)(K_cN - D^2) - K_c{}^2P^2 - 4K_cCDP + 4\ C^2D^2 \geq 0. \tag{3.28}$$

For expression (3.28) never to be negative

$$-K_c{}^2\ P^2 \geq 0.$$

It was shown from (3.23) that K_c is a real number which is greater than or equal to zero, thereby implying

$$P^2 \leq 0. \tag{3.29}$$

Therefore, P=0, for P to be a real constant as assumed.

With proof that P = 0, the last two equations of (3.18) can be simplified to

$$\begin{aligned} p_f &= -\beta Ne + N\Psi \\ p_g &= -\alpha Me + M\zeta \end{aligned} \tag{3.30}$$

We can define, using (3.18), the bulk modulus for an "open system" where $p_g = 0$ and $p_f = 0$ as

$$K = \lambda + 2\mu/3 \tag{3.31}$$

which is the inverse of the "jacketed" compressibility. The open system would correspond to a jacketed compression test where the pore fluids are allowed to escape freely. Carrying Biot's (1956) analogy between a porous media and a thermoelastic solid, it can be concluded that λ_c and λ correspond to the adiabatic and isothermal Lamé coefficients for a nonporous medium.

Solving equations (3.30) for ζ and Ψ yields

$$\begin{aligned} \zeta &= \alpha e + p_g/M \\ \Psi &= \beta e + p_f/N \end{aligned} \tag{3.32}$$

Using

$$\lambda = \lambda_c - \frac{\alpha^2 M}{N} - \frac{\beta^2 N}{M} \tag{3.33}$$

(3.18) can be transformed into the form

$$\begin{aligned} &\tau_{xx} + \alpha p_g + \beta p_f = 2\mu e_x + \lambda e \\ &\tau_{yy} + \alpha p_g + \beta p_f = 2\mu e_y + \lambda e \\ &\tau_{zz} + \alpha p_g + \beta p_f = 2\mu e_z + \lambda e \\ &\tau_{yz} = \mu\gamma_x \qquad \tau_{zx} = \mu\gamma_y \qquad \tau_{xy} = \mu\gamma_z \\ &\qquad\qquad \zeta = \alpha e + p_g/M \\ &\qquad\qquad \Psi + \beta e + p_f/N. \end{aligned} \tag{3.34}$$

Written in abbreviated form

$$\tau_{ij} + \delta_{ij}(\alpha p_g + \beta p_f) = 2\mu e_{ij} + \delta_{ij}\,\lambda e$$

$$\zeta = \alpha e + p_g/M \qquad (3.35)$$

$$\Psi = \beta e + p_f/N.$$

The stress-strain relations can now be written in such a form as to yield the "effective stress," i.e. the total stress in excess of local fluid pressures

$$\tau'_{ij} = \tau_{ij} + \delta_{ij}\,(p_f + p_g) \qquad (3.36)$$

or

$$\tau'_{ij} - \delta_{ij}\,((1 - \alpha)p_g + (1 - \beta)p_f) \qquad (3.37)$$

$$= 2\mu\, e_{ij} + \delta_{ij}\,\lambda e$$

$$\zeta = \alpha e + p_g/M$$

$$\Psi = \beta e + p_f/N.$$

The "effective stress" for a fluid-solid system is commonly used in soil mechanics for the study of fluid saturated clays.

The bulk modulus expressions for the two types of systems defined as the "closed system," (3.19), and the "open system," (3.31), can be combined with (3.33) to give

$$K_c - K = \lambda_c - \lambda = \frac{\alpha^2 M}{N} + \frac{\beta^2 N}{M}. \tag{3.38}$$

We obtain by combining (3.24) and (3.26)

$$(K_c - \beta^2 N)N + (K_c - \alpha^2 M)M \geq 0.$$

Substituting from (3.38) into the above relation

$$K_c(N + M - 2MN) + 2KMN + \alpha^2 M^2 + \beta^2 N^2 \geq 0 \tag{3.39}$$

For expression (3.39) never to be negative, we conclude

$$N + M - 2MN \geq 0 \tag{3.40}$$

In summary, the non-negative nature of the strain energy yields the following limits to the coefficients of equations (3.18) and those derived from it,

$$\begin{aligned} &\mu \geq 0 \qquad K_c \geq 0 \qquad K \geq 0 \\ &M \geq 0 \qquad N \geq 0 \\ &N + M - 2NM \geq 0 \\ &(K_c - \beta^2 N)N \geq 0 \quad (K_c - \alpha^2 M)M \geq 0 \\ &P = 0 \end{aligned} \tag{3.41}$$

3.4 Governing Equations for Transient Phenomena

The equations for the quasi-static theory of a gas-solid-liquid have been established. This theory shall now be extended to cover the transient phenomena. The equations will describe the distribution of stress, fluid contents, and displacements as a function of time under given loads. It is important to note that the time variable t enters the theory through Darcy's law. Therefore, the transient problem in this case refers to a flow problem.

Substitution of (3.35) into the equilibrium equation neglecting body forces as before yields

$$\tau_{ij,j} = 2\mu\, e_{ij,j} - \delta_{ij}(\alpha p_g + \beta p_f - \lambda e_{kk}),_j \qquad (3.42)$$

Substitution of the strain displacement relation

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad (3.43)$$

into (3.42) produces

$$\mu u_{i},_{jj} + (\lambda + \mu)u_{j,ji} - \alpha p_{g},_{i} - \beta p_{f},_{i} = 0. \qquad (3.44)$$

Equations (3.44) are a set of three equations with five unknowns, u, v, w, p_g, p_f. Two additional equations are needed to complete the system. These equations can be obtained by introducing Darcy's Law governing the flow of each of the fluids.

First consider the flow of the liquid, assuming it to be incompressible. Paria (1966) gives a modified form of Darcy's law

$$p_{f,i} = C_f \frac{\partial}{\partial t} (Q_i - u_i) \tag{3.45}$$

where $C_f = \frac{\nu_f f^2}{k_f}$ for an isotropic media, where ν_f is the viscosity of the liquid, k_f the permeability of the medium to liquid, and f the liquid porosity. If the solid displacements are zero, $u_i = 0$, (3.45) reduces to the classical form of Darcy's law for an undeformed medium. Combining the divergence of (3.45) with the p_f relation of (3.18) and (3.12) yields

$$-\beta N u_{k,ki} + Nf(u_k - Q_k)_{,ki} = C_f \frac{\partial}{\partial t} (Q_i - u_i). \tag{3.46}$$

Darcy's equation for the flow of gas in the medium is

$$p_{g,i} = C_g \frac{\partial}{\partial t} (U_i - u_i) \tag{3.47}$$

with $C_g = \frac{\nu_g g^2}{k_g}$ for an isotropic medium, where ν_g is the viscosity of the gas, k_g the gas permeability of the medium, and g the gas porosity. The density, ρ, for isothermal flow of gases is directly proportional to p_g, hence

$$\frac{\partial^2 (p_g{}^2)}{\partial x_i{}^2} = \frac{2C_g}{g} \frac{\partial p_g}{\partial t} \tag{3.48}$$

Carman (1956). Equation (3.48) can be combined with the last two equations of (3.18) to produce

(no sum on j)

$$(-\alpha Me_{kk} + M\zeta)^2{}_{,jj} = \frac{2C_g}{g}\frac{\partial}{\partial t}(-\alpha Me_{kk} + M\zeta) \tag{3.49}$$

where as defined in (3.12)

$$\zeta = g(u_i - U_i)_{,j}. \tag{3.50}$$

Combining these two equations gives

$$(-\alpha Mu_{k,k} + gM(u_i - U_i)_{,i})^2{}_{,jj}$$
$$= 2C_gM\frac{\partial}{\partial t}(u_i - U_i)_{,i} - \frac{2Cg\alpha M}{g}\frac{\partial}{\partial t}u_{k,k}. \tag{3.51}$$

(no sum on j)

Substituting (3.45) and (3.47) into the equilibrium equation (3.44) yields a set of nine differential equations and nine unknown displacements when combined with (3.46) and (3.51). Summarizing, the nine equations are

$$\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ji} - \alpha C_g\frac{\partial}{\partial t}(U_i - u_i)$$
$$-\beta C_f\frac{\partial}{\partial t}(Q_i - u_i) = 0$$
$$-\beta Nu_{k,ki} + N_f\,(u_i - Q_i)_{,ij} \tag{3.52}$$
$$= C_f\frac{\partial}{\partial t}(Q_i - u_i)$$
$$(-\alpha Mu_{k,k} + gM(u_i - U_i)_{,i})^2_{,qq}$$
$$= 2\,C_gM\frac{\partial}{\partial t}((u_i - U_i)_{,i}) + \frac{2C_g\alpha M}{g}\frac{\partial}{\partial t}u_{k,k}.$$

(no sum on q)

Written in conventional notation, these equations are

$$\mu\nabla^2\vec{u} + (\lambda + \mu)\text{grad } e - \alpha C_g \frac{\partial}{\partial t}(\vec{U} - \vec{u})$$
$$- \beta C_f \frac{\partial}{\partial t}(\vec{Q} - \vec{u}) = 0 \tag{3.53}$$
$$-\beta N\nabla e + N_f\nabla^2(\vec{u} - \vec{Q}) = C_f \frac{\partial}{\partial t}(\vec{Q} - \vec{u})$$
$$\nabla^2(-\alpha Me + gM\nabla(\vec{u} - \vec{U})^2$$
$$= 2gM \frac{\partial}{\partial t}(\nabla(\vec{u} - \vec{U}) + \frac{2C_g\alpha M}{g}\frac{\partial e}{\partial t} .$$

The nine equations governing the transient problem are coupled with each other and hence would have to be solved simultaneously. As Paria (1966) points out for a liquid-solid media, this implies that the flow fields and the elastic field are not merely superposed, but that they react upon each other. This could also be concluded directly from the constitutive equations (3.18).

3.5 Determination of Elastic Coefficients

The following series of hypothetical tests are intended to show the physical meaning of the elastic coefficients α, β, M, N, and λ_c. The coefficient μ will be assumed to be determined by standard means.

A jacketed test refers to an experiment during which the sample is placed within an impermeable membrane. The pressure of the gas, p_g, and the pressure of the liquid,

p_f, are assumed controllable through tubes which penetrate the membrane and are connected to the appropriate region. See Figure 3.2.

3.5.1 Jacketed Compressibility Test - No Liquid Flow

Consider a jacketed specimen subjected to a hydrostatic pressure P' such that

$$p_g = P'/a \qquad \tau_{xx} = \tau_{yy} = \tau_{zz} = -P' \qquad \Psi = 0.$$

Define the compressibility under the given test conditions as

$$\delta = -\frac{e}{P'}. \tag{3.54}$$

Two relationships are obtained from (3.18)

$$P' = \lambda_c \ \delta P' + 2\mu/3 \ \delta P' - \alpha M\zeta \tag{3.55}$$

$$P'/a = \alpha M\delta P' + M\zeta. \tag{3.56}$$

Assuming $P' \neq 0$, combining (3.55) and (3.56) gives

$$1 - \alpha/a = (\lambda_c + 2\mu/3 - \alpha^2 M)\delta. \tag{3.57}$$

From (3.24)

$$\lambda_c + 2\mu/3 - \alpha^2 M \geq 0,$$

therefore, α/a approaches unity as δ approaches zero which corresponds to an incompressible media.

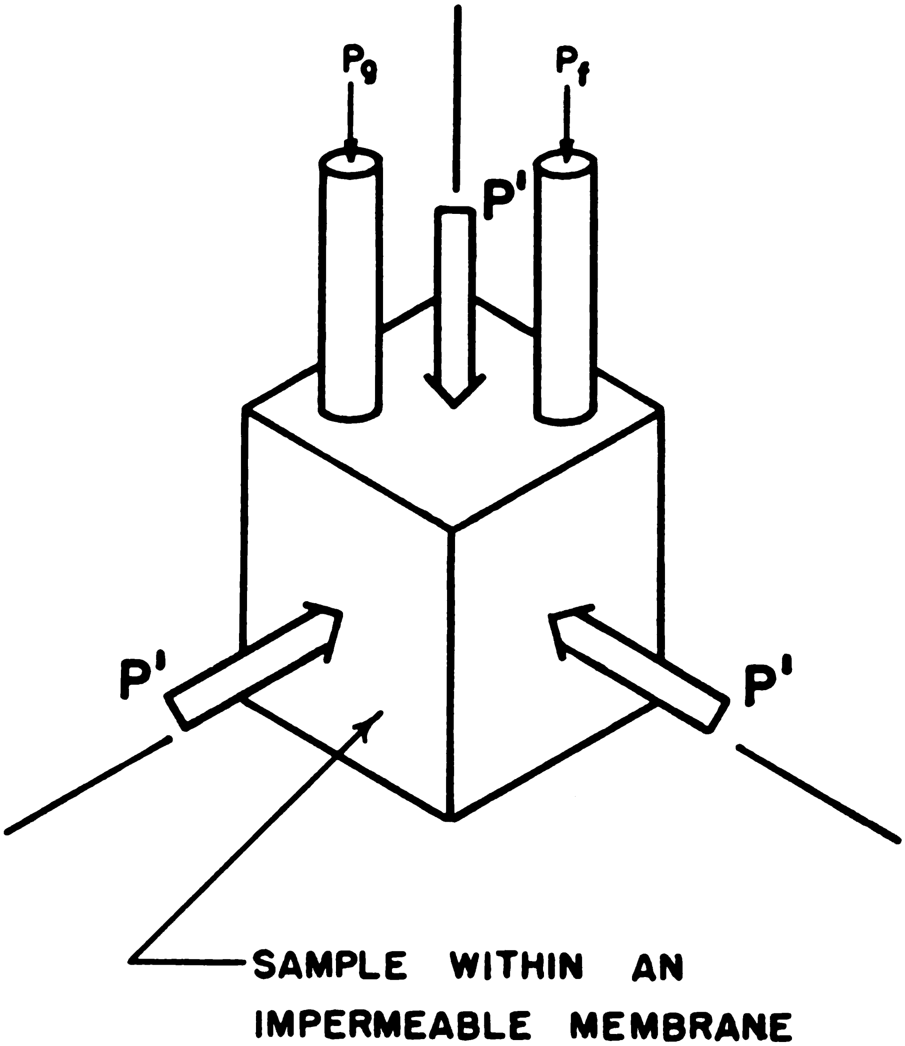

Figure 3.2 Specimen for Hypothetical Tests

3.5.2 Jacketed Compressibility Test - No Gas Flow

Consider that a jacketed specimen subjected to a hydrostatic pressure P' such that

$$p_f = P' \qquad \tau_{xx} = \tau_{yy} = \tau_{zz} = -P' \qquad \zeta = 0.$$

Define the compressibility for these conditions as

$$\Delta = -\frac{e}{P'}. \tag{3.58}$$

Two relations can be obtained from (3.18)

$$-P' = -\lambda_c \Delta P' - 2\mu/3 \; \Delta \; P' - \beta N \Psi \tag{3.59}$$

$$P' = \beta N \Delta P' + N\Psi. \tag{3.60}$$

Assuming $P' \neq 0$ implies

$$1 - \beta = (\lambda_c + 2\mu/3 - \beta^2 N)\Delta. \tag{3.61}$$

Equations (3.26) gives

$$\lambda_c + 2\mu/3 - \beta^2 N \geq 0$$

therefore β equals one in the limit (incompressible case) and $\Delta = 0$.

Combining the results of the first two tests gives the relation

$$\frac{1-\beta}{\Delta} + \beta^2 N = \frac{1 - \alpha/a}{\delta} + \alpha^2 M. \tag{3.62}$$

3.5.3 Jacketed Compressibility Test - No Liquid Flow, No Gas Pressure

Consider a jacketed specimen subjected to a hydrostatic pressure P' such that

$$p_g = 0 \quad \tau_{xx} = \tau_{yy} = \tau_{zz} = -P' \qquad \Psi = 0.$$

Define the compressibility under these conditions as

$$\kappa = -\frac{e}{P'} . \tag{3.63}$$

Equations (3.18) reduce to

$$-P' = -\kappa(\lambda_c + 2\mu/3)P' - \alpha M\zeta \tag{3.64}$$

and

$$0 = \alpha M\kappa P' + M\zeta. \tag{3.65}$$

Assuming $P' \neq 0$, the combining of (3.64) and (3.65) yields

$$1/\kappa = \lambda_c + 2\mu/3 - \alpha^2 M. \tag{3.66}$$

Combining the results of the first and third tests allows κ to be expressed in terms of two measurable compressibilities

$$\alpha = a(1 - \frac{\delta}{\kappa}) \tag{3.67}$$

For a highly compressible gas, the compressibility δ would be very near that of κ implying a small value for α.

When $p_g = 0$, (3.32) reduces to

$$\zeta = \alpha e$$

and α can be interpreted as the ratio of the change in gas volume to dilitation for the jacketed test. If the gas region within the specimen is connected with the atmosphere by a tube, ζ would be the amount of gas flowing through the tube.

Another interpretation of α can be obtained from the first three equations of (3.34), where α is that portion of the gaseous pressure which produces strains.

3.5.4 Jacketed Compressibility Test - No Gas Flow, No Liquid Pressure

Consider a jacketed specimen subjected to a hydrostatic pressure such that

$$p_f = 0 \qquad \tau_{xx} = \tau_{yy} = \tau_{zz} = -P' \qquad \zeta = 0.$$

Define the compressibility under these test conditions as

$$\Theta = -\frac{e}{P'} \; . \tag{3.68}$$

Equations (3.18) again reduce to two equations

$$P' = (\lambda_c + 2\mu/3)\ \theta\ \ P' + \beta N\Psi \tag{3.69}$$

and

$$0 = \beta N \Theta P' + N\Psi. \tag{3.70}$$

Considering $P' \neq 0$, and combining (3.69) and (3.70) produces the equation

$$1/\Theta = \lambda_c + 2\mu/3 - \beta^2 N. \qquad (3.71)$$

When the result (3.71) is combined with that of Test 3.5.2, α can be expressed in terms of two measured compressibilities

$$\beta = 1 - \frac{\Delta}{\Theta} . \qquad (3.72)$$

The combination of (3.71) and the results of Test 3.5.3 yields the relation

$$1/\Theta + \beta^2 N = 1/\kappa + \alpha^2 M \qquad (3.73)$$

which can be shown to be another form of (3.62).

Considering the definition of Ψ for a uniform prorsity, (3.12), and (3.42) can produce the expression

$$\begin{aligned} \Psi &= f(u_{i,i} - Q_{i,i}) = \beta e + p_f/N \\ &= f(e_s - e_f) \end{aligned} \qquad (3.74)$$

where e_s and e_f are the dilitation of the solid and the liquid regions, respectively. If the liquid is incompressible, i.e. $e_f = 0$, then

$$\beta = f \frac{e_s}{e} - \frac{p_f}{N}. \qquad (3.75)$$

It can be seen than $p_f/N \geq 0$ and $e_s/e \leq 1$, therefore

$\beta \geq f$.

The value of Δ could be much smaller than the value of Θ for an incompressible liquid and a highly compressed gas, where the liquid is allowed to escape, leaving the solid and gas portions only to support the load. This implies that β is very close to unity for a soft solid material and air.

3.5.5 Jacketed Compressibility Test - No Liquid Flow, No Gas Flow

Consider a jacketed specimen subjected to a hydrostatic pressure P' such that

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -P' \qquad\qquad \zeta = \Psi = 0.$$

Define the compressibility under these test conditions as

$$\omega = - e/P'. \tag{3.76}$$

Substituting these conditions into (3.18),

$$-P' = - (\lambda_c + 2\mu/3)\omega P'. \tag{3.77}$$

Considering $P' \neq 0$, implies

$$\lambda_c = \frac{1}{\omega} - \frac{2\mu}{3} \tag{3.78a}$$

$$\frac{1}{\omega} = \lambda_c + 2\mu. \tag{3.78b}$$

Combining (3.66) and (3.71) with (3.78b) gives

$$\frac{1}{\omega} = \frac{1}{\Theta} + \beta^2 N \tag{3.79}$$

$$= \frac{1}{\kappa} + \alpha^2 M.$$

M and N can now be solved for in terms of measured compressibilities;

$$M = \left(\frac{1}{\omega} - \frac{1}{\kappa}\right)\frac{1}{\alpha^2} = \left(\frac{1}{\omega} - \frac{1}{\kappa}\right)\left(1 - \frac{\delta}{\kappa}\right)^{-2} \tag{3.80}$$

$$N = \left(\frac{1}{\omega} - \frac{1}{\Theta}\right)\frac{1}{\alpha^2} = \left(\frac{1}{\omega} - \frac{1}{\Theta}\right)\left(1 - \frac{\delta}{\kappa}\right)^{-2}. \tag{3.81}$$

3.5.6 Alternate Procedure for Determination of M

Consider an unjacketed specimen subjected to a hydrostatic pressure P' such that

$$p_g = P' \qquad\qquad \Psi = 0.$$

Define the compressibility under these conditions as

$$\eta = -\frac{e}{P'} . \tag{3.82}$$

Also define a second gas flow related parameter

$$\gamma_1 = \zeta / P'. \tag{3.83}$$

Equations (3.18) can be used to obtain

$$P' = \alpha M \eta P' + M \gamma_1 P'.$$

For $P' \neq 0$, the above relation shows that M can be expressed as

$$M = \frac{1}{\alpha\eta + \gamma_1} . \qquad (3.84)$$

Biot and Willis (1957) describe a possible experimental procedure for determination of γ_1. A unit volume of porous material is placed within a closed chamber. Gas is injected into the chamber under pressure and the volume of injected fluid is measured. The volume of gas injected per unit pressure will be the sum of the solid-liquid compressibility ϕ, the volume of the gas which has entered the pores γ_1, and a fixed quantity describing the elastic properties of the chamber and the gas. The differences between the volumes injected with and without the porous material in the chamber will be given by

$$\Delta V = \phi + \gamma_1 - C_g \qquad (3.85)$$

where C_g is the gas compressibility.

Measurement of the unjacketed compressibility η, and knowledge of the gas compressibility and the gas porosity provide an alternative method for determination of the coefficient M. During the unjacketed test, gas will flow in and out such that the gas pore space and the solid-liquid matrix must undergo the same strain for linearly elastic media. Therefore, the porosity of the gas, g, will not undergo any strain. The dilitation of the gas can be given by

$$e_g = -C_g P' = U_{i,i} \tag{3.86}$$

where C_g is the compressibility of the gas. The dilitation of the solid-liquid region can be expressed as the sum of the solid dilitation plus the liquid dilitation. If the fluid is considered incompressible and the relative flow, Ψ, is zero, the total dilitation of the solid-liquid region is made up of the dilitation of the solid alone, i.e.

$$e_s = -\eta P' = u_{i,i}.$$

γ_1 can now be expressed in terms of compressibilities as

$$\gamma_1 = \frac{\zeta}{P'} = \frac{g(u_{i,i} - U_{i,i})}{P'} = g(C_g - \eta). \tag{3.87}$$

3.5.7 Alternate Procedure for Determination of N

Consider a jacketed specimen subjected to a hydrostatic pressure such that

$$p_f = P' \qquad \tau_{xx} = \tau_{yy} = \tau_{zz} = -P' \qquad\qquad \zeta = 0.$$

As in second test, define the compressibility under these conditions as

$$\Delta = -\frac{e}{P'}. \tag{3.88}$$

Also define a second liquid flow parameter

$$\gamma_2 = \Psi/P'. \tag{3.89}$$

Equations (3.18) can be used to obtain

$$P' = \beta N \Delta P' + \gamma_2 N P'. \tag{3.90}$$

Considering $P' \neq 0$, implies

$$N = \frac{1}{\beta\Delta + \gamma_2} . \tag{3.91}$$

During the jacketed test with p_f equal to the applied pressure, the liquid will flow in and out such that the fluid pore space and the solid-gas matrix must undergo the same strain for a linearly elastic media. Therefore, the porosity, f, will not undergo any strain. If the liquid is considered incompressible, $e_f = 0$, the sum of the solid and gas dilitation must be zero, i.e.

$$e_s = -e_g = -C_g(\Delta p_g). \tag{3.92}$$

Obtaining from (3.18) Δp_g for a change in P' when $\zeta = 0$, e_g can be expressed as

$$e_g = \alpha C_g M e = -\alpha C_g M \Delta P'. \tag{3.93}$$

Using definition (3.12) for Ψ,

$$\gamma_2 = \frac{\Psi}{P'} = \frac{f e_s}{P'} = -\alpha C_g f M \Delta. \tag{3.94}$$

N can now be expressed in terms of the gas compressibility and liquid porosity

$$N = \frac{1}{\Delta(\beta - \alpha C_g f M)} \quad .$$

3.6 Closure

Equations (3.35),

$$\tau_{ij} + \delta_{ij}(\alpha p_g + \beta p_f) = \delta_{ij}\lambda e + 2\mu\, e_{ij}$$

$$\zeta = \alpha e + p_g/M \qquad (3.35)$$

$$\Psi = \beta e + p_f/N,$$

are similar to the conventional Hooke's law for a material made up of gas, solid, and liquid. The equations include the influence on the continuum of the two fluids which are not in the conventional Hooke's law formulation. Four additional material constants, α, β, M, and N, are necessary for complete use of the formulation. The pressure of the gas can be considered zero under most naturally occurring conditions; therefore, only one parameter, β, has a significant effect on the stress values.

Writing the set of equations in matrix form,

$$
\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \\ \zeta \\ \psi \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & \alpha & \beta \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & \alpha & \beta \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ \alpha & \alpha & \alpha & 0 & 0 & 0 & -1/M & 0 \\ \beta & \beta & \beta & 0 & 0 & 0 & 0 & -1/M \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \\ -p_g \\ -p_f \end{bmatrix}
$$

allows one to see they can be used to replace the conventional relations for development of a finite element model. Associating relative flows with the stress vector and pressures with the strain vector is the most convenient formulation.

It is important to note several of the advantages of the present approach. With this approach, it is not necessary to describe the structure of individual cells or the type of interaction which occurs between cells. The model assumes only homogeneity of action in a bulk material sense, not uniformity of cell shape and composition. Measurement of material properties is made on bulk materials not generalized from the action of individual cells. In addition, neither the gas or liquid porosity needs to be determined using this formulation although the effects of both fluids are included in the model.

The governing equations for the transient phenomena, i.e., the equations governing the medium with fluids in motion, were developed by introduction of Darcy's law of flows for the gas and the liquid (see Equation (3.53)). Additional material properties are necessary for these equations including the fluid porosity terms and terms describing the properties of the fluids. These equations imply that the fluid flows and elastic action are not merely superimposed, but are coupled together. Assuming incompressibility of all three materials would uncouple the equations involved in describing the action of the body.

IV. EXPERIMENTAL DETERMINATION OF COEFFICIENTS α AND M FOR APPLE FLESH

A device for applying a hydrostatic pressure to a cylindrical specimen of apple flesh was used to determine the three bulk compressibilities, δ, κ, and ω, as defined in (3.54), (3.63), and (3.76). The change in volume (volumetric strain) of a cylindrical specimen was related to the applied hydrostatic stress while using different values for the gas pressure, p_g. The liquid flow was assumed to be zero in all cases. Equations (3.67) and (3.80) were used to calculate α and M after the three compressibilities were determined. Tests simulating the hypothetical cases discussed in Sections 3.5.2 and 3.5.4 were not attempted because of the difficulty involved in measuring and controlling turgor pressure, p_f, within the tissue.

4.1 Equipment and Procedure

A device developed by Finney (1963) and used by Brusewitz (1969) was used in this work. The pressure chamber was modified to allow the hydrostatic pressure outside the specimen to vary independently of the air pressure within the specimen. The equipment is shown in Figure 4.1 and schematically in Figure 4.2.

A carefully cut cylindrical sample 3.81 cm (1.5 in) long with 2.54 cm (1.0 in) diameter was placed on the sample stand and then covered with a highly flexible membrane. The membrane was sealed around the edge of the stand, thereby prohibiting water from entering the sample. Air pressure and air flow for the sample were controlled through the sample stand. Water was used to apply the external pressure of 137.9 kPa (20 psi). The change in water level within a precision bored glass column was used to measure the change in volume within the system. The column had an inside diameter of 5.00 $\pm$ 0.01 mm (0.1968 $\pm$ 0.0004 in). Volume change could be measured with a sensitivity of $\pm$ 0.00982 cm^3 (0.006 in^3). All water was boiled to remove the air and then allowed to return to room temperature before each test. It was necessary to remove the water from the system between the calibration run and the actual test to insert the test sample.

A calibration curve was determined before each test to account for system characteristics. A steel cylinder was used as the sample during calibration. The system was pressurized to 171.9 kPa (25 psi) before each test to check for leaks.

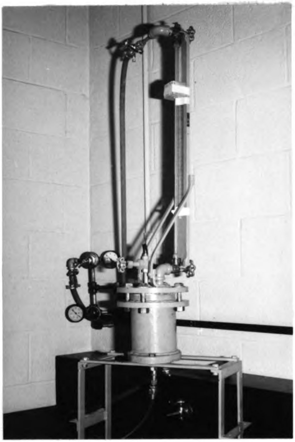

Figure 4.1. Equipment for Measurement of Bulk Compressibility

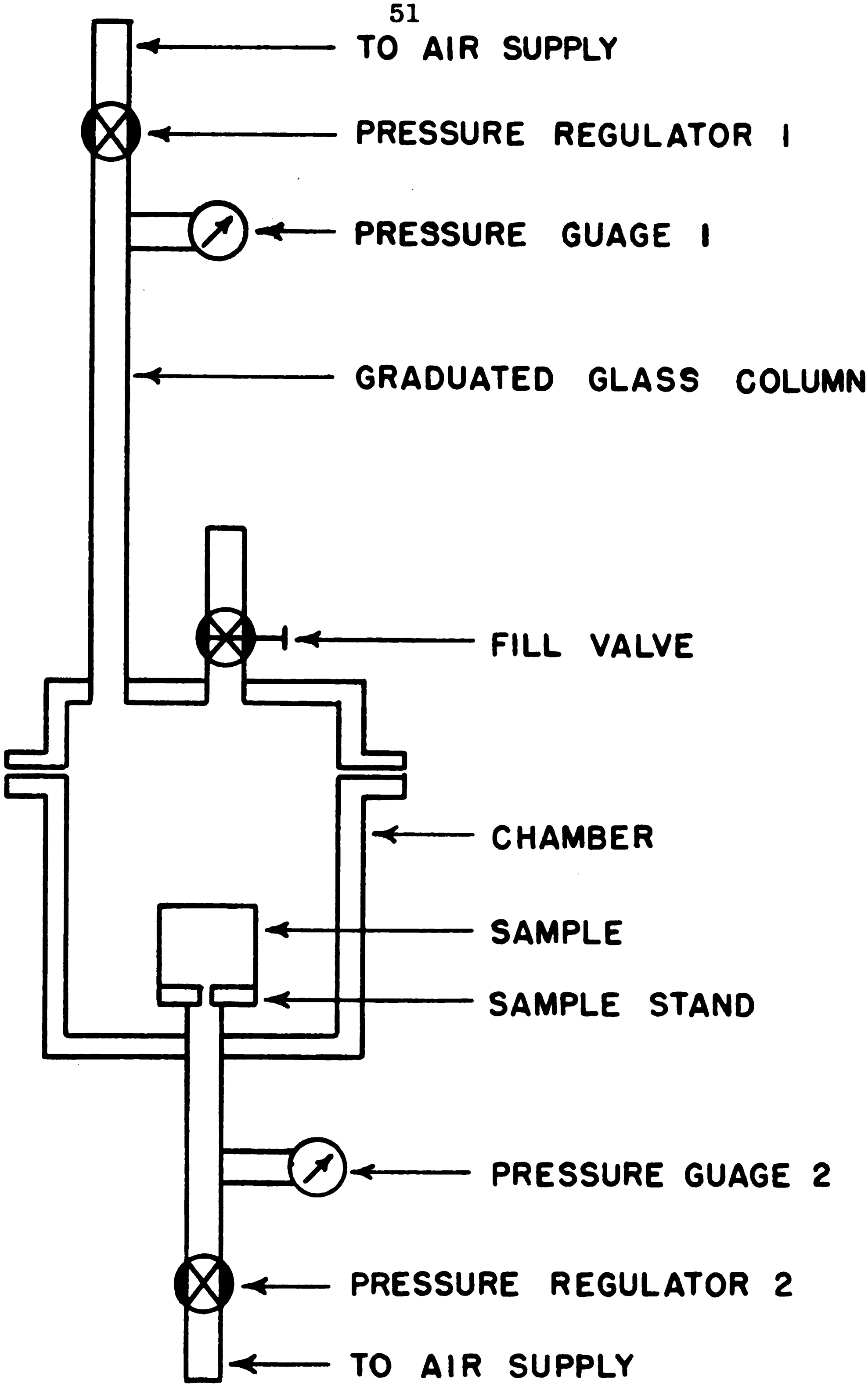

Figure 4.2. Bulk Compression Apparatus

Pressure regulator two (Figure 4.2) was removed for determination of κ making the air pressure within the sample equal to zero gauge pressure. A plug was inserted in the sample stand for determination of ω. The plug prohibited air from flowing out of the sample.

Pressure regulator two (Figure 4.2) was used to vary the internal pressure of the sample for determination of δ. The internal pressure was applied after the external hydrostatic pressure had been applied. Readings were taken for internal pressures of 34.5, 51.7, 68.8, 86.2, and 103.4 kPa (5, 7.5, 10, 12.5, and 15 psi).

The two coefficients, α and M, were calculated once the three compressibilities had been determined. Alpha was plotted as a function of the gas pressure using (3.67)

$$\alpha = a(1 - \delta/\kappa)$$

where a is the ratio of the gas to the applied hydrostatic pressure. The value of α at $p_g = 0$ was determined by projecting the regression line through the vertical axis. M was calculated using (3.81)

$$M = \left(\frac{1}{\omega} - \frac{1}{\kappa}\right)\frac{1}{\alpha^2}$$

using α at $p_g = 0$.

4.2 Results of Experimentation

The equipment was found to be unsatisfactory for the determination of the coefficients α and M. The three major problem areas were repeatability of calibration, resolution, and control of applied pressures. The use of the sample stand and membrane to enclose the sample appears to be a satisfactory technique, but the construction of the pressure chamber has some undesirable characteristics. Consistent measurable differences between the two compressibilities κ and ω were found for the several varieties of apples tested. In all cases, κ was found to be larger than ω. Unsatisfactory repeatability of calibration made determination of the absolute magnitude of the difference unreliable. The coefficient α appeared to be a nearly linear increasing function of the gas pressure, p_g.

Calibration of the system was necessary due to the expansion of the equipment and compression of the fluid. The calibration indicated that the system deformation was of the same order of magnitude as that to be measured in the deformed sample. It would be necessary to reduce the volume change of the apparatus and reduce the variability of the volume change for reliable results.

4.3 Discussion of Apparatus Limitations

The unsatisfactory calibration repeatability of the present system stems from several sources. The large expansion of the apparatus was a major source of error. This expansion was mainly due to the method of attaching the top portion of the apparatus. A rubber gasket, 0.318 cm (0.125 in) thick, was compressed between the upper and lower portions by tightening four bolts. The inability to maintain constant bolt tension created variation in calibration which could not be corrected. The need to remove the water to change samples was a source of error. This disturbance may have created significant variability in the compressibility of the water.

Readings for the fluid level in the glass column could be made to $\pm$ 0.05 mm (0.002 in). That is a volume change of $\pm$ 0.00982 cm^3 (0.0006 in^3). A resolution of $\pm$ 0.005 cm^3 (0.0003 in^3) or less would be necessary to produce satisfactory results. This resolution is equivalent to a diameter of 3 mm (0.118 in) or less. This size of column would be impractical unless the variation of the apparatus itself is reduced.

The system was noted to be very sensitive to changes in room temperature during operation. A change of $1^{\circ}C$ could produce a water volume change of approximately 0.2557 cm^3 (0.00867 in^3) for the amount of water in the present system.

Standard gas pressure regulators were used to control pressures. Difficulty was experienced in holding stable readings with these regulators. Variation in pressure with time at one regulator setting also contributed to the difficulties encountered in determing α and M.

Suggestions for an improved apparatus have been included in Section IV, Suggestions for Further Study.

V. FINITE ELEMENT FORMULATION

The finite element method is a numerical procedure for solving differential equations and can be used in conjunction with the constitutive equations developed for a gas-solid-liquid medium to calculate stresses in a body of arbitrary shape. The element equations can be derived by minimizing the potential energy of the system. The assemblage of the element equations yields a system of algebraic equations which are solved for the desired quantities. A detailed discussion of the general theory of the finite element method is given in Zienkiewicz (1971), Oden (1972), Desai and Abel (1972), and Martin and Carey (1973).

The region under consideration is divided into small elements connected at node points for the finite element formulation. The unknowns, stresses, gas pressure, and liquid pressure, are approximated over each element by polynomials. The polynomials for the simplex triangular axisymmetric element (Figure 5.1) are

$$u = \alpha_1 + \alpha_2 r + \alpha_3 Z$$

$$v = \beta_1 + \beta_2 r + \beta_3 Z \qquad (5.1)$$

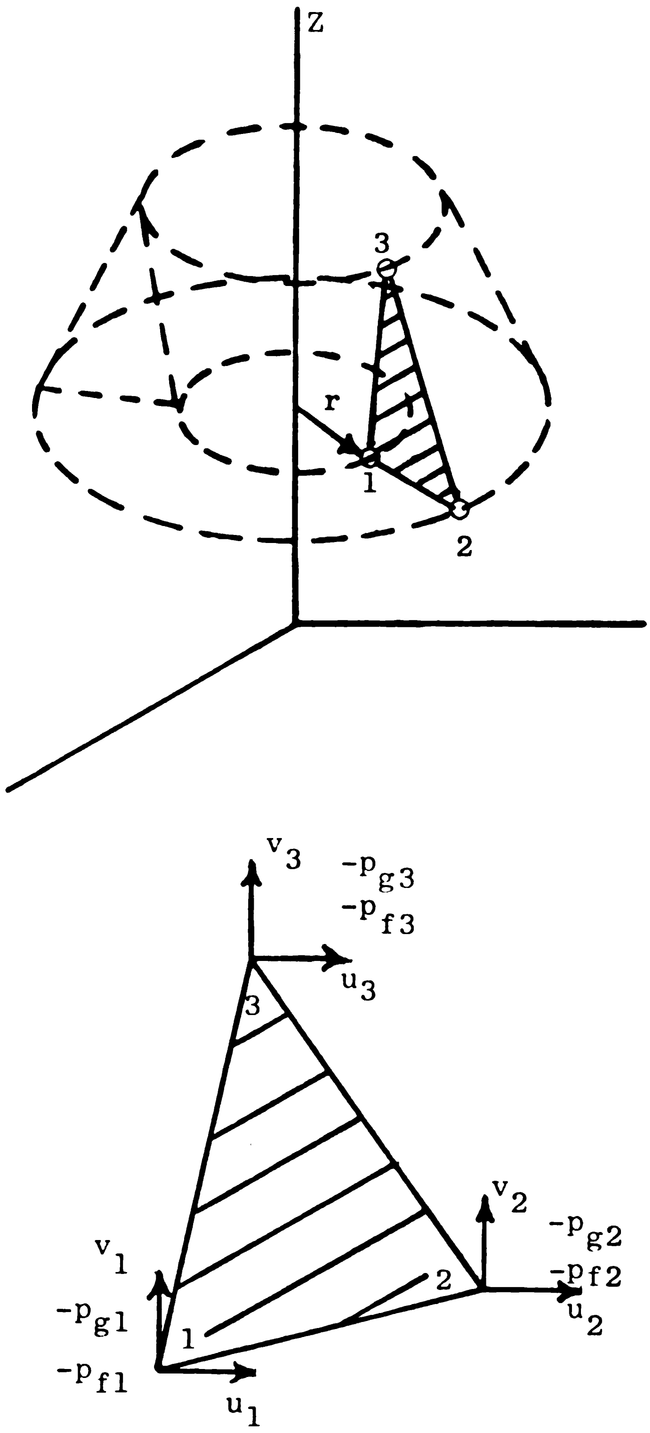

Figure 5.1. Simplex Triangular Axisymmetric Element

$$-p_g = \phi_1 + \phi_2 r + \phi_3 z$$
$$-p_f = \delta_1 + \delta_2 r + \delta_3 z. \qquad (5.1)$$

Solving the equations for the coefficients using the nodal values of u, v, $-p_g$, and $-p_f$ establishes the interpolating polynomials for the region. Interpolating polynomials are expressed as a product of shape functions, N_i's, and the nodal values. In the above case

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$
$$-p_g = N_1(-p_{g1}) + N_2(-p_{g2}) + N_3(-p_{g3}) \qquad (5.2)$$
$$-p_f = N_1(-p_{f1}) + N_2(-p_{f2}) + N_3(-p_{f3})$$

with shape functions

$$N_1 = (a_1 + b_1 r + c_1 z)/2A$$
$$N_2 = (a_2 + b_2 r + c_2 z)/2A$$
$$N_3 = (a_3 + b_3 r + c_3 z)/2A$$

where

$$a_1 = r_2 z_3 - r_3 z_2$$
$$b_1 = z_2 - z_3$$
$$c_1 = r_3 - r_2$$

etc., in cyclic order,

and A is the area of the element.

More complicated shape functions can be derived for polynomials involving higher-order terms.

5.1 Development of the Variational Equations

Variational equations for the present theory were developed following the procedure as outlined by Segerlind (1974) for solid mechanics. Example finite element equations in the following development are for the axisymmetric case. The column vector of nodal values, $\{U\}$[2], consists of the nodal displacements, and pressure terms $-p_f$ and $-p_g$ for the gas-solid-liquid theory. This column vector is

$$\{U\}^T = [u_1, v_1, -p_{g_1}, p_{f_1}, \ . \ . \ . \ , u_n, v_n, -p_{g_n}, -p_{f_n}] \tag{5.3}$$

where n is the number of nodes and u and v the nodal displacements parallel to the R and Z axes, respectively.

The total potential energy of the system can be expressed as

$$\Pi = SE - WL \tag{5.4}$$

---

[2] Standard matrix notation common to finite element work is used in the following development: { }'s denote a column vector, [ ] a matrix, superscript T denotes the transpose of the matrix.

where SE is the strain energy and WL the work done by the internal and applied loads (Segerlind, 1974). For a body subdivided into a number of elements, the expression for the total potential can be written in summation form

$$= \sum_{e=1}^{E} (SE_{(e)} - WL_{(e)}) \tag{5.5}$$

where E is the number of elements and $SE_{(e)}$ and $WL_{(e)}$ are the respective components in each element.

The strain energy for an arbitrary differential volume is

$$d(SE_{(e)}) = \tfrac{1}{2}\{\varepsilon\}^T\{\tau\}dV. \tag{5.6}$$

Equation (3.10) simplifies to the above form by letting

$$\begin{aligned} \{\varepsilon\}^T &= \{\varepsilon_z, \varepsilon_r, \varepsilon_\Theta, \gamma_{rz}, -p_g, -p_f\} \\ \{\tau\}^T &= \{\tau_{zz}, \tau_{rr}, \tau_{\Theta\Theta}, \tau_{rz}, \zeta, \Psi\}. \end{aligned} \tag{5.7}$$

Assuming that each force and the corresponding displacement are linear, the total strain energy is obtained by integrating over the volume of the body giving

$$\begin{aligned} SE_{(e)} &= \int_{V_{(e)}} \tfrac{1}{2}\{\varepsilon\}^T\{\tau\}dV \\ &= \int_{V_{(e)}} \tfrac{1}{2}\{U\}^T[B]^T[D][B]\{U\}dV. \end{aligned} \tag{5.8}$$

The material properties matrix, [D], is defined in $\{\tau\} = [D]\{\varepsilon\}$. This matrix is

$$[D] = \begin{bmatrix} R & \frac{R\mu}{1-\mu} & \frac{R\mu}{1-\mu} & 0 & \alpha & \beta \\ & R & \frac{R\mu}{1-\mu} & 0 & \alpha & \beta \\ & & R & 0 & \alpha & \beta \\ \text{symmetric} & & & \frac{R(1-2\mu)}{2(1-\mu)} & 0 & 0 \\ & & & & -1/M & 0 \\ & & & & & -1/N \end{bmatrix} \tag{5.9}$$

with

$$R = \frac{E\,(1-\mu)}{(1+\mu)(1-2\mu)}$$

The B matrix is defined by the relation $\{\varepsilon\} = [B]\{U\}$ and contains shape functions as well as their derivatives. The general form of [B] is

$$[B] = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial z} & 0 & 0 & 0 & \frac{\partial N_2}{\partial z} & 0 & 0 & \cdots & 0 & \frac{\partial N_n}{\partial z} & 0 & 0 \\ \frac{\partial N_1}{\partial r} & 0 & 0 & 0 & \frac{\partial N_2}{\partial r} & 0 & 0 & 0 & \cdots & \frac{\partial N_n}{\partial r} & 0 & 0 & 0 \\ \frac{N_1}{r} & 0 & 0 & 0 & \frac{N_2}{r} & 0 & 0 & 0 & \cdots & \frac{N_n}{r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & 0 & 0 & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & 0 & 0 & \cdots & \frac{\partial N_n}{\partial z} & \frac{\partial N_n}{\partial r} & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & 0 & N_2 & 0 & \cdots & 0 & 0 & N_n & 0 \\ 0 & 0 & 0 & N_1 & 0 & 0 & 0 & N_2 & \cdots & 0 & 0 & 0 & N_n \end{bmatrix} \quad (5.10)$$

where n is the number of nodes.

The work done by the applied loads can be separated into four components:

1. Work due to concentrated loads,
2. Work resulting from the stress components acting on the outside surface,
3. Work due to movement of liquid across the outside boundary, and
4. Work due to movement of gas across the outside boundary.

The work due to body forces will be neglected in this analysis.

The work done by a concentrated force is the value of the force multiplied by the distance through which it acts. Denoting the nodal forces as

$$\{P\}^T = \{P_{r1}, P_{z1}, 0, 0, \ldots, P_{rn}, P_{zn}, 0, 0\} \quad (5.11)$$

the product becomes

$$W_c = \{U\}^T\{P\} = \{P\}^T\{U\}. \quad (5.12)$$

The zeros placed in the $\{P\}$ matrix make the relation compatible with the displacement matrix as defined in relation (5.3).

Work done by the stress components on the surface is described by

$$W_q = \int_{S_1} (uq_r + vq_z)dS_1 \quad (5.13)$$

where u and v are the displacements and q_r and q_z are the stress components parallel to the coordinate axes. Equation (5.13) can also be written as

$$W_q = 2\kappa \int_{\Delta} \{U\}^T [M_1]^T \begin{Bmatrix} q_r \\ q_z \end{Bmatrix} dL_1. \quad (5.14)$$

The $[M_i]^T$'s are used to denote the shape function matrices.

The work done by the flow of the liquid across the boundary can be expressed using (3.6) and (3.5)

$$W_f = \int_{S_2} v\, p_f dS_2 = \int_{S_2} [M_2]\{U\} v dS_2. \quad (5.15)$$

A similar expression for the gas flow is

$$W_g = \int_{S_3} wp_g dS_3 = \int_{S_3} [M_3]\{U\}w dS_3. \tag{5.16}$$

The total potential energy is

$$\begin{aligned}\Pi = \sum_{e=1}^{E} \int_{V_{(e)}} &(\tfrac{1}{2}\{U\}^T[B]^T[D][B]\{U\})dV_{(e)} \\ -2\pi \int_{\Delta_1} \{U\}^T[M_1]^T \begin{Bmatrix} q_r \\ q_z \end{Bmatrix} r dL_1 &- 2\pi \int_{\Delta_2} \{U\}^T[M_2]^T\{v_{(e)}\} r dL_2 \\ -2\pi \int_{\Delta_3} \{U\}^T[M_3]^T\{w_{(e)}\} r dL_3 &- \{U\}^T\{P\}.\end{aligned} \tag{5.17}$$

Minimization of the energy expression produces

$$\sum_{e=1}^{E} ([k_{(e)}]\{U\} - \{f_{(e)}\}) = \{P\} \tag{5.18}$$

where the element stiffness matrix is

$$[k_{(e)}] = \int_{V_{(e)}} [B]^T[D][B]dV_{(e)}$$

and the element force matrix is

$$\begin{aligned}\{f_{(e)}\} = 2\pi (&\int_{\Delta_1} [M_1]^T \begin{Bmatrix} q_r \\ q_z \end{Bmatrix} dL_1 \\ &+ \int_{\Delta_2} [M_2]^T\{v_{(e)}\}dL_2 \\ &+ \int_{\Delta_3} [M_3]^T\{w_{(e)}\}dL_3) .\end{aligned}$$

5.2 Implementation of the Method

An existing axisymmetric finite element computer program and associated subroutines were modified for use with the present theory. Modifications included incorporating the new constitutive equations and allowing for four unknowns at each node. This program utilized isoparametric elements. Figure 5.2 outlines the structure of the computer solution. The negative values on the main diagonal of the material properties matrix, $[D]$, produced negative terms on the diagonal of the stiffness matrix not found in the conventional formulation. A subroutine, SIGN, was used to change the sign of a negative terms on the main diagonal of the stiffness matrix before solving the system of equations.

A spherical body with a radius of 3.33 cm (1.31 in) was selected for study. The symmetry of the sphere and a load about the vertical axis made it necessary to consider only one quadrant. The body was subdivided into 22 elements. Figure 5.3, using quadratic quadrilaterals (8 nodes/element) and quadratic triangles (6 nodes/element) to fit the curved surfaces of the body and approximate the unknowns within each element. A layer of uniformly thick elements, 0.1524 cm (0.06 in), was placed at the surface to represent the skin of a fruit. Smaller elements

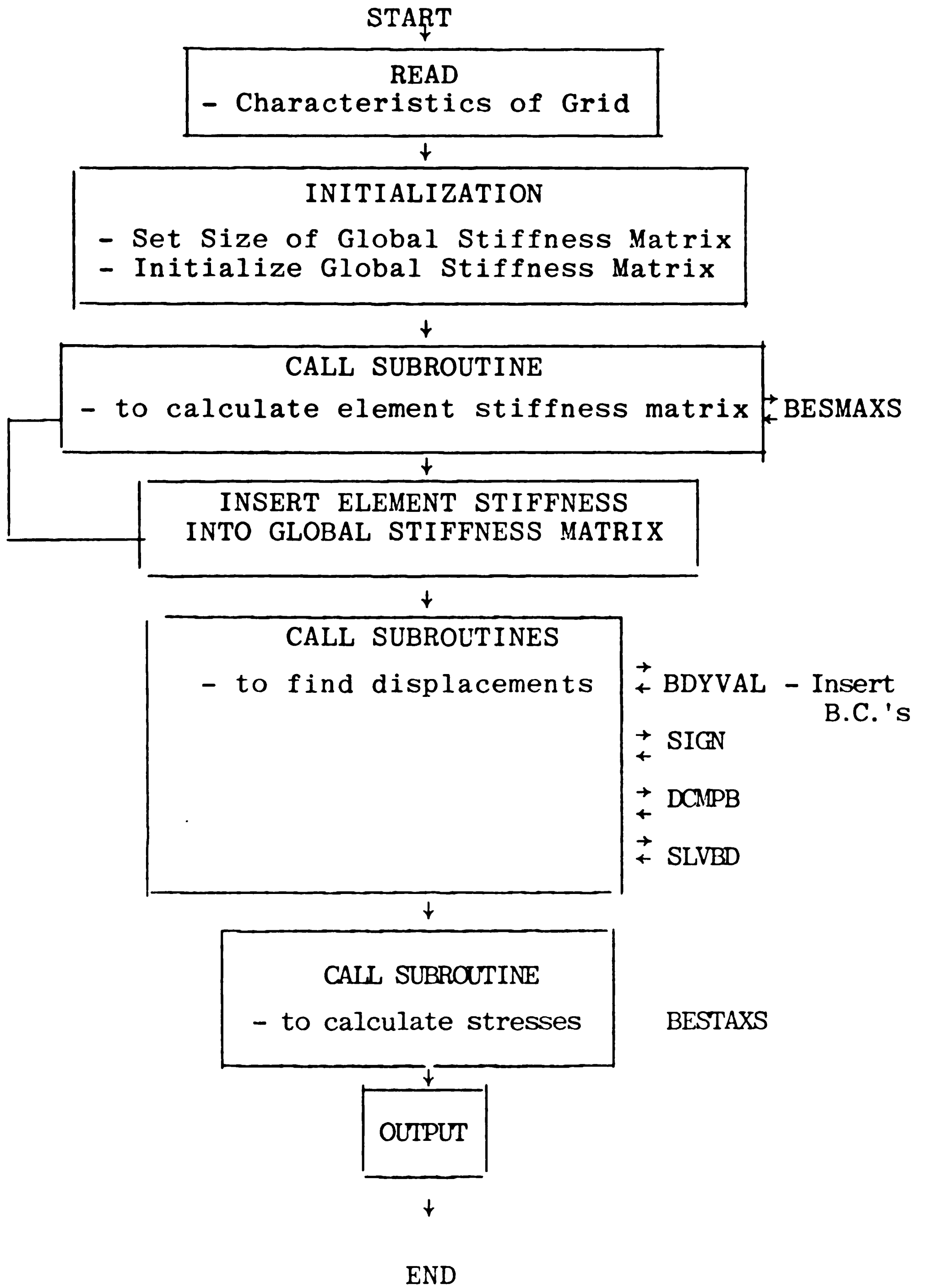

Figure 5.2. Outline of Finite Element Computer Program

No. of elements = 22

No. of Nodes = 83

Degrees of Freedom = 332

Width of Band Matrix = 68

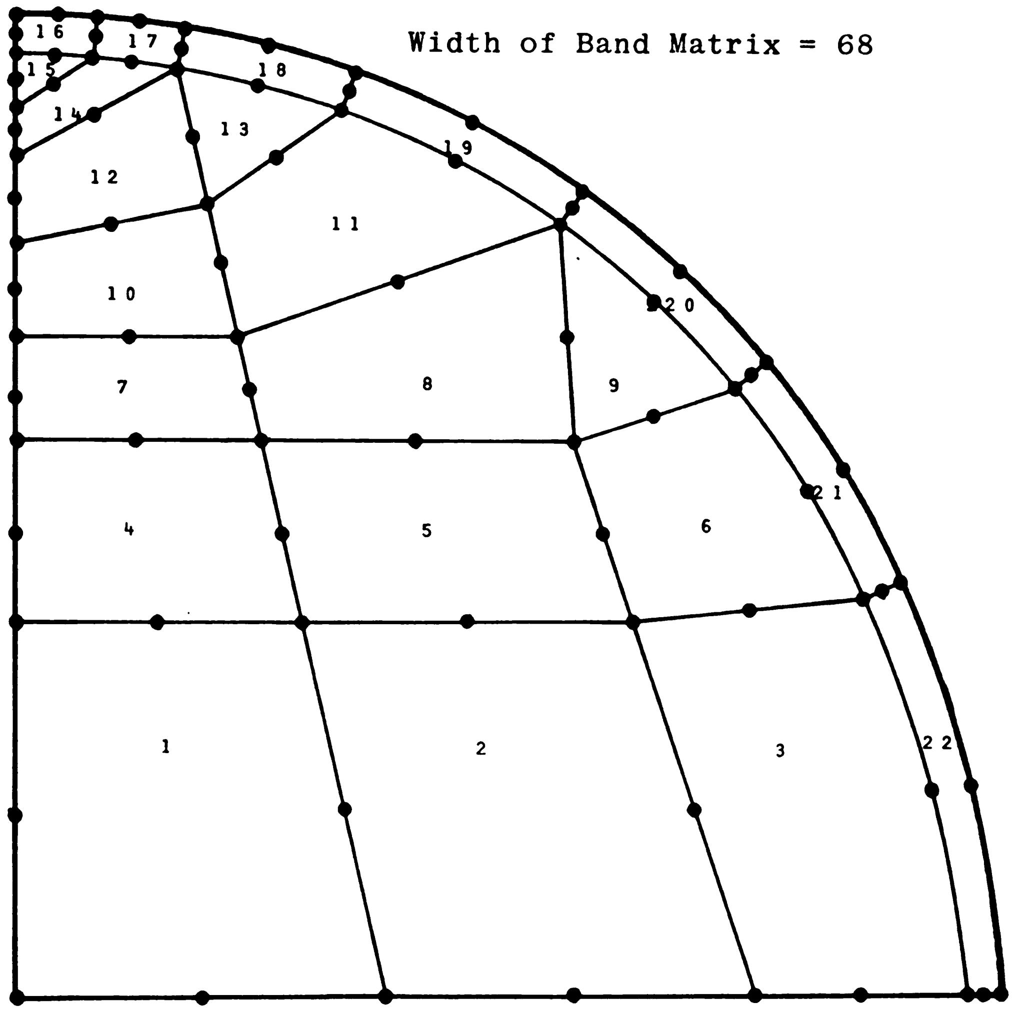

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Figure 5.3. Grid for Finite Element Application

were used at the top to increase the accuracy since the load was applied parallel to the z-axis.

Determination of the element matrix for an isoparametric element requires numerical integration. Nine integration points were used for each quadrilateral and seven for each triangle. Location of the integration points and weighting factors can be found in Zienkiewicz (1971) or Segerlind (1974).

Material properties (Mohsenin, 1970) were chosen to simulate the properties of an apple. An elastic modulus of 13790 kPa (2000 psi) and a Poisson's ratio of 0.30 were used for skin. An elastic modulus of 5171 kPa (750 psi) was used for the parenchyma (flesh). These values for the skin and parenchyma were used in all the numerical problems. Poisson's ratio of the parenchyma was varied to determine its influence on the stress distribution. The following values were used for the other material properties:

Parenchyma

$\alpha = 0.1$	$M = 6895$ kPa (1000 psi)
$\beta = 0.95$	$N = 3.10$ kPa (0.45 psi)

Skin

$\alpha = 0.0$	$M = 6895$ kPa (1000 psi)
$\beta = 0.0$	$N = 3.10$ kPa (0.45 psi).

The values of the gas pressure, $-p_g$, and the liquid pressure, $-p_f$, were assumed constant throughout the whole body. A pressure of 0 kPa was used for the gas pressure in all cases.

5.3 Results of Finite Element Analysis

Three types of numerical problems were run using the quarter sphere grid described in Section 5.2.

5.3.1 Unrestrained Sphere

The first problem simulated a body with no skin. One set of material properties was used for the entire body, which was subjected to a turgor pressure and allowed to expand freely. It was determined that a homogeneous body expands due to turgor pressure, $-p_f$, but no stresses are developed within the body. The 3.33 cm (1.31 in) radius sphere with the following material properties

$E = 5171$ kPa (750 psi)	$\mu = 0.25$
$\alpha = 0.10$	$M = 6895$ kPa (1000 psi)
$\beta = 0.95$	$N = 3.10$ kPa (0.45 psi)

enlarged by 6.34 percent in the radial direction due to a turgor pressure of 689.5 kPa (100 psi).

5.3.2 Restrained Sphere

The second problem simulated the action of an apple with no external restraints. A thin layer with a higher elastic modulus was used to represent the skin. The turgor pressure in this case created stress in both the parenchyma of the apple and the skin. The stress in the parenchyma was compressive and uniform throughout the body. This compressive stress was directly proportional to the turgor pressure for any given Poisson's ratio. The magnitudes of all stress components in both regions, however, were highly dependent on Poisson's ratio. Figure 5.4 shows the hydrostatic stress in the center of the apple for various Poisson's ratios. The turgor pressure was 2758 kPa (400 psi) in each case.

The turgor pressure created tension and shear stresses in the skin. Figure 5.5 shows the relationship between maximum principal stress and maximum shear stress in the skin as related to the Poisson's ratio of the parenchyma. Turgor pressure was 2758 kPa (400 psi) in each case.

5.3.3 Flat Plate Contact

The third numerical study determined the stress distribution within the spherical body due to compression by a flat plate perpendicular to the vertical axis of

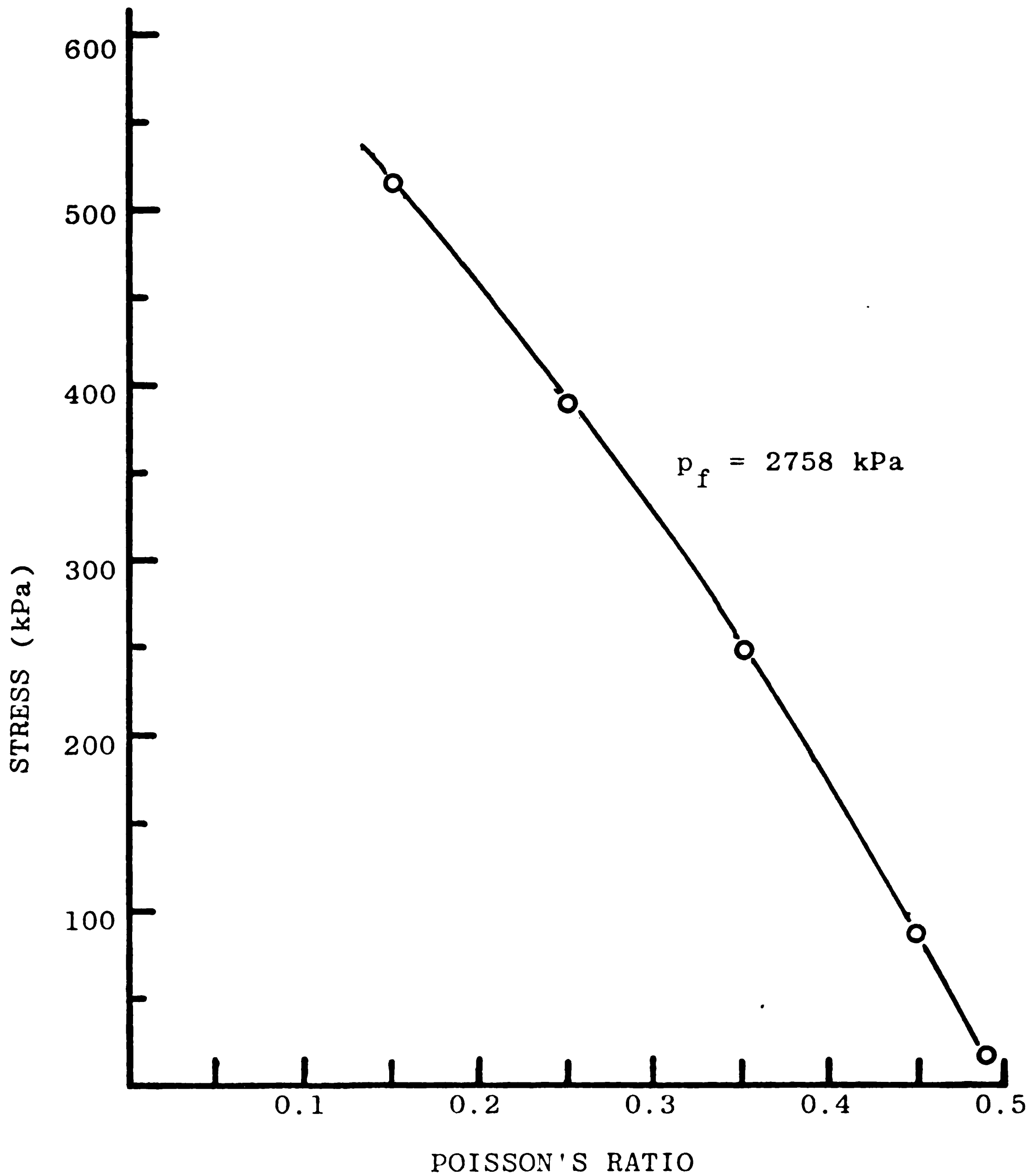

Figure 5.4. Hydrostatic Stress in Parenchyma of Body

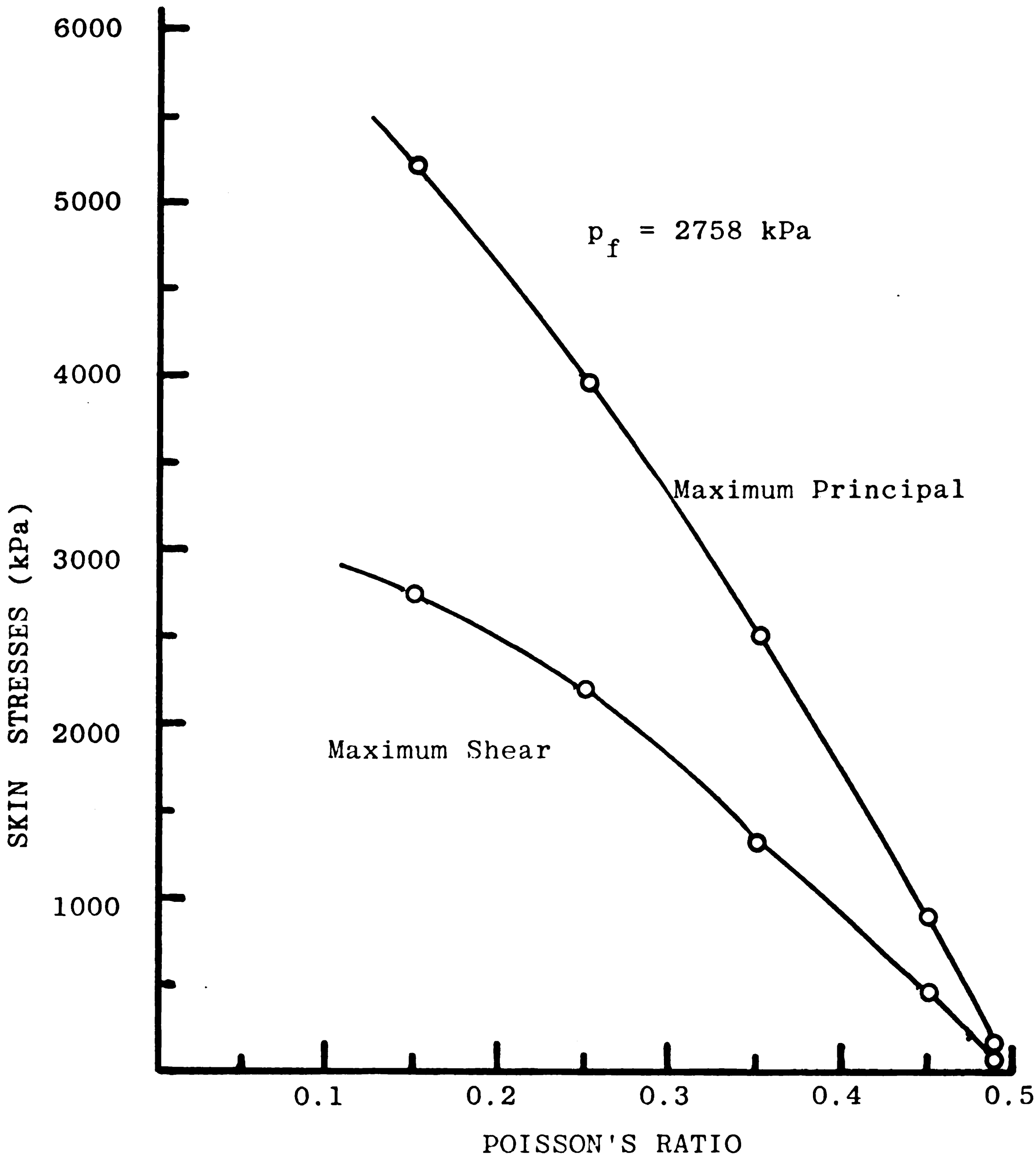

Figure 5.5. Maximum Principal Stress and Shear Stress in the Skin

symmetry. A two-step procedure was necessary to solve the problem. The expansion due to turgor pressure was determined. The set deflection was then subtracted from the expanded equilibrium state and boundary conditions input for the second step. The body was restrained to meet the flat plate deflection conditions during the second run.

The body shown in Figure 5.3 was given the following material properties:

Skin	Parenchyma
E = 13790 kPa (2000 psi)	E = 5171 kPa (750 psi)
μ = 0.30	μ = 0.35
α = 0.0	α = 0.10
β = 0.0	β = 0.95
M = 6895 kPa (1000 psi)	M = 6895 kPa (1000 psi)
N = 3.10 kPa (0.45 psi)	N = 3.10 kPa (0.45 psi)

The body with no external restraints expanded 0.3138 cm (0.1236 in) for a turgor pressure, $-p_f$, of -2068 kPa (300 psi). The parenchyma developed by a hydrostatic stress of -186.8 kPa (-27.1 psi). A maximum principal stress of 2537 kPa (368 psi) and a maximum shear stress of 1324 kPa (192 psi) developed in the center of the skin.

Expansion of the body due to turgor pressure was restrained for the second run to yield a deflection of 0.0762 cm (0.03 in) along the vertical axis of symmetry. The plane of contact to produce a flat surface at that level had a radius of slightly greater than 0.86 cm (0.34 in). The deflection developed sizable maximum shear stresses within the parenchyma, as shown in Figure 5.6. The maximum shear stress of 227.5 kPa (33 psi) appeared approximately 0.318 cm (0.125 in) below the surface along the axis of symmetry. Shear stress in the skin under the area of contact was decreased slightly on the surface and increased slightly on the inner side. Effect on the shear stresses in the skin damped out rapidly beyond the area of contact.

The principal stress of the largest absolute magnitude was a compressive stress. The stress increased in magnitude from -186.8 kPa (-27.1 psi), as it would be due to turgor pressure alone, as shown in Figure 5.7.

5.4 Closure

Fruit splitting generally does not involve external loading but rather variation of the internal conditions. The gas-solid-liquid model developed, used in conjunction with the finite element method, has been shown to be capable of describing the stress distribution within a fruit created by variation of internal conditions such as turgor pressure. Good agreement exists between the

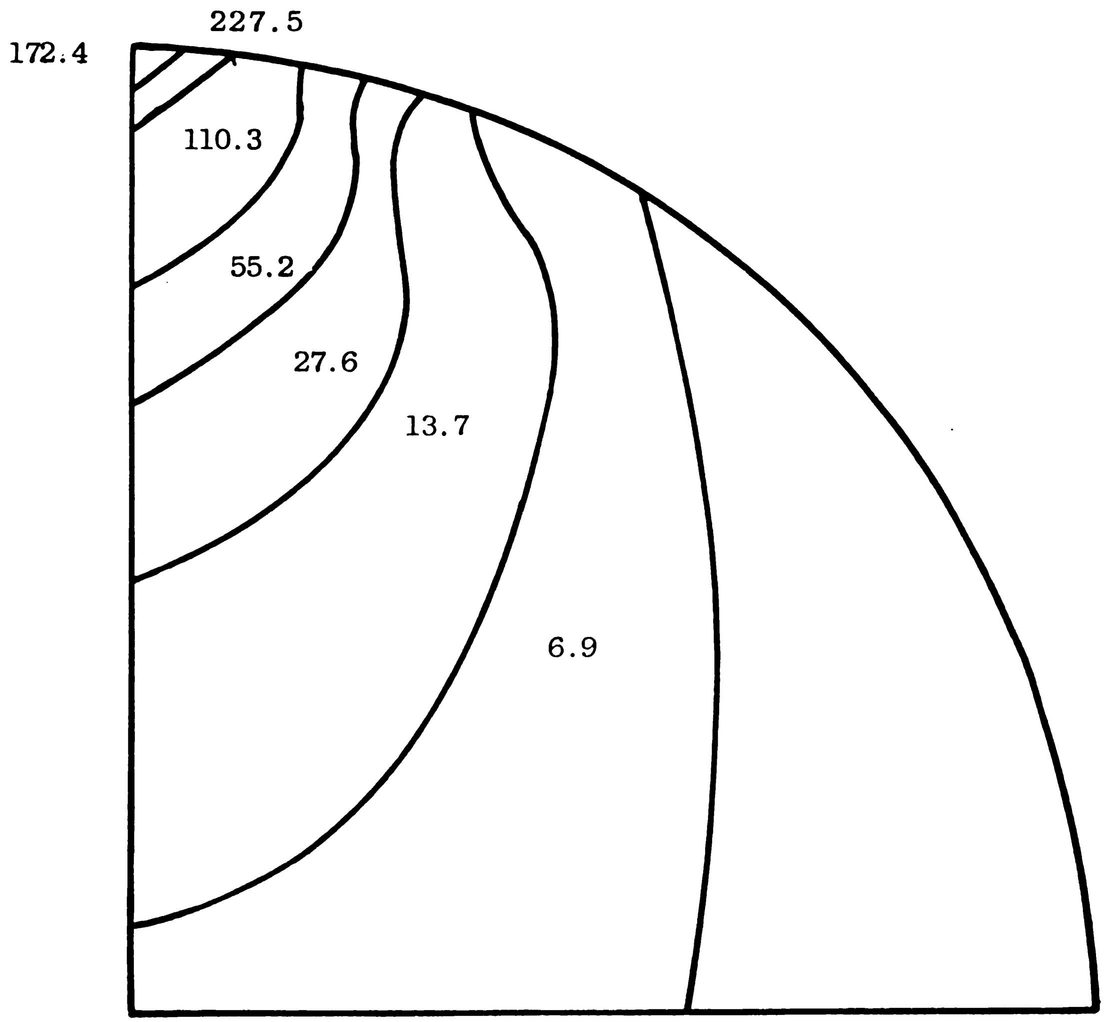

Figure 5.6. Flat Plate Contact - Distribution Curves for τ_{max} (kPa)

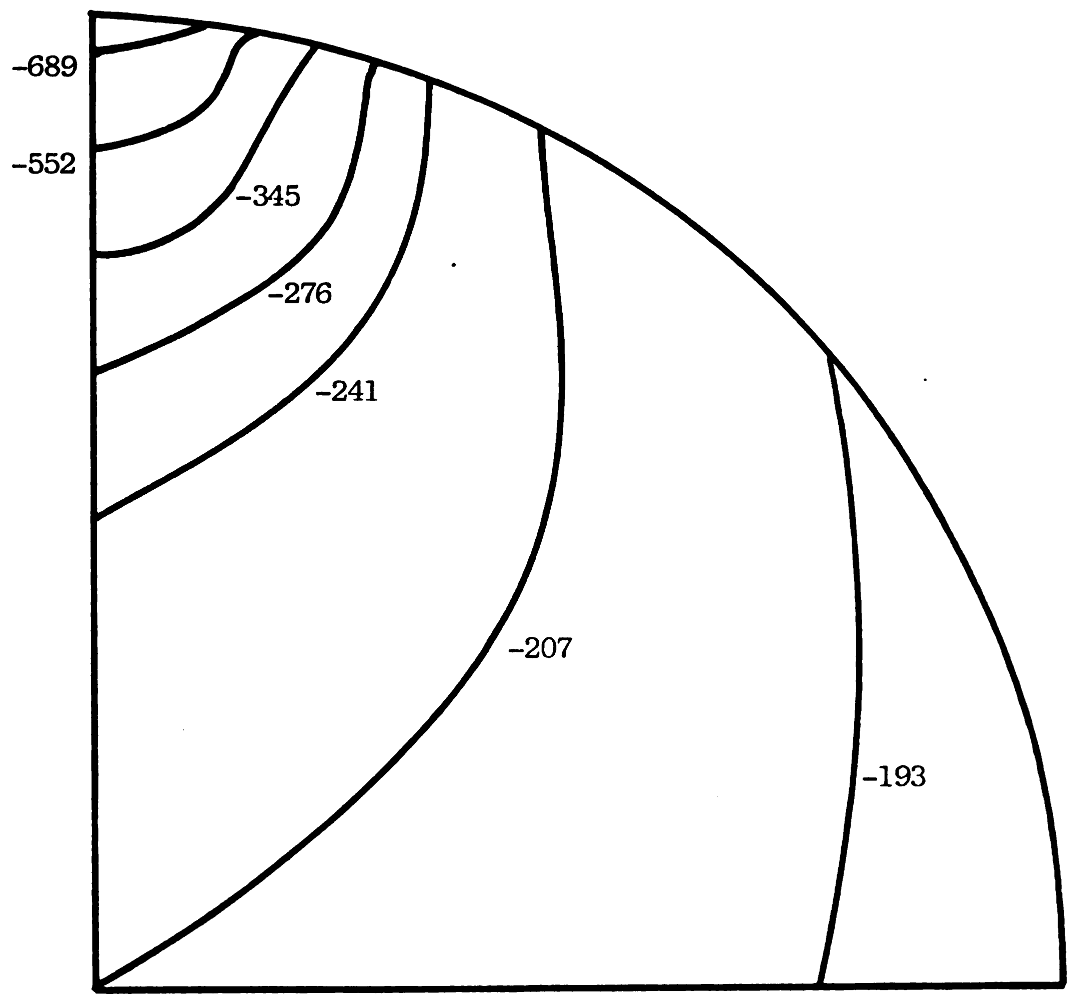

Figure 5.7. Flat Plate Contact - Distribution Curves for Principal Stress (kPa)

results of the numerical problems and the observations of Considine and Kriedeman (1972) (see page 9, Review of Literature). A fruit without skin expanded freely due to increased liquid (turgor) pressure in both cases. The restraint created by the skin produced stresses in the body with turgor pressure.

VI. SUGGESTIONS FOR FURTHER STUDY

This work has established a theory for a gas-solid-liquid medium similar to a biological material such as a fruit. The finite element method was shown applicable for solution of numerical problems. Numerous numerical problems can now be approached using the theory in conjunction with the finite element method. Several areas which should be given high priority are:

1. Study the effect of skin properties; i. e. thickness, elastic modulus, etc., on the stress within the skin and body.

2. Consider the effect of external distributed loads.

3. Consider the case where the body is subjected to large gas pressure, such as in the pressure bomb technique for measuring turgor pressure.

4. Extend the program capabilities to consider the viscoelastic material.

More accurate determination of material properties is needed before more specific use of the model can be made. Equipment needs to be developed which has satisfactory resolution for determining bulk compressibilities. A suggested design is shown in Figure 6.1 for a bulk compressibility apparatus. Pressure sources other than gas regulators should be considered. A column of heavy liquid such as mercury is one possible pressure source.

The present theory should be extended to include viscoelastic materials. Biot (1961) has used the correspondence principle to develop the viscoelastic case for a fluid-solid medium.

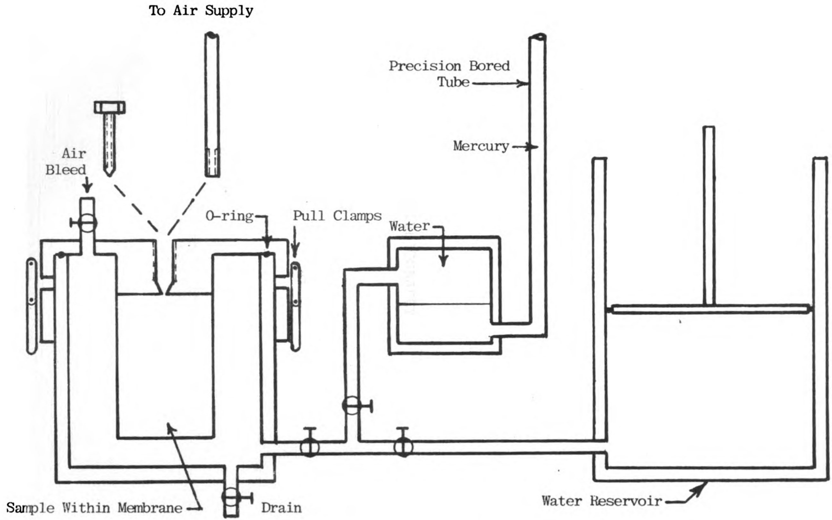

Figure 6.1. Suggested Bulk Compressibility Apparatus

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