ELASTOSTATIC STRESS ANALYSIS OF FINITE ANISOTROPIC PLATES WITH CENTRALLY LOCATED TRACTION-FREE CRACKS

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This is to certify that the

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ABSTRACT

ELASTOSTATIC STRESS ANALYSIS OF FINITE
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LOCATED TRACTION-FREE CRACKS

By

George Stephen Gyékényesi

A mapping-collocation method is developed for the elastostatic stress analysis of finite anisotropic plates with centrally located traction-free cracks. The essence of the method is as follows:

1. The crack is mapped into the unit circle.
2. The boundary conditions on the crack are satisfied exactly by expressing one of the stress potentials in the form of the other.
3. A form of representation is assumed for the remaining unknown stress potential in the parametric plane.
4. The boundary conditions on the outer boundary of the region are satisfied by the method of least squares boundary collocation.

In order to demonstrate the feasibility of the method, rectangular orthotropic plates with centrally located traction-free cracks under constant tensile and shear loads are analyzed. A parametric study of the finite plate stress intensity factors is presented showing the effects of varying material properties, orientation angle, crack length to plate width and plate height to plate width ratios. In general, some of the more significant results can be summarized as follows:

George Stephen Gyékényesi

1. Rectangular orthotropic plate with a centrally located traction-free crack under constant tensile load acting in the direction perpendicular to the crack.
   (a) The opening mode stress intensity factor increases with decreasing E_{22}/E_{11} ratios, where E_{22} and E_{11} are Young's moduli of elasticity for an orthotropic material and $0 < E_{22}/E_{11} < 1$.
   (b) The opening mode stress intensity factor increases as the crack length to plate width ratio increases.
   (c) The opening mode stress intensity factor increases as the plate height to plate width ratio decreases.
   (d) Considering a square plate, the maximum opening mode stress intensity factor occurs at the value of the orientation angle of 90° while the minimum opening mode stress intensity factor is obtained at 0° for any constant crack length to plate width ratio.
   (e) The presence of the sliding mode stress intensity factor is due to values of the orientation angle other than 0° or 90° and to the finite size of the plate.
2. Orthotropic square plate with a centrally located traction-free crack under constant shear loading.
   (a) The sliding mode stress intensity factor increases with decreasing E_{22}/E_{11} ratios.
   (b) The sliding mode stress intensity factor increases as the crack length to plate width ratio increases.

(c) The minimum sliding mode stress intensity factor always occurs at the zero value of the orientation angle while the maximum sliding mode stress intensity factor can occur at various values of the orientation angle for constant crack length to plate width ratios.

(d) The presence of the opening mode stress intensity factor is due to values of the orientation angle different from 0° or 90° and to the finite size of the plate.

In addition to the development of the mapping-collocation method and to the parametric study of stress intensity factors, the infinite anisotropic plate solutions of Savin and Lekhnitskii are derived by considering a finite rectangular anisotropic plate with a centrally located traction-free crack and extending its dimensions to infinity.

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To my parents

Gyékényesi György László

and

Gy. Korcsmár Katalin

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

The following list contains the more commonly used symbols and their representative meaning unless otherwise defined in the text. In general, all symbols are defined when first introduced.

a	half crack length
A_{1n}	n^{th} complex constant in a Laurent series associated with positive exponent terms
B_{1n}	n^{th} complex constant in a Laurent series associated with negative exponent terms
c	half-width of a rectangular plate
C	elastic compliance matrix
C_{ij}	ij^{th} element of elastic compliance matrix
C_1, C_2, C_1^0, C_2^0	real arbitrary integration constants
$\mathscr{C}_{1n}$	n^{th} modified complex constant associated with the positive exponent terms in a Laurent series
$\mathscr{C}_{2n}$	n^{th} modified complex constant associated with the negative exponent terms in a Laurent series
$\mathscr{C}_c$	complex constant depending on material properties
$\mathscr{C}_s$, $\mathscr{C}_d$	modified complex constants depending on the sum and difference of A_{11} and B_{11}, respectively
E_{11}, E_{22}, E_{33}	Young's moduli of elasticity with respect to coordinate directions
F_{1n}, F_{2n}, G_{1n}, G_{2n}	complex quantities utilized in the construction of the coefficient matrix in the least squares boundary collocation method
G_{13}, G_{12}, G_{23}	shear moduli of an anisotropic material

h	half-height of a rectangular plate
K_1, K_2	opening and sliding mode stress intensity factors, respectively
M	total number of collocation points on the outer boundary of a region with a central crack
NN, NP	truncation numbers on infinite sums associated with the negative and positive exponent terms in a Laurent series, respectively
p_1, p_2, q_1, q_2	complex constants depending on material properties
r, θ	crack tip polar coordinates
S	elastic stiffness matrix
S_{ij}	ij^{th} element of elastic stiffness matrix
T_s	constant shear stress applied to the boundary of a square plate
T_t, T_t'	constant tensile stress applied to the boundary of a rectangular plate
u, v	displacements in the x- and y-direction, respectively
x, y	rectangular coordinates
X_n, Y_n	applied boundary stresses acting in the x- and y-direction, respectively
z	conventional complex variable
z_1, z_2	complex variables modified by material properties
α	reduced compliance matrix
α_{ij}	ij^{th} element of reduced compliance matrix
α_1, α_2	real parts of Lekhnitskii's material parameters
β	reduced stiffness matrix
β_{ij}	ij^{th} element of reduced stiffness matrix
β_1, β_2	imaginary parts of Lekhnitskii's material parameters

δ	material orientation angle
ε_{xx}, ε_{yy}, ε_{xy}	components of the strain tensor, rectangular coordinates
ζ	complex variable in parametric plane
Θ	polar angle in parametric plane
$\left.\begin{matrix}\nu_{12}, \nu_{21}, \nu_{23}, \\ \nu_{32}, \nu_{13}, \nu_{31}\end{matrix}\right\}$	Poisson's ratios in an anisotropic material
μ_1, μ_2	Lekhnitskii's material parameters
σ	complex variable on the unit circle
σ_{xx}, σ_{yy}, σ_{xy}	components of the stress tensor, rectangular coordinates
$\phi_1(z_1)$, $\phi_2(z_2)$	complex stress potential functions
ϕ	Airy's stress function
ψ	displacement function

1. INTRODUCTION

The recently extended application of anisotropic materials in various fields of structural design makes one aware of a whole class of new problems in anisotropic elasticity theory. In order to study the fracture phenomena of these materials, the problem of a centrally cracked finite plate seems to be of immediate importance. Methods of solution for centrally cracked finite isotropic plates were developed by A. S. Kobayashi (ref. 1) and O. L. Bowie and D. M. Neal (ref. 2) while the problem of the finite anisotropic plate remained fairly neglected.

The plane problem of a centrally cracked finite anisotropic plate could be considered as a special case of the more general problem of the anisotropic plate containing an elliptic hole. The limiting case of this problem, i. e., the problem of an elliptic hole in an infinite anisotropic plate, has extensively been investigated by various authors over the past thirty years. The complex variable approach of N. I. Muskhelishvili (ref. 3) as adopted by S. G. Lekhnitskii (ref. 4) to plane anisotropic theory was used by both Lekhnitskii and G. N. Savin to solve the problem of the infinite anisotropic plate bounded by an ellipse. Lekhnitskii (refs. 4, 5) employed the method of series expansion while Savin (refs. 6, 7) applied Schwarz's formula in order to obtain the complex stress potentials of Lekhnitskii.

Simultaneously with these achievements, A. E. Green, in a series of articles published in the early 1940's, developed his own method, also solving the problem of an elliptic hole in an infinite anisotropic region. His results are condensed in his Theoretical Elasticity (ref. 8).

Another approach to plane elastic problems is advanced in L. M. Milne-Thomson's Plane Elastic Systems (ref. 9) in handling both isotropic and anisotropic plane problems by a "semi-unified" method. He also discusses the problem of the elliptic hole in an infinite anisotropic region.

An entirely different method is due to D. D. Ang and M. L. Williams (ref. 10). They used a formulation in integral equations for a centrally cracked orthotropic plate. This method, however, applies only for infinite orthotropic plates with zero orientation angle.

The semi-infinite specially orthotropic plate problem with a centrally located crack was first discussed and solved by A. Mendelson and S. W. Spero (ref. 11). The solution was obtained by the application of finite Fourier transforms resulting in an integral equation for the crack opening.

The concept of the extension of Fracture Mechanics from isotropic to anisotropic media was proposed by P. C. Paris and G. C. Sih in 1961 (ref. 12). If their approach is adopted, the stress intensity factors can be determined directly from the anisotropic stress potentials in a manner similar to that introduced by Sih for isotropic materials.

As a further development, E. M. Wu examined the conditions necessary for the application of Fracture Mechanics to anisotropic materials and also verified them experimentally for orthotropic plates (ref. 13). He concluded that the crack tip stress singularity is of the same order as that of isotropic materials and the stress intensity factors are similar to those of the isotropic case.

The subject of this dissertation is a logical continuation of the above listed work as it is applied and extended to the problem of a finite anisotropic region with a centrally located traction-free crack. The proposed method of solution parallels Bowie's modified mapping-collocation technique (ref. 2) which was successfully used in the case of isotropic materials.

Since the presence of flaws in a material is of critical nature, the main objective of the work reported herein was the construction of a practical method toward defining and obtaining the stress intensity factors of finite anisotropic plates with central traction-free cracks. The method of solution can briefly be described as follows:

1. The crack is mapped into the unit circle.
2. The complex stress potentials of Lekhnitskii are found by exactly satisfying the zero traction conditions on the crack with the application of Muskhelishvili's function extension concept across the unit circle.
3. Upon expanding the remaining undefined stress potential in a Laurent series, the conditions on the outer boundary of the region are approximated by the method of least squares boundary collocation.

The stress intensity factors for both the opening and sliding modes are then computed since they are shown to be functions of the coefficients of the Laurent series already defined. In addition, expressions for the stress components and displacements are given in general for the case of the traction-free crack problem.

As an illustration of the practicality of the method, the problem of an orthotropic rectangular plate with a central traction-free crack is solved under constant tension and shear loading. The effects of varying material properties, orientation angles, crack length to plate width and plate height to plate width ratios are investigated and discussed at some length. It is shown for plates of various crack length to plate width ratios that in the tension problem, the maximum opening mode stress intensity factor is obtained in each case when the "strong" axis of the material is perpendicular to the crack. In the shear problem, the maximum sliding mode stress intensity factor can occur at various values of the orientation angle.

In addition to the finite plate results, the problem of infinite anisotropic plates with central cracks is also discussed for the cases of constant tension and shear loading. It is shown that the infinite plate solutions of Lekhnitskii and Savin (refs. 4, 6) can also be obtained by extending the dimensions of a rectangular plate to infinity.

2. PLANE PROBLEM FORMULATIONS FOR ANISOTROPIC MATERIALS

2.1 INTRODUCTION

In linear anisotropic elasticity there are two plane problems which may be discussed from the classical point of view:

1. The state of plane stress, corresponding to a plane plate of constant thickness loaded by forces in the plane of the plate;
2. The state of plane strain, corresponding to a long (theoretically infinite) body acted upon by loads which are uniformly distributed in the infinite direction and have no components normal to the finite planes.

These cases are similar to those in plane linear isotropic elasticity (ref. 15).

It can easily be shown that the difference between the plane stress and the plane strain problems is inherent in the use of elastic constants within the stress-strain relations as given in Appendix A. Therefore, any method of problem formulation for the plane stress problem is also valid for the state of plane strain (or vice versa) with the proper interchange of the elastic constants.

In the following, two methods of problem formulation will be demonstrated:

1. The conventional stress function formulation (ref. 4);

2. The displacement function formulation corresponding to Marguerre's displacement function approach (refs. 15, 16) in linear isotropic elasticity.

The formulations developed on the following pages are given for plane stress problems with no body forces.

2.2 STRESS FUNCTION FORMULATION OF PLANE STRESS PROBLEMS

Assuming that the stress components are derivable from a function such that

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{2.1}$$

one finds that the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned} \tag{2.2}$$

are identically satisfied.

The displacements, u and v, satisfy the strain compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{2.3}$$

also identically.

Therefore, the function, ϕ, can be related to the displacements, u and v, through the use of the strain-displacement relations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad 2.4$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

and Hooke's Law for the state of anisotropic plane stress

$$\sigma_i = \beta_{ij}\varepsilon_j \qquad (i,j = 1,\ 2,\ 6) \qquad 2.5$$

in the following manner:

$$\frac{\partial^2\phi}{\partial y^2} = \left(\beta_{11}\frac{\partial}{\partial x} + \beta_{16}\frac{\partial}{\partial y}\right)u + \left(\beta_{16}\frac{\partial}{\partial x} + \beta_{12}\frac{\partial}{\partial y}\right)v$$

$$\frac{\partial^2\phi}{\partial x^2} = \left(\beta_{12}\frac{\partial}{\partial x} + \beta_{26}\frac{\partial}{\partial y}\right)u + \left(\beta_{26}\frac{\partial}{\partial x} + \beta_{22}\frac{\partial}{\partial y}\right)v \qquad 2.6$$

$$-\frac{\partial^2\phi}{\partial x\partial y} = \left(\beta_{16}\frac{\partial}{\partial x} + \beta_{66}\frac{\partial}{\partial y}\right)u + \left(\beta_{66}\frac{\partial}{\partial x} + \beta_{26}\frac{\partial}{\partial y}\right)v$$

Upon the elimination of u and v from Eqs. 2.6, the following fourth order partial differential equation is obtained:

$$(\beta_{11}\beta_{66} - \beta_{16}^2)\frac{\partial^4\phi}{\partial x^4} + 2(\beta_{11}\beta_{26} - \beta_{12}\beta_{16})\frac{\partial^4\phi}{\partial x^3\partial y}$$

$$+ (2\beta_{16}\beta_{26} - 2\beta_{12}\beta_{66} + \beta_{11}\beta_{22} - \beta_{12}^2)\frac{\partial^4\phi}{\partial x^2\partial y^2}$$

$$+ 2(\beta_{16}\beta_{22} - \beta_{12}\beta_{26})\frac{\partial^4\phi}{\partial x\partial y^3} + (\beta_{22}\beta_{66} - \beta_{26}^2)\frac{\partial^4\phi}{\partial y^4} = 0$$

2.7

Using the relations 42 from Appendix A, this differential equation can be written as,

$$C_{22} \frac{\partial^4 \phi}{\partial x^4} - 2C_{26} \frac{\partial^4 \phi}{\partial x^3 \partial y} + (C_{66} + 2C_{12}) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} - 2C_{16} \frac{\partial^4 \phi}{\partial x \partial y^3} + C_{11} \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{2.8}$$

The displacements are found from the stress-strain relations

$$\varepsilon_i = C_{ij}\sigma_j \qquad (i,j = 1,\ 2,\ 6) \tag{2.9}$$

by substituting the strain-displacement relations and the stress function, ϕ, into the stress-strain relations, and integrating the resulting expressions,

$$\begin{aligned} u &= C_{11} \int_x \frac{\partial^2 \phi}{\partial y^2}\, dx + C_{12} \frac{\partial \phi}{\partial x} - C_{16} \frac{\partial \phi}{\partial y} + u_o - \omega_o y \\ v &= C_{22} \int_y \frac{\partial^2 \phi}{\partial x^2}\, dy + C_{12} \frac{\partial \phi}{\partial y} - C_{26} \frac{\partial \phi}{\partial x} + v_o + \omega_o x \end{aligned} \tag{2.10}$$

In Eqs. 2.10, the terms, u_o, $\omega_o y$, v_o, $\omega_o x$, designate rigid body motions. This formulation was first obtained by S. G. Lekhnitskii in the 1930's (ref. 4).

The general solution of the differential equation (Eq. 2.8) subject to displacement and/or stress boundary conditions would then be the solution of the linear anisotropic plane stress problem in the case of zero body forces.

2.3 DISPLACEMENT FUNCTION FORMULATION OF PLANE STRESS PROBLEMS

The substitution of the strain-displacement relations

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ 2\varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \qquad 2.11$$

into Hooke's Law for the state of anisotropic plane stress

$$\sigma_i = \beta_{ij}\varepsilon_j \qquad (i,j = 1,\ 2,\ 6) \qquad 2.12$$

results in the following expressions:

$$\begin{aligned} \sigma_{xx} &= \beta_{11}\frac{\partial u}{\partial x} + \beta_{12}\frac{\partial v}{\partial y} + \beta_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \sigma_{yy} &= \beta_{12}\frac{\partial u}{\partial x} + \beta_{22}\frac{\partial v}{\partial y} + \beta_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \sigma_{xy} &= \beta_{16}\frac{\partial u}{\partial x} + \beta_{26}\frac{\partial v}{\partial y} + \beta_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned} \qquad 2.13$$

Using the equations of equilibrium

$$\begin{aligned} \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} &= 0 \end{aligned} \qquad 2.14$$

in conjunction with the stresses given in 2.13, results in Navier's equations for the state of anisotropic plane stress,

$$\left(\beta_{11}\frac{\partial^2}{\partial x^2}+2\beta_{16}\frac{\partial^2}{\partial x\partial y}+\beta_{66}\frac{\partial^2}{\partial y^2}\right)u+\left[\beta_{16}\frac{\partial^2}{\partial x^2}+(\beta_{12}\quad\beta_{66})\frac{\partial^2}{\partial x\partial y}+\beta_{26}\frac{\partial^2}{\partial y^2}\right]v=0$$

$$\left[\beta_{16}\frac{\partial^2}{\partial x^2}+(\beta_{12}+\beta_{66})\frac{\partial^2}{\partial x\partial y}+\beta_{26}\frac{\partial^2}{\partial y^2}\right]u+\left(\beta_{66}\frac{\partial^2}{\partial x^2}+2\beta_{26}\frac{\partial^2}{\partial x\partial y}+\beta_{22}\frac{\partial^2}{\partial y^2}\right)v=0$$

2.15

If the displacements, u and v, are defined in terms of an unknown function, ψ, such that

$$u=\left(\beta_{16}\frac{\partial^2}{\partial x^2}+(\beta_{12}+\beta_{66})\frac{\partial^2}{\partial x\partial y}+\beta_{26}\frac{\partial^2}{\partial y^2}\right)\psi$$

2.16

$$v=-\left(\beta_{11}\frac{\partial^2}{\partial x^2}+2\beta_{16}\frac{\partial^2}{\partial x\partial y}+\beta_{66}\frac{\partial^2}{\partial y^2}\right)\psi$$

where $\psi = \psi(x,y)$, then the first equation of 2.15 is identically satisfied and the second equation becomes

$$(\beta_{66}\beta_{11}-\beta_{16}^2)\frac{\partial^4\psi}{\partial x^4}+2(\beta_{11}\beta_{66}-\beta_{16}\beta_{12})\frac{\partial^4\psi}{\partial x^3\partial y}$$
$$+(\beta_{11}\beta_{22}+2\beta_{16}\beta_{26}-2\beta_{12}\beta_{66}-\beta_{12}^2)\frac{\partial^4\psi}{\partial x^2\partial y^2}$$
$$+2(\beta_{22}\beta_{16}-\beta_{26}\beta_{12})\frac{\partial^4\psi}{\partial x\partial y^3}+(\beta_{66}\beta_{22}-\beta_{26}^2)\frac{\partial^4\psi}{\partial y^4}=0$$

2.17

This equation can be written with the elastic compliances as,

$$C_{22}\frac{\partial^4\psi}{\partial x^4}-2C_{26}\frac{\partial^4\psi}{\partial x^3\partial y}+(C_{66}+2C_{12})\frac{\partial^4\psi}{\partial x^2\partial y^2}-2C_{16}\frac{\partial^4\psi}{\partial x\partial y^3}+C_{11}\frac{\partial^4\psi}{\partial y^4}=0$$

2.18

It may be noted that Eqs. 2.18 and 2.8 are of the same form. The stress components can be obtained from 2.13 upon substituting for the displacements.

$$\sigma_{xx} = (\beta_{11}\beta_{66} - \beta_{16}^2)\frac{\partial^3\psi}{\partial x^2\partial y} + (\beta_{11}\beta_{26} - \beta_{16}\beta_{12})\frac{\partial^3\psi}{\partial x\partial y^2} + (\beta_{16}\beta_{26} - \beta_{12}\beta_{66})\frac{\partial^3\psi}{\partial y^3}$$

$$\sigma_{yy} = (\beta_{12}\beta_{16} - \beta_{11}\beta_{26})\frac{\partial^3\psi}{\partial x^3} + (\beta_{12}\beta_{66} + \beta_{12}^2 - \beta_{22}\beta_{11} - \beta_{26}\beta_{16})\frac{\partial^3\psi}{\partial x^2\partial y}$$

$$+ 2(\beta_{12}\beta_{26} - \beta_{16}\beta_{22})\frac{\partial^3\psi}{\partial x\partial y^3} + (\beta_{26}^2 - \beta_{22}\beta_{66})\frac{\partial^3\psi}{\partial y^3}$$

$$\sigma_{xy} = (\beta_{16}^2 - \beta_{11}\beta_{66})\frac{\partial^3\psi}{\partial x^3} + (\beta_{12}\beta_{16} - \beta_{11}\beta_{26})\frac{\partial^3\psi}{\partial x^2\partial y} + (\beta_{12}\beta_{66} - \beta_{16}\beta_{26})\frac{\partial^3\psi}{\partial x\partial y^2} \tag{2.19}$$

In order to compare the above formulation with Marguerre's displacement function approach, consider the case of isotropy

$$C_{11} = C_{22} = \frac{1}{E} \qquad C_{16} = C_{26} = 0$$
$$C_{12} = -\frac{\nu}{E} \qquad C_{66} = \frac{2(1+\nu)}{E} \tag{2.20}$$

Then the displacements are written as,

$$u = \frac{E}{2(1-\nu)}\frac{\partial^2\psi}{\partial x\partial y}$$
$$v = -\frac{E}{1-\nu^2}\frac{\partial^2\psi}{\partial x^2} - \frac{E}{2(1+\nu)}\frac{\partial^2\psi}{\partial y^2} \tag{2.21}$$

With the definition

$$\mathscr{U} = -\frac{E\psi}{2(1+\nu)} \tag{2.22}$$

the displacements,

$$u = - \frac{1 + \nu}{1 - \nu} \frac{\partial^2 \mathcal{M}}{\partial x \partial y}$$

$$v = \frac{2}{1 - \nu} \frac{\partial^2 \mathcal{M}}{\partial x^2} + \frac{\partial^2 \mathcal{M}}{\partial y^2} \qquad 2.23$$

are obtained, where $\mathcal{M}$ is the displacement function which was proposed by Marguerre (ref. 16). Of course, in the case of isotropy, the fourth order partial differential equation with constant coefficients (Eq. 2.18) will simplify to the biharmonic form.

In conclusion it can be stated that the general solution of the differential equation 2.18 subject to the appropriate boundary conditions would constitute the solution of the anisotropic plane stress problem with zero body forces.

The displacement function method was presented only as an interesting point to note in plane anisotropic problem formulation paralleling that of Marguerre's proposed method in isotropic elasticity.

3. GENERAL CONSIDERATIONS FOR PROBLEM SOLUTIONS CONCERNING MULTIPLY-CONNECTED REGIONS

3.1 INTRODUCTION

The problem of multiply connected regions in plane linear anisotropic elasticity is discussed in both Lekhnitskii's and Savin's works (refs. 4, 6). In the following, a brief treatment of the subject matter is given for reasons of completeness.

First, the governing differential equation is solved in terms of two arbitrary complex functions (stress potentials) and expressions for the stress components and displacements are derived in terms of these two functions. The boundary conditions are established for both the First and the Second Fundamental Problems. Lekhnitskii's complex material parameters are examined and the resultant force and moment for a segment of a curve are also found. Finally, the forms of the stress potentials for a finite multiply connected region are defined.

For a more comprehensive discussion, the reader is referred to the references given above.

3.2 SOLUTION OF THE DIFFERENTIAL EQUATION

The fourth order partial differential equation with constant coefficients obtained from the stress function formulation

$$C_{22}\frac{\partial^4\phi}{\partial x^4} - 2C_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (C_{66} + 2C_{12})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2C_{16}\frac{\partial^4\phi}{\partial x\partial y^3} + C_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \tag{3.1}$$

has been extensively studied by Lekhnitskii, Savin, Green and Milne-Thomson (refs. 4, 6, 8, 9). Following Lekhnitskii's method of solution, it is observed that the above differential equation can be written in symbolic form as,

$$D_1 D_2 D_3 D_4 \phi = 0 \tag{3.2}$$

where

$$D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x} \qquad (k = 1,\ 2,\ 3,\ 4)$$

and μ_k are the roots of the characteristic equation given as,

$$C_{22} - 2C_{26}\mu + (C_{66} + 2C_{12})\mu^2 - 2C_{16}\mu^3 + C_{11}\mu^4 = 0 \tag{3.3}$$

Integrating Eq. 3.2, the stress function, ϕ, can be written as the sum of four arbitrary functions,

$$\phi = \sum_{k=1}^{4} F_k(x + \mu_k y) \tag{3.4}$$

Considering the positive definiteness of strain energy, Lekhnitskii proved that all the roots, μ_k, must necessarily be complex. Since the roots, μ_k, are all complex and the coefficients of the characteristic equation defining the roots, μ_k, are all real; the roots, μ_k, must be complex conjugates of each other. So, the roots can be designated as, μ_1, μ_2, $\overline{\mu}_1$ and $\overline{\mu}_2$. These roots, called complex parameters, depend entirely on the material constants.

For real stresses, the stress function, ϕ, must necessarily be real. Therefore, the general solution of Eq. 3.2 must be of the form

$$\phi = 2\mathrm{Re}[F_1(z_1) + F_2(z_2)] \tag{3.5}$$

where

$$z_1 = x + \mu_1 y \qquad \text{and} \qquad \mu_1 \neq \mu_2$$
$$z_2 = x + \mu_2 y$$

If the roots are not distinct, i.e.,

$$\mu_1 = \mu_2; \qquad \bar{\mu}_1 = \bar{\mu}_2$$

then the stress function becomes

$$\phi = 2\text{Re}[F_1(z_1) + \bar{z}_1 F_2(z_1)] \qquad 3.6$$

In particular, consider an isotropic material. The characteristic equation becomes

$$\frac{1}{E} + \left[\frac{2(1 + \nu)}{E} - 2\,\frac{\nu}{E}\right] \mu^2 + \frac{1}{E}\,\mu^4 = 0 \qquad 3.7$$

Solving for the roots, one obtains

$$\mu_1 = \mu_2 = i$$
$$\bar{\mu}_1 = \bar{\mu}_2 = -i$$

Hence

$$z_1 = z_2 = x + iy; \qquad \bar{z}_1 = \bar{z}_2 = x - iy$$

and the well known expression

$$\phi = 2\text{Re}[\bar{z}F_2(z) + F_1(z)] \qquad 3.8$$

is obtained.

3.3 STRESS COMPONENTS

It was shown in the preceding section that the stress function, ϕ, can be expressed in terms of two arbitrary functions of the complex variables z_1 and z_2 when $\mu_1 \neq \mu_2$, in terms of z_1 when $\mu_1 = \mu_2$ and in terms of z when isotropy is considered. Knowing the stress function, one can readily find the stress components in terms of the above arbitrary functions by simple differentiation. Defining

$$\frac{dF_1}{dz_1} = \phi_1(z_1)$$
$$\frac{dF_2}{dz_2} = \phi_2(z_2) \qquad 3.9$$

the stress components will be given by the following equations:

1. The complex parameters are distinct ($\mu_1 \neq \mu_2$):

$$\sigma_{xx} = 2\text{Re}\left[\mu_1^2\phi_1'(z_1) + \mu_2^2\phi_2'(z_2)\right]$$
$$\sigma_{yy} = 2\text{Re}\left[\phi_1'(z_1) + \phi_2'(z_2)\right] \qquad 3.10$$
$$\sigma_{xy} = -2\text{Re}\left[\mu_1\phi_1'(z_1) + \mu_2\phi_2'(z_2)\right]$$

2. The complex parameters are equal ($\mu_1 = \mu_2$):

$$\sigma_{xx} = 2\text{Re}\left[\mu_1^2\phi_1'(z_1) + 2\bar{\mu}_1\mu_1\phi_2(z_1) + \bar{z}_1\mu_1^2\phi_2'(z_1)\right]$$
$$\sigma_{yy} = 2\text{Re}\left[\phi_1'(z_1) + 2\phi_2(z_1) + \bar{z}_1\phi_2'(z_1)\right] \qquad 3.11$$
$$\sigma_{xy} = -2\text{Re}\left[\mu_1\phi_1'(z_1) + (\mu_1 + \bar{\mu}_1)\phi_2(z_1) + \mu_1\bar{z}_1\phi_2'(z_1)\right]$$

3. Isotropic case ($\mu_1 = \mu_2 = i$, $\bar{\mu}_1 = \bar{\mu}_2 = -i$):

$$\sigma_{xx} = 2\text{Re}\left[-\phi_1'(z) + 2\phi_2(z) - \bar{z}\phi_2'(z)\right]$$
$$\sigma_{yy} = 2\text{Re}\left[\phi_1'(z) + 2\phi_2(z) + \bar{z}\phi_2'(z)\right] \qquad 3.12$$
$$\sigma_{xy} = -2\text{Re}\left[i\phi_1'(z) + i\bar{z}\phi_2'(z)\right]$$

The fundamental stress combinations will then become

$$\sigma_{xx} + \sigma_{yy} = 4\phi_2(z) + 4\overline{\phi}_2(\overline{z})$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 4\phi_1'(z) + 4\overline{z}\phi_2'(z) \qquad 3.13$$

3.4 DISPLACEMENTS

The displacements can also be expressed by using Eq. 2.10.

1. The complex parameters are distinct ($\mu_1 \neq \mu_2$):

$$u = 2\mathrm{Re}\left[p_1\phi_1(z_1) + p_2\phi_2(z_2)\right] + u_o - \omega_o y$$

$$v = 2\mathrm{Re}\left[q_1\phi_1(z_1) + q_2\phi_2(z_2)\right] + v_o + \omega_o x \qquad 3.14$$

where

$$p_1 = C_{11}\mu_1^2 + C_{12} - C_{16}\mu_1 \qquad q_1 = \frac{C_{22}}{\mu_1} + C_{12}\mu_1 - C_{26}$$

$$p_2 = C_{11}\mu_2^2 + C_{12} - C_{16}\mu_2 \qquad q_2 = \frac{C_{22}}{\mu_2} + C_{12}\mu_2 - C_{26}$$

2. The complex parameters are equal ($\mu_1 = \mu_2$):

$$u = 2\mathrm{Re}\left\{p_1\phi_1(z_1) + p_1\overline{z}_1\phi_2(z_1) + \left[C_{11}(2\overline{\mu}_1\mu_1 - \mu_1^2) + C_{12} - \overline{\mu}_1 C_{16}\right]F_2(z_1)\right\}$$

$$+ u_o - \omega_o y$$

$$v = 2\mathrm{Re}\left\{q_1\phi_1(z_1) + q_1\overline{z}_1\phi_2(z_1) + \left[\frac{C_{22}}{\mu_1}\left(2 - \frac{\overline{\mu}_1}{\mu_1}\right) + C_{12}\overline{\mu}_1 - C_{26}\right]F_2(z_1)\right\}$$

$$+ v_o + \omega_o x$$

3.15

3. Isotropic case ($\mu_1 = \mu_2 = i$, $\overline{\mu}_1 = \overline{\mu}_2 = -i$):

$$u = 2\mathrm{Re}\left\{-\frac{1+\nu}{E}\left[\phi_1(z) + \overline{z}\phi_2(z)\right] + \frac{3-\nu}{E}F_2(z)\right\} + u_o - \omega_o y$$

$$v = 2\mathrm{Re}\left\{-i\frac{1+\nu}{E}\left[\phi_1(z) + \overline{z}\phi_2(z)\right] - i\frac{3-\nu}{E}F_2(z)\right\} + v_o + \omega_o x$$

The fundamental displacement combination is written as,

$$u + iv = 2\left\{-\frac{1+\nu}{E}\left[\overline{\phi}_1(\overline{z}) + z\overline{\phi}_2(\overline{z})\right] + \frac{3-\nu}{E}F_2(z)\right\}$$

$$+ u_o - \omega_o y + i(v_o + \omega_o x) \qquad 3.17$$

3.5 BOUNDARY CONDITIONS

3.51 First Fundamental Problem

Suppose that the boundary stresses, X_n and Y_n, are given (Figure 1). Then on the boundary, Γ, of the area, L, the following equations must be satisfied:

$$X_n = \sigma_{xx}\cos(n,x) + \sigma_{xy}\cos(n,y)$$
$$Y_n = \sigma_{xy}\cos(n,x) + \sigma_{yy}\cos(n,y) \qquad 3.18$$

Taking into consideration that

$$\cos(n,x) = \frac{dy}{ds}; \qquad \cos(n,y) = -\frac{dx}{ds}$$

and substituting for the stresses, the boundary conditions on the stress function can be written as,

$$X_n = \frac{d}{ds}\frac{\partial\phi}{\partial y}; \qquad Y_n = -\frac{d}{ds}\frac{\partial\phi}{\partial x} \qquad 3.19$$

Upon integration, these two expressions become

$$\int_s X_n ds + C_2 = \frac{\partial\phi}{\partial y}$$

$$- \int_s Y_n ds + C_1 = \frac{\partial\phi}{\partial x} \qquad 3.20$$

where C_1 and C_2 are arbitrary real constants of integration and s is measured from an arbitrarily chosen point on Γ.

Since the stress function, ϕ, can be expressed in terms of two arbitrary complex functions, $F_1(z_1)$ and $F_2(z_2)$, and consequently using the definitions, 3.9; the derivatives, $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$, can be written in terms of the complex functions, $\phi_1(z_1)$ and $\phi_2(z_2)$. Hence, the equations which must be satisfied by the functions, $\phi_1(z_1)$ and $\phi_2(z_2)$, on the boundary, Γ, of the area, L, are as follows:

$$\phi_1(z_1) + \phi_2(z_2) + \overline{\phi}_1(z_1) + \overline{\phi}_2(\overline{z}_2) = - \int_s Y_n dx + C_1$$

$$\mu_1\phi_1(z_1) + \mu_2\phi_2(z_2) + \overline{\mu}_1\overline{\phi}_1(\overline{z}_1) + \overline{\mu}_2\overline{\phi}_2(z_2) = \int_s X_n ds + C_2 \qquad z \text{ in } \Gamma$$

3.21

3.52 Second Fundamental Problem

Suppose that the displacement components $u(x,y)$ and $v(x,y)$ are given in the boundary, Γ, of the area, L. Then the boundary conditions will be given by

$$p_1\phi_1(z_1) + p_2\phi_2(z_2) + \overline{p}_1\overline{\phi}_1(\overline{z}_1) + \overline{p}_2\overline{\phi}_2(\overline{z}_2) - \omega_o y + u_o = u^o(s)$$

$$q_1\phi_1(z_1) + q_2\phi_2(z_2) + \overline{q}_1\overline{\phi}_1(\overline{z}_1) + \overline{q}_2\overline{\phi}_2(\overline{z}_2) + \omega_o x + v_o = v^o(s) \qquad z \text{ in } \Gamma$$

3.22

where $u^o(s)$ and $v^o(s)$ are specified values of the components of displacement on the boundary, Γ, which are functions of the arcs of the contour from an arbitrarily chosen reference point.

Considering both the First and the Second Fundamental Problems, it is observed that both problems can be solved completely in terms of the two arbitrary complex functions, $\phi_1(z_1)$ and $\phi_2(z_2)$.

3.6 ON THE COMPLEX PARAMETERS OF LEKHNITSKII

Lekhnitskii has shown (ref. 4) that the complex parameters, μ_k, in a rotated coordinate system are related to the complex parameters, μ_k^o, in the original system by the equations

$$\mu_k = \frac{\mu_k^o \cos\delta - \sin\delta}{\cos\delta + \mu_k^o \sin\delta} \qquad (k = 1,\ 2) \tag{3.23}$$

where δ is the angle between the x-axes of the original and the rotated systems. These transformation equations are of particular importance when problems of orthotropic media are considered.

For the case when the material and the reference axes coincide (special orthotropy) Eq. 3.3 reduces to

$$C_{22} + (C_{66} + 2C_{12})\mu^2 + C_{11}\mu^4 = 0 \tag{3.24}$$

For the case when the material and reference axes are not aligned along the same directions (general orthotropy), the μ_k^o are found from 3.24 with respect to the material axes first and then in accordance with Eq. 3.23, the values of μ_k can be determined with respect to the reference system.

Upon solving Eq. 3.24, one obtains the roots

$$
\begin{aligned}
\mu_1^o &= i\left\{\frac{C_{66}+2C_{12}}{2C_{11}} + \left[\left(\frac{C_{66}+2C_{12}}{2C_{11}}\right)^2 - \frac{C_{22}}{C_{11}}\right]^{1/2}\right\}^{1/2} \\
\bar{\mu}_1^o &= -\,i\left\{\frac{C_{66}+2C_{12}}{2C_{11}} + \left[\left(\frac{C_{66}+2C_{12}}{2C_{11}}\right)^2 - \frac{C_{22}}{C_{11}}\right]^{1/2}\right\}^{1/2} \\
\mu_2^o &= i\left\{\frac{C_{66}+2C_{12}}{2C_{11}} - \left[\left(\frac{C_{66}+2C_{12}}{2C_{11}}\right)^2 - \frac{C_{22}}{C_{11}}\right]^{1/2}\right\}^{1/2} \\
\bar{\mu}_2^o &= -\,i\left\{\frac{C_{66}+2C_{12}}{2C_{11}} - \left[\left(\frac{C_{66}+2C_{12}}{2C_{11}}\right)^2 - \frac{C_{22}}{C_{11}}\right]^{1/2}\right\}^{1/2}
\end{aligned}
\qquad 3.25
$$

Since the complex parameters, μ_k^o, are of the form

$$\mu_k^o = \alpha_k^o + i\beta_k^o \qquad (k = 1, 2) \qquad 3.26$$

one can immediately conclude that for the condition

$$\left[\frac{C_{66}+2C_{12}}{2C_{11}}\right]^2 - \frac{C_{22}}{C_{11}} \geq 0 \qquad 3.27$$

the complex parameters, μ_k^o, become purely imaginary, i.e.,

$$\mu_k^o = i\beta_k^o \qquad (k = 1, 2) \qquad 3.28$$

where

$$\beta_1^o > 0, \qquad \beta_2^o > 0$$

It also follows from Eq. 3.25 that

$$\beta_1^o \geq \beta_2^o \qquad 3.29$$

Note that condition 3.27 is equivalent to

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \leq 1 \qquad 3.30$$

in terms of engineering constants.

The rest of the range of material orthotropy is covered by the condition

$$\left[\frac{C_{66} + 2C_{12}}{2C_{11}}\right]^2 - \frac{C_{22}}{C_{11}} < 0 \qquad 3.31$$

resulting in $\alpha_1^o = -\alpha_2^o$ and $\beta_1^o = \beta_2^o > 0$. In terms of engineering constants, this condition is equivalent to

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} > 1 \qquad 3.32$$

These dimensionless ratios are convenient to use and cover the whole range of material properties.

The positive definiteness of β_k^o guarantees that β_k in Eq. 3.23 are positive definite. As a matter of fact, one can always construct μ_k such that $\beta_k > 0$, even in the case of complete anisotropy (ref. 6). This fact is of importance if the following affine transformation is considered:

$$\begin{aligned} x_k &= x + \alpha_k y \\ y_k &= \beta_k y \end{aligned} \qquad (k = 1,\ 2) \qquad 3.33$$

where z_k is now defined as,

$$z_k = x_k + iy_k \qquad (k = 1,\ 2) \qquad 3.34$$

The Jacobians of the transformation become

$$J_k = \begin{vmatrix} 1 & \alpha_k \\ 0 & \beta_k \end{vmatrix} = \beta_k > 0 \qquad (k = 1,\ 2) \tag{3.35}$$

Since $\beta_1 > 0$ and $\beta_2 > 0$, the description of any two transformed contours, Γ_1 and Γ_2, in L_1 and L_2, respectively, correspond to Γ in L such that when z describes Γ in L, z_1 describes Γ_1 in L_1 and z_2 describes Γ_2 in L_2, all in the same sense (ref. 9, Figure 2).

3.7 RESULTANT FORCE AND RESULTANT MOMENT

Consider the curve AB (Figure 1) loaded by the complex stress, $X_n + iY_n$. Suppose that one wants to find the resultant force and the resultant moment of the forces acting on the curve AB.

The force vector acting on an element ds of the curve AB can be written as,

$$(X_n + iY_n)ds = d\left[\frac{\partial\phi}{\partial y} - i\,\frac{\partial\phi}{\partial x}\right] = -\,id\left[\frac{\partial\phi}{\partial x} + i\,\frac{\partial\phi}{\partial y}\right] \tag{3.36}$$

Hence, the total force acting on the curve AB becomes

$$X + iY = \int_A^B (X_n + iY_n)ds = -\,i\left[\frac{\partial\phi}{\partial x} + i\,\frac{\partial\phi}{\partial y}\right]_A^B \tag{3.37}$$

where $[\ldots]_A^B$ denotes the increase in the value of the bracketed expression when moving along the curve AB from A to B.

Upon substitution for

$$\frac{\partial\phi}{\partial x} + i\,\frac{\partial\phi}{\partial y}$$

the final form of the force acting on the curve AB is obtained

$$X + iY = -i\Big\{(1 + i\mu_1)\phi_1(z_1) + (1 + i\mu_2)\phi_2(z_2)$$

$$+ (1 + i\bar{\mu}_1)\bar{\phi}_1(\bar{z}_1) + (1 + i\bar{\mu}_2)\bar{\phi}_2(\bar{z}_2)\Big\}_A^B \qquad 3.38$$

The moment, M_o, with respect to the origin acting on the curve AB is given by

$$M_o = \int_A^B (xY_n - yX_n)ds \qquad 3.39$$

After substituting for $X_n ds$ and $Y_n ds$ the moment, M_o, becomes

$$M_o = -\int_A^B \left[xd\left(\frac{\partial\phi}{\partial x}\right) + yd\left(\frac{\partial\phi}{\partial y}\right)\right] \qquad 3.40$$

Upon integration by parts, one obtains

$$M_o = -\left[x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right]_A^B + [\phi]_A^B \qquad 3.41$$

Then the moment of the forces acting on the curve AB can be written in terms of the complex stress potentials as,

$$M_o = 2\mathrm{Re}\Big[F_1(z_1) + F_2(z_2)\Big]_A^B - \mathrm{Re}\Big\{z\Big[(1 - i\mu_1)\phi_1(z_1) + (1 - i\mu_2)\phi_2(z_2)$$

$$+ (1 - i\bar{\mu}_1)\bar{\phi}_1(\bar{z}_1) + (1 - i\bar{\mu}_2)\bar{\phi}_2(\bar{z}_2)\Big]\Big\}_A^B \qquad 3.42$$

3.8 FORMS OF THE STRESS POTENTIALS FOR FINITE MULTIPLY CONNECTED REGIONS

Savin has shown (ref. 6) that the functions, $\phi_1(z_1)$ and $\phi_2(z_2)$ for a multiply connected region are expressed in general as,

$$\phi_1(z_1) = \sum_{\ell=1}^{L} A_\ell \ell n(z_1 - z_{1,\ell}) + \phi_1^*(z_1)$$

$$\phi_2(z_2) = \sum_{\ell=1}^{L} B_\ell \ell n(z_2 - z_{2,\ell}) + \phi_2^*(z_2) \qquad 3.43$$

where $z_{1,\ell}$ and $z_{2,\ell}$ are arbitrarily chosen points inside the boundaries, $\Gamma_{1,\ell}$ and $\Gamma_{2,\ell}$ (Figure 2); $\phi_1^*(z_1)$ and $\phi_2^*(z_2)$ are holomorphic functions in L_1 and L_2, respectively and A_ℓ and B_ℓ are complex constants which have to be determined from some known condition.

In general, the ℓ^{th} boundary, Γ_ℓ, is loaded by the force vector, $X_\ell + iY_\ell$, as given in the preceding section. Due to force equilibrium, any closed curve C_ℓ in L reconciliable with Γ_ℓ will be subject to a load given by

$$-(X_\ell + iY_\ell) = -i\Big\{(1 + i\mu_1)\phi_1(z_1) + (1 + i\mu_2)\phi_2(z_2) + (1 + i\overline{\mu}_1)\overline{\phi}_1(\overline{z}_1) + (1 + i\overline{\mu}_2)\overline{\phi}_2(\overline{z}_2)\Big\}_A^{B=A} \qquad 3.44$$

Considering the forms of $\phi_1(z_1)$ and $\phi_2(z_2)$(Eqs. 3.43), it is noted that the holomorphic functions $\phi_1^*(z_1)$ and $\phi_2^*(z_2)$ do not increase in value over the complete circuit, C_ℓ. Therefore, only the logarithmic terms have to be investigated in accordance with Eq. 3.44. If one carries out this investigation, two equations will result which are complex conjugates of each other. These equations are given as,

$$(1 + i\mu_1)A_\ell - (1 + i\overline{\mu}_1)\overline{A}_\ell + (1 + i\mu_2)B_\ell - (1 + i\overline{\mu}_2)\overline{B}_\ell = -\frac{X_\ell + iY_\ell}{2\pi}$$

$$-(1 - i\mu_1)A_\ell + (1 - i\overline{\mu}_1)\overline{A}_\ell - (1 - i\mu_2)B_\ell + (1 - i\overline{\mu}_2)\overline{B}_\ell = -\frac{X_\ell - iY_\ell}{2\pi} \qquad \text{(ref. 6)}$$

3.45

Obviously, Eqs. 3.45 are not sufficient in themselves to determine the values of the complex constants, A_ℓ and B_ℓ. The other two equations, necessary for the determination of A_ℓ and B_ℓ, are based on the consideration of single valued displacements.

Upon writing the fundamental displacement combination

$$u + iv = p_1\phi_1(z_1) + \overline{p}_1\overline{\phi}_1(\overline{z}_1) + p_2\phi_2(z_2) + \overline{p}_2\overline{\phi}_2(\overline{z}_2)$$

$$+ i\left[q_1\phi_1(z_1) + \overline{q}_1\overline{\phi}_1(\overline{z}_1) + q_2\phi_2(z_2) + \overline{q}_2\overline{\phi}_2(\overline{z}_2)\right] \qquad 3.46$$

one requires that the condition

$$\Big[u + iv\Big]_A^{B=A} = 0$$

be satisfied. The satisfaction of this condition will also result in two complex conjugate equations

$$(p_1 + iq_1)A_\ell - (\overline{p}_1 + i\overline{q}_1)\overline{A}_\ell + (p_2 + iq_2)B_\ell - (\overline{p}_2 + i\overline{q}_2)\overline{B}_\ell = 0$$

$$-(p_1 - iq_1)A_\ell + (\overline{p}_1 - i\overline{q}_1)\overline{A}_\ell - (p_2 - iq_2)B_\ell + (\overline{p}_2 - i\overline{q}_2)\overline{B}_\ell = 0 \qquad \text{(ref. 6)}$$

$$3.47$$

For a solution of the system of equations consisting of Eqs. 3.45 and 3.47, the determinant of the coefficient matrix cannot vanish. In order to prove that this condition is satisfied, the determinant, D, of the coefficient matrix is obtained

$$D = \frac{-16\beta_1\beta_2C_{11}C_{22}}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)}\Big\{2(\alpha_2 - \alpha_1)^2(\beta_2^2 + \beta_1^2)$$

$$+ (\beta_2 - \beta_1)^2(\beta_2 + \beta_1)^2 + (\alpha_2 - \alpha_1)^4\Big\} \qquad 3.48$$

This determinant is the corrected form of that given by Savin which is apparently in error (ref. 6). Obviously, for $\beta_k > 0$, the determinant, D, does not vanish. Hence, the system of equations constructed from Eqs. 3.45 and 3.47 can be solved for the complex constants, A_ℓ, $\overline{A}_\ell$, B_ℓ and $\overline{B}_\ell$.

There are two cases possible:

1. The resultant force vanishes on each boundary, i.e.,

$$X_\ell + iY_\ell = X_\ell - iY_\ell = 0$$

2. The resultant force does not vanish on each individual boundary.

In the first case, the complex constants, A_ℓ, $\overline{A}_\ell$, B_ℓ and $\overline{B}_\ell$, must be taken as zero since $D \neq 0$. Hence, the stress potentials, $\phi_1(z_1)$ and $\phi_2(z_2)$, are holomorphic functions in their respective regions, L_1 and L_2. In the second case, the complex constants, A_ℓ, $\overline{A}_\ell$, B_ℓ and $\overline{B}_\ell$, will have values other than zero. Hence, the complex stress potentials, $\phi_1(z_1)$ and $\phi_2(z_2)$, will become multiple-valued due to the presence of the logarithmic terms.

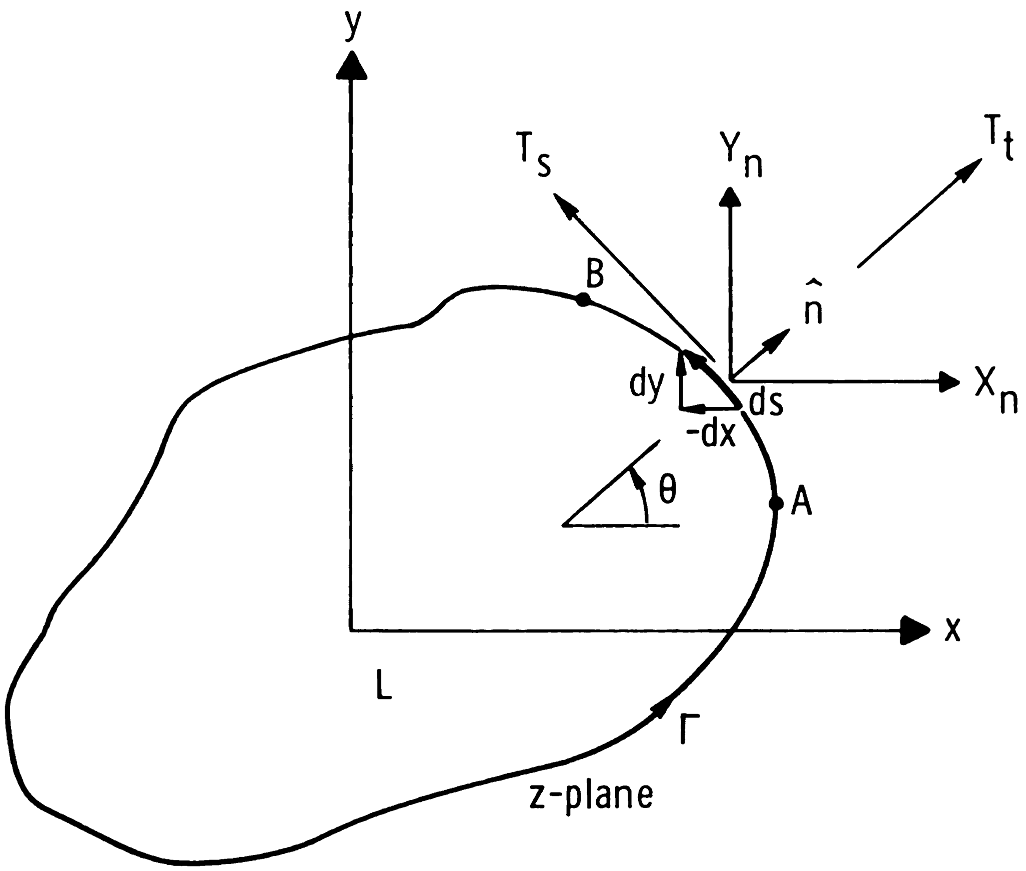


Figure 1. - Curve segment loaded by the complex stress, $X_n + iY_n$.

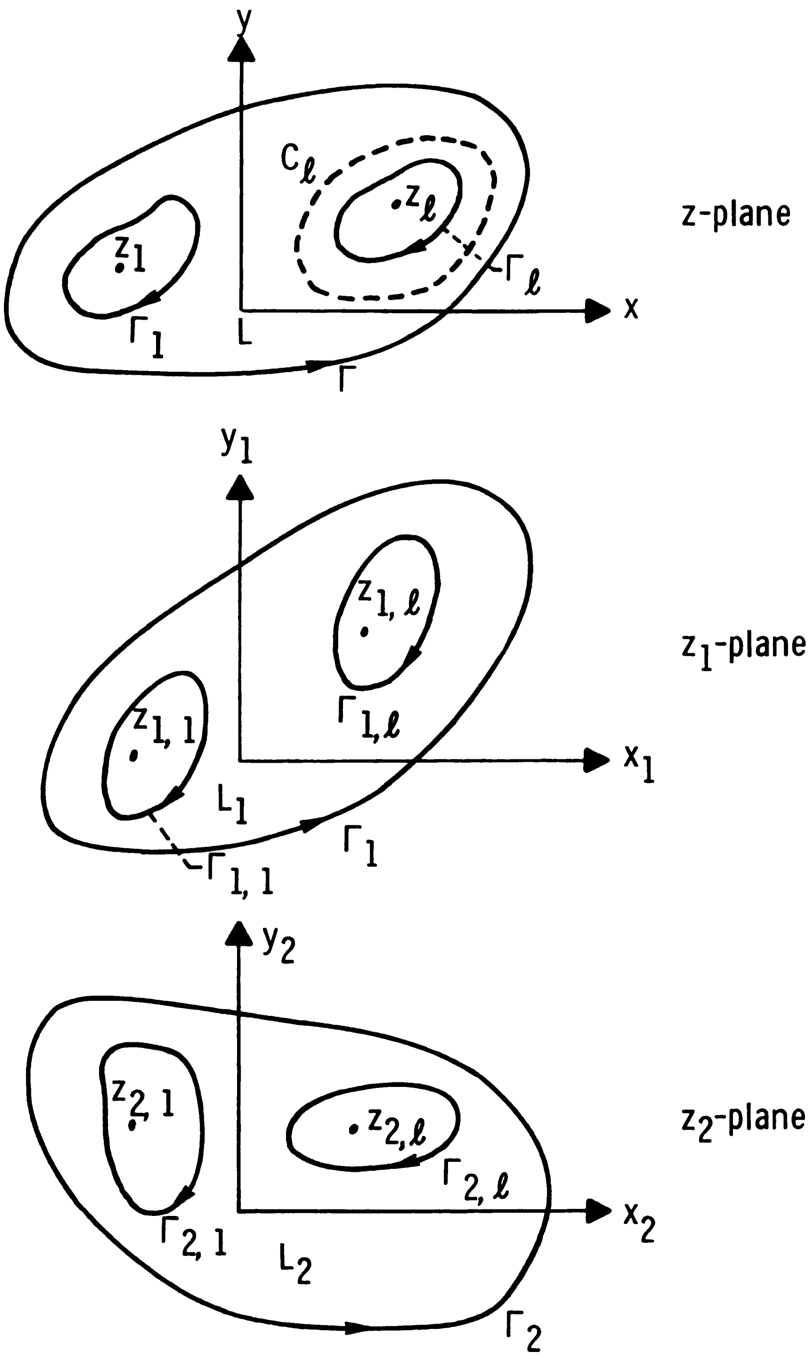


Figure 2. - The multiply connected region, L, in the z-, z_1-, and z_2-planes.

4. AN APPROACH TO THE SOLUTION OF FINITE-GEOMETRY TRACTION-FREE CRACK PROBLEMS

4.1 INTRODUCTION

It was established in the preceding section that two complex holomorphic functions define the solution for problems for which the resultant force vanishes on each individual boundary. Since the solution is defined in terms of two arbitrary complex holomorphic functions, it still remains to find the forms of these functions such that the appropriate boundary conditions are satisfied on the boundaries of the region.

In this section, the problem considered is that of a central, traction-free crack in a finite anisotropic region. The method of solution proposed herein was called by Bowie "A Modified Mapping-Collocation Technique" (ref. 2) and was successfully employed by him for isotropic problems.

As a first step, the crack is mapped into the unit circle. Then the zero traction conditions on the crack are exactly satisfied by using Muskhelishvili's continuation concept (ref. 3) across the unit circle such that the form of either of the two functions is defined in the form of the other. These expressions, will for the first time, be derived in this section.

Upon satisfying the zero traction conditions on the crack, a Laurent series form of representation is assumed for the remaining arbitrary function. The boundary conditions for the outer boundary are then derived in terms of this Laurent series.

In order to satisfy the conditions on the outer boundary, the least squares boundary collocation method is proposed. Bowie (ref. 2) also suggests the possible use of two stress and the moment conditions on the boundary in addition to the force boundary conditions for problems with local stress irregularities. In this dissertation, the force boundary conditions were found to be sufficient for the solution of the problems considered.

In addition to the method of solution described above, the expressions for the stress components are derived and specialized to the neighborhood of the crack tip. It is then shown that the stress intensity factors for a finite plate for both the opening and the shear modes can be defined analogously to that of the infinite plate.

Finally, expressions for the displacements are derived and in particular, the displacements of the crack boundary are also given.

4.2 THE MAPPING OF A CRACK AND ITS EXTERIOR INTO THE UNIT CIRCLE AND ITS EXTERIOR

Let the unit circle, γ, in a ζ-plane be mapped into the crack boundary, Γ_c, in the z-plane by

$$z = \omega(\sigma) \tag{4.1}$$

where

$$\sigma = e^{i\Theta}$$

The unit circle, γ, in the ζ-plane will then correspond to the crack, Γ_c, in the z-plane with the same sense of description (Figure 3).

Since the transformation

$$z_k = x + \mu_k y = x_k + iy_k \qquad (k = 1, 2) \tag{4.2}$$

can be expressed in terms of the conventional complex variables as,

$$z_k = \frac{1 - i\mu_k}{2} z + \frac{1 + i\mu_k}{2} \bar{z} \qquad (k = 1, 2) \tag{4.3}$$

it follows that

$$z_k = \frac{1 - i\mu_k}{2} \omega(\sigma) + \frac{1 + i\mu_k}{2} \bar{\omega}\left(\frac{1}{\sigma}\right) \equiv \omega_k(\sigma) \qquad (k = 1, 2) \tag{4.4}$$

maps the unit circle, γ, in the ζ-plane into the crack boundary, Γ_{ck}, in the z_k-planes. Consequently

$$z_k = \omega_k(\zeta) \tag{4.5}$$

will map the region, L_ζ, determined by the unit circle into the regions, L_k, in the z_k-planes.

The mapping of the unit circle, γ, into the crack boundary, Γ_c, is accomplished by the function

$$z = \frac{a}{2}\left(\sigma + \frac{1}{\sigma}\right) \qquad \text{(ref. 9)} \tag{4.6}$$

where a is the half-length of the crack (Figure 3). Hence, the unit circle, γ, is mapped into the crack boundary, Γ_{ck}, in the z_k-planes by

$$z_k = \frac{a}{2}\left(\sigma + \frac{1}{\sigma}\right) \qquad (k = 1, 2) \tag{4.7}$$

Upon explicitly expressing the variable on the unit circle, one obtains

$$\sigma = \frac{z_k + \sqrt{z_k^2 - a^2}}{a} \qquad (k = 1, 2) \tag{4.8}$$

Therefore, the exterior region, L_ζ, to the unit circle, γ, is given by the mapping

$$\zeta = \frac{z_k + \sqrt{z_k^2 - a^2}}{a} \qquad (k = 1, 2) \tag{4.9}$$

Note that the positive sign preceding the radical guarantees the "exterior to exterior" transformation.

4.3 ON THE CONTINUATION AND FORM OF THE STRESS POTENTIALS

In the following Muskhelishvili's extension concept (ref. 3) will be applied to the complex stress potentials, $\phi_1(z_1)$ and $\phi_2(z_2)$, in the ζ-plane.

Consider that a traction-free crack, Γ_c, in the z-plane is mapped into the unit circle, γ, in the ζ-plane by the functions

$$\zeta = \frac{z + \sqrt{z^2 - a^2}}{a}, \qquad z \text{ in } \Gamma_c \tag{4.10}$$

and

$$\zeta = \frac{z_k + \sqrt{z_k^2 - a^2}}{a}, \qquad z_k \text{ in } \Gamma_{ck} \qquad (k = 1, 2) \tag{4.11}$$

(Figure 4). Then on the unit circle, the following boundary conditions will prevail:

$$\begin{aligned} &\phi_1(\zeta) + \phi_2(\zeta) + \overline{\phi}_1(\overline{\zeta}) + \overline{\phi}_2(\overline{\zeta}) = 0, \qquad \zeta \text{ in } \gamma \\ &\mu_1\phi_1(\zeta) + \mu_2\phi_2(\zeta) + \overline{\mu}_1\overline{\phi}_1(\overline{\zeta}) + \overline{\mu}_2\overline{\phi}_2(\overline{\zeta}) = 0, \qquad \zeta \text{ in } \gamma \end{aligned} \tag{4.12}$$

where

$$\phi_k(z_k) = \phi_k[\omega_k(\zeta)] \equiv \phi_k(\zeta) \qquad (k = 1, 2)$$

The functions, $\phi_1(\zeta)$ and $\phi_2(\zeta)$, are holomorphic in L_ζ and are defined in L_ζ. In order to define $\phi_1(\zeta)$ and $\phi_2(\zeta)$ in R_ζ (inside the unit circle), substitute $1/\zeta$ for $\bar{\zeta}$ into Eq. 4.12 and consider ζ in R_ζ. Then the following expressions will result:

$$\begin{aligned} \phi_1(\zeta) + \phi_2(\zeta) &= -\bar{\phi}_1\left(\frac{1}{\zeta}\right) - \bar{\phi}_2\left(\frac{1}{\zeta}\right), \qquad \zeta \text{ in } \gamma \\ \mu_1\phi_1(\zeta) + \mu_2\phi_2(\zeta) &= -\bar{\mu}_1\bar{\phi}_1\left(\frac{1}{\zeta}\right) - \bar{\mu}_2\bar{\phi}_2\left(\frac{1}{\zeta}\right), \qquad \zeta \text{ in } \gamma \end{aligned} \tag{4.13}$$

Hence

$$\begin{aligned} \bar{\phi}_1\left(\frac{1}{\zeta}\right) &= \frac{\mu_2 - \mu_1}{\mu_1 - \bar{\mu}_1}\,\phi_2(\zeta) + \frac{\bar{\mu}_2 - \mu_1}{\mu_1 - \bar{\mu}_1}\,\bar{\phi}_2\left(\frac{1}{\zeta}\right), \qquad \zeta \text{ in } R_\zeta \\ \bar{\phi}_2\left(\frac{1}{\zeta}\right) &= \frac{\mu_1 - \mu_2}{\mu_2 - \bar{\mu}_2}\,\phi_1(\zeta) + \frac{\bar{\mu}_1 - \mu_2}{\mu_2 - \bar{\mu}_2}\,\bar{\phi}_1\left(\frac{1}{\zeta}\right), \qquad \zeta \text{ in } R_\zeta \end{aligned} \tag{4.14}$$

The substitution $\zeta = 1/\bar{\zeta}$ will change these definitions to

$$\begin{aligned} \bar{\phi}_1(\bar{\zeta}) &= \frac{\mu_2 - \mu_1}{\mu_1 - \bar{\mu}_1}\,\phi_2\left(\frac{1}{\bar{\zeta}}\right) + \frac{\bar{\mu}_2 - \mu_1}{\mu_1 - \bar{\mu}_1}\,\bar{\phi}_2(\bar{\zeta}), \qquad \zeta \text{ in } L_\zeta \\ \bar{\phi}_2(\bar{\zeta}) &= \frac{\mu_1 - \mu_2}{\mu_2 - \bar{\mu}_2}\,\phi_1\left(\frac{1}{\bar{\zeta}}\right) + \frac{\bar{\mu}_1 - \mu_2}{\mu_2 - \bar{\mu}_2}\,\bar{\phi}_1(\bar{\zeta}), \qquad \zeta \text{ in } L_\zeta \end{aligned} \tag{4.15}$$

Upon taking the complex conjugate expressions, one obtains

$$\begin{aligned} \phi_1(\zeta) &= \frac{\bar{\mu}_1 - \bar{\mu}_2}{\mu_1 - \bar{\mu}_1}\,\bar{\phi}_2\left(\frac{1}{\zeta}\right) + \frac{\bar{\mu}_1 - \mu_2}{\mu_1 - \bar{\mu}_1}\,\phi_2(\zeta), \qquad \zeta \text{ in } L_\zeta \\ \phi_2(\zeta) &= \frac{\bar{\mu}_2 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_2}\,\bar{\phi}_1\left(\frac{1}{\zeta}\right) + \frac{\bar{\mu}_2 - \mu_1}{\mu_2 - \bar{\mu}_2}\,\phi_1(\zeta), \qquad \zeta \text{ in } L_\zeta \end{aligned} \tag{4.16}$$

Equations 4.16 show that either $\phi_1(\zeta)$ or $\phi_2(\zeta)$ can be expressed in terms of the other. The zero traction condition on the crack provided the extended definitions of either one of the two functions in terms of the other such that each is continuous across the unit circle in its ζ-plane.

The inverse transformation, back to the z_k-planes, results in

$$\phi_1(z_1) = \frac{\bar{\mu}_1 - \bar{\mu}_2}{\mu_1 - \bar{\mu}_1} \bar{\phi}_2\left(\frac{z_1 - \sqrt{z_1^2 - a^2}}{a}\right) + \frac{\bar{\mu}_1 - \mu_2}{\mu_1 - \bar{\mu}_1} \phi_2\left(\frac{z_1 + \sqrt{z_1^2 - a^2}}{a}\right), \qquad z_1 \text{ in } L_1$$

$$\phi_2(z_2) = \frac{\bar{\mu}_2 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_2} \bar{\phi}_1\left(\frac{z_2 - \sqrt{z_2^2 - a^2}}{a}\right) + \frac{\bar{\mu}_2 - \mu_1}{\mu_2 - \bar{\mu}_2} \phi_1\left(\frac{z_2 + \sqrt{z_2^2 - a^2}}{a}\right), \qquad z_2 \text{ in } L_2$$

4.17

Now, $\phi_1(z_1)$ and $\phi_2(z_2)$ are defined in L_1 and L_2 such that using either one or the other in solving a finite geometry problem, will satisfy the zero traction condition on the crack. Furthermore, if $\phi_1(z_1)$ or $\phi_2(z_2)$ is transformed to the ζ-plane, each can be considered holomorphic within its doubly connected region enclosed by τ and τ' where τ' is obtained by inversion of τ with respect to the unit circle (Figure 5).

The determination of either $\phi_1(\zeta)$ or $\phi_2(\zeta)$ depends on a form of representation and the satisfaction of boundary conditions on τ corresponding to Γ_1 or Γ_2, respectively. It will be assumed that $\phi_1(\zeta)$ or $\phi_2(\zeta)$ can be represented in the form of a Laurent series. This assumption appears to be reasonable for a certain class of problems although the boundaries τ' and τ are not circular (ref. 2). As

Bowie stated in reference 2: "There is no a priori reason to suspect (for a certain class of problems) that the region of convergence of such a series could not extend over the desired parameter range."

In general, assume the form for $\phi_1(\zeta)$ as

$$\phi_1(\zeta) = A_{10} + \sum_{n=1}^{\infty}\left[A_{1n}\zeta^n + B_{1n}\zeta^{-n}\right] \tag{4.18}$$

considering that $\phi_2(z_2)$ will be found from the second equation of 4.17. Hence

$$\phi_1(z_1) = A_{10} + \sum_{n=1}^{\infty}\left\{A_{1n}\left[\frac{z_1 + \sqrt{z_1^2 - a^2}}{a}\right]^n + B_{1n}\left[\frac{z_1 - \sqrt{z_1^2 - a^2}}{a}\right]^n\right\} \tag{4.19}$$

The expressions for $\bar{\phi}_1\left(\frac{z_2 - \sqrt{z_2^2 - a^2}}{a}\right)$ and $\phi_1\left(\frac{z_2 + \sqrt{z_2^2 - a^2}}{a}\right)$ become

$$\bar{\phi}_1\left(\frac{z_2 - \sqrt{z_2^2 - a^2}}{a}\right) = \bar{A}_{10} + \sum_{n=1}^{\infty}\left\{\bar{A}_{1n}\left[\frac{z_2 - \sqrt{z_2^2 - a^2}}{a}\right]^n + \bar{B}_{1n}\left[\frac{z_2 + \sqrt{z_2^2 - a^2}}{a}\right]^n\right\} \tag{4.20}$$

$$\phi_1\left(\frac{z_2 + \sqrt{z_2^2 - a^2}}{a}\right) = A_{10} + \sum_{n=1}^{\infty}\left\{A_{1n}\left[\frac{z_2 + \sqrt{z_2^2 - a^2}}{a}\right]^n + B_{1n}\left[\frac{z_2 - \sqrt{z_2^2 - a^2}}{a}\right]^n\right\} \qquad 4.21$$

Therefore, $\phi_2(z_2)$ is obtained as,

$$\phi_2(z_2) = \frac{\bar{\mu}_2 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_2}\bar{A}_{10} + \frac{\bar{\mu}_2 - \mu_1}{\mu_2 - \bar{\mu}_2}A_{10}$$

$$+ \sum_{n=1}^{\infty}\left\{\left[\frac{\bar{\mu}_2 - \mu_1}{\mu_2 - \bar{\mu}_2}A_{1n} + \frac{\bar{\mu}_2 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_2}\bar{B}_{1n}\right]\left[\frac{z_2 + \sqrt{z_2^2 - a^2}}{a}\right]^n\right.$$

$$\left. + \left[\frac{\bar{\mu}_2 - \bar{\mu}_1}{\mu_2 - \bar{\mu}_2}\bar{A}_{1n} + \frac{\bar{\mu}_2 - \mu_1}{\mu_2 - \bar{\mu}_2}B_{1n}\right]\left[\frac{z_2 - \sqrt{z_2^2 - a^2}}{a}\right]^n\right\} \qquad 4.22$$

It is obvious that this expression is also a Laurent series and it is of the form

$$\phi_2(\zeta) = A_{20} + \sum_{n=1}^{\infty}\left[A_{2n}\zeta^n + B_{2n}\zeta^{-n}\right] \qquad 4.23$$

At this point, it is noted that upon assuming both $\phi_1(\zeta)$ and $\phi_2(\zeta)$ in the form of Laurent series' and satisfying the zero traction conditions on the crack, one obtains the very same form of $\phi_2(z_2)$ as given by 4.22. This result is obtained by the elimination of the constants A_{20}, A_{2n} and B_{2n} which is possible upon the satisfaction of the zero traction conditions on the crack.

4.4 BOUNDARY CONDITIONS FOR FINITE REGIONS CONTAINING A TRACTION-FREE CRACK

Considering the First Fundamental Problem, the boundary conditions 3.21 must be satisfied by the functions $\phi_1(z_1)$ and $\phi_2(z_2)$ on the outer boundary of the region. Upon making the proper substitutions (using the results of the preceding section), the boundary conditions 3.21 are written as,

$$\sum_{n=1}^{\infty}\Bigg[A_{1n}\left\{\left(\frac{z_1+\sqrt{z_1^2-a^2}}{a}\right)^n+\frac{\bar{\mu}_2-\mu_1}{\mu_2-\bar{\mu}_2}\left(\frac{z_2+\sqrt{z_2^2-a^2}}{a}\right)^n+\frac{\mu_2-\mu_1}{\bar{\mu}_2-\mu_2}\left(\frac{\bar{z}_1-\sqrt{\bar{z}_2^2-a^2}}{a}\right)^n\right\}$$

$$+\bar{A}_{1n}\left\{\left(\frac{\bar{z}_1+\sqrt{\bar{z}_1^2-a^2}}{a}\right)^n+\frac{\mu_2-\bar{\mu}_1}{\bar{\mu}_2-\mu_2}\left(\frac{\bar{z}_2+\sqrt{\bar{z}_2^2-a^2}}{a}\right)^n+\frac{\bar{\mu}_2-\bar{\mu}_1}{\mu_2-\bar{\mu}_2}\left(\frac{z_2-\sqrt{z_2^2-a^2}}{a}\right)^n\right\}$$

$$+B_{1n}\left\{\left(\frac{z_1-\sqrt{z_1^2-a^2}}{a}\right)^n+\frac{\bar{\mu}_2-\mu_1}{\mu_2-\bar{\mu}_2}\left(\frac{z_2-\sqrt{z_2^2-a^2}}{a}\right)^n+\frac{\mu_2-\mu_1}{\bar{\mu}_2-\mu_2}\left(\frac{\bar{z}_2+\sqrt{\bar{z}_2^2-a^2}}{a}\right)^n\right\}$$

$$+\bar{B}_{1n}\left\{\left(\frac{\bar{z}_1-\sqrt{\bar{z}_1^2-a^2}}{a}\right)^n+\frac{\mu_2-\bar{\mu}_1}{\bar{\mu}_2-\mu_2}\left(\frac{\bar{z}_2-\sqrt{\bar{z}_2^2-a^2}}{a}\right)^n+\frac{\bar{\mu}_2-\bar{\mu}_1}{\mu_2-\bar{\mu}_2}\left(\frac{z_2+\sqrt{z_2^2-a^2}}{a}\right)^n\right\}\Bigg]$$

$$= -\int_s Y_n ds + C_1^o, \qquad z \text{ in } \Gamma \qquad 4.24$$

and

$$
\begin{aligned}
\sum_{n=1}^{\infty} \Bigg[A_{1n} \Bigg\{ & \mu_1 \left(\frac{z_1 + \sqrt{z_1^2 - a^2}}{a} \right)^n + \frac{\mu_2 (\bar{\mu}_2 - \mu_1)}{\mu_2 - \bar{\mu}_2} \left(\frac{z_2 + \sqrt{z_2^2 - a^2}}{a} \right)^n \\
& + \frac{\bar{\mu}_2 (\mu_2 - \mu_1)}{\bar{\mu}_2 - \mu_2} \left(\frac{\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}}{a} \right)^n \Bigg\} \\
+ \bar{A}_{1n} \Bigg\{ & \bar{\mu}_1 \left(\frac{\bar{z}_1 + \sqrt{\bar{z}_1^2 - a^2}}{a} \right)^n + \frac{\bar{\mu}_2 (\mu_2 - \bar{\mu}_1)}{\bar{\mu}_2 - \mu_2} \left(\frac{\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}}{a} \right)^n \\
& + \frac{\mu_2 (\bar{\mu}_2 - \bar{\mu}_1)}{\mu_2 - \bar{\mu}_2} \left(\frac{z_2 - \sqrt{z_2^2 - a^2}}{a} \right)^n \Bigg\} \\
+ B_{1n} \Bigg\{ & \mu_1 \left(\frac{z_1 - \sqrt{z_1^2 - a^2}}{a} \right)^n + \frac{\mu_2 (\bar{\mu}_2 - \mu_1)}{\mu_2 - \bar{\mu}_2} \left(\frac{z_2 - \sqrt{z_2^2 - a^2}}{a} \right)^n \\
& + \frac{\bar{\mu}_2 (\mu_2 - \mu_1)}{\bar{\mu}_2 - \mu_2} \left(\frac{\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}}{a} \right)^n \Bigg\} \\
+ \bar{B}_{1n} \Bigg\{ & \bar{\mu}_1 \left(\frac{\bar{z}_1 - \sqrt{\bar{z}_1^2 - a^2}}{a} \right)^n + \frac{\bar{\mu}_2 (\mu_2 - \bar{\mu}_1)}{\bar{\mu}_2 - \mu_2} \left(\frac{\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}}{a} \right)^n \\
& + \frac{\mu_2 (\bar{\mu}_2 - \bar{\mu}_1)}{\mu_2 - \bar{\mu}_2} \left(\frac{z_2 + \sqrt{z_2^2 - a^2}}{a} \right)^n \Bigg\} \Bigg]
\end{aligned}
$$

$$
= \int_s X_n \, ds + C_2^o, \qquad z \text{ in } \Gamma \qquad 4.25
$$

where C_1^o and C_2^o are real arbitrary constants of integration corresponding to the outer boundary of the doubly connected region. It may be noted, here, that the complex constants, A_{10} and $\bar{A}_{10}$, remain undetermined. This fact, however, is of no importance since the stresses are not influenced by these constants and the rigid body displacements have already been expressed by other constants (see Eq. 3.14).

Upon examination of the boundary conditions, 4.24 and 4.25, one finds that the $n=1$ terms can be contracted due to the fact that

$$(\mu_2 - \bar{\mu}_2)z_1 + (\bar{\mu}_2 - \mu_1)z_2 + (\mu_1 - \mu_2)\bar{z}_2 = 0$$

and 4.26

$$\mu_1(\mu_2 - \bar{\mu}_2)z_1 + \mu_2(\bar{\mu}_2 - \mu_1)z_2 + \bar{\mu}_2(\mu_1 - \mu_2)\bar{z}_2 = y(\mu_2 - \mu_1)(\mu_2 - \bar{\mu}_2)(\bar{\mu}_2 - \mu_1)$$

Hence, the boundary conditions are written as

$$2\mathrm{Re}\left\{\frac{A_{11} - B_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[(\mu_2 - \bar{\mu}_2)\sqrt{z_1^2 - a^2} + (\bar{\mu}_2 - \mu_1)\sqrt{z_2^2 - a^2} + (\mu_2 - \mu_1)\sqrt{\bar{z}_2^2 - a^2}\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{\infty}\left\{\frac{A_{1n}}{a^n(\mu_2 - \bar{\mu}_2)}\left[(\mu_2 - \bar{\mu}_2)\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n + (\bar{\mu}_2 - \mu_1)\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n + (\mu_1 - \mu_2)\left(\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{\infty}\left\{\frac{B_{1n}}{a^n(\mu_2 - \bar{\mu}_2)}\left[(\mu_2 - \bar{\mu}_2)\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n + (\bar{\mu}_2 - \mu_1)\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n + (\mu_1 - \mu_2)\left(\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}$$

$$= -\int_s Y_n\,ds + C_1^o, \qquad z \text{ in } \Gamma \qquad 4.27$$

and

$$2\mathrm{Re}\left\{\frac{A_{11} + B_{11}}{a(\mu_2 - \bar{\mu}_2)}\; y(\mu_2 - \mu_1)(\mu_2 - \bar{\mu}_2)(\bar{\mu}_2 - \mu_1)\right\}$$

$$+ 2\mathrm{Re}\left\{\frac{A_{11} - B_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[\mu_1(\mu_2 - \bar{\mu}_2)\sqrt{z_1^2 - a^2} + \mu_2(\bar{\mu}_2 - \mu_1)\sqrt{z_2^2 - a^2}\right.\right.$$

$$\left.\left. + \bar{\mu}_2(\mu_2 - \mu_1)\sqrt{\bar{z}_2^2 - a^2}\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{\infty}\left\{\frac{A_{1n}}{a^n(\mu_2 - \bar{\mu}_2)}\left[\mu_1(\mu_2 - \bar{\mu}_2)\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n\right.\right.$$

$$+ \mu_2(\bar{\mu}_2 - \mu_1)\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n$$

$$\left.\left. + \bar{\mu}_2(\mu_1 - \mu_2)\left(\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{\infty}\left\{\frac{B_{1n}}{a^n(\mu_2 - \bar{\mu}_2)}\left[\mu_1(\mu_2 - \bar{\mu}_2)\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n\right.\right.$$

$$+ \mu_2(\bar{\mu}_2 - \mu_1)\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n$$

$$\left.\left. + \bar{\mu}_2(\mu_1 - \mu_2)\left(\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}$$

$$= \int_s X_n ds + C_2^o, \qquad z \text{ in } \Gamma \qquad 4.28$$

At this point, it is convenient to introduce the following definitions:

$$\mathscr{C}_s = \frac{A_{11} + B_{11}}{a(\mu_2 - \overline{\mu}_2)}$$

$$\mathscr{C}_d = \frac{A_{11} - B_{11}}{a(\mu_2 - \overline{\mu}_2)}$$

$$\mathscr{C}_{1n} = \frac{A_{1n}}{a^n(\mu_2 - \overline{\mu}_2)} \qquad 4.29$$

$$\mathscr{C}_{2n} = \frac{B_{1n}}{a^n(\mu_2 - \overline{\mu}_2)}$$

$$\mathscr{C}_c = (\mu_2 - \mu_1)(\mu_2 - \overline{\mu}_2)(\overline{\mu}_2 - \mu_1)$$

$$F_{11} = (\mu_2 - \overline{\mu}_2)\sqrt{z_1^2 - a^2} + (\overline{\mu}_2 - \mu_1)\sqrt{z_2^2 - a^2} + (\mu_2 - \mu_1)\sqrt{\overline{z}_2^2 - a^2}$$

$$G_{11} = \mu_1(\mu_2 - \overline{\mu}_2)\sqrt{z_1^2 - a^2} + \mu_2(\overline{\mu}_2 - \mu_1)\sqrt{z_2^2 - a^2} + \overline{\mu}_2(\mu_2 - \mu_1)\sqrt{\overline{z}_2^2 - a^2}$$

$$F_{1n} = (\mu_2 - \overline{\mu}_2)\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n + (\overline{\mu}_2 - \mu_1)\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n + (\mu_1 - \mu_2)\left(\overline{z}_2 - \sqrt{\overline{z}_2^2 - a^2}\right)^n$$

$$F_{2n} = (\mu_2 - \overline{\mu}_2)\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n + (\overline{\mu}_2 - \mu_1)\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n + (\mu_1 - \mu_2)\left(\overline{z}_2 + \sqrt{\overline{z}_2^2 - a^2}\right)^n$$

$$G_{1n} = \mu_1(\mu_2 - \overline{\mu}_2)\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n + \mu_2(\overline{\mu}_2 - \mu_1)\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n + \overline{\mu}_2(\mu_1 - \mu_2)\left(\overline{z}_2 - \sqrt{\overline{z}_2^2 - a^2}\right)^n$$

$$G_{2n} = \mu_1(\mu_2 - \overline{\mu}_2)\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n + \mu_2(\overline{\mu}_2 - \mu_1)\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n + \overline{\mu}_2(\mu_1 - \mu_2)\left(\overline{z}_2 + \sqrt{\overline{z}_2^2 - a^2}\right)^n$$

Now, observing that

$$F_{11}\mathscr{C}_d + \bar{F}_{11}\bar{\mathscr{C}}_d = 2[\mathrm{Re}F_{11}\mathrm{Re}\mathscr{C}_d - \mathrm{Im}F_{11}\mathrm{Im}\mathscr{C}_d] \qquad \text{etc.,}$$

the boundary conditions become

$$\mathrm{Re}F_{11}\mathrm{Re}\mathscr{C}_d - \mathrm{Im}F_{11}\mathrm{Im}\mathscr{C}_d + \sum_{n=2}^{\infty}[\mathrm{Re}F_{1n}\mathrm{Re}\mathscr{C}_{1n} - \mathrm{Im}F_{1n}\mathrm{Im}\mathscr{C}_{1n}]$$

$$+ \sum_{n=2}^{\infty}[\mathrm{Re}F_{2n}\mathrm{Re}\mathscr{C}_{2n} - \mathrm{Im}F_{2n}\mathrm{Im}\mathscr{C}_{2n}]$$

$$= \frac{1}{2}\left[-\int_s Y_n ds + C_1^o\right], \qquad z \text{ in } \Gamma \qquad 4.30$$

$$y\mathrm{Re}\mathscr{C}_c\mathrm{Re}\mathscr{C}_s - y\mathrm{Im}\mathscr{C}_c\mathrm{Im}\mathscr{C}_s + \mathrm{Re}G_{11}\mathrm{Re}\mathscr{C}_d - \mathrm{Im}G_{11}\mathrm{Im}\mathscr{C}_d$$

$$+ \sum_{n=2}^{\infty}[\mathrm{Re}G_{1n}\mathrm{Re}\mathscr{C}_{1n} - \mathrm{Im}G_{1n}\mathrm{Im}\mathscr{C}_{1n}] + \sum_{n=2}^{\infty}[\mathrm{Re}G_{2n}\mathrm{Re}\mathscr{C}_{2n} - \mathrm{Im}G_{2n}\mathrm{Im}\mathscr{C}_{2n}]$$

$$= \frac{1}{2}\left[\int_s X_n ds + C_2^o\right], \qquad z \text{ in } \Gamma \qquad 4.31$$

4.5 METHOD OF LEAST SQUARES BOUNDARY COLLOCATION

Up to this point, it was shown that the solution of the governing differential equation, 3.1, can be constructed in terms of the stress potentials, $\phi_1(z_1)$ and $\phi_2(z_2)$; and in turn, $\phi_2(z_2)$ can be defined in the form of $\phi_1(z_1)$ such that the zero traction conditions on the crack boundary are exactly satisfied. It still remains to satisfy the boundary conditions, 4.30 and 4.31, on the outer boundary of the region in order to complete the solution of the problem of a finite anisotropic region with a central crack.

Perhaps the best method applicable to this end is the least squares boundary collocation method as given in references 17 and 18. The method is described briefly in the following:

1. The boundary conditions are satisfied at discrete points on the boundary in the least squares sense.
2. One takes an overdetermined system of equations, i.e., the number of equations must exceed the number of unknowns of the system. Then in indicial notation, one has the following system of equations:

$$\mathcal{E}_{ij}\mathcal{C}_j = K_i \qquad 4.32$$

   where K_i depend on the loads acting on the boundary, $\mathcal{E}_{ij}$ are dependent on material properties and the boundary geometry and $\mathcal{C}_j$ are the unknowns of the system.
3. Upon assuming an approximation to the unknowns of the system, one has, say,

$$\mathcal{E}_{ij}\mathcal{C}^*_j - K_i = R_i \qquad 4.33$$

   where R_i are the error terms due to the approximation, $\mathcal{C}^*_j$ to $\mathcal{C}_j$.
4. Then the square of the error,

$$R_iR_i = (\mathcal{E}_{ij}\mathcal{C}^*_j - K_i)(\mathcal{E}_{ik}\mathcal{C}^*_k - K_i) \qquad 4.34$$

is minimized by taking

$$\frac{d(R_i R_i)}{d\mathscr{C}^*_\ell} = 0 \qquad 4.35$$

which in turn results in

$$\mathscr{E}_{i\ell}\mathscr{E}_{ik}\mathscr{C}^*_k = \mathscr{E}_{i\ell}K_i \qquad 4.36$$

Or in matrix notation, one has

$$\mathscr{E}^T\mathscr{E}\mathscr{C}^* = \mathscr{E}^T K \qquad 4.37$$

The applicability of the least squares boundary collocation method to the problem at hand is readily obvious. Considering the truncation of the infinite series' in the boundary conditions at NP corresponding to the series' with the unknown constants, $\mathscr{C}_{1n}$, and at NN for the series' with $\mathscr{C}_{2n}$, one has at a point on the outer boundary two equations in $2(NN + NP + 2)$ unknowns on the left hand side and two arbitrary integration constants on the right hand side. In general, these integration constants can be handled as unknowns in the collocation method; however, for certain orthotropic cases they can readily be determined. Then at a point on the boundary for the anisotropic case, one has the following two equations,

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 & \mathrm{Re}F_{11}\cdots -\mathrm{Im}F_{2NN} \\ 0 & -\frac{1}{2} & y\mathrm{Re}C_c & -y\mathrm{Im}C_c & \mathrm{Re}G_{11}\cdots -\mathrm{Im}G_{2NN} \end{bmatrix} \begin{bmatrix} C^o_1 \\ C^o_2 \\ \mathrm{Re}\mathscr{C}_s \\ \mathrm{Im}\mathscr{C}_s \\ \mathrm{Re}\mathscr{C}_d \\ \vdots \\ \mathrm{Im}\mathscr{C}_{2NN} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\int_s Y_n ds \\ \frac{1}{2}\int_s X_n ds \end{bmatrix}, \quad z \text{ in } \Gamma \qquad 4.38$$

Considering M points on the boundary, the coefficient matrix $\mathscr{S}$ will be of $2M$ rows and $2(NN + NP + 3)$ columns subject to the condition that $2M > 2(NN + NP + 3)$.

4.6 STRESS COMPONENTS

Using expressions 3.10 in conjunction with Eqs. 4.19 and 4.22, the stress components are obtained. These expressions are somewhat modified by noting the contraction of the $n=1$ terms and using some of the definitions given in 4.29:

$$
\begin{aligned}
\sigma_{xx} = {} & 2\mathrm{Re}\left[\mathscr{C}_c\mathscr{C}_s\right] \\
& + 2\mathrm{Re}\left\{\mathscr{C}_d\left[\mu_1^2(\mu_2 - \bar{\mu}_2)\frac{z_1}{\sqrt{z_1^2 - a^2}} + \mu_2^2(\bar{\mu}_2 - \mu_1)\frac{z_2}{\sqrt{z_2^2 - a^2}}\right.\right. \\
& \qquad\qquad \left.\left. + \bar{\mu}_2^2(\mu_2 - \mu_1)\frac{\bar{z}_2}{\sqrt{\bar{z}_2^2 - a^2}}\right]\right\} \\
& + 2\mathrm{Re}\sum_{n=2}^{NP}\left\{n\mathscr{C}_{1n}\left[\frac{\mu_1^2(\mu_2 - \bar{\mu}_2)}{\sqrt{z_1^2 - a^2}}\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n\right.\right. \\
& \left.\left. + \frac{\mu_2^2(\bar{\mu}_2 - \mu_1)}{\sqrt{z_2^2 - a^2}}\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n + \frac{\bar{\mu}_2^2(\mu_2 - \mu_1)}{\sqrt{\bar{z}_2^2 - a^2}}\left(\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\} \\
& - 2\mathrm{Re}\sum_{n=2}^{NN}\left\{n\mathscr{C}_{2n}\left[\frac{\mu_1^2(\mu_2 - \bar{\mu}_2)}{\sqrt{z_1^2 - a^2}}\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n\right.\right. \\
& \left.\left. + \frac{\mu_2^2(\bar{\mu}_2 - \mu_1)}{\sqrt{z_2^2 - a^2}}\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n + \frac{\bar{\mu}_2(\mu_2 - \mu_1)}{\sqrt{\bar{z}_2^2 - a^2}}\left(\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}
\end{aligned}
\qquad 4.39
$$

$$
\begin{aligned}
\sigma_{yy} = 2\mathrm{Re}\Bigg\{\mathscr{C}_d\Bigg[(\mu_2 - \overline{\mu}_2)\frac{z_1}{\sqrt{z_1^2 - a^2}} &+ (\overline{\mu}_2 - \mu_1)\frac{z_2}{\sqrt{z_2^2 - a^2}} \\
&+ (\mu_2 - \mu_1)\frac{\overline{z}_2}{\sqrt{\overline{z}_2^2 - a^2}}\Bigg]\Bigg\} \\
+ 2\mathrm{Re}\sum_{n=2}^{NP}\Bigg\{n\mathscr{C}_{1n}\Bigg[\frac{(\mu_2 - \overline{\mu}_2)}{\sqrt{z_1^2 - a^2}}\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n & \\
&+ \frac{(\overline{\mu}_2 - \mu_1)}{\sqrt{z_2^2 - a^2}}\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n \\
&+ \frac{(\mu_2 - \mu_1)}{\sqrt{\overline{z}_2^2 - a^2}}\left(\overline{z}_2 - \sqrt{\overline{z}_2^2 - a^2}\right)^n\Bigg]\Bigg\} \\
- 2\mathrm{Re}\sum_{n=2}^{NN}\Bigg\{n\mathscr{C}_{2n}\Bigg[\frac{(\mu_2 - \overline{\mu}_2)}{\sqrt{z_1^2 - a^2}}\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n & \\
&+ \frac{(\overline{\mu}_2 - \mu_1)}{\sqrt{z_2^2 - a^2}}\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n \\
&+ \frac{(\mu_2 - \mu_1)}{\sqrt{\overline{z}_2^2 - a^2}}\left(\overline{z}_2 + \sqrt{\overline{z}_2^2 - a^2}\right)^n\Bigg]\Bigg\}
\end{aligned}
\tag{4.40}
$$

$$\sigma_{xy} = - 2\mathrm{Re}\left\{\mathscr{C}_d\left[\mu_1(\mu_2 - \bar{\mu}_2)\frac{z_1}{\sqrt{z_1^2 - a^2}} + \mu_2(\bar{\mu}_2 - \mu_1)\frac{z_2}{\sqrt{z_2^2 - a^2}}\right.\right.$$

$$\left.\left. + \bar{\mu}_2(\mu_2 - \mu_1)\frac{\bar{z}_2^2}{\sqrt{\bar{z}_2^2 - a^2}}\right]\right\}$$

$$- 2\mathrm{Re}\sum_{n=2}^{NP}\left\{n\mathscr{C}_{1n}\left[\frac{\mu_1(\mu_2 - \bar{\mu}_2)}{\sqrt{z_1^2 - a^2}}\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n\right.\right.$$

$$+ \frac{\mu_2(\bar{\mu}_2 - \mu_1)}{\sqrt{z_2^2 - a^2}}\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n$$

$$\left.\left. + \frac{\bar{\mu}_2(\mu_2 - \mu_1)}{\sqrt{\bar{z}_2^2 - a^2}}\left(\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{NN}\left\{n\mathscr{C}_{2n}\left[\frac{\mu_1(\mu_2 - \bar{\mu}_2)}{\sqrt{z_1^2 - a^2}}\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n\right.\right.$$

$$+ \frac{\mu_2(\bar{\mu}_2 - \mu_1)}{\sqrt{z_2^2 - a^2}}\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n$$

$$\left.\left. + \frac{\bar{\mu}_2(\mu_2 - \mu_1)}{\sqrt{\bar{z}_2^2 - a^2}}\left(\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}$$

4.41

4.7 CRACK TIP STRESS COMPONENTS AND STRESS INTENSITY FACTORS

The stress components were expressed, in general, for any traction-free central crack problem in series form as given above. One can specialize these expressions for the neighborhood of the crack tip by considering the transformation,

$$z = a + r(\cos\theta + i\sin\theta) \qquad \text{(Fig. 6)} \qquad 4.42$$

The substitution of 4.42 into 4.3 will result in

$$z_k = a + r(\cos\theta + \mu_k \sin\theta) \qquad (k = 1,\ 2) \qquad 4.43$$

Subjecting the expressions for the stress components to these transformations and observing the condition for the neighborhood of the crack tip, (i.e., $r/a \ll 1$), the following expressions are obtained:

$$\sigma_{xx} = \frac{4}{\sqrt{2ra}}\,\mathrm{Re}\left[(\mu_2 - \mu_1)\sum\right]\mathrm{Re}\left\{\frac{\mu_1\mu_2}{\mu_1 - \mu_2}\left[\frac{\mu_2}{\sqrt{\cos\theta + \mu_2\sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta + \mu_1\sin\theta}}\right]\right\}$$
$$- \frac{4}{\sqrt{2ra}}\,\mathrm{Re}\left[\bar{\mu}_2(\mu_2 - \mu_1)\sum\right]\mathrm{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_2^2}{\sqrt{\cos\theta + \mu_2\sin\theta}} - \frac{\mu_1^2}{\sqrt{\cos\theta + \mu_1\sin\theta}}\right]\right\}$$

4.44

$$\sigma_{yy} = \frac{4}{\sqrt{2ra}}\,\mathrm{Re}\left[(\mu_2 - \mu_1)\sum\right]\mathrm{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_1}{\sqrt{\cos\theta + \mu_2\sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_1\sin\theta}}\right]\right\}$$
$$- \frac{4}{\sqrt{2ra}}\,\mathrm{Re}\left[\bar{\mu}_2(\mu_2 - \mu_1)\sum\right]\mathrm{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{1}{\sqrt{\cos\theta + \mu_2\sin\theta}} - \frac{1}{\sqrt{\cos\theta + \mu_1\sin\theta}}\right]\right\}$$

4.45

$$\sigma_{xy} = \frac{4}{\sqrt{2ra}} \mathrm{Re}\left[(\mu_2 - \mu_1)\sum\right] \mathrm{Re}\left\{\frac{\mu_1\mu_2}{\mu_1 - \mu_2}\left[\frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right]\right\}$$

$$- \frac{4}{\sqrt{2ra}} \mathrm{Re}\left[\overline{\mu}_2(\mu_2 - \mu_1)\sum\right] \mathrm{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right]\right\} \tag{4.46}$$

where

$$\sum = a\mathscr{C}_d + \sum_{n=2}^{NP} na^n\mathscr{C}_{1n} - \sum_{n=2}^{NN} na^n\mathscr{C}_{2n}$$

The forms of the crack tip stress components are the same as that of given by Paris and Sih (ref. 12) for an infinite anisotropic region with a central crack. Hence, the stress intensity factors for a finite anisotropic region with a central crack are automatically defined as

$$K_1 = \frac{4}{\sqrt{a}} \mathrm{Re}\left[(\mu_2 - \mu_1)\sum\right] \tag{4.47}$$

and

$$K_2 = -\frac{4}{\sqrt{a}} \mathrm{Re}\left[\overline{\mu}_2(\mu_2 - \mu_1)\sum\right] \tag{4.48}$$

where K_1 and K_2 are the opening mode and sliding mode stress intensity factors, respectively. A further examination of the stress intensity factors reveals that

$$K_2 + \mu_2 K_1 = -\frac{2}{\sqrt{a}}(\mu_1 - \mu_2) \lim_{\zeta \to 1} \phi_1'(\zeta) \tag{4.49}$$

which form was first published in reference 12.

In addition, one may note that the stress intensity factors depend on the material properties and the geometry of the region, while the stress components at the crack tip have a singularity of the order of $\sqrt{r}$. This singularity is similar to that obtained for isotropic materials.

4.8 DISPLACEMENTS

The substitution of Eqs. 4.19 and 4.22 into 3.14 results in the expressions for the displacements. These expressions are somewhat modified by using the definitions 4.29:

$$
\begin{aligned}
u = {} & 2(xC_{11} - yC_{16})\,\mathrm{Re}\left\{\mathscr{C}_s(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)\right\} \\
& + 2yC_{11}\,\mathrm{Re}\left\{\mathscr{C}_s\left[\mu_1^3(\mu_2 - \bar{\mu}_2) + \mu_2^3(\bar{\mu}_2 - \mu_1) + \bar{\mu}_2^3(\mu_1 - \mu_2)\right]\right\} \\
& + 2\,\mathrm{Re}\left\{\mathscr{C}_d\left[p_1(\mu_2 - \bar{\mu}_2)\sqrt{z_1^2 - a^2} + p_2(\bar{\mu}_2 - \mu_1)\sqrt{z_2^2 - a^2} + \bar{p}_2(\mu_2 - \mu_1)\sqrt{\bar{z}_2^2 - a^2}\right]\right\} \\
& + 2\,\mathrm{Re}\sum_{n=2}^{NP}\left\{\mathscr{C}_{1n}\left[p_1(\mu_2 - \bar{\mu}_2)\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n + p_2(\bar{\mu}_2 - \mu_1)\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n\right.\right. \\
& \qquad \left.\left. + \bar{p}_2(\mu_1 - \mu_2)\left(\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\} \\
& + 2\,\mathrm{Re}\sum_{n=2}^{NN}\left\{\mathscr{C}_{2n}\left[p_1(\mu_2 - \bar{\mu}_2)\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n + p_2(\bar{\mu}_2 - \mu_1)\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n\right.\right. \\
& \qquad \left.\left. + \bar{p}_2(\mu_1 - \mu_2)\left(\bar{z}_2 + \sqrt{\bar{z}_2^2 - a^2}\right)^n\right]\right\}
\end{aligned}
\qquad 4.50
$$

and

$$v = 2yC_{12}\mathrm{Re}\left\{\mathscr{C}_s(\mu_2 - \overline{\mu}_2)(\mu_2 - \mu_1)(\overline{\mu}_2 - \mu_1)\right\}$$

$$+ \frac{2xC_{22}}{\mu_1\overline{\mu}_1\mu_2\overline{\mu}_2}\mathrm{Re}\left\{\mathscr{C}_s(\mu_2 - \overline{\mu}_2)(\mu_2 - \mu_1)(\overline{\mu}_2 - \mu_1)\overline{\mu}_1\right\}$$

$$+ 2\mathrm{Re}\left\{\mathscr{C}_d\left[q_1(\mu_2 - \overline{\mu}_2)\sqrt{z_1^2 - a^2} + q_2(\overline{\mu}_2 - \mu_1)\sqrt{z_2^2 - a^2} + \overline{q}_2(\mu_2 - \mu_1)\sqrt{\overline{z}_2^2 - a^2}\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{NP}\left\{\mathscr{C}_{1n}\left[q_1(\mu_2 - \overline{\mu}_2)\left(z_1 + \sqrt{z_1^2 - a^2}\right)^n + q_2(\overline{\mu}_2 - \mu_1)\left(z_2 + \sqrt{z_2^2 - a^2}\right)^n + \overline{q}_2(\mu_1 - \mu_2)\left(\overline{z}_2 - \sqrt{\overline{z}_2^2 - a^2}\right)^n\right]\right\}$$

$$+ 2\mathrm{Re}\sum_{n=2}^{NN}\left\{\mathscr{C}_{2n}\left[q_1(\mu_2 - \overline{\mu}_2)\left(z_1 - \sqrt{z_1^2 - a^2}\right)^n + q_2(\overline{\mu}_2 - \mu_1)\left(z_2 - \sqrt{z_2^2 - a^2}\right)^n + \overline{q}_2(\mu_1 - \mu_2)\left(\overline{z}_2 + \sqrt{\overline{z}_2^2 - a^2}\right)^n\right]\right\} \qquad 4.51$$

In order to obtain the displacements on the crack boundary, one has to set

$$z_k + \sqrt{z_k^2 - a^2} = ae^{i\Theta} \qquad (k = 1,\ 2) \qquad 4.52$$

where

$$\Theta = \cos^{-1}\frac{x}{a}$$

Further simplification of the crack displacement expressions is possible by noting that

$$p_1 + p_2\frac{\overline{\mu}_2 - \mu_1}{\mu_2 - \overline{\mu}_2} + \overline{p}_2\frac{\mu_2 - \mu_1}{\overline{\mu}_2 - \mu_2} = C_{11}(\mu_2 - \mu_1)(\overline{\mu}_2 - \mu_1)$$

and

$$q_1 + q_2 \frac{\bar{\mu}_2 - \mu_1}{\mu_2 - \bar{\mu}_2} + \bar{q}_2 \frac{\mu_2 - \mu_1}{\bar{\mu}_2 - \mu_2} = \frac{C_{22}}{\mu_1\bar{\mu}_1\mu_2\bar{\mu}_2} \bar{\mu}_1(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)$$

Hence, the crack displacements become

$$u = 2C_{11}\mathrm{Re}\Big\{a\mathscr{C}_s(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)\cos\Theta$$

$$+ ia\mathscr{C}_d(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)\sin\Theta\Big\}$$

$$+ 2C_{11}\mathrm{Re}\sum_{n=2}^{NP}\Big[a^n\mathscr{C}_{1n}(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)(\cos n\Theta + i\sin n\Theta)\Big]$$

$$+ 2C_{11}\mathrm{Re}\sum_{n=2}^{NN}\Big[a^n\mathscr{C}_{2n}(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)(\cos n\Theta - i\sin n\Theta)\Big]$$

4.53

$$v = \frac{2C_{22}}{\mu_1\bar{\mu}_1\mu_2\bar{\mu}_2}\mathrm{Re}\Big\{a\mathscr{C}_s\bar{\mu}_1(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)\cos\Theta$$

$$+ ia\mathscr{C}_d\bar{\mu}_1(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)\sin\Theta\Big\}$$

$$+ \frac{2C_{22}\mathrm{Re}}{\mu_1\bar{\mu}_1\mu_2\bar{\mu}_2}\sum_{n=2}^{NP}\Big[a^n\mathscr{C}_{1n}\bar{\mu}_1(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)(\cos n\Theta + i\sin n\Theta)\Big]$$

$$+ \frac{2C_{22}\mathrm{Re}}{\mu_1\bar{\mu}_1\mu_2\bar{\mu}_2}\sum_{n=2}^{NN}\Big[a^n\mathscr{C}_{2n}\bar{\mu}_1(\mu_2 - \bar{\mu}_2)(\mu_2 - \mu_1)(\bar{\mu}_2 - \mu_1)(\cos n\Theta - i\sin n\Theta)\Big]$$

4.54

With the expressions of the crack displacements, the formulation of the method of solution of problems of central cracks in finite anisotropic regions can be considered complete. As a matter of fact, the method described above could also be used for the Second Fundamental Problem by employing Eqs. 4.50 and 4.51 for the prescribed displacements on the outer boundary.

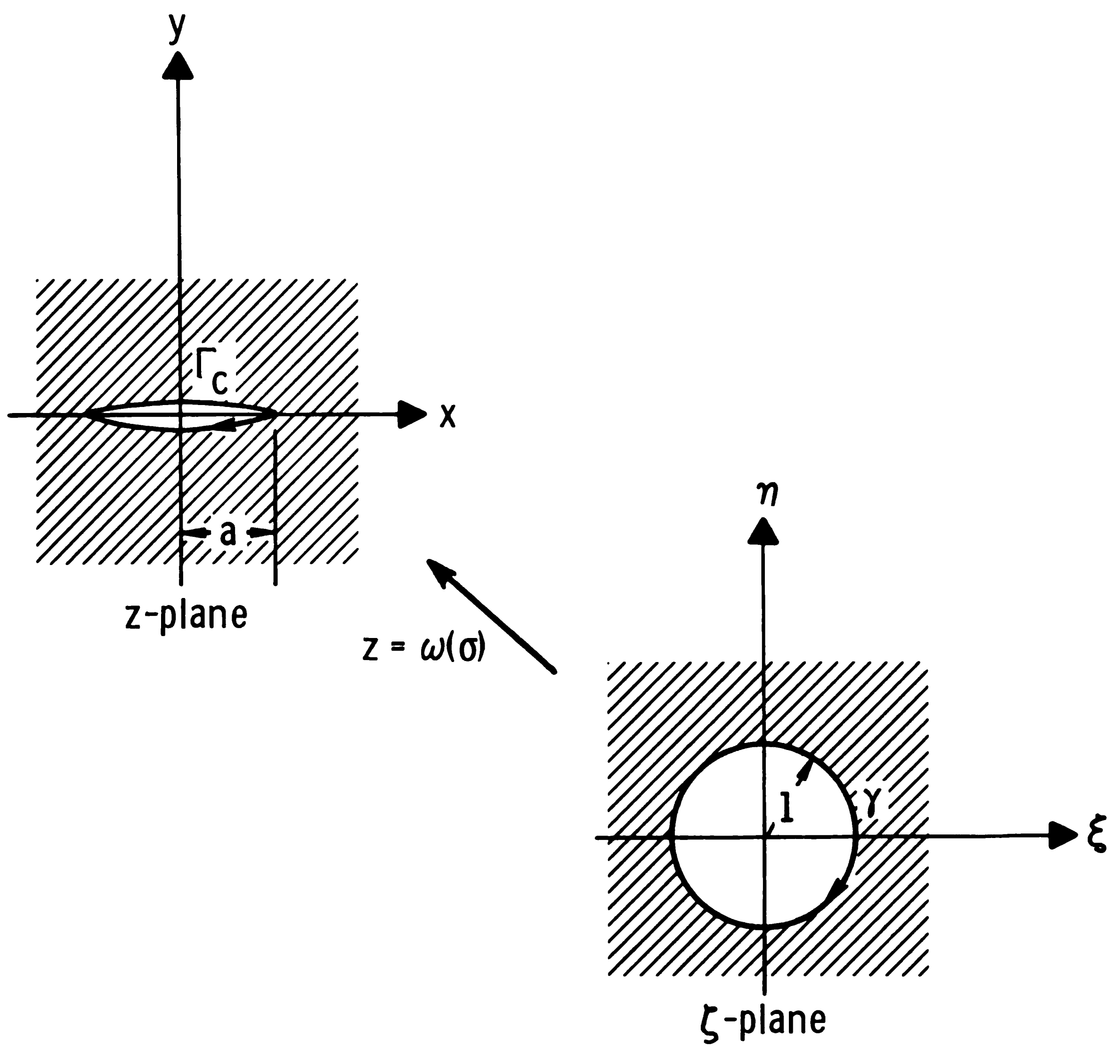


Figure 3. - The mapping of the unit circle into a crack.

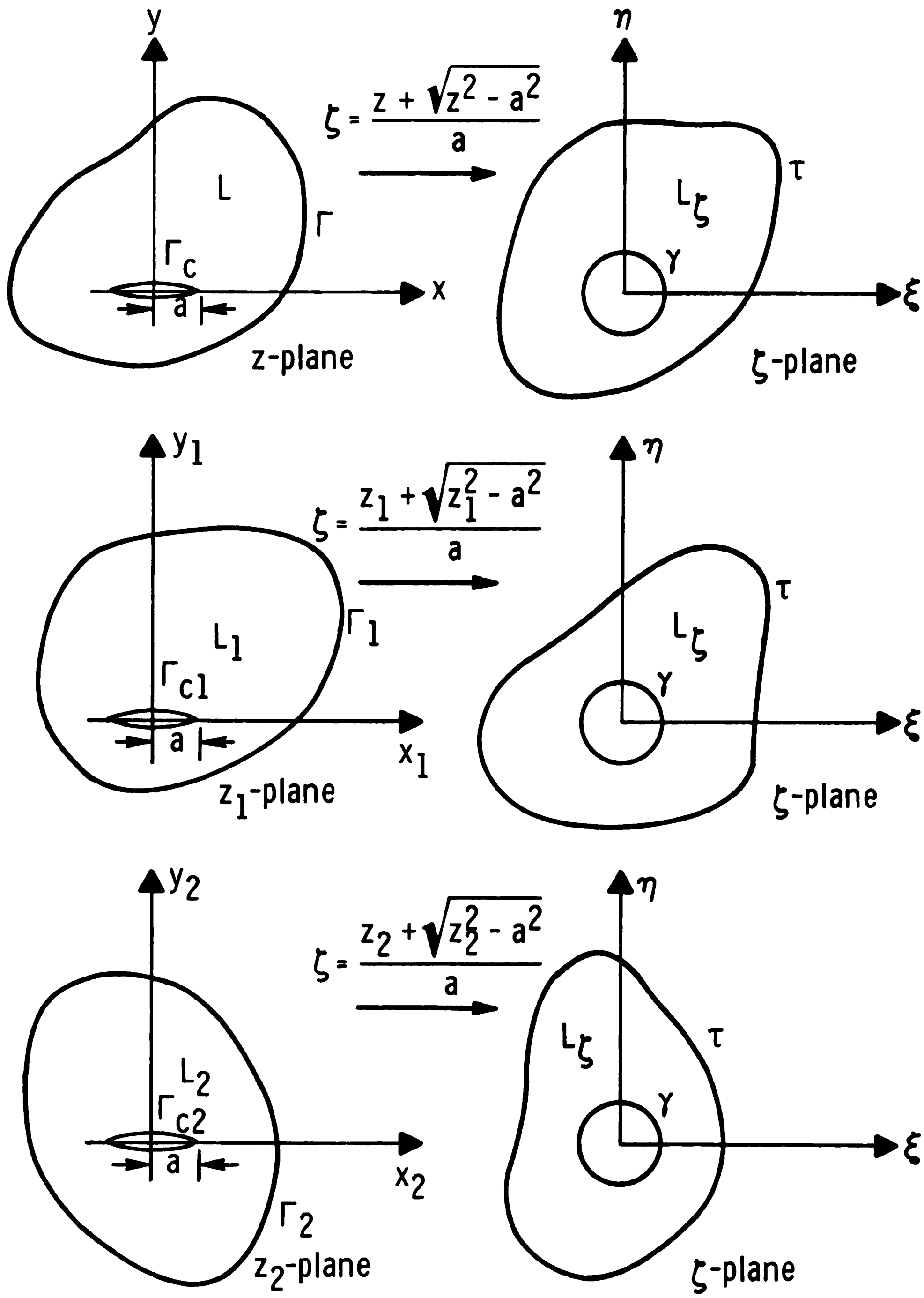


Figure 4. - The mapping of the doubly-connected regions, L, L_1, and L_2 into their corresponding parametric planes.

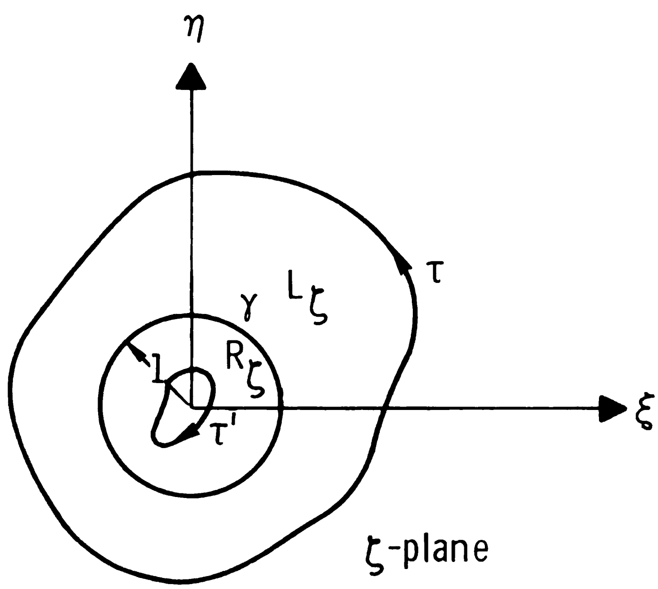


Figure 5. - The region, L_ζ, as continued across the unit circle.

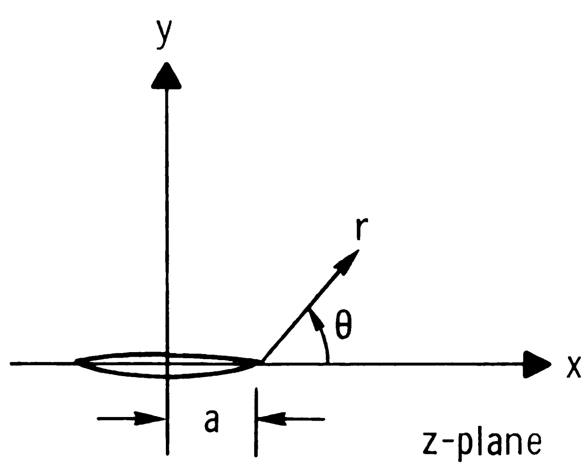


Figure 6. - The crack tip coordinate system.

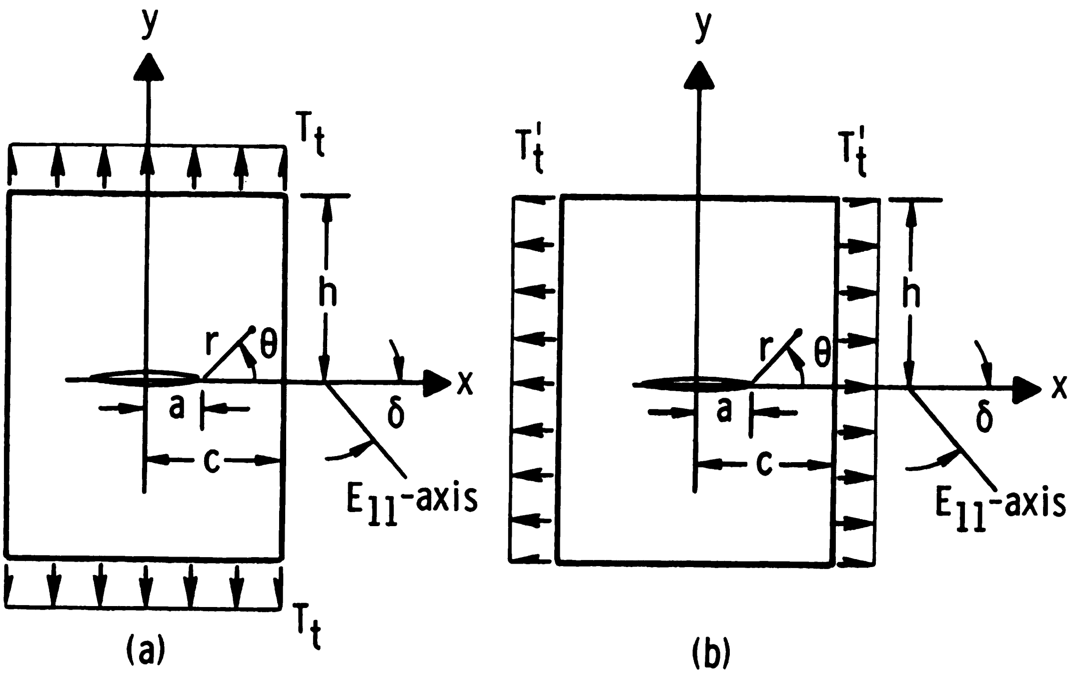


Figure 7. - The tension problems of a rectangular plate with a central crack.

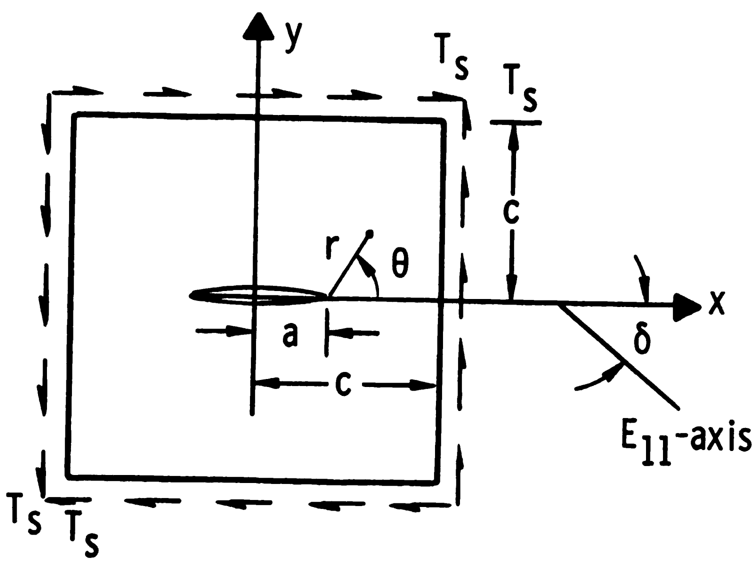


Figure 8. - The shear problem of a square plate with a central crack.

5. THE ANISOTROPIC INFINITE PLATE CONTAINING A CENTRALLY LOCATED TRACTION-FREE CRACK UNDER CONSTANT LOAD

5.1 INTRODUCTION

The conventional method of solving the problem of a centrally cracked infinite anisotropic plate under constant loading is due to Lekhnitskii (ref. 4) and Savin (ref. 6). They consider the superposition of a constant state of stress in an infinite plate and that of the stress disturbance caused by the presence of the crack.

In this section it will be demonstrated that the solutions of Lekhnitskii and Savin can also be obtained by a direct limiting process applied to the dimensions of a finite rectangular plate under constant loading. In the following, three problems will be considered:

1. Central crack in an infinite plate under constant load in the y-direction.
2. Central crack in an infinite plate under constant shear loading.
3. Central crack in an infinite plate under constant load in the x-direction.

5.2 CENTRAL CRACK IN AN INFINITE PLATE UNDER CONSTANT LOAD IN THE y-DIRECTION

The boundary conditions 4.24 and 4.25 are specialized to the finite plate shown in Figure 7(a). The infinite plate can then be considered by letting the dimensions of the plate approach infinity. As a consequence of $z \to \infty$, one must set

$$A_{1n} = B_{1n} = 0 \qquad \text{for} \quad n \geq 2$$

in order to preserve the finiteness of the stress components at infinity. Then the boundary conditions become

$$\frac{2A_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[(\mu_2 - \bar{\mu}_2)z_1 + (\bar{\mu}_2 - \mu_1)z_2\right] + \frac{2\bar{A}_{11}}{a(\bar{\mu}_2 - \mu_2)}\left[(\bar{\mu}_2 - \mu_2)\bar{z}_1 + (\mu_2 - \bar{\mu}_1)\bar{z}_2\right]$$

$$-\frac{2B_{11}}{a(\mu_2 - \bar{\mu}_2)}(\mu_2 - \mu_1)\bar{z}_2 - \frac{2\bar{B}_{11}}{a(\bar{\mu}_2 - \mu_2)}(\bar{\mu}_2 - \bar{\mu}_1)z_2 = \begin{cases} C_1^o & \text{on} \quad x = c \\ -2T_t c + C_1^o & \text{on} \quad x = -c \\ T_t(x - c) + C_1^o & \text{on} \quad y = \pm h \end{cases}$$

5.1

and

$$\frac{2A_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[\mu_1(\mu_2 - \bar{\mu}_2)z_1 + \mu_2(\bar{\mu}_2 - \mu_1)z_2\right] + \frac{2\bar{A}_{11}}{a(\bar{\mu}_2 - \mu_2)}\left[\bar{\mu}_1(\bar{\mu}_2 - \mu_2)\bar{z}_1\right.$$

$$\left. + \bar{\mu}_2(\mu_2 - \bar{\mu}_1)\bar{z}_2\right] - \frac{2B_{11}}{a(\mu_2 - \bar{\mu}_2)}\bar{\mu}_2(\mu_2 - \mu_1)\bar{z}_2 - \frac{2\bar{B}_{11}}{a(\bar{\mu}_2 - \mu_2)}\mu_2(\bar{\mu}_2 - \bar{\mu}_1)z_2$$

$$= \begin{cases} C_2^o & \text{on} \quad x = \pm c \\ C_2^o & \text{on} \quad y = \pm h \end{cases}$$

5.2

Since C_2^o is a finite arbitrary constant, the consideration, $z \to \infty$, applied to Eq. 5.2 results in two independent equations:

$$A_{11}\bar{\mu}_2(\mu_2 - \mu_1) - \bar{A}_{11}\mu_2(\bar{\mu}_2 - \bar{\mu}_1) - B_{11}\bar{\mu}_2(\mu_2 - \mu_1) + \bar{B}_{11}\mu_2(\bar{\mu}_2 - \bar{\mu}_1) = 0$$

5.3

and

$$A_{11}\left[\mu_1^2(\mu_2 - \bar{\mu}_2) + \mu_2^2(\bar{\mu}_2 - \mu_1)\right] - \bar{A}_{11}\left[\bar{\mu}_1^2(\bar{\mu}_2 - \mu_2) + \bar{\mu}_2^2(\mu_2 - \bar{\mu}_1)\right]$$

$$- B_{11}\bar{\mu}_2^2(\mu_2 - \mu_1) + \bar{B}_{11}\mu_2^2(\bar{\mu}_2 - \bar{\mu}_1) = 0 \qquad 5.4$$

on both $x = \pm c$ and $y = \pm h$. Then Eq. 5.1 can be written as,

$$x[A_{11}(\mu_2 - \mu_1) - \bar{A}_{11}(\bar{\mu}_2 - \bar{\mu}_1) - B_{11}(\mu_2 - \mu_1) + \bar{B}_{11}(\bar{\mu}_2 - \bar{\mu}_1)]$$

$$= \begin{cases} \dfrac{C_1^o a(\mu_2 - \bar{\mu}_2)}{2} & \text{on } x = c \\[2ex] -T_t ca(\mu_2 - \bar{\mu}_2) + \dfrac{C_1^o a(\mu_2 - \bar{\mu}_2)}{2} & \text{on } x = -c \\[2ex] \dfrac{T_t(x - c)a(\mu_2 - \bar{\mu}_2)}{2} + \dfrac{C_1^o a(\mu_2 - \bar{\mu}_2)}{2} & \text{on } y = \pm h \end{cases} \qquad 5.5$$

In order to satisfy Eq. 5.5, one must take

$$A_{11}(\mu_2 - \mu_1) - \bar{A}_{11}(\bar{\mu}_2 - \bar{\mu}_1) - B_{11}(\mu_2 - \mu_1) + \bar{B}_{11}(\bar{\mu}_2 - \bar{\mu}_1) = \frac{T_t a(\mu_2 - \bar{\mu}_2)}{2} \qquad 5.6$$

and

$$C_1^o = T_t c \qquad 5.7$$

At this point, it is noted that there are only three equations available (Eqs. 5.3, 5.4 and 5.6) for the determination of four constants. Consequently, one of the constants must be arbitrary. One may also note that these three expressions correspond to the stress components at infinity.

From Eqs. 5.3 and 5.6, one obtains

$$A_{11} - B_{11} = \frac{T_t a\mu_2}{2(\mu_2 - \mu_1)} \qquad 5.8$$

It is then convenient to express the stress components (Eqs. 4.39, 4.40 and 4.41) as follows:

$$\sigma_{xx} = 2\mathrm{Re}\left\{\frac{A_{11} - B_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[\mu_1^2(\mu_2 - \bar{\mu}_2)\frac{z_1 - \sqrt{z_1^2 - a^2}}{\sqrt{z_1^2 - a^2}} + \mu_2^2(\bar{\mu}_2 - \mu_1)\frac{z_2 - \sqrt{z_2^2 - a^2}}{\sqrt{z_2^2 - a^2}}\right.\right.$$

$$\left.\left. + \bar{\mu}_2^2(\mu_2 - \mu_1)\frac{\bar{z}_2 - \sqrt{\bar{z}_2^2 - a^2}}{\sqrt{\bar{z}_2^2 - a^2}}\right]\right\}$$

$$+ 4\mathrm{Re}\left\{\frac{A_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[\mu_1^2(\mu_2 - \bar{\mu}_2) + \mu_2^2(\bar{\mu}_2 - \mu_1)\right] - \frac{B_{11}}{a(\mu_2 - \bar{\mu}_2)}\bar{\mu}_2^2(\mu_2 - \mu_1)\right\} \qquad 5.9$$

$$\sigma_{yy} = 2\mathrm{Re}\left\{\frac{A_{11} - B_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[(\mu_2 - \bar{\mu}_2)\frac{z_1}{\sqrt{z_1^2 - a^2}} + (\bar{\mu}_2 - \mu_1)\frac{z_2}{\sqrt{z_2^2 - a^2}}\right.\right.$$

$$\left.\left. + (\mu_2 - \mu_1)\frac{\bar{z}_2}{\sqrt{\bar{z}_2^2 - a^2}}\right]\right\} \qquad 5.10$$

$$\sigma_{xy} = -2\mathrm{Re}\left\{\frac{A_{11} - B_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[\mu_1(\mu_2 - \bar{\mu}_2)\frac{z_1}{\sqrt{z_1^2 - a^2}} + \mu_2(\bar{\mu}_2 - \mu_1)\frac{z_2}{\sqrt{z_2^2 - a^2}}\right.\right.$$

$$\left.\left. + \bar{\mu}_2(\mu_2 - \mu_1)\frac{\bar{z}_2}{\sqrt{\bar{z}_2^2 - a^2}}\right]\right\} \qquad 5.11$$

Upon applying the boundary conditions in the forms of 5.4 and 5.8, the expressions for the stress components are obtained:

$$\sigma_{xx} = \mathrm{Re}\left\{\frac{T_t\mu_1\mu_2a^2}{\mu_2 - \mu_1}\left[\frac{\mu_1}{\sqrt{z_1^2 - a^2}\left(z_1 + \sqrt{z_1^2 - a^2}\right)} - \frac{\mu_2}{\sqrt{z_2^2 - a^2}\left(z_2 + \sqrt{z_2^2 - a^2}\right)}\right]\right\}$$

5.12

$$\sigma_{yy} = \mathrm{Re}\left\{\frac{T_t}{\mu_2 - \mu_1}\left[\frac{\mu_2 z_1}{\sqrt{z_1^2 - a^2}} - \frac{\mu_1 z_2}{\sqrt{z_2^2 - a^2}}\right]\right\} \qquad 5.13$$

$$\sigma_{xy} = \mathrm{Re}\left\{\frac{T_t\mu_1\mu_2}{\mu_1 - \mu_2}\left[\frac{z_1}{\sqrt{z_1^2 - a^2}} - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right]\right\} \qquad 5.14$$

These expressions agree with either Savin's or Lekhnitskii's results.

In order to obtain the stress field in the neighborhood of the crack tip, the following transformation is used:

$$z_k = a + r(\cos\theta + \mu_k \sin\theta) \qquad (k = 1, 2) \qquad 5.15$$

where r, θ and a are defined in Figure 6.

Considering $\frac{r}{a} << 1$, one obtains

$$\sqrt{z_k^2 - a^2}\left(z_k + \sqrt{z_k^2 - a^2}\right) = a\sqrt{2ra}\sqrt{\cos\theta + \mu_k \sin\theta} \qquad (k = 1, 2)$$

and

$$\frac{z_k}{\sqrt{z_k^2 - a^2}} \quad \frac{a}{\sqrt{2ra}\sqrt{\cos\theta + \mu_k \sin\theta}} \qquad (k = 1, 2)$$

Hence, the stress components are written for the neighborhood of the crack tip as,

$$\sigma_{xx} = \frac{T_t\sqrt{a}}{\sqrt{2r}}\,\mathrm{Re}\left\{\frac{\mu_1\mu_2}{\mu_2-\mu_1}\left[\frac{\mu_1}{\sqrt{\cos\theta+\mu_1\sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta+\mu_2\sin\theta}}\right]\right\} \tag{5.16}$$

$$\sigma_{yy} = \frac{T_t\sqrt{a}}{\sqrt{2r}}\,\mathrm{Re}\left\{\frac{1}{\mu_2-\mu_1}\left[\frac{\mu_2}{\sqrt{\cos\theta+\mu_1\sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta+\mu_2\sin\theta}}\right]\right\} \tag{5.17}$$

$$\sigma_{xy} = \frac{T_t\sqrt{a}}{\sqrt{2r}}\,\mathrm{Re}\left\{\frac{\mu_1\mu_2}{\mu_1-\mu_2}\left[\frac{1}{\sqrt{\cos\theta+\mu_1\sin\theta}} - \frac{1}{\sqrt{\cos\theta+\mu_2\sin\theta}}\right]\right\} \tag{5.18}$$

The stress intensity factors are obviously

$$K_1 = T_t\sqrt{a} \qquad K_2 = 0 \tag{5.19}$$

i.e., the sliding mode stress intensity factor is not present in the solution.

5.3 CENTRAL CRACK IN AN INFINITE PLATE UNDER CONSTANT SHEAR LOADING

According to Figure 8, the boundary conditions, 4.24 and 4.25, with $z \to \infty$, become

$$\frac{2A_{11}}{a(\mu_2-\bar{\mu}_2)}\left[(\mu_2-\bar{\mu}_2)z_1+(\bar{\mu}_2-\mu_1)z_2\right]+\frac{2\bar{A}_{11}}{a(\bar{\mu}_2-\mu_2)}\left[(\bar{\mu}_2-\mu_2)\bar{z}_1+(\mu_2-\bar{\mu}_1)\bar{z}_2\right]$$

$$-\frac{2B_{11}}{a(\mu_2-\bar{\mu}_2)}(\mu_2-\mu_1)\bar{z}_2-\frac{2\bar{B}_{11}}{a(\bar{\mu}_2-\mu_2)}(\bar{\mu}_2-\bar{\mu}_1)z_2 = \begin{cases}-T_s y + C_1^o & \text{on } x = \pm c\\ -T_s c + C_1^o & \text{on } y = h\\ T_s c + C_1^o & \text{on } y = -h\end{cases} \tag{5.20}$$

and

$$\frac{2A_{11}}{a(\mu_2-\bar{\mu}_2)}\left[\mu_1(\mu_2-\bar{\mu}_2)z_1+\mu_2(\bar{\mu}_2-\mu_1)z_2\right]+\frac{2\bar{A}_{11}}{a(\bar{\mu}_2-\mu_2)}\left[\bar{\mu}_1(\bar{\mu}_2-\mu_2)\bar{z}_1\right.$$

$$\left.+\bar{\mu}_2(\mu_2-\bar{\mu}_1)\bar{z}_2\right]-\frac{2B_{11}}{a(\mu_2-\bar{\mu}_2)}\bar{\mu}_2(\mu_2-\mu_1)\bar{z}_2-\frac{2\bar{B}_{11}}{a(\bar{\mu}_2-\mu_2)}\mu_2(\bar{\mu}_2-\bar{\mu}_1)z_2$$

$$=\begin{cases} C_2^o & \text{on} \quad x=c \\ 2T_s c+C_2^o & \text{on} \quad x=-c \\ T_s(c-x)+C_2^o & \text{on} \quad y=\pm h \end{cases} \qquad 5.21$$

The boundary conditions, 5.20 and 5.21 can further be simplified, resulting in the following three equations in four unknowns

$$A_{11}\left[\mu_1^2(\mu_2-\bar{\mu}_2)+\mu_2^2(\bar{\mu}_2-\mu_1)\right]-\bar{A}_{11}\left[\bar{\mu}_1^2(\bar{\mu}_2-\mu_2)+\bar{\mu}_2^2(\mu_2-\bar{\mu}_1)\right]$$

$$-B_{11}\bar{\mu}_2^2(\mu_2-\mu_1)+\bar{B}_{11}\mu_2^2(\bar{\mu}_2-\bar{\mu}_1)=0 \qquad 5.22$$

$$A_{11}(\mu_2-\mu_1)-\bar{A}_{11}(\bar{\mu}_2-\bar{\mu}_1)-B_{11}(\mu_2-\mu_1)+\bar{B}_{11}(\bar{\mu}_2-\bar{\mu}_1)=0 \qquad 5.23$$

$$A_{11}\bar{\mu}_2(\mu_2-\mu_1)-\bar{A}_{11}\mu_2(\bar{\mu}_2-\bar{\mu}_1)-B_{11}\bar{\mu}_2(\mu_2-\mu_1)+\bar{B}_{11}\mu_2(\bar{\mu}_2-\bar{\mu}_1)$$

$$=-\frac{T_s a(\mu_2-\bar{\mu}_2)}{2} \qquad 5.24$$

The unknown integration constant was found to be

$$C_2^o=-T_s c \qquad 5.25$$

One may again note, as in the tension problem above, that one of the four unknown constants is arbitrary.

From Eqs. 5.23 and 5.24, one obtains

$$A_{11} - B_{11} = \frac{T_s a}{2(\mu_2 - \mu_1)} \qquad 5.26$$

Upon substituting Eqs. 5.22 and 5.26 into the expressions for the stress components (Eqs. 5.9, 5.10 and 5.11), the resulting expressions become

$$\sigma_{xx} = \mathrm{Re}\left\{\frac{T_s a^2}{\mu_2 - \mu_1}\left[\frac{\mu_1^2}{\sqrt{z_1^2 - a^2}\left(z_1 + \sqrt{z_1^2 - a^2}\right)} - \frac{\mu_2^2}{\sqrt{z_2^2 - a^2}\left(z_2 + \sqrt{z_2^2 - a^2}\right)}\right]\right\} \qquad 5.27$$

$$\sigma_{yy} = \mathrm{Re}\left\{\frac{T_s}{\mu_2 - \mu_1}\left[\frac{z_1}{\sqrt{z_1^2 - a^2}} - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right]\right\} \qquad 5.28$$

$$\sigma_{xy} = \mathrm{Re}\left\{\frac{T_s}{\mu_1 - \mu_2}\left[\frac{\mu_1 z_1}{\sqrt{z_1^2 - a^2}} - \frac{\mu_2 z_2}{\sqrt{z_2^2 - a^2}}\right]\right\} \qquad 5.29$$

With the transformation, 5.15, the crack tip stress field was found to be

$$\sigma_{xx} = \frac{T_s\sqrt{a}}{\sqrt{2r}}\,\mathrm{Re}\left\{\frac{1}{\mu_2 - \mu_1}\left[\frac{\mu_1^2}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{\mu_2^2}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right]\right\} \qquad 5.30$$

$$\sigma_{yy} = \frac{T_s\sqrt{a}}{\sqrt{2r}} \mathrm{Re}\left\{\frac{1}{\mu_2 - \mu_1}\left[\frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right]\right\} \qquad 5.31$$

$$\sigma_{xy} = \frac{T_s\sqrt{a}}{\sqrt{2r}} \mathrm{Re}\left\{\frac{1}{\mu_1 - \mu_2}\left[\frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right]\right\} \qquad 5.32$$

In this case, the stress intensity factors are

$$K_1 = 0 \qquad K_2 = T_s\sqrt{a} \qquad 5.33$$

i.e., the opening mode stress intensity factor is not present in this solution.

5.4 CENTRAL CRACK IN AN INFINITE PLATE UNDER CONSTANT LOAD IN THE x-DIRECTION

In accordance with Figure 7(b), as $z \to \infty$, the boundary conditions, 4.24 and 4.25, become

$$\frac{2A_{11}}{a(\mu_2 - \bar{\mu}_2)}\left[(\mu_2 - \bar{\mu}_2)z_1 + (\bar{\mu}_2 - \mu_1)z_2\right] + \frac{2\bar{A}_{11}}{a(\bar{\mu}_2 - \mu_2)}\left[(\bar{\mu}_2 - \mu_2)\bar{z}_1 + (\mu_2 - \bar{\mu}_1)\bar{z}_2\right]$$

$$- \frac{2B_{11}}{a(\mu_2 - \bar{\mu}_2)}(\mu_2 - \mu_1)\bar{z}_2 \quad \frac{2\bar{B}_{11}}{a(\bar{\mu}_2 - \mu_2)}(\bar{\mu}_2 - \bar{\mu}_1)z_2 = \begin{cases} C_1^o & \text{on} \quad x = \pm c \\ C_1^o & \text{on} \quad y = \pm h \end{cases} \qquad 5.34$$

$$\frac{2A_{11}}{a(\mu_2-\bar{\mu}_2)}\left[\mu_1(\mu_2-\bar{\mu}_2)z_1+\mu_2(\bar{\mu}_2-\mu_1)z_2\right]+\frac{2\bar{A}_{11}}{a(\bar{\mu}_2-\mu_2)}\left[\bar{\mu}_1(\bar{\mu}_2-\mu_2)\bar{z}_1\right.$$

$$\left.+\bar{\mu}_2(\mu_2-\bar{\mu}_1)\bar{z}_2\right]-\frac{2B_{11}}{a(\mu_2-\bar{\mu}_2)}\bar{\mu}_2(\mu_2-\mu_1)\bar{z}_2-\frac{2\bar{B}_{11}}{a(\bar{\mu}_2-\mu_2)}\mu_2(\bar{\mu}_2-\bar{\mu}_1)z_2$$

$$=\begin{cases} T_t'y+C_2^o & \text{on} \quad x=\pm c \\ T_t'h+C_2^o & \text{on} \quad y=h \\ -T_t'h+C_2^o & \text{on} \quad y=-h \end{cases} \qquad 5.35$$

These conditions will also result in three equations having four unknowns

$$A_{11}\left[\mu_1^2(\mu_2-\bar{\mu}_2)+\mu_2^2(\bar{\mu}_2-\mu_1)\right]-\bar{A}_{11}\left[\bar{\mu}_1^2(\bar{\mu}_2-\mu_2)+\bar{\mu}_2^2(\mu_2-\bar{\mu}_1)\right]$$

$$-B_{11}\bar{\mu}_2^2(\mu_2-\mu_1)+\bar{B}_{11}\mu_2^2(\bar{\mu}_2-\bar{\mu}_1)=\frac{T_t'a(\mu_2-\bar{\mu}_2)}{2} \qquad 5.36$$

$$A_{11}(\mu_2-\mu_1)-\bar{A}_{11}(\bar{\mu}_2-\bar{\mu}_1)-B_{11}(\mu_2-\mu_1)+\bar{B}_{11}(\bar{\mu}_2-\bar{\mu}_1)=0 \qquad 5.37$$

$$A_{11}\bar{\mu}_2(\mu_2-\mu_1)-\bar{A}_{11}\mu_2(\bar{\mu}_2-\bar{\mu}_1)-B_{11}\bar{\mu}_2(\mu_2-\mu_1)+\bar{B}_{11}\mu_2(\bar{\mu}_2-\bar{\mu}_1)=0$$

5.38

From Eqs. 5.37 and 5.38

$$A_{11}-B_{11}=0 \qquad 5.39$$

Hence, upon substituting into the expressions for the stress components, one obtains

$$\sigma_{xx}=T_t' \qquad 5.40$$

$$\sigma_{yy}=\sigma_{xy}=0 \qquad 5.41$$

Therefore, one can conclude that the stress field is not influenced by the crack in this particular case.

6. PROBLEM FORMULATION FOR A FINITE ORTHOTROPIC RECTANGULAR PLATE CONTAINING A CENTRALLY LOCATED TRACTION-FREE CRACK UNDER CONSTANT LOAD

6.1 INTRODUCTION

The motivation for considering the problem of a finite orthotropic plate with a central crack is mainly due to the structural importance of unidirectional materials in present day design. As it will be shown, the handling of such problems is somewhat simpler than that of anisotropic problems with regard to the size of the coefficient matrix in the boundary collocation method. This simplification is a consequence of stress symmetry prevailing in a symmetrically loaded rectangular orthotropic plate.

The consideration of stress symmetry will lead one to the conclusion that only the odd terms are needed in the Laurent series expansion of the stress potentials. One also finds that the arbitrary integration constants, C_1^o and C_2^o, can also be determined before collocation.

For the loads considered in the case of the anisotropic infinite plate (Figures 7(a), 7(b) and 8), the boundary conditions of the finite orthotropic plate will be given in this section in a form readily applicable to the boundary collocation method.

6.2 STRESS SYMMETRY

From physical considerations of the problems at hand (Figures 7(a), 7(b) and 8) in the case of orthotropic materials one can expect symmetry in the stress components such that

$$\sigma_{xx}(z) = \sigma_{xx}(-z)$$

$$\sigma_{yy}(z) = \sigma_{yy}(-z) \qquad 6.1$$

$$\sigma_{xy}(z) = \sigma_{xy}(-z)$$

It can be shown that in order to satisfy these symmetry conditions, one has to set

$$\mathscr{C}_{1n} = \mathscr{C}_{2n} = 0 \qquad \text{for} \quad n \quad \text{even}$$

As a consequence to this relation among the unknown constants, the satisfaction of the boundary conditions is required on the $y > 0$ part of the boundary only.

For the further consideration of special orthotropy, i.e., for $\delta = 0^\circ, 90^\circ$, in addition to conditions 6.1, the following symmetry is also obvious

$$\sigma_{xx}(z) = \sigma_{xx}(-\bar{z})$$

$$\sigma_{yy}(z) = \sigma_{yy}(-\bar{z}) \qquad 6.2$$

$$\sigma_{xy}(z) = \sigma_{xy}(-\bar{z})$$

The symmetry conditions 6.2 will lead one to the conclusion that the constants $\mathscr{C}_s$, $\mathscr{C}_d$, $\mathscr{C}_{1n}$ and $\mathscr{C}_{2n}$ are pure imaginary quantities. Thus, the above problems can be solved with the satisfaction of the boundary conditions on the boundary in the first quadrant for specially orthotropic materials.

6.3 THE DETERMINATION OF THE INTEGRATION CONSTANTS

First, the boundary conditions 4.27 and 4.28 are written at (c,0) and (-c,0):

$$2\text{Re}\sum_{n=1}^{\infty}\left\{\left[\frac{\mu_2-\mu_1}{\mu_2-\bar{\mu}_2}(A_{1n}-B_{1n})\right]\left(\frac{c}{a}\right)^n\left[\left(1+\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n-\left(1-\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n\right]\right\}$$

$$= -\int_s Y_n ds\Big]_{\substack{x=c\\y=0}} + C_1^o \qquad 6.3$$

$$2\text{Re}\sum_{n=1}^{\infty}\left\{\left[\frac{\bar{\mu}_2(\mu_2-\mu_1)}{\mu_2-\bar{\mu}_2}(A_{1n}-B_{1n})\right]\left(\frac{c}{a}\right)^n\left[\left(1+\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n-\left(1+\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n\right]\right\}$$

$$= \int_s X_n ds\Big]_{\substack{x=c\\y=0}} + C_2^o \qquad 6.4$$

$$2\text{Re}\sum_{n=1}^{\infty}\left\{\left[\frac{\mu_2-\mu_1}{\mu_2-\bar{\mu}_2}(A_{1n}-B_{1n})\right]\left(-\frac{c}{a}\right)^n\left[\left(1+\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n-\left(1-\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n\right]\right\}$$

$$= -\int_s Y_n ds\Big]_{\substack{x=-c\\y=0}} + C_1^o \qquad 6.5$$

$$2\text{Re}\sum_{n=1}^{\infty}\left\{\left[\frac{\bar{\mu}_2(\mu_2-\mu_1)}{\mu_2-\bar{\mu}_2}(A_{1n}-B_{1n})\right]\left(-\frac{c}{a}\right)^n\left[\left(1+\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n-\left(1-\sqrt{1-\left(\frac{a}{c}\right)^2}\right)^n\right]\right\}$$

$$= \int_s X_n ds\Big]_{\substack{x=-c\\y=0}} + C_2^o \qquad 6.6$$

Upon considering n odd (orthotropic materials), the addition of Eq. 6.3 to Eq. 6.5 and Eq. 6.4 to Eq. 6.6 will result in

$$C_1^o = \frac{1}{2}\left\{\left.\int_s Y_n ds\right]_{\substack{x=c\\y=0}} + \left.\int_s Y_n ds\right]_{\substack{x=-c\\y=0}}\right\} \tag{6.7}$$

$$C_2^o = \frac{1}{2}\left\{\left.-\int_s X_n ds\right]_{\substack{x=c\\y=0}} - \left.\int_s X_n ds\right]_{\substack{x=-c\\y=0}}\right\} \tag{6.8}$$

Then for the three different cases of loading shown in Figures 7(a), 7(b) and 8, the arbitrary integration constants become

Constant load in y-direction: $C_1^o = T_t c$, $C_2^o = 0$ (6.9)

Constant shear loading: $C_1^o = 0$, $C_2^o = -T_s c$ (6.10)

Constant load in x-direction: $C_1^o = 0$, $C_2^o = 0$ (6.11)

6.4 BOUNDARY CONDITIONS

In addition to the above observations, one also finds that the constant

$$\mathscr{C}_c = (\mu_2 - \mu_1)(\mu_2 - \bar{\mu}_2)(\bar{\mu}_2 - \mu_1)$$

is an imaginary quantity in the case of special orthotropy. Consequently, the square matrix $\mathscr{E}^T\mathscr{E}$ (Eq. 4.37) becomes singular due to the third column of $\mathscr{E}$ (Eq. 4.38) containing zero elements only. In order to account for the case of special orthotropy among the various orthotropic cases, it is noted that

$$y\,\mathrm{Re}\mathscr{C}_c\mathrm{Re}\mathscr{C}_s - y\,\mathrm{Im}\mathscr{C}_c\mathrm{Im}\mathscr{C}_s = y\,\mathrm{Re}[\mathscr{C}_c\mathscr{C}_s]$$

Considering this contraction and the determination of the integration constants, Eq. 4.38 becomes

$$\begin{bmatrix} 0 & \mathrm{Re}F_{11} & \cdots & -\mathrm{Im}F_{2NN} \\ y & \mathrm{Re}G_{11} & \cdots & -\mathrm{Im}G_{2NN} \end{bmatrix} \begin{bmatrix} \mathrm{Re}[\mathscr{C}_c\mathscr{C}_s] \\ \mathrm{Re}\mathscr{C}_d \\ \vdots \\ \mathrm{Im}\mathscr{C}_{2NN} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\int_s Y_n ds + \frac{C_1^o}{2} \\ \frac{1}{2}\int_s X_n ds + \frac{C_2^o}{2} \end{bmatrix},$$

$$z \text{ in } \Gamma \qquad 6.12$$

It can be noted, here, that upon solving Eqs. 6.12 at M points, the stress components are fully determinate, (see Eqs. 4.39, 4.40 and 4.41), however, the displacement expressions only allow the deletion of the first two columns of Eq. 4.38. Since this dissertation primarily deals with stresses and stress intensity factors, the form given in 6.12 will be used.

At this point, the boundary conditions for the three different cases of loading (Figures 7(a), 7(b), and 8) are written in a compact form readily applicable for automatic computation.

$$\begin{bmatrix} 0 & \mathrm{Re}F_{11} & -\mathrm{Im}F_{11} \cdots \mathrm{Re}F_{1NP} & -\mathrm{Im}F_{1NP} \cdots \mathrm{Re}F_{2NN} & -\mathrm{Im}F_{2NN} \\ y & \mathrm{Re}G_{11} & -\mathrm{Im}G_{11} \cdots \mathrm{Re}G_{1NP} & -\mathrm{Im}G_{1NP} \cdots \mathrm{Re}G_{2NN} & -\mathrm{Im}G_{2NN} \end{bmatrix} \begin{bmatrix} \mathrm{Re}[\mathscr{C}_c \mathscr{C}_s] \\ \mathrm{Re}\mathscr{C}_d \\ \mathrm{Im}\mathscr{C}_d \\ \vdots \\ \mathrm{Re}\mathscr{C}_{1NP} \\ \mathrm{Im}\mathscr{C}_{1NP} \\ \vdots \\ \mathrm{Re}\mathscr{C}_{2NN} \\ \mathrm{Im}\mathscr{C}_{2NN} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} T_t x - T_s y \\ T_t' y - T_s x \end{bmatrix}, \qquad z \text{ in } \Gamma \qquad 6.13$$

6.5 SOME REMARKS ABOUT THE COMPUTER PROGRAM

The consideration of Eqs. 6.13 at M points on the boundary of a rectangular plate results in a matrix equation such as 4.32, where

$$\underset{2M \times (NN + NP + 1)}{\mathscr{E}} \cdot \underset{(NN + NP + 1) \times 1}{\mathscr{C}} = \underset{2M \times 1}{K} \qquad 6.14$$

Then for the least squares boundary collocation method, the problem becomes

$$\mathscr{E}^T \mathscr{E} \mathscr{C}^* = \mathscr{E}^T K \qquad 6.15$$

where $\mathscr{C}^*$ is the desired solution vector.

The solution of Eq. 6.15 was carried out on the IBM TSS/360 computer by declaring all quantities and operations in double precision and utilizing the Gauss-elimination technique with complete pivoting. It should also be mentioned that the elements of the coefficient matrix, $\mathcal{E}$, were scaled by the devisor $R^n = \left(\sqrt{c^2 + h^2}\right)^n$.

The listing of the complete program is given in Appendix B.

7. RESULTS AND DISCUSSION OF THE SOLUTIONS OF THE TENSION AND SHEAR PROBLEMS

7.1 INTRODUCTION

In this section, the solutions of the problems shown in Figures 7(a), 7(b) and 8, are presented and discussed. For both the tension and shear problems the effects of the variations of the dimensionless material ratios (3.30, 3.32), the orientation angle and the a/c ratios are considered. For the tension problem, variation in the h/c ratio is also taken into account.

7.2 SPECIFICATIONS OF PARAMETERS TREATED

The solutions of the three problems shown in Figures 7(a), 7(b) and 8 are based on the following specifications:

1. For all cases $a = .5$ inches unless otherwise specified (see Table 1).
2. The problems were non-dimensionalized by considering the cancellation of the constant boundary stresses from both sides of the boundary condition equations. This is effectively equivalent to setting the applied boundary stresses equal to unity. If other values of the applied boundary stresses are used, they are properly indicated (see Table 1)
3. With the exception of the investigation of the effects of material properties on the stress intensity factors, the ratios

$$\frac{E_{22}}{E_{11}} = .366$$

and

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = .257$$

were used throughout the entire analysis. These ratios correspond to the properties of fiberglass.

4. In studying the effects of varying material properties on the stress intensity factors in both the tension and shear problems, the following parameter ranges were considered:

$$.05 \leq \frac{E_{22}}{E_{11}} \leq 1$$

$$.001 \leq \frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \leq \infty$$

$$\frac{a}{c} = \frac{2}{3}; \qquad \frac{h}{c} = 1; \qquad \delta = 45^{o}$$

5. In studying the effects of varying plate aspect ratio, h/c, and crack length to plate width ratio, a/c, on the stress intensity factors in the tension problem, the parameter ranges were chosen as follows:

$$0 < \frac{a}{c} \leq \frac{2}{3}$$

$$.25 \leq \frac{h}{c} \leq 133$$

$$\delta = 45^{o}$$

6. The effects of variation of the orientation angle, δ, were also analyzed for both the tension and shear problems considering the following parameter ranges:

$$0 < \frac{a}{c} \leq \frac{2}{3}$$

$$0 \leq \delta \leq 90^{o}$$

$$\frac{h}{c} = 1$$

7.3 EXAMINATION OF THE MAPPING COLLOCATION METHOD

The available results which could qualify the validity of the mapping-collocation method for centrally cracked finite rectangular orthotropic plates are those of the centrally cracked infinite plates, given in section 5, and the results of Kobayashi, Isida and Sawyer (refs. 1, 19 and 20) for finite isotropic rectangular plates. In order to gain a certain degree of confidence in the mapping-collocation method as developed in the preceeding sections, both the results for the infinite orthotropic and finite isotropic plates were verified.

One knows that in the infinite plate solutions $A_{1n} = B_{1n} = 0$ for $n \geq 2$ and furthermore only the difference $A_{11} - B_{11}$ is needed to express the stress components and the stress intensity factors whether the plate under consideration is anisotropic or orthotropic. Hence, the very first verification of the mapping-collocation method would result from the use of the constants, $\mathrm{Re}[\mathcal{C}_c \mathcal{C}_s]$ and $\mathcal{C}_d$, only, with fairly large plate dimensions as compared to the crack length. However, this type of substantiation of the mapping-collocation method rests on the a priori assumption that a large finite plate is the same as an infinite plate.

A better method to verify the infinite plate results involves the use of the "full" form of the stress potential, $\phi_1(\zeta)$. Upon considering a 100"x100" plate with a one-inch crack, it was found that the stress intensity factors were quite insensitive to the number of terms of the Laurent series expansion of $\phi_1(\zeta)$. For example, the use of $\mathrm{Re}[\mathscr{C}_c \mathscr{C}_s]$ and $\mathscr{C}_d$ (three unknowns) resulted in $K_1 = .70712\ \sqrt{\text{in}}.$ and $K_2 = .00000$ while using 83 unknowns resulted in the stress intensity factors, $K_1 = .70713\ \sqrt{\text{in}}.$ and $K_2 = .00000$ with $T_t = 1$. The corresponding analytical results for an infinite plate are, $K_1 = \sqrt{.5}\ \sqrt{\text{in}}$ and $K_2 = 0$. It was, however, also noted that the satisfaction of the stress boundary conditions markedly improved with the use of more and more terms in the Laurent series expansion of $\phi_1(\zeta)$. Thus, the results given in section 5 were verified for both the tension and shear problems.

The verification of Kobayashi's Isida's and Sawyer's opening mode stress intensity factors for various finite isotropic rectangular plates in tension was carried out by setting the ratios

$$\frac{E_{22}}{E_{11}} = 1$$

and

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = 1$$

The value of the orientation angle had no influence on the results. The cases considered are shown in Table 1, where the various stress intensity factors are tabulated and can readily be compared with each other.

As can be seen from Table 1, agreement with the known results is extremely good, as a matter of fact one cannot speak of a higher degree of accuracy than the one obtained when numerical methods are used.

In order to economize the mapping-collocation method with regard to the use of the number of terms in the Laurent series expansion of $\phi_1(\zeta)$, two different truncation numbers NN and NP were assigned as limits of the sums in the various expressions. The numbers NN and NP are odd for orthotropic materials and they are actually corresponding to the number of the negative and positive exponent terms in the Laurent series expansion of the stress potential function, $\phi_1(\zeta)$. Upon investigating various NN/NP ratios for the finite plate problems, it was found that a ratio of NN/NP = 5/3 resulted in fairly fast convergence with rather well-satisfied boundary conditions. This NN/NP ratio was retained throughout the whole investigation of the problems given in Figures 7(a), 7(b) and 8.

One of the most important indications of the degree of accuracy of the solutions obtained by the mapping-collocation method comes from the examination of the boundary stress between collocation points (ref. 2). Since the least squares method of collocation results in the satisfaction of the boundary conditions only in the approximate sense, the magnitude and frequency of oscillations of the boundary stresses serve as an indicator to the "exactness" of the solution of the stress boundary value problem. Thus, in each case, when the problem of a centrally cracked finite plate was solved, the boundary stresses were also examined. For example, in the case of a square plate in tension with

$a/c = 2/3$ and $\delta = 45^{\circ}$, the largest error in the boundary stresses was found to be about 4% and it occurred in a sudden manner at the corners of the plate.

Since this dissertation deals with stresses and stress intensity factors of centrally cracked finite rectangular orthotropic plates, the pattern of convergence of the stress intensity factors, K_1 and K_2 was also investigated. It was observed that the insensitiveness of K_1 and K_2 to a change in the number of terms of the Laurent series expansion of the stress potential, $\phi_1(\zeta)$, is an excellent indicator to how well the boundary conditions are satisfied by the least squares approximation. Referring to Figures 9, 10 and Table 2, one observes two types of convergences:

1. Using approximately twice the number of equations as unknowns or larger, i.e., $2M \geq 2(NN + NP + 1)$, will result in nonoscillatory, steady values of K_1 and K_2. These values of K_1 and K_2 are still subject to variations as the number of unknowns changes.
2. The second type of convergence is that when K_1 and K_2 become insensitive to an increase in the terms of the Laurent series expansion of the stress potential, $\phi_1(\zeta)$, i.e., when a further increase in the number of unknowns does not change the values of K_1 and K_2.

Thus, it is noted that for a square plate in tension with $a/c = 1/2$ and $\delta = 45^{\circ}$, the number of unknowns $NN + NP + 1 = 41$ is already sufficient to result in converged values of K_1 and K_2. However, one may also note that the boundary conditions are somewhat better satisfied when NN + NP + 1 = 83, despite the fact that the stress intensity factors have already converged. Hence, one can draw the conclusion that for increasing plate dimensions less and less terms of the Laurent series expansion are needed in order to obtain converged values of the stress intensity factors. This fact was also observed rather radically in the case of the 100"x100" plate.

Upon considering various materials, plate sizes, a/c ratios and orientation angles under the loading conditions shown in Figures 7(a), 7(b) and 8, two conclusions became obvious regarding the convergence of the stress intensity factors:

1. Convergence of the stress intensity factors is definitely affected by the material parameters. It was found, for example, that for a square plate in tension with $a/c = 2/3$ and $\delta = 45^{\circ}$ the use of the dimensionless material constant

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = .0001$$

with any $\frac{E_{22}}{E_{11}}$ ratio resulted in oscillatory values of K_1 and K_2. In this case, the number of unknowns was taken as 83 which gave excellent results for higher values of the dimensionless material constant.

2. Convergence of the stress intensity factors is also affected by the a/c ratios. For a square plate in tension with

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = .257 \qquad \frac{E_{22}}{E_{11}} = .366$$

$a/c = .9$, $\delta = 45^o$ with the use of 83 unknowns, it was found that the boundary conditions were rather badly satisfied, the method probably resulting in unconverged stress intensity factors. For smaller a/c ratios well-converged values of K_1 and K_2 were obtained.

7.4 SOLUTION OF THE TENSION PROBLEMS OF AN ORTHOTROPIC RECTANGULAR PLATE WITH A CENTRAL CRACK

As shown in Figures 7(a) and 7(b), two distinct tension problems were considered in this study. The solution of the problem for loading applied in the x-direction (Figure 7(b)) for various h/c and a/c ratios, orientation angles and material constants resulted in zero values for the stress intensity factors. This fact is of significance because it shows that just as in the isotropic case, the stress field in the plate is not disturbed by the presence of the crack. It should furthermore be noted that in the case of Figure 7(b) the crack is aligned with the direction of loading, and for this particular configuration the plate behaves as a finite orthotropic rectangular plate loaded by a constant boundary stress in the x-direction without a crack.

The consideration of the other tension problem, i.e., when the constant boundary stress is applied in the y-direction (Figure 7(a)), resulted in various parametric studies involving material properties, a/c ratios, h/c ratios and orientation angles.

7.41 Effects of Material Properties on the Stress Intensity Factors

According to expressions, 3.30 and 3.32, an orthotropic material can be characterized by two dimensionless ratios constructed from the engineering material constants, E_{11}, E_{22}, G_{12}, and ν_{12}. As a consequence to this method of characterization, a parametric study was undertaken in order to determine the effects of these two dimensionless ratios on the stress intensity factors.

Upon considering a square plate with $a/c = 2/3$, $\delta = 45^{\circ}$ and $T_t = 1$, the dimensionless ratios were varied and Table 3 was obtained. From Table 3, the family of curves shown in Figures 11 and 12 were constructed. Figure 11 demonstrates the effects of the dimensionless material ratios on the opening mode stress intensity factor while Figure 12 shows the sliding mode stress intensity factor as a function of these two ratios.

Considering Figure 11, one arrives at the following conclusions:

1. The complete family of curves is asymptotic to the

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = 0$$

ordinate, i.e., as $\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \to 0$, $K_1 \to \infty$.

2. The curves are asymptotic to various K_1 - values as

$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \to \infty$; and they are asymptotic from below the

asymptotes.

3. As the E_{22}/E_{11} ratio decreases, the opening mode stress intensity factor, K_1, increases in value.

4. For each constant E_{22}/E_{11} ratio, the stress intensity factor, K_1, will attain a minimum value which is not the K_1-asymptote.
5. Upon examining the parametric curve for $E_{22}/E_{11} = 1$, it seems that the smallest possible minimum value which can be attained by K_1 is $K_1 = 1.131 \sqrt{\text{in.}}$ occurring at

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \simeq 3.$$

   This value of K_1 is very close to the value of the opening mode stress intensity factor, K_1, for an isotropic material ($K_1 = 1.134 \sqrt{\text{in.}}$).

The consideration of Figure 12 results in conclusions somewhat different from those for the opening mode stress intensity factor. First of all, one may note that the presence of the sliding mode stress intensity factor, K_2, is strictly due to the orthotropy of the material and to the finiteness of the h/c and the a/c ratios. It may be noted that the maximum magnitude of the sliding mode stress intensity factor shown in Figure 12 is about 14% of the value of the opening mode stress intensity factor for the same E_{22}/E_{11} ratio.

The closer examination of Figure 12 results in the following conclusions:

1. The complete family of curves is asymptotic to the

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = 0$$

ordinate, i.e., as

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \to 0, \qquad K_2 \to -\infty.$$

2. The curves are asymptotic to various K_2 - values as

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \to \infty.$$

3. As the E_{22}/E_{11} ratio decreases, the absolute value of the sliding mode stress intensity factor, K_2, increases.
4. For each constant E_{22}/E_{11} ratio, the sliding mode stress intensity factor attains its minimum absolute value at

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = \infty.$$

5. The smallest possible minimum absolute value of the sliding mode stress intensity factor, K_2, is zero as one would expect, i.e., for an isotropic material $K_2 = 0$.

7.42 Effects of Plate Width and Plate Size on the Stress Intensity Factors

In order to investigate the effects of various plate widths and plate sizes on the stress intensity factors, K_1 and K_2, the dimensionless material ratios were set at

$$\frac{E_{22}}{E_{11}} = .366$$

and

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = .257$$

with an orientation angle, $\delta = 45^{o}$. These ratios correspond to fiberglass properties.

The variation of the h/c and a/c ratios resulted in Table 4. In accordance with Table 4 graphical representations were constructed as shown in Figures 13 and 14.

The family of curves for the opening mode stress intensity factor, K_1, (Figure 13) indicates the following:

1. As $a/c \to 2/3$ the value of the opening mode stress intensity factor becomes larger and larger for each constant h/c ratio.
2. As $a/c \to 0$ the value of the opening mode stress intensity factor approaches $K_1 = \sqrt{.5}\ \sqrt{in.}$, the value of K_1 for an infinitely large plate, also for each constant h/c ratio.
3. As the h/c ratio decreases, the opening mode stress intensity factor, K_1, increases in value.
4. As the h/c ratio becomes very large, the opening mode stress intensity factor approaches $K_1 = \sqrt{.5}\ \sqrt{in.}$ for the range of values $0 < a/c \leq 2/3$. This seems to indicate that for an infinite strip, only very high a/c ratios will effect the stress intensity factor, K_1, such that it will deviate from the K_1-value of the infinite plate.

The consideration of Figure 14 leads one to the following conclusions:

1. As $a/c \to 2/3$ the absolute value of the sliding mode stress intensity factor, K_2, becomes larger and larger for each constant h/c ratio.
2. As $a/c \to 0$ the sliding mode stress intensity factor approaches the value, $K_2 = 0$, i.e., the value of K_2 for an infinitely large plate. This also occurs for each constant h/c ratio.
3. As the h/c ratio decreases, the sliding mode stress intensity factor, K_2, increases in absolute value.
4. As the h/c ratio becomes very large, the sliding mode stress intensity factor approaches the value, $K_2 = 0$, i.e., the value of K_2 for an infinite plate for the range of values $0 < a/c \leq 2/3$.

The presence of significant values of K_2 are due to the value of the orientation angle ($\delta = 45^o$), however, the variations in K_2 are definitely induced by the variations in both the a/c and h/c ratios. Thus, one can conclude that the finiteness of the plate with $a/c > 0$ accounts for the increase in the values of both stress intensity factors as compared to the case of the infinite plate.

7.43 Effects of Orientation Angle and Plate Width on the Stress Intensity Factors

For the investigation of the effects of the orientation angle and the plate width on the stress intensity factors, the h/c ratio was taken as one (square plate) and the dimensionless material ratios were held constant at the values specified above.

The variations of the orientation angle, δ, and the a/c ratio resulted in Table 5. The graphs constructed from Table 5 are presented in

Figures 15 and 16, where Figure 15 shows the effects of various orientation angles, δ, on the opening mode stress intensity factor, K_1, for constant a/c ratios, while Figure 16 shows the sliding mode stress intensity factor, K_2, as affected by changes in δ and a/c.

An examination of Figure 15 results in the following observations:

1. As the a/c ratio increases, the opening mode stress intensity factor, K_1, also increases.
2. As the orientation angle, δ, increases, the opening mode stress intensity factor, K_1, becomes larger and larger for each constant a/c ratio.
3. The curves indicate zero slopes at $\delta = 0^\circ$ and $\delta = 90^\circ$.
4. For each constant a/c ratio, the maximum value of K_1 occurs at $\delta = 90^\circ$ and the minimum value at $\delta = 0^\circ$.
5. The smallest possible minimum value of the opening mode stress intensity factor is $\sqrt{.5}\ \sqrt{\text{in.}}$ which is the K_1-value for an infinite plate.

The conclusions concerning Figure 16 are as follows:

1. The existence of the sliding mode stress intensity factor, K_2, is strictly due to the values of the orientation angle, δ, different from 0° or 90°.
2. As the a/c ratio increases, the sliding mode stress intensity factor, K_2, also increases in absolute value.
3. For each constant a/c ratio, the absolute value of the sliding mode stress intensity factor reaches a maximum at different values of δ. For example, the maximum absolute value of K_2 for $a/c = 2/3$ occurs at about $\delta = 50^\circ$, with K_2 being $-.065\ \sqrt{\text{in.}}$.

4. The smallest possible absolute value which is attained by K_2 is zero as expected for an infinite plate.

7.44 Stress Distribution on the $y = 0$, $x > a$ Line

As an illustration of the typical stress distribution on the line $y = 0$ for the range $1 < x/a \leq 3/2$, the values of the stress components were tabulated in Table 6 for a plate with $h/c = 1$, $a/c = 2/3$, $\delta = 45^{\circ}$ and plotted in Figures 17 and 18. Figure 17 shows that both normal stresses, σ_{xx} and σ_{yy}, become asymptotic to the $x/a = 1$ line and σ_{xx} becomes zero at $x/a = 3/2$ which is a point on the boundary. This is expected since σ_{xx} was specified to be zero on the $x/a = 3/2$ boundary. The other stress component, σ_{yy}, has the value of $\sigma_{yy} = .68$ at $y = 0$, $x/a = 3/2$.

The intersting behavior of the shear stress is shown in Figure 18, where in the neighborhood of the crack tip $\sigma_{xy} < 0$ and $\sigma_{xy} \to -\infty$ as $x/a \to 1$. This behavior of σ_{xy} in the neighborhood of the crack tip could be expected from knowing that K_2 was found to be negative for the same case (see Figure 16). Then as x/a increases σ_{xy} will reach a maximum value ($\sigma_{xy} = .153$) at about $x/a = 1.35$ which is 15.3% of the applied stress. It is noted that the presence of σ_{xy} is due to the general orthotropy of the material. It should also be observed that σ_{xy} becomes effectively zero on the boundary, which it should be for proper satisfaction of the boundary conditions.

It should be mentioned at this point that for each K_1- and K_2-value obtained, the complete boundary value problem of an orthotropic rectangular plate containing a central crack had to be solved. In each case, in addition to the computation of the stress components on the boundary, the stress components on the $y = 0$ line were also recorded. It gives one confidence in the method to know that for isotropic and specially orthotropic materials, the shear stress component, σ_{xy}, was effectively zero on the $y = 0$ line. It should also be noted that for problems with $E_{22}/E_{11} = 1$, the shear stress component, σ_{xy}, on the $y = 0$ line was found to be practically zero for any value of

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2}.$$

7.5 SOLUTION OF THE SHEAR PROBLEM OF AN ORTHOTROPIC SQUARE PLATE WITH A CENTRAL CRACK

In addition to the tension problem discussed in the preceding section, the problem of an orthotropic square plate containing a central crack and loaded by unit shear stress (Figure 8) was also considered. As in the case of the tension problem, the shear problem was also solved for various dimensionless material ratios, a/c ratios and orientation angles. The h/c ratio was held constant ($h/c = 1$) throughout the entire analysis of the shear problem.

7.51 Effects of Material Properties on the Stress Intensity Factors

The results of the parametric study involving variations of the dimensionless material ratios while considering $a/c = 2/3$ and $\delta = 45^o$ are summerized in Table 7 and Figures 19 and 20.

An examination of Figure 19 reveals the following effects of the dimensionless material ratios on the opening mode stress intensity factor:

1. The complete family of curves is asymptotic to the

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = 0$$

ordinate, i.e., as

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \rightarrow 0, \qquad K_1 \rightarrow -\infty.$$

2. The curves are asymptotic to various values of K_1 as

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \rightarrow \infty.$$

3. As the E_{22}/E_{11} ratio decreases, the absolute value of K_1 increases.

4. For each constant E_{22}/E_{11} ratio, the opening mode stress intensity factor, K_1, attains its minimum absolute value at

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = \infty.$$

5. The smallest possible minimum absolute value of the opening mode stress intensity factor, K_1, is zero as one would expect i.e., for an isotropic material $K_1 = 0$.

The effects of the variations of the dimensionless material ratios on the sliding mode stress intensity factor, K_2, are shown in Figure 20. According to Figure 20, one arrives at the following conclusions:

1. The complete family of curves is asymptotic to the

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = 0$$

ordinate, i.e., as

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \to 0, \qquad K_2 \to \infty.$$

2. The curves are asymptotic to various K_2 - values as

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \to \infty.$$

3. As the E_{22}/E_{11} ratio decreases, the value of the sliding mode stress intensity factor, K_2, increases.

4. For each constant E_{22}/E_{11} ratio, the sliding mode stress intensity factor attains its minimum value at

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = \infty.$$

5. It seems that the smallest possible minimum value of K_2 is .982 $\sqrt{\text{in.}}$ and it occurs at

$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = \infty.$$

For an isotropic material $K_2 = 1.022 \sqrt{\text{in.}}$.

7.52 Effects of Orientation Angle and Plate Width on the Stress Intensity Factors

The variation of the orientation angle, δ, and the a/c ratio resulted in Table 8 and Figures 21 and 22. Considering Figure 21, one arrives at the following conclusions:

1. The existence of the opening mode stress intensity factor, K_1, is strictly due to the values of the orientation angle, δ, different from 0^o or 90^o.
2. As the a/c ratio increases, the absolute value of the opening mode stress intensity factor, K_1, also increases.
3. For each constant a/c ratio, the absolute value of the opening mode stress intensity factor, K_1, reaches a maximum at different values of δ. For example, the maximum absolute value of K_1 for $a/c = 2/3$ occurs at about $\delta = 46^o$, with K_1 being $-.073 \sqrt{in}$..
4. The smallest possible absolute value of K_1 is zero as expected in the case of an infinite plate.

The effects of the orientation angle and the a/c ratio on the sliding mode stress intensity factor are shown in Figure 22. According to Figure 22, these effects can be summarized as follows:

1. As the a/c ratio increases, the sliding mode stress intensity factor, K_2, also increases.
2. For each constant a/c ratio, the sliding mode stress intensity factor, K_2, reaches a maximum and this maximum occurs at various values of δ. The minimum of K_2 always occurs at $\delta = 0^o$. For

example, for the curve, $a/c = 2/3$, the maximum value of K_2 is 1.072 $\sqrt{in.}$ occurring at $\delta = 42^o$ while the minimum value of K_2 is .994 $\sqrt{in.}$ at $\delta = 0^o$.

3. The curves indicate zero slopes at $\delta = 0^o$ and $\delta = 90^o$.
4. The smallest possible minimum value of K_2 is $\sqrt{.5}$ $\sqrt{in.}$, which is the K_2-value for an infinite plate.

There is an important observation which concerns the effects of the variation of the orientation angle in the shear problem. As far as δ-dependence is concerned, the orthotropic plate behaves differently in tension and in shear. In the tension problem it was found that the maximum K_1 values always occurred at $\delta = 90^o$, (specially orthotropic material) while in the shear problem it is obvious that the maximum K_2 is δ-dependent. However, it is also observed that in both the tension and shear problems the minimum values of the significant stress intensity factors occurred at $\delta = 0^o$ which designates the other type of special orthotropy.

7.53 Stress Distribution on the $y = 0$, $x > a$ Line

As an illustration of the typical stress distribution on the line $y = 0$, $1 < x/a \leq 3/2$, the values of the stress components were recorded in Table 9 and depicted in Figures 23 and 24 for a plate with $a/c = 2/3$ and $\delta = 45^o$.

Figure 23 shows the normal stress components, σ_{xx} and σ_{yy}, as they vary for $1 < x/a \leq 3/2$. It is noted that $\sigma_{xx} \to -\infty$ and $\sigma_{yy} \to -\infty$ as $x/a \to 1$, but σ_{yy} becomes positive and attains the value of $\sigma_{yy} = .72$ at $x/a = 3/2$. In agreement with the specified boundary condition, σ_{xx} becomes zero at $y = 0$, $x/a = 3/2$.

The distribution of the shear stress, σ_{xy}, is shown in Figure 24, where $\sigma_{xy} \to \infty$ as $x/a \to 1$ and it has the specified value, $\sigma_{xy} = 1$, at $x/a = 3/2$, $y = 0$ on the boundary.

It should also be mentioned that both σ_{xx} and σ_{yy} were found to be zero on the line $y = 0$, $1 < x/a \leq 3/2$ with $E_{22}/E_{11} = 1$ and for any value of $\dfrac{E_{11}/E_{22}}{\left(\dfrac{E_{11}}{2G_{12}} - \nu_{12}\right)^2}$. The special cases of isotropy and special orthotropy resulted also in zero values for the normal stresses, σ_{xx} and σ_{yy}, on the $y = 0$, $1 < x/a \leq 3/2$ line.

Table 1.

Comparison of opening mode stress intensity factors for rectangular isotropic plates with central cracks

T_t = 1000 psi			Kobayashi Ref. 1	Sawyer Ref. 20	Isida Ref. 19	Author
h	c	a	K_1	K_1	K_1	K_1
in.	in.	in.	psi $\sqrt{in.}$	psi $\sqrt{in.}$	psi $\sqrt{in.}$	psi $\sqrt{in.}$
6	3	.25	502.4	504.7	502	502.1
6	3	.50	720.6	725.0	718	719.2
6	3	.75	904.8	910.1	900	900.3
6	3	1.00	1083.0	1073.7	1075	1073.3
6	3	1.50	1496.0	1401.5	1436	1454.6
6	3	2.00	2074.0	1952.2	2000	2002.8

Table 2.

Typical pattern of convergence of the stress intensity factors for tensile loading ($h/c = 1$, $a/c = 1/2$, $\delta = 45^{\circ}$, $T_t = 1$)

a = .5 in.	Number of unknowns 15		Number of unknowns 25		Number of unknowns 41		Number of unknowns 83	
Number of equations	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$
16	.9580	-.0267						
24	.9522	-.0279						
32	.9514	-.0281						
40			.9568	-.0326				
48					.9593	-.0336		
64	.9511	-.0283	.9560	-.0325				
96					.9590	-.0334		
128	.9511	-.0284	.9560	-.0326	.9590	-.0334		
176							.9591	-.0334

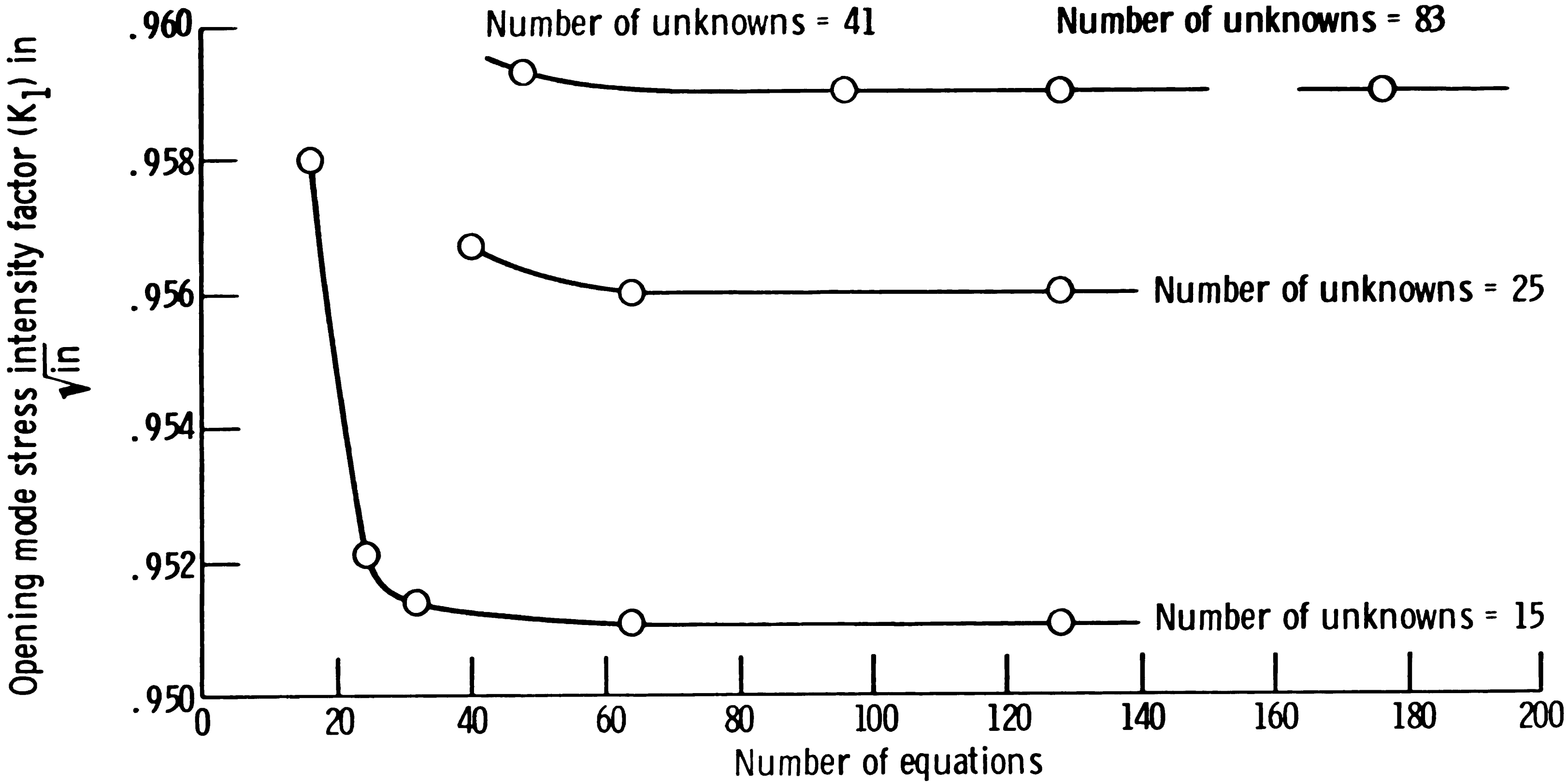


Figure 9. - Typical pattern of convergence of the opening mode stress intensity factor for tensile loading ($h/c = 1$, $a/c = 1/2$, $\delta = 45^0$, $T_t = 1$).

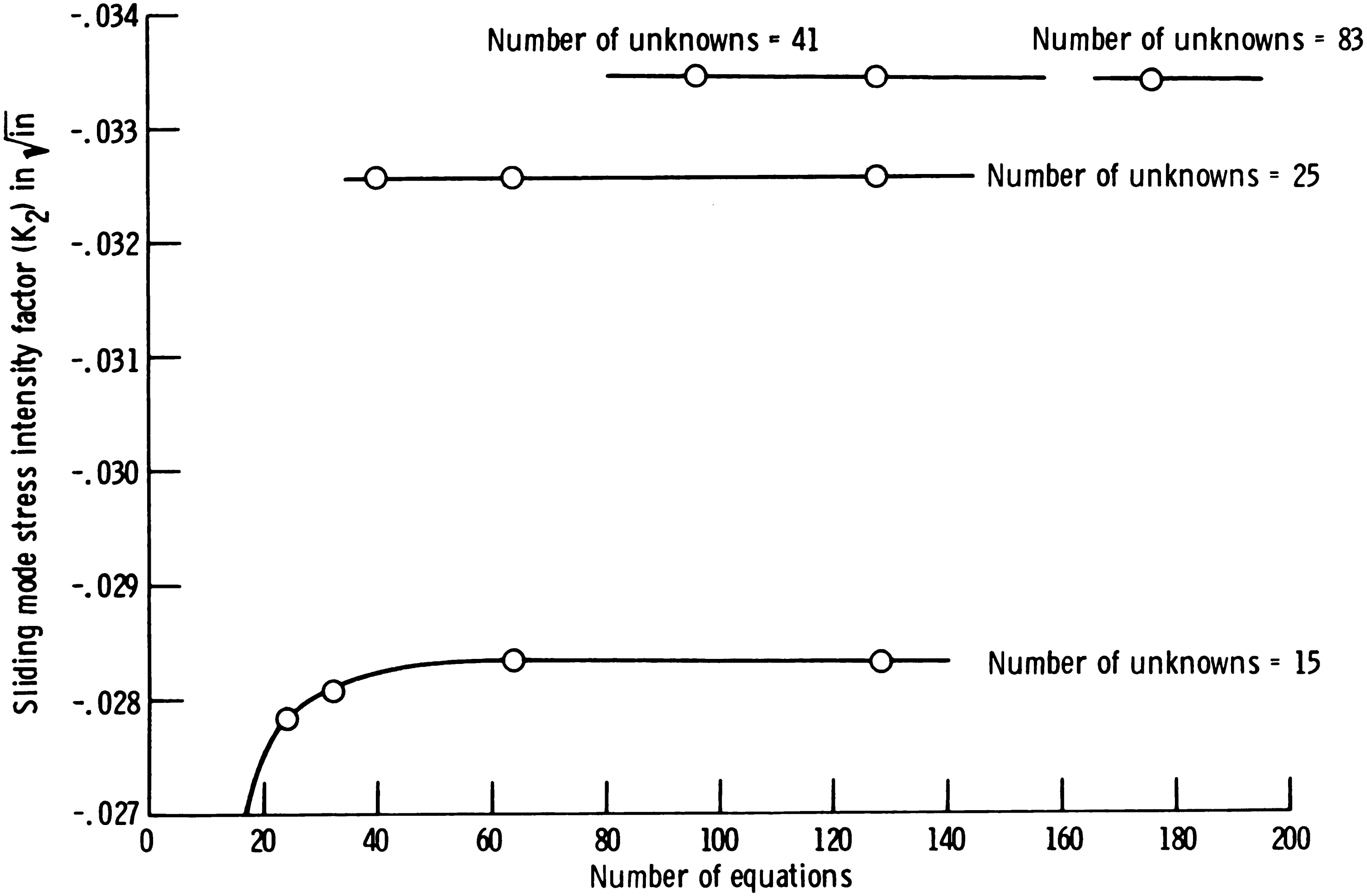


Figure 10. - Typical pattern of convergence of the sliding mode stress intensity factor for tensile loading ($h/c = 1$, $a/c = 1/2$, $\delta = 45^0$, $T_t = 1$).

Table 3.

Effects of material properties on the stress intensity factors for tensile loading
($h/c = 1$, $a/c = 2/3$, $\delta = 45^{o}$, $T_t = 1$)

a = .5 in.	$\frac{E_{22}}{E_{11}} = .05$		$\frac{E_{22}}{E_{11}} = .2$		$\frac{E_{22}}{E_{11}} = .4$		$\frac{E_{22}}{E_{11}} = .6$		$\frac{E_{22}}{E_{11}} = .8$		$\frac{E_{22}}{E_{11}} = 1$	
$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$
.001	1.750	-.460	1.667	-.244	1.654	-.134	1.640	-.072	1.639	-.004	1.691	-.023
.05	1.418	-.274	1.276	-.130	1.231	-.070	1.227	-.040	1.224	-.017	1.223	.000
.25	1.388	-.234	1.220	-.108	1.176	-.059	1.162	-.032	1.157	-.014	1.155	.000
.50	1.388	-.225	1.210	-.102	1.164	-.055	1.148	-.030	1.143	-.013	1.141	.000
.75	1.391	-.220	1.208	-.099	1.159	-.054	1.144	-.029	1.138	-.012	1.136	.000
1.00	1.394	-.217	1.207	-.098	1.157	-.053	1.141	-.029	1.135	-.012	1.134	.000
2.00	1.402	-.212	1.206	-.095	1.155	-.051	1.138	-.028	1.132	-.012	1.131	.000
4.00	1.410	-.209	1.209	-.093	1.156	-.050	1.138	-.028	1.132	-.012	1.133	.000
10.00	1.420	-.206	1.212	-.091	1.158	-.049	1.141	-.027	1.134	-.012	1.133	.000
100.00	1.438	-.203	1.221	-.090	1.165	-.049	1.147	-.027	1.140	-.012	1.139	.000
∞	1.448	-.201	1.227	-.090	1.170	-.049	1.152	-.027	1.145	-.012	1.144	.000

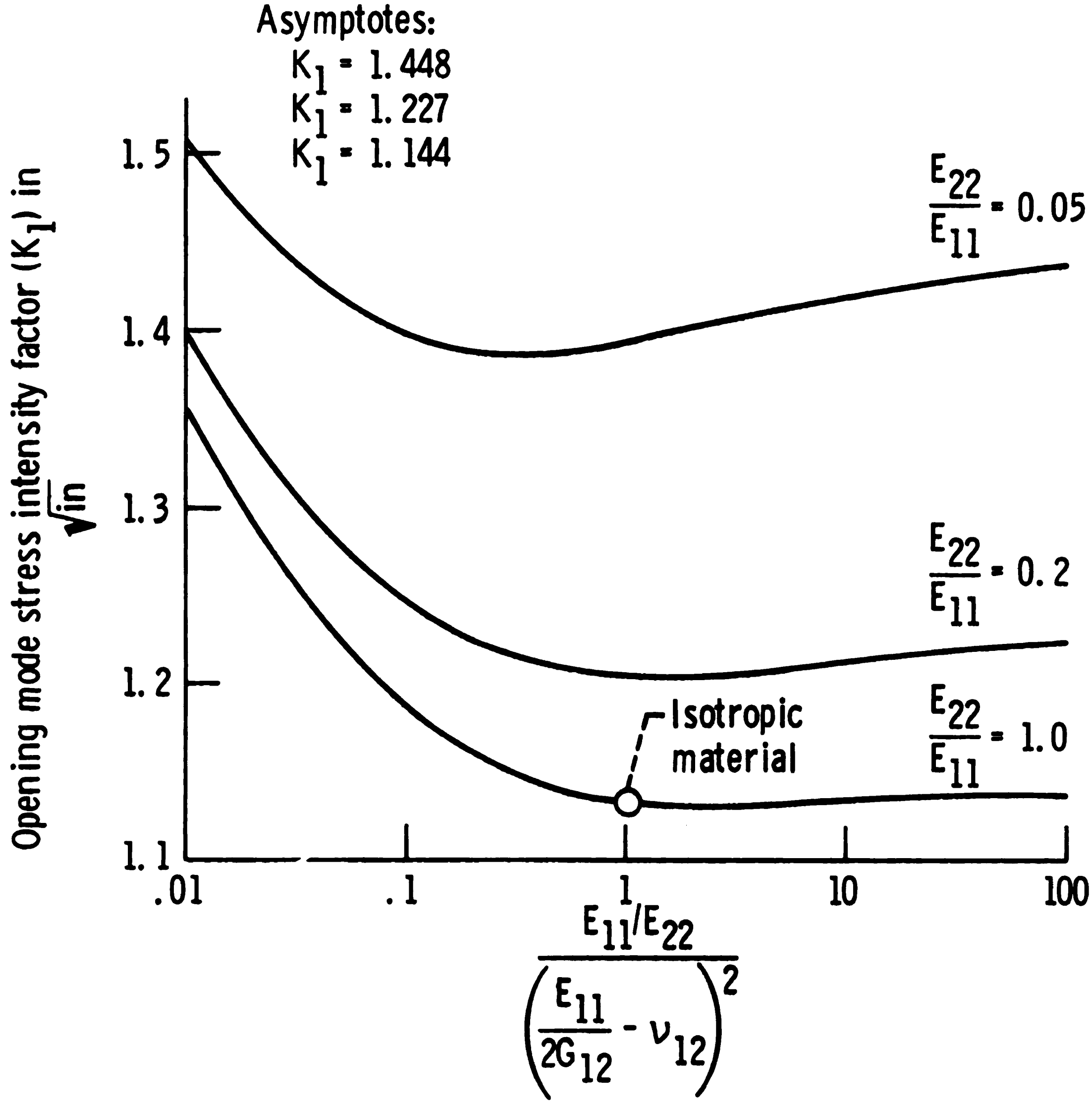


Figure 11. - Effects of material properties on the opening mode stress intensity factor for tensile loading (h/c = 1, a/c = 2/3, δ = 45^0, T_t = 1).

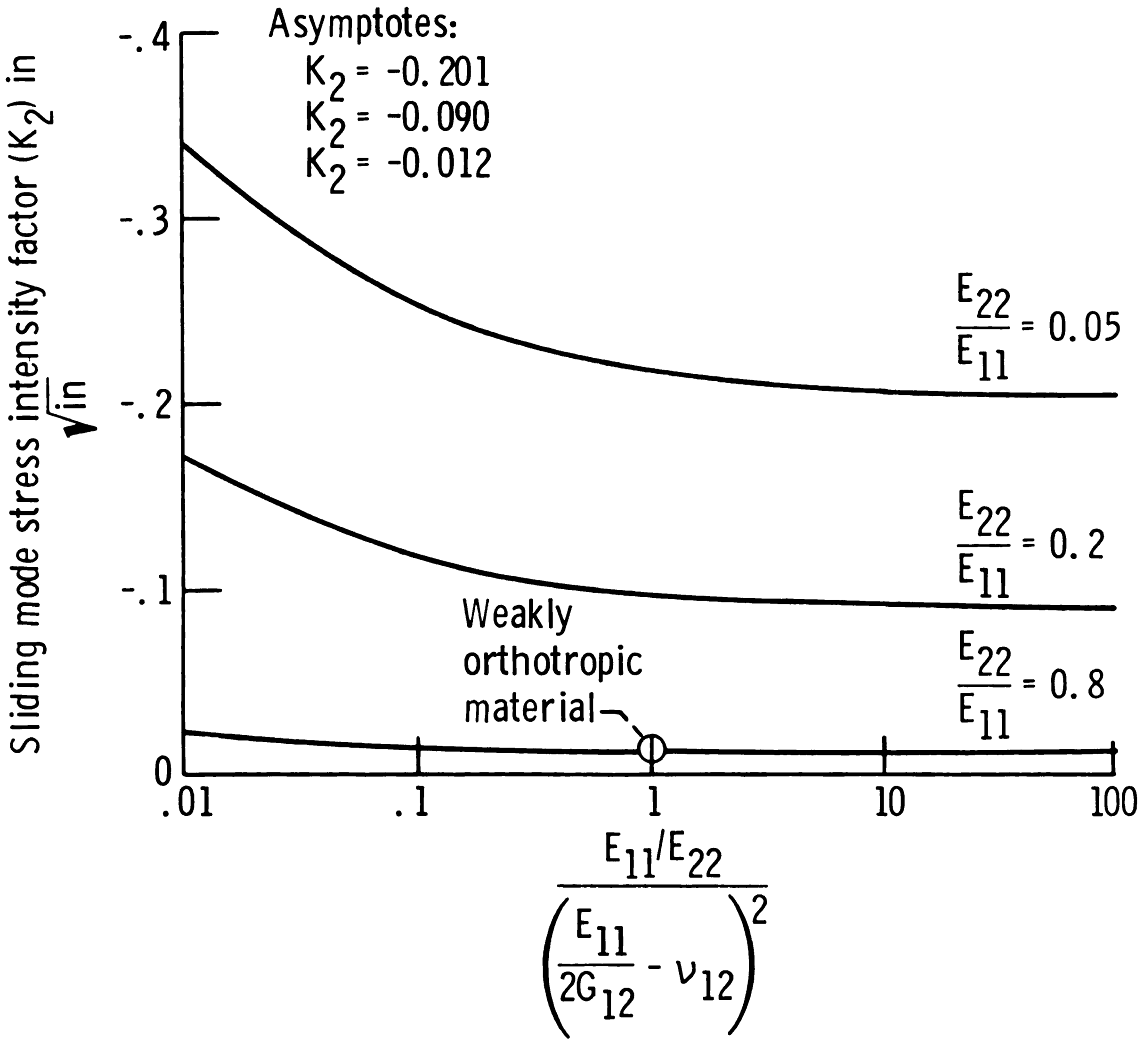


Figure 12. - Effects of material properties on the sliding mode stress intensity factor for tensile loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^0$, $T_t = 1$).

Table 4.

Effects of plate width and plate size on stress intensity factors for tensile loading ($\delta = 45^{\circ}$, $T_t = 1$)

a = .5 in.	$\frac{h}{c} = \frac{1}{4}$		$\frac{h}{c} = \frac{1}{2}$		$\frac{h}{c} = 1$		$\frac{h}{c} = 2$		$\frac{h}{c} = 4$		$\frac{h}{c} = 133$	
$\frac{a}{c}$	$\frac{K_1}{\sqrt{in.}}$	$\frac{K_2}{\sqrt{in.}}$	$\frac{K_1}{\sqrt{in.}}$	$\frac{K_2}{\sqrt{in.}}$	$\frac{K_1}{\sqrt{in.}}$	$\frac{K_2}{\sqrt{in.}}$	$\frac{K_1}{\sqrt{in.}}$	$\frac{K_2}{\sqrt{in.}}$	$\frac{K_1}{\sqrt{in.}}$	$\frac{K_2}{\sqrt{in.}}$	$\frac{K_1}{\sqrt{in.}}$	$\frac{K_2}{\sqrt{in.}}$
0	.707	0	.707	0	.707	0	.707	0	.707	0	.707	0
$\frac{1}{6}$	1.002	-.039	.790	-.011	.735	-.004	.723	-.004	.723	-.004	.711	0
$\frac{1}{3}$	1.648	-.141	1.025	-.044	.818	-.015	.773	-.015	.770	-.015	.719	0
$\frac{1}{2}$	2.450	-.283	1.428	-.107	.959	-.033	.871	-.032	.858	-.032	.734	0
$\frac{2}{3}$	3.497	-.481	2.120	-.204	1.180	-.064	1.058	-.058	1.032	-.054	.756	0

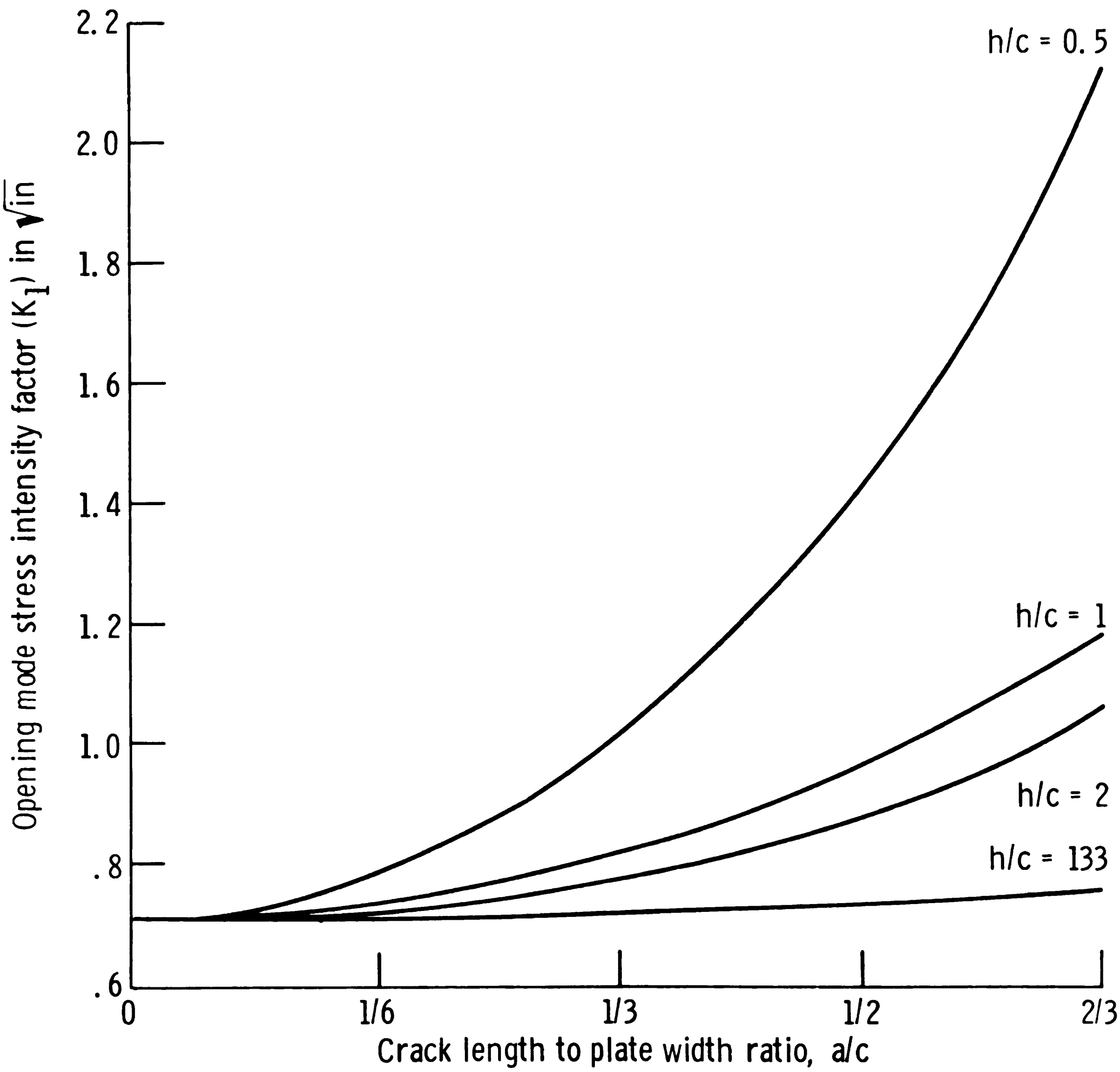


Figure 13. - Effects of **plate width** and plate size on the opening mode stress intensity factor for tensile loading ($\delta = 45^0$, $T_t = 1$).

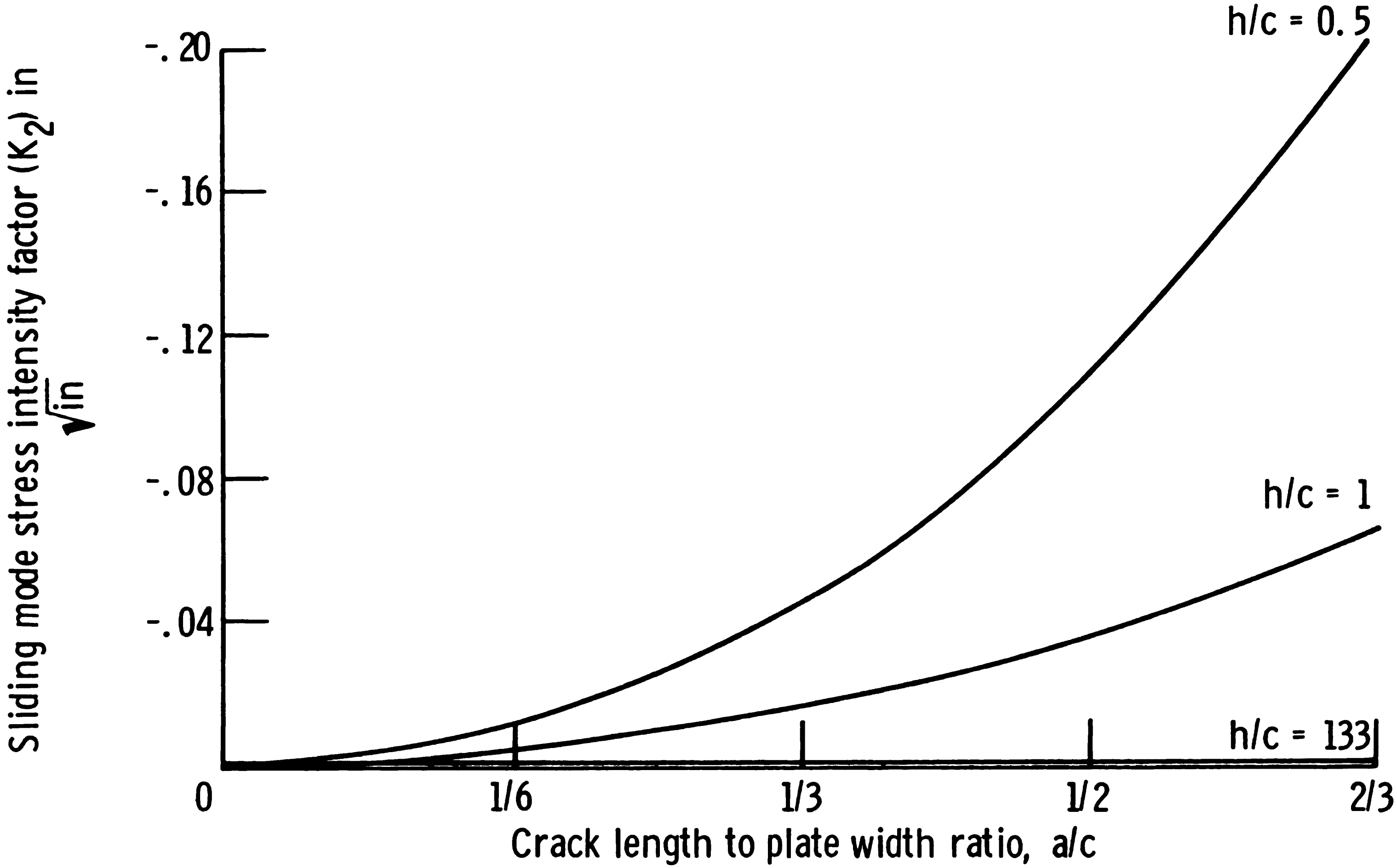


Figure 14. - Effects of plate width and plate size on the sliding mode stress intensity factor for tensile loading ($\delta = 45^0$, $T_t = 1$).

Table 5.

Effects of plate width **and orientation angle on the stress intensity factors for tensile loading ($h/c = 1$, $T_t = 1$)**

$a = .5$ in.	$\frac{a}{c} = 0$		$\frac{a}{c} = \frac{1}{6}$		$\frac{a}{c} = \frac{1}{3}$		$\frac{a}{c} = \frac{1}{2}$		$\frac{a}{c} = \frac{2}{3}$	
δ	K_1	K_2	K_1	K_2	K_1	K_2	K_1	K_2	K_1	K_2
deg	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$
0	.707	0	.727	.000	.783	.000	.876	.000	1.030	.000
2	.707	0	.727	.000	.783	.000	.876	-.002	1.030	-.004
5	.707	0	.727	.000	.784	-.001	.878	-.004	1.032	-.010
20	.707	0	.728	-.001	.790	-.005	.894	-.016	1.067	-.038
36	.707	0							1.135	-.058
40	.707	0							1.155	-.062
45	.707	0	.735	-.004	.813	-.015	.959	-.033	1.180	-.064
50	.707	0							1.205	-.065
52	.707	0			.832	-.018	.985	-.037		
56	.707	0	.741	-.005	.841	-.019	1.001	-.038	1.234	-.064
60	.707	0	.744	-.006	.850	-.020	1.017	-.039		
62	.707	0					1.025	-.039		
64	.707	0	.747	-.006	.859	-.021	1.032	-.038		
68	.707	0	.749	-.006	.867	-.020				
70	.707	0	.750	-.005	.871	-.019	1.053	-.034	1.293	-.048
85	.707	0	.756	-.002	.888	-.006	1.083	-.010	1.330	-.013
88	.707	0	.757	-.001	.890	-.002	1.085	-.004	1.332	-.006
90	.707	0	.757	.000	.890	.000	1.085	.000	1.332	.000

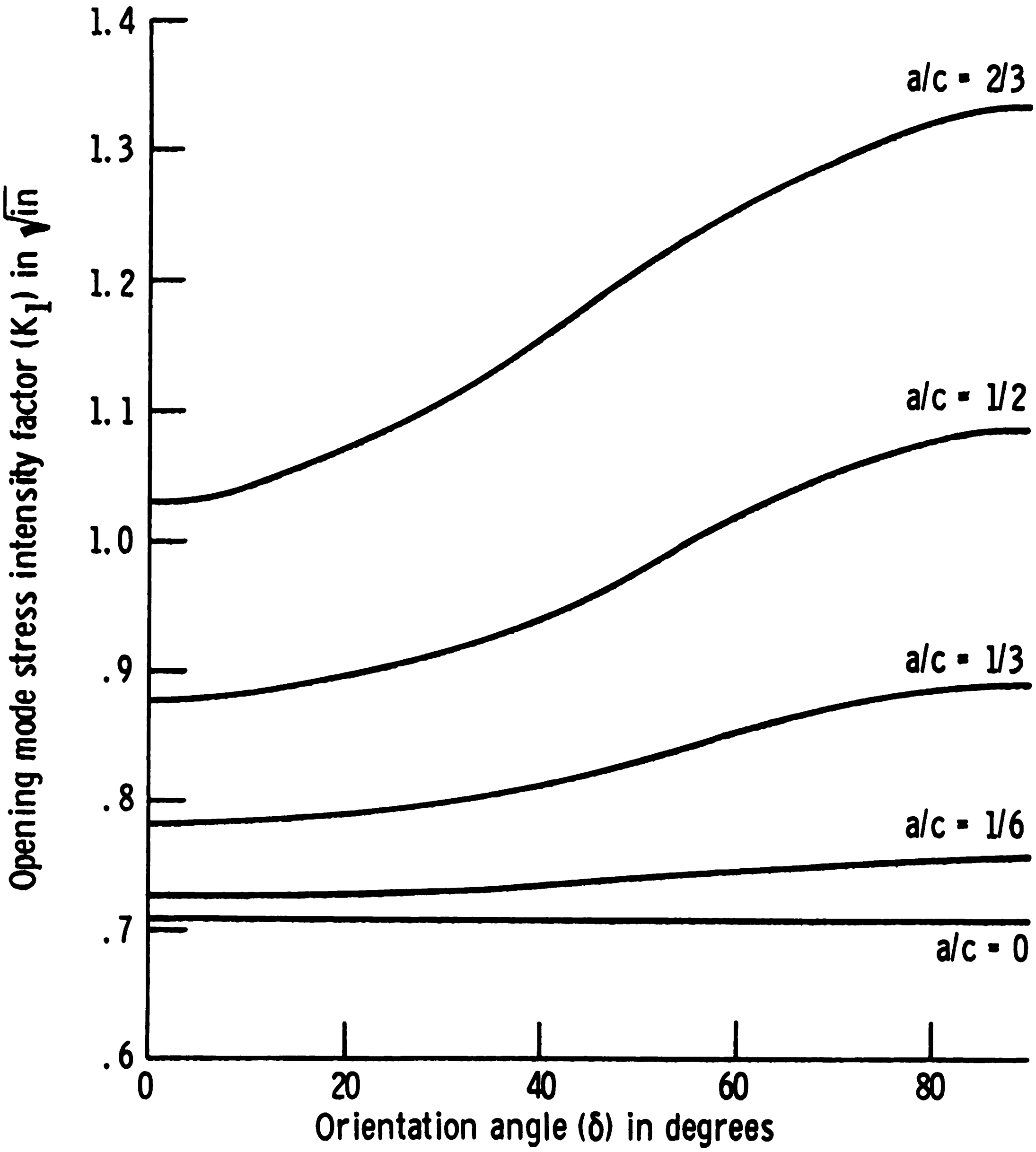


Figure 15. - Effects of orientation angle and plate width on the opening mode stress intensity factor for tensile loading ($h/c = 1$, $T_t = 1$).

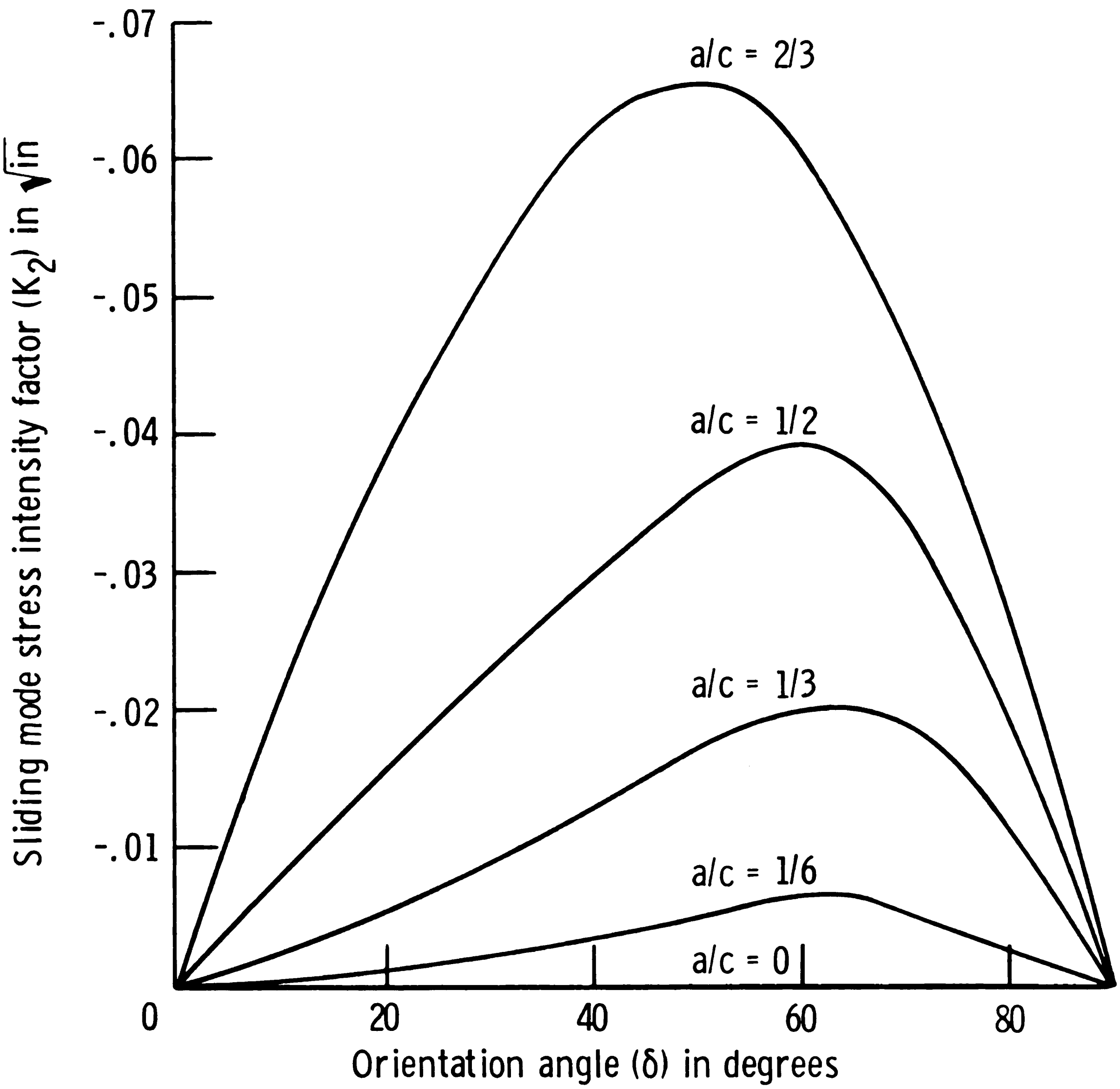


Figure 16. - Effects of orientation angle and plate width on the sliding mode stress intensity factor for tensile loading ($h/c = 1$, $T_t = 1$).

Table 6.

Stress distribution on the $y = 0$, $x > a$ line for tensile loading
($h/c = 1$, $a/c = 2/3$, $\delta = 45^{\circ}$, $T_t = 1$)

$a = .5$ in.

$\frac{x}{a}$	σ_{xx}	σ_{yy}	σ_{xy}
1.016	7.33	9.59	-.482
1.046	3.54	5.53	-.227
1.076	2.37	4.27	-.131
1.106	1.74	3.58	-.069
1.136	1.33	3.12	-.021
1.166	1.04	2.77	.020
1.196	.81	2.50	.055
1.228	.63	2.26	.085
1.258	.49	2.05	.111
1.288	.37	1.86	.131
1.318	.26	1.67	.146
1.348	.18	1.50	.153
1.374	.11	1.33	.150
1.410	.06	1.17	.136
1.440	.03	1.01	.108
1.470	.00	.84	.063
1.500	.00	.68	.000

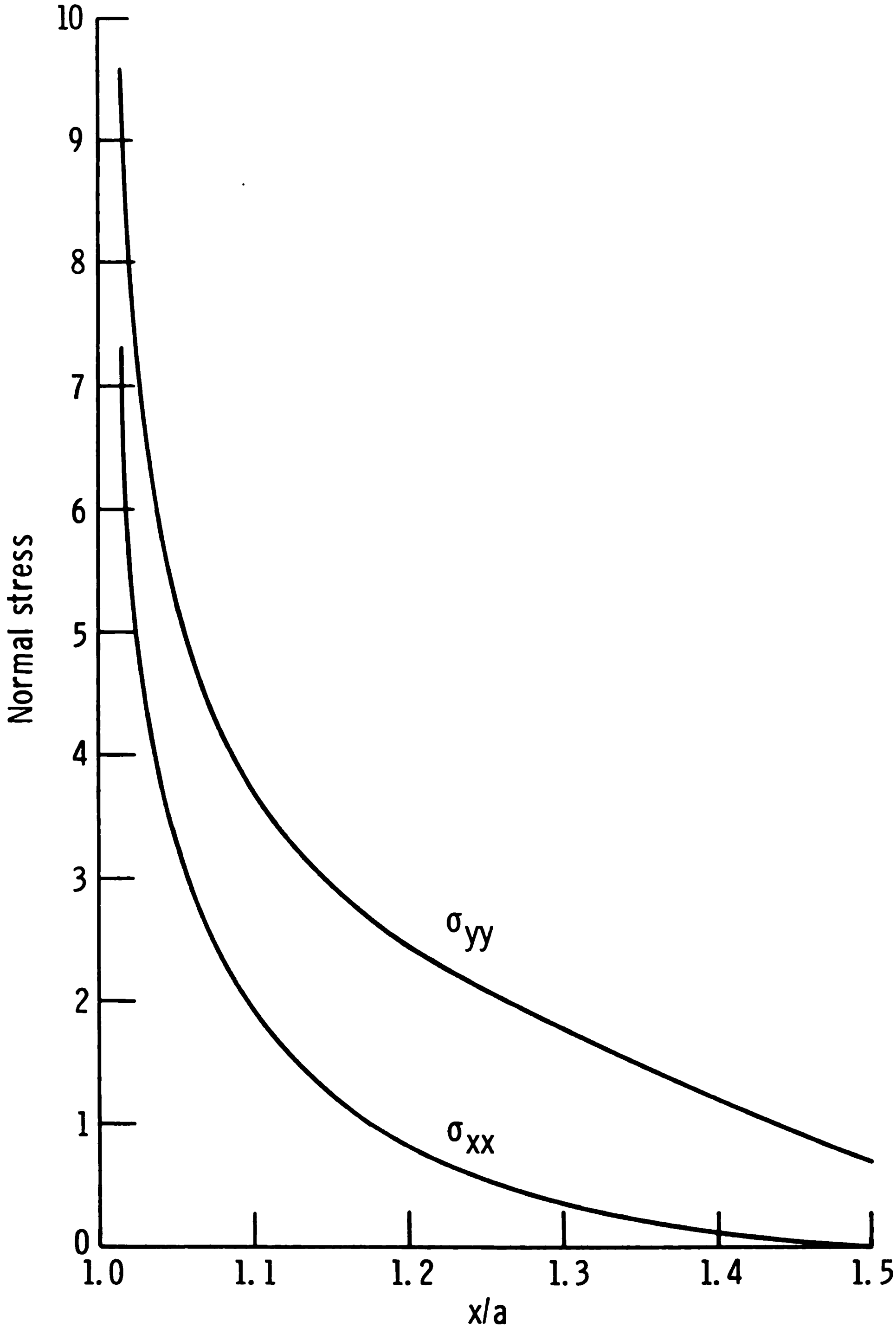


Figure 17. - Normal stress distribution on $y = 0$, $x > a$ line for tensile loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^0$, $T_t = 1$).

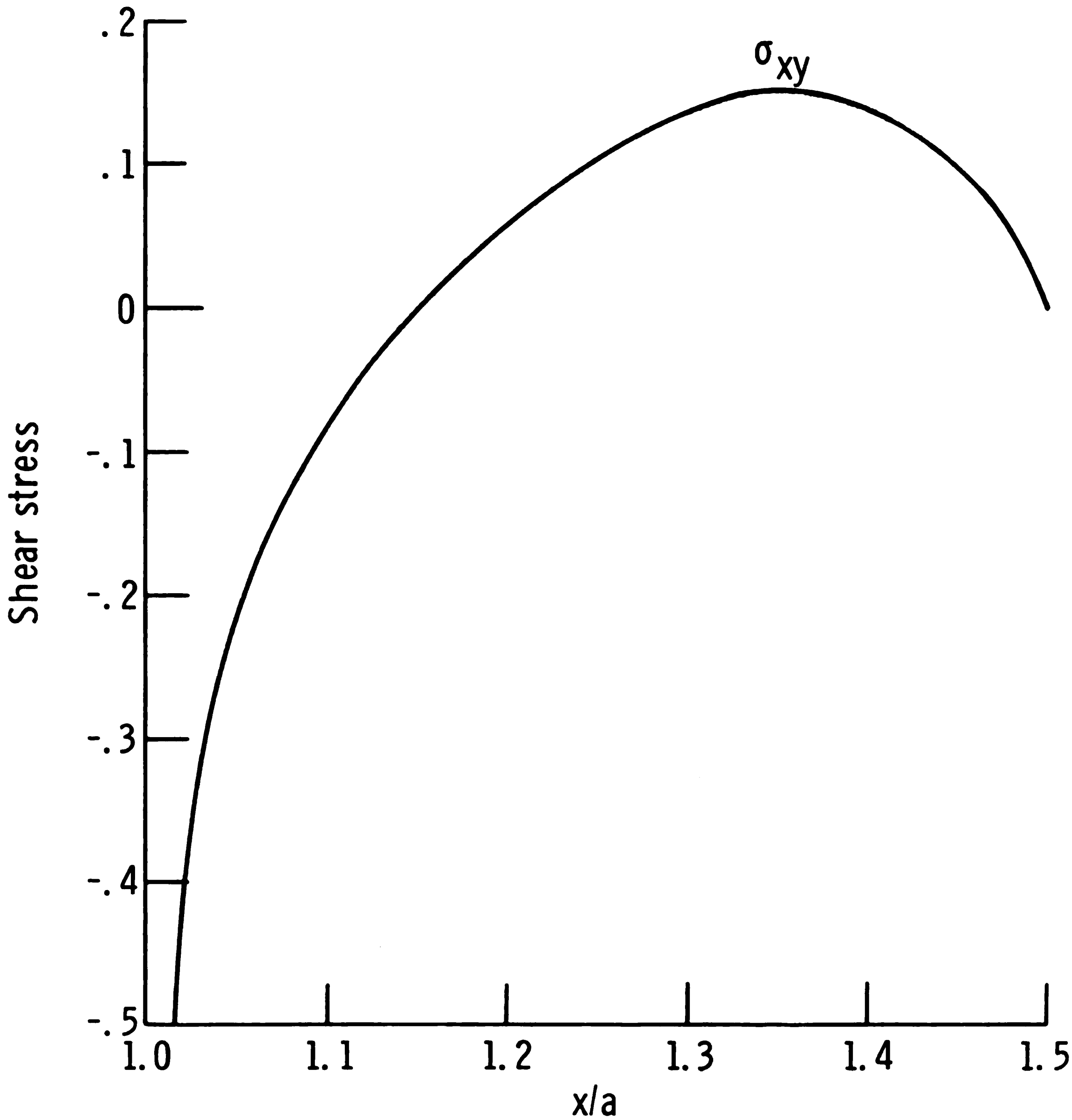


Figure 18. - Shear stress distribution on $y = 0$, $x > a$ line for tensile loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^0$, $T_t = 1$).

Table 7.

Effects of material properties on the stress intensity factors for shear loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^{o}$, $T_s = 1$)

$a = .5$ in.	$\frac{E_{22}}{E_{11}} = .05$		$\frac{E_{22}}{E_{11}} = .2$		$\frac{E_{22}}{E_{11}} = .4$		$\frac{E_{22}}{E_{11}} = .6$		$\frac{E_{22}}{E_{11}} = .8$		$\frac{E_{22}}{E_{11}} = 1$	
$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$	K_1 $\sqrt{in.}$	K_2 $\sqrt{in.}$
.001	-.445	1.577	-.233	1.494	-.134	1.469	-.073	1.462	-.030	1.458	-.012	1.467
.05	-.300	1.272	-.153	1.162	-.083	1.125	-.047	1.123	-.021	1.120	.000	1.119
.25	-.244	1.216	-.120	1.099	-.066	1.069	-.037	1.059	-.016	1.056	.000	1.055
.50	-.223	1.202	-.107	1.082	-.059	1.051	-.033	1.041	-.014	1.037	.000	1.035
.75	-.213	1.196	-.101	1.074	-.055	1.042	-.030	1.032	-.013	1.028	.000	1.027
1.0	-.206	1.193	-.096	1.069	-.053	1.037	-.028	1.026	-.013	1.023	.000	1.022
2.0	-.191	1.187	-.087	1.059	-.047	1.026	-.026	1.016	-.011	1.012	.000	1.011
100	-.152	1.181	-.062	1.038	-.033	1.003	-.019	.991	-.008	.987	.000	.986
∞	-.148	1.180	-.057	1.035	-.030	.999	-.016	.987	-.007	.983	.000	.982

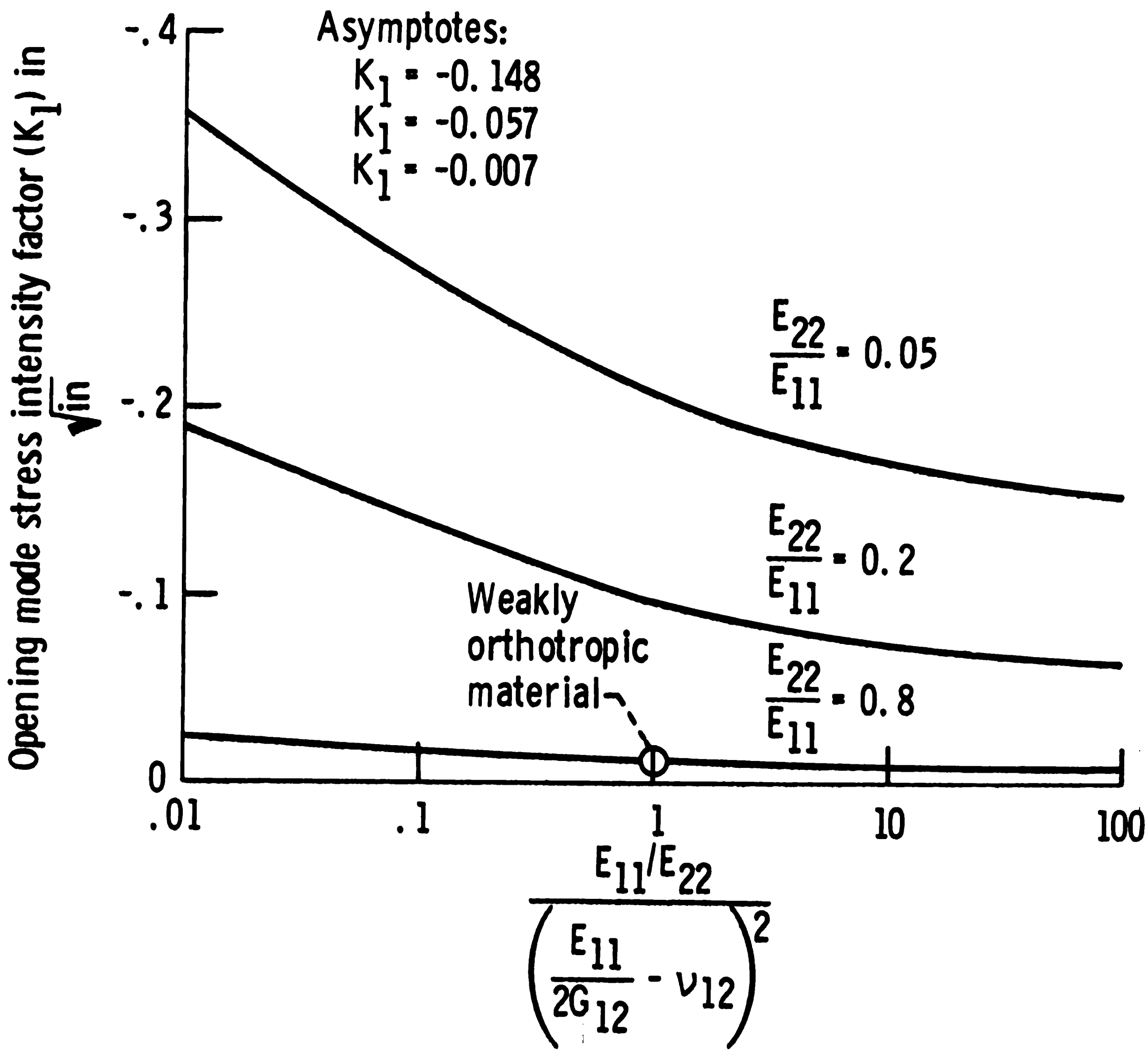
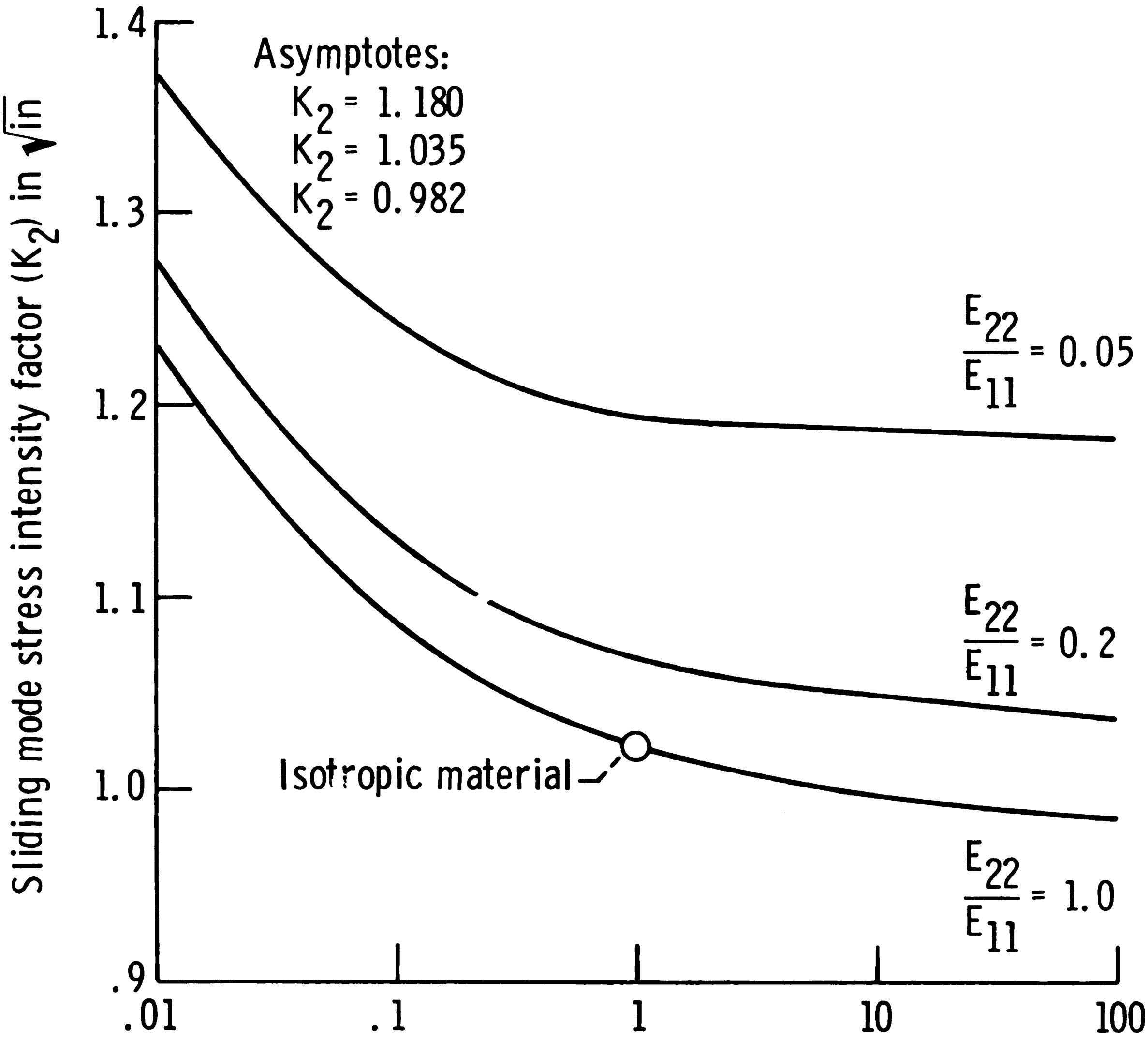


Figure 19. - Effects of material properties on the opening mode stress intensity factor for shear loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^0$, $T_s = 1$).



$$\frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2}$$

Figure 20. - Effects of material properties on the sliding mode stress intensity factor for shear loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^0$, $T_s = 1$).

Table 8.

Effects of plate width and orientation angle on the stress intensity factors for shear loading ($h/c = 1$, $T_s = 1$)

$a = .5$ in.

δ	$\frac{a}{c} = 0$		$\frac{a}{c} = \frac{1}{6}$		$\frac{a}{c} = \frac{1}{3}$		$\frac{a}{c} = \frac{1}{2}$		$\frac{a}{c} = \frac{2}{3}$	
	K_1	K_2	K_1	K_2	K_1	K_2	K_1	K_2	K_1	K_2
deg	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$	$\sqrt{in.}$
0	0	.707	.000	.720	.000	.760	.000	.839	.000	.994
2	0	.707	.000	.720	.000	.760	.000	.839	-.003	.994
5	0	.707	.000	.720	.000	.761	-.002	.841	-.008	.997
20	0	.707	.000	.722	-.002	.770	-.011	.861	-.037	1.037
34	0	.707							-.064	1.068
38	0	.707			-.005	.779	-.022	.881	-.069	1.072
42	0	.707							-.072	1.072
44	0	.707			-.007	.781	-.024	.883	-.073	1.072
45	0	.707	-.001	.725	-.007	.781	-.024	.883	-.073	1.071
46	0	.707							-.073	1.070
48	0	.707					-.025	.882		
50	0	.707			-.008	.781	-.025	.882	-.072	1.066
54	0	.707			-.009	.782	-.024	.879		
58	0	.707			-.010	.782				
62	0	.707			-.011	.782				
64	0	.707	-.004	.727						
66	0	.707			-.012	.782				
68	0	.707			-.012	.781				
70	0	.707	-.004	.727	-.011	.781	-.019	.869	-.041	1.027
85	0	.707	-.001	.727	-.004	.779	-.006	.861	-.010	1.005
88	0	.707	.000	.727	-.002	.779	-.002	.861	-.004	1.005
90	0	.707	.000	.727	.000	.779	.000	.861	.000	1.005

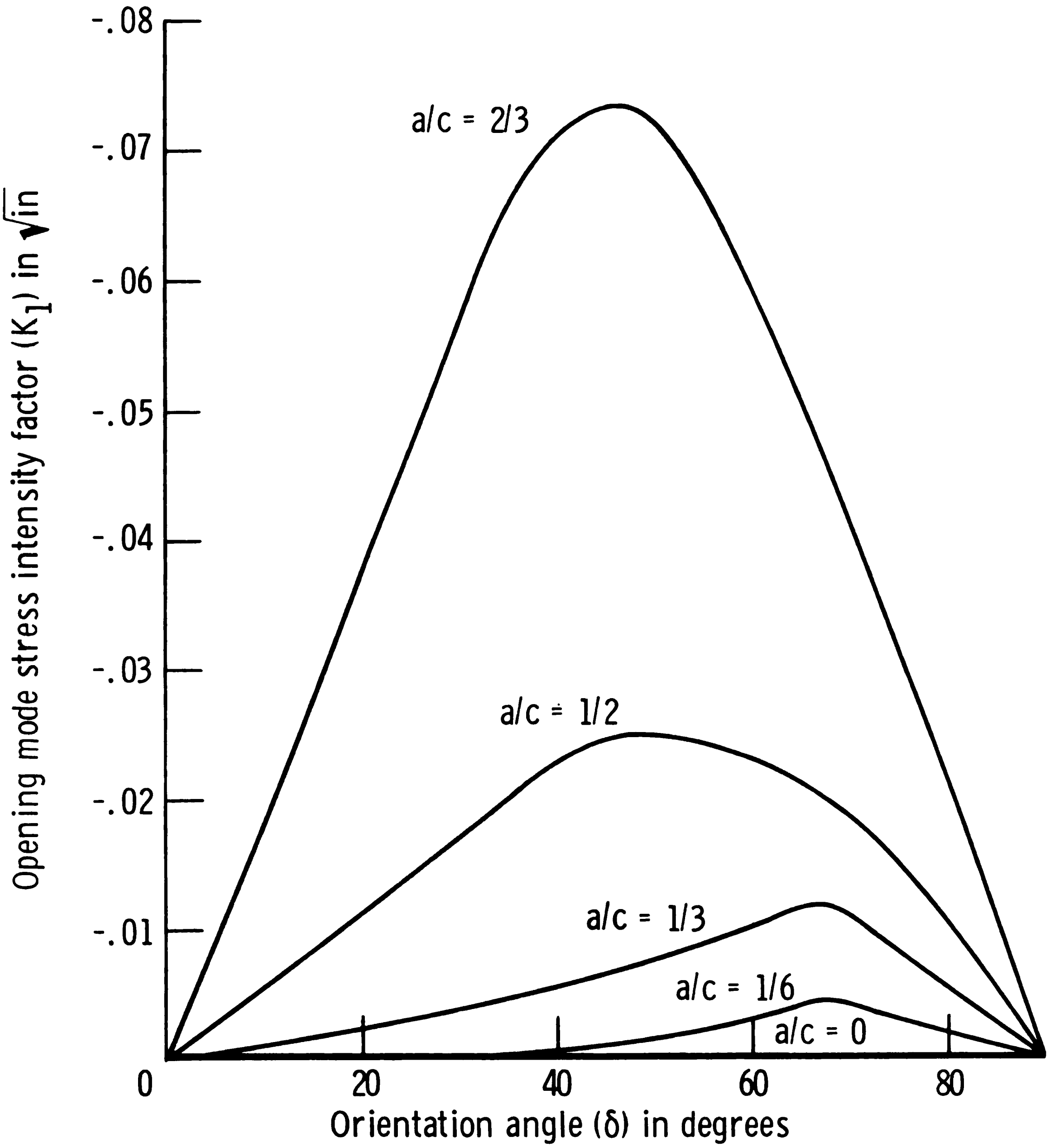


Figure 21. - Effects of orientation angle and plate width on the opening mode stress intensity factor for shear loading ($h/c = 1$, $T_S = 1$).

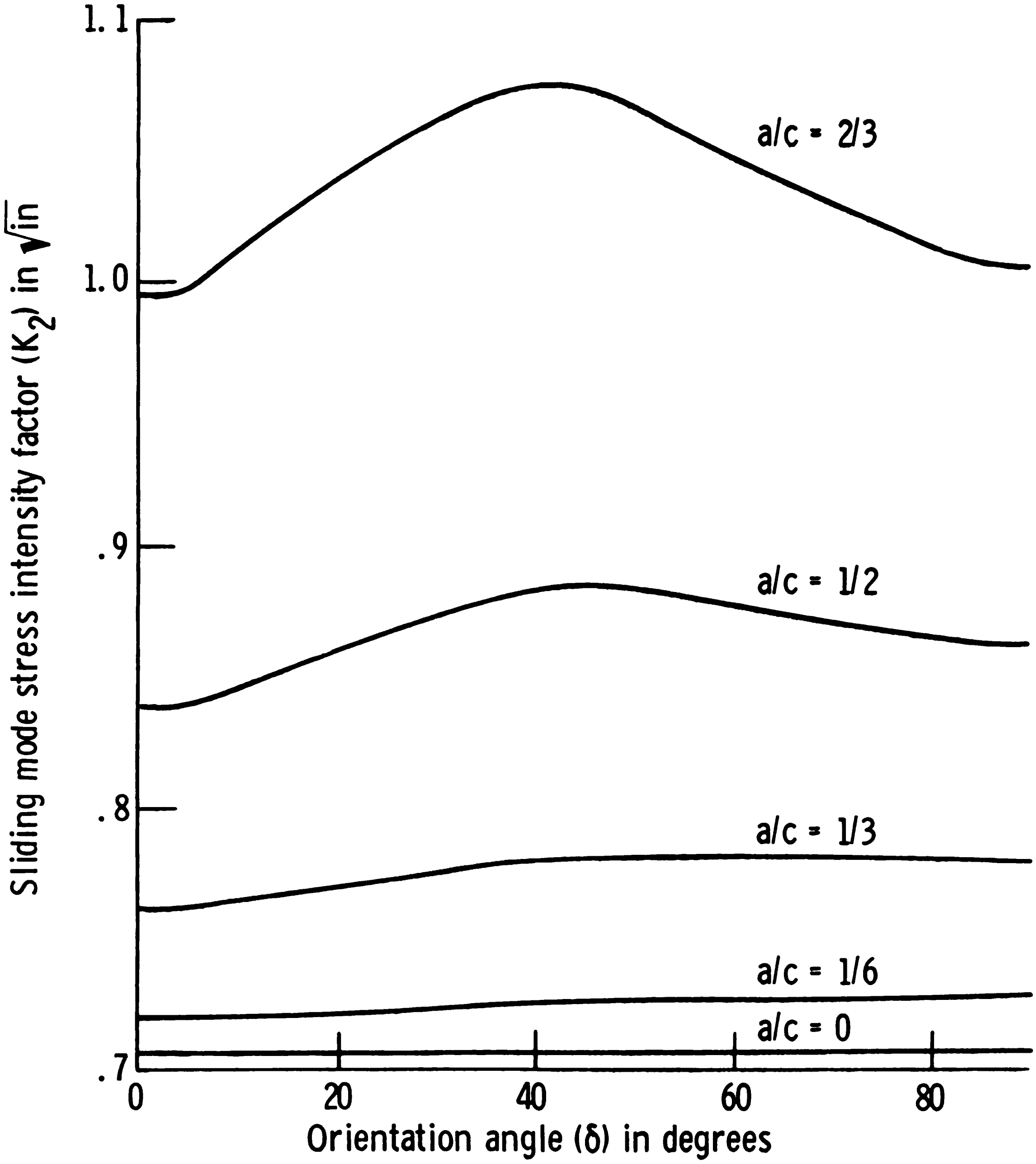


Figure 22. - Effects of orientation angle and plate width on the sliding mode stress intensity factor for shear loading (h/c = 1, T_s = 1).

Table 9.

Stress distribution on the $y = 0$, $x > a_o$ line for shear loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^o$, $T_s = 1$)

$a = .5$ in.

$\frac{x}{a}$	σ_{xx}	σ_{yy}	σ_{xy}
1.016	-3.01	-0.60	8.77
1.046	-1.55	-0.36	5.14
1.076	-1.09	-0.28	4.02
1.106	-.085	-0.23	3.43
1.136	-0.68	-0.19	3.04
1.166	-0.56	-0.15	2.76
1.196	-0.46	-0.11	2.53
1.228	-0.37	-0.07	2.34
1.258	-0.29	-0.03	2.17
1.288	-0.23	0.03	2.01
1.318	-0.18	0.09	1.87
1.348	-0.12	0.17	1.72
1.374	-0.08	0.25	1.58
1.410	-0.04	0.35	1.44
1.440	-0.02	0.46	1.30
1.470	0.00	0.58	1.15
1.500	0.00	0.72	1.00

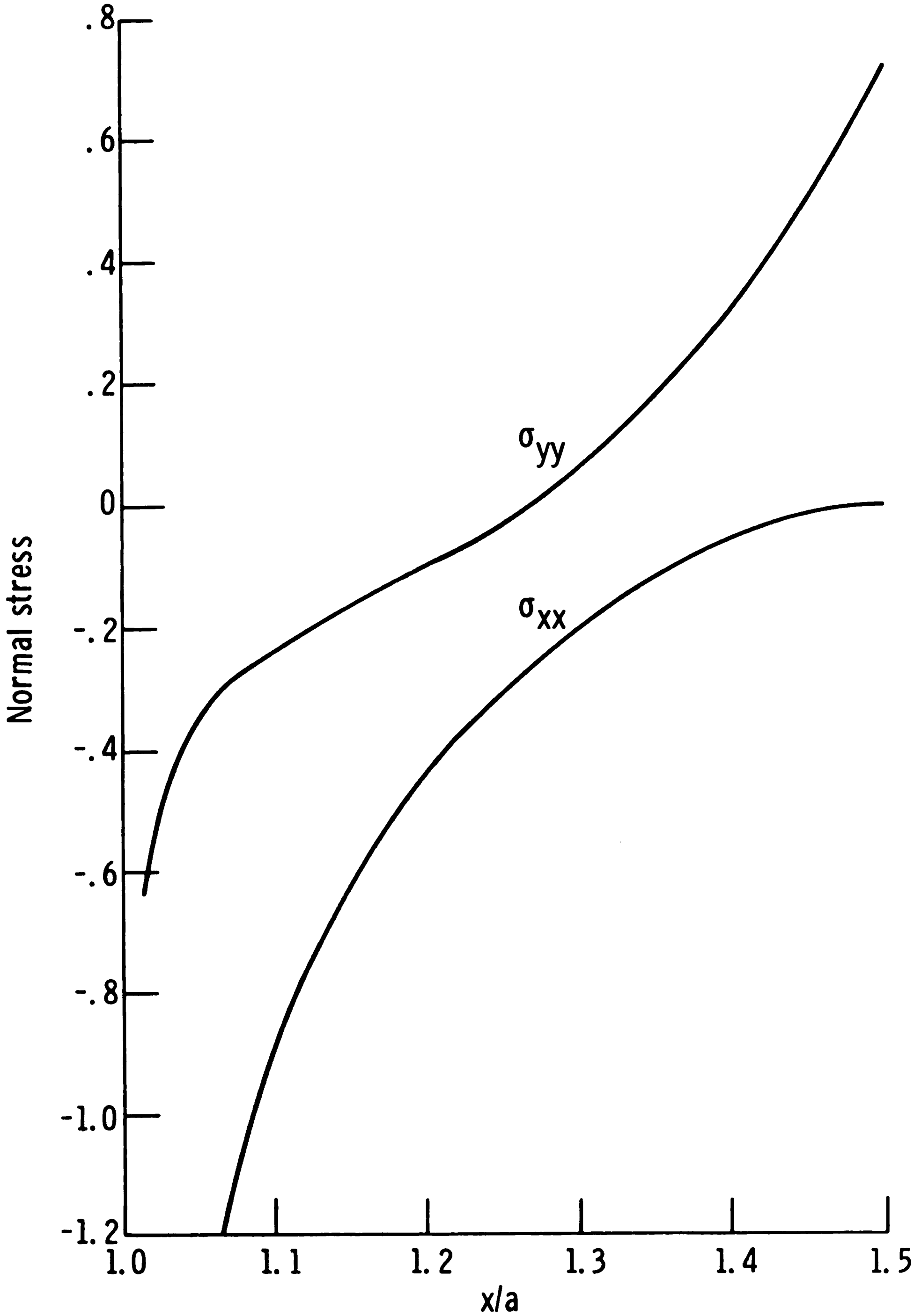


Figure 23. - Normal stress distribution on $y = 0$, $x > a$ line for shear loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^{0}$, $T_s = 1$).

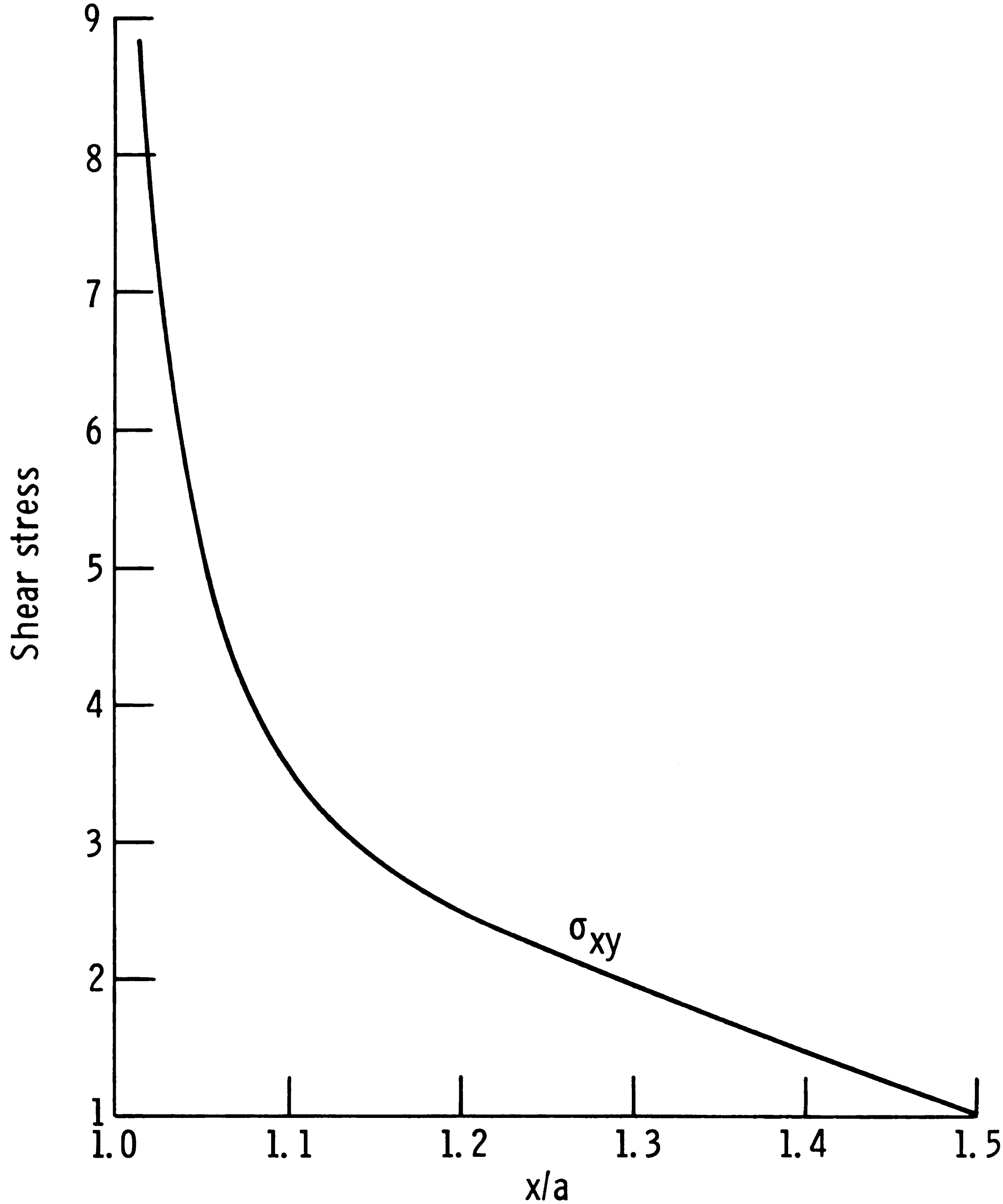


Figure 24. - Shear stress distribution on $y = 0$, $x > a$ line for shear loading ($h/c = 1$, $a/c = 2/3$, $\delta = 45^0$, $T_s = 1$).

8. SUMMARY AND RECOMMENDATIONS

The problem of multiply connected regions in plane linear anisotropic elasticity was discussed by both Lekhnitskii and Savin (refs. 4, 6), however there was no practical method of solution developed from their discussion of the problem. The motivation of the development of the mapping-collocation method for finite anisotropic regions with centrally located traction-free cracks is due to Bowie's successful results (ref. 2) for finite isotropic regions with centrally located traction-free cracks. The method developed herein closely parallels Bowie's method for the isotropic problem, i.e.:

1. The crack is mapped into the unit circle.
2. The boundary conditions on the crack are satisfied exactly by using Muskhelishvili's continuation principle.
3. A form of representation is assumed for the unknown stress potential function.
4. The boundary conditions on the outer boundary are satisfied by the method of least squares boundary collocation.

During the development of this method of solution for finite anisotropic regions, the following intermediate results were obtained:

1. The possibility of a displacement function formulation was clearly demonstrated. It was shown that Marguerre's displacement function is directly derivable from a displacement function in anisotropic elasticity.

2. For orthotropic materials two dimensionless material ratios were defined which were constructed from the engineering material constants, E_{11}, E_{22}, G_{12} and ν_{12}. These ratios are of convenient forms when a parametric study is contemplated involving stresses and stress intensity factors.
3. Upon applying Muskhilishvili's continuation principle across the unit circle, one of the complex stress potentials were expressed in the form of the other such that the zero traction conditions on the crack were satisfied. This result was directly applicable in the mapping-collocation method.
4. It was shown that the infinite region results of Lekhnitskii and Savin can be obtained from considering a finite rectangular region and extending its outer boundaries to infinity.

As final result, the complete formulation of the problem solution of a finite anisotropic region with a centrally located traction-free crack was obtained.

The mapping-collocation method as developed herein for anisotropic regions with centrally located traction-free cracks was then applied to a large number of rectangular orthotropic plate problems. The effects of varying material properties, orientation angles, crack length to plate width and plate height to plate width ratios on the stress intensity factors were thoroughly studied for both tensile and shear loadings. The parameter ranges considered in the solutions of problems of rectangular orthotropic plates with centrally located traction-free cracks are summarized as follows:

1. Variation of Material Properties for both the Tension and Shear Problems;

$$.05 \le \frac{E_{22}}{E_{11}} \le 1$$

$$.001 \le \frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} \le \infty$$

$$\frac{a}{c} = \frac{2}{3}; \qquad \frac{h}{c} = 1; \qquad \delta = 45^{\circ}$$

2. Variation of Crack Length to Plate Width and Crack Height to Plate Width Ratios for the Tension Problem;

$$0 < \frac{a}{c} \le \frac{2}{3}$$

$$.25 \le \frac{h}{c} \le 133$$

$$\frac{E_{22}}{E_{11}} = .366; \qquad \frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = .257; \qquad \delta = 45^{\circ}$$

3. Variation of Orientation Angle and Crack Length to Plate Width Ratio for both the Tension and Shear Problems;

$$0 \le \delta \le 90^{\circ}$$

$$0 < \frac{a}{c} \le \frac{2}{3}$$

$$\frac{E_{22}}{E_{11}} = .366; \qquad \frac{E_{11}/E_{22}}{\left(\frac{E_{11}}{2G_{12}} - \nu_{12}\right)^2} = .257; \qquad \frac{h}{c} = 1$$

A natural extension of the mapping-collocation method as applied to finite anisotropic regions with centrally located traction-free cracks would be to consider various shapes for the inner boundary. For example, triangular, rectangular and elliptical boundaries could be specified by a single general Schwarz-Christoffel transformation and mapped into the unit circle. This possibility would require further research as to the accuracy of the method when applied to various types of doubly connected regions.

Another direction of research presents itself in the consideration of a more detailed parametric study of the effects of dimensionless material constants, orientation angles, a/c ratios and h/c ratios on the stress intensity factors. It is a possibility that for certain parameter ranges "practical forms" of expressions could be obtained for the approximation of the stress intensity factors. These forms would result from curve-fitting the various parametric data.

9. LIST OF REFERENCES

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10. APPENDICES

A. DISCUSSION OF ELASTIC CONSTANTS

INTRODUCTION

In order to formulate problems in linear anisotropic elasticity one is compelled to study the various constitutive relations emanating from the Generalized Hooke's Law. The purpose of such a study is a thorough understanding of the different cases of anisotropy possible with certain specialized forms of the constitutive relations. Although in this dissertation interest is centered on plane problems, a discussion such as this will help in viewing the plane stress and plane strain constitutive relations in their proper perspective.

THE GENERALIZED HOOKE'S LAW

In general, it will be assumed that the stress tensor, σ_{ij}, is related to the strain tensor, $\varepsilon_{k\ell}$, as follows:

$$\sigma_{ij} = \mathcal{F}_{ij}(\varepsilon_{k\ell}) \qquad (i, j, k, \ell = 1, 2, 3) \tag{1}$$

where $\mathcal{F}_{ij}(\varepsilon_{k\ell})$ is an analytic function of the components of the strain tensor, $\varepsilon_{k\ell}$. Considering that the function, $\mathcal{F}_{ij}(\varepsilon_{k\ell})$, is analytic, it can be expanded in Taylor series. Retaining only linear terms in this expansion, the following relationship is obtained:

$$\sigma_{ij} = \mathcal{K} + E_{ijk\ell}\varepsilon_{k\ell} \tag{2}$$

Since the elastic medium is unstressed in the initial unstrained state, the constant, $\mathcal{K}$, will be taken as zero. Hence, the stress-strain relation, called the Generalized Hooke's Law, is written as,

$$\sigma_{ij} = E_{ijk\ell}\varepsilon_{k\ell} \qquad 3$$

In this expression, $E_{ijk\ell}$ is a fourth order tensor, having 81 components; however, all these components are not independent of each other. By taking into consideration the symmetric properties of both the stress and the strain tensors, it can be shown that the tensor, $E_{ijk\ell}$, has only 36 independent components.

At this point, it is convenient to introduce the notation

$$\begin{array}{ll} \sigma_1 = \sigma_{11} & \varepsilon_1 = \varepsilon_{11} \\ \sigma_2 = \sigma_{22} & \varepsilon_2 = \varepsilon_{22} \\ \sigma_3 = \sigma_{33} & \varepsilon_3 = \varepsilon_{33} \\ \sigma_4 = \sigma_{23} & \varepsilon_4 = \varepsilon_{23} \\ \sigma_5 = \sigma_{13} & \varepsilon_5 = \varepsilon_{13} \\ \sigma_6 = \sigma_{12} & \varepsilon_6 = \varepsilon_{12} \end{array} \qquad 4$$

Then the stress-strain relation can be written in indicial notation as,

$$\sigma_i = S_{ij}\varepsilon_j \qquad (i,j = 1, 2, 3, 4, 5, 6) \qquad 5$$

where the conversion of indices is obvious from 4. The elements, S_{ij}, of the matrix, S, are called the elastic stiffnesses or the modili of elasticity.

For a real material, the matrix, S, is never singular; hence, it possesses an inverse. The elements of the inverse of S are customarily called the elastic compliances or the coefficients of deformation. Thus, the inverse stress-strain relation is given by

$$\varepsilon_i = C_{ij}\sigma_j \qquad (i,j = 1, 2, 3, 4, 5, 6) \qquad 6$$

It may be noted that neither S_{ij} nor C_{ij} is a tensor. The indicial notation is adopted only as a matter of convenience.

An additional symmetry property can be deduced from considering the strain energy density of the material. If the existence of an elastic potential is postulated, it can easily be shown that

$$S_{ij} = S_{ji} \tag{7}$$

and

$$C_{ij} = C_{ji} \tag{8}$$

This symmetric property reduces the number of independent components of the tensor, $E_{ijk\ell}$, and also the independent elements of the matrices S and C from 36 to 21 material constants. The material which has 21 independent constants relating its strains to its stresses is called generally anisotropic.

ROTATION OF AXES

The fourth order tensor, $E_{ijk\ell}$, obeys the transformation

$$E'_{ijk\ell} = t_{im}t_{jn}t_{ko}t_{\ell p}E_{mnop} \tag{9}$$

Considering the symmetry properties given above, this transformation consists of 21 equations for an anisotropic material.

For illustration purposes and later use, consider a rotation of the $0\text{-}x_1\text{-}x_2\text{-}x_3$ coordinate system about the x_3-axis defined by the transformation matrix

$$t = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ o & o & 1 \end{bmatrix} \tag{10}$$

where

$$m = \cos\delta$$

$$n = \sin\delta$$

δ being the angle between the x_1-axes of the original and the rotated systems. Such a rotation will somewhat simplify the transformation equations. Noting that

$$t_{13} = t_{23} = t_{31} = t_{32} = 0$$

the 21 equations relating the elastic compliances and the elastic stiffnesses in the original and the rotated systems can be given in matrix form (ref. 14)

$$\begin{bmatrix} C'_{11}(S'_{11}) \\ C'_{12}(S'_{12}) \\ C'_{16}(2S'_{16}) \\ C'_{22}(S'_{22}) \\ C'_{26}(2S'_{26}) \\ C'_{66}(4S'_{66}) \end{bmatrix} = \begin{bmatrix} m^4 & 2m^2n^2 & 2m^3n & n^4 & 2mn^3 & m^2n^2 \\ m^2n^2 & m^4 + n^4 & mn^3 - m^3n & m^2n^2 & m^3n - mn^3 & -m^2n^2 \\ -2m^3n & 2m^3n - 2mn^3 & m^4 - 3m^2n^2 & 2mn^3 & 3m^2n^2 - n^4 & m^3n - mn^3 \\ n^4 & 2m^2n^2 & -2mn^3 & m^4 & -2m^3n & m^2n^2 \\ -2mn^3 & 2mn^3 - 2m^3n & 3m^2n^2 - n^4 & 2m^3n & m^4 - 3m^2n^2 & mn^3 - m^3n \\ 4m^2n^2 & -8m^2n^2 & 4mn^3 - 4m^3n & 4m^2n^2 & 4m^3n - 4mn^3 & (m^2 - n^2)^2 \end{bmatrix} \begin{bmatrix} C_{11}(S_{11}) \\ C_{12}(S_{12}) \\ C_{16}(2S_{16}) \\ C_{22}(S_{22}) \\ C_{26}(2S_{26}) \\ C_{66}(4S_{66}) \end{bmatrix}$$

$$\begin{bmatrix} C'_{14}(S'_{14}) \\ C'_{15}(S'_{15}) \\ C'_{24}(S'_{24}) \\ C'_{25}(S'_{25}) \\ C'_{46}(2S'_{46}) \\ C'_{56}(2S'_{56}) \end{bmatrix} = \begin{bmatrix} m^3 & -m^2n & mn^2 & -n^3 & m^2n & -mn^2 \\ m^2n & m^3 & n^3 & mn^2 & mn^2 & m^2n \\ mn^2 & -n^3 & m^3 & -m^2n & -m^2n & mn^2 \\ n^3 & mn^2 & m^2n & m^3 & -mn^2 & -m^2n \\ -2m^2n & 2mn^2 & 2m^2n & -2mn^2 & m^3 - mn^2 & n^3 - m^2n \\ -2mn^2 & -2m^2n & 2mn^2 & 2m^2n & m^2n - n^3 & m^3 - mn^2 \end{bmatrix} \begin{bmatrix} C_{14}(S_{14}) \\ C_{15}(S_{15}) \\ C_{24}(S_{24}) \\ C_{25}(S_{25}) \\ C_{46}(2S_{46}) \\ C_{56}(2S_{56}) \end{bmatrix} \qquad 13$$

$$\begin{bmatrix} C'_{13}(S'_{13}) \\ C'_{23}(S'_{23}) \\ C'_{36}(2S'_{36}) \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} C_{13}(S_{13}) \\ C_{23}(S_{23}) \\ C_{36}(2S_{36}) \end{bmatrix} \qquad 14$$

$$\begin{bmatrix} C'_{44}(S'_{44}) \\ C'_{45}(S'_{45}) \\ C'_{55}(S'_{55}) \end{bmatrix} = \begin{bmatrix} m^2 & -2mn & n^2 \\ mn & m^2 - n^2 & -mn \\ n^2 & 2mn & m^2 \end{bmatrix} \begin{bmatrix} C_{44}(S_{44}) \\ C_{45}(S_{45}) \\ C_{55}(S_{55}) \end{bmatrix} \qquad 15$$

$$\begin{bmatrix} C'_{34}(S'_{34}) \\ C'_{35}(S'_{35}) \end{bmatrix} = \begin{bmatrix} m & -n \\ n & m \end{bmatrix} \begin{bmatrix} C_{34}(S_{34}) \\ C_{35}(S_{35}) \end{bmatrix} \qquad 16$$

$$C'_{33}(S'_{33}) = C_{33}(S_{33}) \qquad 17$$

It is noted here that the transformation equations are arranged in six groups. The components of each group interact with one another but are completely uncoupled from the other groups. This information is useful in studying elastic symmetries.

AEOLOTROPIC MATERIALS

Aeolotropic or monoclinic elastic symmetry is defined by the invariance of the elastic stiffnesses and compliances under the transformation given by

$$t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{(ref. 14)} \qquad 18$$

This transformation specifies a material which has the x_1-0-x_2 plane as its plane of elastic symmetry, i.e., the properties of the material at $+x_3$ are equal to those at $-x_3$.

In order to derive the elastic stiffness and compliance matrices for an aeolotropic material, consider the stress and the strain tensors under the above transformation.

The stress tensor obeys the transformation

$$\sigma'_{ij} = t_{ik} t_{j\ell} \sigma_{k\ell} \qquad 19$$

The strain tensor transforms in a similar manner

$$\varepsilon'_{ij} = t_{ik} t_{j\ell} \varepsilon_{k\ell} \tag{20}$$

Carrying out these operations, one obtains

$$\begin{array}{llll} \sigma'_{11} = \sigma_{11} & \sigma'_{23} = -\sigma_{23} & \varepsilon'_{11} = \varepsilon_{11} & \varepsilon'_{23} = -\varepsilon_{23} \\ \sigma'_{22} = \sigma_{22} & \sigma'_{31} = -\sigma_{31} & \varepsilon'_{22} = \varepsilon_{22} & \varepsilon'_{31} = -\varepsilon_{31} \\ \sigma'_{33} = \sigma_{33} & \sigma'_{12} = \sigma_{12} & \varepsilon'_{33} = \varepsilon_{33} & \varepsilon'_{12} = \varepsilon_{12} \end{array} \tag{21}$$

Upon changing to the appropriate indices in contracted notation, Hooke's Law is written in both coordinate systems. For example, σ'_1 and σ_1 become

$$\sigma'_1 = S_{11}\varepsilon'_1 + S_{12}\varepsilon'_2 + S_{13}\varepsilon'_3 + S_{14}\varepsilon'_4 + S_{15}\varepsilon'_5 + S_{16}\varepsilon'_6 \tag{22}$$

$$\sigma_1 = S_{11}\varepsilon_1 + S_{12}\varepsilon_2 + S_{13}\varepsilon_3 + S_{14}\varepsilon_4 + S_{15}\varepsilon_5 + S_{16}\varepsilon_6 \tag{23}$$

Substituting the relations 21 into 22 and comparing 22 with 23, one must take for an aeolotropic material

$$S_{14} = S_{15} = 0.$$

By considering all the equations of Hooke's Law, it can be shown that for an aeolotropic material the elastic stiffness and compliance matrices must be of the form

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \qquad 24$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

Hence, it is concluded that an aeolotropic material is characterized by 13 independent constants. It may also be noted that for the state of plane stress or plane strain, aeolotropy is the first assumption.

ANISOTROPIC PLANE STRAIN

The state of generalized plane strain is obtained by assuming

$$\begin{aligned} u_1 &= u_1(x_1, x_2) \\ u_2 &= u_2(x_1, x_2) \qquad 26 \\ u_3 &= u_0 \end{aligned}$$

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \qquad 24$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

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where u_0 is a constant. Upon using the infinitesimal strain-displacement relations in conjunction with the generalized plane strain assumption, one obtains

$$\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0 \tag{27}$$

When Hooke's Law for aeolotropic materials is applied, one finds that the normal stress in the x_3 - direction can be expressed in terms of the other stress components. Consequently, it can be eliminated from Hooke's Law, and the resulting stress-strain relations can be written for the state of anisotropic plane strain as,

$$\varepsilon_i = \alpha_{ij}\sigma_j \tag{28}$$

where

$$\alpha_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \qquad (i,j = 1,\ 2,\ 6)$$

The constants, α_{ij}, are called the reduced compliance coefficients. If the elastic stiffnesses are used, Hooke's Law becomes

$$\sigma_i = S_{ij}\varepsilon_j \qquad (i,j = 1,\ 2,\ 6) \tag{29}$$

i.e., the elements of the elastic stiffness matrix remain unchanged.

Considering both forms of the stress-strain relations, the relationships among the elastic constants can immediately be obtained. Taking

$$\sigma_i = S_{ij}\varepsilon_j = S_{ij}\alpha_{jk}\sigma_k \tag{30}$$

and using Kronecker's delta, results in

$$S_{ij}\alpha_{jk} = \delta_{ik} \tag{31}$$

Hence, for the state of anisotropic plane strain, the expressions which relate the various elements of the reduced elastic compliance and the elastic stiffness matrices are given as follows:

Elastic stiffnesses in terms of reduced elastic compliances:

$$S_{11} = \frac{\alpha_{22}\alpha_{66} - \alpha_{26}^2}{|\alpha|} \qquad S_{22} = \frac{\alpha_{11}\alpha_{66} - \alpha_{16}^2}{|\alpha|}$$

$$S_{12} = \frac{\alpha_{16}\alpha_{26} - \alpha_{66}\alpha_{12}}{|\alpha|} \qquad S_{26} = \frac{\alpha_{16}\alpha_{12} - \alpha_{11}\alpha_{26}}{|\alpha|} \qquad 32$$

$$S_{16} = \frac{\alpha_{12}\alpha_{26} - \alpha_{22}\alpha_{16}}{|\alpha|} \qquad S_{66} = \frac{\alpha_{11}\alpha_{22} - \alpha_{12}^2}{|\alpha|}$$

Reduced elastic compliances in terms of elastic stiffnesses:

$$\alpha_{11} = \frac{S_{22}S_{66} - S_{26}^2}{|S|} \qquad \alpha_{22} = \frac{S_{11}S_{66} - S_{16}^2}{|S|}$$

$$\alpha_{12} = \frac{S_{16}S_{26} - S_{12}S_{66}}{|S|} \qquad \alpha_{26} = \frac{S_{16}S_{12} - S_{11}S_{26}}{|S|} \qquad 33$$

$$\alpha_{13} = \frac{S_{12}S_{26} - S_{22}S_{16}}{|S|} \qquad \alpha_{66} = \frac{S_{11}S_{22} - S_{12}^2}{|S|}$$

In these expressions $|\alpha|$ and $|S|$ designate the determinants of the reduced elastic compliance and the elastic stiffness matrices, respectively.

For a stable solid, the strain energy density function must always be positive definite for any value of the strain (stress) components, except when all the strain (stress) components are zero. From matrix theory, the necessary and sufficient condition for the quadratic form i.e., the strain energy density function,

$$W = \frac{1}{2} \alpha_{ij}\sigma_i\sigma_j \qquad 34$$

or

$$W = \frac{1}{2} S_{ij}\varepsilon_i\varepsilon_j \qquad 35$$

to be positive definite is that all the leading principal minors of the matrices, α and S must be positive. Hence, the material constants must satisfy the following inequalities:

$$\alpha_{11} > 0, \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{16} \\ \alpha_{12} & \alpha_{22} & \alpha_{26} \\ \alpha_{16} & \alpha_{26} & \alpha_{66} \end{vmatrix} > 0 \qquad 36$$

and

$$S_{11} > 0, \quad \begin{vmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{vmatrix} > 0 \qquad 37$$

These relations in conjunction with relations 32 and 33 imply that the diagonal elements are all positive.

ANISOTROPIC PLANE STRESS

The state of anisotropic plane stress is defined by the following assumption:

$$\sigma_3 = \sigma_4 = \sigma_5 = 0 \qquad 38$$

Upon the use of these assumptions in Hooke's Law for aeolotropic materials, it is found that the normal strain in the x_3 -direction can be eliminated from the strain relations. The resulting stress-strain relation for the state of anisotropic plane stress becomes

$$\sigma_i = \beta_{ij}\varepsilon_j \qquad 39$$

where

$$\beta_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}} \qquad (i,j = 1, 2, 6)$$

In this expression, the constants, β_{ij}, are called the reduced elastic stiffnesses. When writing Hooke's Law in terms of the elastic compliances, one obtains

$$\varepsilon_i = C_{ij}\sigma_j \qquad (i,j = 1, 2, 6) \qquad 40$$

Hence, in the case of anisotropic plane stress, the elastic compliances are not reduced.

The relationships among the reduced elastic stiffnesses and the elastic compliances are given as follows:

Reduced elastic stiffnesses in terms of elastic compliances:

$$\beta_{11} = \frac{C_{22}C_{66} - C_{26}^2}{|C|} \qquad \beta_{22} = \frac{C_{11}C_{66} - C_{16}^2}{|C|}$$

$$\beta_{12} = \frac{C_{16}C_{26} - C_{12}C_{66}}{|C|} \qquad \beta_{26} = \frac{C_{16}C_{12} - C_{11}C_{26}}{|C|} \qquad 41$$

$$\beta_{16} = \frac{C_{12}C_{26} - C_{22}C_{16}}{|C|} \qquad \beta_{66} = \frac{C_{11}C_{22} - C_{12}^2}{|C|}$$

Elastic compliances in terms of reduced elastic stiffnesses:

$$C_{11} = \frac{\beta_{22}\beta_{66} - \beta_{26}^2}{|\beta|} \qquad C_{22} = \frac{\beta_{11}\beta_{66} - \beta_{16}^2}{|\beta|}$$

$$C_{12} = \frac{\beta_{16}\beta_{26} - \beta_{12}\beta_{66}}{|\beta|} \qquad C_{26} = \frac{\beta_{16}\beta_{12} - \beta_{11}\beta_{26}}{|\beta|} \qquad 42$$

$$C_{16} = \frac{\beta_{12}\beta_{26} - \beta_{22}\beta_{16}}{|\beta|} \qquad C_{66} = \frac{\beta_{11}\beta_{22} - \beta_{12}^2}{|\beta|}$$

The consideration of the positive definiteness of the strain energy density function results in inequalities similar to those obtained for the state of plane strain.

$$\beta_{11} > 0, \qquad \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{12} & \beta_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{vmatrix} > 0 \qquad 43$$

and

$$C_{11} > 0, \qquad \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{vmatrix} > 0 \qquad 44$$

In this case, it may also be concluded that the diagonal elements of the reduced elastic stiffness and the elastic compliance matrices are positive.

ENGINEERING CONSTANTS

From simple tensile and torsion tests, one can obtain constants which can readily be related to the elastic stiffnesses and compliances. These constants, customarily called engineering constants, are given in ref. 4, and they are reproduced here for reasons of completeness.

For an aeolotropic material, the elastic compliances are expressed in terms of engineering constants as follows:

$$C_{11} = \frac{1}{E_{11}} \qquad C_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \qquad C_{44} = \frac{1}{G_{23}}$$

$$C_{22} = \frac{1}{E_{22}} \qquad C_{23} = -\frac{\nu_{23}}{E_{22}} = -\frac{\nu_{13}}{E_{33}} \qquad C_{55} = \frac{1}{G_{13}} \tag{45}$$

$$C_{33} = \frac{1}{E_{33}} \qquad C_{13} = -\frac{\nu_{31}}{E_{33}} = -\frac{\nu_{13}}{E_{11}} \qquad C_{66} = \frac{1}{G_{12}}$$

$$C_{45} = \frac{\mu_{31,12}}{G_{23}} = \frac{\mu_{23,31}}{G_{13}} \qquad C_{16} = \frac{\eta_{12,1}}{E_{11}} = \frac{\eta_{1,12}}{G_{12}}$$

$$C_{36} = \frac{\eta_{12,3}}{E_{33}} = \frac{\eta_{3,12}}{G_{12}} \qquad C_{26} = \frac{\eta_{12,2}}{E_{22}} = \frac{\eta_{2,12}}{G_{12}}$$

where:

E_{11}, E_{22} and E_{33} are Young's moduli of elasticity with respect to the coordinate directions.

G_{23}, G_{13} and G_{12} are the shear moduli for planes which are parallel to the coordinate planes, x_2-0-x_3, x_1-0-x_3 and x_1-0-x_2.

ν_{21}, ν_{32}, ν_{13}, ν_{12}, ν_{23} and ν_{31} are the Poisson ratios which characterize contraction in the direction given by the second subscript due to extension in the direction given by the first subscript.

The constants, $\mu_{31,12}$ and $\mu_{23,31}$, are called the coefficients of Chentsov. These constants characterize shear in the coordinate planes. For example, $\mu_{31,12}$ characterizes shear in the planes parallel to the x_1 and x_2 coordinates inducing shear stresses in planes parallel to the x_3 and x_1 coordinates.

$\eta_{23,1}$, etc., are called the coefficients of mutual influence of the first kind and $\eta_{1,23}$, etc., are called the coefficients of mutual influence of the second kind. The mutual influence coefficients of the first kind characterize stretching in the directions parallel to the x_1 coordinte, etc., induced by shear stresses in the x_2-0-x_3 plane, etc. The mutual influence coefficients of the second kind characterize the shears in the planes parallel to the x_2 and x_3 coordintes, etc., which are due to normal stresses along the x_1 coordinte, etc..

Using these engineering constants, the elastic compliance and the reduced elastic compliance matrices for the anisotropic plane stress and plante strain states become

$$\alpha = \begin{bmatrix} \dfrac{1 - \nu_{13}\nu_{31}}{E_{11}} & -\dfrac{\nu_{21} + \nu_{31}\nu_{23}}{E_{22}} & \dfrac{\eta_{12,1} + \nu_{13}\eta_{12,3}}{E_{11}} \\ -\dfrac{\nu_{12} + \nu_{13}\nu_{32}}{E_{11}} & \dfrac{1 - \nu_{32}\nu_{23}}{E_{22}} & \dfrac{\eta_{12,2} + \nu_{23}\eta_{12,3}}{E_{22}} \\ \dfrac{\eta_{12,1} + \nu_{13}\eta_{12,3}}{E_{11}} & \dfrac{\eta_{12,2} + \nu_{23}\eta_{12,3}}{E_{22}} & \dfrac{1 - \eta_{3,12}\eta_{12,3}}{G_{12}} \end{bmatrix} \qquad 46$$

and

$$C = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & \frac{\eta_{12,1}}{E_{11}} \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & \frac{\eta_{12,2}}{E_{22}} \\ \frac{\eta_{1,12}}{G_{12}} & \frac{\eta_{2,12}}{G_{12}} & \frac{1}{G_{12}} \end{bmatrix} \qquad 47$$

The corresponding reduced elastic stiffness and elastic stiffness matrices can readily be obtained from relations 32 and 41.

INVARIANTS

For a generally anisotropic material there are a number of invariants associated with the compliance and stiffness matrices with respect to a rotation about the x_3-axis (ref. 14). The proofs of these invariants follow directly from the transformation equations 12, 13, 14, 15, 16 and 17. These invariants can, of course, be specialized for the various cases discussed by taking the appropriate terms as zero.

$$\begin{aligned} I_1 &= C_{11} + C_{22} + 2C_{12} \\ I_2 &= C_{66} - 4C_{12} \\ I_3 &= C_{44} + C_{55} \\ I_4 &= C_{13} + C_{23} \\ I_5 &= C_{34}^2 + C_{35}^2 \\ I_6 &= C_{33} \end{aligned} \qquad 48$$

and

$$
\begin{aligned}
J_1 &= S_{11} + S_{22} + 2S_{12} \\
J_2 &= S_{66} - S_{12} \\
J_3 &= S_{44} + S_{55} \\
J_4 &= S_{23} + S_{13} \\
J_5 &= S_{34}^2 + S_{35}^2 \\
J_6 &= S_{33}
\end{aligned} \qquad 49
$$

It may also be noted that the compressibility of an anisotropic material given by

$$\frac{1}{I_1 + I_6 + 2I_4} \qquad 50$$

is also invariant.

ORTHOTROPIC MATERIALS

An anisotropic material possessing two perpendicular planes of elastic symmetry, say, $x_3 = 0$ and $x_1 = 0$, will automatically have the $x_2 = 0$ plane also as its plane of elastic symmetry. Such a material is called orthotropic; characterized by 9 independent material constants. The compliance and stiffness matrices formed by these constants must be of the form

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \qquad 51$$

and

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \qquad 52$$

ORTHOTROPIC PLANE STRAIN

The consideration of an orthotropic material in the state of plane strain results in

$$C_{16} = C_{26} = C_{36} = C_{45} = S_{16} = S_{26} = S_{36} = S_{45} = 0 \qquad 53$$

consequently

$$\alpha_{16} = \alpha_{26} = 0 \qquad 54$$

Then the reduced elastic compliance matrix, α, and the elastic stiffness matrix, S, will be of the form

$$\alpha = \begin{bmatrix} C_{11} - \dfrac{C_{13}^2}{C_{33}} & C_{12} - \dfrac{C_{13}C_{23}}{C_{33}} & 0 \\ C_{12} - \dfrac{C_{13}C_{23}}{C_{33}} & C_{22} - \dfrac{C_{23}^2}{C_{33}} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \qquad 55$$

and

$$S = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \qquad 56$$

In terms of engineering constants, these two matrices become

$$\alpha = \begin{bmatrix} \dfrac{1 - \nu_{13}\nu_{31}}{E_{11}} & -\dfrac{\nu_{21} + \nu_{31}\nu_{23}}{E_{22}} & 0 \\ -\dfrac{\nu_{12} + \nu_{13}\nu_{32}}{E_{11}} & \dfrac{1 - \nu_{32}\nu_{23}}{E_{22}} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \qquad 57$$

and

$$S = \begin{bmatrix} K_\nu E_{11}(1 - \nu_{23}\nu_{32}) & K_\nu E_{11}(\nu_{21} + \nu_{23}\nu_{31}) & 0 \\ K_\nu E_{22}(\nu_{12} + \nu_{13}\nu_{32}) & K_\nu E_{22}(1 - \nu_{31}\nu_{13}) & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \qquad 58$$

where

$$K_\nu = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31})^{-1}$$

ORTHOTROPIC PLANE STRESS

For the state of plane stress, the reduced elastic stiffness and the elastic compliance matrices satisfy the following forms:

$$\beta = \begin{bmatrix} S_{11} - \dfrac{S_{13}^2}{S_{33}} & S_{12} - \dfrac{S_{13}S_{23}}{S_{33}} & 0 \\ S_{12} - \dfrac{S_{13}S_{23}}{S_{33}} & S_{22} - \dfrac{S_{23}^2}{S_{33}} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}. \qquad 59$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \qquad 60$$

These matrices can be given in terms of engineering constants as,

$$\beta = \begin{bmatrix} \dfrac{E_{11}}{1 - \nu_{12}\nu_{21}} & \dfrac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} & \dfrac{E_{22}}{1 - \nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \qquad 61$$

and

$$C = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & 0 \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \qquad 62$$

GENERAL ORTHOTROPY IN PLANE PROBLEMS

In orthotropic problems, the consideration of different material and reference axes forces one to use the equations transforming the material constants from the material axis system to the reference coordinate system. The material constants are customarily defined with respect to the material axes; hence, the elements of the elastic stiffness and compliance matrices will vary in accordance with the prescribed transformations between the two coordinate systems.

In the case of plane orthotropy, one finds that the elastic stiffness and compliance matrices will become full matrices upon transformation; however, the number of independent material constants remains unchanged. In the literature, one speaks of general orthotropy when the reference axes do not coincide with the axes of orthotropy.

The transformation equations which apply for plane orthotropy are given in group 12 with the specialization

$$C_{16} = C_{26} = S_{16} = S_{26} = 0$$

Consequently, the elements of the elastic compliance matrix can easily be obtained for plane orthotropy as,

$$C'_{11} = \frac{\cos^4\delta}{E_{11}} + \frac{\sin^4\delta}{E_{22}} + \frac{\sin^2 2\delta}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)$$

$$C'_{12} = -\frac{\nu_{21}}{E_{22}} + \frac{\sin^2 2\delta}{4}\left(\frac{1+\nu_{12}}{E_{11}} + \frac{1+\nu_{21}}{E_{22}} - \frac{1}{G_{12}}\right)$$

$$C'_{22} = \frac{\cos^4\delta}{E_{22}} + \frac{\sin^4\delta}{E_{11}} + \frac{\sin^2 2\delta}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{21}}{E_{22}}\right)$$

63

$$C'_{66} = \sin^2 2\delta\left(\frac{1}{E_{11}} + \frac{1}{E_{22}} + \frac{2\nu_{21}}{E_{22}}\right) + \frac{\cos^2 2\delta}{G_{12}}$$

$$C'_{16} = \sin 2\delta\left[\frac{\sin^2\delta}{E_{22}} - \frac{\cos^2\delta}{E_{11}} + \left(\frac{1}{2G_{12}} - \frac{\nu_{21}}{E_{22}}\right)\cos 2\delta\right]$$

$$C'_{26} = \sin 2\delta\left[\frac{\cos^2\delta}{E_{22}} - \frac{\sin^2\delta}{E_{11}} - \left(\frac{1}{2G_{12}} - \frac{\nu_{21}}{E_{22}}\right)\cos 2\delta\right]$$

where δ is the angle between the material axes of orthotropy and the reference coordinate system. The elastic stiffness, the reduced elastic compliance and the reduced elastic stiffness matrices can then be constructed in accordance with the appropriate relations given before.

B. COMPUTER PROGRAM

The computer program whose listing is reproduced on the following pages, will perform the required operations in the following order:

1. The coordinates of the collocation points are calculated.
2. The elements of the load vector, K, are computed for each collocation point.
3. Lekhnitskii's complex material parameters are obtained.
4. The elements of the coefficient matrix, $\mathcal{C}$, are calculated and the multiplication, $\mathcal{C}^T\mathcal{C}$, is accomplished.
5. The modified load vector, $\mathcal{C}^T K$, is obtained.
6. By calling the subroutine, DGELG, the least squares solution vector, $\mathcal{C}^*$, is found. The DGELG subroutine utilizes Gauss' method with complete pivoting.
7. The stress intensity factors, K_1 and K_2 are obtained.
8. Finally, the stress components are calculated on the boundary and on the $y=0$ line.

The input variables are designated as follows:

TX, SY	one half of the constant specified boundary tensile stress; TX on the $y=h$, $-c < x < c$ and SY on the $x = \pm c$, $0 < y < h$ part of the boundary.
SX, TY	minus one half of the constant specified boundary shear stress; SX on the $y=h$, $-c < x < c$ and TY on the $x = \pm c$, $0 < y < h$ part of the boundary.

NYM, NXM, NXP, NYP	number of collocation points on the following sections of the boundary: NYM for $x = -c$, $0 \leq y \leq h$; NXM for $y=h$, $-c < x \leq 0$; NXP for $y=h$, $0 < x \leq c$; NYP for $x=c$, $0 < y < h$.
A	half-length of crack
C	half-width of plate
H	half-height of plate
DELTA	orientation angle
V12	Poisson's ratio
E_{11}, E_{22}	Young's moduli of elasticity
G_{12}	shear modulus
NN, NP	truncation numbers of infinite sums associated with the negative and positive exponent terms of a Laurent series, respectively

The results recorded in the output are the coordinates of the collocation points, the elements of the least squares solution vector, the stress intensity factors, K_1 and K_2, the stress components on the boundary and on the $y=0$ line at discrete points.

```
      COMMON/DATA/Y(200),X(200),ASIGXX(200),ASIGYY(200),ASIGXY(200),SY,SX,TY,TX,C,H,A,DELTA,
     *V12,G12,E22,E11,HEAD(7),NYP,NYM,NXP,NXM,NN,NP
      REAL*8 X,Y,SY,SX,TX,TY,C,H,A,DELTA,V12,G12,E22,E11,ASIGXX,ASIGYY,ASIGXY
      DIMENSION ETEM(100,100)
      REAL*8 ETEM
      NAMELIST/OUTCON/NP,NN,NYM,NXM,NXP,NYP,-
     *TX,TY,SX,SY,C,H,A,DELTA,V12,G12,E22,E11
      READ(5,OUTCON)
      N4=NP+NN+1
      WRITE(6,600)
  600 FORMAT(1H1)
      WRITE(7,OUTCON)
      CALL CALCN(ETEM,N4)
      STOP
      END
```

```
      SUBROUTINE CALCN(ETEM,N4X)
      DIMENSION ETEMH(100,100),XKM(200),AUX(100)
      DIMENSION ETEM(N4X,N4X)
      DIMENSION EM(200,100),CM(100),KM(200),ZCON(6),ZCONN(6),
     *ETKM(100),ZCM(100),ZCM2(100)
      COMMON/DATA/Y(200),X(200),ASIGXX(200),ASIGYY(200),ASIGXY(200),SY,SX,TY,TX,C,H,A,DELTA,
     * V12,G12,E22,E11,HEAD(7),NNYP,NNYM,NNXP,NNXM,NN,NP
      REAL*8  F11,E22,G12,V12,DELTA,A,A2,H,C,X,Y,TX,TY,SX,SY,ETEMH,      -
     * EM,XKM,CM,KM,AUX,ETKM,TEM1,TEM2,C22,C11,C66,C12,NDAN,R,DELYM,     -
     * DELXM,DELXP,DELYP,DELX,DELTAR,DR,TEMC,TEMS,RK,EPS,CMK,CMKP1,      -
     * CMKP2,CMKP3,DELY,XX,YY,SIGXX,SIGXY,SIGYY,DREAL,DAIMAG,ETEM,SUM    -
     *,ALPHA2,BETA2,K1,K2,ADR,RECCCS,ASIGXX,ASIGYY,ASIGXY
      COMPLEX*16 U1D,U2D,U1,U2,U1B,U2B,U1DB,U2DB,IMAG,F1N,F2N,G1N,G2N,Z1-
     *,Z2,Z1B,Z2B,UCON1,UCON2,UCON3,UCON4,ZCON1,ZCON,ZTEM,ZCONN,ZCM,ZCM2-
     *,SIGXXS,SIGYYS,SIGXYS,ZTEM3,TERM1,TERM2,UCON5,U1SQ,U2SQ,ZSTRS-
     *,U2BMU1,U1MU2,U2MU2B,HI1N,HI2N,XJ1N,XJ2N,UCON6,UCON7,UCON8         -
     *,SOTZ1,SOTZ2,SOTZ2B,ZCONN1,ZCONN2,ZCONN3,ZCONN4,ZCONN5,ZCONN6,CSUM-
     *,F11,G11,CC,ZCD,KSUMNP,KSUMNN,Z1D,Z2D,Z2BD,SXXNN,SYYNN,SXYNN,SXXNP-
     *,SYYNP,SXYNP,TEM1C,TEM2C
      DATA DR /.0174532925   /
      DATA IMAG/(0.,1.)/
      NAMELIST/BUG0/N4,NP,NN,M,M2,C,H,A,R,DELTA,V12,G12,E22,E11,-
     *TX,TY,SX,SY,C11,C12,C22,C66,U1D,U1DB,U2D,U2DB,U1,U1B,U2,U2B,-
     *U1MU2,U2MU2B,U2BMU1,UCON2,UCON3,UCON4,UCON5,UCON6,UCON7,CC
      NAMELIST/BUG1/Z1,Z2,ZTEM,ZCON
      NAMELIST/BUG2/F1N,F2N,G1N,G2N,ZCONN
      ITRIG=0
      IOTRIG=0
      N4=N4X
      N=NN
      NYM=NNYM
      NXP=NNXP
      NXM=NNXM
      NYP=NNYP
      A2 = A*A
  100 N1S = NYM +1
      N2S = N1S+NXM
      N3S = N2S+NXP
      DELYM = H/FLOAT(NYM)
      DELXM = C/FLOAT(NXM)
      DELXP = C/FLOAT(NXP)
      DELYP = H/FLOAT(NYP   )
      X(1) = -C
      Y(1) = 0.D+0
      M=NYM
      DO 1 I=2,M
      X(I)= -C
    1 Y(I) = Y(I-1)+DELYM
      Y(N1S) =H
      X(N1S) = -C
      NSP1=N1S+1
      M = M+NXM
      DO 2  I=NSP1,M
      Y(I)= H
    2 X(I) = X(I-1)+DELXM
      Y(N2S) = H
      X(N2S) = 0.D+0
      NSP1 = N2S+1
      M = M+NXP
      DO 3 I=NSP1,M
      Y(I)=H
    3 X(I)= X(I-1) + DELXP
      Y(N3S) = H
      X(N3S) = C
      IF((DELTA.GE. 1.).AND.(DELTA .LE.89.)) GO TO 350
```

```
      DELYP=H/FLOAT(NYP+1)
      Y(N3S)=H-DELYP
  350 NSP1=N3S+1
      M = M+NYP
      DO 4 I=NSP1,M
      Y(I)= Y(I-1)-DELYP
      X(I) = C
    4 CONTINUE
      J=1
      DO 5 I=1,M
      KM(J)=TX*X(I)+TY*Y(I)
      J=J+1
      KM(J)=SX*X(I)+SY*Y(I)
    5 J=J+1
      M2 = 2*M
      IF(ITRIG.EQ.1) GO TO 200
      C11= 1./E11
      C12=-V12/E11
      C22= 1./E22
      C66 = 1./G12
      R=DSQRT(C**2+H**2)
      TEM1C = (C66+2.*C12)/(2.*C11)
      TEM2C =CDSQRT(TEM1C**2-C22/C11)
      U1D= IMAG* CDSQRT(TEM1C+TEM2C)
      U1DB= -U1D
      U2D=IMAG*CDSQRT(TEM1C-TEM2C)
      U2DB= -U2D
      DELTAR= DELTA*DR
      TEMC = DCOS(DELTAR)
      TEMS = DSIN(DELTAR)
      U1 = (U1D*TEMC - TEMS)/(TEMC+U1D*TEMS)
      U1SQ= U1*U1
      U2 = (U2D*TEMC - TEMS)/(TEMC+U2D*TEMS)
      U2SQ= U2*U2
      U1B=DCONJG(U1)
      U2B=DCONJG(U2)
      U2MU2B=U2-U2B
      UCON1 = (U2B-U1)/(U2MU2B)
      U2BMU1=U2B-U1
      U1MU2=U1-U2
      UCON2=U1*U2MU2B
      UCON3=U2*U2BMU1
      UCON4=U2B*U1MU2
      UCON5=UCON2*U1
      UCON6=UCON3*U2
      UCON7=UCON4*U2B
      UCON8= (U2B-U1B)/(U2MU2B)
      CC=UCON5+UCON6+UCON7
      WRITE(6,BUGO)
      J=1
      DO 25  I=1,M
      Z1 = X(I) + U1*Y(I)
      Z2=  X(I) + U2*Y(I)
      Z1B=DCONJG(Z1)
      Z2B=DCONJG(Z2)
      SQTZ1= CDSQRT(Z1-A)*CDSQRT(Z1+A)
      ZCON(1)= (Z1+SQTZ1)
      ZCON(4)= (Z1-SQTZ1)
      SQTZ2= CDSQRT(Z2-A)*CDSQRT(Z2+A)
      ZCON(2)= (Z2+SQTZ2)
      ZCON(5)= (Z2-SQTZ2)
      SQTZ2B= DCONJG( SQTZ2)
      ZCON(3)= (Z2B-SQTZ2B)
      ZCON(6)= (Z2B+SQTZ2B)
      K=4
      JP1 = J+1
      F11=(U2MU2B*SQTZ1+U2BMU1*SQTZ2-U1MU2*SQTZ2B)/R
      G11=(UCON2*SQTZ1+UCON3*SQTZ2-UCON4*SQTZ2B)/R
      EM(J,1)=0.D+0
      EM(JP1,1)=Y(I)/R
      EM(J,2)=DREAL(F11)
      EM(JP1,2)=DREAL(G11)
      EM(J,3)=-DAIMAG(F11)
      EM(JP1,3)=-DAIMAG(G11)
      IF(NP.EQ.1) GO TO 17
      DO 15 KK=3,NP,2
      DO 16  JJ=1,3
   16 ZCONN(JJ)=(ZCON(JJ)/R)**KK
      F1N = U2MU2B*ZCONN(1) +U2BMU1*ZCONN(2) +U1MU2 *ZCONN(3)
      G1N=UCON2*ZCONN(1) + UCON3*ZCONN(2) + UCON4*ZCONN(3)
      EM(J,K)= DREAL(F1N)
      EM(JP1,K)= DREAL(G1N)
      K=K+1
      EM(J,K) = -DAIMAG(F1N)
      EM(JP1,K)= -DAIMAG(G1N)
   15 K=K+1
```

```
17     IF(NN.EQ.1)GO TO 25
       DO 1015 KK=3,NN,2
       DO 1016  JJ=4,6
1016   ZCONN(JJ)=(ZCON(JJ)/R)**KK
       F2N = U2MU2B*ZCONN(4) +U2BMU1*ZCONN(5) +U1MU2 *ZCONN(6)
       G2N= UCON2*ZCONN(4)+UCON3*ZCONN(5) + UCON4*ZCONN(6)
       EM(J,K) = DREAL(F2N)
       EM(JP1,K) = DREAL(G2N)
       K=K+1
       EM(J,K) = - DAIMAG(F2N)
       EM(JP1,K) = -DAIMAG(G2N)
1015   K=K+1
25     J=J+2
6501   FORMAT(1HL, 8HK VECTOR /1HK)
       WRITE(6,6497)
6497   FORMAT(1HL,'X ARRAY' /1HK)
       WRITE(6,6001) (X(I),I=1,M)
       WRITE(6,6498)
 6498   FORMAT(1HL,'Y ARRAY'/1HK)
       WRITE(6,6001) (Y(I),I=1,M)
       WRITE(6,6501)
        WRITE(6,6001) (KM(I),I=1,M2)
6001   FORMAT(1X,8G15.7)
       DO 265  J=1,N4
       DO 265 I=J,N4
       SUM = 0.
       DO 26  K=1,M2
26     SUM = SUM + EM(K,I) * EM(K,J)
       ETEM(J,I)=SUM
265    ETEM(I,J)=SUM
       DO 275 I=1,N4
       SUM=0.
       DO 27 J=1,M2
27     SUM= SUM+  EM(J,I)*KM(J)
275    ETKM(I)=SUM
       IF(IOTRIG.EQ.0) GO TO 6512
       WRITE(6,6502)
6502   FORMAT (1H1,27H  E TRANSPOSE *  E  MATRIX // )
       DO 890 I=1,N4
890    WRITE(6,6000) (ETEM(I,J),J=1,N4)
       WRITE(6,6503)
6503   FORMAT (1HL,27H  E TRANSPOSE *  K  VECTOR // )
       WRITE(6,6000)(ETKM(I),I=1,N4)
       DO 999 I=1,N4
       DO 999 J=1,N4
999    ETEMH(I,J)=ETEM(I,J)
6512   DO 60 I=1,N4
60     CM(I)=ETKM(I)
       DATA EPS/1.E-15  /
       CALL DGELG(CM,ETEM ,N4,1,EPS,IER)
640    FORMAT(1HL,'IER= ',I5)
       WRITE(6,640) IER
       IF(IOTRIG.EQ.0) GO TO 6511
       DO 9975 I=1,M2
       SUM=0.
       DO 997 J=1,N4
997    SUM=SUM+EM(I,J)*CM(J)
9975   XKM(I)=SUM
       WRITE(6,6508)
6508   FORMAT(1HL 22H  E  * C   VECTOR     //)
       WRITE(6,6000)(XKM(I),I=1,M2)
       DO 9985 I=1,N4
       SUM=0.
       DO 998 J=1,N4
998    SUM=SUM+CM(J)*ETEMH(I,J)
9985   AUX(I)=SUM
       WRITE(6,6509)
       WRITE(6,6000)(AUX(I),I=1,N4)
6509   FORMAT(1H1,12HETE*C MATRIX   )
6511   K=3
       ADR=A/R
       KSUMNP=0.D+0
       RECCCS=CM(1)
       ZCD=CM(2)+CM(3)*IMAG
       WRITE(6,619) RECCCS,ZCD
       WRITE(7,619) RECCCS,ZCD
       IF(NP.EQ.1) GO TO 32
       WRITE(6,620)
       K=4
       DO 30 J=3,NP,2
       KP1=K+1
       ZCM(J)=CM(K)+IMAG*CM(KP1)
       WRITE(6,621) J,ZCM(J)
       KSUMNP=KSUMNP+ADR**J*FLOAT(J)*ZCM(J)
30     K=K+2
32     KSUMNN=0.D+0
       IF(NN.EQ.1) GO TO 33
```

```
      WRITE(6,622)
      DO 31 J=3,NN,2
      KP1=K+1
      ZCM2(J)=CM(K)+IMAG*CM(KP1)
      WRITE(6,621) J,ZCM2(J)
      KSUMNN=KSUMNN+ADR**J*FLOAT(J)*ZCM2(J)
31    K=K+2
620   FORMAT(1HL/6X,'N',16X,'C1')
621   FORMAT(1X,I2,7X,2G15.7)
622   FORMAT(1HL/ 6X,'N',16X,'C2')
619   FORMAT(1H1/1HL,4X,'REAL(CC*CS)=',G15.7,4X,'CD=',2G15.7)
33    ALPHA2=DREAL(U2)
      BETA2=DAIMAG(U2)
      TEM1=2./DSQRT(A)
      ZTEM=U1MU2*U2MU2B*(ADR*ZCD+KSUMNP-KSUMNN)
      TEMS=DAIMAG(ZTEM)
      K1=-TEM1/BETA2*TEMS
      K2=-TEM1*DREAL(ZTEM)+TEM1*ALPHA2/BETA2*TEMS
      WRITE(6,644)K1,K2
      WRITE(7,644)K1,K2
644   FORMAT(1HL,'K1=',G15.7,'K2=',G15.7/(1X,I5,2G15.7))
      DATA NYMS,NXMS,NXPS,NYPS/50,50,50,50/
      NYM=NYMS
      NXM=NXMS
      NXP=NXPS
      NYP=NYPS
      ITRIG =1
      GO TO 100
200   WRITE(6,625)
625   FORMAT(1H1/1HL,19X,'Z',18X,'SIGMA XX',8X,'SIGMA YY',8X,'SIGMA XY'/-
     */)
      DO 50 I=1,M
      XX=X(I)
      YY=Y(I)
      Z1 = XX+ U1*YY
      Z2 = XX+U2*YY
      Z1B=DCONJG(Z1)
      Z2B=DCONJG(Z2)
      SQTZ1= CDSQRT(Z1-A)*CDSQRT(Z1+A)
      ZCON(1)= (Z1+SQTZ1)
      ZCON(4)= (Z1-SQTZ1)
      SQTZ2= CDSQRT(Z2-A)*CDSQRT(Z2+A)
      ZCON(2)= (Z2+SQTZ2)
      ZCON(5)= (Z2-SQTZ2)
      SQTZ2B= DCONJG( SQTZ2)
      ZCON(3)= (Z2B-SQTZ2B)
      ZCON(6)= (Z2B+SQTZ2B)
      SXXNN=0.D+0
      SXYNN=0.D+0
      SYYNN=0.D+0
      SXXNP=0.D+0
      SXYNP=0.D+0
      SYYNP=0.D+0
      IF(NP.EQ.1) GO TO 45
      DO 41 J=3,NP,2
      DO 40 JJ=1,3
40    ZCONN(JJ)=(ZCON(JJ)/R)**J
      ZCONN(1)=ZCONN(1)/SQTZ1
      ZCONN(2)=ZCONN(2)/SQTZ2
      ZCONN(3)=ZCONN(3)/SQTZ2B
      XJ=J
      SXXNP = SXXNP+XJ*ZCM(J)*(UCON5*ZCONN(1)+UCON6*ZCONN(2)-UCON7*ZCONN-
     *(3))
      SYYNP = SYYNP+XJ*ZCM(J)*(U2MU2B*ZCONN(1)+U2BMU1*ZCONN(2)-U1MU2*ZCO-
     *NN(3))
41    SXYNP = SXYNP+XJ*ZCM(J)*(UCON2*ZCONN(1)+UCON3*ZCONN(2)-UCON4*ZCONN-
     *(3))
45    IF(NN.EQ.1) GO TO 46
      DO 43 J=3,NN,2
      DO 42 JJ=4,6
42    ZCONN(JJ)=(ZCON(JJ)/R)**J
      ZCONN(4)=ZCONN(4)/SQTZ1
      ZCONN(5)=ZCONN(5)/SQTZ2
      ZCONN(6)=ZCONN(6)/SQTZ2B
      XJ=J
      SXXNN = SXXNN+XJ*ZCM2(J)*(UCON5*ZCONN(4)+UCON6*ZCONN(5)-UCON7*ZCON-
     *N(6))
      SYYNN = SYYNN+XJ*ZCM2(J)*(U2MU2B*ZCONN(4)+U2BMU1*ZCONN(5)-U1MU2*ZC-
     *ONN(6))
43    SXYNN = SXYNN+XJ*ZCM2(J)*(UCON2*ZCONN(4)+UCON3*ZCONN(5)-UCON4*ZCON-
     *N(6))
46    Z1D=Z1/SQTZ1
      Z2D=Z2/SQTZ2
      Z2BD=Z2B/SQTZ2B
      ZTEM=ZCD*(UCON5*Z1D+UCON6*Z2D-UCON7*Z2BD)/R
      SIGXX=2.*(RECCCS/R+DREAL(ZTEM+SXXNP-SXXNN))
```

```
      ZTEM=ZCD*(U2MU2B*Z1D+U2BMU1*Z2D-U1MU2*Z2BD)/R
      SIGYY=2.*DREAL(ZTEM+SYYNP-SYYNN)
      ZTEM=ZCD*(UCON2*Z1D+UCON3*Z2D-UCON4*Z2BD)/R
      SIGXY=-2.*DREAL(ZTEM+SXYNP-SXYNN)
      ASIGXX(I)=SIGXX
      ASIGYY(I)=SIGYY
   50 ASIGXY(I)=SIGXY
      WRITE(6,630)(I,X(I),Y(I),ASIGXX(I),ASIGYY(I),ASIGXY(I),I=1,M)
  630 FORMAT(1X,I4,5E15.7)
      DATA NCP/100/
      IF(ITRIG.EQ.2)RETURN
      M=NCP
      DEL=2.*C/FLOAT(NCP-1)
      Y(1)=0.
      X(1)=-C
      DO 77 I=2,NCP
      Y(I)=0.
   77 X(I)=X(I-1)+DEL
      ITRIG=2
      GO TO 200
 6000 FORMAT(     8E15.7)
      END
```

```
C     ..................................................................
C
C        SUBROUTINE DGELG
C
C          PURPOSE
C             TO SOLVE A GENERAL SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS.
C
C          USAGE
C        CALL DGELG(R,A,M,N,EPS,IER)
C
C          DESCRIPTION OF PARAMETERS
C             R      - DOUBLE PRECISION M BY N RIGHT HAND SIDE MATRIX
C                      (DESTROYED). ON RETURN R CONTAINS THE SOLUTIONS
C                      OF THE EQUATIONS.
C             A      - DOUBLE PRECISION M BY N COEFFICIENT MATRIX
C                      (DESTROYED).
C             M      - THE NUMBER OF EQUATIONS IN THE SYSTEM.
C             N      - THE NUMBER OF RIGHT HAND SIDE VECTORS.
C             EPS    - AN INPUT CONSTANT WHICH IS USED AS RELATIVE
C                      TOLERANCE FOR TEST ON LOSS OF SIGNIFICANCE.
C             IER    - RESULTING ERROR PARAMETER CODED AS FOLLOWS
C                        IER=0  - NO ERROR,
C                        IER=-1 - NO RESULT BECAUSE OF M LESS THAN 1 OR
C                                 PIVOT ELEMENT AT ANY ELIMINATION STEP
C                                 EQUAL TO 0,
C                        IER=K  - WARNING DUE TO POSSIBLE LOSS OF SIGNIFI-
C                                 CANCE INDICATED AT ELIMINATION STEP K+1,
C                                 WHERE PIVOT ELEMENT WAS LESS THAN OR
C                                 EQUAL TO THE INTERNAL TOLERANCE EPS TIMES
C                                 ABSOLUTELY GREATEST ELEMENT OF MATRIX A.
C
C          REMARKS
C             INPUT MATRICES R AND A ARE ASSUMED TO BE STORED COLUMNWISE
C             IN M*N RESP. M*M SUCCESSIVE STORAGE LOCATIONS. ON RETURN
C             SOLUTION MATRIX R IS STORED COLUMNWISE TOO.
C             THE PROCEDURE GIVES RESULTS IF THE NUMBER OF EQUATIONS M IS
C             GREATER THAN 0 AND PIVOT ELEMENTS AT ALL ELIMINATION STEPS
C             ARE DIFFERENT FROM 0. HOWEVER WARNING IER=K - IF GIVEN -
C             INDICATES POSSIBLE LOSS OF SIGNIFICANCE. IN CASE OF A WELL
C             SCALED MATRIX A AND APPROPRIATE TOLERANCE EPS, IER=K MAY BE
C             INTERPRETED THAT MATRIX A HAS THE RANK K. NO WARNING IS
C             GIVEN IN CASE M=1.
C
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C             NONE
C
C          METHOD
C             SOLUTION IS DONE BY MEANS OF GAUSS-ELIMINATION WITH
C             COMPLETE PIVOTING.
C
C     ..................................................................
C
      SUBROUTINE DGELG(R,A,M,N,EPS,IER)
C
C
      DIMENSION A(1),R(1)
      DOUBLE PRECISION R,A,PIV,TB,TOL,PIVI
      IF(M)23,23,1
C
```

```
C      SEARCH FOR GREATEST ELEMENT IN MATRIX A                                  DGEL  58
     1 IER=0
       PIV = 0.D0
       MM=M*M
       NM=N*M
       DO 3 L=1,MM
       TB = DABS(A(L))
       IF(TB-PIV)3,3,2
     2 PIV=TB
       I=L
     3 CONTINUE
       TOL=EPS*PIV
C      A(I) IS PIVOT ELEMENT. PIV CONTAINS THE ABSOLUTE VALUE OF A(I).          DGEL  70
C                                                                               DGEL  71
C                                                                               DGEL  72
C      START ELIMINATION LOOP                                                   DGEL  73
       LST=1
       DO 17 K=1,M
C                                                                               DGEL  76
C      TEST ON SINGULARITY                                                      DGEL  77
       IF(PIV)23,23,4
     4 IF(IER)7,5,7
     5 IF(PIV-TOL)6,6,7
     6 IER=K-1
     7 PIVI = 1.D0/A(I)
       J=(I-1)/M
       I=I-J*M-K
       J=J+1-K
C      I+K IS ROW-INDEX, J+K COLUMN-INDEX OF PIVOT ELEMENT                      DGEL  86
C                                                                               DGEL  87
C      PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT HAND SIDE R             DGEL  88
       DO 8 L=K,NM,M
       LL=L+I
       TB=PIVI*R(LL)
       R(LL)=R(L)
     8 R(L)=TB
C                                                                               DGEL  94
C      IS ELIMINATION TERMINATED                                                DGEL  95
       IF(K-M)9,18,18
C                                                                               DGEL  97
C      COLUMN INTERCHANGE IN MATRIX A                                           DGEL  98
     9 LEND=LST+M-K
       IF(J)12,12,10
    10 II=J*M
       DO 11 L=LST,LEND
       TB=A(L)
       LL=L+II
       A(L)=A(LL)
    11 A(LL)=TB
C                                                                               DGEL 107
C      ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A                      DGEL 108
    12 DO 13 L=LST,MM,M
       LL=L+I
       TB=PIVI*A(LL)
       A(LL)=A(L)
    13 A(L)=TB
C                                                                               DGEL 114
C      SAVE COLUMN INTERCHANGE INFORMATION                                      DGEL 115
       A(LST)=J
C                                                                               DGEL 117
C      ELEMENT REDUCTION AND NEXT PIVOT SEARCH                                  DGEL 118
       PIV = 0.D0
       LST=LST+1
       J=0
       DO 16 II=LST,LEND
       PIVI=-A(II)
       IST=II+M
       J=J+1
       DO 15 L=IST,MM,M
       LL=L-J
       A(L)=A(L)+PIVI*A(LL)
       TB = DABS(A(L))
       IF(TB-PIV)15,15,14
    14 PIV=TB
       I=L
    15 CONTINUE
       DO 16 L=K,NM,M
       LL=L+J
    16 R(LL)=R(LL)+PIVI*R(L)
    17 LST=LST+M
C      END OF ELIMINATION LOOP                                                  DGEL 138
C                                                                               DGEL 139
C                                                                               DGEL 140
C      BACK SUBSTITUTION AND BACK INTERCHANGE                                   DGEL 141
    18 IF(M-1)23,22,19
    19 IST=MM+M
```

```
      LST=M+1
      DO 21 I=2,M
      II=LST-I
      IST=IST-LST
      L=IST-M
      L = A(L) + .5D0
      DO 21 J=II,NM,M
      TB=R(J)
      LL=J
      DO 20 K=IST,MM,M
      LL=LL+1
   20 TB=TB-A(K)*R(LL)
      K=J+L
      R(J)=R(K)
   21 R(K)=TB
   22 RETURN
C
C
C     ERROR RETURN
   23 IER=-1
      RETURN
      END
```

```
      FUNCTION DREAL(Z)
      DIMENSION Z(2)
      DOUBLE PRECISION Z,DREAL
      DREAL=Z(1)
      RETURN
      END
```

```
      FUNCTION DAIMAG(Z)
      DIMENSION Z(2)
      DOUBLE PRECISION Z,DAIMAG
      DAIMAG=Z(2)
      RETURN
      END
```