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ABSTRACT

SOLUTION OF SOME MIXED BOUNDARY VALUE PROBLEMS
OF THREE-DIMENSIONAL ELASTICITY BY THE METHOD OF LINES

By

John Paul Gyekenyesi

A semi-numerical method is developed for solving a set of coupled partial differential equations subject to mixed and possibly coupled boundary conditions. The application of this method to these equations leads to coupled sets of simultaneous ordinary differential equations. Their solutions are obtained along sets of continuous lines in a discretized region. When de-coupling of the equations and their boundary conditions is not possible, the use of a successive approximation method permits the analytical solution of the resulting ordinary differential equations.

The use of this method is illustrated by presenting previously unavailable solutions for a number of mixed boundary value problems in three-dimensional elasticity. Stress and displacement distributions are obtained for two finite geometry, rectangular bars which are loaded by a uniform surface stress distribution. The first bar contains a through-thickness central crack while the second bar has double edge cracks. Stress intensity factors K_I for both configurations are presented.

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John Paul Gyekenyesi

An independent treatment of problems in cylindrical coordinates is also included. Stress and displacement distributions are calculated for finite circular bars each with a penny shaped crack. Approximate results for an annular plate containing internal surface cracks are also presented.

Displacement distributions for each problem are calculated by applying the method of lines to the Navier-Cauchy equations. By comparing them to known solutions, the results of the circular bar are used to examine the rate of convergence and the accuracy of this method.

The results obtained show that the method of lines presents a systematic approach to the solution of some three-dimensional elasticity problems with arbitrary boundary conditions. The advantage of this method over other numerical solutions is that good results are obtained even from the use of a relatively coarse grid.

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SOLUTION OF SOME MIXED BOUNDARY VALUE PROBLEMS
OF THREE-DIMENSIONAL ELASTICITY BY THE METHOD OF LINES

By

John Paul Gyekenyesi

A THESIS

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Metallurgy, Mechanics and Materials Science

1972

To my Mother a

To my Mother and Father, Katalin and György László

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Dr. A. Mendelson, the

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ACKNOWLEDGMENTS

The author wishes to express his deepest appreciation to Dr. A. Mendelson, the writer's thesis advisor, for his assistance during the course of this research and for his encouragement and continuing guidance throughout the past years. Thanks are also due to the other members of the graduate committee, Professors R. W. Little, W. N. Sharpe, W. A. Bradley and N. L. Hills for reviewing this manuscript.

Very special thanks are due my wife, Erika, for her help during my doctoral studies.

The help provided by Mr. J. Kring of the Lewis Research Center in setting up and running the computer programs will always be appreciated.

Finally, I wish to thank Lewis Research Center of the National Aeronautics and Space Administration for providing generous support for this project.

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LIST OF SYMBOLS

a	half crack width or crack width in rectangular coordinates, crack radius in cylindrical coordinates
$[A(K_x,x)]$, $[A(K_r,r)]$	coefficient matrices of first order differential equations
$[A_{ij}]$	partitioned submatrices of $[A]$ for $i = 1, 2$ and $j = 1, 2$
A, $\bar{A}$	constants in difference equation solution
b	half bar width in rectangular coordinates, outside radius in cylindrical coordinates
$\{B_i\}$	particular integrals of first order differential equations for $i = 1, 2, 3, 4$
B, $\bar{B}$	constants in difference equation solution
$\bar{C}_1$, $\bar{C}_2$	constants of integration for modified Bessel's equation
c	surface crack length emanating from internal hole
c_i	elements of coefficient matrix $[K_{zc}]$, $i = 1, 2, \ldots, 3NR-4$
d	depth of surface crack along internal hole surface
$[D_1]$	diagonal matrix for transforming $[K_1]$ into a symmetric matrix
$[D]$	diagonal matrix
$[D_{ij}]$	similar matrix functions to $[A_{ij}]$ associated with (K_y,y) for $i = 1, 2$, $j = 1, 2$

$[D_{ij\alpha k}]$, $[D_{ij\beta k}]$	partitioned matrix functions
$[D_a]$, $[D_b]$, $[D_c]$	products of partitioned matrix functions
e	Naperian or natural logarithm base
e_r, e_c, e_s	dilatation in rectangular, cylindrical and axisymmetric cylindrical coordinate problems
E	Young's modulus of elasticity
[ER]	error matrix for accuracy check of matrix functions
f_i, $\bar{f}_i$	functions of given variables
$\{F_i\}$	initial value vectors for the first order differential equations i = 1, 2, 3, 4
$\{f\}_i$	partitioned subvectors of the coupling vectors {r}, {s} and {t}
G	shear modulus of elasticity
h_x, h_y, h_z	increments along Cartesian coordinate axes
h_r, h_θ, h_z	increments along cylindrical coordinate axes
$\bar{i}$	square root of minus one
[I], $[I_i]$	identity matrices of different order, i = 1, 2
i, j, k	integers
I_0, I_1	modified Bessel functions of the first kind
K_0, K_1	modified Bessel functions of the second kind
K_I	stress intensity factor for opening mode
k_i	elements of coefficient matrices i = 1, 2, . . ., 18
$[K_x]$, $[K_y]$, $[K_z]$	coefficient matrices of second order differential equations in Cartesian coordinates

$[K_r]$, $[K_\theta]$, $[K_{zc}]$	coefficient matrices of second order differential equations in cylindrical coordinates
$[K_{rs}]$, $[K_{zs}]$, $[K_{zsh}]$	coefficient matrices of second order differential equations for axisymmetric cylindrical coordinates
$[K_1]$, $[K_2]$,	tri-diagonal component matrices
$[K_{xi}]$, $[K_{yi}]$, $[K_{zi}]$	component matrices of $[K_x]$, $[K_y]$ and $[K_z]$
$[K_{ri}]$, $[K_{\theta i}]$, $[K_{zci}]$	component matrices of $[K_r]$, $[K_\theta]$ and $[K_{zc}]$
ℓ	number of lines in x or radial directions
L	half bar length in Cartesian coordinates
m	number of lines in y or circumferential directions
n	number of lines in z or axial directions
NX, NY, NZ, NR, Nθ	number of lines in a given plane
NOC	number of normal lines outside crack surface
NIC	number of normal lines inside crack surface
$[P]$, $[P_1]$, $[P_2]$, $[P_s]$	modal matrices of $[K_x]$, $[K_1]$, $[K_2]$ and $[K_{rs}]$
r_o	internal hole radius
r, θ, z	cylindrical coordinate directions
$\{r\}$	coupling vector for x-directional or radial second order differential equations
R	distance from crack edge
$\{\bar{r}\}$	coupling vector for x-directional or radial first order differential equations

x, y, z

$\{X1\}, \{X2\}, \{X3\}$

$\{s\}$	coupling vector for y-directional or circumferential second order differential equations
$\{\bar{s}\}$	coupling vector for y-directional or circumferential first order differential equations
t	half bar or plate thickness
$\{t\}$	coupling vector for z-directional or axial second order differential equations
$\{\bar{t}\}$	coupling vector for z-directional or axial first order differential equations
$\bar{u}_i$	components of displacement along the coordinates
u	x-directional or radial displacement
U	x-directional or radial dependent variable in the first order differential equations
v	y-directional or circumferential displacement
V	y-directional or circumferential dependent variable in the first order differential equations
w	z-directional or axial displacement
W	z-directional or axial dependent variable in the first order differential equations
x, y, z	rectangular Cartesian coordinate directions
$\{X1\}, \{X2\}, \{X3\}$	eigenvectors of $[K_1]$, $[K_2]$ and $[K_x]$ respectively
α_i	solutions of second order difference equation
δ'_{ij}	Kroenecker delta
δ_i	eigenvalues of $[K_2]$
ε_{ij}	components of the strain tensor
λ	Lame's constant
$\bar{\lambda}_{ij}, \bar{\lambda}_{is}$	eigenvalues of $[K_x]$ and $[K_{rs}]$ respectively

μ_j	eigenvalues of $[K_1]$
γ	hyperbolic trigonometric function variable
$\bar{\eta}$	transformed radial displacements
η	variable of integration
ν	Poisson's ratio
$[\Omega_o^r(A)]$	matrizant of [A]
$[\Omega_{ij}]$	partitioned submatrices of the matrizant of [A] $i = 1, 2$, $j = 1, 2$
ρ_i	variable of integration $i = 1, 2, 3 \ldots$
2ρ	coefficient in difference equation
ϕ	direction angle in complex plane of α_i
σ_o	applied uniform surface stress
σ_{ij}	components of the stress tensor
$\sigma_x, \sigma_y, \sigma_z, \sigma_{xy}, \sigma_{zx}, \sigma_{yz}$	components of the stress tensor in Cartesian coordinates
$\sigma_r, \sigma_\theta, \sigma_z, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z}$	components of the stress tensor in cylindrical coordinates
$\sigma_{rs}, \sigma_{\theta s}, \sigma_{zs}, \sigma_{rzs}$	stress components for axisymmetric cylindrical coordinate problems
θ_o	circumferential length of annular plate cutout
$\nabla^2, \nabla_c^2, \nabla_s^2$	Laplacian in Cartesian, cylindrical and axisymmetric cylindrical coordinates respectively
$[\Lambda]$	diagonal matrix having eigenvalues of $[K_x]$ for its elements
$[\Lambda_{rs}]$	diagonal matrix having eigenvalues of $[K_{rs}]$ for its elements
$[\Lambda_i]$	diagonal matrices with eigenvalues as their elements

Subscripts:

x, y, z

Superscripts:

$[\Lambda_{11}]$, $[\Lambda_{12}]$	diagonalized matrix functions $[A_{11}]$ and $[A_{12}]$

Subscripts:

x, y, z	refer to rectangular Cartesian coordinates
r, θ, zc	refer to cylindrical coordinates
α, β	refer to partitioning of variables for treating mixed boundary conditions
i, j, k	integers
s	refers to axisymmetric cylindrical problems
c	refers to cylindrical coordinates
$\vert_i$	denotes lines along which coupling variables are written

Superscripts:

~	non-dimensionalized variables
•	derivative of displacements with respect to independent variable in corresponding direction
T	transpose of a matrix
s	denotes symmetric

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CHAPTER 1

INTRODUCTION

The object of a problem in elasticity is usually to calculate the displacement and stress distributions in an elastic body which is subject to given body forces or surface conditions. These distributions are the solutions of the applicable field equations which mathematically describe the behavior of engineering materials. The solution of this general system of equations is, however, usually too difficult to evaluate. For many problems of practical interest, some simplifying assumptions can be made regarding the displacement or stress distributions which then make the solution of these equations relatively simple. These assumptions are generally based on the geometry and loading of the problem at hand. Plane stress or plane strain solutions are two examples that result from these simplifying assumptions.

Since all real bodies are three-dimensional in nature, situations arise when these simplifying assumptions cannot be validly accepted. One important class of elasticity problems falling in this category is found in the theory of fracture mechanics. It is essential, therefore, that a method be developed that makes the solution of the general field equations possible.

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dimensional problems exist, and even these solutions are frequently based on some symmetry condition required to simplify the governing equations. Recently, with the availability of large digital computers, the use of a number of approximate methods was attempted, but these methods yielded only partial results for these problems. Among these approximate methods are the finite difference, direct potential, finite element, eigenfunction expansion and the line method of analysis. Of all these solution techniques, the line method of analysis is probably the least known and least used method in three-dimensional elasticity. Although the concept of this method for solving partial differential equations is not new, its useful application in the past has been limited to simple examples. Because of their practical importance and inherent singularities, the work in this dissertation will concentrate on three-dimensional bodies containing flaws or cracks. Assuming that the method of lines can be successfully applied to these solids which contain large stress and strain gradients, its use for less complicated, three-dimensional, elasticity problems should present little difficulty.

The phenomenon of structural failure by catastrophic crack propagation at average stresses well below the yield strength has been known for many years. Large scale failures have occurred in such diverse structures as ships, storage tanks, aircraft and rocket motors. These brittle failures have occurred with increasing frequ-

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existing matheincreasing frequency as the strength and size of our structures have increased (1). Recent military and aerospace requirements for very high strength, light weight hardware have given added importance to the problem of brittle fracture.

Brittle fracture, it should be noted, refers to a material failure with negligible plastic deformation while materials with ratios of yield strength to Young's modulus greater than 5×10^{-3} are regarded as high strength materials (2). In general, when high strength materials contain small cracks or flaws they are found to behave in a brittle manner and fail prematurely at low design stress levels.

The main goal of fracture mechanics is the prediction of the load at which a structure weakened by a crack will fail. Knowledge of the stress and displacement distributions near the crack tip is of fundamental importance in evaluating this load at failure. Ludwig (3) has pointed out that the state of stress near the crack front is triaxial in nature which today is recognized as the main cause of brittle behavior in some materials.

It is important, therefore, that for structures with cracks or inherent flaws, a three-dimensional solution of the stresses be obtained. Although certain analytic methods for the solution of three-dimensional elasticity problems have been developed (4), this is a very difficult problem. This difficulty arises because existing mathematical techniques are not suitable for solving the equations of el

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equations of elasticity when solids with geometric singularities are involved. The lack of a systematic approach to the solution of three-dimensional crack problems has limited the previously obtained solutions to a particular crack geometry under simple types of loading.

In the past, most of the stress analysis of cracked bodies has been based on the plane theory of elasticity. A good summary of the results of this work has been presented by Paris and Sih (5). Most of these two-dimensional results come from the application of eigenfunction expansions, the complex variable formulation and boundary point collocation. It must be recognized that these solutions are all based on linear elasticity even though small amounts of plasticity and other non-linear effects arise near the crack tip. These non-linearities arise because linear elastic analysis of crack problems will always predict stresses near the crack tip which approach infinity as the inverse square root of the distance from the crack tip. Real materials cannot possibly sustain such a stress state and hence must develop a small plastic zone in which the linear elastic solution is not valid. Irwin (6), however, showed that even though linear elastic analysis does not admit these non-linearities, the elastic solution of the gross displacement and stress fields has practical importance in estimating the onset of fracture.

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Early three-dimensional solutions of crack problems involved an extension of the so-called Griffith crack, that is a central crack in an infinite plate under uniform tension, to three-dimensions. These analytical solutions usually described the stresses near circular or ellipsoidal cavities enclosed in infinitely large solids. In 1946, Sack (7) presented the solution for a flat circular crack in an infinite solid under uniform tension. Following Sack, other solutions for circular and ellipsoidal cavities were obtained by Sneddon (8), Sternberg and Sadowsky (9), Green and Sneddon (10). In reference (8), Sneddon applied Hankel transform methods to Love's biharmonic strain function and reduced the mixed boundary value problem of the axisymmetric half-space to a set of dual integral equations. More recently, Smith (11) solved the problem of a semi-circular edge crack in a semi-infinite body. In his solution, Smith used superposition methods in conjunction with an iteration technique to satisfy the stress free boundary conditions of the crack and side faces. Using Smith's method, Alavi (12) obtained the solution for a circular crack embedded in a semi-infinite solid. Kassir and Sih (13, 14) solved the more general problem of an embedded elliptical crack subject to prescribed shear and linearly varying normal pressure on the crack surface. Solutions of three-dimensional crack problems with quadratic and higher order loadings were presented by Segedin (15). Shah (16) solved the problems of embedded hyper-

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Common to all the work cited above is that they represent solutions of crack problems in infinite or semi-infinite solids. Only limited work has been done on three-dimensional solutions of crack problems in finite geometry solids. Irwin (18) in 1962 estimated the stress intensity factor for a part through elliptical crack in a flat plate. Some experimental investigation of fracture in thin sections containing through and part-through cracks was performed by Orange, Sullivan and Calfo (19) and Kuhn (20). After making certain assumptions on the nature of the thickness variation of the stresses, Hartranft and Sih were successful in obtaining partial results for cracks in finite thickness bodies using variational methods (21) and eigenfunction expansions (22). Studies at NASA, Lewis Research Center, are currently being conducted using three-dimensional scattered light photoelasticity to evaluate this solution. In 1966, Sih, Williams and Swedlow (23) attempted the use of Galerkin's biharmonic stress functions together with the eigenfunction expansion of Williams in obtaining the solution for a cracked plate with finite thickness. However, they were unable to obtain complete results.

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of three-dimensional elasticity problems, Walker (24) attempted to solve the problem of a rectangular bar with a central crack. After a successful plane elasticity solution, he extended the direct potential method of Rizzo to three dimensions but was unable to obtain meaningful results. In using the direct potential method, Walker obtained singular integral equations in terms of boundary tractions and displacements. The numerical solution of these equations was then used in an analogous manner to Green's boundary formula in potential theory to express the interior stresses and displacements. In 1970, Cruse and Van Buren (25) were more successful in applying this method to a finite rectangular bar with a single edge crack. Recently, Ayres (26) presented a complete finite difference formulation for the computation of stresses and deformations in a three-dimensional elastic-plastic solid. However, the inherent inaccuracies in a complete finite difference solution of the Navier-Cauchy equations of three-dimensional elasticity are well known, especially when mixed boundary conditions are involved.

Other possible numerical solution of three-dimensional crack problems may involve the use of three-dimensional finite elements such as the tetrahedron (27) or the isoparametric hexahedron elements (28). At present, however, the finite element method has not been extensively investigated for the solution of three-dimensional crack problems.

Another method of solution may be obtained by applying the method of lines (29) to the partial differential equations of equilibrium. Faddeva (30) discusses the application of this method for the solution of a single elliptic, hyperbolic or parabolic partial differential equation. In a similar manner, Henrici (31) discusses the solution of the one-dimensional conduction equation obtained through the use of this technique. For three-dimensional elasticity problems, the essence of this semi-analytical procedure is to reduce the three Navier-Cauchy equations to three sets of simultaneous ordinary differential euqations, whose solutions can then be obtained in closed form. Since the dependent variables in the resulting equations and their boundary conditions are coupled, the use of a successive approximation procedure becomes necessary. In addition, the closed form solution of the resulting ordinary differential equations may, in some cases, be difficult to evaluate. However, the usefulness of this method in three-dimensional elasticity has been demonstrated by Irobe (32).

It is the primary objective of this dissertation to present a simple and systematic approach to the solution of three-dimensional elasticity problems with mixed boundary conditions. These problems are more complex then those in (32), since the geometric singularities involved require a large number of lines. Hence, an extension of the solution methods presented in (32) becomes necessary.

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Problems in both rectangular Cartesian and cylindrical coordinates are considered since they result in different types of ordinary differential equations. Detailed results are given for two rectangular bars which are loaded by a uniform surface stress distribution. The first bar contains a through-thickness central crack while the second bar has double edge cracks. Stresses and displacements are also listed for a cylindrical solid with a penny shaped crack. Similar numerical results for the case of a central hole along the cylinder axis are also included. In addition, approximate results for an annular plate containing internal surface cracks are presented.

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CHAPTER 2

SOLUTION OF THE NAVIER-CAUCHY EQUATIONS IN RECTANGULAR CARTESIAN COORDINATES BY THE METHOD OF LINES

Let us consider a finite solid with rectangular boundaries which is loaded by a given boundary stress or displacement distribution. For all the problems discussed in this thesis, the following assumptions shall apply:

a. The deformations are infinitesimal, that is products of displacement gradients can be neglected.

b. All deformations are elastic.

c. All materials are homogeneous and isotropic.

d. Body forces will be neglected.

2.1 Governing Equations

Problems satisfying the first three of the above assumptions fall in the general class of linearized elasticity, for which the field equations neglecting body forces are listed below. Using the standard summation convention, the equilibrium equations are

$$\sigma_{ji,j} = 0 \qquad i, j = 1, 2, 3 \tag{2.1}$$

Hooke's law is,

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$$\sigma_{ij} = \lambda\delta'_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} \qquad (2.2)$$

Strain-displacement relations are

$$\varepsilon_{ij} = \frac{1}{2}(\overline{u}_{i,j} + \overline{u}_{j,i}) \qquad (2.3)$$

The solution must satisfy these equations at all interior points of the body and, in addition, prescribed conditions on stress and/or displacements must be met on the bounding surfaces. For mixed boundary value problems, displacements are prescribed over a portion of the boundary while stresses are prescribed over the remaining part. The above three sets of equations may be combined to form three partial differential equations in terms of displacements by substituting the strain-displacement relations (2.3) into Hooke's law (2.2) and the result in turn being substituted into (2.1). The resulting equations

$$G\overline{u}_{i,jj} + (\lambda + G)\overline{u}_{j,ji} = 0 \quad i,j = 1, 2, 3 \qquad (2.4)$$

are called the Navier-Cauchy equations of elasticity.

For problems formulated in rectangular Cartesian coordinates, these equations can be written as

$$(\lambda + G)\frac{\partial e_r}{\partial x} + G\nabla^2 u = 0 \qquad (2.5)$$

$$(\lambda + G)\frac{\partial e_r}{\partial y} + G\nabla^2 v = 0 \qquad (2.6)$$

$$(\lambda + G)\frac{\partial e_r}{\partial z} + G\nabla^2 w = 0 \qquad (2.7)$$

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Solution

where the dilatation, e_r, is given by

$$e_r = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (2.8)$$

and the Laplacian, ∇^2, is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (2.9)$$

The stress-displacement relations, which are needed in satisfying the boundary stress distributions and in expressing the interior stresses, can be obtained from substituting the strain-displacement relations into Hooke's law. The following equations, listed in the form used for Cartesian coordinate problems, are obtained:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] \qquad (2.10)$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial v}{\partial y} + \nu\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\right] \qquad (2.11)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \qquad (2.12)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \qquad (2.13)$$

$$\sigma_{zx} = \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \qquad (2.14)$$

$$\sigma_{yz} = \frac{E}{2(1+\nu)}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \qquad (2.15)$$

Solution of these equations will be obtained by applying the method of lines, which is described in the following section, to equations (2.5) through (2.7) and by satisfying all relevant boundary conditions.

2.2 Method of Lines

Approximate solutions for second order, elliptic and linear partial differential equations are frequently obtained by the finite difference technique (33). For three-dimensional elasticity problems, this may involve the solution of an enormous number of algebraic equations in order that reasonable accuracy may be attained. This will be particularly true for problems involving steep stress and strain gradients which arise at geometric singularities and thus require close grid spacing in these regions.

An approximate solution with greater accuracy and much fewer equations to solve can be constructed, however, by applying the method of lines to these equations. The line method lies midway between completely analytical and grid methods. The basis of the method is substitution of finite differences for the derivatives with respect to all the independent variables except one, with respect to which the derivatives are retained. This approach replaces a given partial differential equation with a system of simultaneous ordinary differential equations. These equations describe the dependent variable along lines which are parallel to the coordinate in whose direction the derivatives were retained.

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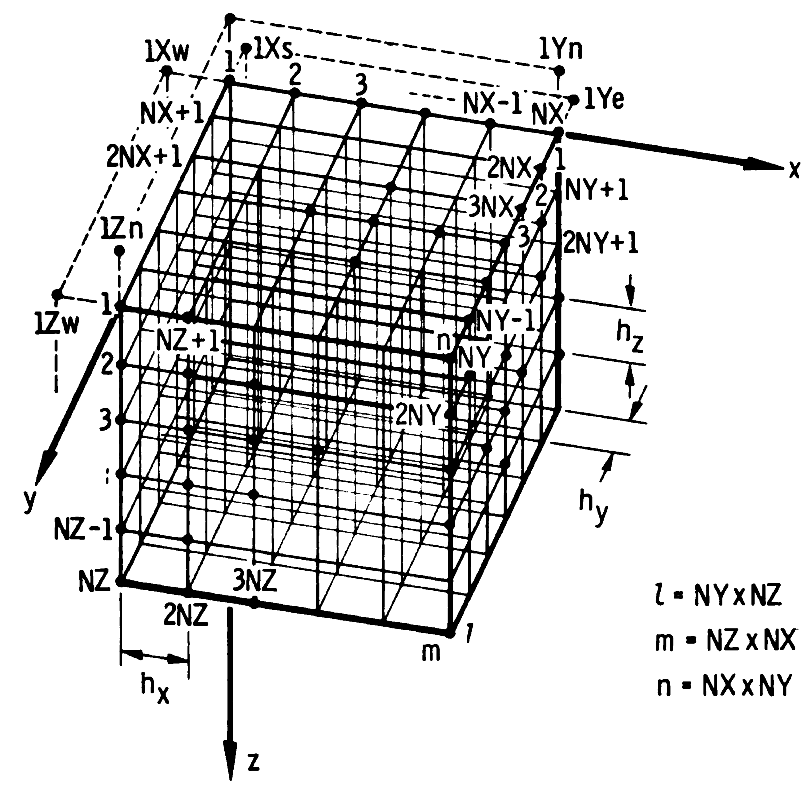
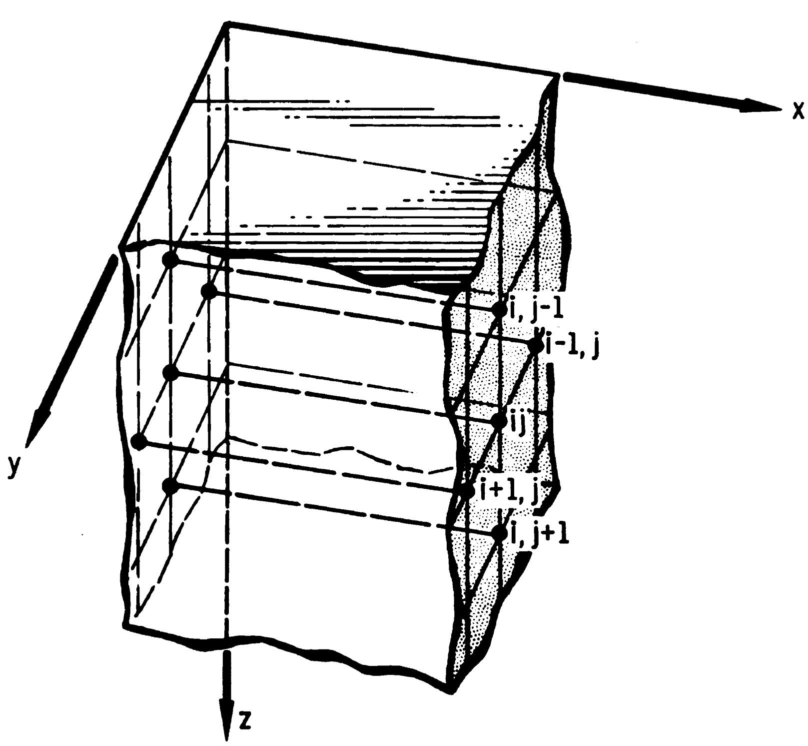
It can be noted that this method can be applied to a higher-order linear (or nonlinear) partial differential equation or a system of partial differential equations. Application of the line method is most useful when the resulting ordinary differential equations are linear and have constant coefficients. Equations (2.5) through (2.7) lead to linear differential equations with constant coefficients.

Since in three-dimensional elasticity problems solutions of three partial differential equations are desired, three sets of parallel lines must be constructed. An arbitrary rectangular grid consisting of these three sets of parallel lines is shown in Figure 1(a). It is assumed, of course, that a grid of this type will sufficiently describe the geometry of the problem in question. The lines parallel to the x axis are numbered as 1, 2, 3---NY, NY+1---2NY, 2NY+1---3NY, 3NY+1---ℓ. The lines parallel to the y axis are numbered 1, 2, 3---NZ, NZ+1---2NZ, 2NZ+1---3NZ, 3NZ+1---m. Finally, lines parallel to the z axis are numbered as 1, 2, 3---NX, NX+1---2NX, 2NX+1---3NX, 3NX+1---n. This numbering system is chosen so that the resulting variables in the computer listings are identified through double subscripts only. The first subscript identifies the line along which the variables are calculated while the second subscript indicates the position along that line. For convenience, the lines are evenly spaced with h_x, h_y and h_z each equal to some different



(a) Three sets of lines parallel to x-y-z coordinates..



(b) Set of interior lines parallel to x-coordinate.

Figure 1. - Sets of lines parallel to Cartesian coordinates.

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constant, although this is not absolutely necessary. The advantage of uniform line spacing is that the resulting ordinary differential equations can be solved more easily than those that are derived from non-uniform line spacing.

Inherent to coupled systems of partial differential equations, such as the Navier-Cauchy equations, is that solutions for the dependent variables are possible only at points where the three sets of parallel lines intersect. These points are usually called nodes in a discretized body. This limitation is the result of coupling among the equations, which makes the particular solution of the ordinary differential equations valid only at the nodes.

2.2.1 Reduction of the First Navier-Cauchy Equation and Associated Boundary Conditions.

For the solution of equation (2.5), the lines parallel to the x-axis in Figure 1(a) are considered. The x-directional displacements of points along these lines will be denoted as u_1, u_2, . . ., u_ℓ. We define $\mathring{v}|_1$, $\mathring{v}|_2$, $\mathring{v}|_3$, . . ., $\mathring{v}|_\ell$ as the derivatives of the y-directional displacements of the same points on these lines with respect to y and $\mathring{w}|_1$, $\mathring{w}|_2$, $\mathring{w}|_3$, . . ., $\mathring{w}|_\ell$ as the derivatives of the z-directional displacements of the same points on these lines with respect to z. These displacements and derivatives can then be regarded as functions of x only since they are variables upon lines which are parallel to the x-axis. Substituting equations (2.8) and (2.9) into equation (2.5) and expressing this equation along a general, x-directional line, denoted as (ij) in Figure 1(b) gives

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]_{ij}+(1-2\nu)\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]_{ij}=0 \quad (2.16)$$

By the above given definitions, we write (2.16) as

$$\frac{d^2u_{ij}}{dx^2}+\frac{d}{dx}\dot{v}\Big|_{ij}+\frac{d}{dx}\dot{w}\Big|_{ij}+(1-2\nu)\left[\frac{d^2u_{ij}}{dx^2}+\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)_{ij}\right]=0 \quad (2.17)$$

where an expression for the last term in (2.17) is still needed. Introducing finite difference calculus (33), the partial derivatives of u along the x-directional line (ij) of Figure 1(b) can be written as follows:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{ij} \simeq \frac{1}{h_y^2}(u_{i+1,j}-2u_{ij}+u_{i-1,j}) \quad (2.18)$$

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_{ij} \simeq \frac{1}{h_z^2}(u_{i,j+1}-2u_{ij}+u_{i,j-1}) \quad (2.19)$$

Using equations (2.18) and (2.19) in equation (2.17), the general equation along interior lines is obtained. Thus,

$$\frac{d^2u_{ij}}{dx^2}+\frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{h_y^2}+\frac{2}{h_z^2}\right)u_{ij}+\frac{u_{i+1,j}+u_{i-1,j}}{h_y^2}\right.$$

$$\left.+\frac{u_{i,j+1}+u_{i,j-1}}{h_z^2}\right]+\frac{f_{ij}(x)}{2(1-\nu)}=0 \quad (2.20)$$

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where $f_{ij}(x) = \left.\frac{d\hat{v}}{dx}\right|_{ij} + \left.\frac{d\hat{w}}{dx}\right|_{ij}$ (2.21)

Similar differential equations are obtained for the displacements u_{ij} of the points on the other x-directional lines. Since each equation has the terms of the displacements of the points on the surrounding lines, these equations constitute a system of ordinary differential equations for the displacements $u_1, u_2, \ldots, u_\ell$.

Equations (2.20) and (2.21) are applicable to interior lines only, since for bounding surface lines the central difference expressions for the required second derivatives involve imaginary lines. In order that central difference derivative approximations may be used, expressions for these fictitious line displacements must be found which are independent of the other differential equations. Since three-dimensional elasticity problems have three boundary conditions at every point of the bounding surface and a second order ordinary differential equation can satisfy only a total of two conditions, some of the boundary data can be used to find expressions for these imaginary displacements. Hence, conditions of normal stress and displacement will be enforced through the constants of the homogeneous solutions while shear stress boundary data will be incorporated into the differential equations of the surface lines.

As an example of this procedure, let us take the first x-directional line which is formed by the intersection of the x-z and x-y coord

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$$\sigma_{zx}\Big|_{x\text{-}y\text{ coordinate plane}} = 0 \tag{2.22}$$

$$\sigma_{yx}\Big|_{x\text{-}z\text{ coordinate plane}} = 0 \tag{2.23}$$

Using equation (2.14) in equation (2.22) we obtain along line 1 that

$$\left(\frac{\partial u}{\partial z}\right)_1 = -\left(\frac{\partial w}{\partial x}\right)_1$$

In terms of the discretized displacements shown in Figure 1(a), this equation gives

$$u_{1Yn} = u_{NY+1} + 2h_z \left.\frac{dw}{dx}\right|_1 \tag{2.24}$$

In a similar manner, equation (2.23) leads to

$$u_{1Ye} = u_2 + 2h_y \left.\frac{dv}{dx}\right|_1 \tag{2.25}$$

Note that the x-directional fictitious lines are numbered 1Yn and 1Ye with n and e indicating adjacent north and east line positions to the 1st x-directional line. Substitution of equations (2.24) and (2.25) into the general equation (2.20), leads to the following equation along line 1:

$$\frac{d^2 u}{dx^2} + \frac{(1-}{\ldots}$$

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$$\frac{d^2u_1}{dx^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{h_y^2}+\frac{2}{h_z^2}\right)u_1 + \frac{2}{h_y^2}u_2 + \frac{2}{h_z^2}u_{NY+1}\right] + \frac{\mathcal{F}_1(x)}{2(1-\nu)} = 0 \qquad (2.26)$$

where

$$\mathcal{F}_1(x) = \left.\frac{d\dot{v}}{dx}\right|_1 + \left.\frac{d\dot{w}}{dx}\right|_1 + (1-2\nu)\left[\frac{2}{h_y}\left.\frac{dv}{dx}\right|_1 + \frac{2}{h_z}\left.\frac{dw}{dx}\right|_1\right] \qquad (2.27)$$

Equations (2.26) and (2.27) are typical of the corner line equations with the exception of some sign changes in (2.27) at the other corners.

It will be convenient to non-dimensionalize the above equations with respect to some characteristic dimension of the problem at hand. For the numerical examples presented, the following variables were introduced:

$$\left.\begin{array}{lll} \tilde{u} = \frac{u}{a} & \tilde{x} = \frac{x}{a} & \tilde{h}_x = \frac{h_x}{a} \\ \tilde{v} = \frac{v}{a} & \tilde{y} = \frac{y}{a} & \tilde{h}_y = \frac{h_y}{a} \\ \tilde{w} = \frac{w}{a} & \tilde{z} = \frac{z}{a} & \tilde{h}_z = \frac{h_z}{a} \end{array}\right\} \qquad (2.28)$$

where a is the crack dimension.

Equations (2.26) and (2.27) can then be written as

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$$\frac{d^2\tilde{u}_1}{d\tilde{x}^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{\tilde{h}_y^2} + \frac{2}{\tilde{h}_z^2}\right)\tilde{u}_1 + \frac{2}{\tilde{h}_y^2}\tilde{u}_2 + \frac{2}{\tilde{h}_z^2}\tilde{u}_{NY+1}\right]$$

$$+ \frac{\bar{\mathcal{F}}_1(\tilde{x})}{2(1-\nu)} = 0 \tag{2.29}$$

$$\bar{\mathcal{F}}_1(\tilde{x}) = \left.\frac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_1 + \left.\frac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_1 + (1-2\nu)\left[\frac{2}{\tilde{h}_y}\left.\frac{d\tilde{v}}{d\tilde{x}}\right|_1 + \frac{2}{\tilde{h}_z}\left.\frac{d\tilde{w}}{d\tilde{x}}\right|_1\right] \tag{2.30}$$

Introducing matrix notation, the differential equations along the x-directional lines can be expressed in the form

$$\underset{\ell\times 1}{\frac{d^2}{d\tilde{x}^2}\{\tilde{u}\}} = \underset{\ell\times\ell}{[K_x]}\,\underset{\ell\times 1}{\{\tilde{u}\}} + \underset{\ell\times 1}{\{r(\tilde{x})\}} \tag{2.31}$$

where the non-dimensionalized coefficient matrix $[K_x]$ and the column vectors $\{\tilde{u}\}$ and $\{r(\tilde{x})\}$ are given below.

$$\underset{\ell\times\ell}{[K_x]} = \left[\begin{array}{c|c|c|c|c}
\underset{NY\times NY}{[K_{x1}]} & \underset{NY\times NY}{2[K_{x2}]} & 0 & 0 & 0 \\ \hline
\underset{NY\times NY}{[K_{x2}]} & \underset{NY\times NY}{[K_{x1}]} & \underset{NY\times NY}{[K_{x2}]} & 0 & 0 \\ \hline
0 & \ddots & \ddots & \ddots & 0 \\ \hline
0 & 0 & \underset{NY\times NY}{[K_{x2}]} & \underset{NY\times NY}{[K_{x1}]} & \underset{NY\times NY}{[K_{x2}]} \\ \hline
0 & 0 & 0 & \underset{NY\times NY}{2[K_{x2}]} & \underset{NY\times NY}{[K_{x1}]}
\end{array}\right] \tag{2.32}$$

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$$\frac{d^2\tilde{u}_1}{d\tilde{x}^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{\tilde{h}_y^2} + \frac{2}{\tilde{h}_z^2}\right)\tilde{u}_1 + \frac{2}{\tilde{h}_y^2}\tilde{u}_2 + \frac{2}{\tilde{h}_z^2}\tilde{u}_{NY+1}\right]$$

$$+ \frac{\bar{\mathcal{F}}_1(\tilde{x})}{2(1-\nu)} = 0 \qquad (2.29)$$

$$\bar{\mathcal{F}}_1(\tilde{x}) = \left.\frac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_1 + \left.\frac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_1 + (1-2\nu)\left[\frac{2}{\tilde{h}_y}\left.\frac{d\tilde{v}}{d\tilde{x}}\right|_1 + \frac{2}{\tilde{h}_z}\left.\frac{d\tilde{w}}{d\tilde{x}}\right|_1\right] \qquad (2.30)$$

Introducing matrix notation, the differential equations along the x-directional lines can be expressed in the form

$$\frac{d^2}{d\tilde{x}^2}\underset{\ell\times 1}{\{\tilde{u}\}} = \underset{\ell\times\ell}{[K_x]}\,\underset{\ell\times 1}{\{\tilde{u}\}} + \underset{\ell\times 1}{\{r(\tilde{x})\}} \qquad (2.31)$$

where the non-dimensionalized coefficient matrix $[K_x]$ and the column vectors $\{\tilde{u}\}$ and $\{r(\tilde{x})\}$ are given below.

$$\underset{\ell\times\ell}{[K_x]} = \left[\begin{array}{c|c|c|c|c}
\underset{NY\times NY}{[K_{x1}]} & \underset{NY\times NY}{2[K_{x2}]} & 0 & 0 & 0 \\ \hline
\underset{NY\times NY}{[K_{x2}]} & \underset{NY\times NY}{[K_{x1}]} & \underset{NY\times NY}{[K_{x2}]} & 0 & 0 \\ \hline
0 & \ddots & \ddots & \ddots & 0 \\ \hline
0 & 0 & \underset{NY\times NY}{[K_{x2}]} & \underset{NY\times NY}{[K_{x1}]} & \underset{NY\times NY}{[K_{x2}]} \\ \hline
0 & 0 & 0 & \underset{NY\times NY}{2[K_{x2}]} & \underset{NY\times NY}{[K_{x1}]}
\end{array}\right] \qquad (2.32)$$

where the

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$[K_{x2}] =$

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where the submatrices $[K_{x1}]$ and $[K_{x2}]$ are

$$\underset{NYxNY}{[K_{x1}]} = \begin{bmatrix} k_1 & -2k_2 & 0 & 0 & 0 \\ -k_2 & k_1 & -k_2 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_2 & k_1 & -k_2 \\ 0 & 0 & 0 & -2k_2 & k_1 \end{bmatrix}$$

$$k_1 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{2}{\tilde{h}_y^2} + \frac{2}{\tilde{h}_z^2}\right]$$

$$k_2 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_y^2}\right]$$

$$\underset{NYxNY}{[K_{x2}]} = \begin{bmatrix} -k_3 & 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & 0 & -k_3 \end{bmatrix}$$

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$$k_3 = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{1}{\tilde{h}_z^2} \right]$$

Note that $k_1 = 2(k_2 + k_3)$.

$$\underset{\ell x1}{\{\tilde{u}\}} = \begin{Bmatrix} \{\tilde{u}\}_1 \\ \{\tilde{u}\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{\tilde{u}\}_{NZ-1} \\ \{\tilde{u}\}_{NZ} \end{Bmatrix} \qquad \underset{\ell x1}{\{r(\tilde{x})\}} = \begin{Bmatrix} \{f(\tilde{x})\}_1 \\ \{f(\tilde{x})\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{f(\tilde{x})\}_{NZ-1} \\ \{f(\tilde{x})\}_{NZ} \end{Bmatrix} \tag{2.33}$$

where the partitioned column vectors are

$$\underset{NYx1}{\{\tilde{u}\}_1} = \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{u}_{NY-1} \\ \tilde{u}_{NY} \end{Bmatrix} \qquad \underset{NYx1}{\{\tilde{u}\}_2} = \begin{Bmatrix} \tilde{u}_{NY+1} \\ \tilde{u}_{NY+2} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{u}_{2NY-1} \\ \tilde{u}_{2NY} \end{Bmatrix}$$

$\{U\}_{NZ-1} =$

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$\{f(x)\}_1 =$

$NY \times 1$

$$\underset{NY\times 1}{\{\tilde{u}\}_{NZ-1}} = \begin{Bmatrix} \tilde{u}_{\ell-2NY+1} \\ \tilde{u}_{\ell-2NY+2} \\ \vdots \\ \tilde{u}_{\ell-NY-1} \\ \tilde{u}_{\ell-NY} \end{Bmatrix} \qquad \underset{NY\times 1}{\{\tilde{u}\}_{NZ}} = \begin{Bmatrix} \tilde{u}_{\ell-NY+1} \\ \tilde{u}_{\ell-NY+2} \\ \vdots \\ \tilde{u}_{\ell-1} \\ \tilde{u}_{\ell} \end{Bmatrix}$$

$$\underset{NY\times 1}{\{f(x)\}_1} = -\frac{1}{2(1-\nu)} \begin{Bmatrix} \left.\frac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_1 + \left.\frac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_1 + (1-2\nu)\left[\frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{x}} + \frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{x}}\right]_1 \\ \left.\frac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_2 + \left.\frac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_2 + (1-2\nu)\left[\frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{x}}\right]_2 \\ \vdots \qquad \vdots \qquad \vdots \\ \left.\frac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{NY-1} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{NY-1} + (1-2\nu)\left[\frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{x}}\right]_{NY-1} \\ \left.\frac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{NY} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{NY} + (1-2\nu)\left[-\frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{x}} + \frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{x}}\right]_{NY} \end{Bmatrix}$$

$$\underset{NY\times 1}{\{\mathcal{f}(\tilde{x})\}_2} = -\frac{1}{2(1-\nu)}\left\{\begin{array}{l} \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{NY+1} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{NY+1} + (1-2\nu)\left[\dfrac{2}{\tilde{h}_y}\,\dfrac{d\tilde{v}}{d\tilde{x}}\right]_{NY+1} \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{NY+2} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{NY+2} \\ \quad\vdots \qquad\qquad\quad \vdots \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{2NY-1} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{2NY-1} \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{2NY} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{2NY} + (1-2\nu)\left[\dfrac{-2}{\tilde{h}_y}\,\dfrac{d\tilde{v}}{d\tilde{x}}\right]_{2NY} \end{array}\right\}$$

$$\{f(\tilde{x})\}_{\substack{NZ-1 \\ NY\times 1}} = -\frac{1}{2(1-\nu)} \left\{ \begin{array}{l} \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-2NY+1} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-2NY+1} + (1-2\nu)\left[\dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{x}}\right]_{\ell-2NY+1} \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-2NY+2} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-2NY+2} \\ \vdots \qquad\qquad\quad \vdots \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-NY-1} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-NY-1} \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-NY} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-NY} + (1-2\nu)\left[-\dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{x}}\right]_{\ell-NY} \end{array} \right\}$$

$$\{f(\tilde{x})\}_{\substack{NZ \\ NY\times 1}} = -\frac{1}{2(1-\nu)} \left\{ \begin{array}{l} \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-NY+1} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-NY+1} + (1-2\nu)\left[\dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{x}} - \dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{x}}\right]_{\ell-NY+1} \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-NY+2} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-NY+2} + (1-2\nu)\left[-\dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{x}}\right]_{\ell-NY+2} \\ \vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\qquad\quad \vdots \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell-1} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell-1} + (1-2\nu)\left[-\dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{x}}\right]_{\ell-1} \\ \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{x}}\right|_{\ell} + \left.\dfrac{d\dot{\tilde{w}}}{d\tilde{x}}\right|_{\ell} - (1-2\nu)\left[\dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{x}} + \dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{x}}\right]_{\ell} \end{array} \right\}$$

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obtained:

Since the elements of the coefficient matrix $[K_x]$ are all constants, equations (2.31) are differential equations with constant coefficients. The coupling terms from the second and third Navier-Cauchy equations appear only in the vector $\{r(\tilde{x})\}$ which makes the particular solution of (2.31) a function of this coupling. Assuming that $\{r(\tilde{x})\}$ is known, solutions of (2.31) can be obtained in closed form when boundary data at the end points of the x-directional lines are given.

2.2.2 Reduction of the Second Navier-Cauchy Equation and Associated Boundary Conditions

For the solution of equation (2.6), the lines parallel to the y-axis in Figure 1(a) must be utilized. The y-directional displacements of points along these lines will be denoted as v_1, v_2, v_3, . . ., v_m. We define $\dot{u}|_1$, $\dot{u}|_2$, $\dot{u}|_3$, . . ., $\dot{u}|_m$ as the derivatives of the x-directional displacements of the same points on these lines with respect to x and $\dot{w}|_1$, $\dot{w}|_2$, $\dot{w}|_3$, . . ., $\dot{w}|_m$ as the derivatives of the z-directional displacements of the same points on these lines with respect to z. These displacements and derivatives can then be regarded as functions of y only since they are variables upon lines which are parallel to the y axis. By substituting equations (2.8) and (2.9) into equation (2.6) and analogously expressing this equation along a general y-directional line (ij) to equation (2.20), the following equation will be obtained:

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]_{ij} + (1-2\nu)\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]_{ij} = 0 \qquad (2.34)$$

$$\frac{d^2 v_{ij}}{dy^2} + \left.\frac{d\dot{u}}{dy}\right|_{ij} + \left.\frac{d\dot{w}}{dy}\right|_{ij} + (1-2\nu)\left[\frac{d^2 v_{ij}}{dy^2} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right)_{ij}\right] = 0 \qquad (2.35)$$

In a similar manner to equations (2.18) and (2.19), we have

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{ij} \simeq \frac{1}{h_x^2}\left(v_{i+1,j} - 2v_{ij} + v_{i-1,j}\right) \qquad (2.36)$$

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_{ij} \simeq \frac{1}{h_z^2}\left(v_{i,j+1} - 2v_{ij} + v_{i,j-1}\right) \qquad (2.37)$$

Combining equations (2.36), (2.37) and (2.35), the general equation along y-directional, interior lines is obtained. Thus, we have

$$\frac{d^2 v_{ij}}{dy^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{h_x^2} + \frac{2}{h_z^2}\right)v_{ij} + \frac{v_{i+1,j} + v_{i-1,j}}{h_x^2} + \frac{v_{i,j+1} + v_{i,j-1}}{h_z^2}\right] + \frac{f_{ij}(y)}{2(1-\nu)} = 0 \qquad (2.38)$$

where $$f_{ij}(y) = \left.\frac{d\dot{u}}{dy}\right|_{ij} + \left.\frac{d\dot{w}}{dy}\right|_{ij} \qquad (2.39)$$

Similar differential equations are obtained for the displacements along the other y-directional lines. Since each equation contains displacements of the surrounding lines, these equations

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]$$

$$\frac{\partial^2 v_{ij}}{\partial y^2}+\frac{\partial f}{\partial y_{ij}}+$$

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$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{ij}$$

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_{ij}$$

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$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]_{ij} + (1-2\nu)\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]_{ij} = 0 \qquad (2.34)$$

$$\frac{d^2 v_{ij}}{dy^2} + \left.\frac{d\dot{u}}{dy}\right|_{ij} + \left.\frac{d\dot{w}}{dy}\right|_{ij} + (1-2\nu)\left[\frac{d^2 v_{ij}}{dy^2} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right)_{ij}\right] = 0 \qquad (2.35)$$

In a similar manner to equations (2.18) and (2.19), we have

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_{ij} \simeq \frac{1}{h_x^2}(v_{i+1,j} - 2v_{ij} + v_{i-1,j}) \qquad (2.36)$$

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_{ij} \simeq \frac{1}{h_z^2}(v_{i,j+1} - 2v_{ij} + v_{i,j-1}) \qquad (2.37)$$

Combining equations (2.36), (2.37) and (2.35), the general equation along y-directional, interior lines is obtained. Thus, we have

$$\frac{d^2 v_{ij}}{dy^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{h_x^2} + \frac{2}{h_z^2}\right)v_{ij} + \frac{v_{i+1,j} + v_{i-1,j}}{h_x^2} + \frac{v_{i,j+1} + v_{i,j-1}}{h_z^2}\right] + \frac{f_{ij}(y)}{2(1-\nu)} = 0 \qquad (2.38)$$

$$\text{where} \quad f_{ij}(y) = \left.\frac{d\dot{u}}{dy}\right|_{ij} + \left.\frac{d\dot{w}}{dy}\right|_{ij} \qquad (2.39)$$

Similar differential equations are obtained for the displacements along the other y-directional lines. Since each equation contains displacements of the surrounding lines, these equations form a set of c

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form a set of ordinary differential equations for $v_1, v_2, \ldots,$ v_m.

Noting that equations (2.38) and (2.39) are defined fully for interior lines only, expressions for the imaginary displacements v_{1Zw} and v_{1Zn} must be obtained from the shear stress conditions in case of the first y-directional boundary line. Since the first y-directional line is formed by the intersection of the y-z and y-x coordinate planes, shear stresses on these planes in the y direction are utilized. For shear stress free surfaces, the boundary data gives

$$\sigma_{zy}\Big|_{\text{y-x coordinate plane}} = 0 \qquad (2.40)$$

$$\sigma_{xy}\Big|_{\text{y-z coordinate plane}} = 0 \qquad (2.41)$$

Using equations (2.13) and (2.15) the following equations will be obtained in terms of the discretized displacements shown in Figure 1(a):

$$v_{1Zw} = v_{NZ+1} + 2h_x \frac{du}{dy}\Big|_1 \qquad (2.42)$$

$$v_{1Zn} = v_2 + 2h_z \frac{dw}{dy}\Big|_1 \qquad (2.43)$$

Note that subscripts w and n again describe west and north adjacent line positions with respect to the first y-directional line. Substitution of equations (2.42) and (2.43) into the general equation (2.38) results in the following equation along

The 1:

$$\frac{d^2 v}{dy^2} + \frac{(1-2\nu)}{2(1-\nu)} \Bigg[$$

$$+ \frac{\bar{F}(\,}{2(1-}$$

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$$\underset{m\times 1}{\frac{d^2}{dy^2}\{\bar{v}\}}$$

The coefficient

$$\underset{m\times m}{[K_y]} = \left[\begin{array}{l} \underset{NZ\times}{[K_y} \\ \underset{NZ}{[K_y} \\ \\ \\ \\ \end{array} \right.$$

line 1:

$$\frac{d^2 v_1}{dy^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{h_z^2} + \frac{2}{h_x^2}\right)v_1 + \frac{2v_2}{h_z^2} + \frac{2v_{NZ+1}}{h_x^2}\right]$$

$$+ \frac{\bar{\mathcal{F}}_1(y)}{2(1-\nu)} = 0 \qquad (2.44)$$

$$\mathcal{F}_1(y) = \left.\frac{d\dot{u}}{dy}\right|_1 + \left.\frac{d\dot{w}}{dy}\right|_1 + (1-2\nu)\left[\frac{2}{h_x}\frac{du}{dy} + \frac{2}{h_z}\frac{dw}{dy}\right]_1 \qquad (2.45)$$

Non-dimensionalizing these equations according to (2.28) and introducing matrix notation, the ordinary differential equations along the y-directional lines can be written as

$$\underset{m\times 1}{\frac{d^2}{d\tilde{y}^2}\{\tilde{v}\}} = \underset{m\times m}{[K_y]}\ \underset{m\times 1}{\{\tilde{v}\}} + \underset{m\times 1}{\{s(\tilde{y})\}} \qquad (2.46)$$

The coefficient matrix $[K_y]$ is given by

$$\underset{m\times m}{[K_y]} = \begin{bmatrix} \underset{NZ\times NZ}{[K_{y1}]} & \underset{NZ\times NZ}{2[K_{y2}]} & 0 & 0 & 0 \\ \underset{NZ\times NZ}{[K_{y2}]} & \underset{NZ\times NZ}{[K_{y1}]} & \underset{NZ\times NZ}{[K_{y2}]} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \underset{NZ\times NZ}{[K_{y2}]} & \underset{NZ\times NZ}{[K_{y1}]} & \underset{NZ\times NZ}{[K_{y2}]} \\ 0 & 0 & 0 & \underset{NZ\times NZ}{2[K_{y2}]} & \underset{NZ\times NZ}{[K_{y1}]} \end{bmatrix} \qquad (2.47)$$

where the submat

$$\underset{12\times12}{[K_{y1}]} = \begin{bmatrix} k_4 \\ -k_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$k_3 =$

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Note that k_4

$$\underset{12\times12}{[K_{y2}]} = \begin{bmatrix} -k_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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The vectors

where the submatrices $[K_{y1}]$ and $[K_{y2}]$ are

$$\underset{NZ\times NZ}{[K_{y1}]} = \begin{bmatrix} k_4 & -2k_3 & 0 & 0 & 0 \\ -k_3 & k_4 & -k_3 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_3 & k_4 & -k_3 \\ 0 & 0 & 0 & -2k_3 & k_4 \end{bmatrix}$$

$$k_3 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_z^2}\right]$$

$$k_4 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{2}{\tilde{h}_z^2} + \frac{2}{\tilde{h}_x^2}\right]$$

Note that $k_4 = 2(k_3 + k_5)$.

$$\underset{NZ\times NZ}{[K_{y2}]} = \begin{bmatrix} -k_5 & 0 & 0 & 0 & 0 \\ 0 & -k_5 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_5 & 0 \\ 0 & 0 & 0 & 0 & -k_5 \end{bmatrix}$$

$$k_5 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_x^2}\right]$$

The vectors $\{\tilde{v}\}$ and $\{s(\tilde{y})\}$ can be written as here the subma

$$[K_{y2}] = \begin{bmatrix} k_4 \\ -k_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
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$$[K_{y2}] = \begin{bmatrix} -k_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
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where the submatrices $[K_{y1}]$ and $[K_{y2}]$ are

$$\underset{NZ\times NZ}{[K_{y1}]} = \begin{bmatrix} k_4 & -2k_3 & 0 & 0 & 0 \\ -k_3 & k_4 & -k_3 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_3 & k_4 & -k_3 \\ 0 & 0 & 0 & -2k_3 & k_4 \end{bmatrix}$$

$$k_3 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_z^2}\right]$$

$$k_4 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{2}{\tilde{h}_z^2} + \frac{2}{\tilde{h}_x^2}\right]$$

Note that $k_4 = 2(k_3 + k_5)$.

$$\underset{NZ\times NZ}{[K_{y2}]} = \begin{bmatrix} -k_5 & 0 & 0 & 0 & 0 \\ 0 & -k_5 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_5 & 0 \\ 0 & 0 & 0 & 0 & -k_5 \end{bmatrix}$$

$$k_5 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_x^2}\right]$$

The vectors $\{\tilde{v}\}$ and $\{s(\tilde{y})\}$ can be written as

$$\underset{nx1}{\{\tilde{V}\}} = \begin{Bmatrix} \{\tilde{V}\}_1 \\ \{\tilde{V}\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{\tilde{V}\}_{N\ldots} \\ \{\tilde{V}\}_{\ldots} \end{Bmatrix}$$

where the part

$$\underset{NZx1}{\{\tilde{V}\}_1} = \begin{Bmatrix} \tilde{V}\ldots \\ \tilde{V}\ldots \\ \cdot \\ \cdot \\ \cdot \\ \ldots \end{Bmatrix}$$

$$\underset{NZx1}{\{\tilde{V}\}_{NX-1}} = \begin{Bmatrix} \ldots \end{Bmatrix}$$

$$\underset{m\times 1}{\{\tilde{v}\}} = \begin{Bmatrix} \{\tilde{v}\}_1 \\ \{\tilde{v}\}_2 \\ \vdots \\ \{\tilde{v}\}_{NX-1} \\ \{\tilde{v}\}_{NX} \end{Bmatrix} \qquad \underset{m\times 1}{\{s(\tilde{y})\}} = \begin{Bmatrix} \{f(\tilde{y})\}_1 \\ \{f(\tilde{y})\}_2 \\ \vdots \\ \{f(\tilde{y})\}_{NX-1} \\ \{f(\tilde{y})\}_{NX} \end{Bmatrix} \tag{2.48}$$

where the partitioned column vectors are

$$\underset{NZ\times 1}{\{\tilde{v}\}_1} = \begin{Bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_{NZ-1} \\ \tilde{v}_{NZ} \end{Bmatrix} \qquad \underset{NZ\times 1}{\{\tilde{v}\}_2} = \begin{Bmatrix} \tilde{v}_{NZ+1} \\ \tilde{v}_{NZ+2} \\ \vdots \\ \tilde{v}_{2NZ-1} \\ \tilde{v}_{2NZ} \end{Bmatrix}$$

$$\underset{NZ\times 1}{\{\tilde{v}\}_{NX-1}} = \begin{Bmatrix} \tilde{v}_{m-2NZ+1} \\ \tilde{v}_{m-2NZ+2} \\ \vdots \\ \tilde{v}_{m-NZ-1} \\ \tilde{v}_{m-NZ} \end{Bmatrix} \qquad \underset{NZ\times 1}{\{\tilde{v}\}_{NX}} = \begin{Bmatrix} \tilde{v}_{m-NZ+1} \\ \tilde{v}_{m-NZ+2} \\ \vdots \\ \tilde{v}_{m-1} \\ \tilde{v}_{m} \end{Bmatrix}$$

$$\underset{NZ\times 1}{\{f(\tilde{y})\}_1} = \frac{-1}{2(1-\nu)} \left\{ \begin{array}{l} \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_1 + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_1 + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{y}} + \frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{y}}\right]_1 \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_2 + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_2 + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{y}}\right]_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_{NZ-1} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_{NZ-1} + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{y}}\right]_{NZ-1} \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_{NZ} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_{NZ} + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{y}} - \frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{y}}\right]_{NZ} \end{array} \right\}$$

$$\underset{NZ\times 1}{\{f(\tilde{y})\}_2} = \frac{-1}{2(1-\nu)} \left\{ \begin{array}{l} \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_{NZ+1} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_{NZ+1} + (1-2\nu)\left[\frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{y}}\right]_{NZ+1} \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_{NZ+2} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_{NZ+2} \\ \quad\vdots \qquad\qquad \vdots \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_{2NZ-1} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_{2NZ-1} \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{y}}\right|_{2NZ} + \left.\frac{d\dot{\tilde{w}}}{d\tilde{y}}\right|_{2NZ} + (1-2\nu)\left[-\frac{2}{\tilde{h}_z}\frac{d\tilde{w}}{d\tilde{y}}\right]_{2NZ} \end{array} \right\}$$

$$
\{f(\bar{y})\}_{\substack{NX-1 \\ NZ\times 1}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-2NZ+1} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-2NZ+1} + (1-2\nu)\left[\dfrac{2}{\bar{h}_z}\dfrac{d\bar{w}}{d\bar{y}}\right]_{m-2NZ+1} \\[2ex]
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-2NZ+2} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-2NZ+2} \\[2ex]
\quad\vdots \qquad\qquad\qquad \vdots \\[2ex]
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-NZ-1} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-NZ-1} \\[2ex]
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-NZ} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-NZ} + (1-2\nu)\left[\dfrac{-2}{\bar{h}_z}\dfrac{d\bar{w}}{d\bar{y}}\right]_{m-NZ}
\end{array}
\right\}
$$

$$
\{f(\bar{y})\}_{\substack{NX \\ NZ\times 1}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-NZ+1} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-NZ+1} + (1-2\nu)\left[-\dfrac{2}{\bar{h}_x}\dfrac{d\bar{u}}{d\bar{y}} + \dfrac{2}{\bar{h}_z}\dfrac{d\bar{w}}{d\bar{y}}\right]_{m-NZ+1} \\[2ex]
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-NZ+2} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-NZ+2} + (1-2\nu)\left[-\dfrac{2}{\bar{h}_x}\dfrac{d\bar{u}}{d\bar{y}}\right]_{m-NZ+2} \\[2ex]
\quad\vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\[2ex]
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m-1} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m-1} + (1-2\nu)\left[-\dfrac{2}{\bar{h}_x}\dfrac{d\bar{u}}{d\bar{y}}\right]_{m-1} \\[2ex]
\left.\dfrac{d\dot{\bar{u}}}{d\bar{y}}\right|_{m} + \left.\dfrac{d\dot{\bar{w}}}{d\bar{y}}\right|_{m} - (1-2\nu)\left[\dfrac{2}{\bar{h}_x}\dfrac{d\bar{u}}{d\bar{y}} + \dfrac{2}{\bar{h}_z}\dfrac{d\bar{w}}{d\bar{y}}\right]_{m}
\end{array}
\right\}
$$

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Equations (2.46) form a set of equations that are similar to equations (2.31). Assuming that $\{s(\tilde{y})\}$ is known, solutions of (2.46) can be obtained in closed form when boundary data at the end points of the y-directional lines are given.

2.2.3 Reduction of the Third Navier-Cauchy Equations and Associated Boundary Conditions

For the solution of equation (2.7), the lines parallel to the z-axis in Figure 1(a) must be employed. The z-directional displacements of points along these lines will be denoted as $w_1, w_2, w_3, \ldots, w_n$. We define $\dot{u}|_1, \dot{u}|_2, \dot{u}|_3, \ldots, \dot{u}|_n$ as the derivatives of the x-directional displacements of the same points on these lines with respect to x and $\dot{v}|_1, \dot{v}|_2, \dot{v}|_3, \ldots, \dot{v}|_n$ as the derivatives of the y-directional displacements of the same points on these lines with respect to y. These displacements and derivatives can then be regarded as functions of z only since they are variables upon lines which are parallel to the z axis. Following the same procedure as in the derivation of equations (2.20) and (2.38), the differential equation for a general interior line (ij) along the z direction in Figure 1(a) is

$$\frac{d^2 w_{ij}}{dz^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{h_x^2} + \frac{2}{h_y^2}\right) w_{ij} + \frac{w_{i+1,j} + w_{i-1,j}}{h_x^2} + \frac{w_{i,j+1} + w_{i,j-1}}{h_y^2}\right] + \frac{\mathcal{B}_{ij}(z)}{2(1-\nu)} = 0 \qquad (2.49)$$

where

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$$\frac{d^2 w_1}{dz^2} + \frac{(1-2\nu}{2(1-\nu}$$

$+$

where

$$f_{ij}(z) = \left.\frac{d\mathring{u}}{dz}\right|_{ij} + \left.\frac{d\mathring{v}}{dz}\right|_{ij} \qquad (2.50)$$

Similar differential equations are obtained for the other z-directional lines. Since each equation has the terms of the displacements w_i of the points on the surrounding lines, a system of ordinary differential equations is obtained for w_1, w_2, w_3, . . ., w_n. Noting again that equations (2.49) and (2.50) are defined fully for interior lines only, expressions for the fictitious displacements w_{1xw} and w_{1xs} must be obtained from the shear stress conditions in case of the first z-directional boundary line. Since line 1 in the z-direction is formed by the intersection of the z-x and z-y coordinate planes, shear stresses on these planes in the z-direction are utilized. For shear stress free surfaces we obtain

$$w_{1Xw} = w_2 + 2h_x \left.\frac{du}{dz}\right|_1 \qquad (2.51)$$

$$w_{1Xs} = w_{NX+1} + 2h_y \left.\frac{dv}{dz}\right|_1 \qquad (2.52)$$

Substituting equations (2.51) and (2.52) back into the general equation (2.49) and non-dimensionalizing the result according to (2.28), leads to the following equation along line 1:

$$\frac{d^2\tilde{w}_1}{d\tilde{z}^2} + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{\tilde{h}_x^2} + \frac{2}{\tilde{h}_y^2}\right)\tilde{w}_1 + \frac{2}{\tilde{h}_x^2}\tilde{w}_2 + \frac{2}{\tilde{h}_y^2}\tilde{w}_{NX+1}\right]$$

$$+ \frac{\bar{f}_1(z)}{2(1-\nu)} = 0$$

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$$\bar{\mathcal{F}}_1(\tilde{z}) = \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_1 + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_1 + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{z}} + \frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{z}}\right]_1$$

Introducing matrix notation, the differential equations along the z-directional lines can be written as

$$\frac{d^2}{d\tilde{z}^2}\underset{n\times 1}{\{\tilde{w}\}} = \underset{n\times n}{[K_z]}\,\underset{n\times 1}{\{\tilde{w}\}} + \underset{n\times 1}{\{t(\tilde{z})\}} \qquad (2.53)$$

where the coefficient matrix $[K_z]$ and the column vectors $\{\tilde{w}\}$ and $\{t(\tilde{z})\}$ are given below.

$$\underset{n\times n}{[K_z]} = \left[\begin{array}{c|c|c|c|c}
\underset{NX\times NX}{[K_{z1}]} & \underset{NX\times NX}{2[K_{z2}]} & 0 & 0 & 0 \\ \hline
\underset{NX\times NX}{[K_{z2}]} & \underset{NX\times NX}{[K_{z1}]} & \underset{NX\times NX}{[K_{z2}]} & 0 & 0 \\ \hline
0 & \ddots & \ddots & \ddots & 0 \\ \hline
0 & 0 & \underset{NX\times NX}{[K_{z2}]} & \underset{NX\times NX}{[K_{z1}]} & \underset{NX\times NX}{[K_{z2}]} \\ \hline
0 & 0 & 0 & \underset{NX\times NX}{2[K_{z2}]} & \underset{NX\times NX}{[K_{z1}]}
\end{array}\right] \qquad (2.54)$$

where the submatrices $[K_{z1}]$ and $[K_{z2}]$ are

$[K_{22}] =$

SYMM

$[K_{22}] =$

SYMM

Note

$$
\underset{NX \times NX}{[K_{z1}]} = \begin{bmatrix} k_6 & -2k_5 & 0 & 0 & 0 \\ -k_5 & k_6 & -k_5 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_5 & k_6 & -k_5 \\ 0 & 0 & 0 & -2k_5 & k_6 \end{bmatrix}
$$

$$
k_5 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_x^2}\right]
$$

$$
k_6 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{2}{\tilde{h}_x^2} + \frac{2}{\tilde{h}_y^2}\right]
$$

$$
\underset{NX \times NX}{[K_{z2}]} = \begin{bmatrix} -k_2 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & -k_2 \end{bmatrix}
$$

$$
k_2 = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{1}{\tilde{h}_y^2}\right]
$$

Note that $k_6 = 2(k_2 + k_5)$.

$\{W\}_i =$

$Nx1$

$\{W\}_{i-1} =$

$NNx1$

$\{W\}_{NT-1} =$

$NNx1$

$$\underset{n\times 1}{\{\tilde{w}\}} = \begin{Bmatrix} \{\tilde{w}\}_1 \\ \{\tilde{w}\}_2 \\ \vdots \\ \{\tilde{w}\}_{NY-1} \\ \{\tilde{w}\}_{NY} \end{Bmatrix} \qquad \underset{n\times 1}{\{t(\tilde{z})\}} = \begin{Bmatrix} \{f(\tilde{z})\}_1 \\ \{f(\tilde{z})\}_2 \\ \vdots \\ \{f(\tilde{z})\}_{NY-1} \\ \{f(\tilde{z})\}_{NY} \end{Bmatrix} \tag{2.55}$$

$$\underset{NX\times 1}{\{\tilde{w}\}_1} = \begin{Bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \tilde{w}_{NX-1} \\ \tilde{w}_{NX} \end{Bmatrix} \qquad \underset{NX\times 1}{\{\tilde{w}\}_2} = \begin{Bmatrix} \tilde{w}_{NX+1} \\ \tilde{w}_{NX+2} \\ \vdots \\ \tilde{w}_{2NX-1} \\ \tilde{w}_{2NX} \end{Bmatrix}$$

$$\underset{NX\times 1}{\{\tilde{w}\}_{NY-1}} = \begin{Bmatrix} \tilde{w}_{n-2NX+1} \\ \tilde{w}_{n-2NX+2} \\ \vdots \\ \tilde{w}_{n-NX-1} \\ \tilde{w}_{n-NX} \end{Bmatrix} \qquad \underset{NX\times 1}{\{\tilde{w}\}_{NY}} = \begin{Bmatrix} \tilde{w}_{n-NX+1} \\ \tilde{w}_{n-NX+2} \\ \vdots \\ \tilde{w}_{n-1} \\ \tilde{w}_{n} \end{Bmatrix}$$

$$\underset{NX\times 1}{\{f(\tilde{z})\}_1} = \frac{-1}{2(1-\nu)} \begin{Bmatrix} \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_1 + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_1 + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{z}} + \frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{z}}\right]_1 \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_2 + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_2 + (1-2\nu)\left[\frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{z}}\right]_2 \\ \vdots \qquad \vdots \qquad \vdots \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{NX-1} + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{NX-1} + (1-2\nu)\left[\frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{z}}\right]_{NX-1} \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{NX} + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{NX} + (1-2\nu)\left[-\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{z}} + \frac{2}{\tilde{h}_y}\frac{d\tilde{v}}{d\tilde{z}}\right]_{NX} \end{Bmatrix}$$

$$\underset{NX\times 1}{\{f(\tilde{z})\}_2} = \frac{-1}{2(1-\nu)} \begin{Bmatrix} \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{NX+1} + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{NX+1} + (1-2\nu)\left[\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{z}}\right]_{NX+1} \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{NX+2} + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{NX+2} \\ \vdots \qquad \vdots \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{2NX-1} + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{2NX-1} \\ \left.\frac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{2NX} + \left.\frac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{2NX} + (1-2\nu)\left[-\frac{2}{\tilde{h}_x}\frac{d\tilde{u}}{d\tilde{z}}\right]_{2NX} \end{Bmatrix}$$

$$\underset{NX\mathrm{x}1}{\{f(\tilde{z})\}_{NY-1}} = \frac{-1}{2(1-\nu)} \left\{ \begin{array}{l} \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-2NX+1} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-2NX+1} + (1-2\nu)\left[\dfrac{2}{\tilde{h}_x}\dfrac{d\tilde{u}}{d\tilde{z}}\right]_{n-2NX+1} \\ \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-2NX+2} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-2NX+2} \\ \quad\vdots \qquad\qquad\quad \vdots \\ \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-NX-1} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-NX-1} \\ \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-NX} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-NX} + (1-2\nu)\left[-\dfrac{2}{\tilde{h}_x}\dfrac{d\tilde{u}}{d\tilde{z}}\right]_{n-NX} \end{array} \right\}$$

$$\underset{NX\mathrm{x}1}{\{f(\tilde{z})\}_{NY}} = \frac{-1}{2(1-\nu)} \left\{ \begin{array}{l} \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-NX+1} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-NX+1} + (1-2\nu)\left[\dfrac{2}{\tilde{h}_x}\dfrac{d\tilde{u}}{d\tilde{z}} - \dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{z}}\right]_{n-NX+1} \\ \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-NX+2} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-NX+2} + (1-2\nu)\left[-\dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{z}}\right]_{n-NX+2} \\ \quad\vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\qquad\qquad\quad \vdots \\ \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n-1} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n-1} + (1-2\nu)\left[-\dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{z}}\right]_{n-1} \\ \left.\dfrac{d\dot{\tilde{u}}}{d\tilde{z}}\right|_{n} + \left.\dfrac{d\dot{\tilde{v}}}{d\tilde{z}}\right|_{n} - (1-2\nu)\left[\dfrac{2}{\tilde{h}_x}\dfrac{d\tilde{u}}{d\tilde{z}} + \dfrac{2}{\tilde{h}_y}\dfrac{d\tilde{v}}{d\tilde{z}}\right]_{n} \end{array} \right\}$$

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Equations (2.53) are a set of equations that are analogous to the previously obtained two sets of differential equations. Assuming that $\{t(\tilde{z})\}$ is known, closed form solutions of (2.53) can be obtained when boundary conditions at the end points of the z-directional lines are given. Finally, it can be noted that the three coefficient matrices obtained are all singular.

2.3 Solution of Simultaneous Differential Equations With Constant Coefficients

Solution methods for a system of ordinary differential equations with constant coefficients have been investigated extensively and are available in the literature (34). One relatively simple method is that of the power series solution (34). In applying this method to a set of second order differential equations, we must, by suitable transformations, reduce the given simultaneous equations to a new set of first order differential equations. The solution of this set of first order equations can then be expressed in terms of a power series. The advantage of this method is that it avoids the problem of finding the eigenvalues and eigenvectors of a given coefficient matrix. However, in the application of this method serious convergence difficulties may arise in evaluating the power series at large values of the independent variable.

An entirely different method of solution involves the transformation of the original coordinates into a new set of coordinates in

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coordinates in which the differential equations become uncoupled. In this new set of coordinates, usually called principal coordinates, the matrix equations are reduced to a set of scalar equations whose solutions can be easily evaluated. The required transformation, however, is only possible when the matrix of eigenvectors, usually denoted as the modal matrix, can be accurately constructed. The solution method employed in this dissertation is a combination of the above described two procedures.

Since equations (2.31), (2.46) and (2.53) are all of the same type it will be sufficient to discuss equations (2.31) only. Hence, let us define the following new variables:

$$\left.\begin{array}{llllll} U_1 = \tilde{u}_1 & U_2 = \tilde{u}_2 & \cdots & U_\ell = \tilde{u}_\ell \\ U_{\ell+1} = \dfrac{d\tilde{u}_1}{d\tilde{x}} & U_{\ell+2} = \dfrac{d\tilde{u}_2}{d\tilde{x}} & \cdots & U_{2\ell} = \dfrac{d\tilde{u}_\ell}{d\tilde{x}} \end{array}\right\} \quad (2.56)$$

In terms of these variables, equations (2.31) can be written as

$$\underset{2\ell x1}{\frac{d}{d\tilde{x}}\{U\}} = \underset{2\ell x2\ell}{[A]} \; \underset{2\ell x1}{\{U\}} + \underset{2\ell x1}{\{\bar{r}(\tilde{x})\}} \quad (2.57)$$

where

$$\underset{2\ell x2\ell}{[A]} = \begin{bmatrix} \underset{\ell x\ell}{[0]} & \underset{\ell x\ell}{[I]} \\ \underset{\ell x\ell}{[K_x]} & \underset{\ell x\ell}{[0]} \end{bmatrix} \quad (2.58)$$

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$[A]^{2n+1}$

$$\underset{2\ell x1}{\{\bar{r}(\tilde{x})\}} = \begin{Bmatrix} \underset{\ell x1}{\{0\}} \\ \\ \underset{\ell x1}{\{r(\tilde{x})\}} \end{Bmatrix} \qquad (2.59)$$

The solution of equation (2.57) is well known and can be written as (34)

$$\underset{2\ell x1}{\{U(\tilde{x})\}} = \underset{2\ell x2\ell}{e^{[A]\tilde{x}}} \underset{2\ell x1}{\{U(o)\}} + \underset{2\ell x2\ell}{e^{[A]\tilde{x}}} \int_o^{\tilde{x}} \underset{2\ell x2\ell}{e^{-[A]\eta}} \underset{2\ell x1}{\{\bar{r}(\eta)\}} d\eta \qquad (2.60)$$

where $\{U(o)\}$ is a vector which consists of the boundary values of $\tilde{u}$ and $\frac{d\tilde{u}}{d\tilde{x}}$ at $\tilde{x} = 0$ and $e^{[A]\tilde{x}}$ is a matrix series given by the following equation:

$$e^{[A]\tilde{x}} = [I] + \frac{[A]\tilde{x}}{1!} + \frac{[A]^2\tilde{x}^2}{2!} + \frac{[A]^3\tilde{x}^3}{3!} + \ldots \qquad (2.61)$$

Using equation (2.58), the powers of matrix $[A]$ can be expressed as

$$[A]^{2n} = \begin{bmatrix} [K_x]^n & [0] \\ \\ [0] & [K_x]^n \end{bmatrix} \qquad n = 0, 1, 2, 3, \ldots$$

$$[A]^{2n+1} = \begin{bmatrix} [0] & [K_x]^n \\ \\ [K_x]^{n+1} & [0] \end{bmatrix} \qquad n = 0, 1, 2, \ldots$$

Substituting

yields

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$\underset{\ell x \ell}{[A_{12}]}$

Substituting the powers of matrix [A] into equation (2.61) yields

$$e^{[A]\tilde{x}} = \begin{bmatrix} \sum_{m=0}^{\infty} \frac{\tilde{x}^{2m}}{(2m)!} [K_x]^m & \sum_{m=0}^{\infty} \frac{\tilde{x}^{2m+1}}{(2m+1)!} [K_x]^m \\ \sum_{m=0}^{\infty} \frac{\tilde{x}^{2m+1}}{(2m+1)!} [K_x]^{m+1} & \sum_{m=0}^{\infty} \frac{\tilde{x}^{2m}}{(2m)!} [K_x]^m \end{bmatrix}$$

$$\underset{2\ell \times 2\ell}{e^{[A]\tilde{x}}} = \begin{bmatrix} \underset{\ell \times \ell}{[A_{11}(K_x,\tilde{x})]} & \underset{\ell \times \ell}{[A_{12}(K_x,\tilde{x})]} \\ \underset{\ell \times \ell}{[A_{21}(K_x,\tilde{x})]} & \underset{\ell \times \ell}{[A_{22}(K_x,\tilde{x})]} \end{bmatrix} \tag{2.62}$$

Since $\cosh\gamma = 1 + \frac{\gamma^2}{2!} + \frac{\gamma^4}{4!} + \frac{\gamma^6}{6!} + \ldots$

$$\sinh\gamma = \gamma + \frac{\gamma^3}{3!} + \frac{\gamma^5}{5!} + \frac{\gamma^7}{7!} + \ldots$$

the submatrices $[A_{ij}]$ $i,j = 1,2$ can be expressed as the matrix functions shown below.

$$\underset{\ell \times \ell}{[A_{11}]} = \underset{\ell \times \ell}{[A_{22}]} = \cosh [K_x]^{1/2}\tilde{x} \tag{2.63}$$

$$\underset{\ell \times \ell}{[A_{12}]} = \left([K_x]^{1/2}\right)^{-1} \sinh [K_x]^{1/2}\tilde{x} \tag{2.64}$$

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$$[A_{21}] = [K_x][A_{12}] \qquad (2.65)$$

where $[K_x]^{1/2}$ is a matrix whose square is equal to $[K_x]$. A good summary of matrix function definitions and theory can be found in (35).

2.3.1 Evaluation of Matrix Functions

As can be noted from equation (2.60), the accuracy of the solution for a given problem will depend largely on the number of terms retained in (2.61). In the course of this work, a number of different methods were employed in evaluating (2.61). A short description of each technique is given below. The accuracy of each method can be checked by substituting equation (2.58) into the identity of

$$e^{[A]\tilde{x}} \cdot e^{-[A]\tilde{x}} = [I] \qquad (2.66)$$

In terms of the submatrices $[A_{ij}]$ this equation yields

$$\underset{\ell\times\ell}{[A_{11}(K_x,\tilde{x})]^2} - \underset{\ell\times\ell}{[K_x]}\underset{\ell\times\ell}{[A_{12}(K_x,\tilde{x})]^2} = \underset{\ell\times\ell}{[I]} \qquad (2.67)$$

The most straightforward calculation procedure for evaluating the submatrices $[A_{ij}]$ for each value of the independent variable $\tilde{x}$ is to use the matrix series definitions (2.62) and truncate them at some given values. However, for increasing values of $\tilde{x}$, a large number of terms must be taken and if matrices with orders of 20 or larger are involved, the numerical procedure becomes inefficient. In order to avoid the computation of a large number of terms, additive formulas for these matrix functions may be obtained from using the identity of

$$e^{[A]\tilde{x}_1} \cdot e^{[A]\tilde{x}_2} = e^{[A](\tilde{x}_1+\tilde{x}_2)}$$

In terms of the submatrices $[A_{ij}]$ this identity yields the following two equations:

$$\left.\begin{aligned}
[A_{11}(K_x,\tilde{x}_1+\tilde{x}_2)] &= [A_{11}(K_x,\tilde{x}_1)][A_{11}(K_x,\tilde{x}_2)] \\
&\quad + [K_x][A_{12}(K_x,\tilde{x}_1)][A_{12}(K_x,\tilde{x}_2)] \\
[A_{12}(K_x,\tilde{x}_1+\tilde{x}_2)] &= [A_{12}(K_x,\tilde{x}_1)][A_{11}(K_x,\tilde{x}_2)] \\
&\quad + [A_{11}(K_x,\tilde{x}_1)][A_{12}(K_x,\tilde{x}_2)]
\end{aligned}\right\} \qquad (2.68)$$

Alternate methods used in evaluating (2.62) involved the use of Runge-Kutta integration formulas (33) and the conjoint algorithm of Frame and Needler (36). An examination of equation (2.62) will show that the matrix functions $[A_{ij}]$ satisfy the following two sets of simultaneous matrix differential equations:

$$\left.\begin{aligned}
\frac{d}{d\tilde{x}}[A_{11}(K_x,\tilde{x})] &= [A_{21}(K_x,\tilde{x})] \\
\frac{d}{d\tilde{x}}[A_{21}(K_x,\tilde{x})] &= [A_{11}(K_x,\tilde{x})][K_x]
\end{aligned}\right\} \qquad (2.69)$$

$$\left.\begin{aligned} \frac{d}{d\tilde{x}}[A_{12}(K_x,\tilde{x})] &= [A_{22}(K_x,\tilde{x})] \\ \frac{d}{d\tilde{x}}[A_{22}(K_x,\tilde{x})] &= [A_{12}(K_x,\tilde{x})][K_x] \end{aligned}\right\} \qquad (2.70)$$

The initial conditions needed for the Runge-Kutta integration of these equations are respectively

$$\left.\begin{aligned} [A_{11}(K_x,\tilde{x})]_{\tilde{x}=0} &= [I] \\ [A_{21}(K_x,\tilde{x})]_{\tilde{x}=0} &= [0] \end{aligned}\right\} \qquad (2.71)$$

$$\left.\begin{aligned} [A_{12}(K_x,\tilde{x})]_{\tilde{x}=0} &= [0] \\ [A_{22}(K_x,\tilde{x})]_{\tilde{x}=0} &= [I] \end{aligned}\right\} \qquad (2.72)$$

Details of Frame's and Needler's solution may be found in (36) and will not be repeated here. In comparing the results of these three methods, it was found that using the first approach with additive formulas gave the most accurate results. The relative merit of each method was established by its ability to evaluate the submatrices $[A_{ij}]$ for large order coefficient matrices at increasing values of the independent variable.

Common to all these techniques is that they do not require the solution of an eigenvalue problem. However, as the number of differential equations increases, these matrix series methods place a serious limitation on the value of the independent variable for

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variable for which equation (2.69) can be determined. Therefore, the use of an improved matrix function evaluation routine is needed. Inspection of equations (2.63), (2.64) and (2.65) shows that if the coefficient matrix $[K_x]$ could be diagonalized the matrix functions would be replaced by simple hyperbolic functions which could then be evaluated with better accuracy. This diagonalization of $[K_x]$ is possible only if the associated eigenvalues and eigenvectors can be accurately determined. Hence, a closed form solution of the associated eigenvalue problem is necessary.

Appendix A gives the details of the development showing how the eigenvalues and eigenvectors of a matrix having the form of $[K_x]$ can be analytically evaluated. The equation for the eigenvalues of $[K_x]$ is

$$\bar{\lambda}_{ij} = 2k_3\left[1 - \cos\left(\frac{i-1}{NZ-1}\right)\pi\right] + 2k_2\left[1 - \cos\left(\frac{j-1}{NY-1}\right)\pi\right] \tag{2.73}$$

where $i = 1, 2, \ldots NZ$ and $j = 1, 2, \ldots, NY$. Note that these eigenvalues are ordered by fixing i first and then varying j. Since $\ell = NZ \times NY$, the number of eigenvalues associated with $[K_x]$ is ℓ.

The modal matrix or the matrix of eigenvectors corresponding to the above ordered eigenvalues is given by

$$\underset{\ell\times\ell}{[P]} = \underset{NZ\times NZ}{[P_2]} \otimes \underset{NY\times NY}{[P_1]} \tag{2.74}$$

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where $\otimes$ denotes the Kroenecker product of two component modal matrices (37) whose elements are given by

$$\left[P_{1_{sj}}\right] = \left[\cos \frac{(s-1)(j-1)}{NY-1} \pi\right] \quad s,j = 1, 2, \ldots, NY \tag{2.75}$$

$$\left[P_{2_{ri}}\right] = \left[\cos \frac{(r-1)(i-1)}{NZ-1} \pi\right] \quad r,i = 1, 2, \ldots, NZ \tag{2.76}$$

The similarity transformation, diagonalizing the coefficient matrix $[K_x]$ is (35)

$$[\Lambda] = [P]^{-1} [K_x][P] \tag{2.77}$$

where $[\Lambda]$ is a diagonal matrix whose elements are the eigenvalues of $[K_x]$. The matrix functions (2.63) and (2.64) can now be evaluated in diagonalized form and retransformed according to the transformation

$$[A_{11}] = [P][\Lambda_{11}][P]^{-1} \tag{2.78}$$

$$[A_{12}] = [P][\Lambda_{12}][P]^{-1} \tag{2.79}$$

Note that the inverse of the modal matrix $[P]$ can also be evaluated in closed form and the details of the derivations are shown in Appendix A.

2.3.2 Evaluation of the Particular Integral

In the explicit evaluation of equation (2.60), the value of the particular integral cannot be obtained until the vector

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$\{\overline{r}(\eta)\}$ is known along the x-directional lines. We define column vectors $\{B_1(\tilde{x})\}$ and $\{B_2(\tilde{x})\}$ as follows:

$$\underset{2\ell x1}{\begin{Bmatrix} \{B_1(\tilde{x})\} \\ \\ \{B_2(\tilde{x})\} \end{Bmatrix}} = \int_0^{\tilde{x}} \underset{2\ell x 2\ell}{e^{-[A]\eta}} \underset{2\ell x1}{\{\overline{r}(\eta)\}} d\eta \tag{2.80}$$

Using equations (2.59) and (2.62) in the above equation gives

$$\begin{Bmatrix} \{B_1\} \\ \\ \{B_2\} \end{Bmatrix} = \int_0^{\tilde{x}} \begin{bmatrix} [A_{11}(K_x,\eta)][A_{12}(K_x,\eta)] \\ [A_{21}(K_x,\eta)][A_{22}(K_x,\eta)] \end{bmatrix}^{-1} \begin{Bmatrix} \{0\} \\ \{r(\eta)\} \end{Bmatrix} d\eta$$

Taking the inverse of the partitioned matrix leads to the following equations.

$$\{B_1(\tilde{x})\} = -\int_0^{\tilde{x}} [A_{11}][A_{12}]([A_{22}] - [A_{21}][A_{11}]^{-1}[A_{12}])^{-1} \cdot \{r(\eta)\}\, d\eta \tag{2.81}$$

$$\{B_2(\tilde{x})\} = \int_0^{\tilde{x}} ([A_{22}] - [A_{21}][A_{11}]^{-1}[A_{12}])^{-1} \cdot \{r(\eta)\}\, d\eta \tag{2.82}$$

where the arguments of $[A_{ij}]$ are understood to be (K_x,η).

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Equations

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From considering the even and odd character of the matrix functions $[A_{ij}]$, it can be shown that

$$[A_{11}] = ([A_{22}] - [A_{21}][A_{11}]^{-1}[A_{12}])^{-1}$$
$$[A_{12}] = [A_{11}][A_{12}]([A_{22}] - [A_{21}][A_{11}]^{-1}[A_{12}])^{-1} \tag{2.83}$$

The use of equations (2.83) allows us to write

$$\{B_1(\tilde{x})\} = -\int_o^{\tilde{x}} [A_{12}(K_x,\eta)]\ \{r(\eta)\}\ d\eta \tag{2.84}$$

$$\{B_2(\tilde{x})\} = \int_o^{\tilde{x}} [A_{11}(K_x,\eta)]\ \{r(\eta)\}\ d\eta \tag{2.85}$$

Similar expressions are obtained for the particular integrals of the solutions along the y and z directional lines. Since $\{r(\eta)\}$ is unknown, we start the solution of the problem by assuming values for $\frac{d\tilde{v}}{d\tilde{x}}$, $\frac{d\dot{\tilde{v}}}{d\tilde{x}}$, $\frac{d\tilde{w}}{d\tilde{x}}$ and $\frac{d\dot{\tilde{w}}}{d\tilde{x}}$ along the x directional lines. Generally, using zero values for these derivatives is a good place to start. Using the partitioned form of the matrices, equation (2.60) can be written as

$$\begin{Bmatrix} \{\tilde{u}(\tilde{x})\} \\ \\ \{\dot{\tilde{u}}(\tilde{x})\} \end{Bmatrix} = \begin{pmatrix} [A_{11}(K_x,\tilde{x})] & [A_{12}(K_x,\tilde{x})] \\ \\ [A_{21}(K_x,\tilde{x})] & [A_{22}(K_x,\tilde{x})] \end{pmatrix} \left(\begin{Bmatrix} \{\tilde{u}(o)\} \\ \\ \{\dot{\tilde{u}}(o)\} \end{Bmatrix} + \begin{Bmatrix} \{B_1(\tilde{x})\} \\ \\ \{B_2(\tilde{x})\} \end{Bmatrix} \right) \tag{2.86}$$

Equations (2.86) give us the first estimate for the vectors $\{\tilde{u}(\tilde{x})\}^{(1)}$ and $\{\dot{\tilde{u}}(\tilde{x})\}^{(1)}$. It is assumed, of course, that the boundary con

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boundary condition vectors $\{\tilde{u}(o)\}$ and $\{\dot{\tilde{u}}(o)\}$ are specified or known. Using these calculated values of $\{\tilde{u}\}^{(1)}$ and $\{\dot{\tilde{u}}\}^{(1)}$ we can evaluate the vector $\{s(\tilde{y})\}^{(1)}$ where we still use the originally assumed values of $\frac{d\tilde{w}}{d\tilde{y}}$ and $\frac{d\dot{\tilde{w}}}{d\tilde{y}}$. An analogous equation to (2.86) will then give us the first value of $\{\tilde{v}(\tilde{y})\}^{(1)}$ and $\{\dot{\tilde{v}}(\tilde{y})\}^{(1)}$. First values of $\{\tilde{w}(\tilde{z})\}^{(1)}$ and $\{\dot{\tilde{w}}(\tilde{z})\}^{(1)}$ can then be calculated by using the first estimates of the x and y directional displacements and their derivatives in the vector $\{t(\tilde{z})\}$. At this point we return to equation (2.86) and calculate the second value of $\{\tilde{u}(\tilde{x})\}^{(2)}$ and $\{\dot{\tilde{u}}(\tilde{x})\}^{(2)}$ based on the first values of the y and z directional solutions.

If the values of $\{\tilde{u}(\tilde{x})\}^{(n)}$, $\{\dot{\tilde{u}}(\tilde{x})\}^{(n)}$, $\{\tilde{v}(\tilde{y})\}^{(n)}$, $\{\dot{\tilde{v}}(\tilde{y})\}^{(n)}$, $\{\tilde{w}(\tilde{z})\}^{(n)}$ and $\{\dot{\tilde{w}}(\tilde{z})\}^{(n)}$ converge with the repetition of this procedure, an approximate solution of a given problem will be determined.

Based on the accumulated experience with this method, certain comments can be made regarding this convergence. It was found that the major factor controlling the convergence rate is the accuracy to which the matrix functions $[A_{ij}]$ could be determined. Using equation (2.67), an error matrix [ER] can be constructed for each value of the independent variable. This matrix is

$$[ER] = [A_{11}(K_x,\tilde{x})]^2 - [K_x][A_{12}(K_x,\tilde{x})]^2 - [I] \qquad (2.87)$$

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elements in [ER] should be less than about 3.5×10^{-4}. The rate of convergence increases as the error decreases. Errors on the order of 10^{-3} or larger will generally lead to divergence in the successive approximation procedure.

Since the vector $\{r(\eta)\}$ in equations (2.84) and (2.85) involves displacements and their derivatives that are defined only at the nodes, finite difference calculus must be used in evaluating its elements. Hence, integrals $\{B_1(\tilde{x})\}$ and $\{B_2(\tilde{x})\}$ are evaluated by a suitable numerical integration technique. For the numerical examples presented, the trapezoidal rule (33) is used to evaluate the necessary particular integrals. In addition, the lack of displacement data at some boundary points necessitates the use of first order forward and backward differences in evaluating certain coupling term derivatives. The use of higher order finite difference formulas at these points, however, gave no noticeable improvement in accuracy. We note that similar conclusions can be made about the particular integrals involving the vectors $\{s(\tilde{y})\}$ and $\{t(\tilde{z})\}$.

2.4 Application to Specific Geometries

A great amount of experimental work has been done in fracture mechanics (2) through the use of crack-notched specimens. In the past, many different types of specimens have been utilized for determining a material's fracture toughness. The most common early specimens employed were the center cracked and double edge notched bar

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notched bar specimens. It was recognized early, however, that the obtained results were largely thickness dependent and available theoretical solutions did not account for this variation.

A prerequisite to the application of fracture testing is the knowledge of the stress distribution in the specimen. Based on this theoretical solution, the stress intensity factor K can be calculated as a function of the load, crack and specimen dimensions. With this information, it is possible then to determine a material's fracture toughness through the use of crack-notched specimen tests. The lack of a valid theoretical solution has forced researchers in the past to make certain empirical assumptions about the stress states in their specimens. So that these assumptions may be analytically verified, the stress and displacement fields in both the center cracked and double edge cracked bar specimens will be investigated.

2.4.1 Bar With Through-Thickness Central Crack

In order to demonstrate the use of this analysis for the solution of some three-dimensional, mixed boundary value, elasticity problems, a number of previously unavailable solutions will be investigated. Figure 2(a) shows a finite rectangular bar with a through-thickness central crack which is loaded by a uniform normal stress, σ_o. Because of the symmetric geometry and loading, only one-eighth of the bar has to be discretized as shown in Figure 2(b). The existing displacement fields in the

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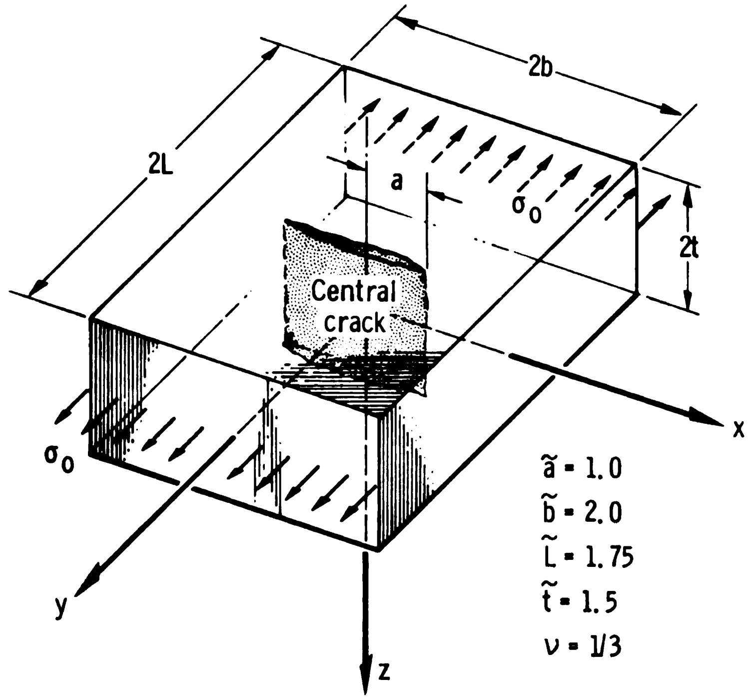
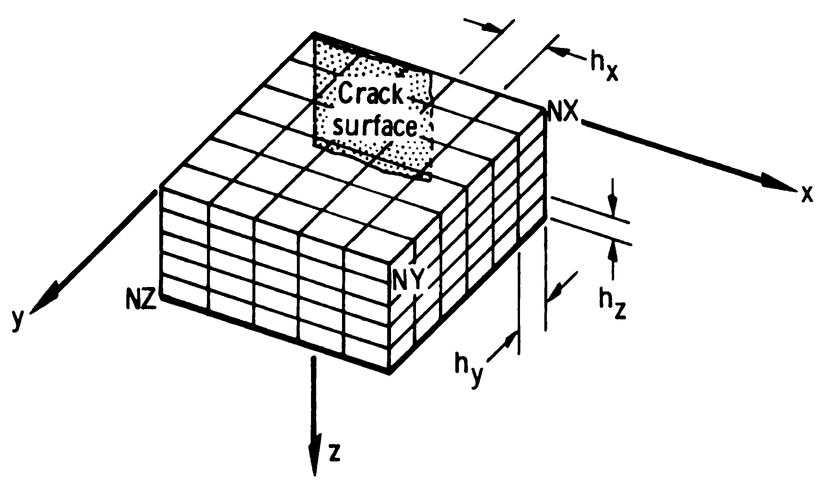
Figure 2. - Rec



(a) Rectangular bar with through-thickness central crack.



(b) Discretized region of rectangular bar with through-thickness central crack.

Figure 2. - Rectangular bar with through-thickness central crack under uniform tension.

bar will be described by the solutions of the three sets of ordinary differential equations. In this case, the previously obtained plane elasticity solutions are not applicable because the associated assumptions are not valid. Values of the non-dimensionalized variables $\tilde{a}$, $\tilde{b}$, $\tilde{L}$ and $\tilde{t}$ used for this problem are shown in Figure 2(a). Inspection of the derived ordinary differential equations also shows that for any numerical computations, a value of Poisson's ratio must be selected. For all the following examples, a Poisson's ratio of 1/3 will be used as shown in the attached figures.

At this time we return to solution (2.86) and note that the constants of the homogeneous solution were expressed in terms of initial values. Since the problem in Figure 2(a) is a two point boundary value problem, the initial values of both $\{\tilde{u}\}$ and $\{\dot{\tilde{u}}\}$ are usually not available. A method for evaluating the initial value vector for equations (2.31) when the x-directional lines in Figure 2(b) are involved will now be investigated. From symmetry conditions, we can immediately conclude that

$$\underset{\ell \times 1}{\{\tilde{u}(o)\}} = 0 \qquad (2.88)$$

The zero normal stress boundary condition on the face $\tilde{x} = \tilde{b}$ will be used to evaluate the vector $\{\dot{\tilde{u}}(o)\}$. From equation (2.86) we have at $\tilde{x} = \tilde{b}$

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$$\{\mathring{\tilde{u}}(\tilde{b})\} = [A_{21}(K_x,\tilde{b})](\{\tilde{u}(o)\} + \{B_1(\tilde{b})\})$$

$$+ [A_{22}(K_x,\tilde{b})](\{\mathring{\tilde{u}}(o)\} + \{B_2(\tilde{b})\}) \qquad (2.89)$$

where $\sigma_x\Big|_{\tilde{x}=\tilde{b}} = 0$ gives

$$\{\mathring{\tilde{u}}(\tilde{b})\} = \frac{-\nu}{1-\nu}(\{\dot{\tilde{v}}\} + \{\dot{\tilde{w}}\})_{\tilde{x}=\tilde{b}}$$

Provided that $[A_{22}(K_x,\tilde{b})]$ is not singular and using (2.88) we find from (2.89) that

$$\underset{\ell\times 1}{\{\mathring{\tilde{u}}(o)\}} = \frac{-\nu}{1-\nu}\underset{\ell\times\ell}{[A_{22}(K_x,\tilde{b})]^{-1}}\underset{\ell\times 1}{(\{\dot{\tilde{v}}\} + \{\dot{\tilde{w}}\})_{\tilde{x}=\tilde{b}}}$$

$$- \underset{\ell\times\ell}{[A_{22}(K_x,\tilde{b})]^{-1}}\ \underset{\ell\times\ell}{[A_{21}(K_x,\tilde{b})]}\ \underset{\ell\times 1}{\{B_1(\tilde{b})\}} - \underset{\ell\times 1}{\{B_2(\tilde{b})\}}$$

$$(2.90)$$

Similar equations are obtained for the initial value vector of equations (2.53) along the z-directional lines.

Along the y-directional lines the boundary conditions are somewhat more involved since in the crack plane mixed boundary conditions are specified. Denoting the number of y-directional lines following over the crack surface as NIC and those falling outside as NOC, the zero normal stress condition over the crack face and the symmetry condition in the crack plane result in the following equations:

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$$\underset{NIC\times 1}{\{\dot{\tilde{v}}(o)\}_{\substack{over\\crack}}} = \frac{-\nu}{1-\nu}\left(\{\dot{\tilde{u}}\} + \{\dot{\tilde{w}}\}\right)_{\substack{\tilde{y}=o\\ \text{over crack}}}$$

$$\underset{NOC\times 1}{\{\tilde{v}(o)\}_{\substack{outside\\crack}}} = \{0\}_{\substack{\tilde{y}=o\\ \text{outside crack}}} \qquad (2.91)$$

Equations (2.91) specify m elements of the initial value vector $\{V(o)\}$. The value of the rest of the elements in $\{V(o)\}$ can be found from using the given surface stress conditions at $\tilde{y} = \tilde{L}$. Thus, from $\sigma_y\big|_{\tilde{y}=\tilde{L}} = \sigma_o$ we have

$$\underset{m\times 1}{\{\dot{\tilde{v}}(\tilde{L})\}} = \underset{m\times 1}{\left\{\frac{\sigma_o}{E}\,\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\right\}} - \underset{m\times 1}{\frac{\nu}{1-\nu}\left(\{\dot{\tilde{u}}\} + \{\dot{\tilde{w}}\}\right)_{\tilde{y}=\tilde{L}}} \qquad (2.92)$$

This vector can be suitably partitioned into vectors $\underset{NIC\times 1}{\{\dot{\tilde{v}}_\alpha\}_{\tilde{y}=\tilde{L}}}$ and $\underset{NOC\times 1}{\{\dot{\tilde{v}}_\beta\}_{\tilde{y}=\tilde{L}}}$.

For convenience of matrix manipulations, we partition the initial value vector $\{V(o)\}$ as

$$\underset{2m\times 1}{\{V(o)\}} = \left\{\begin{matrix}\{\tilde{v}(o)\}\\ m\times 1\\ \{\dot{\tilde{v}}(o)\}\\ m\times 1\end{matrix}\right\} = \left\{\begin{matrix}\{F_3\}\\ m\times 1\\ \{F_4\}\\ m\times 1\end{matrix}\right\} = \left\{\begin{matrix}\{F_{3\alpha}\}\\ NIC\times 1\\ \{F_{3\beta}\}\\ NOC\times 1\\ \{F_{4\alpha}\}\\ NIC\times 1\\ \{F_{4\beta}\}\\ NOC\times 1\end{matrix}\right\} \qquad (2.93)$$

Values of $\{F_{4\alpha}\}$ and $\{F_{3\beta}\}$ are given by equations (2.91) respective

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$$\begin{Bmatrix} \{V(\tilde{y})\} \\ \\ \{\dot{V}(\tilde{y})\} \end{Bmatrix}$$

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respectively. An analogous solution to equations (2.86) along the y-directional lines can be written as

$$\begin{Bmatrix} \{\tilde{v}(\tilde{y})\} \\ \\ \{\dot{\tilde{v}}(\tilde{y})\} \end{Bmatrix} = \begin{pmatrix} [D_{11}(K_y,\tilde{y})][D_{12}(K_y,\tilde{y})] \\ \\ [D_{21}(K_y,\tilde{y})][D_{22}(K_y,\tilde{y})] \end{pmatrix} \left(\begin{Bmatrix} \{\tilde{v}(o)\} \\ \\ \{\dot{\tilde{v}}(o)\} \end{Bmatrix} + \begin{Bmatrix} \{B_3(\tilde{y})\} \\ \\ \{B_4(\tilde{y})\} \end{Bmatrix} \right) \tag{2.94}$$

where $[D_{ij}]$ are similar matrix functions to $[A_{ij}]$ and $\{B_3\}$ $\{B_4\}$ are analogous particular integrals to (2.84) and (2.85). From equation (2.94), we can express $\{\dot{\tilde{v}}(L)\}$ in a partitioned form consistent with (2.93) as

$$\begin{Bmatrix} \{\dot{\tilde{v}}_\alpha\} \\ NICx1 \\ \{\dot{\tilde{v}}_\beta\} \\ NOCx1 \end{Bmatrix}_{\tilde{y}=\tilde{L}} = \begin{bmatrix} [D_{21\alpha1}] & [D_{21\alpha2}] \\ NICxNIC & NICxNOC \\ [D_{21\beta1}] & [D_{21\beta2}] \\ NOCxNIC & NOCxNOC \end{bmatrix}_{\tilde{y}=\tilde{L}} \left(\begin{Bmatrix} \{F_{3\alpha}\} \\ NICx1 \\ \{F_{3\beta}\} \\ NOCx1 \end{Bmatrix} + \begin{Bmatrix} \{B_{3\alpha}\} \\ NICx1 \\ \{B_{3\beta}\} \\ NOCx1 \end{Bmatrix}_{\tilde{y}=\tilde{L}} \right)$$

$$+ \begin{bmatrix} [D_{22\alpha1}] & [D_{22\alpha2}] \\ NICxNIC & NICxNOC \\ [D_{22\beta1}] & [D_{22\beta2}] \\ NOCxNIC & NOCxNOC \end{bmatrix}_{\tilde{y}=\tilde{L}} \left(\begin{Bmatrix} \{F_{4\alpha}\} \\ NICx1 \\ \{F_{4\beta}\} \\ NOCx1 \end{Bmatrix} \quad \begin{Bmatrix} \{B_{4\alpha}\} \\ NICx1 \\ \{B_{4\beta}\} \\ NOCx1 \end{Bmatrix}_{\tilde{y}=\tilde{L}} \right) \tag{2.95}$$

Equation (2.95) leads to two matrix equations involving the two unknown vectors $\{F_{3\alpha}\}$ and $\{F_{4\beta}\}$. Solution of these equations yields

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$$\underset{NOCx1}{\{F_{4\beta}\}} = \underset{NOCxNOC}{[D_a]^{-1}} \underset{NOCx1}{\{\dot{\tilde{v}}_\beta\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[D_a]^{-1}} \underset{NOCxNIC}{[D_{21\beta 1}]_{\tilde{y}=\tilde{L}}} \underset{NICxNIC}{[D_{21\alpha 1}]_{\tilde{y}=\tilde{L}}} \underset{NICx1}{\{\dot{\tilde{v}}_\alpha\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[D_a]^{-1}} \underset{NOCxNOC}{[D_b]} \underset{NOCx1}{\{B_{3\beta}\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[D_a]^{-1}} \underset{NOCxNIC}{[D_c]} \left(\underset{NICx1}{\{F_{4\alpha}\}} + \underset{NICx1}{\{B_{4\alpha}\}_{\tilde{y}=\tilde{L}}}\right)$$

$$- \underset{NOCx1}{\{B_{4\beta}\}_{\tilde{y}=\tilde{L}}} \tag{2.96}$$

where

$$\underset{NOCxNOC}{[D_a]} = \left(\underset{NOCxNOC}{[D_{22\beta 2}]} - \underset{NOCxNIC}{[D_{21\beta 1}]} \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}} \underset{NICxNOC}{[D_{22\alpha 2}]}\right)_{\tilde{y}=\tilde{L}}$$

$$\underset{NOCxNOC}{[D_b]} = \left(\underset{NOCxNOC}{[D_{21\beta 2}]} - \underset{NOCxNIC}{[D_{21\beta 1}]} \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}} \underset{NICxNOC}{[D_{21\alpha 2}]}\right)_{\tilde{y}=\tilde{L}}$$

$$\underset{NOCxNIC}{[D_c]} = \left(\underset{NOCxNIC}{[D_{22\beta 1}]} - \underset{NOCxNIC}{[D_{21\beta 1}]} \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}} \underset{NICxNIC}{[D_{22\alpha 1}]}\right)_{\tilde{y}=\tilde{L}}$$

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\underset{NICx1}{\{F_{3\alpha}\}} = & \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \underset{NICx1}{\{\dot{\tilde{v}}_\alpha\}_{\tilde{y}=\tilde{L}}} - \underset{NICx1}{\{B_{3\alpha}\}_{\tilde{y}=\tilde{L}}} \\
& - \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \underset{NICxNOC}{[D_{21\alpha 2}]_{\tilde{y}=\tilde{L}}} \underset{NOCx1}{\{B_{3\beta}\}_{\tilde{y}=\tilde{L}}} \\
& - \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \underset{NICxNIC}{[D_{22\alpha 1}]_{\tilde{y}=\tilde{L}}} \left(\underset{NICx1}{\{F_{4\alpha}\}} + \underset{NICx1}{\{B_{4\alpha}\}_{\tilde{y}=\tilde{L}}}\right) \\
& - \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \underset{NICxNOC}{[D_{22\alpha 2}]_{\tilde{y}=\tilde{L}}} \left(\underset{NOCx1}{\{F_{4\beta}\}} + \underset{NOCx1}{\{B_{4\beta}\}_{\tilde{y}=\tilde{L}}}\right) \quad (2.97)
\end{aligned}
$$

where $\{F_{4\beta}\}$ is given by (2.96). Note that although the full matrix $[D_{21}]_{\tilde{y}=\tilde{L}}$ is singular, the partitioned submatrices are not. Equations (2.96) and (2.97) together with the given boundary data completely specify the initial value vector needed for the solution of equations (2.46).

Certain conclusions using the shear stress conditions can also be noted for problems involving zero or uniform normal displacements in a given plane. Using symmetry conditions (2.88), for example, we have

$$
\left.\frac{d\tilde{u}}{d\tilde{z}}\right|_{\substack{\tilde{x}=0 \\ \text{plane}}} = \left.\frac{d\tilde{u}}{d\tilde{y}}\right|_{\substack{\tilde{x}=0 \\ \text{plane}}} = 0 \qquad (2.98)
$$

The zero shear stress conditions on $\tilde{\sigma}_{xy}$ and $\tilde{\sigma}_{xz}$, together with equation (2.98), will lead to

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$$\left.\frac{\partial \tilde{v}}{\partial \tilde{x}}\right|_{\substack{\tilde{x}=0 \\ \text{plane}}} = \left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_{\substack{\tilde{x}=0 \\ \text{plane}}} = 0 \qquad (2.99)$$

Since $\frac{d\dot{\tilde{v}}}{d\tilde{x}} = \frac{d}{d\tilde{y}}\frac{\partial \tilde{v}}{\partial \tilde{x}}$ and $\frac{d\dot{\tilde{w}}}{d\tilde{x}} = \frac{d}{d\tilde{z}}\frac{\partial \tilde{w}}{\partial \tilde{x}}$, we can conclude at once that

$$\underset{\ell \times 1}{\{r(\tilde{x})\}}_{\tilde{x}=0} = 0 \qquad (2.100)$$

Equation (2.100) shows that all the elements of vector $\{r(\tilde{x})\}$ in the plane of zero x-directional displacements are zero. Similar conclusions can also be obtained for the elements of the vectors $\underset{m\times 1}{\{s(\tilde{y})\}}$ and $\underset{n\times 1}{\{t(\tilde{z})\}}$.

There are problems when it is more convenient to apply a given boundary displacement, $\{\tilde{v}(\tilde{L})\}$, rather than a uniform normal stress, $\{\sigma_o(\bar{L})\}$. Although we shall not present detailed results for this case, the analogous boundary equations to equations (2.96) and (2.97) are listed below.

$$\underset{NOC\times 1}{\{F_{4\beta}\}} = \underset{NOC\times NOC}{[\bar{D}_a]^{-1}} \underset{NOC\times 1}{\{\tilde{v}_\beta\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOC\times NOC}{[\bar{D}_a]^{-1}} \underset{NOC\times NIC}{[D_{11\beta 1}]_{\tilde{y}=\tilde{L}}} \underset{NIC\times NIC}{[D_{11\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \underset{NIC\times 1}{\{\tilde{v}_\alpha\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOC\times NOC}{[\bar{D}_a]^{-1}} \underset{NOC\times NOC}{[\bar{D}_b]} \underset{NOC\times 1}{\{B_{3\beta}\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[\bar{D}_a]^{-1}} \; \underset{NOCxNIC}{[\bar{D}_c]} \left(\underset{NICx1}{\{F_{4\alpha}\}} + \underset{NICx1}{\{B_{4\alpha}\}_{\tilde{y}=\tilde{L}}} \right)$$

$$- \underset{NOCx1}{\{B_{4\beta}\}_{\tilde{y}=\tilde{L}}} \tag{2.101}$$

where

$$\underset{NOCxNOC}{[\bar{D}_a]} = \left(\underset{NOCxNOC}{[D_{12\beta 2}]} - \underset{NOCxNIC}{[D_{11\beta 1}]} \; \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}} \; \underset{NICxNOC}{[D_{12\alpha 2}]} \right)_{\tilde{y}=\tilde{L}}$$

$$\underset{NOCxNOC}{[\bar{D}_b]} = \left(\underset{NOCxNOC}{[D_{11\beta 2}]} - \underset{NOCxNIC}{[D_{11\beta 1}]} \; \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}} \; \underset{NICxNOC}{[D_{11\alpha 2}]} \right)_{\tilde{y}=\tilde{L}}$$

$$\underset{NOCxNIC}{[\bar{D}_c]} = \left(\underset{NOCxNIC}{[D_{12\beta 2}]} - \underset{NOCxNIC}{[D_{11\beta 1}]} \; \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}} \; \underset{NICxNIC}{[D_{12\alpha 1}]} \right)_{\tilde{y}=\tilde{L}}$$

$$\underset{NICx1}{\{F_{3\alpha}\}} = \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \; \underset{NICx1}{\{\tilde{v}_\alpha\}_{\tilde{y}=\tilde{L}}} - \underset{NICx1}{\{B_{3\alpha}\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \; \underset{NICxNOC}{[D_{11\alpha 2}]_{\tilde{y}=\tilde{L}}} \; \underset{NOCx1}{\{B_{3\beta}\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \; \underset{NICxNIC}{[D_{12\alpha 1}]_{\tilde{y}=\tilde{L}}} \left(\underset{NICx1}{\{F_{4\alpha}\}} + \underset{NICx1}{\{B_{4\alpha}\}_{\tilde{y}=\tilde{L}}} \right)$$

$$- \underset{NICxNIC}{[D_{11\alpha 1}]^{-1}_{\tilde{y}=\tilde{L}}} \; \underset{NICxNOC}{[D_{12\alpha 2}]_{\tilde{y}=\tilde{L}}} \left(\underset{NOCx1}{\{F_{4\beta}\}} + \underset{NOCx1}{\{B_{4\beta}\}_{\tilde{y}=\tilde{L}}} \right) \tag{2.102}$$

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With these equations and the given boundary data, the initial vector of equations (2.46) can be evaluated for the case of a displacement loaded rectangular bar containing a central crack.

Once the displacement field in the bar has been calculated and the successive approximation procedure has converged, the normal stress distributions along the sets of parallel lines can be obtained from the following equations:

$$\underset{\ell \times 1}{\{\sigma_x\}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \underset{\ell \times 1}{\{\dot{\tilde{u}}\}}_{\text{along x lines}} + \frac{\nu E}{(1+\nu)(1-2\nu)} \left(\underset{\ell \times 1}{\{\dot{\tilde{v}}\}} + \underset{\ell \times 1}{\{\dot{\tilde{w}}\}} \right)_{\text{along x lines}} \quad (2.103)$$

$$\underset{m \times 1}{\{\sigma_y\}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \underset{m \times 1}{\{\dot{\tilde{v}}\}}_{\text{along y lines}} + \frac{\nu E}{(1+\nu)(1-2\nu)} \left(\underset{m \times 1}{\{\dot{\tilde{u}}\}} + \underset{m \times 1}{\{\dot{\tilde{w}}\}} \right)_{\text{along y lines}} \quad (2.104)$$

$$\underset{n \times 1}{\{\sigma_z\}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \underset{n \times 1}{\{\dot{\tilde{w}}\}}_{\text{along z lines}} + \frac{\nu E}{(1+\nu)(1-2\nu)} \left(\underset{n \times 1}{\{\dot{\tilde{u}}\}} + \underset{n \times 1}{\{\dot{\tilde{v}}\}} \right)_{\text{along z lines}} \quad (2.105)$$

Note that the above equations involve only derivatives that can be evaluat

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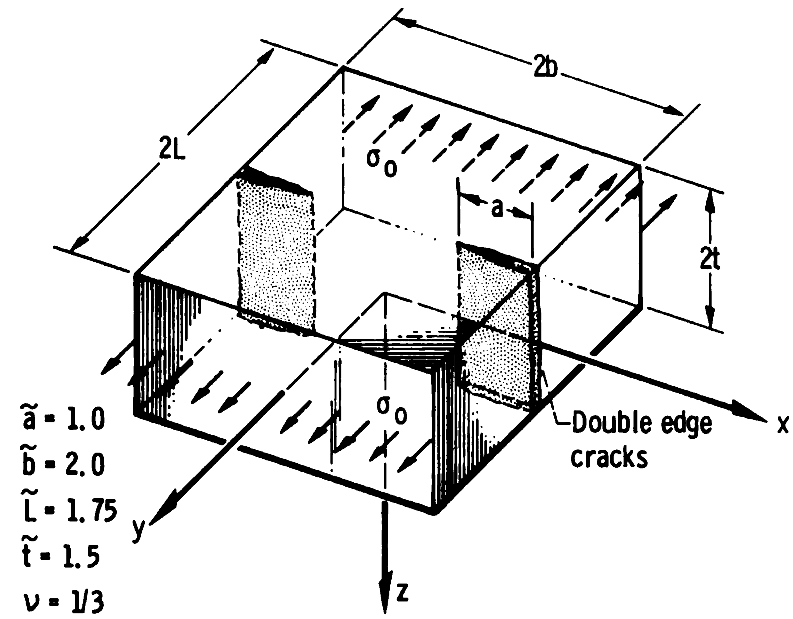
be evaluated in closed form. Hence, we expect that the normal stress boundary conditions will be accurately enforced. The shear stresses at each node can be obtained from applying equations (2.13) through (2.15). These equations, however, involve derivatives that can only be evaluated through the use of finite difference calculus. In general, this presents no important loss of accuracy since values of the shear stresses are an order of magnitude smaller than the normal stresses (25). This same conclusion was obtained when the shear stresses in our examples were investigated. Numerical results for the problem of Figure 2(a) are listed and discussed in Chapter 4.

2.4.2 Bar With Through-Thickness Double Edge Cracks

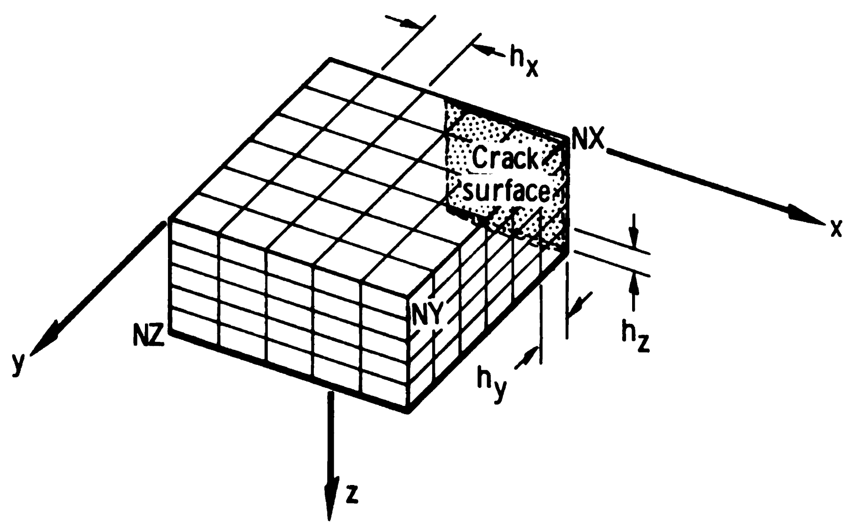
A problem closely related to the previously described central crack solution is that of a rectangular bar with through-thickness double edge cracks. The configuration and applied loading of this problem are shown in Figure 3. For the geometry shown, similar conclusions can be drawn about symmetry conditions and validity of previous solutions as for the central crack problem. The non-dimensionalized variables of the bar in Figure 3(a) are made identical to those of Figure 2(a) so that comparison between the two solutions will be possible.

Most of the equations and their boundary conditions for this problem are identical to those of the central crack configuration. The only difference is in the mixed boundary conditions

Figure 3. -



(a) Rectangular bar with through-thickness double edge cracks.



(b) Discretized region of rectangular bar with double edge cracks.

Figure 3. - Rectangular bar with through-thickness double edge cracks under uniform tension.

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of the crack plane. Equations for the initial value vector $\{V(o)\}$, in the partitioned form of (2.93), are listed below.

$$\underset{NOCx1}{\{F_{3\alpha}\}} = 0$$

$$\underset{NICx1}{\{F_{4\beta}\}} = -\frac{\nu}{1-\nu}\,(\underset{NICx1}{\{\tilde{u}\}} + \underset{NICx1}{\{\tilde{w}\}})_{\tilde{y}=0} \text{ over crack} \qquad (2.106)$$

Note that these given boundary conditions are the same as those in (2.91) except that the partitioning subscripts α and β have been interchanged. Using the condition of $\sigma_y\big|_{\tilde{y}=\tilde{L}} = \sigma_o$ we obtain

$$\underset{NOCx1}{\{F_{4\alpha}\}} = \underset{NOCxNOC}{[\bar{\bar{D}}_a]^{-1}}\ \underset{NOCx1}{\{\mathring{v}_\alpha\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[\bar{\bar{D}}_a]^{-1}}\ \underset{NOCxNIC}{[D_{21\alpha2}]_{\tilde{y}=\tilde{L}}}\ \underset{NICxNIC}{[D_{21\beta2}]_{\tilde{y}=\tilde{L}}}\ \underset{NICx1}{\{\mathring{v}_\beta\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[\bar{\bar{D}}_a]^{-1}}\ \underset{NOCxNOC}{[\bar{\bar{D}}_b]}\ \underset{NOCx1}{\{B_{3\alpha}\}_{\tilde{y}=\tilde{L}}}$$

$$- \underset{NOCxNOC}{[\bar{\bar{D}}_a]^{-1}}\ \underset{NOCxNIC}{[\bar{\bar{D}}_c]}\ \Big(\underset{NICx1}{\{F_{4\beta}\}} + \underset{NICx1}{\{B_{4\beta}\}_{\tilde{y}=\tilde{L}}}\Big)$$

$$- \underset{NOCx1}{\{B_{4\alpha}\}_{\tilde{y}=\tilde{L}}} \qquad (2.107)$$

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$$\underset{NOCxNOC}{[\overline{\overline{D}}_a]} = \Big(\underset{NOCxNOC}{[D_{22\alpha 1}]} - \underset{NOCxNIC}{[D_{21\alpha 2}]}\ \underset{NICxNIC}{[D_{21\beta 2}]^{-1}}\ \underset{NICxNOC}{[D_{22\beta 1}]}\Big)_{\tilde{y}=\tilde{L}}$$

$$\underset{NOCxNOC}{[\overline{\overline{D}}_b]} = \Big(\underset{NOCxNOC}{[D_{21\alpha 1}]} - \underset{NOCxNIC}{[D_{21\alpha 2}]}\ \underset{NICxNIC}{[D_{21\beta 2}]^{-1}}\ \underset{NICxNOC}{[D_{21\beta 1}]}\Big)_{\tilde{y}=\tilde{L}}$$

$$\underset{NOCxNIC}{[\overline{\overline{D}}_c]} = \Big(\underset{NOCxNIC}{[D_{22\alpha 2}]} - \underset{NOCxNIC}{[D_{21\alpha 2}]}\ \underset{NICxNIC}{[D_{21\beta 2}]^{-1}}\ \underset{NICxNIC}{[D_{22\beta 2}]}\Big)_{\tilde{y}=\tilde{L}}$$

$$\begin{aligned}\underset{NICx1}{\{F_{3\beta}\}} = {} & \underset{NICxNIC}{[D_{21\beta 2}]^{-1}_{\tilde{y}=\tilde{L}}}\ \underset{NICx1}{\{\dot{\tilde{v}}_\beta\}_{\tilde{y}=\tilde{L}}} \\ & - \underset{NICxNIC}{[D_{21\beta 2}]^{-1}_{\tilde{y}=\tilde{L}}}\ \underset{NICxNOC}{[D_{21\beta 1}]_{\tilde{y}=\tilde{L}}}\ \underset{NOCx1}{\{B_{3\alpha}\}_{\tilde{y}=\tilde{L}}} - \underset{NICx1}{\{B_{3\beta}\}_{\tilde{y}=\tilde{L}}} \\ & - \underset{NICxNIC}{[D_{21\beta 2}]^{-1}_{\tilde{y}=\tilde{L}}}\ \underset{NICxNOC}{[D_{22\beta 1}]_{\tilde{y}=\tilde{L}}}\ \Big(\underset{NOCx1}{\{F_{4\alpha}\}} + \underset{NOCx1}{\{B_{4\alpha}\}_{\tilde{y}=\tilde{L}}}\Big) \\ & - \underset{NICxNIC}{[D_{21\beta 2}]^{-1}_{\tilde{y}=\tilde{L}}}\ \underset{NICxNIC}{[D_{22\beta 2}]_{\tilde{y}=\tilde{L}}}\ \Big(\underset{NICx1}{\{F_{4\beta}\}} + \underset{NICx1}{\{B_{4\beta}\}_{\tilde{y}=\tilde{L}}}\Big) \qquad (2.108)\end{aligned}$$

Similar equations can be derived for a displacement loaded double edge crack bar by using the given displacements rather than their derivatives at the $\tilde{y} = \tilde{L}$ plane. The details of that problem are not considered in this paper. Numerical results for the problem of

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problem of Figure 3(a) are presented and discussed in Chapter IV along with the solutions of the other examples.

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CHAPTER 3

SOLUTION OF THE NAVIER-CAUCHY EQUATIONS IN CYLINDRICAL COORDINATES BY THE METHOD OF LINES

3.1 Governing Equations

The discretization technique presented in the previous chapter cannot be readily extended to problems having circular boundaries. For problems of this geometry, it is more convenient to formulate the field equations and associated boundary conditions in cylindrical coordinates. In this coordinate system, the Navier-Cauchy equations (2.4) can be written as follows:

$$\frac{\partial e_c}{\partial r} + (1-2\nu)\left[\left(\nabla_c^2 - \frac{1}{r^2}\right)u - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right] = 0 \qquad (3.1)$$

$$\frac{1}{r}\frac{\partial e_c}{\partial \theta} + (1-2\nu)\left[\left(\nabla_c^2 - \frac{1}{r^2}\right)v + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right] = 0 \qquad (3.2)$$

$$\frac{\partial e_c}{\partial z} + (1-2\nu)\,\nabla_c^2 w = 0 \qquad (3.3)$$

where the dilatation, e_c, is given by

$$e_c = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{\partial w}{\partial z} \qquad (3.4)$$

and the Laplacian, ∇_c^2, is

$$\nabla_c^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{3.5}$$

The stress-displacement relations, obtained by substituting the strain-displacement relations into Hooke's law, can be written in the following form:

$$\sigma_r = \lambda e_c + 2G\frac{\partial u}{\partial r} \tag{3.6}$$

$$\sigma_\theta = \lambda e_c + 2G\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) \tag{3.7}$$

$$\sigma_z = \lambda e_c + 2G\frac{\partial w}{\partial z} \tag{3.8}$$

$$\sigma_{r\theta} = G\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right) \tag{3.9}$$

$$\sigma_{rz} = G\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \tag{3.10}$$

$$\sigma_{\theta z} = G\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{3.11}$$

Solution of these equations can again be obtained by using the line method together with the successive approximation procedure and the applicable boundary conditions.

3.2 Ordinary Differential Equations and Boundary Conditions in the Radial Direction

Following the line method as discussed in the previous chapter, we construct three sets of lines in the direction of the cylindrical coordinate axes. An arbitrary cylindrical grid consisting of these three sets of lines is shown in Figure 4. The numbering of the lines is analogous to that shown in Figure

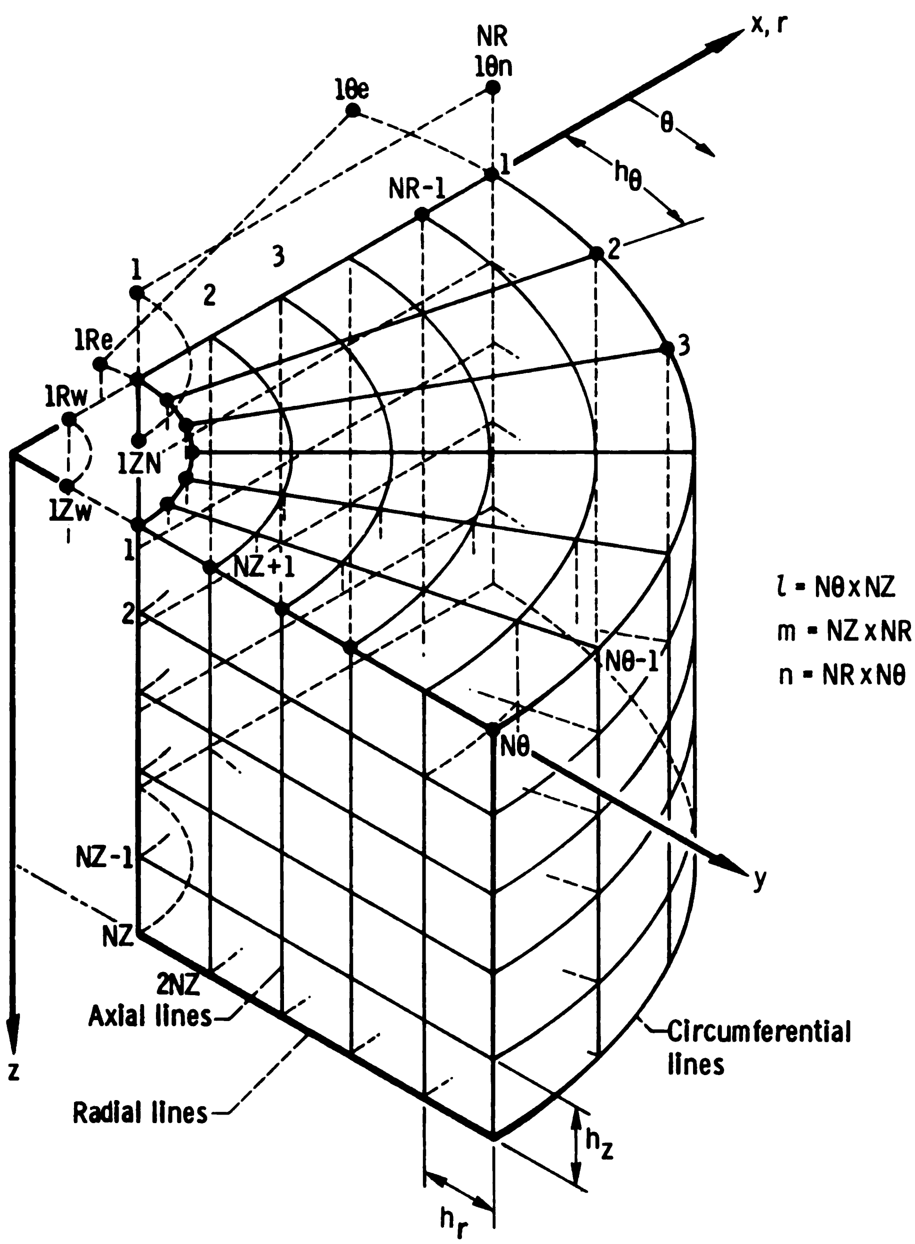

Figure 4. - Sets of lines in the direction of cylindrical coordinates.

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1. For convenience, the lines are evenly spaced with h_r, h_θ and h_z each equal to some given constant. The advantage of a more easily calculated solution with even line spacing is not obtained in fully three-dimensional cylindrical coordinates because some of the resulting differential equations have variable coefficients. In addition, a closed form solution of the eigenvalue problem for the remaining equations is also impractical. The previously discussed limitations on the validity of particular solutions also apply to problems in cylindrical coordinates.

For the solution of equation (3.1), the radial lines of Figure 4 must be utilized. The radial displacement of points along these lines will be denoted as $u_1, u_2, \ldots, u_\ell$. We define $\dot{v}|_1, \dot{v}|_2, \ldots, \dot{v}|_\ell$ now as the derivatives of the circumferential displacements of the same points on these lines with respect to θ and $\dot{w}|_1, \dot{w}|_2, \ldots, \dot{w}|_\ell$ as the derivatives of the axial displacements of the same points on these lines with respect to z. These displacements and derivatives can then be regarded as functions of the radius only. From equations (3.1), (3.4) and (3.5), the following equation is obtained along the first radial line:

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$$\frac{d^2u_1}{dr^2} + \frac{1}{r}\frac{d\dot{v}}{dr}\bigg|_1 - \frac{1}{r^2}\dot{v}\bigg|_1 + \frac{1}{r}\frac{du_1}{dr} - \frac{u_1}{r^2} + \frac{d\dot{w}}{dr} + (1-2\nu)\Bigg[\frac{d^2u_1}{dr^2}$$

$$+ \frac{1}{r}\frac{du_1}{dr} + \frac{1}{r^2}\frac{\partial^2 u_1}{\partial\theta^2} + \frac{\partial^2 u_1}{\partial z^2} - \frac{u_1}{r^2} - \frac{2}{r^2}\dot{v}\bigg|_1\Bigg] = 0 \tag{3.12}$$

where, by using finite difference calculus, we have

$$\frac{\partial^2 u_1}{\partial\theta^2} = \frac{1}{h_\theta^2}(u_2 - 2u_1 + u_{1\theta e}) \tag{3.13}$$

$$\frac{\partial^2 u_1}{\partial z^2} = \frac{1}{h_z^2}(u_{N\theta+1} - 2u_1 + u_{1\theta n}) \tag{3.14}$$

The use of zero shear stress boundary conditions in the radial direction on the r-z and r-θ coordinate planes gives respectively

$$u_{1\theta e} = u_2 + 2h_\theta r\frac{dv}{dr}\bigg|_1 - 2h_\theta v\bigg|_1 \tag{3.15}$$

$$u_{1\theta n} = u_{N\theta+1} + 2h_z\frac{dw}{dr}\bigg|_1 \tag{3.16}$$

Substituting these equations into equation (3.12) leads to the following ordinary differential equation:

$$\left[\frac{d^2u_1}{dr^2} + \frac{1}{r}\frac{du_1}{dr} - \frac{u_1}{r^2}\right] + \frac{(1-2\nu)}{2(1-\nu)}\left[-\left(\frac{2}{r^2h_\theta^2} + \frac{2}{h_z^2}\right)u_1 + \frac{2u_2}{r^2h_\theta^2} + \frac{2u_{N\theta+1}}{h_z^2}\right] + \frac{f_1(r)}{2(1-\nu)} = 0 \tag{3.17}$$

where

$$f_1(r) = \frac{(4\nu-3)}{r^2}\dot{v}\Big|_1 + \frac{1}{r}\frac{d\dot{v}}{dr}\Big|_1 + \frac{d\dot{w}}{dr}\Big|_1 + (1-2\nu)\left[\frac{2}{rh_\theta}\frac{dv}{dr} - \frac{2}{r^2h_\theta}v + \frac{2}{h_z}\frac{dw}{dr}\right]_1 \tag{3.18}$$

Similar differential equations are obtained for the displacements u_i of the points on the other radial lines. Since each equation contains the displacements of the surrounding lines, these equations constitute a system of ordinary differential equations. Noting that equation (3.17) was derived for a corner line, the form of the equations for interior lines and surface lines will differ according to the application of known shear stress conditions.

It will be convenient to non-dimensionalize equations (3.17) and (3.18) with respect to some characteristic dimension. For the penny shaped crack problems, which are to be discussed later in this report, the same variables can be used as in (2.28) with the following modifications:

$$\tilde{r} = \frac{r}{a} \qquad \tilde{h}_r = \frac{h_r}{a}$$
$$\tilde{\theta} = \theta \qquad \tilde{h}_\theta = h_\theta \tag{3.19}$$

Introducing matrix notation, the differential equations along the radial lines can be expressed in the form shown below.

$$\underset{\ell\times 1}{\left(\frac{d^2}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}} - \frac{1}{\tilde{r}^2}\right)\{\tilde{u}\}} = \underset{\ell\times\ell}{[K_r(\tilde{r})]}\ \underset{\ell\times 1}{\{\tilde{u}\}} + \underset{\ell\times 1}{\{r(\tilde{r})\}} \tag{3.20}$$

where the coefficient matrix $[K_r(\tilde{r})]$ and the column vectors $\{\tilde{u}\}$ and $\{r(\tilde{r})\}$ are given below.

$$\underset{\ell\times\ell}{[K_r(\tilde{r})]} = \left[\begin{array}{c|c|c|c|c}
\underset{N\theta\times N\theta}{[K_{r1}]} & \underset{N\theta\times N\theta}{2[K_{r2}]} & 0 & 0 & 0 \\ \hline
\underset{N\theta\times N\theta}{[K_{r2}]} & \underset{N\theta\times N\theta}{[K_{r1}]} & \underset{N\theta\times N\theta}{[K_{r2}]} & 0 & 0 \\ \hline
0 & \ddots & \ddots & \ddots & 0 \\ \hline
0 & 0 & \underset{N\theta\times N\theta}{[K_{r2}]} & \underset{N\theta\times N\theta}{[K_{r1}]} & \underset{N\theta\times N\theta}{[K_{r2}]} \\ \hline
0 & 0 & 0 & \underset{N\theta\times N\theta}{2[K_{r2}]} & \underset{N\theta\times N\theta}{[K_{r1}]}
\end{array}\right] \tag{3.21}$$

where the

$[K_{r1}(\bar{r})]$

$N6xN6$

$[K_{r2}]$ =

$N6xN6$

where the submatrices $[K_{r1}]$ and $[K_{r2}]$ are

$$
\underset{N\theta \times N\theta}{[K_{r1}(\tilde{r})]} = \begin{bmatrix} k_7 & -2k_8 & 0 & 0 & 0 \\ -k_8 & k_7 & -k_8 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_8 & k_7 & -k_8 \\ 0 & 0 & 0 & -2k_8 & k_7 \end{bmatrix}
$$

$$
k_7 = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2}{\tilde{r}^2 \tilde{h}_\theta^2} + \frac{2}{\tilde{h}_z^2} \right]
$$

$$
k_8 = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{1}{\tilde{r}^2 \tilde{h}_\theta^2} \right]
$$

$$
\underset{N\theta \times N\theta}{[K_{r2}]} = \begin{bmatrix} -k_3 & 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & 0 & -k_3 \end{bmatrix}
$$

$$
k_3 = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{1}{\tilde{h}_z^2} \right]
$$

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Note that the coefficient matrix $[K_r(\tilde{r})]$ is a function of the radius and since the sum of all the elements in any given row is zero, it is also singular. The column vectors are written as

$$\underset{\ell x1}{\{\tilde{u}\}} = \begin{Bmatrix} \{\tilde{u}\}_1 \\ \{\tilde{u}\}_2 \\ \vdots \\ \{\tilde{u}\}_{NZ-1} \\ \{\tilde{u}\}_{NZ} \end{Bmatrix} \qquad \underset{\ell x1}{\{r(\tilde{r})\}} = \begin{Bmatrix} \{f(\tilde{r})\}_1 \\ \{f(\tilde{r})\}_2 \\ \vdots \\ \{f(\tilde{r})\}_{NZ-1} \\ \{f(\tilde{r})\}_{NZ} \end{Bmatrix} \tag{3.22}$$

where the partitioned column vectors of $\{\tilde{u}\}$ are the same as those in the previous chapter except that they are of order $N\theta x1$.

The partitioned vectors $\{f(\tilde{r})\}_i$ $i = 1, 2, \ldots, NZ$, are given by

$$
\underset{N\theta\times 1}{\{f(\tilde{r})\}_1} = \frac{-1}{2(1-\nu)} \left\{ \begin{matrix}
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_1 + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_1 + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_1 + (1-2\nu)\left[\dfrac{2}{\tilde{r}\tilde{h}_\theta}\dfrac{d\tilde{v}}{d\tilde{r}} - \dfrac{2}{\tilde{r}^2\tilde{h}_\theta}\tilde{v} + \dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{r}}\right]_1 \\
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_2 + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_2 + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_2 + (1-2\nu)\left[\dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{r}}\right]_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_{N\theta-1} + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_{N\theta-1} + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_{N\theta-1} + (1-2\nu)\left[\dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{r}}\right]_{N\theta-1} \\
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_{N\theta} + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_{N\theta} + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_{N\theta} + (1-2\nu)\left[\dfrac{-2}{\tilde{r}\tilde{h}_\theta}\dfrac{d\tilde{v}}{d\tilde{r}} + \dfrac{2\tilde{v}}{\tilde{r}^2\tilde{h}_\theta} + \dfrac{2}{\tilde{h}_z}\dfrac{d\tilde{w}}{d\tilde{r}}\right]_{N\theta}
\end{matrix} \right\}
$$

$$
\underset{N\theta\times 1}{\{f(\tilde{r})\}_2} = \frac{-1}{2(1-\nu)} \left\{ \begin{matrix}
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_{N\theta+1} + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_{N\theta+1} + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_{N\theta+1} + (1-2\nu)\left[\dfrac{2}{\tilde{r}\tilde{h}_\theta}\dfrac{d\tilde{v}}{d\tilde{r}} - \dfrac{2}{\tilde{r}^2\tilde{h}_\theta}\tilde{v}\right]_{N\theta+1} \\
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_{N\theta+2} + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_{N\theta+2} + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_{N\theta+2} \\
\vdots \qquad \vdots \qquad \vdots \\
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_{2N\theta-1} + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_{2N\theta-1} + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_{2N\theta-1} \\
\dfrac{(4\nu-3)}{\tilde{r}^2}\,\dot{\tilde{v}}\Big|_{2N\theta} + \dfrac{1}{\tilde{r}}\dfrac{d\dot{\tilde{v}}}{d\tilde{r}}\Big|_{2N\theta} + \dfrac{d\dot{\tilde{w}}}{d\tilde{r}}\Big|_{2N\theta} + (1-2\nu)\left[\dfrac{-2}{\tilde{r}\tilde{h}_\theta}\dfrac{d\tilde{v}}{d\tilde{r}} + \dfrac{2\tilde{v}}{\tilde{r}^2\tilde{h}_\theta}\right]_{2N\theta}
\end{matrix} \right\}
$$

$$\underset{N\theta\times 1}{\{f(\tilde r)\}_{NZ-1}} = \frac{-1}{2(1-\nu)} \left\{ \begin{array}{l} \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-2N\theta+1} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-2N\theta+1} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-2N\theta+1} + (1-2\nu)\left[\frac{2}{\tilde r\tilde h_\theta}\frac{d\tilde v}{d\tilde r} - \frac{2}{\tilde r^2\tilde h_\theta}\tilde v\right]_{\ell-2N\theta+1} \\ \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-2N\theta+2} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-2N\theta+2} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-2N\theta+2} \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-N\theta-1} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-N\theta-1} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-N\theta-1} \\ \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-N\theta} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-N\theta} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-N\theta} + (1-2\nu)\left[\frac{-2}{\tilde r\tilde h_\theta}\frac{d\tilde v}{d\tilde r} + \frac{2}{\tilde r^2\tilde h_\theta}\tilde v\right]_{\ell-N\theta} \end{array} \right\}$$

$$\underset{N\theta\times 1}{\{f(\tilde r)\}_{NZ}} = \frac{-1}{2(1-\nu)} \left\{ \begin{array}{l} \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-N\theta+1} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-N\theta+1} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-N\theta+1} + (1-2\nu)\left[\frac{2}{\tilde r\tilde h_\theta}\frac{d\tilde v}{d\tilde r} - \frac{2}{\tilde r^2\tilde h_\theta}\tilde v - \frac{2}{\tilde h_z}\frac{d\tilde w}{d\tilde r}\right]_{\ell-N\theta+1} \\ \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-N\theta+2} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-N\theta+2} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-N\theta+2} + (1-2\nu)\left[-\frac{2}{\tilde h_z}\frac{d\tilde w}{d\tilde r}\right]_{\ell-N\theta+2} \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell-1} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell-1} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell-1} + (1-2\nu)\left[-\frac{2}{\tilde h_z}\frac{d\tilde w}{d\tilde r}\right]_{\ell-1} \\ \frac{(4\nu-3)}{\tilde r^2}\dot{\tilde v}\Big|_{\ell} + \frac{1}{\tilde r}\frac{d\dot{\tilde v}}{d\tilde r}\Big|_{\ell} + \frac{d\dot{\tilde w}}{d\tilde r}\Big|_{\ell} + (1-2\nu)\left[\frac{2}{\tilde r^2\tilde h_\theta}\tilde v - \frac{2}{\tilde r\tilde h_\theta}\frac{d\tilde v}{d\tilde r} - \frac{2}{\tilde h_z}\frac{d\tilde w}{d\tilde r}\right]_{\ell} \end{array} \right\}$$

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Assuming that $\{r(\tilde{r})\}$ is known, solutions of equations (3.20) can be obtained in closed form using the given boundary data at the end points of the radial lines.

3.3 Ordinary Differential Equations and Boundary Conditions in the Circumferential Direction

For the solution of equation (3.2), ordinary differential equations are developed along the circumferential lines of Figure 4. The displacements along these lines will be denoted as $v_1, v_2, \ldots, v_m$. We define $\mathring{u}|_1, \mathring{u}|_2, \ldots, \mathring{u}|_m$ as the derivatives of the radial displacements of points on these lines with respect to r, and $\dot{w}|_1, \hat{w}|_2, \ldots, \hat{w}|_m$ as the derivatives of the axial displacements of the same points on these lines with respect to z. $\tilde{u}$ and $\tilde{w}$ are defined, of course, as the radial and axial displacements respectively. These displacements and derivatives can then be regarded as functions of θ only, since they are variables along circumferential lines. Following a similar procedure to that used in the previous section, the set of differential equations obtained along these lines is listed below.

$$\underset{m\times 1}{\frac{d^2}{d\tilde{\theta}^2}\{\tilde{v}\}} = \underset{m\times m}{[K_\theta]}\ \underset{m\times 1}{\{\tilde{v}\}} + \underset{m\times 1}{\{s(\tilde{\theta})\}} \tag{3.23}$$

We note that for equations (3.23), the shear stress boundary conditions in the θ direction were utilized. An inspection of

Figure 4

lines, th

$\{s(\tilde{\theta})\}$ b

we define

NR. Then

$\{s(\bar{\theta})\}$ c

$[K_6]$ =
$m \times m$

where the

and i,

Figure 4 shows that for any given subset of NZ circumferential lines, the radius is a constant. In order that $[K_\theta]$ and $\{s(\tilde{\theta})\}$ be expressed in a form similar to (3.21) and (3.22), we define the radius for each subset as r_i $i = 1, 2, 3, \ldots,$ NR. Then the coefficient matrix $[K_\theta]$ and the vectors $\{\tilde{v}\}$ and $\{s(\tilde{\theta})\}$ can be written as follows:

$$\underset{m\times m}{[K_\theta]} = \left[\begin{array}{c|c|c|c|c}
\underset{NZ\times NZ}{[K_{\theta 1}(\tilde{r}_1)]} & \underset{NZ\times NZ}{[K_{\theta 4}(\tilde{r}_1)]} & 0 & 0 & 0 \\ \hline
\underset{NZ\times NZ}{[K_{\theta 5}(\tilde{r}_i)]} & \underset{NZ\times NZ}{[K_{\theta 2}(\tilde{r}_i)]} & \underset{NZ\times NZ}{[K_{\theta 6}(\tilde{r}_i)]} & 0 & 0 \\ \hline
0 & \ddots & \ddots & \ddots & 0 \\ \hline
0 & 0 & \underset{NZ\times NZ}{[K_{\theta 5}(\tilde{r}_i)]} & \underset{NZ\times NZ}{[K_{\theta 2}(\tilde{r}_i)]} & \underset{NZ\times NZ}{[K_{\theta 6}(\tilde{r}_i)]} \\ \hline
0 & 0 & 0 & \underset{NZ\times NZ}{[K_{\theta 4}(\tilde{r}_{NR})]} & \underset{NZ\times NZ}{[K_{\theta 3}(\tilde{r}_{NR})]}
\end{array}\right] \tag{3.24}$$

where the submatrices $[K_{\theta 1}]$ through $[K_{\theta 6}]$ are expressed below and i, as used above, varies from 2 to NR-1.

$[K_{\theta 1}(\bar{r}_1)]$

$NZ \times NZ$

$[K_{\theta 2}(\bar{r}_i)]$

$NZ \times NZ$

$$
\underset{NZxNZ}{[K_{\theta 1}(\tilde{r}_1)]} = \begin{bmatrix} k_9 & -2k_{10} & 0 & 0 & 0 \\ -k_{10} & k_9 & -k_{10} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_{10} & k_9 & -k_{10} \\ 0 & 0 & 0 & -2k_{10} & k_9 \end{bmatrix}
$$

$$
k_9 = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2\tilde{r}_1^2}{\tilde{h}_r^2} + \frac{2\tilde{r}_1^2}{\tilde{h}_z^2} + \frac{2\tilde{r}_1}{\tilde{h}_r} \right]
$$

$$
k_{10} = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{\tilde{r}_1^2}{\tilde{h}_z^2} \right]
$$

$$
\underset{NZxNZ}{[K_{\theta 2}(\tilde{r}_i)]} = \begin{bmatrix} k_{11}(\tilde{r}_i) & -2k_{12}(\tilde{r}_i) & 0 & 0 & 0 \\ -k_{12}(\tilde{r}_i) & k_{11}(\tilde{r}_i) & -k_{12}(\tilde{r}_i) & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_{12}(\tilde{r}_i) & k_{11}(\tilde{r}_i) & -k_{12}(\tilde{r}_i) \\ 0 & 0 & 0 & -2k_{12}(\tilde{r}_i) & k_{11}(\tilde{r}_i) \end{bmatrix}
$$

$$
k_{11}(\tilde{r}_i) = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2\tilde{r}_i^2}{\tilde{h}_r^2} + \frac{2\tilde{r}_i^2}{\tilde{h}_z^2} + 1 \right]
$$

$$
k_{12}(\tilde{r}_i) = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{\tilde{r}_i^2}{\tilde{h}_z^2} \right]
$$

$[K_{83}(\bar{r}_{NB})]$

NZxNZ

$[K_{84}(\bar{r}_j)]$

NZxNZ

$$[K_{\theta 3}(\tilde{r}_{NR})]_{NZxNZ} = \begin{bmatrix} k_{13} & -2k_{14} & 0 & 0 & 0 \\ -k_{14} & k_{13} & -k_{14} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_{14} & k_{13} & -k_{14} \\ 0 & 0 & 0 & -2k_{14} & k_{13} \end{bmatrix}$$

$$k_{13} = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2\tilde{r}_{NR}^2}{\tilde{h}_r^2} + \frac{2\tilde{r}_{NR}^2}{\tilde{h}_z^2} - \frac{2\tilde{r}_{NR}}{\tilde{h}_r} \right]$$

$$k_{14} = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{\tilde{r}_{NR}^2}{\tilde{h}_z^2} \right]$$

$$[K_{\theta 4}(\tilde{r}_j)]_{NZxNZ} = \begin{bmatrix} -k_{15}(\tilde{r}_j) & 0 & 0 & 0 & 0 \\ 0 & -k_{15}(\tilde{r}_j) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_{15}(\tilde{r}_j) & 0 \\ 0 & 0 & 0 & 0 & -k_{15}(\tilde{r}_j) \end{bmatrix}$$

$$k_{15}(\tilde{r}_j) = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2\tilde{r}_j^2}{\tilde{h}_r^2} \right]$$

j = 1 and NR only

$[K_{gg}(\bar{F}_i)]$

NZxNZ

$[K_{gg}\,\bar{F}_i)$

NZxNZ

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equations

$$[K_{\theta 5}(\tilde{r}_i)]_{NZ\times NZ} = \begin{bmatrix} -k_{16}(\tilde{r}_i) & 0 & 0 & 0 & 0 \\ 0 & -k_{16}(\tilde{r}_i) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_{16}(\tilde{r}_i) & 0 \\ 0 & 0 & 0 & 0 & -k_{16}(\tilde{r}_i) \end{bmatrix}$$

$$k_{16}(\tilde{r}_i) = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{\tilde{r}_i^2}{\tilde{h}_r^2} - \frac{\tilde{r}_i}{2\tilde{h}_r} \right]$$

$$[K_{\theta 6}(\tilde{r}_i)]_{NZ\times NZ} = \begin{bmatrix} -k_{17}(\tilde{r}_i) & 0 & 0 & 0 & 0 \\ 0 & -k_{17}(\tilde{r}_i) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -k_{17}(\tilde{r}_i) & 0 \\ 0 & 0 & 0 & 0 & -k_{17}(\tilde{r}_i) \end{bmatrix}$$

$$k_{17}(\tilde{r}_i) = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{\tilde{r}_i^2}{\tilde{h}_r^2} + \frac{\tilde{r}_i}{2\tilde{h}_r} \right]$$

Note that one set of submatrices $[K_{\theta 2}]$, $[K_{\theta 5}]$ and $[K_{\theta 6}]$ is constructed for each value of the radius and then they are assembled according to equation (3.24).

The displacement vector $\{\bar{v}\}_{m\times 1}$ can be written as in equation (2.48) with the exception of replacing NX by NR in those equations. The vector $\{\tilde{s}(\theta)\}$ becomes

$\{s(\ldots$

where the

$$\{s(\tilde{\theta})\} = \begin{Bmatrix} \{f(\tilde{\theta})\}_1 \\ \{f(\tilde{\theta})\}_2 \\ \vdots \\ \{f(\tilde{\theta})\}_{NR-1} \\ \{f(\tilde{\theta})\}_{NR} \end{Bmatrix} \tag{3.25}$$

where the partitioned column vectors are

$$
\underset{NZ\times 1}{\{f(\tilde\theta)\}_1} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_1 + \tilde r_1 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_1 + \tilde r_1 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_1 + (1-2\nu)\left[\frac{2\tilde r_1}{\tilde h_r}\frac{d\tilde u}{d\tilde\theta} - \frac{d\tilde u}{d\tilde\theta} + \frac{2\tilde r_1}{\tilde h_z}\frac{d\tilde w}{d\tilde\theta}\right]_1 \\
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_2 + \tilde r_1 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_2 + \tilde r_1 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_2 + (1-2\nu)\left[\frac{2\tilde r_1}{\tilde h_r}\frac{d\tilde u}{d\tilde\theta} - \frac{d\tilde u}{d\tilde\theta}\right]_2 \\
\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_{NZ-1} + \tilde r_1 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_{NZ-1} + \tilde r_1 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_{NZ-1} + (1-2\nu)\left[\frac{2\tilde r_1}{\tilde h_r}\frac{d\tilde u}{d\tilde\theta} - \frac{d\tilde u}{d\tilde\theta}\right]_{NZ-1} \\
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_{NZ} + \tilde r_1 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_{NZ} + \tilde r_1 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_{NZ} + (1-2\nu)\left[\frac{2\tilde r_1}{\tilde h_r}\frac{d\tilde u}{d\tilde\theta} - \frac{d\tilde u}{d\tilde\theta} - \frac{2\tilde r_1}{\tilde h_z}\frac{d\tilde w}{d\tilde\theta}\right]_{NZ}
\end{array}
\right\}
$$

$$
\{f(\tilde\theta)\}_2 = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_{NZ+1} + \tilde r_2 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_{NZ+1} + \tilde r_2 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_{NZ+1} + (1-2\nu)\left[\frac{2\tilde r_2}{\tilde h_z}\frac{d\tilde w}{d\tilde\theta}\right]_{NZ+1} \\
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_{NZ+2} + \tilde r_2 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_{NZ+2} + \tilde r_2 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_{NZ+2} \\
\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_{2NZ-1} + \tilde r_2 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_{2NZ-1} + \tilde r_2 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_{2NZ-1} \\
(3-4\nu)\left.\frac{d\tilde u}{d\tilde\theta}\right|_{2NZ} + \tilde r_2 \left.\frac{d\dot{\tilde u}}{d\tilde\theta}\right|_{2NZ} + \tilde r_2 \left.\frac{d\dot{\tilde w}}{d\tilde\theta}\right|_{2NZ} + (1-2\nu)\left[\frac{-2\tilde r_2}{\tilde h_z}\frac{d\tilde w}{d\tilde\theta}\right]_{2NZ}
\end{array}
\right\}
$$

$$
\{f(\tilde{\theta})\}_{\substack{NR-1\\ NZ\times 1}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-2NZ+1} + \tilde{r}_{NR-1}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-2NZ+1} + \bar{r}_{NR-1}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-2NZ+1} + (1-2\nu)\left[\dfrac{2\bar{r}_{NR-1}}{\tilde{h}_z}\dfrac{d\omega}{d\tilde{\theta}}\right]_{m-2NZ+1} \\
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-2NZ+2} + \tilde{r}_{NR-1}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-2NZ+2} + \bar{r}_{NR-1}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-2NZ+2} \\
\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-NZ-1} + \bar{r}_{NR-1}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-NZ-1} + \bar{r}_{NR-1}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-NZ-1} \\
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-NZ} + \tilde{r}_{NR-1}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-NZ} + \bar{r}_{NR-1}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-NZ} + (1-2\nu)\left[\dfrac{-2\bar{r}_{NR-1}}{\tilde{h}_z}\dfrac{d\omega}{d\tilde{\theta}}\right]_{m-NZ}
\end{array}
\right\}
$$

$$
\{f(\tilde{\theta})\}_{\substack{NR\\ NZ\times 1}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-NZ+1} + \tilde{r}_{NR}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-NZ+1} + \bar{r}_{NR}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-NZ+1} + (1-\nu)\left[\dfrac{-2\bar{r}_{NR}}{\tilde{h}_r}\dfrac{d\Omega}{d\tilde{\theta}} - \dfrac{d\Omega}{d\tilde{\theta}} + \dfrac{2\bar{r}_{NR}}{\tilde{h}_z}\dfrac{d\omega}{d\tilde{\theta}}\right]_{m-NZ+1} \\
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-NZ+2} + \bar{r}_{NR}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-NZ+2} + \bar{r}_{NR}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-NZ+2} + (1-2\nu)\left[\dfrac{-2\bar{r}_{NR}}{\tilde{h}_r}\dfrac{d\Omega}{d\tilde{\theta}} - \dfrac{d\Omega}{d\tilde{\theta}}\right]_{m-NZ+2} \\
\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
(3-4\nu)\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m-1} + \bar{r}_{NR}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m-1} + \tilde{r}_{NR}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m-1} + (1-2\nu)\left[\dfrac{-2\bar{r}_{NR}}{\tilde{h}_r}\dfrac{d\Omega}{d\tilde{\theta}} - \dfrac{d\Omega}{d\tilde{\theta}}\right]_{m-1} \\
(3-4\ )\left.\dfrac{d\Omega}{d\tilde{\theta}}\right|_{m} + \bar{r}_{NR}\left.\dfrac{d\dot{\Omega}}{d\tilde{\theta}}\right|_{m} + \tilde{r}_{NR}\left.\dfrac{d\dot{\omega}}{d\tilde{\theta}}\right|_{m} + (1-2\nu)\left[\dfrac{-2\tilde{r}_{NR}}{\tilde{h}_r}\dfrac{d\Omega}{d\tilde{\theta}} - \dfrac{d\Omega}{d\tilde{\theta}} - \dfrac{2\bar{r}_{NR}}{\tilde{h}_z}\dfrac{d\omega}{d\tilde{\theta}}\right]_{m}
\end{array}
\right\}
$$

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Assuming that $\{s(\tilde{\theta})\}$ is known, solutions of equations (3.23) can be obtained in closed form when boundary data at the end points of the circumferential lines are specified.

Note that even though the elements of $[K_\theta]$ are constants, a closed form solution of the associated eigenvalue problem in accordance with the decomposition methods of Appendix A is not possible. In addition, we find that $[K_\theta]$ is also singular although this is not as evident from its elements as in the case of the other coefficient matrices.

3.4 Ordinary Differential Equations and Boundary Conditions in the Axial Direction

Application of the line method to equation (3.3) will result in a set of ordinary differential equations along the axial lines of Figure 4. The displacements along these lines will be denoted as $w_1, w_2, \ldots, w_n$. We define $\dot{u}|_1, \dot{u}|_2, \ldots, \dot{u}|_n$ as the derivatives of the radial displacements of points on these lines with respect to r and $\dot{v}|_1, \dot{v}|_2, \ldots, \dot{v}|_n$ as the derivatives of the circumferential displacements of the same points on these lines with respect to θ. These displacements and derivatives are then functions of z only, since they are variables along axial lines. Using z-directional shear stress boundary conditions for the corner and surface lines, the simultaneous differential equations along the axial lines can be written as follows:

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$[K_{zc}$

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$$\underset{n\times 1}{\frac{d^2}{d\tilde{z}^2}\{\tilde{w}\}} = \underset{n\times n}{[K_{zc}]}\ \underset{n\times 1}{\{\tilde{w}\}} + \underset{n\times 1}{\{t_c(\tilde{z})\}} \tag{3.26}$$

From Figure 4, one can note that the radius varies with the line position within each subset of NR axial lines. Using the notation of the previous section, these radii are denoted as r_i $i = 1, 2, 3, \ldots, NR$. The coefficient matrix $[K_{zc}]$ is then given by

$$\underset{n\times n}{[K_{zc}]} = \begin{bmatrix} \underset{NR\times NR}{[K_{zc1}]} & \underset{NR\times NR}{2[K_{zc2}]} & 0 & 0 & 0 \\ \underset{NR\times NR}{[K_{zc2}]} & \underset{NR\times NR}{[K_{zc1}]} & \underset{NR\times NR}{[K_{zc2}]} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \underset{NR\times NR}{[K_{zc2}]} & \underset{NR\times NR}{[K_{zc1}]} & \underset{NR\times NR}{[K_{zc2}]} \\ 0 & 0 & 0 & \underset{NR\times NR}{2[K_{zc2}]} & \underset{NR\times NR}{[K_{zc1}]} \end{bmatrix} \tag{3.27}$$

The submatrices $[K_{zc1}]$ and $[K_{zc2}]$ can be written as shown below.

$[K_{zc1}]$

$NR \times NR$

c

c

c

$[K_{zc2}]$

$NR \times NR$

c_j

$$[K_{zc1}]_{NR \times NR} = \begin{bmatrix} c_1 & -k_{18} & 0 & 0 & 0 \\ -c_{2NR-1} & c_2 & -c_{NR+1} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -c_{3NR-4} & c_{NR-1} & -c_{2NR-2} \\ 0 & 0 & 0 & -k_{18} & c_{NR} \end{bmatrix}$$

$$k_{18} = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2}{\tilde{h}_r^2} \right]$$

$$c_i = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{2}{\tilde{h}_r^2} + \frac{2}{\tilde{r}_i^2 \tilde{h}_\theta^2} \right] \qquad \text{for } i = 1, 2, \ldots, NR$$

$$c_i = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{1}{\tilde{h}_r^2} + \frac{1}{2\tilde{h}_r \tilde{r}_j} \right] \qquad \begin{aligned} \text{for } i &= NR+1, NR+2, \ldots, \\ & 2NR-2 \\ j &= i-NR+1 \end{aligned}$$

$$c_i = \frac{(1-2\nu)}{2(1-\nu)} \left[\frac{1}{\tilde{h}_r^2} - \frac{1}{2\tilde{h}_r \tilde{r}_k} \right] \qquad \begin{aligned} \text{for } i &= 2NR-1, 2NR, \ldots, \\ & 3NR-4 \\ k &= i-2NR+3 \end{aligned}$$

$$[K_{zc2}]_{NR \times NR} = \begin{bmatrix} c_{3NR-3} & 0 & 0 & 0 & 0 \\ 0 & c_{3NR-2} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & c_{4NR-3} & 0 \\ 0 & 0 & 0 & 0 & c_{4NR-4} \end{bmatrix}$$

$$c_i = \frac{(1-2\nu)}{2(1-\nu)} \left[- \frac{1}{\tilde{r}_\ell^2 \tilde{h}_\theta^2} \right] \qquad \begin{aligned} \text{for } i &= 3NR-3, 3NR-2, \ldots, \\ & 4NR-4 \\ \ell &= i-3NR+4 \end{aligned}$$

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Note that submatrix $[K_{zc1}]$ will always contain 3NR-4 constants denoted as c_i while the matrix $[K_{zc2}]$ will have only NR elements on its diagonal. The coefficient matrix $[K_{zc}]$ has elements that are all constants which shows that equations (3.26) are also differential equations with constant coefficients. An investigation of its inverse will also show that $[K_{zc}]$ is singular. The displacement vector $\{\tilde{w}\}$ can be written as in equation (2.55) with exception of replacing NY by $N\theta$ and the partitioned $\{\tilde{w}\}_i$ vectors are of order NRx1 instead of NXx1. The vector $\{t_c(\tilde{z})\}$ becomes,

$$\underset{n\times 1}{\{t_c(\tilde{z})\}} = \begin{Bmatrix} \{f(\tilde{z})\}_1 \\ \{f(\tilde{z})\}_2 \\ \cdot \\ \cdot \\ \cdot \\ \{f(\tilde{z})\}_{N\theta-1} \\ \{f(\tilde{z})\}_{N\theta} \end{Bmatrix} \tag{3.28}$$

$$
\underset{NR\times 1}{\{f(\bar z)\}_1} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_1 + \frac{1}{\bar r_1}\left.\frac{d\bar u}{d\bar z}\right|_1 + \frac{1}{\bar r_1}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_1 + (1-2\nu)\left[\left(\frac{2}{\bar h_r} - \frac{1}{\bar r_1}\right)\frac{d\bar u}{d\bar z} + \frac{2}{\bar r_1 \bar h_\theta}\frac{d\bar v}{d\bar z}\right]_1 \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_2 + \frac{1}{\bar r_2}\left.\frac{d\bar u}{d\bar z}\right|_2 + \frac{1}{\bar r_2}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_2 + (1-2\nu)\left[\frac{2}{\bar r_2 \bar h_\theta}\frac{d\bar v}{d\bar z}\right]_2 \\[2ex]
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\bar u}{d\bar z}\right|_{NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{NR-1} + (1-2\nu)\left[\frac{2}{\bar r_{NR-1}\bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{NR-1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\bar u}{d\bar z}\right|_{NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{NR} + (1-2\nu)\left[\left(\frac{-2}{\bar h_r} - \frac{1}{\bar r_{NR}}\right)\frac{d\bar u}{d\bar z} + \frac{2}{\bar r_{NR}\bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{NR}
\end{array}
\right\}
$$

$$
\underset{NR\times 1}{\{f(\bar z)\}_2} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{NR+1} + \frac{1}{\bar r_1}\left.\frac{d\bar u}{d\bar z}\right|_{NR+1} + \frac{1}{\bar r_1}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{NR+1} + (1-2\nu)\left[\left(\frac{2}{\bar h_r} - \frac{1}{\bar r_1}\right)\frac{d\bar u}{d\bar z}\right]_{NR+1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{NR+2} + \frac{1}{\bar r_2}\left.\frac{d\bar u}{d\bar z}\right|_{NR+2} + \frac{1}{\bar r_2}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{NR+2} \\[2ex]
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{2NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\bar u}{d\bar z}\right|_{2NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{2NR-1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{2NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\bar u}{d\bar z}\right|_{2NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{2NR} + (1-2\nu)\left[\left(\frac{-2}{\bar h_r} - \frac{1}{\bar r_{NR}}\right)\frac{d\bar u}{d\bar z}\right]_{2NR}
\end{array}
\right\}
$$

$$
\underset{NR\times 1}{\{f(\bar z)\}_{N\theta-1}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-2NR+1} + \frac{1}{\bar r_1}\left.\frac{d\bar u}{d\bar z}\right|_{n-2NR+1} + \frac{1}{\bar r_1}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-2NR+1} + (1-2\nu)\left[\left(\frac{2}{\bar h_r}-\frac{1}{\bar r_1}\right)\frac{d\bar u}{d\bar z}\right]_{n-2NR+1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-2NR+2} + \frac{1}{\bar r_2}\left.\frac{d\bar u}{d\bar z}\right|_{n-2NR+2} + \frac{1}{\bar r_2}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-2NR+2} \\[2ex]
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR-1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR} + (1-2\nu)\left[\left(\frac{-2}{\bar h_r}-\frac{1}{\bar r_{NR}}\right)\frac{d\bar u}{d\bar z}\right]_{n-NR}
\end{array}
\right\}
$$

$$
\underset{NR\times 1}{\{f(\bar z)\}_{N\theta}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR+1} + \frac{1}{\bar r_1}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR+1} + \frac{1}{\bar r_1}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR+1} + (1-2\nu)\left[\left(\frac{2}{\bar h_r}-\frac{1}{\bar r_1}\right)\frac{d\bar u}{d\bar z} - \frac{2}{\bar r_1\bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n-NR+1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR+2} + \frac{1}{\bar r_2}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR+2} + \frac{1}{\bar r_2}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR+2} + (1-2\nu)\left[\frac{-2}{\bar r_2\bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n-NR+2} \\[2ex]
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\bar u}{d\bar z}\right|_{n-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-1} + (1-2\nu)\left[\frac{-2}{\bar r_{NR-1}\bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n-1} \\[2ex]
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n} + \frac{1}{\bar r_{NR}}\left.\frac{d\bar u}{d\bar z}\right|_{n} + \frac{1}{\bar r_{NR}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n} + (1-2\nu)\left[\left(\frac{-2}{\bar h_r}-\frac{1}{\bar r_{NR}}\right)\frac{d\bar u}{d\bar z} - \frac{2}{\bar r_{NR}\bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n}
\end{array}
\right\}
$$

$$
\{f(\bar z)\}_{N\theta-1} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-2NR+1} + \frac{1}{\bar r_1}\left.\frac{d\bar u}{d\bar z}\right|_{n-2NR+1} + \frac{1}{\bar r_1}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-2NR+1} + (1-2\nu)\left[\left(\frac{2}{\bar h_r}-\frac{1}{\bar r_1}\right)\frac{d\bar u}{d\bar z}\right]_{n-2NR+1} \\
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-2NR+2} + \frac{1}{\bar r_2}\left.\frac{d\bar u}{d\bar z}\right|_{n-2NR+2} + \frac{1}{\bar r_2}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-2NR+2} \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR-1} \\
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR} + \frac{1}{\bar r_{NR}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR} + (1-2\nu)\left[\left(\frac{-2}{\bar h_r}-\frac{1}{\bar r_{NR}}\right)\frac{d\bar u}{d\bar z}\right]_{n-NR}
\end{array}
\right\}
$$

NRx1

$$
\{f(\bar z)\}_{N\theta} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR+1} + \frac{1}{\bar r_1}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR+1} + \frac{1}{\bar r_1}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR+1} + (1-2\nu)\left[\left(\frac{2}{\bar h_r}-\frac{1}{\bar r_1}\right)\frac{d\bar u}{d\bar z} - \frac{2}{\bar r_1 \bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n-NR+1} \\
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-NR+2} + \frac{1}{\bar r_2}\left.\frac{d\bar u}{d\bar z}\right|_{n-NR+2} + \frac{1}{\bar r_2}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-NR+2} + (1-2\nu)\left[\frac{-2}{\bar r_2 \bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n-NR+2} \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\bar u}{d\bar z}\right|_{n-1} + \frac{1}{\bar r_{NR-1}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n-1} + (1-2\nu)\left[\frac{-2}{\bar r_{NR-1} \bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n-1} \\
\left.\frac{d\dot{\bar u}}{d\bar z}\right|_{n} + \frac{1}{\bar r_{NR}}\left.\frac{d\bar u}{d\bar z}\right|_{n} + \frac{1}{\bar r_{NR}}\left.\frac{d\dot{\bar v}}{d\bar z}\right|_{n} + (1-2\nu)\left[\left(\frac{-2}{\bar h_r}-\frac{1}{\bar r_{NR}}\right)\frac{d\bar u}{d\bar z} - \frac{2}{\bar r_{NR} \bar h_\theta}\frac{d\bar v}{d\bar z}\right]_{n}
\end{array}
\right\}
$$

NRx1

Assuming that $\{t_c(\tilde{z})\}$ is known, we can solve equations (3.26) in closed form provided that boundary conditions at the end points of all the axial lines are given.

Inspection of equations (3.23) and (3.26) shows that they are differential equations with constant coefficients whose solutions can be obtained by the methods of Section 2.3. In evaluating the matrix functions for equation (2.60), it is most convenient to use the definitions (2.62) together with the additive formulas (2.68). The difficulty in solving the eigenvelue problems for equations (3.23) and (3.26) arises because the decomposition of (A.2) does not lead to tri-diagonal matrices of the form (A.1). Instead of equation (A.9), the methods of Appendix A will lead to a difference equation with variable coefficients whose solution is difficult to evaluate. Hence, a closed form solution for the eigenvalues and eigenvectors will not be possible.

An alternate approach to the eigenvalue problem would be the use of a suitable numerical method. Since the decomposition of equation (A.2) will always reduce the order of the matrices involved, only the component matrix eigenvalues and eigenvectors need be evaluated numerically. One such suitable numerical technique is the Rutishauser left-right transformation method (38). However, care should be exercised in using this method because of the zero eigenvalue in each coefficient matrix. Doust and

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Price (39) discuss a technique for readily adopting the Rutishauser method to singular matrices. At this time, results of the numerical analysis of this eigenvalue problem are not yet available.

The solutions of equations (3.20) cannot be readily obtained by the above discussed matrix series or normal mode analysis. Since the coefficient matrix $[K_r(\tilde{r})]$ is a function of the independent variable and the differential operator in (3.20) contains additional terms to the second derivative, a solution technique for this type of equations will be presented in the following section.

3.5 Solution of Simultaneous Differential Equations With Variable Coefficients

Solution methods for a system of ordinary differential equations with variable coefficients are, in general, more involved and complicated than those discussed in the previous sections. One technique in solving a system of higher order differential equations of this type is to first reduce the given equations by suitable transformations to a set of first order differential equations. The solution of this set of first order equations can then be expressed in terms of an infinite series constructed by means of repeated integrations of the reduced coefficient matrix whose elements are also functions of the independent variable. The advantages of this method are its simplicity ar

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$$U_1 = r$$

$$U_{l+1} = \frac{1}{r}$$

In terms of

shown below

simplicity and the fact that it avoids the generally difficult problem of solving the determinantal equation. On the other hand, the process often has the disadvantage of slow convergence in numerical applications (34).

Application of the normal mode method, as discussed in Section 2.3, to equations with variable coefficients does not present a practical approach to these equations because the eigenvalues and eigenvectors are all functions of the independent variable. This functional dependency would require a different modal matrix along each point of the independent variable necessitating a possible averaging technique at each node. Because of these difficulties with the normal mode method, we shall use the integral series approach in obtaining the solutions of equations (3.20).

Noting that the differential operator of equations (3.20) can be written as $\frac{d}{d\tilde{r}}\left[\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}\tilde{u})\right]$, the following variables are introduced in order to obtain a system of first order equations:

$$\left.\begin{array}{llll} U_1 = \tilde{r}\,\tilde{u}_1 & U_2 = \tilde{r}\,\tilde{u}_2 & \ldots & U_\ell = \tilde{r}\tilde{u}_\ell \\ \\ U_{\ell+1} = \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}\tilde{u}_1) & U_{\ell+2} = \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}\tilde{u}_2) & \ldots & U_{2\ell} = \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}\tilde{u}_\ell) \end{array}\right\} \tag{3.29}$$

In terms of these variables, equations (3.20) can be written as shown below.

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$\{U(\bar{r})\}$ =
$2\ell x1$

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$$\underset{2\ell x1}{\frac{d}{d\tilde{r}}\{U\}} = \underset{2\ell x2\ell}{[A(\tilde{r})]}\ \underset{2\ell x1}{\{U\}} + \underset{2\ell x1}{\{\bar{r}(\tilde{r})\}} \tag{3.30}$$

where

$$\underset{2\ell x2\ell}{[A(\tilde{r})]} = \begin{bmatrix} \underset{\ell x\ell}{[0]} & \underset{\ell x\ell}{\tilde{r}[I]} \\ \underset{\ell x\ell}{\frac{1}{\tilde{r}}[K_r(\tilde{r})]} & \underset{\ell x\ell}{[0]} \end{bmatrix} \tag{3.31}$$

$$\underset{2\ell x1}{\{\bar{r}(\tilde{r})\}} = \begin{Bmatrix} \underset{\ell x1}{\{0\}} \\ \underset{\ell x1}{\{r(\tilde{r})\}} \end{Bmatrix} \tag{3.32}$$

Following the Peano-Baker form of integration (34), the solution of equation (3.30) can be written as

$$\underset{2\ell x1}{\{U(\tilde{r})\}} = \underset{2\ell x2\ell}{[\Omega_o^{\tilde{r}}(A)]}\ \underset{2\ell x1}{\{U(\tilde{o})\}} + \underset{2\ell x2\ell}{[\Omega_o^{\tilde{r}}(A)]} \int_o^{\tilde{r}} \underset{2\ell x2\ell}{[\Omega_o^{\eta}]^{-1}}\ \underset{2\ell x1}{\{\bar{r}(\eta)\}}d\eta \tag{3.33}$$

where $\{U(\tilde{o})\}$ is a vector which consists of the boundary values of $(\tilde{r}\tilde{u})$ and $\left[\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}\tilde{u})\right]$ at $\tilde{r} = 0$. The matrizant of $[A]$ is an infinite matrix integral series given by

$[\Phi^{\dot{r}}_{o}(A)] = [I$

$\int_{0}^{\rho_2} [A(\rho$

$\int_{0}^{\rho_2} [A(\rho$

Substituting

$[\Phi^{\dot{r}}_{o}(A)] =$

$+\int_{0}$

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$$[\Omega_o^{\tilde{r}}(A)] = [I] + \int_o^{\tilde{r}} [A(\rho_1)]d\rho_1 + \int_o^{\tilde{r}} [A(\rho_2)]d\rho_2$$

$$\cdot \int_o^{\rho_2} [A(\rho_1)]d\rho_1 + \int_o^{\tilde{r}} [A(\rho_3)]d\rho_3 \int_o^{\rho_3} [A(\rho_2)]d\rho_2$$

$$\cdot \int_o^{\rho_2} [A(\rho_1)]d\rho_1 + \ldots \tag{3.34}$$

Substituting equation (3.31) into equation (3.34) yields

$$[\Omega_o^{\tilde{r}}(A)] = \begin{bmatrix} [I] & 0 \\ 0 & [I] \end{bmatrix} + \int_o^{\tilde{r}} \begin{bmatrix} [0] & \rho_1[I] \\ \frac{1}{\rho_1}[K_r] & [0] \end{bmatrix} d\rho_1$$

$$+ \int_o^{\tilde{r}} \begin{bmatrix} [0] & \rho_2[I] \\ \frac{1}{\rho_2}[K_r] & [0] \end{bmatrix} d\rho_2 \int_o^{\rho_2} \begin{bmatrix} [0] & \rho_1[I] \\ \frac{1}{\rho_1}[K_r] & [0] \end{bmatrix} d\rho_1$$

$$+ \int_o^{\tilde{r}} \begin{bmatrix} [0] & \rho_3[I] \\ \frac{1}{\rho_3}[K_r] & [0] \end{bmatrix} d\rho_3 \int_o^{\rho_3} \begin{bmatrix} [0] & \rho_2[I] \\ \frac{1}{\rho_2}[K_r] & [0] \end{bmatrix} d\rho_2$$

$$\cdot \int_o^{\rho_2} \begin{bmatrix} [0] & \rho_1[I] \\ \frac{1}{\rho_1}[K_r] & [0] \end{bmatrix} d\rho_1 + \ldots$$

Partitioning the matrizant of $[A]$ according to (2.62) gives in terms of the coefficient matrix $[K_r]$ the following four matrix integral series:

$[\Omega_{11}] =$
$\ell \times \ell$

$[\Omega_{12}] =$
$\ell \times \ell$

$\cdot \int_c$

$[\Omega_{21}] =$
$\ell \times \ell$

$\cdot \int_c$

$[\Omega_{22}] =$
$\ell \times \ell$

Differentia

$\frac{d}{dr} [\Omega_{11}]$

$$\underset{\ell\times\ell}{[\Omega_{11}]} = [I] + \int_0^{\tilde{r}} \rho_2[I]d\rho_2 \int_0^{\rho_2} \frac{1}{\rho_1}[K_r(\rho_1)]d\rho_1 + \ldots$$

$$\underset{\ell\times\ell}{[\Omega_{12}]} = [I] \int_0^{\tilde{r}} \rho_1 d\rho_1 + \int_0^{\tilde{r}} [I]\rho_3 d\rho_3$$

$$\cdot \int_0^{\rho_3} \frac{1}{\rho_2}[K_r(\rho_2)]d\rho_2 \int_0^{\rho_2} \rho_1[I]d\rho_1 + \ldots$$

$$\underset{\ell\times\ell}{[\Omega_{21}]} = \int_0^{\tilde{r}} \frac{1}{\rho_1}[K_r(\rho_1)]d\rho_1 + \int_0^{\tilde{r}} \frac{1}{\rho_3}[K_r(\rho_3)]d\rho_3$$

$$\cdot \int_0^{\rho_3} [I]\rho_2 d\rho_2 \int_0^{\rho_2} \frac{1}{\rho_1}[K_r(\rho_1)]d\rho_1 + \ldots$$

$$\underset{\ell\times\ell}{[\Omega_{22}]} = [I] + \int_0^{\tilde{r}} \frac{1}{\rho_2}[K_r(\rho_2)]d\rho_2 \int_0^{\rho_2} \rho_1[I]d\rho_1 + \ldots$$

Differentiating these integral series with respect to $\tilde{r}$ gives

$$\frac{d}{d\tilde{r}}[\Omega_{11}] = \tilde{r}[I]\int_0^{\tilde{r}} \frac{1}{\rho_1}[K_r(\rho_1)]d\rho_1 + \ldots$$

$$\frac{d}{d\tilde{r}}[\Omega_{12}] = \tilde{r}[I] + \tilde{r}[I]\int_0^{\tilde{r}} \frac{1}{\rho_2}[K_r(\rho_2)]d\rho_2$$

$$\cdot \int_0^{\rho 2} \rho_1[I]d\rho_1 + \ldots$$

$$\frac{d}{d\tilde{r}}[\Omega_{21}] = \frac{1}{\tilde{r}}[K_r] + \frac{1}{\tilde{r}}[K_r]\int_0^{\tilde{r}} [I]\rho_2 d\rho_2$$

$$\cdot \int_0^{\rho 2} \frac{1}{\rho_1}[K_r(\rho_1)]d\rho_1 + \ldots$$

$$\frac{d}{d\tilde{r}}[\Omega_{22}] = \frac{1}{\tilde{r}}[K_r]\int_0^{\tilde{r}} \rho_1[I]d\rho_1 + \ldots$$

Inspection of these equations shows that the following relationships exist among these four submatrices:

$$\left.\begin{aligned} \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}[\Omega_{11}] &= [\Omega_{21}] \\ \tilde{r}\frac{d}{d\tilde{r}}[\Omega_{21}] &= [\Omega_{11}][K_r] \end{aligned}\right\} \tag{3.35}$$

$$\left.\begin{aligned} \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}[\Omega_{12}] &= [\Omega_{22}] \\ \tilde{r}\frac{d}{d\tilde{r}}[\Omega_{22}] &= [\Omega_{12}][K_r] \end{aligned}\right\} \tag{3.36}$$

$$\frac{d}{d\tilde{r}}[\Omega_{12}] = \tilde{r}[I] + \tilde{r}[I]\int_{o}^{\tilde{r}} \frac{1}{\rho_2}[K_r(\rho_2)]d\rho_2$$

$$\cdot \int_{o}^{\rho 2} \rho_1[I]d\rho_1 + \ldots$$

$$\frac{d}{d\tilde{r}}[\Omega_{21}] = \frac{1}{\tilde{r}}[K_r] + \frac{1}{\tilde{r}}[K_r]\int_{o}^{\tilde{r}} [I]\rho_2 d\rho_2$$

$$\cdot \int_{o}^{\rho 2} \frac{1}{\rho_1}[K_r(\rho_1)]d\rho_1 + \ldots$$

$$\frac{d}{d\tilde{r}}[\Omega_{22}] = \frac{1}{\tilde{r}}[K_r]\int_{o}^{\tilde{r}} \rho_1[I]d\rho_1 + \ldots$$

Inspection of these equations shows that the following relationships exist among these four submatrices:

$$\left.\begin{aligned} \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}[\Omega_{11}] &= [\Omega_{21}] \\ \tilde{r}\frac{d}{d\tilde{r}}[\Omega_{21}] &= [\Omega_{11}][K_r] \end{aligned}\right\} \qquad (3.35)$$

$$\left.\begin{aligned} \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}[\Omega_{12}] &= [\Omega_{22}] \\ \tilde{r}\frac{d}{d\tilde{r}}[\Omega_{22}] &= [\Omega_{12}][K_r] \end{aligned}\right\} \qquad (3.36)$$

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From the definition of these matrix series we can conclude at once that their initial values are

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\begin{aligned}
[\Omega_{11}]_{\tilde{r}=0} &= [I] \\
[\Omega_{12}]_{\tilde{r}=0} &= [0] \\
[\Omega_{21}]_{\tilde{r}=0} &= [0] \\
[\Omega_{22}]_{\tilde{r}=0} &= [I]
\end{aligned}
\tag{3.37}
$$

Since the values of the submatrices $[\Omega_{ij}]$ are difficult to obtain from their series definitions, the above simultaneous matrix equations may be evaluated by a suitable numerical technique such as the Runge-Kutta method. The necessary initial conditions for this numerical solution are listed in (3.37).

In using a numerical method for the solution of a given differential equation, it is usually necessary to solve for the derivative of the dependent variable at the initial point. If the region of interest includes the point $\tilde{r} = 0$, equation (3.35) shows that $\frac{d}{d\tilde{r}}[\Omega_{21}]$ at $\tilde{r} = 0$ is not finite. This problem can be avoided by defining a new variable $[\Omega^*_{21}]$ as

$$[\Omega^*_{21}] = \tilde{r}^3[\Omega_{21}] \tag{3.38}$$

Using equation (3.38) in (3.35), the following simultaneous matrix differential equations will be obtained:

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$$\frac{d}{d\tilde{r}}[\Omega_{11}] = \frac{1}{\tilde{r}^2}[\Omega^*_{21}]; \quad [\Omega_{11}]_{r=0} = [I]$$
$$(3.39)$$
$$\frac{d}{d\tilde{r}}[\Omega^*_{21}] = \frac{3}{\tilde{r}}[\Omega^*_{21}] + [\Omega_{11}][\bar{K}_r]; \quad [\Omega^*_{21}]_{\tilde{r}=0} = [0]$$

By applying L'Hospital's rule, all the necessary derivatives in equations (3.36) and (3.39) can now be evaluated. Note that the matrix $[\bar{K}_r]$ is obtained from multiplying the coefficient matrix $[K_r]$ by $\tilde{r}^2$ such that no element of $[\bar{K}_r]$ contains an $\tilde{r}$ in its denominator.

If one were to use the series definitions for evaluating the matrix functions $[\Omega_{ij}]$, similar difficulties would be encountered since at $\tilde{r} = 0$ some of the integrals would diverge. This divergence is the result of trying to evaluate improper integrals of the second kind.

At this time, the analogy between solutions (3.33) and (2.60) may be noted. Reference (34) shows in detail that the matrizant of a matrix of constants is identically equal to the exponential matrix series (2.61). However, indirect methods such as additive formulas and accuracy checks such as (2.87) for evaluating and checking these matrix functions are not available when variable coefficient differential equations are treated.

It is evident from the above discussion that for specific examples a closed form solution of equations (3.20) is not possible. However, the advantage of the line method over complete finite

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finite difference solutions is still obtained in that higher order numerical solutions are utilized for the computations.

Additional comments on the selection of the matrizant method over a direct numerical solution of equations (3.20) can now be made. The advantage of using equation (3.33) is in its ability to express two point differential equation solutions directly in terms of given boundary data. Direct numerical techniques, such as the single-step Runge-Kutte method or the multi-step predictor-corrector methods (33) usually require that all initial point data be known. Since in a two point boundary value problem some of the initial data is unknown, indirect methods such as "shooting" and successive approximations must be employed (33). This, of course, leads to an inefficient use of the computer and should be avoided whenever possible. In addition, it is well known that direct numerical solutions for a system of two point differential equations are not available.

3.5.1 Evaluation of the Particular Integral for the Radial Differential Equations

In a similar manner to equation (2.80) we represent the particular integral in partitioned form as

$$\left\{\begin{matrix} \underset{\ell x1}{\{B_1(\tilde{r})\}} \\ \\ \underset{\ell x1}{\{B_2(\tilde{r})\}} \end{matrix}\right\} = \int_o^{\tilde{r}} \underset{2\ell x2\ell}{[\Omega_o^{\eta}]^{-1}} \; \underset{2\ell x1}{\{\Upsilon(\eta)\}} d\eta \qquad (3.40)$$

$$\begin{Bmatrix} \{B_1(\bar{r})\} \\ \\ \{B_2(\bar{r})\} \end{Bmatrix}$$

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$$\begin{Bmatrix} \{B_1(\tilde{r})\} \\ \\ \{B_2(\tilde{r})\} \end{Bmatrix} = \int_o^{\tilde{r}} \begin{bmatrix} [\Omega_{11}(K_r,\eta)] & [\Omega_{12}(K_r,\eta)] \\ \\ [\Omega_{21}(K_r,\eta)] & [\Omega_{22}(K_r,\eta)] \end{bmatrix}^{-1} \begin{Bmatrix} \{0\} \\ \\ \{r(\eta)\} \end{Bmatrix} d\eta$$

It was found that equations (2.83) also apply to the matrix functions $[\Omega_{ij}]$ and thus the above integrals may be written as

$$\{B_1(\tilde{r})\} = -\int_o^{\tilde{r}} [\Omega_{12}(K_r,\eta)] \ \{r(\eta)\}d\eta \tag{3.41}$$

$$\{B_2(\tilde{r})\} = \int_o^{\tilde{r}} [\Omega_{11}(K_r,\eta)] \ \{r(\eta)\}d\eta \tag{3.42}$$

However, the simple relationships of (2.63) between matrices $[\Omega_{11}]$ and $[\Omega_{22}]$ and of (2.65) between matrices $[\Omega_{21}]$ and $[\Omega_{12}]$ in an analogous manner to $[A_{ij}]$ are not valid. Note that the particular integrals in the circumferential and axial directions can also be expressed in a similar manner to equations (2.84) and (2.85). Since $\{r(\eta)\}$ in the above integrals is unknown, we start the solution of the problem by assuming zero values for the required quantities. Using the partitioned form of the matrices, the solution of equations (3.30) can be written as

$$\underset{\ell\times 1}{\{\tilde{u}(\tilde{r})\}} = \frac{1}{\tilde{r}}\underset{\ell\times\ell}{[\Omega_{11}(K_r,\tilde{r})]}(\underset{\ell\times 1}{\{F_1\}} + \underset{\ell\times 1}{\{B_1(\tilde{r})\}})$$

$$+ \frac{1}{\tilde{r}}\underset{\ell\times\ell}{[\Omega_{12}(K_r,\tilde{r})]}(\underset{\ell\times 1}{\{F_2\}} + \underset{\ell\times 1}{\{B_2(\tilde{r})\}}) \qquad (3.43)$$

where

$$\{F_1\} = \{\tilde{r}\tilde{u}\}_{\tilde{r}=\tilde{r}_{initial}}$$

$$\{F_2\} = \left\{\frac{1}{\tilde{r}}\frac{d}{d\tilde{r}}(\tilde{r}\tilde{u})\right\}_{\tilde{r}_{initial}} = \left\{\dot{\tilde{u}} + \frac{\tilde{u}}{\tilde{r}}\right\}_{\tilde{r}=\tilde{r}_{initial}}$$

$$\underset{\ell\times 1}{\{\dot{\tilde{u}}(\tilde{r})\}} = \left([\Omega_{21}(K_r,\tilde{r})] - \frac{1}{\tilde{r}^2}[\Omega_{11}(K_r,\tilde{r})]\right)(\{F_1\} + \{B_1(\tilde{r})\})$$

$$+ ([\Omega_{22}(K_r,\tilde{r})] - \frac{1}{\tilde{r}^2}[\Omega_{12}(K_r,\tilde{r})])(\{F_2\} + \{B_2(\tilde{r})\}) \qquad (3.44)$$

Equations (3.43) and (3.44) give us the first estimate for the vectors $\{\tilde{u}(\tilde{r})\}^{(1)}$ and $\{\dot{\tilde{u}}(\tilde{r})\}^{(1)}$. It is assumed of course that the boundary vectors $\{F_1\}$ and $\{F_2\}$ are known. Using the calculated values of $\{\tilde{u}\}^{(1)}$ and $\{\dot{\tilde{u}}\}^{(1)}$ we can evaluate the vector $\{s(\tilde{\theta})\}^{(1)}$. An analogous equation to (2.86) will then give us the first value of $\{\tilde{v}(\tilde{\theta})\}^{(1)}$ and $\{\dot{\tilde{v}}(\tilde{\theta})\}^{(1)}$. Using the first solutions along the radial and circumferential directions in the vector $\{t_c(\tilde{z})\}$, the axial solution is obtained from a similar equation to (2.86). If with the repetition of this procedure convergence of the calculated variables occurs, an approximate solution of a given problem can be determined.

Regarding the convergence of this process, the comments and error checks of the previous chapter apply only to the axial and circumferential equations. Since equation (2.87) is not applicable to the matrix functions in the radial direction, their accuracy can only be checked by varying the Runge-Kutta integration increment. Since the homogeneous solutions of equations (3.30) are independent of the other two sets of differential equations, this integration step can be arbitrarily small. The values of the coupling terms in $\{\dot{r}(\tilde{r})\}$, $\{s(\tilde{\theta})\}$ and $\{t_c(\tilde{z})\}$ can only be determined by the use of finite difference calculus, since they involve displacements and derivatives that are defined only at the nodes. Similar approximations of derivatives near boundaries must be made as in the case of rectangular coordinate problems.

3.6 Application to Specific Geometries - Annular Plate With Internal Surface Cracks

A problem of some practical importance is that of an annular plate containing part-through cracks on the inside surface and which is loaded by a uniform radial stress of σ_o on the outside surface. In order to minimize the numerical computations, we have assumed four internal cracks located symmetrically at ninety degrees to each other. The closed form solution of this problem is extremely difficult to evaluate because the stress and displacements fields of a circular hole interact with the singular stress fields of the cracks. Figure 5 shows the geometry and loading of the problem under investigation. Because of the symmetric geometry and loading, only one-sixteenth of the original plate has to be discretized. Figure 6 shows this region of interest and the assumed crack geometry. The displacement fields in the plate are described by the solutions of the three sets of simultaneous ordinary differential equations. Inspection of Figure 6 shows that non-dimensionalization with respect to the outside radius is more convenient in this case since the crack has two characteristic dimensions. Values of the non-dimensionalized variables $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{t}$ and $\tilde{\theta}_o$ used for this problem are also shown in Figure 6.

At this time we return to solutions (3.43) and (3.44) and note that the initial value vectors $\{F_1\}$ and $\{F_2\}$ are unknown. Using equation (3.6), the given normal stresses on the inside and outside surfaces of the plate can be written in discretized form as follows:

$$\{\dot{\tilde{u}}\}_{\tilde{r}=\tilde{r}_o} = -\frac{\lambda}{\lambda+2G}\left(\frac{1}{\tilde{r}_o}\{\dot{\tilde{v}}\} + \frac{1}{\tilde{r}_o}\{\tilde{u}\} + \{\dot{\tilde{w}}\}\right)_{\tilde{r}=\tilde{r}_o} \qquad (3.45)$$

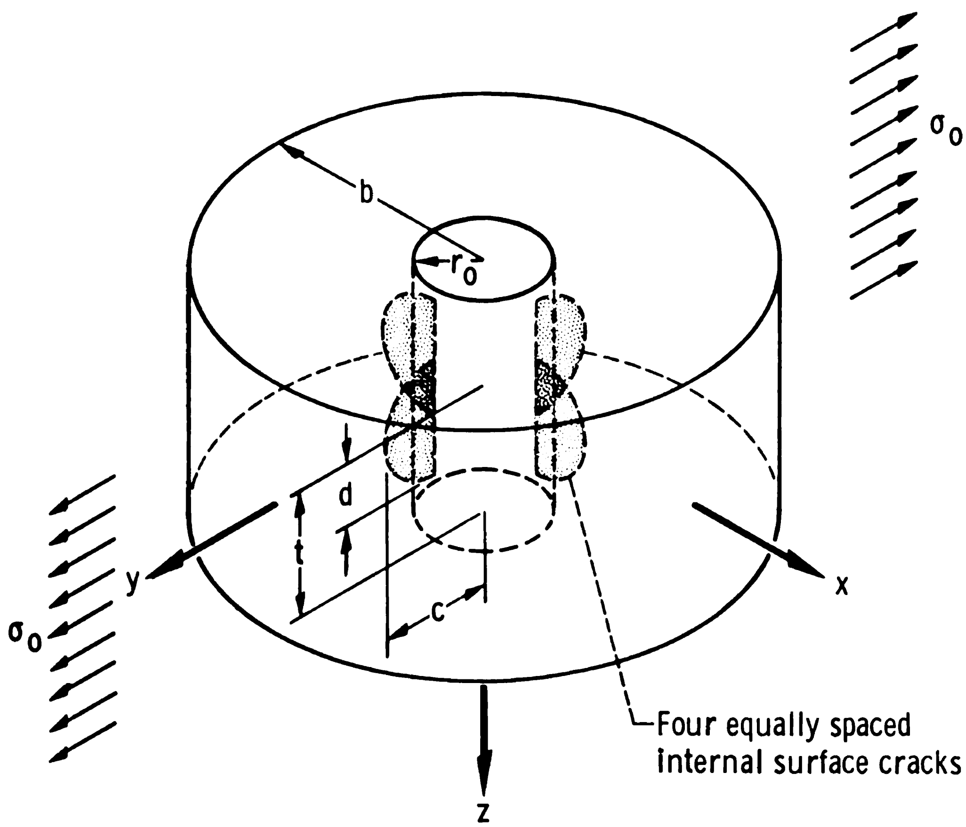

Figure 5. - Annular plate with internal surface cracks under uniform external tension.

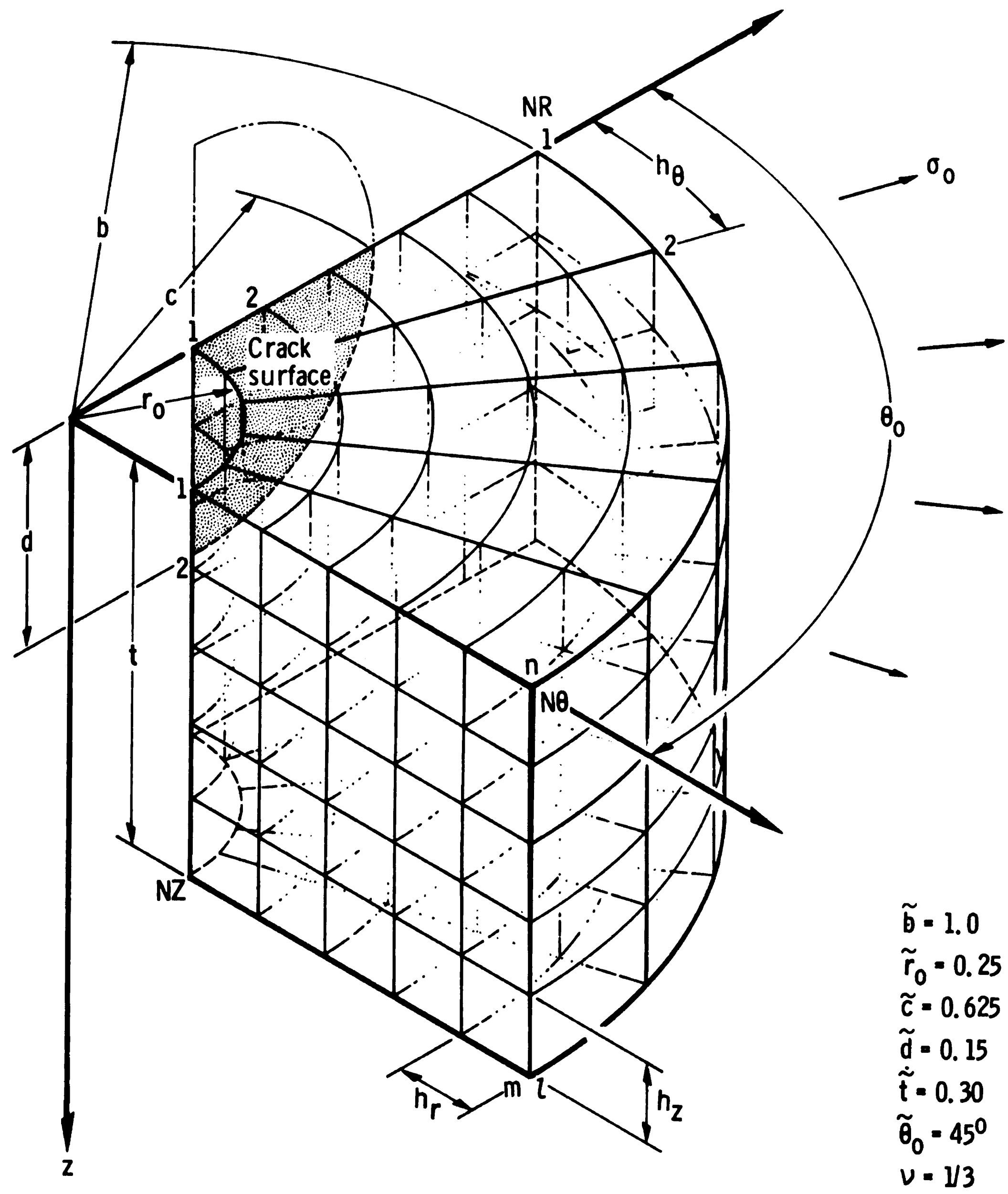

Figure 6. - Part of annular plate with internal surface cracks.

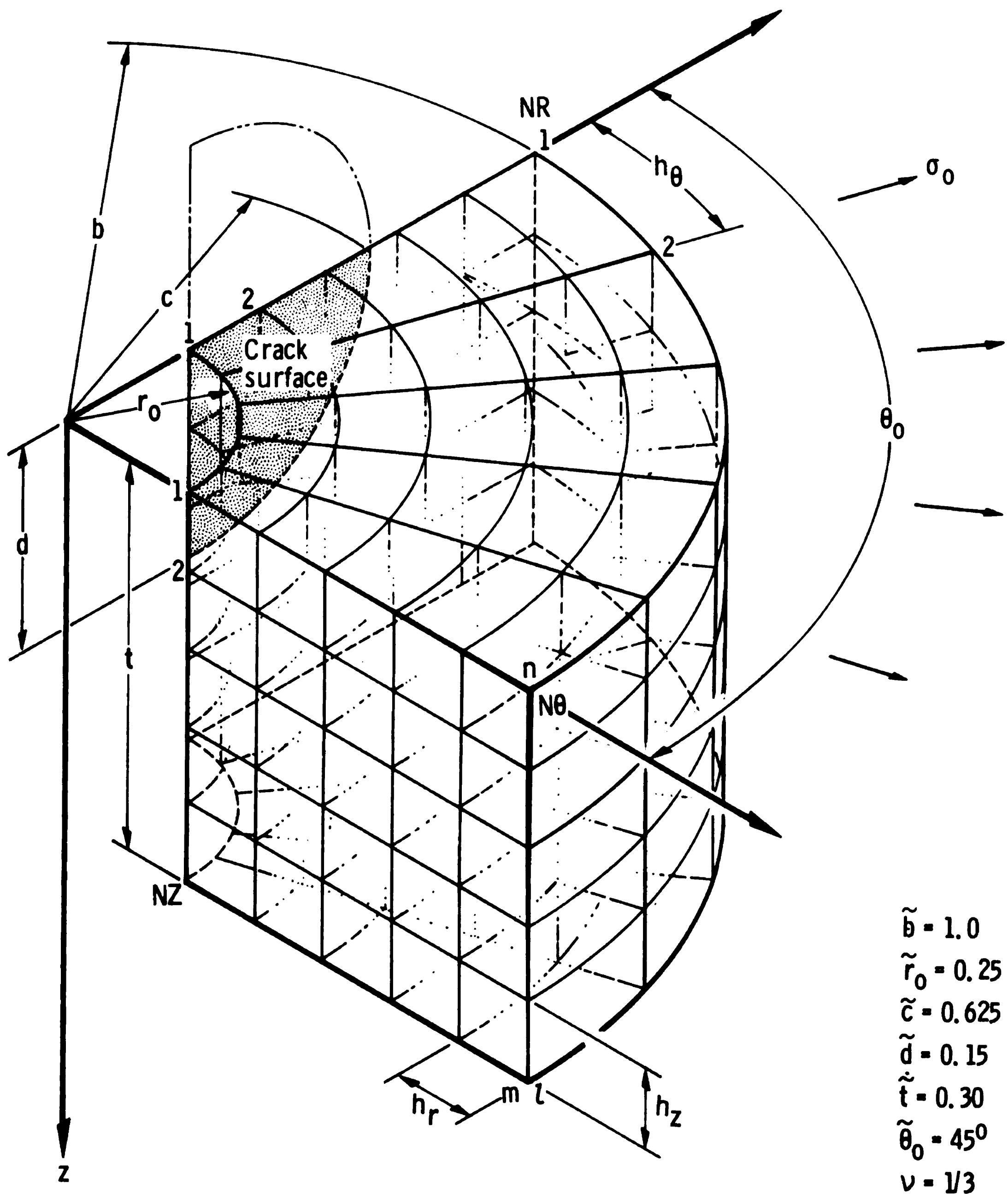

Figure 6. - Part of annular plate with internal surface cracks.

$$\{\dot{\tilde{u}}\}_{\tilde{r}=\tilde{b}} = \frac{1}{\lambda+2G}\{\sigma_o\} - \frac{\lambda}{\lambda+2G}\left(\frac{1}{\tilde{b}}\{\dot{\tilde{v}}\} + \frac{1}{\tilde{b}}\{\tilde{u}\} + \{\dot{\tilde{w}}\}\right)_{\tilde{r}=\tilde{b}} \qquad (3.46)$$

Using equation (3.45) in the definition of $\{F_2\}$ and combining this with the definition of $\{F_1\}$, the following equation will be obtained:

$$\{F_2\} = -\frac{\lambda}{\lambda+2G}\left\{\frac{\dot{\tilde{v}}}{\tilde{r}_o}\right\}_{\tilde{r}=\tilde{r}_o} - \frac{\lambda}{\lambda+2G}\{\dot{\tilde{w}}\}_{\tilde{r}=\tilde{r}_o} + \frac{2G}{(\lambda+2G)\tilde{r}_o^2}\{F_1\} \qquad (3.47)$$

In the partitioned matrix form, solution (3.33) at $\tilde{r} = \tilde{b}$ can be written as

$$\begin{Bmatrix} \{\tilde{b}\tilde{u}\} \\ \\ \left\{\dot{\tilde{u}} + \frac{\tilde{u}}{\tilde{b}}\right\} \end{Bmatrix} = \begin{bmatrix} [\Omega_{11}(K_r,\tilde{b})] & [\Omega_{12}(K_r,\tilde{b})] \\ \\ [\Omega_{21}(K_r,\tilde{b})] & [\Omega_{22}(K_r,\tilde{b})] \end{bmatrix} \begin{pmatrix} \{F_1\} + \{B_1(\tilde{b})\} \\ \\ \{F_2\} + \{B_2(\tilde{b})\} \end{pmatrix} \qquad (3.48)$$

From equation (3.48) two matrix equations can be constructed which when combined with equation (3.46) lead to a similar equation to (3.47) relating the vectors $\{F_1\}$ and $\{F_2\}$. The result of this manipulation is

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$$\{F_1\} = \frac{\lambda}{\lambda+2G}[\Omega_a]^{-1}\left(\left\{\frac{\dot{\tilde{v}}}{\tilde{b}}\right\}+\left\{\dot{\tilde{w}}\right\}\right)_{\tilde{r}=\tilde{b}} - [\Omega_a]^{-1}\left\{\frac{\sigma_o}{\lambda+2G}\right\}$$

$$- \{B_1(\tilde{b})\} - [\Omega_a]^{-1}[\Omega_b]\{F_2\} - [\Omega_a]^{-1}[\Omega_b]\{B_2(\tilde{b})\} \quad (3.49)$$

where

$$[\Omega_a] = \frac{2G}{(\lambda+2G)\tilde{b}^2}[\Omega_{11}(K_r,\tilde{b})] - [\Omega_{21}(K_r,\tilde{b})]$$

$$[\Omega_b] = \frac{2G}{(\lambda+2G)\tilde{b}^2}[\Omega_{12}(K_r,\tilde{b})] - [\Omega_{22}(K_r,\tilde{b})]$$

Simultaneous solution of equations (3.47) and (3.49) gives us the following results:

$$\{F_1\} = \frac{\lambda}{\lambda+2G}[\Omega_c]^{-1}[\Omega_a]^{-1}\left(\left\{\frac{\dot{\tilde{v}}}{\tilde{b}}\right\}+\left\{\dot{\tilde{w}}\right\}\right)_{\tilde{r}=\tilde{b}}$$

$$+ \frac{\lambda}{\lambda+2G}[\Omega_c]^{-1}[\Omega_a]^{-1}[\Omega_b]\left(\left\{\frac{\dot{\tilde{v}}}{\tilde{r}_o}\right\}+\left\{\dot{\tilde{w}}\right\}\right)_{\tilde{r}=\tilde{r}_o}$$

$$- [\Omega_c]^{-1}[\Omega_a]^{-1}\left\{\frac{\sigma_o}{\lambda+2G}\right\} - [\Omega_c]^{-1}\{B_1(\tilde{b})\}$$

$$- [\Omega_c]^{-1}[\Omega_a][\Omega_b]\{B_2(\tilde{b})\} \quad (3.50)$$

where

$$[\Omega_c] = \left([I] + \frac{2G}{(\lambda+2G)\tilde{r}_o^{\,2}}[\Omega_a]^{-1}[\Omega_b]\right)$$

$$\{F_2\} = \left(\frac{2G\lambda}{(\lambda+2G)^2\tilde{r}_o^{\,2}}[\Omega_c]^{-1}[\Omega_a]^{-1}[\Omega_b] - \frac{\lambda}{\lambda+2G}[I]\right)\left(\left\{\frac{\mathring{\tilde{v}}}{\tilde{r}_o}\right\} + \left\{\overset{c}{\tilde{w}}\right\}\right)_{\tilde{r}=\tilde{r}_o}$$

$$- \frac{2G}{(\lambda+2G)\tilde{r}_o^{\,2}}[\Omega_c]^{-1}[\Omega_a]^{-1}\left\{\frac{\sigma_o}{\lambda+2G}\right\}$$

$$+ \frac{2G\lambda}{(\lambda+2G)^2\tilde{r}_o^{\,2}}[\Omega_c]^{-1}[\Omega_a]^{-1}\left(\left\{\frac{\mathring{\tilde{v}}}{\tilde{b}}\right\} + \left\{\overset{c}{\tilde{w}}\right\}\right)_{\tilde{r}=\tilde{b}}$$

$$- \frac{2G}{(\lambda+2G)\tilde{r}_o^{\,2}}\left([\Omega_c]^{-1}\{B_1(\tilde{b})\} + [\Omega_c]^{-1}[\Omega_a]^{-1}[\Omega_b]\{B_2(\tilde{b})\}\right) \tag{3.51}$$

Equations (3.50) and (3.51) define the necessary initial vectors for the case of zero inside surface radial stress and for a given radial stress σ_o on the outside.

For the problem shown in Figure 6, the boundary conditions in the z direction are analogous to those developed in Chapter 2 for the x-directional boundary conditions. As a result of using equation (3.8), the analogous equation to (2.90), of course, will contain additional terms because of the cylindrical coordinates.

Along the circumferential direction, the following boundary conditions are enforced in the crack plane through the use of vectors $\{F_{3c}\}$ and $\{F_{4c}\}$ which are defined analogously to those in equation (2.93):

$$\{\tilde{v}(o)\}_{\text{outside crack}} = \{0\} \tag{3.52}$$

$$\{\tilde{\dot{v}}(o)\}_{\substack{\text{over}\\ \text{crack}}} = -\{\tilde{u}\}_{\substack{\text{over}\\ \text{crack}}} - \frac{\lambda}{\lambda+2G}(\{\tilde{\dot{ru}}\} + \{\tilde{\dot{rw}}\})_{\substack{\text{over}\\ \text{crack}}} \qquad (3.53)$$

and from the symmetry of the problem

$$\{\tilde{v}(\theta_o)\} = \{0\} \qquad (3.54)$$

A similar partitioning technique to that described in Chapter 2 must be followed to correctly evaluate the elements of $\{F_{3c}\}$ and $\{F_{4c}\}$. Care must be exercised in performing this operation since for surface cracks the elements of the associated matrices must be reordered to arrive at the desired form of (2.93).

Investigation of the shear stress boundary conditions involving planes of zero or uniform normal displacements will lead to similar conclusions about the coupling vectors $\{r(\tilde{r})\}$, $\{s(\tilde{\theta})\}$ and $\{t_c(\tilde{z})\}$ as in rectangular coordinate problems. As an example, we consider the symmetry condition (3.54). The zero shear stress conditions in that plane are

$$\left.\sigma_{\theta r}\right|_{\theta=\theta_o} = 0$$
$$\left.\sigma_{\theta z}\right|_{\theta=\theta_o} = 0 \qquad (3.55)$$

Using equations (3.9) and (3.11) we obtain

$$\frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{\theta}}\bigg|_{\tilde{\theta}=\tilde{\theta}_o} = \frac{\tilde{v}}{\tilde{r}}\bigg|_{\tilde{\theta}=\tilde{\theta}_o} - \frac{\partial \tilde{v}}{\partial \tilde{r}}\bigg|_{\tilde{\theta}=\tilde{\theta}_o}$$

$$\frac{1}{\tilde{r}}\frac{\partial \tilde{w}}{\partial \tilde{\theta}}\bigg|_{\tilde{\theta}=\tilde{\theta}_o} = -\frac{\partial \tilde{v}}{\partial \tilde{z}}\bigg|_{\tilde{\theta}=\tilde{\theta}_o} \tag{3.56}$$

Since $\tilde{v}$, $\frac{\partial \tilde{v}}{\partial \tilde{r}}$ and $\frac{\partial \tilde{v}}{\partial \tilde{z}}$ are all zero in the plane $\tilde{\theta} = \tilde{\theta}_o$, we find from equation (3.25) that $\{s(\tilde{\theta})\}_{\tilde{\theta}=\tilde{\theta}_o} = 0$.

Once the successive approximation procedure has converged and the displacement fields in the plate have been determined, the normal stress distributions along the sets of parallel lines can be calculated from the following equations:

$$\{\sigma_r\} = (\lambda+2G)\{\mathring{\tilde{u}}\} + \lambda\left(\left\{\frac{\tilde{u}}{\tilde{r}}\right\} + \frac{1}{\tilde{r}}\{\mathring{\tilde{v}}\} + \{\mathring{\tilde{w}}\}\right)_{\substack{\text{along radial}\\ \text{lines}}} \tag{3.57}$$

$$\{\sigma_\theta\} = (\lambda+2G)\left(\left\{\frac{1}{\tilde{r}}\,\mathring{\tilde{v}}\right\} + \left\{\frac{\tilde{u}}{\tilde{r}}\right\}\right) + \lambda(\{\mathring{\tilde{u}}\} + \{\mathring{\tilde{w}}\})_{\substack{\text{along}\\ \text{circum. lines}}} \tag{3.58}$$

$$\{\sigma_z\} = (\lambda+2G)\{\mathring{\tilde{w}}\} + \lambda\left(\{\mathring{\tilde{u}}\} + \left\{\frac{\tilde{u}}{\tilde{r}}\right\} + \left\{\frac{\tilde{v}}{\tilde{r}}\right\}\right)_{\substack{\text{along axial}\\ \text{lines}}} \tag{3.59}$$

Noting the terms in the above equations, we expect to satisfy normal stress boundary conditions again with good accuracy. The shear stresses at each node can be obtained from using equations (3.9) through (3.11). These equations, however, can be evaluated only through the use of finite differences. Computed stress and displacement results for the problem of Figure 6 are tabulated in the following chapter.

3.7 Axisymmetric Problems

The field equations (3.1) through (3.3) and the stress-displacements relations (3.6) through (3.11) are greatly simplified for problems that have circular symmetry. It is known that for problems of this geometry, the circumferential displacement is inherently zero at every point and all the remaining variables are independent of the circumferential variable θ. Equations (3.1) through (3.11) are then reduced to the following form:

$$\frac{\partial e_s}{\partial r} + (1-2\nu)\left[\left(\nabla_s^2 - \frac{1}{r^2}\right)\right]u = 0 \qquad (3.60)$$

$$\frac{\partial e_s}{\partial z} + (1-2\nu)\,\nabla_s^2 w = 0 \qquad (3.61)$$

where the dilatation, e_s, and the Laplacian, ∇_s^2, are given by

$$e_s = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \qquad (3.62)$$

$$\nabla_s^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad (3.63)$$

The stresses in terms of displacement variables are

$$\sigma_{rs} = (\lambda+2G)\,\frac{\partial u}{\partial r} + \lambda\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right) \qquad (3.64)$$

$$\sigma_{\theta s} = (\lambda+2G)\,\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right) \qquad (3.65)$$

$$\sigma_{zs} = (\lambda+2G)\,\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) \qquad (3.66)$$

$$\sigma_{rzs} = G\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \qquad (3.67)$$

The other two shear stresses are zero at every point in the body. Solution of these equations can again be obtained by the method of lines together with the successive approximation procedure and the applicable boundary conditions.

3.7.1 Ordinary Differential Equations and Boundary Conditions in the Radial Direction

Since at this time the solution of only two partial differential equations is desired, two sets of parallel lines are constructed along the axes of the independent variables. Figure 7 shows an arbitrary axisymmetric cylindrical body with the necessary discretization. For convenience the lines are evenly spaced with h_r and h_z each equal to some given constant. The radius of the solid is assumed to be uniform so that the length of all radial and axial lines is the same. The uniform spacing of lines again provides an advantage for more easily evaluating the resulting differential equations. This follows from the axisymmetric condition since for these problems both sets of ordinary differential equations have constant coefficients. However, the particular solutions for these problems are limited in the same manner as those for the previous cases.

For the solution of equation (3.60), the ordinary differential equations are developed along the radial lines in Figure 7.

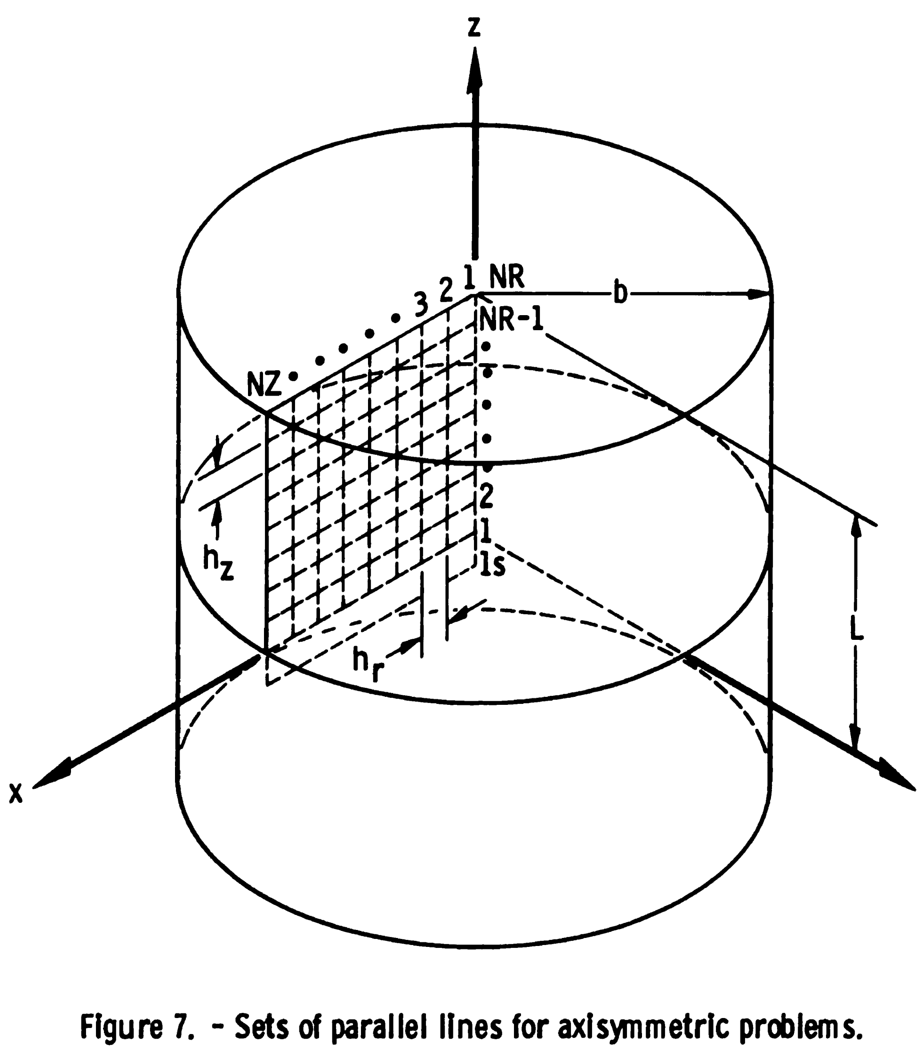


Figure 7. - Sets of parallel lines for axisymmetric problems.

The radial displacements along these lines will be denoted as $u_1, u_2, \ldots, u_{NR}$. We define $\dot{w}|_1, \dot{w}|_2, \ldots, \dot{w}|_{NR}$ as the derivatives of the axial displacements of the same points on these lines with respect to z. Using equation (3.62) and (3.63) in equation (3.60), the following equation is obtained along the first radial line of Figure 7:

$$\left[\frac{d^2u_1}{dr^2} + \frac{1}{r}\frac{du_1}{dr} - \frac{u_1}{r^2}\right] + \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{\partial^2 u_1}{\partial z^2}\right] + \frac{1}{2(1-\nu)}\left.\frac{d\dot{w}}{dr}\right|_1 = 0 \quad (3.68)$$

where

$$\frac{\partial^2 u_1}{\partial z^2} = \frac{u_2 - 2u_1 + u_{1s}}{h_z^2} \quad (3.69)$$

The zero shear stress condition on the plane $z = 0$, gives

$$u_{1s} = 2h_z \left.\frac{dw}{dr}\right|_1 + u_2 \quad (3.70)$$

Combining equations (3.68), (3.69) and (3.70) yields

$$\left[\frac{d^2u_1}{dr^2} + \frac{1}{r}\frac{du_1}{dr} - \frac{u_1}{r^2}\right] + \frac{2(1-2\nu)}{2(1-\nu)h_z^2}\,u_2 - \frac{2(1-2\nu)}{2(1-\nu)h_z^2}\,u_1 + \frac{f_1(r)}{2(1-\nu)} = 0 \quad (3.71)$$

where

$$f_1(r) = \left.\frac{d\dot{w}}{dr}\right|_1 + (1-2\nu)\left(\frac{2}{h_z}\right)\left.\frac{dw}{dr}\right|_1 \quad (3.72)$$

Similar differential equations are obtained for the other radial lines. Noting that equation (3.71) was developed for a surface line, the form of the interior line equations will differ according to the application of known shear stress conditions. The above equation can again be non-dimensionalized with respect to a crack dimension "a" in accordance with equations (2.28) and (3.19). Using matrix notation, the system of differential equations along the radial lines can be written as

$$\left(\frac{d^2}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}} - \frac{1}{\tilde{r}^2}\right) \underset{NRx1}{\{\tilde{u}\}} = \underset{NRxNR}{[K_{rs}]}\ \underset{NRx1}{\{\tilde{u}\}} + \underset{NRx1}{\{r_s(\tilde{r})\}} \tag{3.73}$$

where the coefficient matrix $[K_{rs}]$ and the column vectors $\{\tilde{u}\}$ and $\{r_s(\tilde{r})\}$ are given below.

$$\underset{NRxNR}{[K_{rs}]} = \begin{bmatrix} 2k_3 & -2k_3 & 0 & 0 & 0 \\ -k_3 & 2k_3 & -k_3 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -k_3 & 2k_3 & -k_3 \\ 0 & 0 & 0 & -2k_3 & 2k_3 \end{bmatrix} \tag{3.74}$$

$$k_3 = \frac{(1-2\nu)}{2(1-\nu)}\left(\frac{1}{\tilde{h}_z^2}\right)$$

$$\underset{NR\times 1}{\{\tilde{u}\}} = \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_{NR-1} \\ \tilde{u}_{NR} \end{Bmatrix};\underset{NR\times 1}{\{r_s(\tilde{r})\}} = \frac{-1}{2(1-\nu)} \begin{Bmatrix} \left.\frac{d\dot{\tilde{w}}}{d\tilde{r}}\right|_1 + (1-2\nu)\frac{2}{\tilde{h}_z}\left.\frac{d\tilde{w}}{d\tilde{r}}\right|_1 \\ \left.\frac{d\dot{\tilde{w}}}{d\tilde{r}}\right|_2 \\ \vdots \\ \left.\frac{d\dot{\tilde{w}}}{d\tilde{r}}\right|_{NR-1} \\ \left.\frac{d\dot{\tilde{w}}}{d\tilde{r}}\right|_{NR} - (1-2\nu)\frac{2}{\tilde{h}_z}\left.\frac{d\tilde{w}}{d\tilde{r}}\right|_{NR} \end{Bmatrix} \quad (3.75)$$

In contrast to the general radial coefficient matrix (3.21), the matrix (3.74) has constants for its elements. Note that it is also singular.

Although the above matrices can also be obtained by reducing the general cylindrical coordinate problem of Section 3.2, this reduction is not so evident. Assuming that $\{r_s(\tilde{r})\}$ is known, closed form solutions of equations (3.73) can be obtained.

3.7.2 Ordinary Differential Equations and Boundary Conditions in the Axial Direction

Solution of equation (3.61) is obtained by constructing a set of ordinary differential equations along the axial lines of Figure 7. The axial displacements along these lines are denoted as $w_1, w_2, \ldots, w_{NZ}$. We define $\dot{u}|_1, \dot{u}|_2, \ldots, \dot{u}|_{NZ}$ as the derivatives of the radial displacements of the same points on these lines with respect to r. Using equation (3.62) and

and (3.63) in equation (3.61) the following equation is developed along the first axial line of Figure 7:

$$\left.\frac{d\dot{u}}{dz}\right|_1 + \frac{1}{r_1}\left.\frac{du}{dz}\right|_1 + \frac{d^2w_1}{dz^2} + (1-2\nu)\left[\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r_1}\frac{\partial w_1}{\partial r} + \frac{d^2w_1}{dz^2}\right] = 0 \tag{3.76}$$

Using equation (3.67) and the known zero shear stress condition along the axis of the cylinder, we have

$$\frac{1}{r_1}\frac{\partial w_1}{\partial r} = -\frac{1}{r_1}\left.\frac{du}{dz}\right|_1 \tag{3.77}$$

Symmetry of the problem also leads to

$$\frac{\partial^2 w_1}{\partial r^2} = \frac{1}{h_r^2}(w_2 - 2w_1 + w_2) \tag{3.78}$$

Since $r_1 = 0$ along the axis of the cylinder, the terms containing r_1 in the denominator must be investigated. For a uniformly loaded axisymmetric problem, the radial displacements along the cylinder axis are inherently zero. This implies that the term $\frac{1}{r_1}\left.\frac{du}{dz}\right|_1$ is indeterminate and L'Hospital's rule can be used to find its limit. Thus, we have

$$\lim_{r \to r_1}\left(\frac{1}{r}\left.\frac{du}{dz}\right|_1\right) = \frac{\frac{\partial}{\partial r}\left(\left.\frac{du}{dz}\right|_1\right)}{1} = \left.\frac{d\dot{u}}{dz}\right|_1 \tag{3.79}$$

Substituting equations (3.77), (3.78) and (3.79) into equation (3.76) gives

$$\frac{d^2w_1}{dz^2} - \frac{2(1-2\nu)}{2(1-\nu)h_r^2} w_1 + \frac{2(1-2\nu)}{2(1-\nu)h_r^2} w_2 + \frac{f_1(z)}{2(1-\nu)} = 0 \qquad (3.80)$$

where

$$f_1(z) = (1+2\nu) \left.\frac{d\dot{u}}{dz}\right|_1 \qquad (3.81)$$

Similar differential equations are obtained for the other axial lines. Since equation (3.80) was developed for the special case of a line along the cylinder axis, the form of the equations along the interior and surface lines will differ according to the position of the line. It will be convenient to non-dimensionalize these equations according to the method of the previous section. Using matrix notation, the system of differential equations along the axial lines of Figure 7 can be written as

$$\underset{NZx1}{\frac{d^2}{d\tilde{z}^2}\{\tilde{w}\}} = \underset{NZxNZ}{[K_{zs}]} \underset{NZx1}{\{\tilde{w}\}} + \underset{NZx1}{\{t_s(\tilde{z})\}} \qquad (3.82)$$

From Figure 7 it can be noted that for any given axial line the radius is a constant. We define the radius at each axial line as $\tilde{r}_i$, $i = 1, 2, \ldots NZ$, where $\tilde{r}_1 = 0$ and $\tilde{r}_{NZ} = \tilde{b}$. For a uniform radial increment $\tilde{h}_r$, the radius at any point is $(i-1)\tilde{h}_r$. Using this notation, the coefficient matrix $[K_{zs}]$ can be written as follows:

$$\underset{NZ\times NZ}{[K_{zs}]} = \frac{k_{18}}{2}\left[\begin{array}{c|c|c|c|c|c} 2 & -2 & 0 & 0 & 0 & 0 \\ \hline -\frac{1}{2} & 2 & -\frac{3}{2} & 0 & 0 & 0 \\ \hline 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \hline 0 & 0 & -\left(\frac{2i-3}{2i-2}\right) & \ddots & -\left(\frac{2i-1}{2i-2}\right) & 0 \\ \hline 0 & 0 & 0 & \ddots & 2 & \ddots \\ \hline 0 & 0 & 0 & 0 & -2 & 2 \end{array}\right] \qquad (3.83)$$

$$k_{18} = \frac{(1-2\nu)}{2(1-\nu)}\left[\frac{2}{h_r^2}\right]$$

$$i = 2, 3, \ldots, NZ-1$$

The upper diagonal elements of $[K_{zs}]$ are given by $-\left(\frac{2i-1}{2i-2}\right)\frac{k_{18}}{2}$ while its lower diagonal elements are given by $-\left(\frac{2i-3}{2i-2}\right)\frac{k_{18}}{2}$ for $i = 2, 3, \ldots, NZ-1$. Note that the elements of $[K_{zs}]$ are all constants and since the sum of the elements in any given row is zero, the coefficient matrix is also singular. The column vectors in equation (3.82) are listed below.

$$\underset{NZ\times 1}{\{\tilde{w}\}^T} = \lfloor\, \tilde{w}_1 \quad \tilde{w}_2 \quad \tilde{w}_3 \quad \ldots \quad \tilde{w}_{NZ} \,\rfloor$$

$$
\underset{NZ\times 1}{\{t_s(\tilde z)\}} = \frac{-1}{2(1-\nu)}
\left\{
\begin{array}{l}
(1+2\nu)\left.\dfrac{d\dot{\tilde u}}{d\tilde z}\right|_1 \\[2ex]
\left.\dfrac{d\dot{\tilde u}}{d\tilde z}\right|_2 + \dfrac{1}{\bar h_r}\left.\dfrac{d\tilde u}{d\tilde z}\right|_2 \\[2ex]
\left.\dfrac{d\dot{\tilde u}}{d\tilde z}\right|_3 + \dfrac{1}{2\bar h_r}\left.\dfrac{d\tilde u}{d\tilde z}\right|_3 \\[1ex]
\quad\vdots \qquad\qquad \vdots \\[1ex]
\left.\dfrac{d\dot{\tilde u}}{d\tilde z}\right|_i + \dfrac{1}{(i-1)\bar h_r}\left.\dfrac{d\tilde u}{d\tilde z}\right|_i \\[1ex]
\quad\vdots \qquad\qquad \vdots \\[1ex]
\left.\dfrac{d\dot{\tilde u}}{d\tilde z}\right|_{NZ-1} + \dfrac{1}{(NZ-2)\bar h_r}\left.\dfrac{d\tilde u}{d\tilde z}\right|_{NZ-1} \\[2ex]
\left.\dfrac{d\dot{\tilde u}}{d\tilde z}\right|_{NZ} + \dfrac{1}{\bar b}\dfrac{d\tilde u}{d\tilde z} + (1-2\nu)\left[\dfrac{-2}{\bar h_r}\dfrac{d\tilde u}{d\tilde z} - \dfrac{1}{\bar b}\dfrac{d\tilde u}{d\tilde z}\right]_b
\end{array}
\right\}
\tag{3.84}
$$

$$i = 2, 3, \ldots, NZ-1$$

Assuming that $\{t_s(\tilde z)\}$ is known, the solution of equations (3.82) can be obtained by the previously discussed matrix methods.

3.7.3 Solutions of the Ordinary Differential Equations for Axisymmetric Problems

Although the methods discussed previously can be employed to solve equations (3.73) and (3.82), there are certain simplifications that are possible. First of all we note that the coefficient matrix $[K_{rs}]$ is a tri-diagonal matrix having the desired form of (A.1). Thus, a closed form solution of the eigenvalues and eigenvectors of $[K_{rs}]$ is possible and from equations (A.16) and (A.18) these eigenvalues and eigenvectors are respectively

$$\bar{\lambda}_i = 2k_3 \left[1 - \cos\left(\frac{i-1}{NR-1}\right)\pi\right] \quad i = 1, 2, \ldots, NR \tag{3.85}$$

$$[P_{sjn}] = \left[\cos \frac{(j-1)n\pi}{NR-1}\right] \quad \begin{array}{l} j = 1, 2, \ldots, NR \\ n = 0, 1, 2, \ldots, NR-1 \end{array} \tag{3.86}$$

Using the above matrix of eigenvectors, we diagonalize the coefficient matrix as in equation (A.22). Following the solution of (3.33), the matrizant of [A] can still be found by solving the matrix equations (3.35) and (3.36) subject to the initial conditions (3.37). However, these matrix functions are all in diagonalized form now, since $[K_{rs}]$ is a diagonal matrix. The amount of numerical computations in the Runge-Kutta integration procedure is greatly reduced. The diagonalized matrix functions are then transformed into full matrices by the appropriate similarity transformations at each position along the radius. It must be noted that for axisymmetric problems an analogous variable to equation (3.38) must be introduced. Since $[K_{rs}]$ is a matrix of constants and the matrix [A] has the form of

(3.31) this variable is $\tilde{r}[\Omega_{21}]$ rather than $\tilde{r}^3[\Omega_{21}]$. The analogous matrix equations to (3.39) are

$$\frac{d}{d\tilde{r}}[\Omega_{11}] = [\Omega_{21}^*]; \qquad [\Omega_{11}]_{\tilde{r}=o} = [I]$$

$$\frac{d}{d\tilde{r}}[\Omega_{21}^*] = \frac{1}{\tilde{r}}[\Omega_{21}^*] + [\Omega_{11}][K_{rs}]; \; [\Omega_{21}^*]_{\tilde{r}=o} = [0] \tag{3.87}$$

Once the matrix functions $[\Omega_{ij}]$ have been determined, the particular integrals can be obtained from analogous equations to (2.84) and (2.85). The solution of equations (3.73) are then found from using equations (3.43) and (3.44). In the numerical examples for the cylinder with a penny shaped crack, the above procedure was followed in evaluating the radial solutions.

An alternate approach to the solution of equations (3.73) may be used which avoids the problem of evaluating the matrizant of [A]. Let us define a set of variables, $\{\overline{\eta}(\tilde{r})\}$, by the transformation of

$$\{\tilde{u}(\tilde{r})\} = [P_s]\{\overline{\eta}(\tilde{r})\} \tag{3.88}$$

where $[P_s]$ is the modal matrix of $[K_{rs}]$. Substituting equation (3.88) into (3.73) and pre-multiplying the result by $[P_s]^{-1}$ we obtain a set of uncoupled second order differential equations in $\{\overline{\eta}\}$.

$$\left(\frac{d^2}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{d}{d\tilde{r}} - \frac{1}{\tilde{r}^2}\right)\{\overline{\eta}\} = [\Lambda_{rs}]\{\overline{\eta}\} + [P_s]^{-1}\{r_s(\tilde{r})\} \tag{3.89}$$

The elements of $[\Lambda_{rs}]$ are the eigenvalues of $[K_{rs}]$ and $[\Lambda_{rs}]$ is a diagonal matrix. Equations (3.89) then constitute an uncoupled set of modified Bessel's equations of the first order containing parameters $\overline{\lambda}_i$ (40). The solution of an equation of this type is well known and may be written as shown below (41). The solution is written, say for the first equation of the above set.

$$\overline{\eta}_1(\tilde{r}) = \int_0^{\tilde{r}} f_1(\rho_1)[I_1(\overline{\lambda}_1\rho_1)K_1(\overline{\lambda}_1\tilde{r}) - K_1(\overline{\lambda}_1\rho_1)I_1(\overline{\lambda}_1\tilde{r})]d\rho_1$$

$$\div \frac{1}{I_1(\overline{\lambda}_1\rho_1)[-\overline{\lambda}_1K_o(\overline{\lambda}_1\rho_1)-\frac{1}{\rho_1}K_1(\overline{\lambda}_1\rho_1)]-K_1(\overline{\lambda}_1\rho_1)[\overline{\lambda}_1I_o(\overline{\lambda}_1\rho_1)-\frac{1}{\rho_1}I_1(\overline{\lambda}_1\rho_1)]}$$

$$+ \overline{C}_1 I_1(\overline{\lambda}_1\tilde{r}) + \overline{C}_2 K_1(\overline{\lambda}_1\tilde{r}) \qquad (3.90)$$

where $I_o(\overline{\lambda}_1\tilde{r})$, $K_o(\overline{\lambda}_1\tilde{r})$, $I_1(\overline{\lambda}_1\tilde{r})$ and $K_1(\overline{\lambda}_1\tilde{r})$ are modified Bessel functions of the first and second kind. $\overline{C}_1$ and $\overline{C}_2$ are constants of integration which must be evaluated from the boundary conditions and $f_1(\rho_1)$ is the non-homogeneous term in equations (3.89) obtained from the product of the first row in $[P_s]^{-1}$ by $\{r_s(\tilde{r})\}$. Given boundary conditions may be also transformed according to (3.88) in order to obtain the necessary conditions on $\{\overline{\eta}\}$. Once the above solutions have been evaluated, an inverse transformation will give us the closed form solutions of $\{\tilde{u}\}$. This solution method, however, is only possible for axisymmetric problems.

Since a closed form solution of the associated eigenvalue problem of $[K_{zs}]$ is not possible, the matrix series methods of Chapter 2 must be used to find the solutions of equations (3.82). Because the two sets of differential equations in the radial and axial directions cannot be decoupled, the previously described successive approximation procedure must also be utilized.

3.8 Application to Specific Geometries

Until this section, the results of the numerical examples presented could not be checked against known solutions since none were previously available. At this time, however, we propose to solve the problem of a cylindrical solid containing a penny shaped crack and loaded by uniform tension normal to the crack plane. A closed form solution for a similar geometry and loading has been obtained previously by Sneddon (42) with the exception that his cylinder was infinitely large. This problem can be regarded as the three-dimensional extension of the Griffith crack problem and its solution is one of the few analytical solutions available today. Hence, as the length and diameter of a finite geometry cylinder are increased, the solutions obtained from the line method should approach Sneddon's solution.

This problem is also representative in that the less accurate numerical solution is used to evaluate some of the required matrix functions. A relatively coarse grid is selected so that the previously mentioned advantage of the line method can be established. It is the primary purpose of this example to examine the rate of convergence and the accuracy of the line method in solving three-dimensional, mixed boundary condition, elasticity problems.

3.8.1 Solid Cylindrical Bar With a Penny Shaped Crack

Figure 8 shows a cylindrical bar containing a penny shaped crack and loaded by a uniform normal stress distribution. Any numerical approach, of course, is restricted to a finite geometry problem and strictly speaking Sneddon's solution cannot be duplicated by an analysis using the line method. In the following analysis, only a few convenient values of $\tilde{b}$ and $\tilde{L}$ will be used to show that Sneddon's solution is approached as the length and diameter are increased. An examination of Sneddon's solution shows that a change in radius is more significant for the agreement of the two solutions, than a change in bar length.

In constructing a set of lines for a given crack geometry, the question of the crack edge location relative to the nodes must be considered. Since we specify boundary conditions at the nodes only and it is through given boundary data that a crack surface is specified, we shall assume throughout this paper that the crack edge is located midway between adjacent nodes. Then

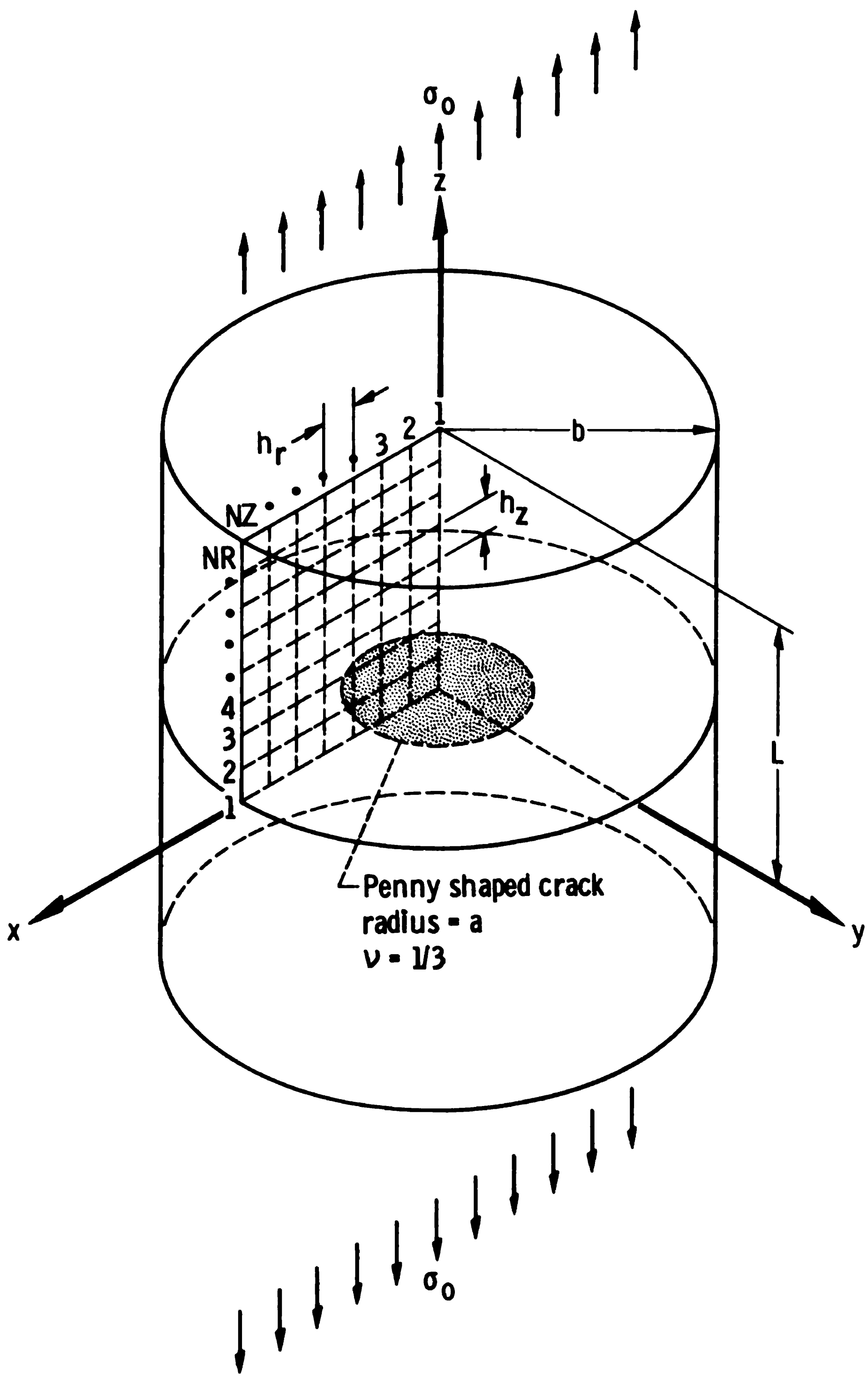


Figure 8. - Solid cylindrical bar with a penny shaped crack.

as we refine the selected grid, the exact crack edge location becomes more accurately established. Note that in the limit, this grid refinement would require the impossible task of satisfying mixed boundary conditions at the same point. In addition, plane elasticity solutions have shown that displacement gradients become singular at that point.

Since the solution of the axisymmetric problem is but a subcase of the more general cylindrical coordinate problem, the radial displacement solutions can be obtained from the following equations:

$$\{\tilde{u}(\tilde{r})\} = \frac{1}{\tilde{r}}[\Omega_{11}(K_{rs},\tilde{r})]\{B_{1s}(\tilde{r})\}$$

$$+ \frac{1}{\tilde{r}}[\Omega_{12}(K_{rs},\tilde{r})](\{F_{2s}\} + \{B_{2s}(\tilde{r})\}) \tag{3.91}$$

$$\{\dot{\tilde{u}}(\tilde{r})\} = ([\Omega_{22}(K_{rs},\tilde{r})] - \frac{1}{\tilde{r}^2}[\Omega_{12}(K_{rs},\tilde{r})])(\{F_{2s}\} + \{B_{2s}(\tilde{r})\})$$

$$+ ([\Omega_{21}(K_{rs},\tilde{r})] - \frac{1}{\tilde{r}^2}[\Omega_{11}(K_{rs},\tilde{r})])\{B_{1s}(\tilde{r})\} \tag{3.92}$$

where $\tilde{r} \neq 0$.

Since we have zero radial displacements along the cylinder axis, the vector $\{F_{1s}\} = \{\tilde{r}\tilde{u}(o)\}$ is equal to zero in the general equations (3.43) and (3.44). Using the zero radial stress condition on the outside surface, the vector $\{F_{2s}\}$ can be expressed analogously to equation (3.51) as

$$\{F_{2s}\} = \frac{\lambda}{\lambda+2G}[\Omega_{bs}]^{-1}\{\dot{\tilde{w}}\}_{\tilde{r}=\tilde{b}}$$

$$- [\Omega_{bs}]^{-1}[\Omega_{as}]\{B_{1s}(\tilde{b})\} - \{B_{2s}(\tilde{b})\} \tag{3.93}$$

where matrices $[\Omega_{as}]$ and $[\Omega_{bs}]$ are defined to be the same as those in equation (3.49) except that the matrix functions are dependent on $[K_{rs}]$ rather then $[K_r]$ as before. Since at this time the point $\tilde{r} = 0$ is in the region of interest, equations (3.91) and (3.92) become at $\tilde{r} = 0$ respectively

$$\begin{aligned} \{\tilde{u}(o)\} &= 0 \\ \{\dot{\tilde{u}}(o)\} &= \frac{1}{2}\{F_{2s}\} \end{aligned} \tag{3.94}$$

The first of equations (3.94) was obtained by applying L'Hospital's rule to equations (3.91). The second equation comes from the definition of $\{F_{2s}\}$ and L'Hospital's rule.

After the necessary transformation of variables, the solutions of equations (3.82) can be expressed in a similar manner to equation (2.94). The associated matrix series, $[\tilde{D}_{ij}(K_{zs},\tilde{z})]$ and the particular integrals $\{B_{3s}(\tilde{z})\}$ and $\{B_{4s}(\tilde{z})\}$ are found the same way as those in Chapter 2. We define the initial value vectors in the z direction as follows:

$$\{W(\tilde{z})\}_{\tilde{z}=o} = \left\{ \begin{array}{c} \{\tilde{w}(o)\} \\ \{\dot{\tilde{w}}(o)\} \end{array} \right\} = \left\{ \begin{array}{c} \{F_{3s}\} \\ \{F_{4s}\} \end{array} \right\} = \left\{ \begin{array}{c} \underset{NIC\times 1}{\{F_{3s\alpha}\}} \\ \\ \underset{NOC\times 1}{\{F_{3s\beta}\}} \\ \\ \underset{NIC\times 1}{\{F_{4s\alpha}\}} \\ \\ \underset{NOC\times 1}{\{F_{4s\beta}\}} \end{array} \right\} \tag{3.95}$$

where we adopted the partioning scheme of (2.93) in order that the mixed boundary conditions in the crack plane can be conveniently handled. We follow an analogous development as in Chapter 2 from equations (2.93) through (2.97). The results of this development completely define the initial vector (3.95) and they are listed below.

$$\begin{array}{lll} \underset{NOC\times 1}{\{F_{3s\beta}\}} & = & \{0\}_{\substack{\tilde{z}=o \\ \text{outside crack}}} \\ \underset{NIC\times 1}{\{F_{4s\alpha}\}} & = & \dfrac{-\lambda}{\lambda+2G} \left(\{\dot{\tilde{u}}\} + \left\{ \dfrac{\tilde{u}}{\tilde{r}} \right\} \right)_{\substack{\tilde{z}=o \\ \text{over crack}}} \end{array} \tag{3.96}$$

Equations (3.96) are specified boundary conditions in the crack plane. We also have

$$\underset{NOCx1}{\{F_{4s\beta}\}} = \frac{1}{\lambda+2G}\left(\underset{NOCxNOC}{[D_a]^{-1}}\ \underset{NOCx1}{\{\sigma_{o\beta}\}} - \lambda\underset{NOCxNOC}{[D_a]^{-1}}\left(\underset{NOCx1}{\{\dot{\tilde{u}}_\beta\}} + \underset{NOCx1}{\left\{\frac{\tilde{u}_\beta}{\tilde{r}}\right\}}\right)_{\tilde{z}=\tilde{L}}\right)$$

$$- \frac{1}{\lambda+2G}\left(\underset{NOCxNOC}{[D_a]^{-1}}\ \underset{NOCxNIC}{[D_{21\beta1}]}\ \underset{NICxNIC}{[D_{21\alpha1}]^{-1}}\ \underset{NICx1}{\{\sigma_{o\alpha}\}}\right.$$

$$\left. - \lambda\underset{NOCxNOC}{[D_a]^{-1}}\ \underset{NOCxNIC}{[D_{21\beta1}]}\ \underset{NICxNIC}{[D_{21\alpha1}]^{-1}}\left(\underset{NICx1}{\{\dot{\tilde{u}}_\alpha\}} + \underset{NICx1}{\left\{\frac{\tilde{u}_\alpha}{\tilde{r}}\right\}}\right)_{\tilde{z}=\tilde{L}}\right)$$

$$- \underset{NOCxNOC}{[D_a]^{-1}}\ \underset{NOCxNOC}{[D_b]}\ \underset{NOCx1}{\{B_{3s\beta}(\tilde{L})\}} - \underset{NOCx1}{\{B_{4s\beta}(\tilde{L})\}}$$

$$+ \frac{\lambda}{\lambda+2G}\underset{NOCxNOC}{[D_a]^{-1}}\ \underset{NOCxNIC}{[D_c]}\left(\underset{NICx1}{\{\dot{\tilde{u}}_\alpha\}} + \underset{NICx1}{\left\{\frac{\tilde{u}_\alpha}{\tilde{r}}\right\}}\right)_{\tilde{z}=o}$$

$$- \underset{NOCxNOC}{[D_a]^{-1}}\ \underset{NOCxNIC}{[D_c]}\ \underset{NOCx1}{\{B_{4s\alpha}(\tilde{L})\}} \tag{3.97}$$

Matrices $[D_a]$, $[D_b]$, $[D_c]$ and the partitioned submatrices of $[D_{ij}]$ are the same as those in equations (2.96) and (2.97) except that in this case they are functions of $(K_{zs}, \tilde{z}=\tilde{L})$. Note that the particular integrals, the applied stress vector, the radial displacements and its derivatives are also partitioned according to their location with respect to the crack. The crack opening displacement is given by

$$
\begin{aligned}
\underset{NICx1}{\{F_{3s\alpha}\}} = \frac{1}{\lambda+2G}\Bigg(&\underset{NICxNIC}{[D_{21\alpha 1}]^{-1}}\ \underset{NICx1}{\{\sigma_{o\alpha}\}} - \lambda\underset{NICxNIC}{[D_{21\alpha 1}]^{-1}}\left(\underset{NICx1}{\{\dot{\tilde{u}}_\alpha\}} + \underset{NICx1}{\left\{\frac{\tilde{u}_\alpha}{\tilde{r}}\right\}}\right)_{\tilde{z}=\tilde{L}}\Bigg) \\
&- \underset{NICx1}{\{B_{3s\alpha}(\tilde{L})\}} - \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}}\ \underset{NICxNOC}{[D_{21\alpha 2}]}\ \underset{NOCx1}{\{B_{3s\beta}(\tilde{L})\}} \\
&- \underset{NOCxNIC}{[D_{21\alpha 1}]^{-1}}\ \underset{NICxNIC}{[D_{22\alpha 1}]}\ \underset{NICx1}{\{B_{4s\alpha}(\tilde{L})\}} \\
&+ \frac{\lambda}{\lambda+2G}\underset{NICxNIC}{[D_{21\alpha 1}]^{-1}}\ \underset{NICxNIC}{[D_{22\alpha 1}]}\left(\underset{NICx1}{\{\dot{\tilde{u}}_\alpha\}} + \underset{NICx1}{\left\{\frac{\tilde{u}}{\tilde{r}}\right\}}\right)_{\tilde{z}=o} \\
&- \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}}\ \underset{NICxNOC}{[D_{22\alpha 2}]}\ \underset{NOCx1}{\{F_{4s\beta}\}} \\
&- \underset{NICxNIC}{[D_{21\alpha 1}]^{-1}}\ \underset{NICxNOC}{[D_{22\alpha 2}]}\ \underset{NOCx1}{\{B_{4s\beta}(\tilde{L})\}} \qquad (3.98)
\end{aligned}
$$

where $\{F_{4s\beta}\}$ is given by equation (3.97). Although the matrix $[D_{21}(K_{zs},\tilde{L})]$ is singular, the partitioned matrices are not. In calculating the above equations, the indeterminate terms at $\tilde{r} = 0$ must be carefully considered.

Similar conclusions regarding the elements of the coupling vectors $\{r_s(\tilde{r})\}$ and $\{t_s(\tilde{z})\}$ can also be noted from the zero shear stress conditions along lines of uniform normal displacements as in equation (2.100). Equation (3.67), when applied to the coupling vectors, gives

$$
\begin{aligned}
\underset{NR\times 1}{\{r_s(\tilde{r})\}_{\tilde{r}=o}} &= \{0\} \\
\underset{NOC\times 1}{\{t_s(\tilde{z})\}_{\tilde{z}=o}} &= \{0\} \quad \text{outside crack}
\end{aligned}
\tag{3.99}
$$

Once the displacement field in the bar has been calculated and the successive approximation procedure has converged, the normal stress distributions along the radial lines can be obtained from

$$
\begin{aligned}
\{\sigma_{rs}\} &= (\lambda+2G)\{\dot{\tilde{u}}\} + \lambda\left(\left\{\frac{\tilde{u}}{\tilde{r}}\right\} + \{\dot{\tilde{w}}\}^T\right) \quad \text{for } \tilde{r} \neq o \\
\{\sigma_{rs}\}_{\tilde{r}=o} &= 2(\lambda+G)\{\dot{\tilde{u}}\}_{\tilde{r}=o} + \lambda\{\dot{\tilde{w}}\}_{\tilde{r}=o} \quad \text{for } \tilde{r} = o
\end{aligned}
\tag{3.100}
$$

$$
\begin{aligned}
\{\sigma_{\theta s}\} &= (\lambda+2G)\left\{\frac{\tilde{u}}{\tilde{r}}\right\} + \lambda(\{\dot{\tilde{u}}\} + \{\dot{\tilde{w}}\}^T) \quad \text{for } \tilde{r} \neq o \\
\{\sigma_{\theta s}\}_{\tilde{r}=o} &= 2(\lambda+G)\{\dot{\tilde{u}}\}_{\tilde{r}=o} + \{\dot{\tilde{w}}\}_{\tilde{r}=o} \quad \text{for } \tilde{r} = o
\end{aligned}
\tag{3.101}
$$

$$\{\sigma_{zs}\} = (\lambda+2G)\{\dot{\tilde{w}}\}^T + \lambda\left(\{\dot{\tilde{u}}\} + \left\{\frac{\tilde{u}}{\tilde{r}}\right\}\right) \quad \text{for } \tilde{r} \neq 0$$

$$\{\sigma_{zs}\} = (\lambda+2G)\{\dot{\tilde{w}}\}_{\tilde{r}=o} + 2\lambda\{\dot{\tilde{u}}\}_{\tilde{r}=o} \quad \text{for } \tilde{r} = 0 \tag{3.102}$$

For convenience, all the normal stresses are expressed along radial lines in the above euqations. Note that at $\tilde{r} = 0$, the radial and circumferential stresses are equal. The shear stress at each node can be calculated from

$$\{\sigma_{rzs}\} = \frac{E}{2(1-\nu)}\left(\left\{\frac{\partial w}{\partial r}\right\} + \left\{\frac{\partial u}{\partial z}\right\}\right)_{\text{along radial lines}} \tag{3.103}$$

where for the required derivatives, finite difference calculus must be used.

In using the given stress equations in cylindrical coordinates, the relationships between Lame's constants, Young's modulus and Poisson's ratio are needed, since the non-dimensionalized displacements and their derivatives are expressed in terms of some constant times $\frac{\sigma_o}{E}$. The needed identities are

$$(\lambda+2G) = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \tag{3.104}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{3.105}$$

$$(\lambda+G) = \frac{E}{2(1+\nu)(1-2\nu)} \tag{3.106}$$

Numerical values of the stress and displacement distributions for the problem of Figure 8 were calculated for a number of different geometry cylinders. These results are given and discussed in the following chapter.

3.8.2 Hollow Cylindrical Bar With a Penny Shaped Crack

A problem very similar to that described in the previous section is that shown in Figure 9. As shown in this figure, an internal hole is assumed through the center of the penny shaped crack, which introduces stress free surfaces on both sides of the crack edge. For convenience in the manipulations, the radius of this hole is taken as h_r, that is, the radial increment. The differential equations in the radial direction are identical to equations (3.73) and only their boundary conditions need be modified. Since the radius at any point in this problem can be expressed as $r_i = i \cdot h_r \quad i = 1, 2, 3, \ldots, NZ,$ the form of the coefficient matrix (3.83) will be different than for the solid cylinder. The coupling vector $\{t_s(\tilde{z})\}$ must also be modified accordingly. The results of these modifications are shown below.

$$\underset{NZx1}{\{t_{sh}(\tilde{z})\}} = \frac{-1}{2(1-\nu)} \begin{Bmatrix} \left.\dfrac{d\mathring{u}}{d\tilde{z}}\right|_1 + \dfrac{1}{\tilde{r}_1}\dfrac{d\tilde{u}}{d\tilde{z}} + (1-2\nu)\left[\left(\dfrac{2}{\tilde{h}_r} - \dfrac{1}{\tilde{r}_1}\right)\dfrac{d\tilde{u}}{d\tilde{z}}\right]_1 \\ \vdots \qquad \vdots \\ \left.\dfrac{d\mathring{u}}{d\tilde{z}}\right|_i + \dfrac{1}{\tilde{r}_i} + \left.\dfrac{d\tilde{u}}{d\tilde{z}}\right|_i \\ \vdots \qquad \vdots \\ \left.\dfrac{d\mathring{u}}{d\tilde{z}}\right|_{NZ} + \dfrac{1}{\tilde{b}}\dfrac{d\tilde{u}}{d\tilde{z}} - (1-2\nu)\left[\left(\dfrac{2}{\tilde{h}_r} + \dfrac{1}{\tilde{b}}\right)\dfrac{d\tilde{u}}{d\tilde{z}}\right]_{NZ} \end{Bmatrix} \qquad (3.107)$$

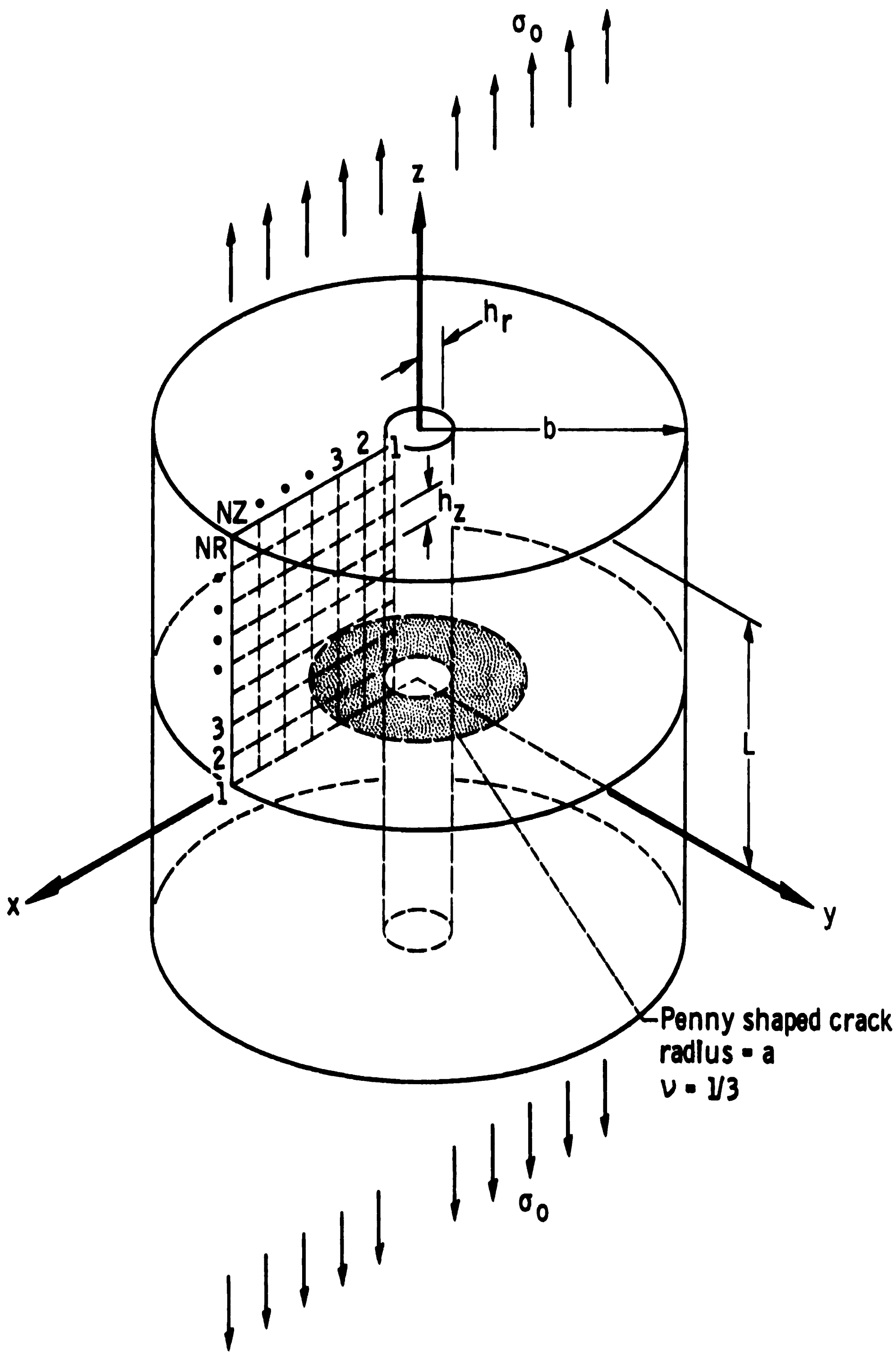


Figure 9. - Hollow cylindrical bar with a penny shaped crack.

We also have

$$[K_{zsh}]_{NZxNZ} = \frac{k_{18}}{2}\begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 2 & -\frac{5}{4} & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & -\left(\frac{2i-1}{2i}\right) & \ddots & -\left(\frac{2i+1}{2i}\right) & 0 \\ 0 & 0 & 0 & \ddots & 2 & \ddots \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix} \qquad (3.108)$$

where $i = 2, 3, \ldots, NZ-1$ for both (3.107) and (3.108).

Solutions of the radial differential equations for this problem can be obtained from similar equations to (3.43) and (3.44). The initial condition vectors $\{F_{1sh}\}$ and $\{F_{2sh}\}$ are obtained from the zero radial stress conditions on the inside and outside surfaces. The results can be derived by setting $\{\mathring{v}\}$ and $\{\sigma_o\}$ equal to zero and setting $\tilde{r}_o = \tilde{h}_r$ in equations (3.50) and (3.51).

Solutions of the axial differential equations are expressed by a similar equation to (2.94). The initial condition vectors needed in these solutions are identical to those listed in equations (3.96), (3.97) and (3.98). The normal stress distributions are found by using equations (3.100) through (3.102) where the equations for $\tilde{r} = 0$ are not applicable.

Equations similar to those described by (3.97) and (3.98) can also be developed for a displacement loaded cylinder. An applied external radial surface load could also be easily incorporated into the modified form of equations (3.50) and (3.51). Numerical results for the problem of Figure 9 are presented in the next chapter. The case of an additional external load of σ_o on the cylinder of Figure 9 was also investigated. Detailed results for this problem are of no specific interest but some general conclusions about the effect of this radial load will be reported in Section 4.2.

3.9 Stress Intensity Factor

Most of the early analytical work in fracture mechanics was based on the plane theory of elasticity. Consequently, the elastic stress field equations usually employed in the engineering analysis of real problems are based on the plane elasticity assumptions. It is customary in fracture mechanics to describe the crack opening displacement as a superposition of three basic deformation modes (5). The first mode, mode I defines an opening mode where the crack surfaces are displaced normal to the crack plane. The second mode, mode II, is described by displacements in which the crack surfaces slide over one another perpendicular to the leading edge of the crack. The third mode, mode III, defines a tearing displacement where the crack surfaces slide with respect to one another parallel to the leading edge of the crack. Superposition of these three modes is sufficient to describe the most general case of crack tip stress and displacement fields. The stress intensity factors for these three modes are designated K_I, K_{II} and K_{III} respectively. Since the examples discussed in this dissertation have geometric symmetry and are symmetrically loaded, only the opening mode of crack displacement will be obtained. Values of K_I, as subsequently defined in this section, are calculated from the obtained crack opening displacements. Knowledge of this factor for a given geometry and loading can then be possibly used to predict failure of a structural component.

Since three-dimensional problems are neither in a state of plane strain or plane stress, the definition of a stress intensity factor for these problems must be first established. The first problem to be considered in detail is Sneddon's penny shaped crack solution. Reference (42) gives the crack opening displacement as

$$w\Big|_{z=o} = \frac{4\sigma_o(1-\nu^2)}{\pi E} \sqrt{a - \text{[illegible]}} \qquad (3.109)$$

which for small values of R, where $R = a - r$, becomes

$$w\Big|_{z=o} = \frac{4\sigma_o(1-\nu^2)}{\pi E}\sqrt{2aR} \qquad (3.110)$$

Paris and Sih (5) list the stress intensity factor for this problem as follows:

$$K_I = \frac{2}{\sqrt{\pi}} \sigma_o \sqrt{a} \qquad (3.111)$$

In terms of this stress intensity factor, the crack opening displacement (3.110) becomes

$$w|_{z=o} = K_I \frac{2(1-\nu^2)\sqrt{2R}}{E\sqrt{\pi}} \qquad (3.112)$$

Rearranging this equation in terms of the known dimensionless displacements $\frac{E}{\sigma_o}\frac{w|_{z=o}}{a}$ gives

$$\frac{E}{\sigma_o} C_I K_I = \frac{\frac{E}{\sigma_o}\frac{w|_{z=o}}{a}}{\sqrt{\frac{R}{a}}} \qquad (3.113)$$

where

$$C_I = \frac{4(1-\nu^2)}{E\sqrt{2\pi a}} \qquad (3.114)$$

Then a plot of equation (3.113) as the $\sqrt{\frac{R}{a}} \rightarrow 0$ gives us the desired value of $\frac{E}{\sigma_o} C_I K_I$ from which an equivalent stress intensity factor for finite geometry cylinders can be calculated.

For the rectangular Cartesian coordinate problems, the crack opening displacement near the crack tip is given by (5)

$$v|_{y=o} = \frac{2(1-\nu)}{G} K_I \sqrt{\frac{R}{2\pi}} \quad \text{plane strain} \qquad (3.115)$$

$$v|_{y=o} = \frac{2}{(1+\nu)G} K_I \sqrt{\frac{R}{2\pi}} \quad \text{plane stress} \qquad (3.116)$$

Note that by definition the plane strain and plane stress stress intensity factors are equal while the above displacements are approximately 12.5% different for $\nu = 1/3$. Since the results, to be discussed later, indicate that most of the bar in the thickness direction is approximately in a state of plane strain, equation (3.115) is selected to calculate the stress intensity factor. Rearranging equation (3.115) so that the dimensionless crack opening displacements can be utilized leads to

$$\frac{E}{\sigma_o} C_I K_I = \frac{\frac{E}{\sigma_o} \frac{v}{a}\Big|_{y=o}}{\sqrt{\frac{R}{a}}} \tag{3.117}$$

where C_I is given by (3.114). A plot of the above equation as $\sqrt{\frac{R}{a}} \to 0$ can then be used to calculate K_I. Since the crack opening displacement is a function of the thickness variable, the stress intensity factor obtained above will vary in the z direction. However, if we were to account for the non-plane strain conditions near the surface by using equation (3.116) or a corrected equation (3.115) for the definition of K_I, this variation in the z direction would be minimized and the stress intensity factor would become a constant across the thickness of the bar by definition. This approach would essentially result in a continuously varying definition of K_I across the thickness.

It must be noted that the above description of K_I is completely arbitrary and it is questionable if it has any real significance in three-dimensional elasticity problems. However, values of K_I, based on equation (3.117), are still presented so that comparison between the calculated results and the published plane strain solutions (5) will be possible.

CHAPTER 4

RESULTS AND DISCUSSION

A computer program has been written for each of the numerical examples using the solution techniques of the preceeding chapters to calculate the stress and displacement fields in each case. With the exception of the annular plate with internal surface cracks example, all numerical computations were performed for two arbitrary grid increments so that the convergence of the finite difference approximations could be checked. In a given direction, uniform line spacing was used in all calculations with no other restriction being placed on the selection of the grid size. In general, an attempt was made to use a finer grid along the direction of largest variable change. Previously given numerical differentiation formulas show that their truncation errors are of Oh^2 with the exception of some boundary terms in the coupling vectors $\{r\}$, $\{s\}$ and $\{t\}$ where Oh derivative approximations were used. However, the use of parabolic differentiation formulas in these isolated cases led to essentially identical results. Since numerical differentiation is inherently an inaccurate procedure, analytic differentiation is used wherever possible such as in evaluating the normal stress distributions.

An inspection of the ordinary differential equations and their boundary conditions shows that decoupling of the dependent variables is impossible and the previously discussed successive approximation procedure must be employed. Since the derivatives $\{\dot{u}\}$, $\{\dot{v}\}$ and $\{\dot{w}\}$ depend on similar matrix functions, particular integrals and initial value vectors as the corresponding displacement vectors, convergence of the displacements will assure convergence of their corresponding derivatives. This conclusion has been confirmed by the results of the numerical examples presented below.

The computations for all the examples were performed on an IBM-360 time sharing digital computer using double precision arithmetic. Since the storage capacity of these computers is essentially unlimited, no attempt was made to optimize the use of the involved arrays.

4.1 Solid Cylindrical Bar With a Penny Shaped Crack

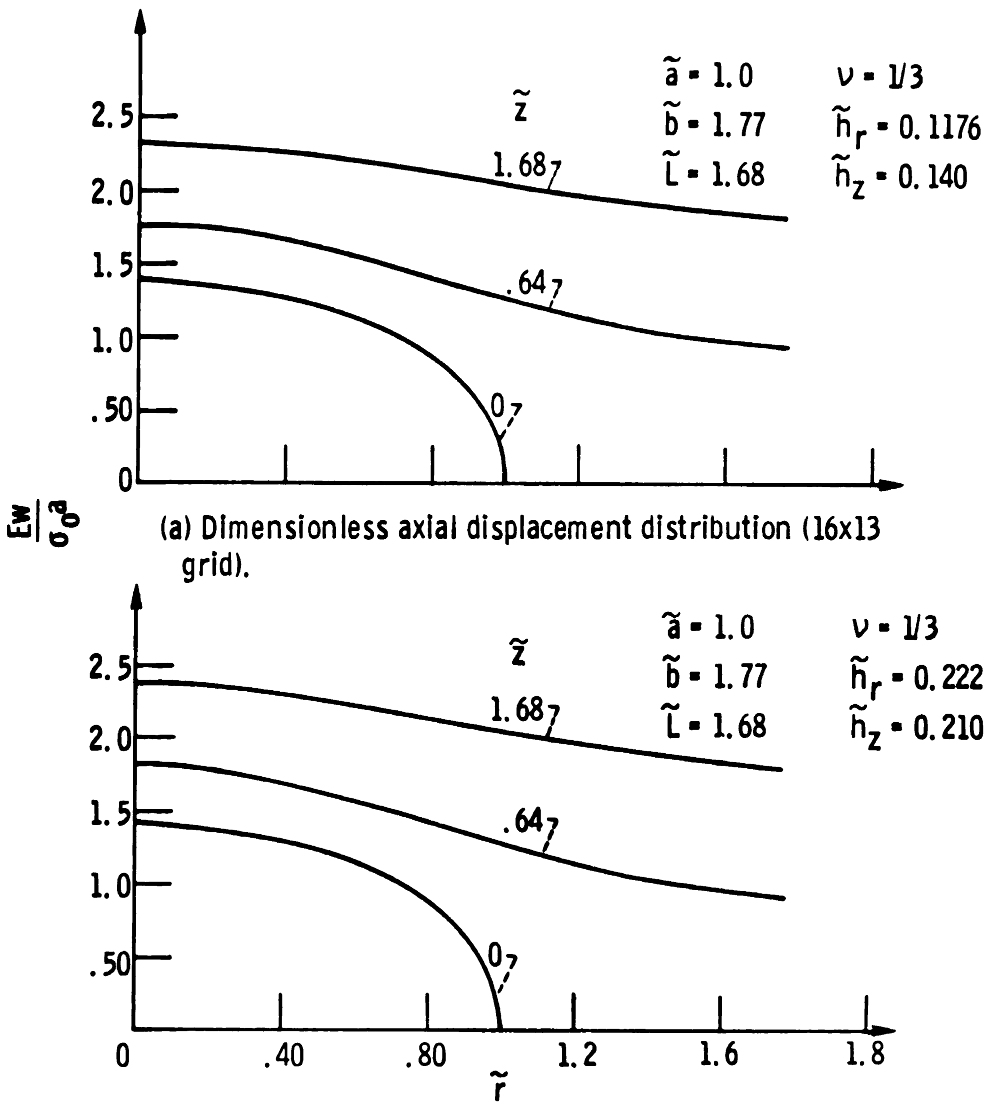
In order to establish the validity of the line method, detailed results will be first presented for the solid cylindrical bar containing a penny shaped crack and loaded normal to the crack plane. A convenient combination of the non-dimensionalized variables for which the evaluation of the required matrix functions presents no difficulty is as follows:

$$\tilde{a} = 1 \qquad \tilde{h}_z = .14$$
$$\tilde{L} = 1.68 \qquad \tilde{h}_r = \frac{2}{17} = .1176 \qquad (4.1)$$
$$\tilde{b} = 1.77$$

The selection of these dimensions results in the construction of 16 axial and 13 radial lines in Figure 8. According to the above choice of $\tilde{h}_r$, the number of lines inside the crack surface must be 9.

The computations for this problem involved 22 iterations in the successive approximation procedure using an arbitrary convergence criterion of 10^{-6}. This convergence criterion is defined as the maximum difference in the absolute values between successively calculated displacements at any point. The largest element of the error matrix (2.87) was .0001 at $\tilde{z} = 1.68$. The Runge-Kutta integration increment in evaluating the diagonalized matrix functions $[\Omega_{ij}]$ was .001. The approximate execution time for a problem of this size using the above given data is 3 minutes on an IBM-360 computer.

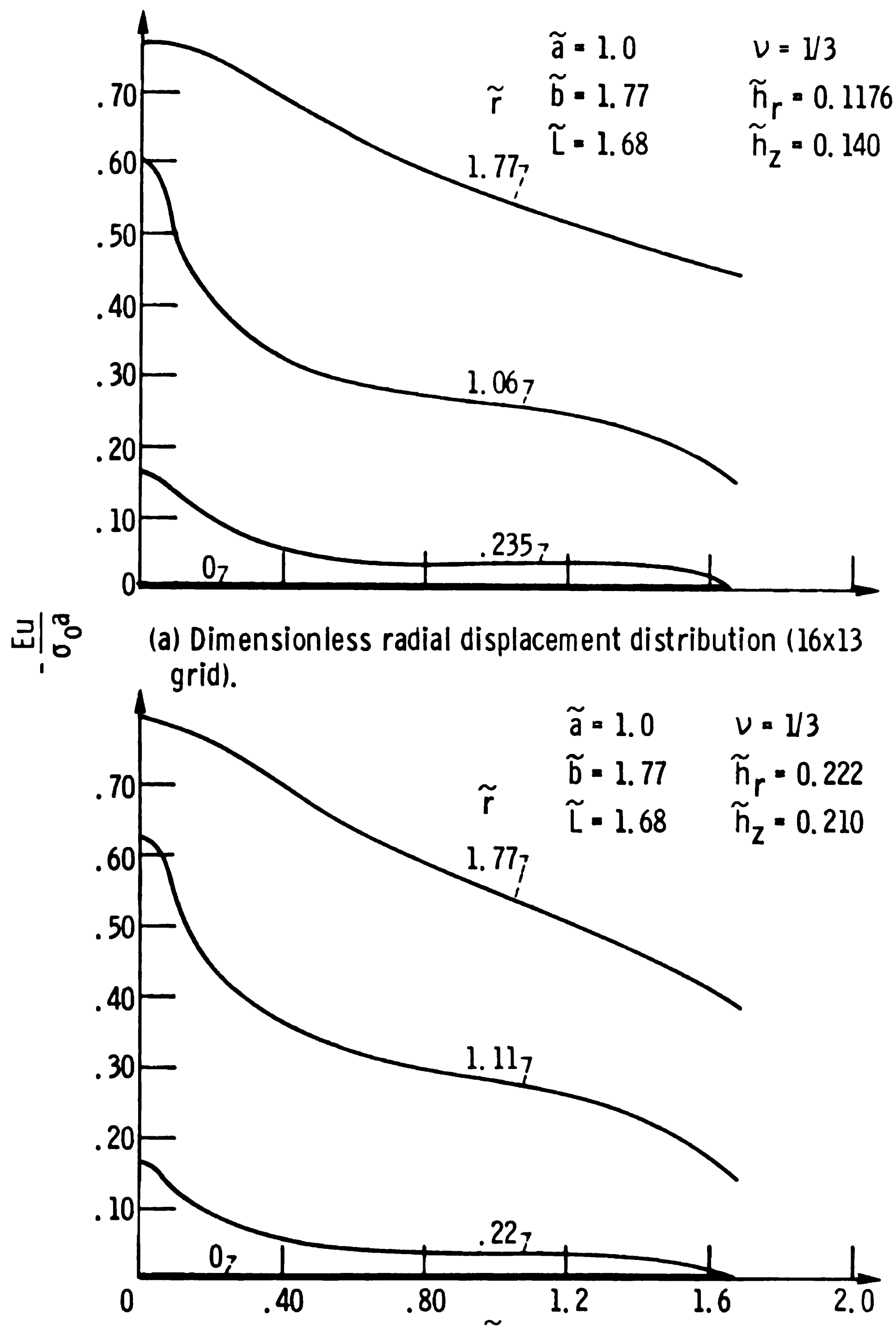
The results of these computations are presented in Figures 10 through 14 and Tables 1 through 5. For easy comparison of data, some of these figures include Sneddon's results for an infinite solid. Figures 10(b) and 11(b) show displacement distributions for an identical bar which were calculated from a grid having only 9 radial and 9 axial lines. Figure 14 contains



(a) Dimensionless axial displacement distribution (16x13 grid).

(b) Dimensionless axial displacement distribution (9x9 grid).

Figure 10. - Dimensionless axial displacement distribution for a solid cylindrical bar with a penny shaped crack.



(b) Dimensionless radial displacement distribution (9x9 grid).

Figure 11. - Dimensionless radial displacement distribution for a solid cylindrical bar with a penny shaped crack.

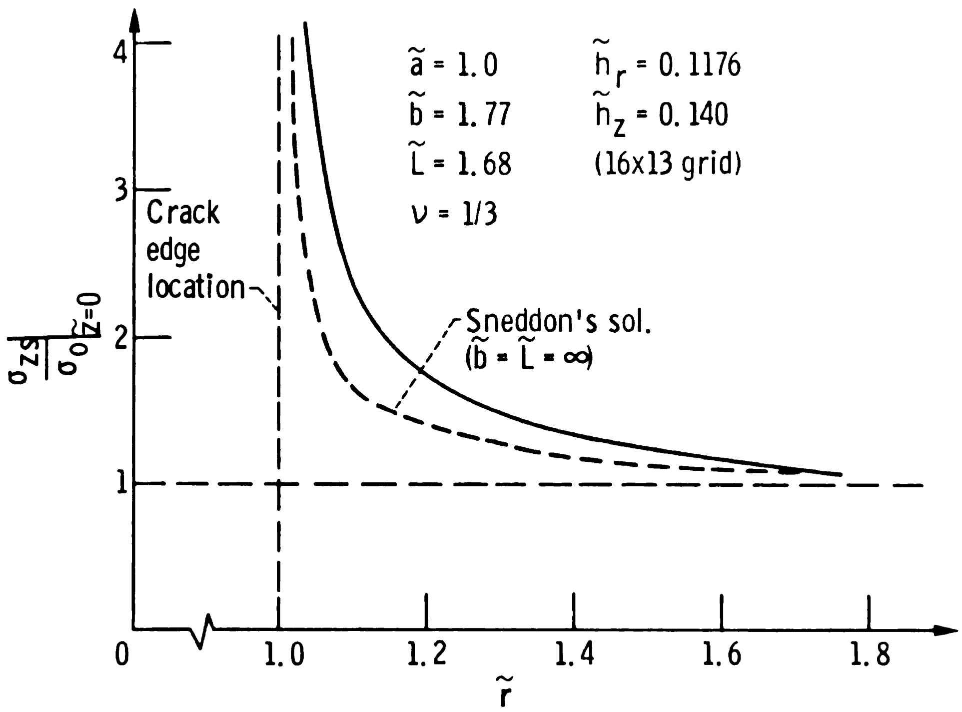


Figure 12. - Dimensionless axial stress distribution for a solid cylindrical bar with a penny shaped crack at $\tilde{z} = 0$.

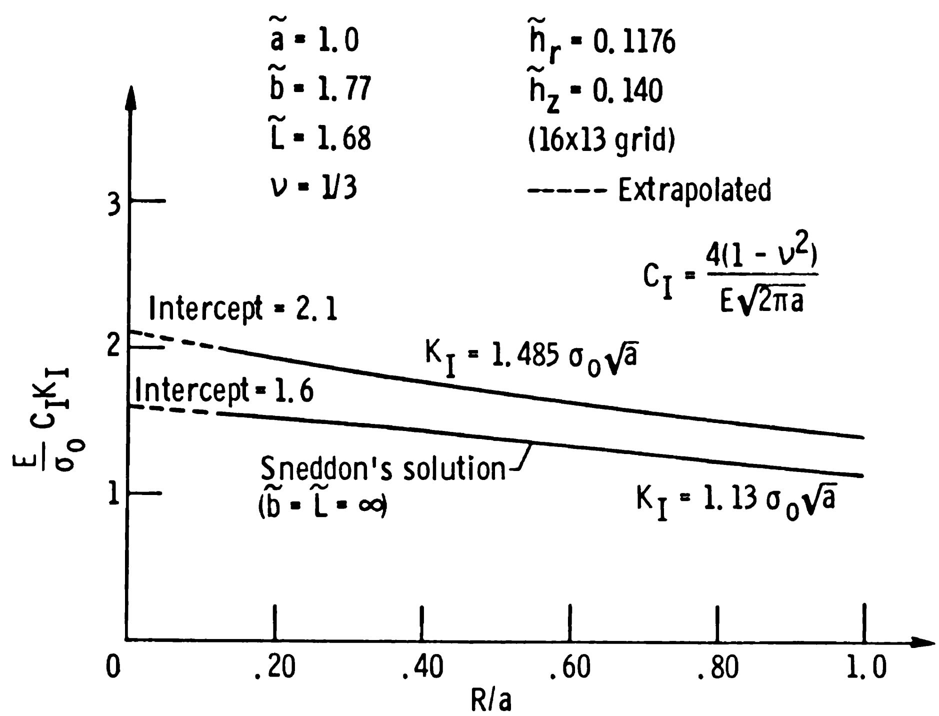


Figure 13. - Calculation of the stress intensity factor K_I for a solid cylindrical bar with a penny shaped crack.

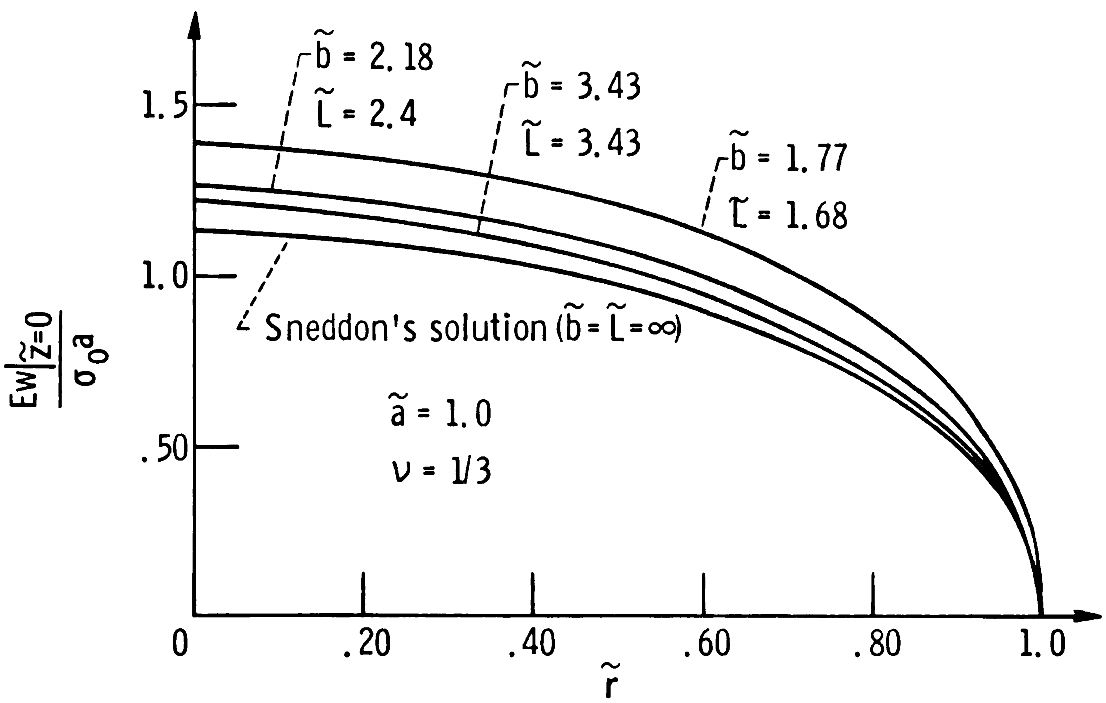


Figure 14. - Dimensionless crack opening displacements for solid cylindrical bars with penny shaped cracks of various lengths and radii.

Table 1. - Non-dimensionalized Radial Displacements $\frac{E}{\sigma_o}\frac{u}{a}$ for a Solid Cylindrical Bar with a Penny Shaped Crack Under Uniform Normal Tension. $\tilde{a} = 1.0$, $\tilde{b} = 1.77$, $\tilde{L} = 1.68$ (16 axial and 13 radial lines)

$\tilde{z}$ \ $\tilde{r}$	.00	.235	.471	.706	.941	1.06	1.294	1.53	1.77
.00	.000	-.168	-.333	-.488	-.579	-.609	-.653	-.705	-.773
.28	.000	-.080	-.153	-.217	-.297	-.371	-.529	-.639	-.725
.56	.000	-.045	-.091	-.149	-.239	-.300	-.433	-.549	-.646
.84	.000	-.038	-.083	-.142	-.223	-.273	-.379	-.482	-.530
1.12	.000	-.039	-.085	-.142	-.214	-.255	-.344	-.435	-.530
1.40	.000	-.034	-.074	-.124	-.188	-.224	-.303	-.388	-.483
1.68	.000	-.006	-.023	-.058	-.114	-.150	-.233	-.325	-.425

Table 2. - Non-dimensionalized Axial Displacements $\frac{E}{\sigma_o}\frac{w}{a}$ for a Solid Cylindrical Bar with a Penny Shaped Crack Under Uniform Normal Tension. $\tilde{a} = 1.0$, $\tilde{b} = 1.77$, $\tilde{L} = 1.68$ (16 axial and 13 radial lines)

$\tilde{r}$ \ $\tilde{z}$	.00	.28	.56	.84	1.12	1.40	1.68
.00	1.394	1.544	1.646	1.766	1.928	2.126	2.329
.235	1.351	1.495	1.596	1.722	1.892	2.097	2.306
.471	1.219	1.359	1.472	1.622	1.812	2.029	2.250
.706	.973	1.113	1.273	1.473	1.696	1.933	2.167
.941	.515	.746	1.029	1.302	1.565	1.823	2.072
1.06	.000	.542	.913	1.221	1.502	1.768	2.024
1.294	.000	.384	.758	1.090	1.390	1.670	1.937
1.53	.000	.334	.673	.999	1.304	1.592	1.866
1.77	.000	.291	.610	.935	1.245	1.535	1.812

Table 3. - Non-dimensionalized Radial Stress $\frac{\sigma_{rs}}{\sigma_o}$ for a Solid Cylindrical Bar with a Penny Shaped Crack Under Uniform Normal Tension. $\tilde{a} = 1.0$, $\tilde{b} = 1.77$, $\tilde{L} = 1.68$ (16 axial and 13 radial lines)

$\tilde{z}$ \ $\tilde{r}$	.00	.235	.471	.706	.941	1.06	1.244	1.53	1.77
.00	-1.060	-1.043	-1.017	-.883	-.522	1.134	.376	.116	.000
.28	-.479	-.451	-.375	-.270	-.262	-.138	-.095	-.017	.000
.56	-.148	-.141	-.123	-.124	-.155	-.150	-.113	-.044	.000
.84	.0211	.0144	.002	-.0251	-.058	-.065	-.056	-.026	.000
1.12	.127	.116	.093	.058	.023	.009	-.004	-.003	.000
1.40	.246	.230	.193	.142	.088	.064	.028	.012	.000
1.68	.470	.429	.352	.247	.136	.085	.008	-.021	.000

Table 4. - Non-dimensionalized-Circumferential Stress $\frac{\sigma_{\theta s}}{\sigma_o}$ for a Solid Cylindrical Bar with a Penny Shaped Crack Under Uniform Normal Tension. $\tilde{a} = 1.0$, $\tilde{b} = 1.77$, $\tilde{L} = 1.68$ (16 axial and 13 radial lines)

$\tilde{z}$ \ $\tilde{r}$	.00	.235	.471	.706	.941	1.06	1.294	1.53	1.77
.00	-1.060	-1.061	-1.047	-.986	-.790	.909	.125	-.020	-.108
.28	-.479	-.461	-.401	-.291	-.112	.108	.015	-.023	-.051
.56	-.148	-.138	-.106	-.055	.011	.049	.037	.022	.024
.84	.021	.023	.032	.047	.062	.067	.062	.055	.057
1.12	.127	.12[illegible]	.120	.113	.104	.099	.086	.075	.064
1.40	.246	.2[illegible]	.218	.191	.160	.144	.116	.096	.072
1.68	.470	.45	.401	.333	.257	.220	.154	.112	.089

Table 5. - Non-dimensionalized Axial Stress $\frac{\sigma_{zs}}{\sigma_o}$ for a Solid Cylindrical Bar with a Penny Shaped Crack Under Uniform Normal Tension. $\tilde{a} = 1.0$, $\tilde{b} = 1.77$, $\tilde{L} = 1.68$ (16 axial and 13 radial lines)

$\tilde{z}$ \ $\tilde{r}$	.00	.235	.471	.706	.941	1.06	1.294	1.53	1.77
.00	.000	.000	.000	.000	.000	3.320	1.512	1.206	.989
.28	.095	.093	.146	.319	.874	1.513	1.365	1.201	1.082
.56	.277	.295	.386	.592	.950	1.150	1.230	1.186	1.172
.84	.515	.542	.624	.770	.957	1.041	1.122	1.136	1.159
1.12	.736	.759	.808	.885	.972	1.010	1.059	1.082	1.092
1.40	.900	.915	.933	.960	.990	1.003	1.022	1.036	1.037
1.68	.999	.998	.999	.999	1.000	.999	.995	.993	.990

the crack opening displacements of several cylindrical bars with different geometry. Figure 13 shows the method for calculating an equivalent plain strain stress intensity factor K_I for three-dimensional problems from knowing the crack opening displacement.

Inspection of Figures 10(a) and 10(b) clearly shows the advantage of the line method over other numerical solutions. A relatively coarse grid of 9 axial and 9 radial lines gave almost identical results to that obtained by using a 16 by 13 grid. Note that an approximate 100% change in $\tilde{h}_r$ and a 50% change in $\tilde{h}_z$ resulted in about a 2% change in the axial displacements. Since the bar is of finite size, the crack opening displacement is expected to be higher than Sneddon's solution. Consistency of the results with this conclusion is obvious from Figure 14. Figures 10 also show that the axial displacement is highest at the end of the bar above the center of the crack. If the bar were of sufficient length, the highest displacement curve would be a straight line with a constant axial displacement. Note that the crack opening displacement curve is assumed to be zero at the crack edge rather than at the first adjacent node outside the crack plane.

Figures 11(a) and 11(b) show the radial contraction of the bar as a function of the length and radius. The contraction is the largest, as one would expect it, at the outside surface in the plane of the crack. Note that symmetry of the problem requires zero slopes along the displacement axes in both Figures 10 and 11. In the plane of the crack, a sudden increase in contraction takes place as can be seen from the curves shown. This increase is due to the constraint free crack surface which permits the higher contraction rate of the bar outside the crack radius to exert a strong influence on the material inside the crack. Similar arguments about the agreement of the radial displacements in Figures 11(a) and 11(b) for corresponding positions can also be made as for Figure 10.

Figure 12 shows the stress distribution normal to the crack plane as a function of the distance from the crack edge. It is known that for linear elastic problems, this stress distribution approaches infinity near the crack tip as the inverse square root of the distance from the crack edge. Establishment of this type of singularity is, however, difficult when numerical methods are used because values of the normal stress are needed within a distance of .05a or less of the crack edge. With the equal spacing of lines used throughout the examples, the minimum node location for these examples is about .06a. For the range of $\dot{\tilde{r}}$ shown in Figure 12, this inverse square root singularity is not valid. However, for the range shown, the obtained stress curve closely resembles Sneddon's solution as can be noted. Obviously, the absolute value of this stress is greater for a finite size bar than for Sneddon's infinite solid. Note that the stress near the outside radius rapidly approaches the value of the applied dimensionless stress of unity.

Figure 13 shows the calculation of the stress intensity factor K_I according to equation (3.113). The intercept of Sneddon's solution is 1.6 which gives a stress intensity factor of 1.13 $\sigma_o\sqrt{a}$. The stress intensity factor for an infinite cylindrical solid containing a penny shaped crack is given by equation (3.111) as 1.13 $\sigma_o\sqrt{a}$. Thus, the validity of using the method of Section 3.9 for calculating these values of K_I is established. The stress intensity factor obtained from the 2.1 intercept is 1.485 $\sigma_o\sqrt{a}$. Hence, the finite bar discussed in Figure 13 has an approximately 31% higher stress intensity factor than the infinite solid.

Figure 14 shows the crack opening displacement for several cylindrical bars with different lengths and radii. The obtained results clearly show that as the length and diameter of the bars are increased, Sneddon's solution for an infinite bar is rapidly approached. For a bar with $\tilde{b} = \tilde{L} = 3.43$, the maximum difference is only about 7%.

Tables 1 through 5 show selected results from the computer listings. The accuracy of the normal stress and displacement boundary conditions can easily be noted from the numerical data listed.

4.2 Hollow Cylindrical Bar With a Penny Shaped Crack

In this example the dimensions of the bar were intentionally increased along with the inclusion of a central hole through the cylinder axis. The expected result of these changes from the problem of (4.1) would be the minimization of the crack influence upon the calculated displacement and stress fields. Selected results of the computations for this problem are shown in Figures 15 through 18. The physical dimensions of the problem are also listed in these figures.

Figure 15 shows the dimensionless axial displacement distribution as a function of the radius and axial position. Note that the maximum crack opening displacement is less than that shown in Figure 14 for solid cylinders. The reason for this is that there is no load applied over the central hole surface and the effect of this is to offset the weakening influence of the hole. As expected, the displacement curves are essentially constant once the results are plotted beyond the vicinity of the crack.

Figure 16 shows the radial contraction of the bar. This contraction is maximum at the outside surface and its variation along the z direction is only about 9%. For a bar without a crack, this outside contraction would be a constant along the z direction. Comparison of the curves in Figure 16 to those in Figures 11 clearly shows that the crack effect on the radial

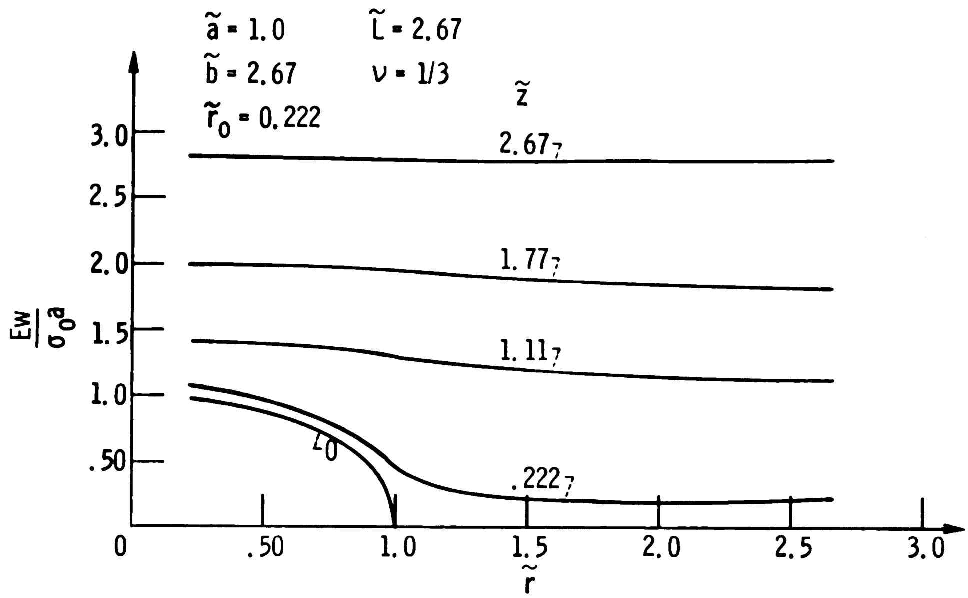


Figure 15. - Dimensionless axial displacement distribution for a hollow cylindrical bar with a penny shaped crack.

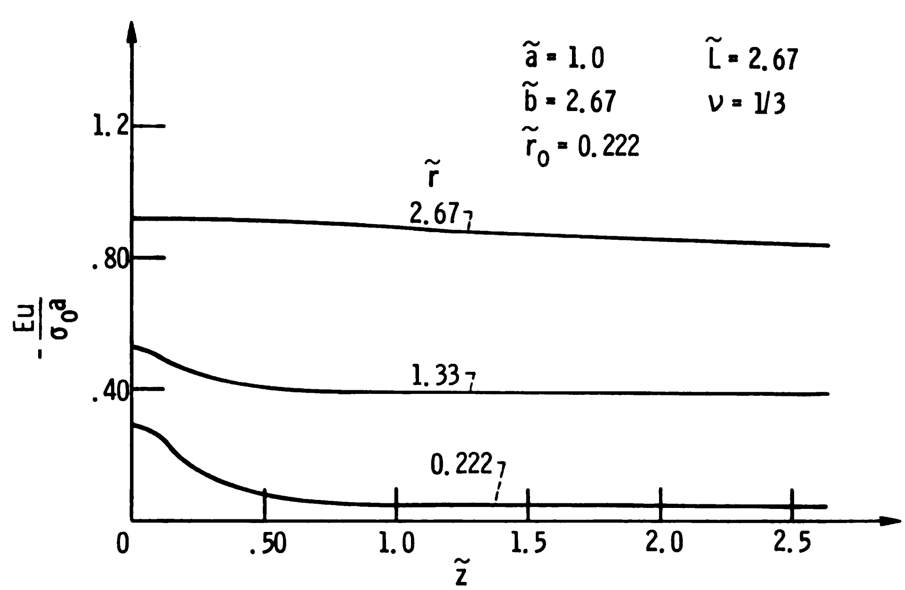


Figure 16. - Dimensionless radial displacement distribution for a hollow cylindrical bar with a penny shaped crack.

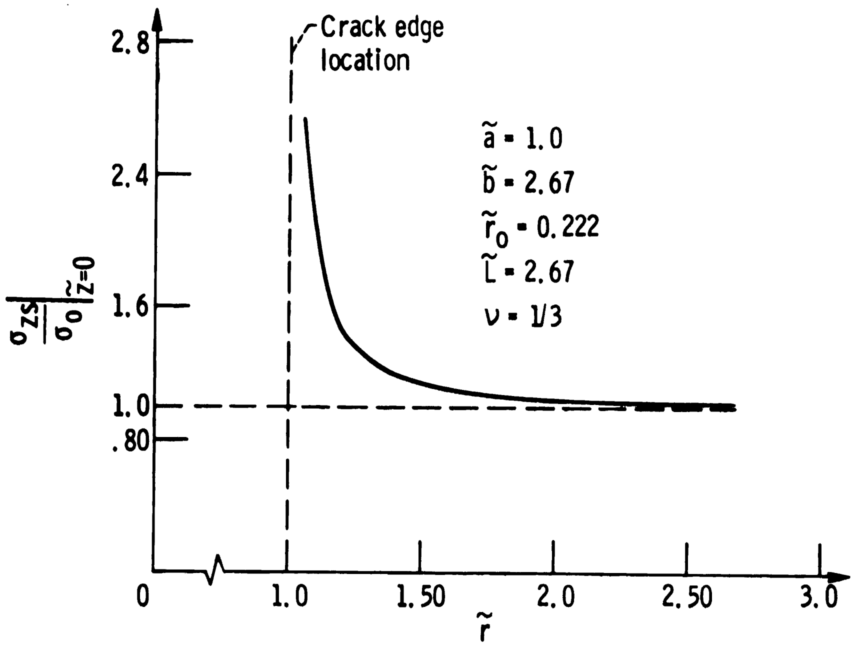


Figure 17. - Dimensionless axial stress distribution for a hollow cylindrical bar with a penny shaped crack at $\tilde{z} = 0$.

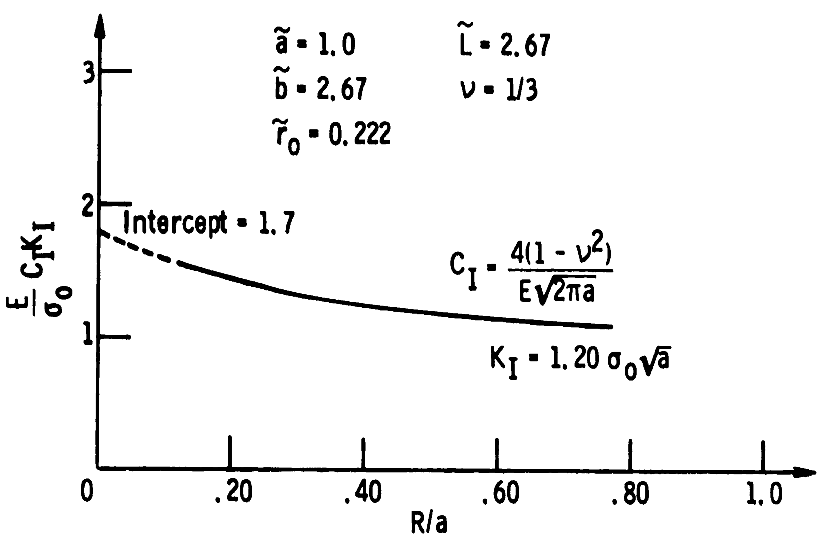


Figure 18. - Calculation of the stress intensity factor K_I for a hollow cylindrical bar with a penny shaped crack.

contraction is more pronounced for bars with smaller overall dimensions relative to the crack radius. Note the sudden increase in contraction near the crack plane along the inside hole radius. This increase is caused again by the crack plane which through its constraint free surface permits a contraction rate approaching the contraction near the crack edge.

The dimensionless axial stress distribution in the crack plane is shown in Figure 17. The shape of this curve is similar to those shown in Figure 12. Note the relatively constant stress beyond a radius of 2 which shows that the region of largest stress variation is between $\tilde{r} = 1$ and $\tilde{r} = 2$. Similar comments about the axial stress singularity can also be made in this case as for the solid cylindrical bar problem.

The calculation of the stress intensity factor K_I using the crack opening displacement is shown in Figure 18. Note that even though the crack opening of Figure 15 is smaller than Sneddon's solution in Figure 14, the calculated stress intensity factor of $1.20\ \sigma_o\sqrt{a}$ is somewhat greater than the $1.13\ \sigma_o\sqrt{a}$ obtained in Figure 13. This is due to the more negative slopes obtained for finite sized bars in Figures 13 and 18. However, the stress intensity factor of Figure 18 is considerably less than that calculated for problem (4.1) in Figure 13.

The application of a uniform radial stress, σ_o, to the outside surface of this hollow cylinder was found to have no effect on the crack opening displacement. However, the axial displacements beyond the crack plane were considerably lowered. The radial displacements for this problem were all in the outward direction rather than inward as shown in Figure 16. The results of Figure 17 were also found to be independent of this radial surface load. As expected, the main effect of a radial surface load was observed to be in the calculated circumferential and radial stress fields.

4.3 Annular Plate With Internal Surface Cracks

As a first attempt at the solution of a general three-dimensional problem in cylindrical coordinates, the problem of Figure 6 was solved using a grid of 16 lines in all directions. Because of the relatively coarse grid involved, the calculated data is listed in Tables 6 through 11. The construction of meaningful figures from these results is obviously difficult. It must be noted, however, that due to the unknown nature of the resulting solutions, the use of a coarse grid is always recommended in generating the first set of displacements. This practice may greatly complicate the programming of the necessary equations, but it has the advantage of providing results quickly from which the numerical limitations can be immediately recognized. Since the construction of a general computer program for this problem requires a great amount of effort, the listings of Appendix B-3 apply only to the specific case when $NR = N\theta = NZ = 4$

Table 6. - Dimensionless Radial Displacements $\frac{Eu}{\sigma_o b}$ for an Annular Plate with Internal Surface Cracks Under Uniform Radial Tension on the Outside Surface

$\tilde{\theta}$ \ $\tilde{r}$	.25	.50	.75	1.00	
0°	.435	.671	.786	.930	$\tilde{z}$ = .00
15°	.940	.890	.896	.998	
30°	1.168	.992	.968	1.058	
45°	1.240	1.026	.994	1.081	
0°	.506	.680	.789	.930	$\tilde{z}$ = .10
15°	.907	.860	.886	.994	
30°	1.092	.953	.957	1.055	
45°	1.152	.986	.984	1.078	
0°	.724	.656	.779	.927	$\tilde{z}$ = .20
15°	.795	.761	.856	.985	
30°	.898	.856	.932	1.048	
45°	.942	.892	.961	1.073	
0°	.716	.659	.769	.915	$\tilde{z}$ = .30
15°	.752	.727	.845	.978	
30°	.815	.809	.926	1.049	
45°	.845	.843	.957	1.077	

Table 7. - Dimensionless Circumferential Displacements $\frac{Ev}{\sigma_o b}$ for an Annular Plate with Internal Surface Cracks Under Uniform Radial Tension on the Outside Surface

$\tilde{z}$ \ $\tilde{\theta}$	0°	15°	30°	45°	
.00	.718	.566	.302	.000	$\tilde{r}$ = .25
.10	.592	.450	.238	.000	
.20	.000	.073	.063	.000	
.30	.000	.009	.012	.000	
.00	.425	.285	.138	.000	$\tilde{r}$ = .50
.10	.350	.232	.115	.000	
.20	.000	.075	.057	.000	
.30	.000	.023	.026	.000	
.00	.000	.037	.029	.000	$\tilde{r}$ = .75
.10	.000	.033	.026	.000	
.20	.000	.023	.020	.000	
.30	.000	.014	.014	.000	
.00	.000	.006	.004	.000	$\tilde{r}$ = 1.0
.10	.000	.007	.005	.000	
.20	.000	.009	.007	.000	
.30	.000	.010	.007	.000	

Table 8. - Dimensionless Axial Displacements $\frac{Ew}{\sigma_o b}$ for an Annular Plate with Internal Surface Cracks Under Uniform Radial Tension on the Outside Surface

$\tilde{r}$ \ $\tilde{z}$	.00	.10	.20	.30	
.25	.000	-.218	-.342	-.416	$\tilde{\theta} = 0°$
.50	.000	-.108	-.217	-.322	
.75	.000	-.089	-.175	-.258	
1.00	.000	-.076	-.152	-.226	
.25	.000	-.098	-.216	-.338	$\tilde{\theta} = 15°$
.50	.000	-.049	-.131	-.232	
.75	.000	-.075	-.154	-.237	
1.00	.000	-.074	-.149	-.226	
.25	.000	-.068	-.167	-.281	$\tilde{\theta} = 30°$
.50	.000	-.040	-.106	-.195	
.75	.000	-.067	-.141	-.222	
1.00	.000	-.075	-.152	-.231	
.25	.000	-.061	-.153	-.2[illegible]2	$\tilde{\theta} = 45°$
.50	.000	-.039	-.101	-.186	
.75	.000	-.065	-.137	-.218	
1.00	.000	-.075	-.153	-.233	

Table 9. - Dimensionless Radial Stress Distribution $\frac{\sigma_r}{\sigma_o}$ for an Annular Plate with Internal Surface Cracks Under Uniform Radial Tension on the Outside Surface

$\tilde{\theta}$ \ $\tilde{r}$	.25	.50	.75	1.00	
0°	.000	.020	1.062	1.00	$\tilde{z}$ = .00
15°	.000	.346	.837	1.00	
30°	.000	.293	.748	1.00	
45°	.000	.282	.726	1.00	
0°	.000	.027	1.048	1.00	$\tilde{z}$ = .10
15°	.000	.405	.872	1.00	
30°	.000	.424	.801	1.00	
45°	.000	.424	.783	1.00	
0°	.000	1.305	1.026	1.00	$\tilde{z}$ = .20
15°	.000	.968	.980	1.00	
30°	.000	.840	.932	1.00	
45°	.000	.801	.915	1.00	
0°	.000	.872	1.005	1.00	$\tilde{z}$ = .30
15°	.000	.950	1.004	1.00	
30°	.000	.966	.986	1.00	
45°	.000	.956	.975	1.00	

Table 10. - Dimensionless Circumferential Stress $\frac{\sigma_\theta}{\sigma_o}$ for an Annular Plate with Internal Surface Cracks Under Uniform Radial Tension on the Outside Surface

$\tilde{\theta}$ \ $\tilde{r}$	.25	.50	.75	1.00	
0°	.00	.00	1.702	1.251	$\tilde{z}$ = .00
15°	.308	.803	1.542	1.339	
30°	.209	1.043	1.460	1.416	
45°	.240	1.145	1.441	1.443	
0°	.00	.00	1.659	1.248	$\tilde{z}$ = .10
15°	.703	.957	1.534	1.331	
30°	.803	1.196	1.472	1.410	
45°	.863	1.270	1.456	1.439	
0°	4.569	2.821	1.506	1.236	$\tilde{z}$ = .20
15°	3.609	2.011	1.526	1.313	
30°	2.971	1.706	1.506	1.400	
45°	2.748	1.629	1.498	1.432	
0°	3.050	1.858	1.442	[illegible]	$\tilde{z}$ = .30
15°	3.110	1.879	1.500	1.301	
30°	3.171	1.825	1.523	1.397	
45°	3.167	1.775	1.524	1.432	

Table 11. - Dimensionless Axial Stress $\frac{\sigma_z}{\sigma_o}$ for an Annular Plate with Internal Surface Cracks Under Uniform Radial Tension on the Outside Surface

$\tilde{\theta}$ \ $\tilde{r}$	.25	.50	.75	1.00	
0°	-1.974	-1.049	.030	-.014	$\tilde{z}$ = .00
15°	- .939	- .126	.043	.040	
30°	- .631	.041	.064	.060	
45°	- .544	.082	.070	.062	
0°	-1.597	-1.073	.028	-.010	$\tilde{z}$ = .10
15°	- .867	- .210	.032	.031	
30°	- .579	.006	.053	.046	
45°	- .487	.055	.059	.048	
0°	.500	.313	.002	-.004	$\tilde{z}$ = .20
15°	-3.135	-1.996	-1.504	-1.310	
30°	- .077	.070	.039	.019	
45°	- .097	.073	.042	.020	
0°	.000	.000	.000	.000	$\tilde{z}$ = .30
15°	.000	.000	.000	.000	
30°	.000	.000	.000	.000	
45°	.000	.000	.000	.000	

in Figure 6. Although the results in Tables 6 through 13 are preliminary, they still demonstrate that the previously presented solution techniques permit the computation of the required displacement and stress fields.

As a result of having 16 lines in all directions, the coordinate increments used in this problem were $\tilde{h}_r = .25$, $\tilde{h}_z = .10$ and $\tilde{h}_\theta = 15°$. Using a convergence criterion of 10^{-6}, the number of successive calculations in the iteration procedure was 41. The Runge-Kutta increment in evaluating the matrix functions Ω_{ij} was .001 while the largest error matrix element in both the circumferential and axial directions was less than 10^{-5} at all points.

Table 6 shows the dimensionless radial displacement distribution as a function all three coordinates. It can be noted that the outward radial displacement increases as a function of angular position while changes in the z direction are most pronounced at the inside radius. Table 7 lists the calculated circumferential displacements. These results show that below the crack plane, the circumferential displacements are essentially zero while the maximum crack opening is at $\tilde{\theta} = \tilde{z} = 0$ and at $\tilde{r} = .25$ as expected. Note that the crack opening decreases with an increase in radius and thickness. Table 8 displays similar results for the calculated axial displacements. While both radial and circumferential displacements are extensional, the axial displacements are negative indicating a contraction in that direction. This contraction is a maximum at the plate surface and in the crack plane decreases with an increase in radius. Displacement Tables 6, 7 and 8 clearly show the accuracy of the enforced displacement boundary conditions and that the number of normal lines inside the crack plane for this example is 4.

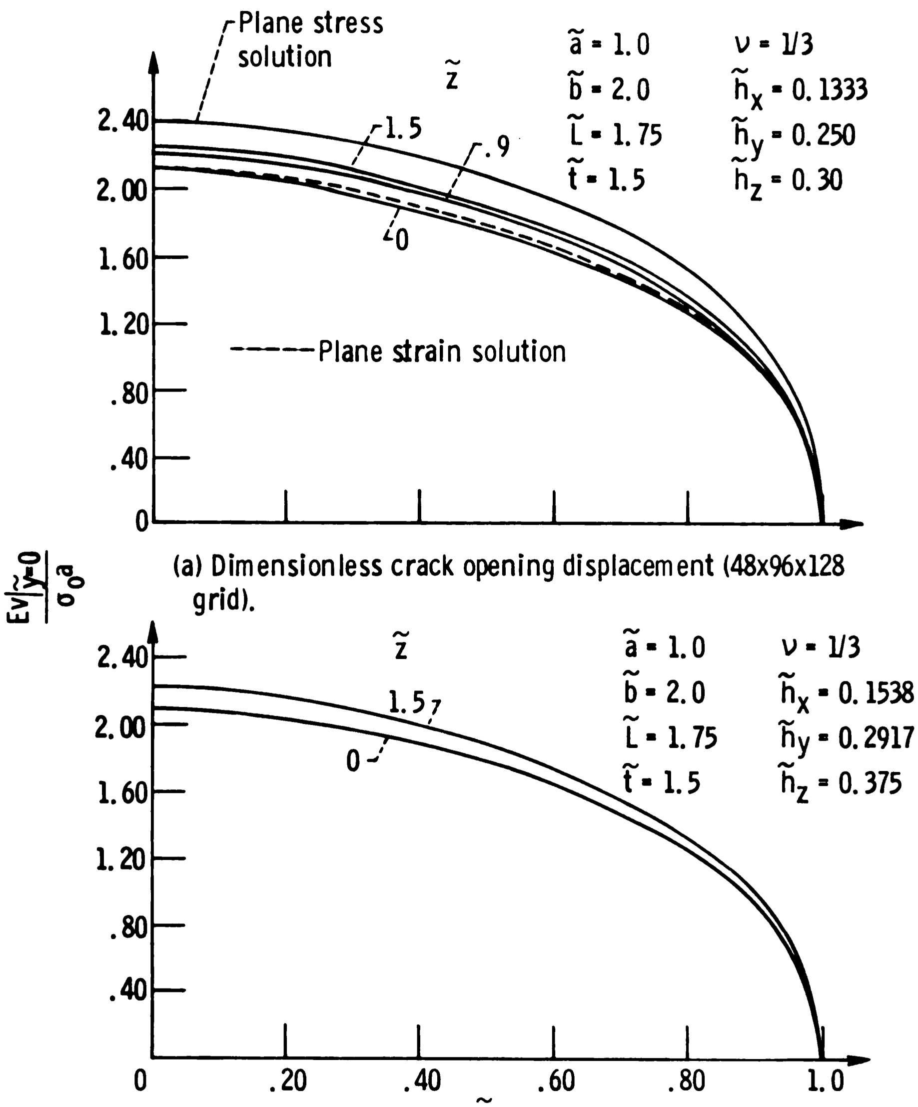
Table 9 contains the dimensionless radial stress distribution as a function of the coordinates. The zero normal stress boundary condition on the inside surface and the applied radial stress of unity on the outside surface are evident in the results of this table. Note that in the plane of the crack, that is for $0 < \tilde{z} < .15$, the radial stress varies gradually between 0 and 1 while below the crack plane it is close to unity everywhere except on the inside surface. The maximum value of this stress occurs in the crack plane at $\tilde{r} = .5$ just outside the crack edge. As expected, the radial stress is tensile or zero everywhere in the body.

The dimensionless circumferential stress distribution is shown in Table 10. It is expected that this stress be the maximum tensile stress in the body since the crack plane is normal to the circumferential axis. Because the interaction between the hole and crack stress fields is most pronounced at the inside radius, this stress should approach infinity most rapidly at the inside boundary just below the crack edge. The results of Table 10 seem to confirm this observation, although we must also remember that the nodes are closer to the crack boundary below the crack surface than along the radius. Note that at $\tilde{r} = 1$, the stresses are essentially constant along the z direction.

Table 11 contains the dimensionless axial stress distribution as function of the coordinates. The zero normal stress boundary condition at $\tilde{z} = .3$ is obvious from the results in this table. Note that the axial stress is either zero or compressive in most of the region except just below the crack edge where a tensile axial stress seems to be generated. Because of the coarse grid used, the numerical results presented for this example are probably somewhat inaccurate in magnitude, but they do indicate some previously unknown variations in the stress and displacement fields for this problem. This conclusion is possible in that the line method does not usually require a fine grid for good results as was shown in Section 4.1.

4.4 Bar With Through-Thickness Central Crack

The solution of the problem shown in Figure 2 was obtained by using two different sets of lines along the coordinate axes. Selected results of these computations are shown in Figures 19 through 26 and Tables 12 through 14. The computation time for the example containing a 35 x 70 x 98 line grid was about 30



(a) Dimensionless crack opening displacement (48x96x128 grid).

(b) Dimensionless crack opening displacement (35x70x98 grid).

Figure 19. - Dimensionless crack opening displacement for a rectangular bar under uniform tension containing a through-thickness central crack.

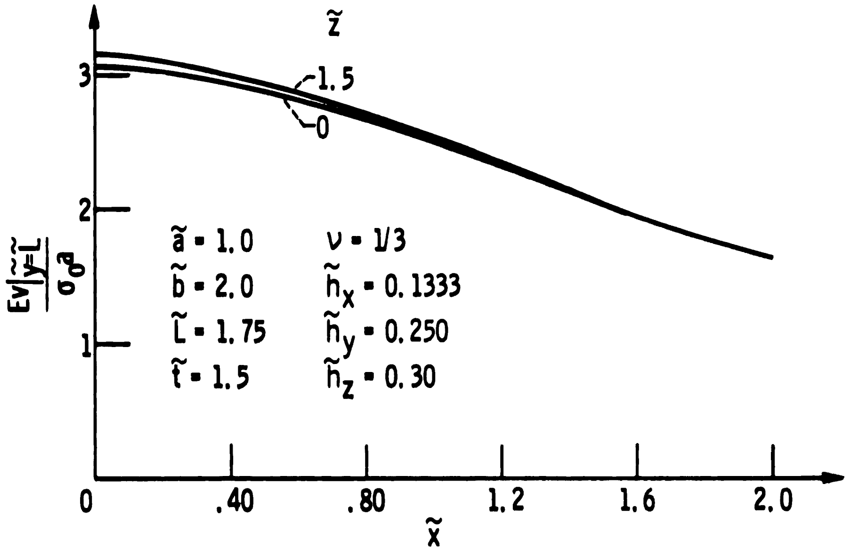


Figure 20. - Dimensionless bar end extension for a rectangular bar under uniform tension containing a through-thickness central crack.

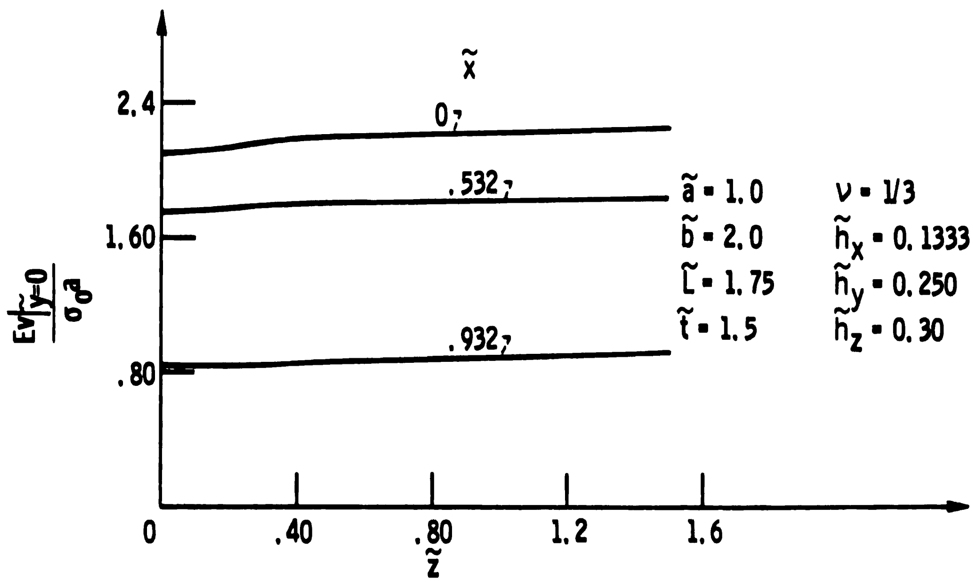
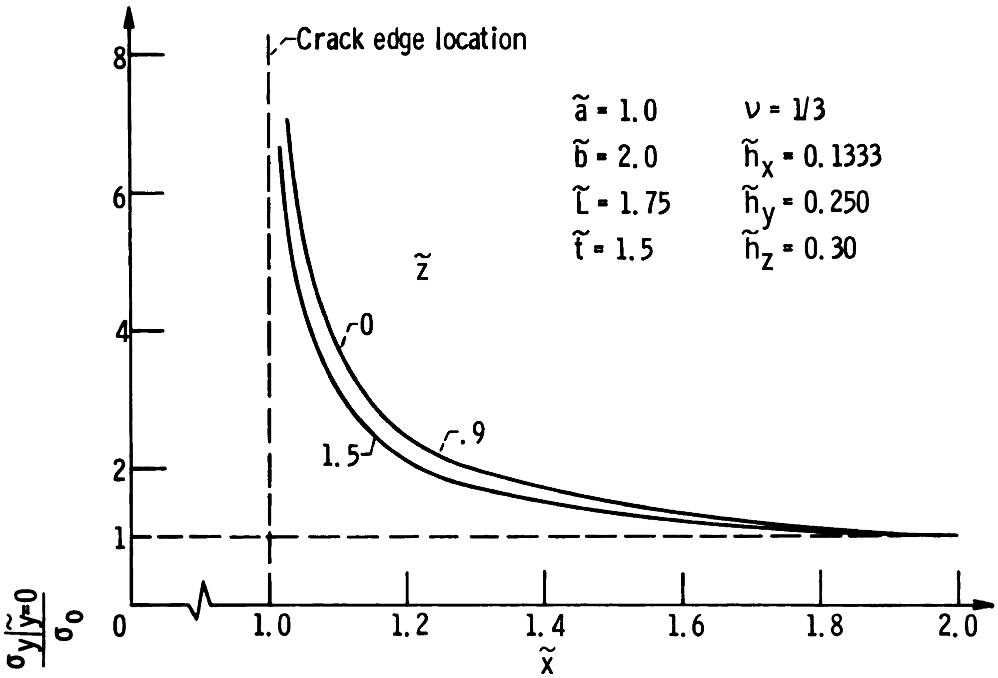
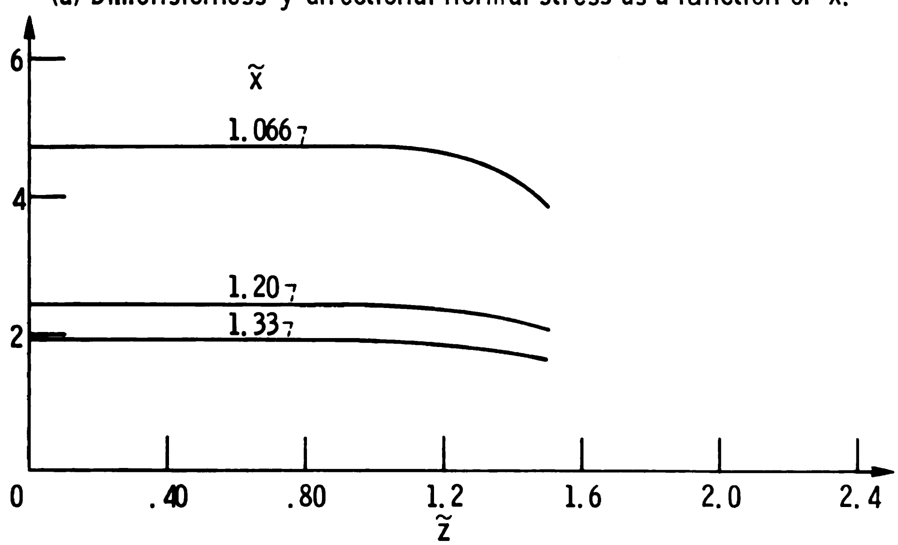


Figure 21. - Dimensionless normal displacement distribution in the crack plane for a rectangular bar under uniform tension containing a through-thickness central crack.

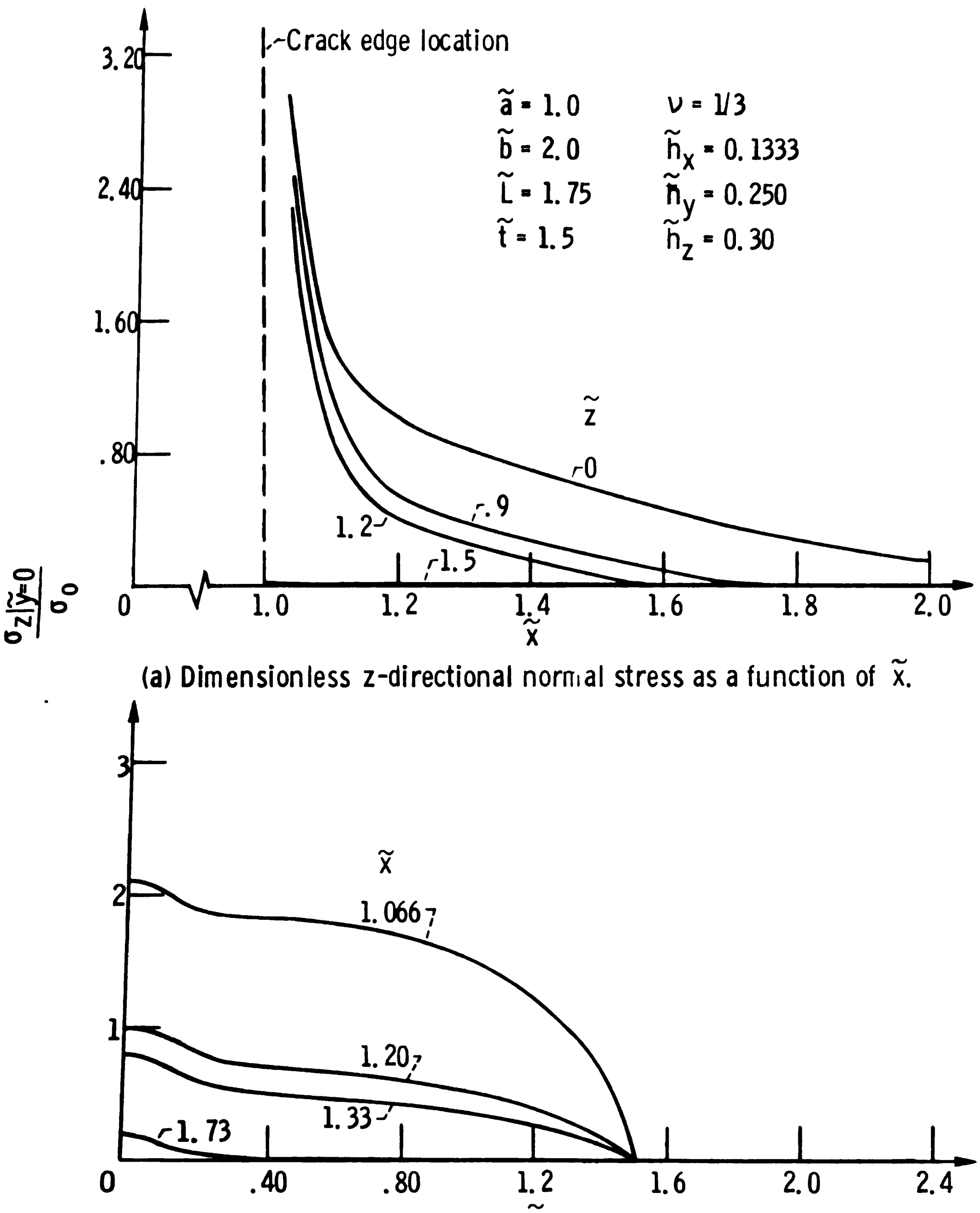


(a) Dimensionless y-directional normal stress as a function of $\tilde{x}$.



(b) Dimensionless y-directional normal stress as a function of $\tilde{z}$.

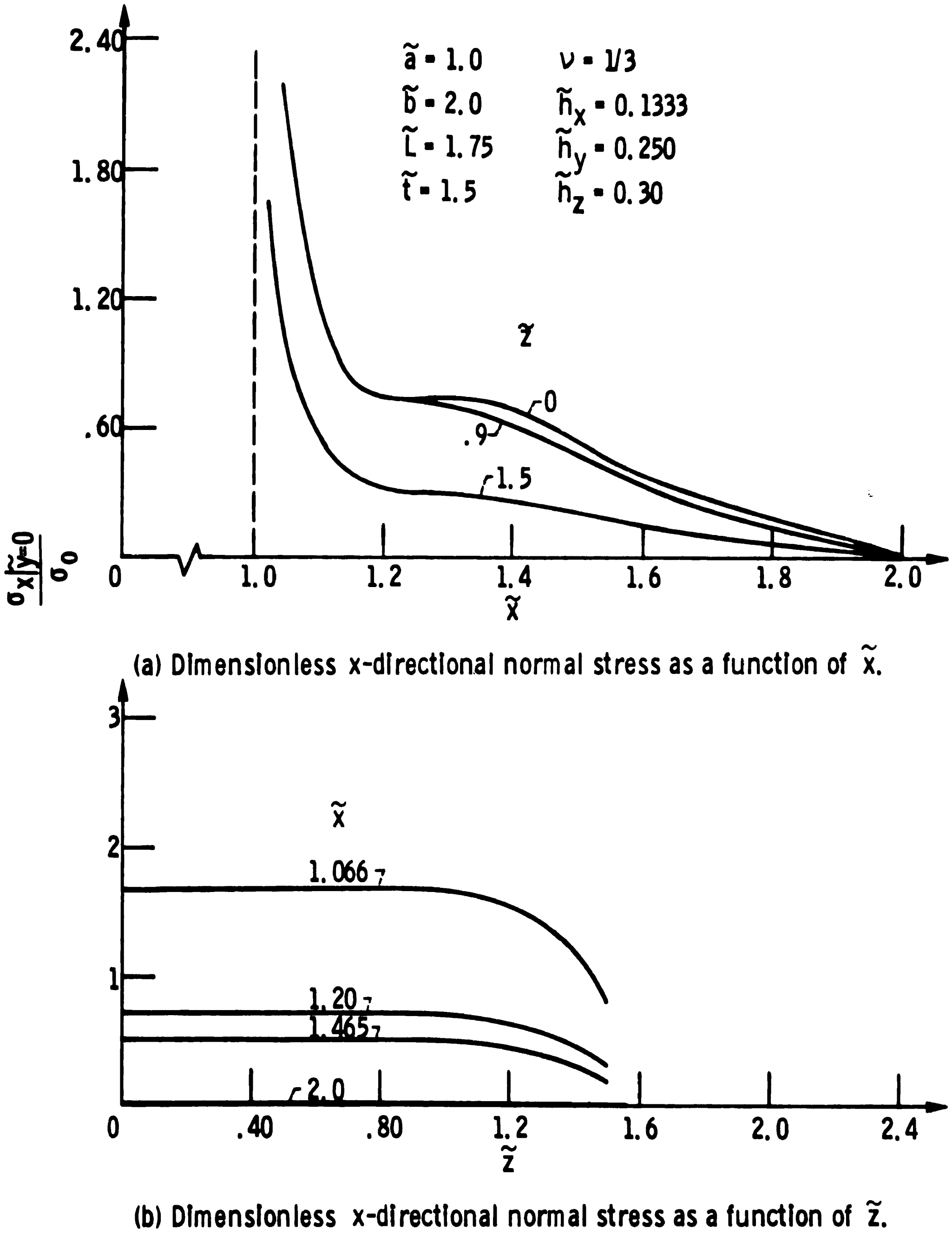
Figure 22. - Dimensionless y-directional normal stress distribution in the crack plane for a rectangular bar under uniform tension containing a through-thickness central crack.



(a) Dimensionless z-directional normal stress as a function of $\tilde{x}$.

(b) Dimensionless z-directional normal stress as a function of $\tilde{z}$.

Figure 23. - Dimensionless z-directional normal stress distribution in the crack plane for a rectangular bar under uniform tension containing a through-thickness central crack.



(a) Dimensionless x-directional normal stress as a function of $\tilde{x}$.

(b) Dimensionless x-directional normal stress as a function of $\tilde{z}$.

Figure 24. - Dimensionless x-directional normal stress distribution in the crack plane for a rectangular bar under uniform tension containing a through-thickness central crack.

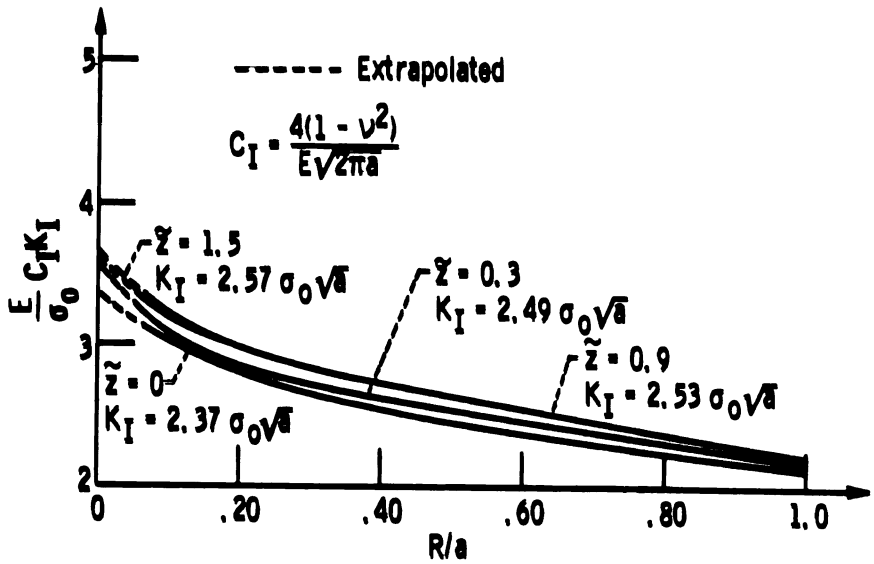


Figure 25. - Calculation of the stress intensity factors K_I for a rectangular bar under uniform tension containing a through-thickness central crack.

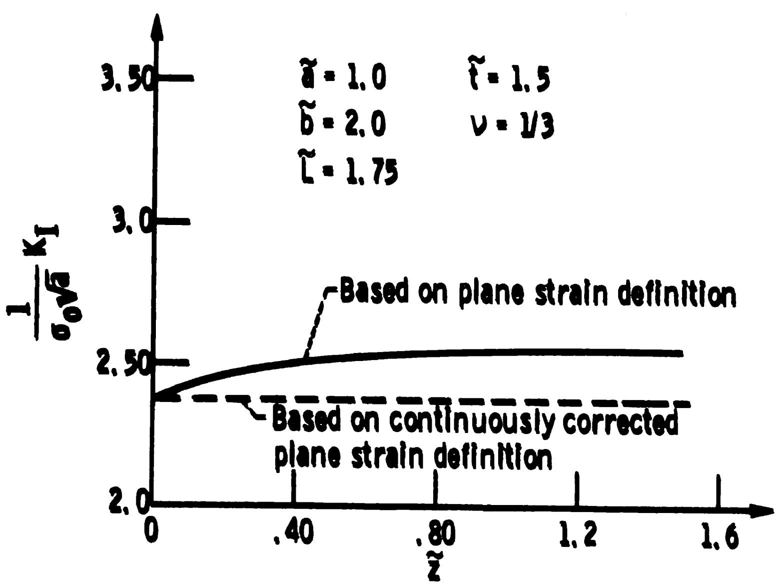


Figure 26. - Variation of the stress intensity factor K_I across the thickness for a rectangular bar under uniform tension containing a through-thickness central crack.

Table 12. - Dimensionless x-Directional Displacements $\frac{E}{\sigma_o}\frac{u}{a}$ for a Rectangular Bar Under Uniform Tension Containing a Through-Thickness Central Crack. $\tilde{a}$ = 1.0, $\tilde{b}$ = 2.0, $\tilde{L}$ = 1.75, $\tilde{t}$ = 1.5 (48-96-128 x-y-z Directional Lines Respectively)

$\tilde{y}$ \ $\tilde{x}$	0.0	.40	.80	1.20	1.60	2.00	
0.0	.000	-.474	-.828	-1.026	-1.095	-1.192	$\tilde{z}$ = .00
.50	.000	-.143	-.276	- .541	- .783	- .920	
1.0	.000	-.025	-.092	- .242	- .420	- .565	
1.75	.000	.169	.257	.238	.148	.029	
0.0	.000	-.479	-.846	-1.00	-1.030	-1.105	$\tilde{z}$ = .90
.50	.000	-.119	-.244	- .498	- .742	- .888	
1.0	.000	-.015	-.076	- .220	- .393	- .540	
1.75	.000	.180	.276	.260	.163	.042	
0.0	.000	-.395	-.713	- .890	- .991	-1.091	$\tilde{z}$ = 1.5
.50	.000	-.062	-.156	- .422	- .699	- .862	
1.0	.000	.009	-.035	- .182	- .367	- .520	
1.75	.000	.179	.278	.263	.168	.050	

Table 13. - Dimensionless y-Directional Displacements $\frac{Ev}{\sigma_o a}$ for a Rectangular Bar Under Uniform Tension Containing a Through-Thickness Central Crack. $\tilde{a}$ = 1.0, $\tilde{b}$ = 2.0, $\tilde{L}$ = 1.75, $\tilde{t}$ = 1.5 (48-96-128 x-y-z Directional Lines Respectively)

$\tilde{z}$ \ $\tilde{y}$	.00	.25	.50	1.00	1.50	1.75	
.00	2.112	2.227	2.315	2.578	2.893	3.048	$\tilde{x}$ = .00
.60	2.197	2.322	2.415	2.638	2.952	3.107	
.90	2.218	2.334	2.420	2.638	2.956	3.117	
1.50	2.249	2.323	2.402	2.631	2.945	3.125	
.00	1.252	1.354	1.552	2.014	2.453	2.660	$\tilde{x}$ = .80
.60	1.287	1.424	1.622	2.072	2.512	2.716	
.90	1.297	1.431	1.627	2.074	2.515	2.721	
1.50	1.313	1.411	1.591	2.029	2.471	2.686	
.00	.000	.270	.572	1.173	1.713	1.966	$\tilde{x}$ = 1.60
.60	.000	.288	.599	1.202	1.736	1.986	
.90	.000	.291	.604	1.206	1.739	1.986	
1.50	.000	.290	.589	1.179	1.716	1.965	
.00	.000	.139	.351	.868	1.372	1.620	$\tilde{x}$ = 2.0
.60	.000	.174	.384	.883	1.385	1.630	
.90	.000	.176	.388	.890	1.390	1.638	
1.50	.000	.167	.377	.881	1.380	1.626	

Table 14. - Dimensionless z-Directional Displacements $\frac{Ew}{\sigma_o a}$ for a Rectangular Bar Under Uniform Tension Containing a Through-Thickness Central Crack. $\bar{a}$ = 1.0, $\bar{b}$ = 2.0, $\bar{L}$ = 1.75, $\bar{t}$ = 1.5 (48-96-128 x-y-z Directional Lines Respectively)

$\bar{x}$ \ $\bar{z}$	.00	.30	.60	.90	1.20	1.50	
.00	.000	-.008	-.034	-.053	-.041	.025	$\bar{y}$ = 0.0
.80	.000	-.054	-.128	-.205	-.266	-.267	
1.60	.000	-.105	-.225	-.355	-.492	-.628	
2.00	.000	-.119	-.237	-.356	-.465	-.554	
.00	.000	-.048	-.094	-.134	-.167	-.194	$\bar{y}$ = .50
.80	.000	-.078	-.156	-.234	-.309	-.377	
1.60	.000	-.110	-.223	-.340	-.458	-.657	
2.00	.000	-.117	-.231	-.343	-.451	-.549	
.00	.000	-.080	-.160	-.241	-.324	-.408	$\bar{y}$ = 1.0
.80	.000	-.093	-.185	-.278	-.374	-.472	
1.60	.000	-.109	-.218	-.328	-.436	-.543	
2.00	.000	-.114	-.226	-.335	-.440	-.544	
.00	.000	-.117	-.224	-.334	-.461	-.614	$\bar{y}$ = 1.75
.80	.000	-.116	-.218	-.319	-.429	-.558	
1.60	.000	-.113	-.219	-.321	-.422	-.526	
2.00	.000	-.116	-.224	-.327	-.427	-.526	

minutes while for the 48 x 96 x 128 grid it was about 50 minutes. The number of iterations for these problems was 28 and 29 respectively. The largest element of the error matrix was .0001 in the y direction at the end of the bar when the matrix functions associated with the larger grid were calculated. A convergence criterion of 10^{-6} was applied to all three displacements in calculating the successive approximations.

The dimensionless crack opening displacements are shown in Figure 19. Inspection of Figures 19(a) and 19(b) shows that the finite difference approximations have been sufficiently refined when the results of the 48 x 96 x 128 grid were calculated. An approximate 15% change in $\tilde{h}_x$, 17% change in $\tilde{h}_y$ and a 25% in $\tilde{h}_z$ resulted in a maximum of 1.5% change in the crack opening displacement. Figure 19(a) also shows the results of the plane elasticity solutions obtained by Mendelson (43) for the problem shown. The plane stress solution gives the highest crack opening displacement while the plane strain solution is very close to the curve obtained at the center of the bar. Values at $\tilde{z} = .9$ and $\tilde{z} = 1.5$ are between the two plane solutions as can be expected. The shapes of the obtained curves are all elliptical which can be easily shown by plotting the equation of an ellipse with the coordinate intercepts as the major and minor axes. Note that the maximum change in the crack opening is about 6.5% across the given thickness.

A plot of the bar end extension is shown in Figure 20. This displacement is maximum at the center of the bar with the variation in the $\tilde{z}$ direction being less pronounced than in the crack plane. Because the center of the bar is more constrained, the y-directional displacements are maximum on the surface of the bar in both Figures 19 and 20. Figure 21 shows the variation of the crack opening displacements across the thickness of the bar. Near the crack edge, that is at $\tilde{x} = .932$, the displacement curve is almost a constant while at $\tilde{x} = 0$ the changes in the z direction are most rapid for $0 \leq z \leq .5$. Similar conclusion is possible from Figure 19(a) where the curves at $\tilde{z} = .9$ and $\tilde{z} = 1.5$ are very close to each other.

Figure 22 contains a plot of the stress distribution normal to the crack plane as a function of both bar width and thickness. Figure 22(a) shows the same general stress distribution as Figure 12, indicating that the normal stress is singular, as expected, at the crack edge. However, the resolution of our grid near the crack is not sufficient to establish an inverse square root singularity. Inspection of these curves shows that the stress is highest near the center of the bar and it rapidly approaches the applied stress for values of $\tilde{x} > 1.5$. Figure 22(b) shows that the variation in the $\tilde{z}$ direction is largest near the crack edge and as $\tilde{x}$ increases, the stress curves become more constant. Note that the stress at $\tilde{x} = 1.066$

indicates a central region of uniform stress for $0 < \tilde{z} < 1.1$ and a boundary region of $1.1 < \tilde{z} < 1.5$ where the stress drops significantly to the surface value. Similar results were obtained by Cruse and Van Buren (25) for a single edge crack bar specimen.

Figures 23 describe the σ_z stress distribution in the crack plane. The results of Figure 23(a) indicate that this normal stress is also singular near the crack edge and its value approaches zero with increasing $\tilde{z}$. This is expected since at $\tilde{z} = 1.5$, the surface is free of this normal stress. Note that for $\tilde{z} = 1.2$, the stress becomes zero at $\tilde{x} = 1.6$ indicating that this normal stress also vanishes on part of the $\tilde{x} = 2.0$ face. This result seems consistent since the stress field is expected to be three-dimensional in the vicinity of the crack and approach the crack-free bar solution at locations far from the crack. Figure 23(b) shows this σ_z normal stress mainly as a function of the thickness variable, $\tilde{z}$. An unexpected increase in this stress near the center of the bar can be noticed from the results shown. This sudden increase in σ_z appears to be the result of the way this stress increases as $\tilde{x}$ approaches the crack edge ($\tilde{x} = 1.0$). Note that the variation across the thickness in σ_z is more continuous than in Figure 22(b) with no noticeable central or boundary region shown. However, the curve at $\tilde{x} = 1.066$ seems to indicate that as the crack edge is being approached, most of the variation will occur near the surface of the bar.

Figures 24 show the σ_x normal stress distribution in the crack plane as a function of both $\tilde{x}$ and $\tilde{z}$. Figure 24(a) indicates that this stress is also singular at the crack edge and it is maximum near the center of the bar. Results in this figure also show that σ_x is zero only on the face $\tilde{x} = 2.0$ and has a given, non-zero value everywhere on the face $\tilde{z} = 1.5$. This is contrary to the results shown in Figure 23 for σ_z and as a consequence the sudden increase in σ_x near the center of the bar has not been obtained in Figure 24(b). Figure 24(b) shows a central region of uniform stress and a boundary layer through which the stress decreases to the surface values. Note that the variation in σ_x across the thickness decreases as $\tilde{x}$ increases and approaches a constant value of zero at $\tilde{x} = 2.0$. Figure 24(a) also indicates that the value of σ_x remains essentially constant for $1.2 < \tilde{x} < 1.35$. This constant σ_x stress is different, of course, as $\tilde{z}$ increases from zero to 1.5.

The results in Figures 19 through 24 cannot be checked against known data but the crack opening displacements of Figure 19 indicate reliability of the reported stress and displacement distributions. Limited results reported in (25) agree with some of our conclusions regarding the normal stress distributions but a detailed comparison between results is not possible because of the problems treated.

Figure 25 shows the calculation of the opening mode stress intensity factor from the plane strain crack opening displacement. In terms of the ordinate intercept, the stress intensity factor is given by K_I = INTERCEPT $\cdot$ (.704) $\sigma_o \sqrt{a}$. Using the extrapolated values in Figure 25, the stress intensity factors K_I for selected values of $\bar{z}$ were calculated. A plot of these stress intensity factors as a function of $\bar{z}$ is shown in Figure 26. Since the stress intensity factor is proportional to the crack opening displacement, the value of K_I from the plane strain definition is maximum at the surface and minimum near the center. Similar results were obtained in reference (25) for a single edge crack specimen where it was shown that as plane stress conditions are approached, the stress intensity factor increases. Srawley and Brown (5) have also found that the plane strain fracture toughness, which is directly related to the stress intensity factor, is considerably less than the plane stress fracture toughness.

Figure 19(a) shows that at the center of the bar plane strain conditions exist while at the surface $\bar{z}$ = 1.5, the crack opening is about half-way between the plane strain and plane stress solutions. Since the plane stress crack opening is about 12.5% higher than the plane strain crack displacement, it is proposed that equation (3.115) be revised as follows:

$$v\Big|_{\substack{y=0\\ z=1.5a}} = 1.0625 \left[\frac{2(1-\nu)}{G} K_I \sqrt{\frac{R}{2\pi}}\right] \quad (4.2)$$

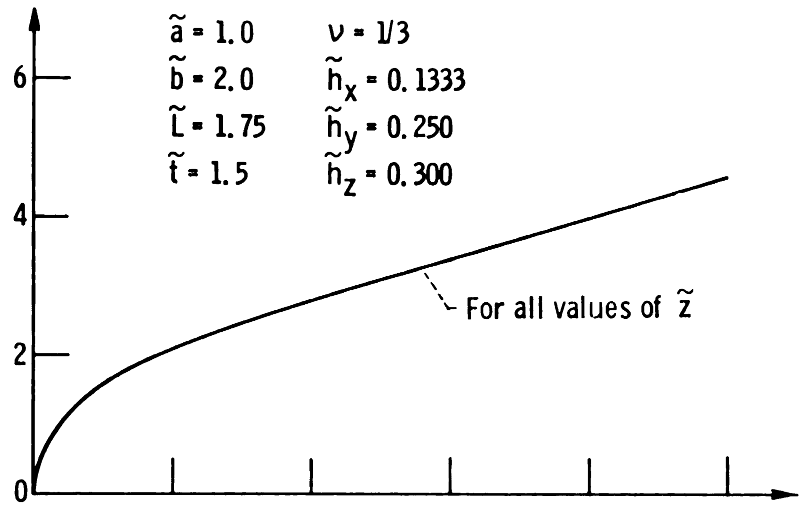
Equation (4.2) will now be used to predict the corrected value of K_I at $z = 1.5$. A similar corrected plane strain crack opening displacement equation may be written for each position along the $\tilde{z}$ axis and a continuously corrected value of K_I can be calculated. Figure 26 also shows a plot of this corrected stress intensity factor. The value of this corrected stress intensity factor is a constant across the thickness which then agrees with the fact that plane stress and plane strain stress intensity factors are equal. Figure 26 shows that this corrected value of K_I is 2.37 $\sigma_o\sqrt{a}$. In reference (2), Brown and Srawley report a plane elasticity stress intensity factor of 2.11 $\sigma_o\sqrt{a}$ for a bar having an identical width and half crack length. Note that the result of Figure 26 is about 12 percent higher than this value which must be attributed to the finite length and thickness of the bar in Figure 2. The solution in reference (2), of course, is for a bar with infinite length and thickness.

Tables 12 through 14 show selected results from the displacements obtained in the computations. The y-directional displacements are all extensional while in the z-direction only contraction is possible. A somewhat unexpected result is obtained in Table 12, which indicates that parts of the bar contract while other parts expand in the x-direction.

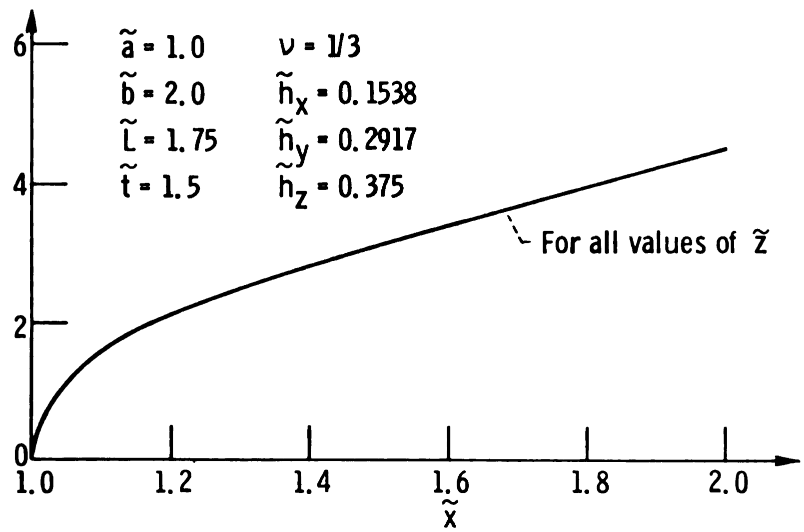
4.5 Bar With Through-Thickness Double Edge Cracks

Selected results for the problem of Figure 3 are presented in Figures 27 through 34 and Tables 15 through 17. These results were obtained with two different sets of lines along the coordinate axes which were identical to the two sets used for the problem of Section 4.4. The convergence criteria and error matrix elements were also the same as those for the central crack problem. However, the number of required iterations increased greatly with the fine grid using 60 iterations while the more coarse grid requiring 48 successive calculations. As a result, the execution time for the finer, 48 x 96 x 128 line grid problem was about 80 minutes.

The dimensionless crack opening displacements are plotted in Figure 27. Comparison of Figures 27(a) and 27(b) shows that the finite difference approximations have sufficiently converged when the 48 x 96 x 128 line grid was used in the final calculations. Contrary to the central crack problem, the crack opening displacements in Figure 27 are independent of the $\bar{z}$ coordinate. Similar results were obtained by Cruse and Van Buren (25) for the single edge crack bar specimen. One must also note that for this problem the crack opening curve is no longer elliptical



(a) Dimensionless crack opening displacement (48x96x128 grid).



(b) Dimensionless crack opening displacement (35x70x98 grid).

Figure 27. - Dimensionless crack opening displacement for a rectangular bar under uniform tension containing through-thickness double edge cracks.

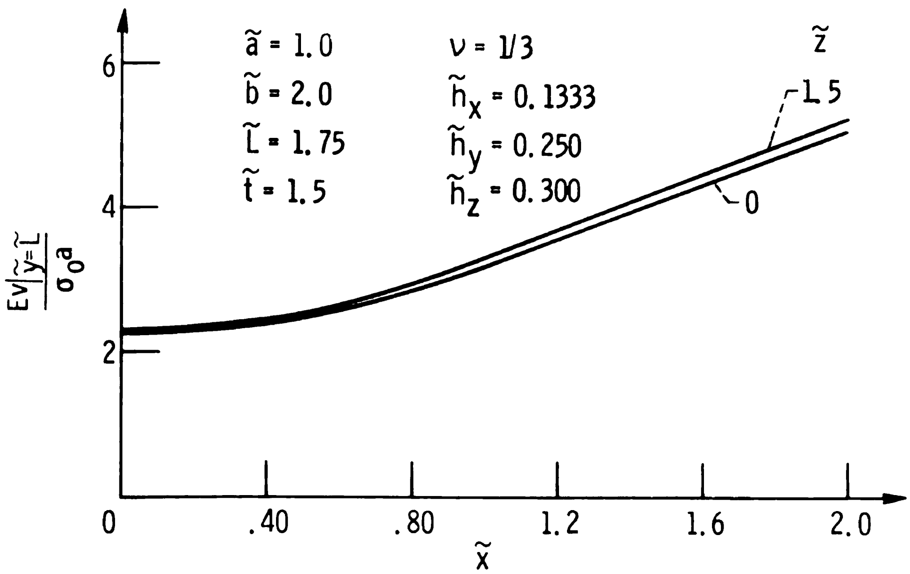


Figure 28. - Dimensionless bar end extension for a rectangular bar under uniform tension containing through-thickness double edge cracks.

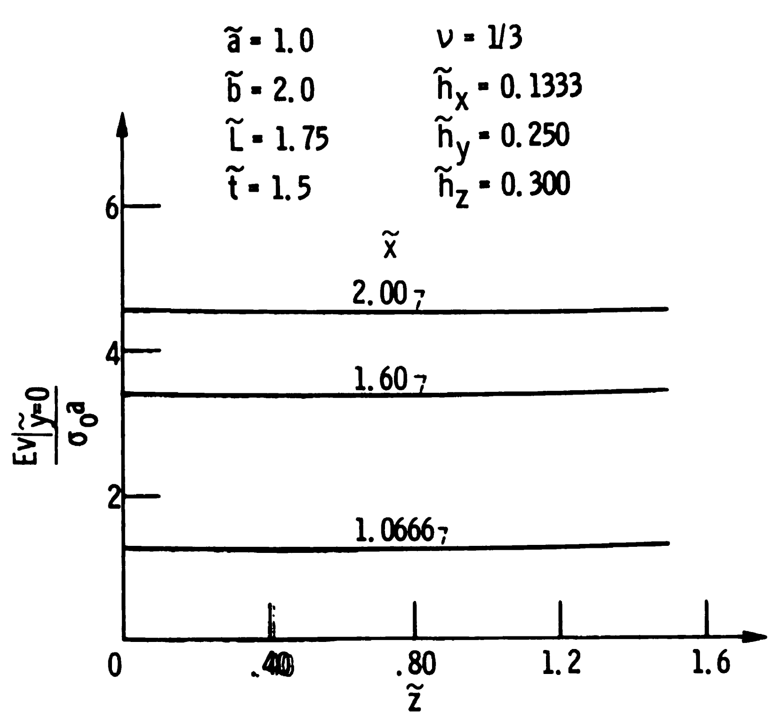
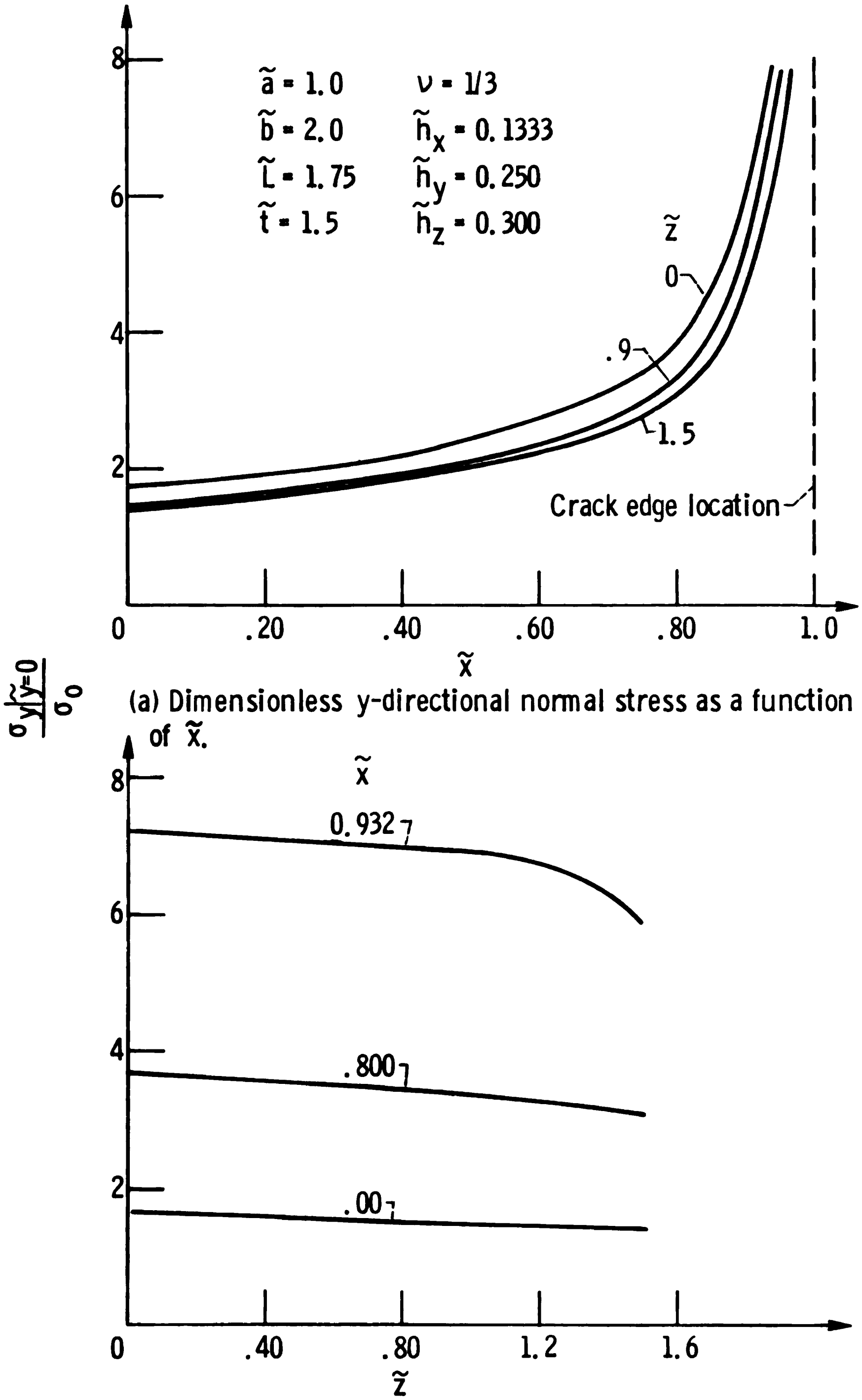


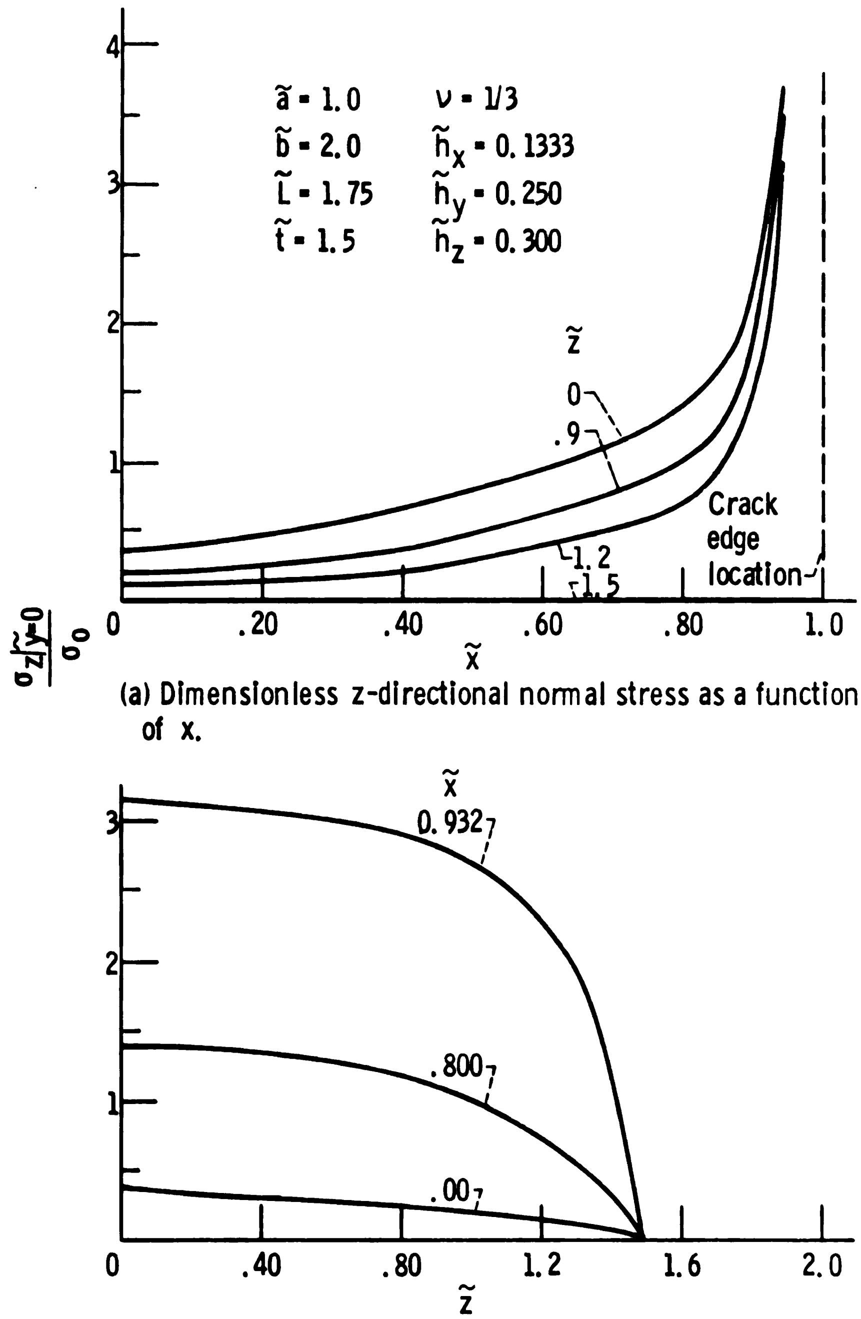
Figure 29. - Dimensionless normal displacement distribution in the crack plane for a rectangular bar under uniform tension containing through-thickness double edge cracks.



(a) Dimensionless y-directional normal stress as a function of $\tilde{x}$.

(b) Dimensionless y-directional normal stress as a function of $\tilde{z}$.

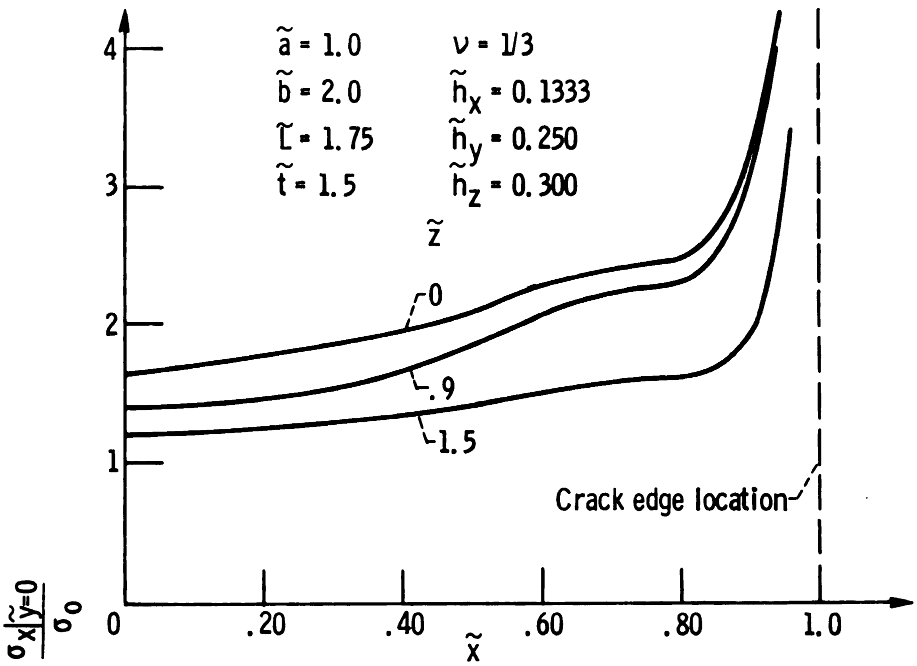
Figure 30. - Dimensionless y-directional normal stress distribution in the crack plane for a rectangular bar under uniform tension containing through-thickness double edge cracks.



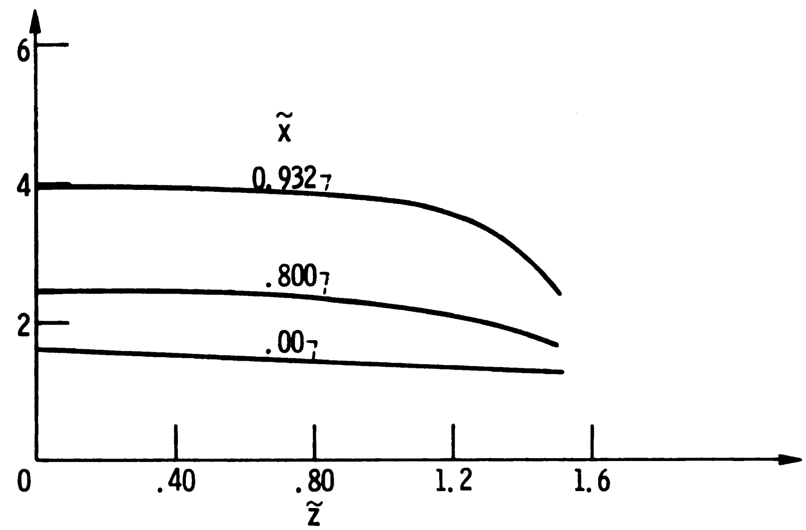
(a) Dimensionless z-directional normal stress as a function of x.

(b) Dimensionless z-directional normal stress as a function of z.

Figure 31. - Dimensionless z-directional normal stress distribution in the crack plane for a rectangular bar under uniform tension containing through-thickness double edge cracks.



(a) Dimensionless x-directional normal stress as a function of x̃.



(b) Dimensionless x-directional normal stress as a function of z̃.

Figure 32. - Dimensionless x-directional normal stress distribution in the crack plane for a rectangular bar under uniform tension containing through-thickness double edge cracks.

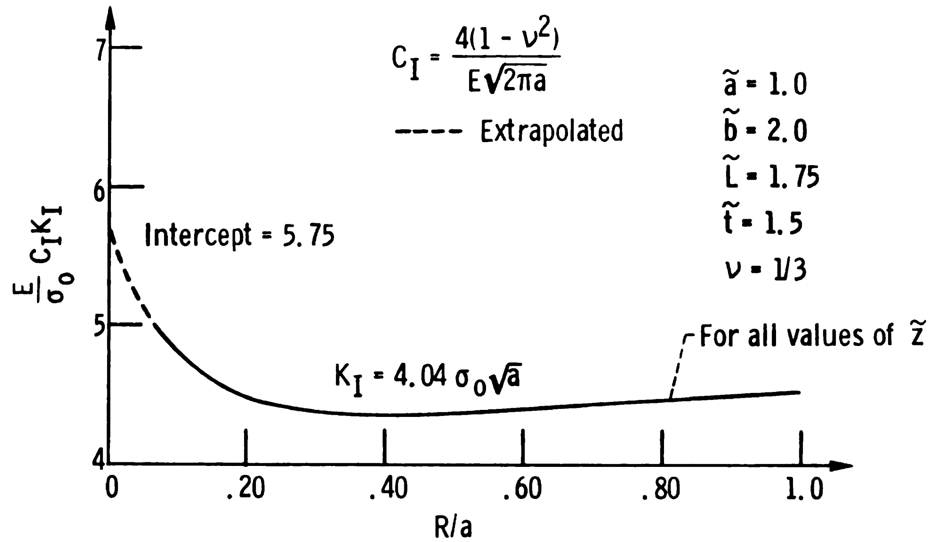


Figure 33. - Calculation of the stress intensity factors K_I for a rectangular bar under uniform tension containing through-thickness double edge cracks.

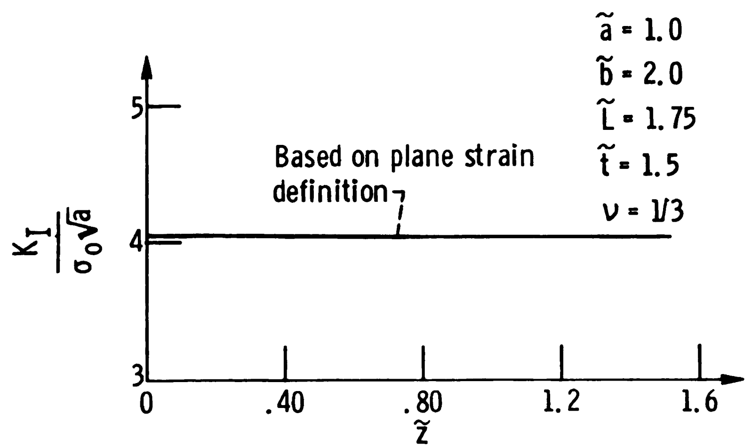


Figure 34. - Variation of the stress intensity factor K_I across the thickness for a rectangular bar under uniform tension containing through-thickness double edge cracks.

Table 15. - Dimensionless x-Directional Displacements $\frac{E}{\sigma_o}\frac{u}{\tilde{a}}$ for a Rectangular Bar Under Uniform Tension Containing Through-Thickness Double Edge Cracks. $\tilde{a} = 1.0$, $\tilde{b} = 2.0$, $\tilde{L} = 1.75$, $\tilde{t} = 1.5$ (48-96-128 x-y-z Directional Lines Respectively)

$\tilde{y}$ \ $\tilde{x}$	0.0	.40	.80	1.20	1.60	2.00	
0.0	.000	.389	.817	1.092	1.218	1.268	$\tilde{z}$ = .00
.50	.000	.054	- .018	- .128	- .082	- .054	
1.00	.000	-.224	- .478	- .700	- .852	- .946	
1.75	.000	-.646	-1.231	-1.688	-1.989	-2.181	
0.0	.000	.348	.784	1.078	1.152	1.169	$\tilde{z}$ = .90
.50	.000	.030	- .060	- .187	- .158	- .133	
1.00	.000	-.256	- .537	- .784	- .924	- .999	
1.75	.000	-.683	-1.295	-1.766	-2.066	-2.261	
0.0	.000	.292	.599	.831	.911	.941	$\tilde{z}$ = 1.5
.50	.000	-.028	- .196	- .374	- .344	- .314	
1.00	.000	-.293	- .616	- .892	-1.038	-1.113	
1.75	.000	-.693	-1.317	-1.801	-2.106	-2.291	

Table 16. - Dimensionless y-Directional Displacements $\frac{Ev}{\sigma_o a}$ for a Rectangular Bar Under Uniform Tension Containing Through-Thickness Double Edge Cracks. $\tilde{a}$ = 1.0, $\tilde{b}$ = 2.0, $\tilde{L}$ = 1.75, $\tilde{t}$ = 1.5 (48-96-128 x-y-z Directional Lines Respectively)

$\tilde{z}$ \ $\tilde{y}$	.00	.25	.50	1.00	1.50	1.75	
.00	.000	.258	.549	1.238	1.879	2.264	$\tilde{x}$ = 0.0
.60	.000	.222	.493	1.146	1.856	2.249	
.90	.000	.228	.503	1.159	1.869	2.257	
1.50	.000	.248	.529	1.182	1.907	2.277	
.00	.000	.588	1.081	1.916	2.474	2.850	$\tilde{x}$ = .80
.60	.000	.509	1.011	1.825	2.521	2.878	
.90	.000	.520	1.025	1.840	2.536	2.891	
1.50	.000	.601	1.086	1.874	2.587	2.931	
.00	3.374	3.432	3.500	3.785	4.036	4.305	$\tilde{x}$ = 1.60
.60	3.369	3.393	3.442	3.698	[illegible].131	4.387	
.90	3.370	3.390	3.440	3.700	4.139	4.403	
1.50	3.412	3.460	3.497	3.721	4.161	4.428	
.00	4.547	4.536	4.512	4.649	4.799	5.060	$\tilde{x}$ = 2.0
.60	4.515	4.498	4.463	4.546	4.938	5.186	
.90	4.512	4.491	4.453	4.538	4.941	5.198	
1.50	4.572	4.536	4.475	4.537	4.938	5.196	

Table 17. - Dimensionless z-Directional Displacements $\frac{Ew}{\sigma_o a}$ for a Rectangular Bar Under Uniform Tension Containing Through-Thickness Double Edge Cracks. $\tilde{a}$ = 1.0, $\tilde{b}$ = 2.0, $\tilde{L}$ = 1.75, $\tilde{t}$ = 1.5 (48-96-128 x-y-z Directional Lines Respectively)

$\tilde{x}$ \ $\tilde{z}$	.00	.30	.60	.90	1.20	1.50	
.00	.000	-.228	-.436	-.647	-.871	-1.120	$\tilde{y}$ = 0.0
.80	.000	-.201	-.394	-.609	-.888	-1.288	
1.60	.000	-.100	-.179	-.238	-.274	- .293	
2.00	.000	-.059	-.102	-.123	-.123	- .119	
.00	.000	-.189	-.379	-.566	-.748	- .928	$\tilde{y}$ = .50
.80	.000	-.167	-.337	-.513	-.702	- .888	
1.60	.000	-.090	-.172	-.249	-.317	- .375	
2.00	.000	-.052	-.110	-.164	-.205	- .219	
.00	.000	-.134	-.270	-.400	-.519	- .625	$\tilde{y}$ = 1.0
.80	.000	-.118	-.241	-.363	-.479	- .584	
1.60	.000	-.057	-.137	-.222	-.301	- .370	
2.00	.000	-.038	-.097	-.165	-.231	- .287	
.00	.000	-.095	-.188	-.267	-.315	- .308	$\tilde{y}$ = 1.75
.80	.000	-.097	-.186	-.264	-.323	- .343	
1.60	.000	-.081	-.149	-.221	-.297	- .374	
2.00	.000	-.067	-.123	-.190	-.270	- .361	

and for $1.3 < \tilde{x} < 2.0$ the displacement curve is essentially a straight line. These results appear to be consistent since at the bar edge no constraints exist against this mode of deformation.

Figure 28 shows the dimensionless bar end extension. This extension is maximum, as expected, at the edge of the bar and it is also somewhat dependent on the $\tilde{z}$ coordinate. The small variation in the $\tilde{z}$ direction is the result of the uneven constraints developed in the central region of the bar, which in planes far from the crack spread over the entire cross section of the problem. As can be noted, this end extension is somewhat higher near the surface than at the center of the bar. Figure 29 shows the variation of the crack opening displacements across the thickness of the bar. As discussed previously, these curves also show constant displacements along the $\tilde{z}$ axis.

Figure 30 contains a plot of the stress distribution normal to the crack plane. Inspection of these figures shows that this stress is maximum at the center of the bar and is singular near the crack edge. As previously mentioned, the type of singularity is difficult to establish but the shape of these curves is similar to that obtained in the other examples. The minimum value of this stress occurs at $\tilde{x} = 0$ but even at this point the stress is at least 40% higher than the applied stress. Figure 30(b) shows that the variation in the $\tilde{z}$ direction is largest near the crack edge and becomes more gradual with decreasing values of $\tilde{x}$. Note that the stress near the crack edge again indicates a central region of approximately uniform stress and a boundary region beyond $\tilde{z} = 1.1$ where the stress drops significantly to the surface values. This agrees with the results obtained for the central crack problem.

Figures 31 describe the dimensionless σ_z stress distribution in the crack plane. The results of Figure 31 are very similar to those shown in Figure 23 for the central crack problem with one major difference. Figure 31(a) indicates that for all values of $\tilde{x}$ this stress has a given, non-zero value and it vanishes only on the surface $\tilde{z} = 1.5$. This result, of course, follows from the relationship between the crack and bar widths for the problem described in Figure 3. The singular nature of this stress near the crack edge is evident from Figure 31(a). The variation of σ_z across the thickness is shown in Figure 31(b). Note that the curve near the crack edge, that is at $\tilde{x} = .932$, begins to display an internal region of uniform stress and a boundary region with significant variation. The variation across the thickness becomes more gradual as the value of $\tilde{x}$ decreases.

Figures 32 contain the σ_x normal stress distribution in the crack plane as a function of both bar width and thickness. Figure 32(a) shows that this stress is also maximum at the center of the bar and is singular near the crack edge. The constant value of this stress just outside the crack edge vicinity is similar to the results obtained for the central crack problem. Figures 32 also indicate that this normal stress is greater than the applied stress at all points in the central portion of the problem. Figure 32(b) displays the σ_x stress distribution across the thickness of the bar. Note the central region of uniform stress and a boundary region with significant stress variation. As the value of $\tilde{x}$ decreases, the stress distribution in the $\tilde{z}$ direction becomes a constant. This is expected since far from the crack edge, the stress field should approach a one-dimensional state of stress.

Figure 33 shows the calculation of the opening mode stress intensity factor from the plane strain crack opening displacement. Since the obtained displacements are independent of the $\tilde{z}$ coordinate, the stress intensity factor shown is a constant across the thickness of the bar. In addition, note that the continuous correction for the changing conditions from plane strain to plane stress, as applied to the central crack problem, has no meaning for the crack opening displacement of Figure 27. As a consequence, Figures 33 and 34 each contain only a single curve. The results of Figure 33 lead to a stress intensity factor of $4.04\ \sigma_o \sqrt{a}$. Brown and Srawley in reference (2) report a plane elasticity stress intensity factor of $2.05\ \sigma_o \sqrt{a}$

which is lower than that reported for the central crack problem. In view of the fact that the crack opening displacements and normal stress distributions at corresponding locations are considerably higher for the double edge crack problem than for the central crack problem, the value of $4.04\ \sigma_o \sqrt{a}$ seems to be the more realistic solution for this stress intensity factor.

Tables 15 through 17 show selected values of the displacements obtained in the computations. Table 15 shows that in the crack plane, the x-directional displacements are outward while in the other planes along the y-axis they are inward. Table 16 shows that all the y-directional displacements are extensional and that the crack opening displacement is essentially constant across the thickness. Table 17 shows that in the z-direction only contraction is possible which is maximum on the surface of the bar.

CHAPTER 5

SUMMARY AND CONCLUSIONS

The line method of analysis was investigated for the solution of coupled partial differential equations which were subject to coupled and mixed boundary conditions. The use of this method was illustrated by solving the Navier-Cauchy equations of elastic equilibrium for a number of mixed boundary value problems in three-dimensional elasticity. Problems in both rectangular Cartesian and cylindrical coordinates were investigated.

The application of the line method to the Navier-Cauchy equations in Cartesian coordinates led to coupled sets of ordinary differential equations with constant coefficients. In cylindrical coordinates, this same solution technique results in coupled sets of ordinary differential equations, some of which have variable coefficients. Analytical methods, in conjunction with a successive approximation procedure, were used to obtain the solution of these resulting ordinary differential equations.

One advantage of solving directly for displacements in solids containing geometric singularities is that the displacements are not singular. In addition, stresses are expressed in terms of first order partial derivatives only, which minimizes inherent inaccuracy in higher order numerical differentiation. It is for this reason, that numerical solution of displacement potentials or the Galerkin vector should be avoided since the stresses are expressed as second and third derivatives of these quantities respectively. The advantage of the line method over other numerical solutions is that it minimizes the required numerical differentiations and thus, it may be considered as a semi-analytical approach to the solution of a problem.

Stress and displacement distributions were calculated in two rectangular bars, one of which contained a through-thickness central crack while the other had double edge cracks. The need for these specific solutions has existed for a number of years in fracture toughness testing. As expected, the results of the central crack problem indicate that near the center of the given geometry solid, the conditions are approximately in a state of plane strain. As one proceeds from the center of the bar to its surface in the thickness direction, plane stress conditions are approached. Hence, displacements are maximum near the surface of the bar while normal stresses are maximum near its center. The singular nature of the normal stresses near the crack edge was established and three-dimensional stress and displacement distributions were successfully calculated. An equivalent plane strain stress intensity factor for three-dimensional problems was introduced. Similar results are reported for a double edge crack bar which indicate that changes along the thickness direction in the displacements parallel to the applied load are less significant. The calculated normal stress distributions, however, led to identical conclusions in this case as for the central crack problem.

Solutions in cylindrical coordinates were obtained for an annular plate containing internal surface cracks. The axisymmetric problem of a solid cylinder with a penny shaped crack was used to check the convergence and accuracy of this method. Results with good accuracy were obtained even from the use of a relatively coarse grid. The stress and displacement solutions of the above examples show that the method of lines provides a simple and systematic approach to the solution of some three-dimensional, mixed boundary value, elasticity problems.

At this time, some improvement in the solution techniques and the use of the computer for Cartesian coordinate problems may be indicated. Since the resulting ordinary differential equations are readily solved by the normal mode method, the numerical computations may be minimized by performing the successive approximation calculations in principal coordinates. Manipulations of diagonalized matrices should minimize both the round-off and inherent error which necessarily arise in all numerical computations. In addition, considerable savings in the cost of the required computer time will be possible.

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LIST OF REFERENCES

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APPENDIX A

EVALUATION OF THE COEFFICIENT MATRIX EIGENVALUES AND EIGENVECTORS

A close investigation of equation (2.32) shows that the coefficient matrix $\underset{\ell\times\ell}{[K_x]}$ can be decomposed into component matrices having the following tri-diagonal format:

$$\underset{M\times M}{\begin{bmatrix} 2 & -2 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix}} \qquad \text{(A.1)}$$

It is a simple matter then to find the eigenvalues and eigenvectors of this type of matrix. Noting that in equation (2.32) we have NZ rows of submatrices each of order NY, we express the coefficient matrix as

$$[K_x] = k_2\,[I_1] \otimes [K_1] + k_3\,[K_2] \otimes [I_2] \qquad (A.2)$$
$$\ell x \ell \qquad NZxNZ \quad NYxNY \quad NZxNZ \quad NYxNY$$

where $\otimes$ denotes the Kroenecker product of two matrices (37). Matrices $[K_1]$ and $[K_2]$ have the desired form of (A.1) but are of different order. Associated with the matrices $[K_1]$ and $[K_2]$ are the following two eigenvalue problems:

$$[K_1]\ \{X1\} = \mu\{X1\} \qquad (A.3)$$
$$NYxNY \quad NYx1 \quad NYx1$$

$$[K_2]\ \{X2\} = \delta\{X2\} \qquad (A.4)$$
$$NZxNZ \quad NZx1 \quad NZx1$$

where μ_j, $j = 1, 2, \ldots, NY$ denote the eigenvalues of $[K_1]$ and δ_i, $i = 1, 2, \ldots, NZ$ represent the eigenvalues of $[K_2]$. The original eigenvalue problem associated with the coefficient matrix $[K_x]$ can be written as

$$[K_x]\ \{X3\} = \bar{\lambda}\{X3\} \qquad (A.5)$$
$$\ell x \ell \quad \ell x1 \quad \ell x1$$

After some matrix manipulations involving Kroenecker products (37), it can be shown that the eigenvalues $\bar{\lambda}_{ij}$ and the corresponding matrix of eigenvectors $\{X3\}^{ij}$ can be expressed in terms of the component matrix eigenvalues and eigenvectors. The results of these manipulations are

$$\overline{\lambda}_{ij} = k_3\delta_i + k_2\mu_j \tag{A.6}$$

$$\underset{\ell\times\ell}{\{X3\}^{ij}} = \underset{NZ\times NZ}{\{X2\}^{i}} \otimes \underset{NY\times NY}{\{X1\}^{j}} \tag{A.7}$$

where $i = 1, 2, \ldots, NZ$ and $j = 1, 2, \ldots, NY$. Equations (A.6) and (A.7) reduce the problem of (A.5) to that of finding the eigenvalues and eigenvectors of (A.1).

The eigenvalues of the tri-diagonal matrix (A.1) can be obtained by using difference equation theory. Let us consider the case when the eigenvalues are denoted as μ_j. Then the j^{th} difference equation can be written as

$$x_{j+1} + (\mu-2)x_j + x_{j-1} = 0 \tag{A.8}$$

We define $2\rho = \mu-2$, where by Gersgorin theorem on bounds of eigenvalues (44), we must have $|\rho| \leqq 1$. Equation (A.8) can now be written as a linear, second order difference equation with constant coefficients, that is

$$x_{j-1} + 2\rho x_j + x_{j+1} = 0 \tag{A.9}$$

Following standard solution techniques (45), we assume that

$$x_j = \alpha^j \tag{A.10}$$

Substituting equation (A.10) into equation (A.9) and noting that $|\rho| \leq 1$ we find that the values of α are given by

$$\alpha_1 = \cos\phi + \bar{i}\sin\phi = e^{\bar{i}\phi}$$

$$\alpha_2 = \cos\phi - \bar{i}\sin\phi = e^{-\bar{i}\phi}$$

where

$$\cos\phi = -\rho$$

$$\pm\,\bar{i}\sin\phi = \pm\sqrt{\rho^2-1}$$

$$\bar{i} = \sqrt{-1}$$

The solution of (A.9) can now be written as

$$\left.\begin{aligned} x_j &= A(e^{\bar{i}\phi})^j + B(e^{-\bar{i}\phi})^j \\ \text{or} \qquad x_j &= \bar{A}\cos j\phi + \bar{B}\,\bar{i}\sin j\phi \end{aligned}\right\} \qquad \text{(A.11)}$$

Constants A and B can be evaluated from the following boundary conditions:

$$\begin{aligned} x_o &= x_2 \\ x_{M-1} &= x_{M+1} \end{aligned} \qquad \text{(A.12)}$$

where for the matrix $[K_1]$, $M = NY$. Applying equations (A.12) to equations (A.11) gives

$$
\begin{aligned}
[(e^{\bar{i}\phi})^2 - 1]A + \quad [(e^{-\bar{i}\phi})^2 - 1]B \quad &= 0 \\
(e^{\bar{i}\phi})^{M-1}[(e^{\bar{i}\phi})^2 - 1]A + (e^{-\bar{i}\phi})^{M-1}[(e^{-\bar{i}\phi})^2 - 1]B \quad &= 0
\end{aligned}
\tag{A.13}
$$

For a non-trivial solution of equations (A.13), the determinant of the coefficients must vanish. This condition leads to the following characteristic equation:

$$e^{(M-1)\bar{i}\phi} - e^{-(M-1)\bar{i}\phi} = 0 \tag{A.14}$$

which can be written as

$$\sin\,(M-1)\,\phi = 0 \tag{A.15}$$

Equation (A.15) will be satisfied if

$$\phi_n = \frac{n\pi}{M-1} \quad \text{where} \quad n = 0, 1, 2, \ldots, M-1$$

Using the definition of ρ and μ, the eigenvalues of matrix $[K_1]$ are given by

$$\mu_j = 2\left[1 - \cos\left(\frac{j-1}{M-1}\right)\pi\right] \quad \text{where} \quad \begin{array}{l} j = 1, 2, \ldots, M \\ M = NY \end{array} \tag{A.16}$$

Similarly, the eigenvalues of $[K_2]$ become

$$\delta_i = 2\left[1 - \cos\left(\frac{i-1}{M-1}\right)\pi\right] \quad \text{where} \quad \begin{array}{l} i = 1, 2, \ldots, M \\ M = NZ \end{array} \tag{A.17}$$

The eigenvectors corresponding to equation (A.16) can be found by substituting the eigenvalues back into (A.11) and using boundary conditions (A.12). For the n^{th} eigenvalue we have

$$x_j^{(n)} = \overline{A}^{(n)} \cos j \frac{n\pi}{M-1} + \overline{B}^{(n)} \overline{i} \sin j \frac{n\pi}{M-1}$$

$$\text{for } n = 0, 1, \ldots, M-1$$

Enforcing the condition of $x_o^{(n)} = x_2^{(n)}$ yields

$$\overline{B}^{(n)} = \frac{\left(1 - \cos \frac{2n\pi}{M-1}\right) \overline{A}^{(n)}}{\overline{i} \sin \frac{2n\pi}{M-1}}$$

Since equations (A.13) are linearly dependent, we may select a convenient value for $\overline{A}^{(n)}$. Let us take $\overline{A}^{(n)}$ as

$$\overline{A}^{(n)} = \cos \frac{n\pi}{M-1}$$

Then the n^{th} eigenvector is given by

$$x_j^{(n)} = \cos \frac{(j-1)n\pi}{M-1} \qquad \begin{array}{l} j = 1, 2, \ldots, M \\ n = 0, 1, 2, \ldots, M-1 \end{array}$$

and the elements of eigenvectors $\{X1\}^j$, corresponding to the eigenvalues μ_j, in equation (A.3) can be written as

$$[P_{1_{sj}}] = \{X1\}_s^j = \left[\cos \frac{(s-1)(j-1)}{NY-1}\pi\right] \quad s = j = 1, 2, \ldots, NY \tag{A.18}$$

Similarly, we find that

$$[P_{2_{ri}}] = \{X2\}_r^i = \left[\cos \frac{(r-1)(i-1)}{NZ-1}\,\pi\right]$$

$$r = i = 1, 2, \ldots, NZ \qquad (A.19)$$

Substituting equations (A.16) and (A.17) into equation (A.6) yields for the eigenvalues of $[K_x]$ the following:

$$\overline{\lambda}_{ij} = 2k_3\left[1 - \cos\left(\frac{i-1}{NZ-1}\right)\pi\right] + 2k_2\left[1 - \cos\left(\frac{j-1}{NY-1}\right)\pi\right] \qquad (A.20)$$

where $i = 1, 2, \ldots, NZ$ and $j = 1, 2, \ldots, NY$.

Note that the smallest eigenvalue is zero while the largest eigenvalue is $(4k_2 + 4k_3)$. The zero eigenvalue is consistent with the inherent singular character of the coefficient matrices obtained.

The modal matrices of $[K_1]$ and $[K_2]$, denoted by $[P_1]$ and $[P_2]$ respectively, are constructed next according to equations (A.18) and (A.19). Since the matrix in (A.1) is non-symmetric, the eigenvectors (A.18) or (A.19) are not orthogonal. The modal matrix of the coefficient matrix $[K_x]$ is given by the Kroenecker product of the component modal matrices. Thus, from equation (A.7) we have

$$\underset{\ell\times\ell}{[P]} = \underset{NZ\times NZ}{[P_2]} \otimes \underset{NY\times NY}{[P_1]} \qquad (A.21)$$

The similarity transformations, diagonalizing the submatrices $[K_1]$ and $[K_2]$, can be written as

$$[K_1]\,[P_1] = [P_1]\,[\Lambda_1] \qquad (A.22)$$

$$\text{NYxNY NYxNY} \quad \text{NYxNY NYxNY}$$

$$[K_2]\,[P_2] = [P_2]\,[\Lambda_2] \qquad (A.23)$$

$$\text{NZxNZ NZxNZ} \quad \text{NZxNZ NZxNZ}$$

where $[\Lambda_1]$ and $[\Lambda_2]$ are diagonal matrices having the eigenvalues of $[K_1]$ and $[K_2]$ as their elements. The similarity transformation of the matrix $[K_x]$, using the decomposition (A.2) and equation (A.21), gives

$$[K_x]([P_2] \otimes [P_1]) = k_5([I_1] \otimes [K_1])([P_2] \otimes [P_1])$$

$$+ \; k_6([K_2] \otimes [I_2])([P_2] \otimes [P_1]) \qquad (A.24)$$

Following the matrix manipulations in (37), equation (A.24) can be reduced to the form shown below.

$$[K_x]\,[P] = [P]\,[\Lambda] \qquad (A.25)$$

where the diagonalized form of $[K_x]$ is given by

$$[\Lambda] = k_2([I_2] \otimes [\Lambda_1]) + k_3([\Lambda_2] \otimes [I_1]) \qquad (A.26)$$

The inverse of the modal matrix becomes

$$[P]^{-1} = [P_2]^{-1} \otimes [P_1]^{-1} \qquad (A.27)$$

Inspection of equations (A.22) and (A.23) shows that for an accurate evaluation of the component diagonalized matrices, the closed form inverses of the modal matrices $[P_1]$ and $[P_2]$ are needed.

Let us consider then the inverse of, say modal matrix $[P_1]$, in detail. Since $[P_1]$ was obtained from the component matrix $[K_1]$, we shall construct a diagonal matrix $[D_1]$ that transforms the non-symmetric matrix $[K_1]$ into a symmetric matrix $[K_1]^S$. The form of this transformation is

$$[D_1]^{1/2}\,[K_1]\left([D_1]^{1/2}\right)^{-1} = [K_1]^S \qquad \text{(A.28)}$$

where by definition the square root of a diagonal matrix is also a diagonal matrix whose elements are the square roots of the elements in the original matrix. For the tri-diagonal matrix $[K_1]$, the diagonal matrix $[D_1]$ is of the form

$$\underset{M\times M}{[D_1]} = \underset{M\times M}{\begin{bmatrix} 1/2 & & & & & & \\ & 1 & & & & & \\ & & 1 & & 0 & & \\ & & & \ddots & & & \\ & & 0 & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1/2 \end{bmatrix}} \quad M=NY \qquad \text{(A.29)}$$

Equation (A.22) will not be changed if we write it as

$$[K_1]\left([D_1]^{1/2}\right)^{-1}[D_1]^{1/2}[P_1] = [P_1][\Lambda_1] \tag{A.30}$$

Pre-multiplying equation (A.30) by $[D_1]^{1/2}$ gives

$$\left([D_1]^{1/2}[K_1]\left([D_1]^{1/2}\right)^{-1}\right)[D_1]^{1/2}[P_1]$$

$$= [D_1]^{1/2}[P_1][\Lambda_1]$$

$$[K_1]^s([D_1]^{1/2}[P_1]) = ([D_1]^{1/2}[P_1])[\Lambda_1] \tag{A.31}$$

Since $[K_1]^s$ is a symmetric matrix, the transformation of (A.31) is orthogonal and the columns of $([D_1]^{1/2}[P_1])$ must be orthogonal. For an orthogonal transformation, it is known that

$$([D_1]^{1/2}[P_1])^T([D_1]^{1/2}[P_1]) = [D] \tag{A.32}$$

where T denotes the transpose and $[D]$ is diagonal. Since $[D_1]$ is a diagonal matrix, the first term in (A.32) can be written as

$$([D_1]^{1/2}[P_1])^T = [P_1]^T[D_1]^{1/2} \tag{A.33}$$

Using equation (A.33) in equation (A.32) yields

$$[P_1]^T[D_1][P_1] = [D] \tag{A.34}$$

Pre-multiplying this equation by $[D]^{-1}$ gives

$$([D]^{-1}[P_1]^T[D_1])[P_1] = [I] \tag{A.35}$$

from which we find that

$$[P_1]^{-1} = [D]^{-1} [P_1]^T [D_1] \tag{A.36}$$

Equation (A.36) requires the inverse of an unknown matrix [D]. However, from equation (A.34) we can show that the matrix [D] always has the following form:

$$\underset{M\times M}{[D]} = \begin{bmatrix} M-1 & & & & \\ & \frac{M-1}{2} & & 0 & \\ & & \ddots & & \\ & 0 & & \frac{M-1}{2} & \\ & & & & M-1 \end{bmatrix} \tag{A.37}$$

Comparing matrices (A.37) and (A.29) we can conclude that

$$\underset{M\times M}{[D]} = \frac{M-1}{2} [D_1]^{-1} \tag{A.38}$$

Substituting this equation into equation (A.36) gives the final closed form inverse of $[P_1]$ as

$$\underset{M\times M}{[P_1]^{-1}} = \frac{2}{M-1} \underset{M\times M}{[D_1]} \underset{M\times M}{[P_1]^T} \underset{M\times M}{[D_1]} \tag{A.39}$$

Similar equation is written for the inverse of the modal matrix $[P_2]$. With the closed form solution of the component modal matrices and their inverses, the diagonalized form of $[K_x]$ is obtained from equation (A.26). The matrix functions $[A_{11}]$ and $[A_{12}]$ can now be evaluated as scalars and then re-transformed to full matrix forms according to the following similarity transformations:

$$[A_{11}] = [P]\,[\Lambda_{11}]\,[P]^{-1} \tag{A.40}$$

$$[A_{12}] = [P]\,[\Lambda_{12}]\,[P]^{-1} \tag{A.41}$$

APPENDIX B

FOURTH-ORDER RUNGE-KUTTA INTEGRATION FORMULAS

In Appendix B, we list the Runge-Kutta integration formulas which are used to solve the simultaneous first order matrix differential equations shown below:

$$\frac{d}{dr}[\Omega_{11}] = r[\Omega_{21}] = [f_1(r,[\Omega_{21}])] \tag{B.1}$$

$$\frac{d}{dr}[\Omega_{21}] = \frac{1}{r}[\Omega_{11}][K_r] = [f_2(r,[\Omega_{11}])] \tag{B.2}$$

The initial conditions for these equations are,

$$[\Omega_{11}(r_o)] = [I] \tag{B.3}$$

$$[\Omega_{21}(r_o)] = [0] \tag{B.4}$$

A step-by-step procedure starting at r_o is applied and the following formulas are used to predict values of the matrices along the independent variable axis (46):

$$[\Omega_{11}]_{n+1} = [\Omega_{11}]_n + \frac{h}{6}([Q_{11}] + 2[Q_{12}] + 2[Q_{13}] + [Q_{14}]) \tag{B.5}$$

$$[\Omega_{21}]_{n+1} = [\Omega_{21}]_n + \frac{h}{6}([Q_{21}] + 2[Q_{22}] + 2[Q_{23}] + [Q_{24}]) \tag{B.6}$$

where the approximate Runge-Kutta derivatives are

$$[Q_{11}] = [f_1(r_n,[\Omega_{21}]_n)] \tag{B.7}$$

$$[Q_{12}] = [f_1(r_n + \frac{h}{2}, [\Omega_{11}]_n + \frac{h}{2}[Q_{11}], [\Omega_{21}]_n + \frac{h}{2}[Q_{21}])] \tag{B.8}$$

$$[Q_{13}] = [f_1(r_n + \frac{h}{2}, [\Omega_{11}]_n + \frac{h}{2}[Q_{12}], [\Omega_{21}]_n + \frac{h}{2}[Q_{22}])] \tag{B.9}$$

$$[Q_{14}] = [f_1(r_n + h, [\Omega_{11}]_n + h[Q_{13}], [\Omega_{21}]_n + h[Q_{23}])] \tag{B.10}$$

$$[Q_{21}] = [f_2(r_n, [\Omega_{11}]_n)] \tag{B.11}$$

$$[Q_{22}] = [f_2(r_n + \frac{h}{2}, [\Omega_{11}]_n + \frac{h}{2}[Q_{11}], [\Omega_{21}]_n + \frac{h}{2}[Q_{21}])] \tag{B.12}$$

$$[Q_{23}] = [f_2(r_n + \frac{h}{2}, [\Omega_{11}]_n + \frac{h}{2}[Q_{12}], [\Omega_{21}]_n + \frac{h}{2}[Q_{22}])] \tag{B.13}$$

$$[Q_{24}] = [f_2(r_n + h, [\Omega_{11}]_n + h[Q_{13}], [\Omega_{21}]_n + h[Q_{23}])] \tag{B.14}$$

In the above formulas, h is the arbitrary integration increment and n denotes the instantaneous position along the path of integration.

APPENDIX C

Appendix C includes a copy of each of the computer programs prepared for the five numerical examples presented. Since the work was performed on an IBM-360 digital computer, the enclosed listings use Fortran IV algebraic type language. A brief description of the required inputs, subroutine functions and a short flow diagram for each problem is also given.

C.1 Solid Cylindrical Bar With a Penny Shaped Crack

A schematic representation of the main subroutines of the computer program is shown in Figure 35. The program is divided into five parts four of which are called from the main program. The main program is denoted as S5 while the subroutines are designated as S1, S2, S3 and S4. Subroutine S1 - RKGS uses the Runge-Kutta algorithm to calculate the diagonalized radial matrix functions. Subroutine S3 - FCT generates the needed first derivatives for the Runge-Kutta solution. Subroutine S2 - MATINV uses the Gauss-Jordan maximum pivot strategy (33) to numerically generate all the required matrix inverses. Subroutine S4 - MAPOWR evaluates the required matrix series for the solution of the constant coefficient differential equations.

The input for each routine is typed in at the beginning of

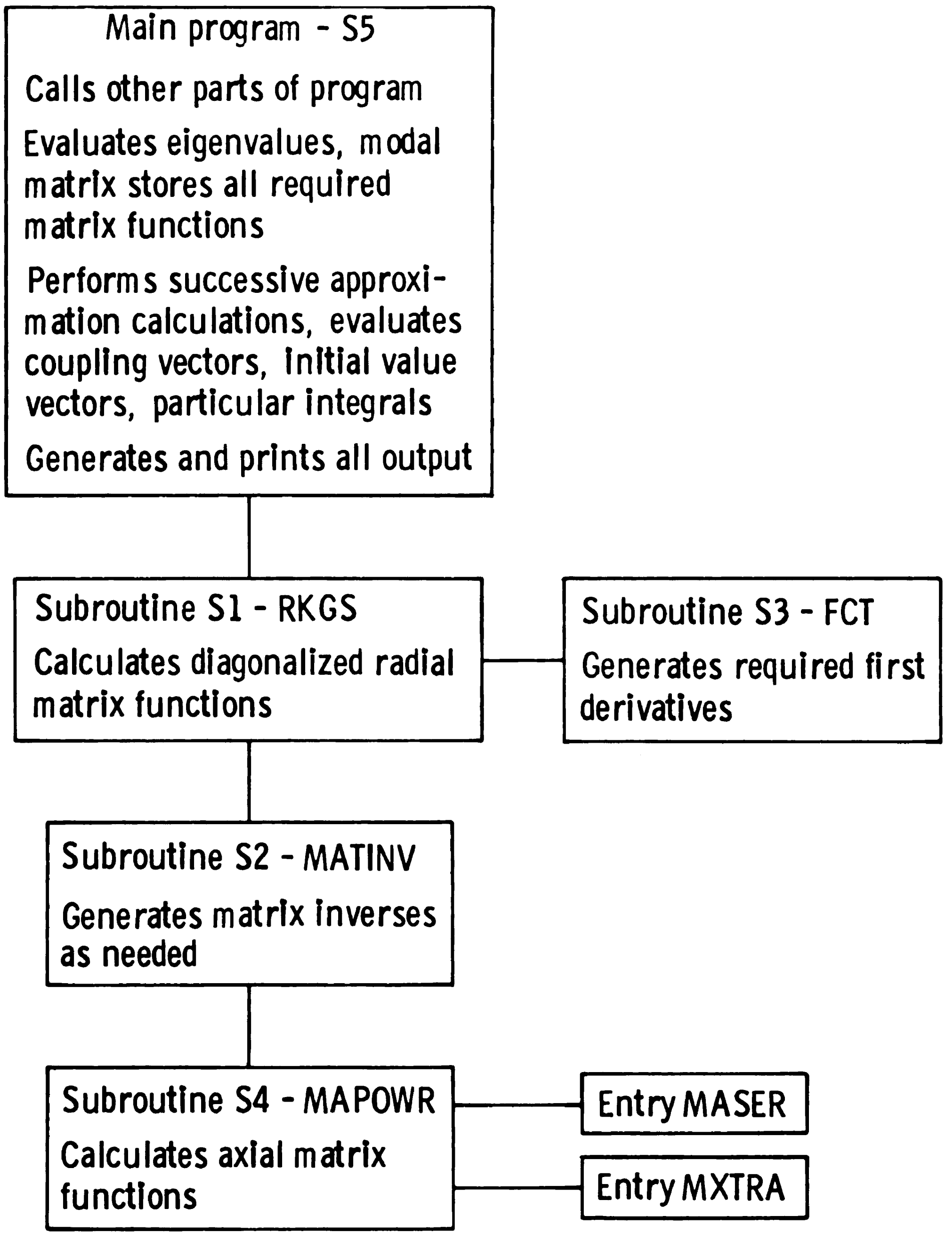

Figure 35. - Schematic representation of the solid cylindrical bar with a penny shaped crack computer program.

each part using the word DATA for identification. The eigenvalues of the radial and the elements of the axial coefficient matrices are given in S5 and S4 respectively. The enclosed program is generalized in that the grid size can be modified by suitable changes in the DATA, COMMON and DIMENSION statements only. The successive approximation procedure is performed in the main program S5 and the symbol KTR is used to follow the number of repeated calculations. Information about matrix function error checks, singularity of matrices and number of iterations is printed at the terminal while the bulk of the output is being stored in the computer for a more efficient printing operation. The output for this problem includes the displacements $\{u\}$ and $\{w\}$, their derivatives $\{\mathring{u}\}$ and $\{\mathring{w}\}$ and the stresses $\{\sigma_{rs}\}$, $\{\sigma_{\theta s}\}$, $\{\sigma_{zs}\}$ and $\{\sigma_{rzs}\}$. The displacements and their derivatives are printed at the end of each iteration, so that their convergence can easily be followed.

```
S1   0000100.     SUBROUTINE RKGS (O1,O2,XO)
S1   0000200.     IMPLICIT REAL*8 (A-H,O-Z)
S1   0000300.     DATA NR,HR,JJ,XJ /13,8.3333333333333333D-2,64,6.4D1/
S1   0000400.     COMMON /RUNGE/ O3(13),O4(13),O7(13),O8(13)
S1   0000500.     DIMENSION O5(13),O6(13),O1(13),O2(13)
S1   0000600.     H=HR/XJ
S1   0000700.     HH=H/2.0D0
S1   0000800.     DO 6 I=1,JJ
S1   0001000.     DO 1 J=1,NR
S1   0001100.     O7(J)=O1(J)
S1   0001200.1    O8(J)=O2(J)
S1   0001300.     CALL FCT (XO)
S1   0001500.     DO 2 J=1,NR
S1   0001600.     O5(J)=O3(J)
S1   0001700.     O6(J)=O4(J)
S1   0001800.     O7(J)=O1(J)+HH*O3(J)
S1   0001900.2    O8(J)=O2(J)+HH*O4(J)
S1   0002000.     XO=XO+HH
S1   0002100.     CALL FCT (XO)
S1   0002300.     DO 3 J=1,NR
S1   0002400.     O7(J)=O1(J)+HH*O3(J)
S1   0002500.     O8(J)=O2(J)+HH*O4(J)
S1   0002600.     O5(J)=O5(J)+2.0D0*O3(J)
S1   0002700. 3   O6(J)=O6(J)+2.0D0*O4(J)
S1   0002800.     CALL FCT (XO)
S1   0003000.     DO 4 J=1,NR
S1   0003100.     O5(J)=O5(J)+2.0D0*O3(J)
S1   0003200.     O6(J)=O6(J)+2.0D0*O4(J)
S1   0003300.     O7(J)=O1(J)+H*O3(J)
S1   0003400. 4   O8(J)=O2(J)+H*O4(J)
S1   0003500.     XO=XO+HH
S1   0003600.     CALL FCT (XO)
S1   0003800.     DO 5 J=1,NR
S1   0003900.     O5(J)=O5(J)+O3(J)
S1   0004000.     O6(J)=O6(J)+O4(J)
S1   0004100.     O1(J)=O1(J)+H*O5(J)/6.0D0
S1   0004200.5    O2(J)=O2(J)+H*O6(J)/6.0D0
S1   0004300.6    CONTINUE
S1   0004400.     RETURN
S1   0004500.     END
S2   0000100.     SUBROUTINE MATINV (X,MMM)
S2   0000200.     DOUBLE PRECISION X,Y,W,A,TEST,R,DELT
S2   0000300.     DIMENSION X(MMM,MMM), Y(13,13), K(13), W(13,13)
S2   0000400.     DATA DELT,LOOPS/1.0D-8,2/
S2   0000500.     KSIG=0
S2   0000600.     M=MMM
S2   0000700.     MM=M-1
S2   0000800.     DO 1 J=1,M
S2   0000900.     K(J)=J
S2   0001000.     DO 1 I=1,M
S2   0001100.1    Y(I,J)=X(I,J)
S2   0001200.     DO 6 I=1,M
S2   0001300.     A=0.D0
S2   0001400.     DO 2 J=I,M
```

```
S2      0001500.      IF (DABS(Y(J,1)).LE.A) GO TO 2
S2      0001600.      A=DABS(Y(J,1))
S2      0001700.      L=J
S2      0001800.2     CONTINUE
S2      0001900.      IF (A.EQ.0.D0) GO TO 18
S2      0002000.      N=K(L)
S2      0002100.      K(L)=K(I)
S2      0002200.      K(I)=N
S2      0002300.      DO 3 J=1,M
S2      0002400.      A=Y(I,J)
S2      0002500.      Y(I,J)=Y(L,J)
S2      0002600.3     Y(L,J)=A
S2      0002700.      A=Y(I,1)
S2      0002800.      DO 4 J=1,MM
S2      0002900.4     Y(I,J)=Y(I,J+1)/A
S2      0003000.      Y(I,M)=1.D0/A
S2      0003100.      DO 6 L=1,M
S2      0003200.      IF (L.EQ.I) GO TO 6
S2      0003300.      A=Y(L,1)
S2      0003400.      DO 5 N=1,MM
S2      0003500.5     Y(L,N)=Y(L,N+1)-A*Y(I,N)
S2      0003600.      Y(L,M)=-A*Y(I,M)
S2      0003700.6     CONTINUE
S2      0003800.      DO 10 I=1,M
S2      0003900.      IF (K(I).EQ.I) GO TO 10
S2      0004000.      DO 7 J=I,M
S2      0004100.      IF (K(J).EQ.I) GO TO 8
S2      0004200.7     CONTINUE
S2      0004300.      GO TO 19
S2      0004400.8     DO 9 L=1,M
S2      0004500.      A=Y(L,I)
S2      0004600.      Y(L,I)=Y(L,J)
S2      0004700.9     Y(L,J)=A
S2      0004800.      K(J)=K(I)
S2      0004900.10    CONTINUE
S2      0005000.      DO 15 N=1,LOOPS
S2      0005100.      TEST=0.D0
S2      0005200.      DO 12 J=1,M
S2      0005300.      DO 12 I=1,M
S2      0005400.      R=0.D0
S2      0005500.      DO 11 L=1,M
S2      0005600.11    R=R-X(I,L)*Y(L,J)
S2      0005700.      IF (I.EQ.J) R=R+1.0D0
S2      0005800.      TEST=DMAX1(TEST,DABS(R))
S2      0005900.12    W(I,J)=R
S2      0006000.      IF (TEST.LE.DELT) GO TO 16
S2      0006100.      DO 14 J=1,M
S2      0006200.      DO 14 I=1,M
S2      0006300.      A=0.D0
S2      0006400.      DO 13 L=1,M
S2      0006500.13    A=A+Y(I,L)*W(L,J)
S2      0006600.14    Y(I,J)=Y(I,J)+A
S2      0006700.15    CONTINUE
S2      0006800.      KSIG=3
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```
S2    0006900.        PRINT 21, TEST
S2    0007000.16      DO 17 J=1,M
S2    0007100.        DO 17 I=1,M
S2    0007200.17      X(I,J)=Y(I,J)
S2    0007300.        GO TO 20
S2    0007400.18      KSIG=1
S2    0007500.        GO TO 20
S2    0007600.19      KSIG=2
S2    0007700.20      IF (KSIG.NE.0) PRINT 22, KSIG
S2    0007800.        RETURN
S2    0007900.C
S2    0008000.21      FORMAT (' ERROR = ',D14.6)
S2    0008100.22      FORMAT (' ERROR INDICATOR NUMBER ',I1)
S2    0008200.        END
S3    0000100.        SUBROUTINE FCT(X)
S3    0000200.        IMPLICIT REAL*8 (A-H,O-Z)
S3    0000300.        COMMON /RUNGE/ O3(13),O4(13),O7(13),O8(13)
S3    0000400.        COMMON /EIGEN/ XL(13)
S3    0000500.        DATA NR /13/
S3    0001600.     40 DO 45 I=1,NR
S3    0001700.        O3(I)=X*O8(I)
S3    0001800.     45 O4(I)=XL(I)*O7(I)/X
S3    0001900.        RETURN
S3    0002000.        END
S4    0000100.        SUBROUTINE MAPOWR
S4    0000200.        IMPLICIT REAL*8 (A-H,O-Z)
S4    0000300.        DOUBLE PRECISION K,MAX
S4    0000400.        DIMENSION XK(12,12), YK(12,12), W(12,12), T(12,12), DXX(12,12), DYK(12,12,35)
S4    0000500.        COMMON /SERIES/ D1(12,12),D2(12,12),D3(12,12)
S4    0000600.        DATA NZ,K,MSER /12,3.6D1,35/
S4    0000700.        NZM1=NZ-1
S4    0000800.        DO 1 M=1,NZ
S4    0000900.        DO 1 L=1,NZ
S4    0001000.1       XK(L,M)=0.D0
S4    0001100.        XK(1,1)=2.0D0*K
S4    0001200.        XK(1,2)=-2.0D0*K
S4    0001300.        XK(NZ,NZM1)=-2.0D0*K
S4    0001400.        XK(NZ,NZ)=2.0D0*K
S4    0001500.        DO 2 L=2,NZM1
S4    0001600.        Y1=2*L
S4    0001700.        XK(L,L-1)=-K*(Y1-3.0D0)/(Y1-2.0D0)
S4    0001800.        XK(L,L)=2.0D0*K
S4    0001900.      2 XK(L,L+1)=-K*(Y1-1.0D0)/(Y1-2.0D0)
S4    0002000.        DO 4 M=1,NZ
S4    0002100.        DO 3 L=1,NZ
S4    0002200.3       YK(L,M)=0.D0
S4    0002300.4       YK(M,M)=1.0D0
S4    0002400.        DO 5 N=1,NZ
S4    0002500.        DO 5 M=1,NZ
S4    0002600.        T(M,N)=XK(M,N)
S4    0002700.5       XK(M,N)=XK(M,N)/T(1,1)
S4    0002800.        DO 9 J=1,MSER
S4    0002900.        DO 6 N=1,NZ
S4    0003000.        DO 6 M=1,NZ
```

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S4       0003100.       W(M,N)=0.D0
S4       0003200.       DO 6 L=1,NZ
S4       0003300.6      W(M,N)=W(M,N)+YK(M,L)*XK(L,N)
S4       0003400.       DO 7 N=1,NZ
S4       0003500.       DO 7 M=1,NZ
S4       0003600.7      YK(M,N)=W(M,N)
S4       0003700.       DO 8 N=1,NZ
S4       0003800.       DO 8 M=1,NZ
S4       0003900.8      DYK(M,N,J)=YK(M,N)
S4       0004000.9      CONTINUE
S4       0004100.       RETURN
S4       0004200.       ENTRY MASER(X)
S4       0004300.       DO 11 N=1,NZ
S4       0004400.       DO 10 M=1,NZ
S4       0004500.       D1(M,N)=0.D0
S4       0004600.10     D2(M,N)=0.D0
S4       0004700.       D1(N,N)=1.0D0
S4       0004800.11     D2(N,N)=X
S4       0004900.       C4=1.0D0
S4       0005000.       C5=X
S4       0005100.       DO 13 J=1,MSER
S4       0005200.       Y2=2*J*(2*J-1)
S4       0005300.       Y3=2*J*(2*J+1)
S4       0005400.       C4=C4*T(1,1)*X*X/Y2
S4       0005500.       C5=C5*T(1,1)*X*X/Y3
S4       0005600.       IFLAG=0
S4       0005700.       DO 12 N=1,NZ
S4       0005800.       DO 12 M=1,NZ
S4       0005900.       TEMP=C4*DYK(M,N,J)
S4       0006000.       IF (TEMP.GT.1.0D-24) IFLAG=1
S4       0006100.       D1(M,N)=D1(M,N)+TEMP
S4       0006200.12     D2(M,N)=D2(M,N)+C5*DYK(M,N,J)
S4       0006300.       IF (IFLAG.EQ.0) GO TO 14
S4       0006400.13     CONTINUE
S4       0006500.       PRINT 19
S4       0006600.14     PRINT 20, X,J
S4       0006700.       ENTRY MXTRA
S4       0006800.       DO 15 N=1,NZ
S4       0006900.       DO 15 M=1,NZ
S4       0007000.       D3(M,N)=0.D0
S4       0007100.       DO 15 L=1,NZ
S4       0007200.15     D3(M,N)=D3(M,N)+T(M,L)*D2(L,N)
S4       0007300.       DO 17 N=1,NZ
S4       0007400.       DO 16 M=1,NZ
S4       0007500.       DXX(M,N)=0.D0
S4       0007600.       DO 16 L=1,NZ
S4       0007700.16     DXX(M,N)=DXX(M,N)+D1(M,L)*D1(L,N)-D3(M,L)*D2(L,N)
S4       0007800.17     DXX(N,N)=DXX(N,N)-1.0D0
S4       0007900.       MAX=0.D0
S4       0008000.       DO 18 N=1,NZ
S4       0008100.       DO 18 M=1,NZ
S4       0008200.18     MAX=DMAX1(DXX(M,N),MAX)
S4       0008300.       IF (MAX.GT.1.0D-9) PRINT 21, X,MAX
S4       0008400.       RETURN
```

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S4      0008500.C
S4      0008600.19     FORMAT (' WARNING -- NOT ENOUGH TERMS TAKEN ')
S4      0008700.20     FORMAT (' FOR ',F8.4,' TERMS TAKEN = ',I2)
S4      0008800.21     FORMAT (' FOR ',F8.4,' ERROR = ',F15.9)
S4      0008900.       END
S5      0000100.       IMPLICIT REAL*8 (A-H,O-Z)
S5      0000200.       COMMON /EIGEN/ XL(13)
S5      0000300.       COMMON /SERIES/ D1(12,12),D2(12,12),D3(12,12)
S5      0000325.       DIMENSION OV1(13),OV2(13),OCA(13,13)
S5      0000335.       DIMENSION O1(13,13),O2(13,13),O3(13,13),O4(13,13),O7(13,13)
S5      0000350.       DIMENSION P(13,13),PIV(13,13),PO1(13),PO2(13),PO3(13),PO4(13),PT1(13,13),PT2(13,13),PT3(13,13),PT4(13,13)
S5      0000400.       DIMENSION D14(12,12,13),WDOT(12,13),UDOT(13,12),O14(13,13,12),O15(13,13,12),-
S5      0000500.      1 D4(12,12),D5(12,12),W(13,13),O5(13,13),O6(13,13),R(13,12),F1(13),F2(13),-
S5      0000600.      2 F4(12),UX(13,12),B1(13,12),F7(12),B2(13,12),B3(12,13),B4(12,13),D10(12,12,13),-
S5      0000700.      1 D11(12,12,13),F6(13),O10(13,13,12),O11(13,13,12),F5(13),WX(12,13),UY(13,12),WY(12,13),-
S5      0000800.      4 T(12,13),SIGR(13,12),SIGT(13,12),SIGZ(13,12),G(13,12),DX2(8,4),DX3(8,8),DX4(8,4),-
S5      0000900.      5 DX6(4,8),DX7(4,4),DX8(4,8),DW2(8,4),F3(12),WAL(4),WBE(8),DA(8,8),DB(8,8),-
S5      0001000.      6 DC(8,4),D6(12,12),DX1(4,4),F8(12)
S5      0001300.       DATA C1,C2,C3,C3IV,XLAM/5.0D-1,5.0D-1,6.666666666666667D-1,1.5D0,7.5D-1/
S5      0001350.       DATA C5 /0.D0/
S5      0001400.       DATA NR,NZ,NIC,HR,HZ/13,12,4,8.333333333333333D-2,8.333333333333333D-2/
S5      0001450.       DATA XKON,PI,IF1,IF2 /3.6D1,3.141592653589793D0,1,2/
S5      0001470.       C4=C2/(HR*HR)
S5      0001490.       C4C1=C4*C1
S5      0001500.       NZM1=NZ-1
S5      0001600.       NRM1=NR-1
S5      0001700.       NOC=NZ-NIC
S5      0001800.       X1=7.5D-1/HR
S5      0001900.       X2=X1/2.0D0
S5      0002000.       X3=X2/2.0D0
S5      0002100.       X4=7.5D-1/HZ
S5      0002200.       X5=X4/2.0D0
S5      0002300.       X6=X5/2.0D0
S5      0002400.       X7=1.0D0/(HR*HZ)
S5      0002500.       X8=X7/2.0D0
S5      0002600.       X9=X8/2.0D0
S5      0002700.       X10=3.0D0*X9
S5      0002800.       X11=X10/2.0D0
S5      0002900.       X12=X8-5.0D-1/HZ
S5      0003000.       X13=X12/2.0D0
S5      0003100.       X14=1.25D0/HZ
S5      0003200.       X15=X14/2.0D0
S5      0003203.       YP2=2.0D0*XKON
S5      0003206.       YP4=DFLOAT(NRM1)
S5      0003209.       YP1=PI/YP4
S5      0003212.       DO 201 I=1,NR
S5      0003215.   201 XL(I)=YP2*(1.0D0-DCOS(YP1*DFLOAT(I-1)))
S5      0003218.       DO 202 J=1,NR
S5      0003221.       DO 202 I=1,NR
S5      0003224.       YP3=YP1*DFLOAT((I-1)*(J-1))
S5      0003227.       P(I,J)=DCOS(YP3)
S5      0003230.   202 PIV(I,J)=P(I,J)/YP4
S5      0003233.       PIV(1,1)=PIV(1,1)/2.0D0
S5      0003236.       PIV(1,NR)=PIV(1,NR)/2.0D0
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S5      0003239.       PIV(NR,1)=PIV(NR,1)/2.0D0
S5      0003242.       PIV(NR,NR)=PIV(NR,NR)/2.0D0
S5      0003245.       DO 203 J=2,NRM1
S5      0003248.       DO 203 I=2,NRM1
S5      0003251.   203 PIV(I,J)=PIV(I,J)*2.0D0
S5      0003254.       DO 204 I=1,NR
S5      0003257.       PO1(I)=1.0D0
S5      0003260.       PO2(I)=0.D0
S5      0003263.       PO3(I)=0.D0
S5      0003266.   204 PO4(I)=1.0D0
S5      0003300.       X=HR
S5      0003400.       XX=HR
S5      0003500.       DO 2 J=1,NR
S5      0003600.       DO 1 I=1,NR
S5      0004000.       O10(I,J,1)=0.D0
S5      0004100.       O11(I,J,1)=0.D0
S5      0004200.       O14(I,J,1)=0.D0
S5      0004500.     1 O15(I,J,1)=0.D0
S5      0004600.       O10(J,J,1)=1.0D0
S5      0004800.     2 O15(J,J,1)=1.0D0
S5      0004900.       DO 11 II=2,NZ
S5      0005000.       XF=X+HR
S5      0005100.       XXF=XX+HR
S5      0005200.       CALL RKGS(PO1,PO2,X)
S5      0005300.       CALL RKGS(PO3,PO4,XX)
S5      0005400.       X=XF
S5      0005500.       XX=XXF
S5      0005600.       DO 205 I=1,NR
S5      0005610.       DO 205 J=1,NR
S5      0005620.       PT1(J,I)=P(J,I)*PO1(I)
S5      0005630.       PT2(J,I)=P(J,I)*PO2(I)
S5      0005640.       PT3(J,I)=P(J,I)*PO3(I)
S5      0005650.   205 PT4(J,I)=P(J,I)*PO4(I)
S5      0005660.       DO 206 J=1,NR
S5      0005670.       DO 206 I=1,NR
S5      0005680.       SUM1=0.D0
S5      0005690.       SUM2=0.D0
S5      0005700.       SUM3=0.D0
S5      0005710.       SUM4=0.D0
S5      0005720.       DO 207 L=1,NR
S5      0005730.       SUM1=SUM1+PT1(I,L)*PIV(L,J)
S5      0005740.       SUM2=SUM2+PT2(I,L)*PIV(L,J)
S5      0005750.       SUM3=SUM3+PT3(I,L)*PIV(L,J)
S5      0005760.   207 SUM4=SUM4+PT4(I,L)*PIV(L,J)
S5      0005770.       O10(I,J,II)=SUM1
S5      0005780.       O14(I,J,II)=SUM2
S5      0005790.       O11(I,J,II)=SUM3
S5      0005800.   206 O15(I,J,II)=SUM4
S5      0006500.11     CONTINUE
S5      0006600.       DO 12 N=1,NR
S5      0006700.       DO 12 M=1,NR
S5      0006800.       O5(M,N)=C2*O11(M,N,NZ)-O15(M,N,NZ)
S5      0006900.    12 O6(M,N)=C2*O10(M,N,NZ)-O14(M,N,NZ)
S5      0007000.       CALL MATINV (O6,NR)
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S5      0007020.       DO 15 N=1,NR
S5      0007040.       DO 14 M=1,NR
S5      0007060.       W(M,N)=0.D0
S5      0007080.       DO 13 J=1,NR
S5      0007100.13     W(M,N)=W(M,N)+O6(M,J)*O5(J,N)
S5      0007120.14     W(M,N)=C4*W(M,N)
S5      0007140.15     W(N,N)=W(N,N)+1.0D0
S5      0007160.       CALL MATINV (W,NR)
S5      0007180.       DO 17 N=1,NR
S5      0007200.       DO 17 M=1,NR
S5      0007220.       OCA(M,N)=0.D0
S5      0007240.       DO 16 J=1,NR
S5      0007260.16     OCA(M,N)=OCA(M,N)+W(M,J)*O6(J,N)
S5      0007280.17     O2(M,N)=C1*OCA(M,N)
S5      0007300.       DO 20 N=1,NR
S5      0007320.       DO 19 M=1,NR
S5      0007340.       O3(M,N)=0.D0
S5      0007360.       DO 18 J=1,NR
S5      0007380.18     O3(M,N)=O3(M,N)+OCA(M,J)*O5(J,N)
S5      0007400.19     O4(M,N)=C4C1*O3(M,N)
S5      0007420.20     O4(N,N)=O4(N,N)-C1
S5      0007440.       DO 21 N=1,NR
S5      0007460.       DO 21 M=1,NR
S5      0007480.       O1(M,N)=C1*O3(M,N)
S5      0007500.       O5(M,N)=C4*O2(M,N)
S5      0007520.       O7(M,N)=C4*W(M,N)
S5      0007540.21     O6(M,N)=C4*O3(M,N)
S5      0007560.       DO 200 N=1,NR
S5      0007580.   200 OV1(N)=0.D0
S5      0007600.       DO 601 N=1,NR
S5      0007620.       DO 601 M=1,NR
S5      0007640.   601 OV1(M)=OV1(M)+OCA(M,N)*C5
S5      0007660.       DO 602 M=1,NR
S5      0007680.   602 OV2(M)=C2*OV1(M)/HR**2
S5      0007900.       CALL MAPOWR
S5      0008000.       DO 23 L=1,NZ
S5      0008100.       DO 22 J=1,NZ
S5      0008200.       D10(J,L,1)=0.D0
S5      0008300.       D11(J,L,1)=0.D0
S5      0008400. 22    D14(J,L,1)=0.D0
S5      0008500. 23    D10(L,L,1)=1.0D0
S5      0008600.       DO 32 I=1,NRM1
S5      0008700.       X=HZ*DFLOAT(I)
S5      0008800.       IF(I.LE.11) GO TO 101
S5      0008900.       DO 103 N=1,NZ
S5      0009000.       DO 103 M=1,NZ
S5      0009100.       D1(M,N)=0.D0
S5      0009200.       D2(M,N)=0.D0
S5      0009300.       DO 103 L=1,NZ
S5      0009400.       D1(M,N)=D1(M,N)+D10(M,L,12)*D10(L,N,I-10)+D14(M,L,12)*D11(L,N,I-10)
S5      0009500.   103 D2(M,N)=D2(M,N)+D11(M,L,12)*D10(L,N,I-10)+D10(M,L,12)*D11(L,N,I-10)
S5      0009600.       CALL MXTRA
S5      0009700.       GO TO 102
S5      0009800.   101 CALL MASER(X)
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S5      0009900.  102 DO 24 N=1,NZ
S5      0010000.      DO 24 M=1,NZ
S5      0010100.      D10(M,N,I+1)=D1(M,N)
S5      0010200.      D14(M,N,I+1)=D3(M,N)
S5      0010300.24    D11(M,N,I+1)=D2(M,N)
S5      0010400.32    CONTINUE
S5      0010500.      DO 33 N=1,NIC
S5      0010550.      DO 33 M=1,NIC
S5      0010600.33    DX1(M,N)=D3(M,N)
S5      0010650.      CALL MATINV (DX1,NIC)
S5      0010700.      DO 34 N=1,NOC
S5      0010750.      NIX=N+NIC
S5      0010800.      DO 34 M=1,NIC
S5      0010850.      DX6(M,N)=0.D0
S5      0010900.      DX8(M,N)=0.D0
S5      0010950.      DO 34 J=1,NIC
S5      0011000.      DX8(M,N)=DX8(M,N)+DX1(M,J)*D1(J,NIX)
S5      0011050.34    DX6(M,N)=DX6(M,N)+DX1(M,J)*D3(J,NIX)
S5      0011100.      DO 35 N=1,NIC
S5      0011150.      DO 35 M=1,NIC
S5      0011200.      DX7(M,N)=0.D0
S5      0011250.      DO 35 J=1,NIC
S5      0011300.35    DX7(M,N)=DX7(M,N)+DX1(M,J)*D1(J,N)
S5      0011350.      DO 37 N=1,NOC
S5      0011400.      NIX=N+NIC
S5      0011450.      DO 37 M=1,NOC
S5      0011500.      MIX=M+NIC
S5      0011550.      DA(M,N)=0.D0
S5      0011600.      DB(M,N)=0.D0
S5      0011650.      DO 36 J=1,NIC
S5      0011700.      DA(M,N)=DA(M,N)+D3(MIX,J)*DX8(J,N)
S5      0011750.36    DB(M,N)=DB(M,N)+D3(MIX,J)*DX6(J,N)
S5      0011800.      DA(M,N)=D1(MIX,NIX)-DA(M,N)
S5      0011850.37    DB(M,N)=D3(MIX,NIX)-DB(M,N)
S5      0011900.      DO 39 N=1,NIC
S5      0011950.      DO 39 M=1,NOC
S5      0012000.      MIX=M+NIC
S5      0012050.      DC(M,N)=0.D0
S5      0012100.      DW2(M,N)=0.D0
S5      0012150.      DO 38 J=1,NIC
S5      0012200.      DW2(M,N)=DW2(M,N)+D3(MIX,J)*DX1(J,N)
S5      0012250.38    DC(M,N)=DC(M,N)+D3(MIX,J)*DX7(J,N)
S5      0012300.39    DC(M,N)=D1(MIX,N)-DC(M,N)
S5      0012350.      CALL MATINV (DA,NOC)
S5      0012400.      DO 40 N=1,NOC
S5      0012450.      DO 40 M=1,NOC
S5      0012500.      DX3(M,N)=0.D0
S5      0012550.      DO 40 J=1,NOC
S5      0012600.40    DX3(M,N)=DX3(M,N)+DA(M,J)*DB(J,N)
S5      0012650.      DO 41 N=1,NIC
S5      0012700.      DO 41 M=1,NOC
S5      0012750.      DX2(M,N)=0.D0
S5      0012800.      DX4(M,N)=0.D0
S5      0012850.      DO 41 J=1,NOC
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S5      0012900.        DX2(M,N)=DX2(M,N)+DA(M,J)*DW2(J,N)
S5      0012950.41      DX4(M,N)=DX4(M,N)+DA(M,J)*DC(J,N)
S5      0013000.        KTR=0
S5      0013050.        DO 42 M=1,NR
S5      0013100.        DO 42 N=1,NZ
S5      0013150.42      WDOT(N,M)=0.DO
S5      0013200.        DO 43 N=1,NZ
S5      0013250.        DO 43 M=1,NR
S5      0013300.43      R(M,N)=0.DO
S5      0013350.44      KTR=KTR+1
S5      0013400.        DO 45 I=1,NR
S5      0013450.        DO 45 J=1,NZ
S5      0013500.45      WY(J,I)=WX(J,I)
S5      0013550.        DO 46 J=1,NZ
S5      0013600.        DO 46 I=1,NR
S5      0013650.46      UY(I,J)=UX(I,J)
S5      0013700.        DO 47 I=1,NZ
S5      0013750.        DO 47 J=1,NR
S5      0013800.        UX(J,I)=0.DO
S5      0013850.        UDOT(J,I)=0.DO
S5      0013900.        B1(J,I)=0.DO
S5      0013950.47      B2(J,I)=0.DO
S5      0014000.        DO 48 I=1,NR
S5      0014050.        F1(I)=0.DO
S5      0014100.48      F2(I)=0.DO
S5      0017800.        DO 50 M=2,NZ
S5      0017900.        DO 50 J=1,NR
S5      0018000.        SUM1=0.DO
S5      0018100.        SUM2=0.DO
S5      0018200.        DO 49 L=1,NR
S5      0018300.        SUM1=SUM1-O11(J,L,M-1)*R(L,M-1)-O11(J,L,M)*R(L,M)
S5      0018400.49      SUM2=SUM2+O10(J,L,M-1)*R(L,M-1)+O10(J,L,M)*R(L,M)
S5      0018500.        B1(J,M)=B1(J,M-1)+HR*SUM1/2.0DO
S5      0018600.50      B2(J,M)=B2(J,M-1)+HR*SUM2/2.0DO
S5      0018700.        DO 51 L=1,NR
S5      0018750.        DO 51 J=1,NR
S5      0018800.        F1(J)=F1(J)+O2(J,L)*WDOT(NZ,L)+O1(J,L)*WDOT(1,L)-W(J,L)*B1(L,NZ)-O3(J,L)*B2(L,NZ)
S5      0018850.51      F2(J)=F2(J)+O4(J,L)*WDOT(1,L)+O5(J,L)*WDOT(NZ,L)-O7(J,L)*B1(L,NZ)-O6(J,L)*B2(L,NZ)
S5      0018900.        DO 603 L=1,NR
S5      0018950.        F1(L)=F1(L)-OV1(L)
S5      0019000.    603 F2(L)=F2(L)-OV2(L)
S5      0019050.        X=HR
S5      0019100.        XX=X*X
S5      0019150.        DO 54 I=1,NZ
S5      0019200.        DO 52 J=1,NR
S5      0019250.        F5(J)=F1(J)+B1(J,I)
S5      0019300.52      F6(J)=F2(J)+B2(J,I)
S5      0019350.        DO 53 N=1,NR
S5      0019400.        DO 53 M=1,NR
S5      0019450.        UDOT(M,I)=UDOT(M,I)+(O14(M,N,I)-O10(M,N,I)/XX)*F5(N)+(O15(M,N,I)-O11(M,N,I)/XX)*F6(N)
S5      0019500.53      UX(M,I)=UX(M,I)+(O10(M,N,I)*F5(N)+O11(M,N,I)*F6(N))/X
S5      0019550.        X=X+HR
S5      0019600.54      XX=X*X
S5      0019650.        WRITE (6,87)
```

```
S5   0019700.      DO 55 KA=1,NR
S5   0019750.55    WRITE (6,93) (UX(KA,K),K=1,NZ)
S5   0019800.      WRITE (6,88)
S5   0019850.      DO 56 KA=1,NR
S5   0019900.56    WRITE (6,93) (UDOT(KA,K),K=1,NZ)
S5   0019950.      T(1,1)=X4*(UDOT(1,1)-UDOT(2,1))+X7*(UX(1,1)-UX(2,1))
S5   0020000.      DO 57 I=2,NZ
S5   0020050.      T(I,1)=0.D0
S5   0020100.57    T(I,NR)=X4*(UDOT(NRM1,I)-UDOT(NR,I))+X10*(UX(NRM1,I)-UX(NR,I))/DFLOAT(I)
S5   0020150.      DO 58 I=2,NIC
S5   0020200.58    T(I,1)=X4*(UDOT(1,I)-UDOT(2,I))+X10*(UX(1,I)-UX(2,I))/DFLOAT(I)
S5   0020250.      T(1,NR)=X4*(UDOT(NRM1,1)-UDOT(NR,1))+X7*(UX(NRM1,1)-UX(NR,1))
S5   0020300.      T(NZ,NR)=X4*(UDOT(NRM1,NZ)-UDOT(NR,NZ))+X12*(UX(NR,NZ)-UX(NRM1,NZ))
S5   0020350.      DO 59 J=2,NRM1
S5   0020400.      T(1,J)=X5*(UDOT(J-1,1)-UDOT(J+1,1))+X8*(UX(J-1,1)-UX(J+1,1))
S5   0020450.      T(NZ,J)=X5*(UDOT(J-1,NZ)-UDOT(J+1,NZ))+X13*(UX(J+1,NZ)-UX(J-1,NZ))
S5   0020500.      DO 59 I=2,NZM1
S5   0020550.59    T(I,J)=X5*(UDOT(J-1,I)-UDOT(J+1,I))+X11*(UX(J-1,I)-UX(J+1,I))/DFLOAT(I)
S5   0020600.      DO 60 I=1,NR
S5   0020650.      DO 60 J=1,NZ
S5   0020700.      WX(J,I)=0.D0
S5   0020750.      WDOT(J,I)=0.D0
S5   0020800.      B3(J,I)=0.D0
S5   0020850.60    B4(J,I)=0.D0
S5   0020900.      DO 61 I=1,NZ
S5   0020950.      F3(I)=0.D0
S5   0021000.61    F4(I)=0.D0
S5   0023500.      DO 63 M=2,NR
S5   0023600.      DO 63 J=1,NZ
S5   0023700.      SUM1=0.D0
S5   0023800.      SUM2=0.D0
S5   0023900.      DO 62 L=1,NZ
S5   0024000.      SUM1=SUM1-D11(J,L,M-1)*T(L,M-1)-D11(J,L,M)*T(L,M)
S5   0024100.62    SUM2=SUM2+D10(J,L,M-1)*T(L,M-1)+D10(J,L,M)*T(L,M)
S5   0024200.      B3(J,M)=B3(J,M-1)+HZ*SUM1/2.0D0
S5   0024300.63    B4(J,M)=B4(J,M-1)+HZ*SUM2/2.0D0
S5   0024400.      DO 64 I=1,NIC
S5   0024500.      WAL(I)=C3-C1*(UDOT(NR,I)+UX(NR,I)/(HR*DFLOAT(I)))
S5   0024600.64    F4(I)=-C1*(UDOT(1,I)+UX(1,I)/(HR*DFLOAT(I)))
S5   0024700.      DO 65 I=1,NOC
S5   0024800.      NIX=I+NIC
S5   0024900.65    WBE(I)=C3-C1*(UDOT(NR,NIX)+UX(NR,NIX)/(HR*DFLOAT(NIX)))
S5   0025000.      DO 68 M=1,NOC
S5   0025100.      MIX=M+NIC
S5   0025200.      SUM1=0.D0
S5   0025300.      SUM2=0.D0
S5   0025400.      DO 66 N=1,NOC
S5   0025500.      NIX=N+NIC
S5   0025600.66    SUM1=SUM1+DA(M,N)*WBE(N)-DX3(M,N)*B3(NIX,NR)
S5   0025700.      DO 67 N=1,NIC
S5   0025800.      NIX=N+NIC
S5   0025900.67    SUM2=SUM2+DX2(M,N)*WAL(N)+DX4(M,N)*(F4(N)+B4(N,NR))
S5   0026000.68    F4(MIX)=SUM1-SUM2-B4(MIX,NR)
S5   0026100.      DO 71 M=1,NIC
```

```
      SUM1=0.D0
      SUM2=0.D0
      DO 69 N=1,NIC
69    SUM1=SUM1+DX1(M,N)*WAL(N)-DX7(M,N)*(F4(N)+B4(N,NR))
      DO 70 N=1,NOC
      NIX=N+NIC
70    SUM2=SUM2+DX6(M,N)*B3(NIX,NR)+DX8(M,N)*(F4(NIX)+B4(NIX,NR))
71    F3(M)=SUM1-SUM2-B3(M,NR)
      DO 73 I=1,NR
      DO 72 J=1,NZ
      F7(J)=F3(J)+B3(J,I)
72    F8(J)=F4(J)+B4(J,I)
      DO 73 N=1,NZ
      DO 73 M=1,NZ
      WDOT(M,I)=WDOT(M,I)+D10(M,N,I)*F8(N)+D14(M,N,I)*F7(N)
73    WX(M,I)=WX(M,I)+D11(M,N,I)*F8(N)+D10(M,N,I)*F7(N)
      WRITE (6,89)
      DO 74 KA=1,NZ
74    WRITE (6,93) (WX(KA,K),K=1,NR)
      WRITE (6,90)
      DO 75 KA=1,NZ
75    WRITE (6,93) (WDOT(KA,K),K=1,NR)
      R(1,1)=X1*(WDOT(1,1)-WDOT(2,1))+X8*(WX(1,1)-WX(2,1))
      R(1,NZ)=X1*(WDOT(NZM1,1)-WDOT(NZ,1))+X8*(WX(NZM1,1)-WX(NZ,1))
      R(NR,1)=X1*(WDOT(1,NR)-WDOT(2,NR))+X8*(WX(2,NR)-WX(1,NR))
      R(NR,NZ)=X1*(WDOT(NZM1,NR)-WDOT(NZ,NR))+X8*(WX(NZ,NR)-WX(NZM1,NR))
      DO 76 I=2,NRM1
      R(I,1)=X1*(WDOT(1,I)-WDOT(2,I))
76    R(I,NZ)=X1*(WDOT(NZM1,I)-WDOT(NZ,I))
      DO 77 J=2,NZM1
      R(1,J)=X2*(WDOT(J-1,1)-WDOT(J+1,1))+X9*(WX(J-1,1)-WX(J+1,1))
      R(NR,J)=X2*(WDOT(J-1,NR)-WDOT(J+1,NR))+X9*(WX(J+1,NR)-WX(J-1,NR))
      DO 77 I=2,NRM1
77    R(I,J)=X2*(WDOT(J-1,I)-WDOT(J+1,I))
      PRINT 91, KTR
      IF (DABS(WX(1,1)).GT.5.0D0) STOP
      DO 78 I=1,NR
      DO 78 J=1,NZ
      IF (DABS(WX(J,I)-WY(J,I)).GT.1.0D-6) GO TO 44
78    CONTINUE
      DO 79 J=1,NZ
      DO 79 I=1,NR
      IF (DABS(UX(I,J)-UY(I,J)).GT.1.0D-6) GO TO 44
79    CONTINUE
      WRITE (6,92) KTR
      DO 80 J=1,NZ
      YJ=HR*DFLOAT(J)
      DO 80 I=1,NR
      SIGR(I,J)=C3IV*UDOT(I,J)+XLAM*(UX(I,J)/YJ+WDOT(J,I))
      SIGT(I,J)=C3IV*UX(I,J)/YJ+XLAM*(UDOT(I,J)+WDOT(J,I))
80    SIGZ(I,J)=C3IV*WDOT(J,I)+XLAM*(UDOT(I,J)+UX(I,J)/YJ)
      G(1,1)=X2*(WX(2,1)-WX(1,1))+X5*(UX(2,1)-UX(1,1))
      G(1,NZ)=X2*(WX(NZ,1)-WX(NZM1,1))+X5*(UX(2,NZ)-UX(1,NZ))
      G(NR,1)=X2*(WX(2,NR)-WX(1,NR))+X5*(UX(NR,1)-UX(NRM1,1))
```

```
S5     0031600.      G(NR,NZ)=X2*(WX(NZ,NR)-WX(NZ-1,NR))+X5*(UX(NR,NZ)-UX(NRM1,NZ))
S5     0031700.      DO 81 J=2,NZM1
S5     0031800.      G(1,J)=X3*(WX(J+1,1)-WX(J-1,1))+X5*(UX(2,J)-UX(1,J))
S5     0031900.      G(NR,J)=X3*(WX(J+1,NR)-WX(J-1,NR))+X5*(UX(NR,J)-UX(NRM1,J))
S5     0032000.      DO 81 I=2,NRM1
S5     0032100.81    G(I,J)=X3*(WX(J+1,I)-WX(J-1,I))+X6*(UX(I+1,J)-UX(I-1,J))
S5     0032200.      DO 82 I=2,NRM1
S5     0032300.      G(I,1)=X2*(WX(2,I)-WX(1,I))+X6*(UX(I+1,1)-UX(I-1,1))
S5     0032400.82    G(I,NZ)=X2*(WX(NZ,I)-WX(NZM1,I))+X6*(UX(I+1,NZ)-UX(I-1,NZ))
S5     0032500.      WRITE (6,94)
S5     0032600.      DO 83 I=1,NR
S5     0032700.83    WRITE (6,93) (SIGR(I,J),J=1,NZ)
S5     0032800.      WRITE (6,95)
S5     0032900.      DO 84 I=1,NR
S5     0033000.84    WRITE (6,93) (SIGT(I,J),J=1,NZ)
S5     0033100.      WRITE (6,96)
S5     0033200.      DO 85 I=1,NR
S5     0033300.85    WRITE (6,93) (SIGZ(I,J),J=1,NZ)
S5     0033400.      WRITE (6,97)
S5     0033500.      DO 86 I=1,NR
S5     0033600.86    WRITE (6,93) (G(I,J),J=1,NZ)
S5     0033700.      STOP
S5     0033800.C
S5     0033900.87    FORMAT (///' U-MATRIX ')
S5     0034000.88    FORMAT (///' U-DOT MATRIX ')
S5     0034100.89    FORMAT (///' W-MATRIX ')
S5     0034200.90    FORMAT (///' W-DOT MATRIX ')
S5     0034300.91    FORMAT (' KTR = ',I2)
S5     0034400.92    FORMAT (///' ITERATION NUMBER ',I2,///)
S5     0034500.93    FORMAT (9(2X,F10.6))
S5     0034600.94    FORMAT (///' SIGMA-R MATRIX ')
S5     0034700.95    FORMAT (///' SIGMA-THETA MATRIX ')
S5     0034800.96    FORMAT (///' SIGMA-Z MATRIX ')
S5     0034900.97    FORMAT (///' SIGMA-RZ MATRIX ')
S5     0035000.      END
```

C.2 Hollow Cylindrical Bar With a Penny Shaped Crack

Since the solution of this problem is very similar to that discussed in Appendix C.1, the enclosed computer program also follows the schematic representation of Figure 35. The input and output data are handled identically to those for the solid cylindrical bar and only the initial value vectors that are derived from given boundary conditions are different. This program, in addition to the axial load C2, includes the possibility of having an outside surface load of C5.

```
S1      0000100.        SUBROUTINE RKGS (O1,O2,XO,IFL)
S1      0000200.        IMPLICIT REAL*8 (A-H,O-Z)
S1      0000300.        DATA NR,HR,JJ,XJ /13,8.3333333333333333D-2,64,6.4D1/
S1      0000400.        COMMON /RUNGE/ O3(13),O4(13),O7(13),O8(13)
S1      0000500.        DIMENSION O5(13),O6(13),O1(13),O2(13)
S1      0000600.        H=HR/XJ
S1      0000700.        HH=H/2.0D0
S1      0000800.        DO 6 I=1,JJ
S1      0001000.        DO 1 J=1,NR
S1      0001100.        O7(J)=O1(J)
S1      0001200.1       O8(J)=O2(J)
S1      0001300.        CALL FCT (XO,IFL)
S1      0001500.        DO 2 J=1,NR
S1      0001600.        O5(J)=O3(J)
S1      0001700.        O6(J)=O4(J)
S1      0001800.        O7(J)=O1(J)+HH*O3(J)
S1      0001900.2       O8(J)=O2(J)+HH*O4(J)
S1      0002000.        XO=XO+HH
S1      0002100.        CALL FCT (XO,IFL)
S1      0002300.        DO 3 J=1,NR
S1      0002400.        O7(J)=O1(J)+HH*O3(J)
S1      0002500.        O8(J)=O2(J)+HH*O4(J)
S1      0002600.        O5(J)=O5(J)+2.0D0*O3(J)
S1      0002700. 3      O6(J)=O6(J)+2.0D0*O4(J)
S1      0002800.        CALL FCT (XO,IFL)
S1      0003000.        DO 4 J=1,NR
S1      0003100.        O5(J)=O5(J)+2.0D0*O3(J)
S1      0003200.        O6(J)=O6(J)+2.0D0*O4(J)
S1      0003300.        O7(J)=O1(J)+H*O3(J)
S1      0003400. 4      O8(J)=O2(J)+H*O4(J)
S1      0003500.        XO=XO+HH
S1      0003600.        CALL FCT (XO,IFL)
S1      0003800.        DO 5 J=1,NR
S1      0003900.        O5(J)=O5(J)+O3(J)
S1      0004000.        O6(J)=O6(J)+O4(J)
S1      0004100.        O1(J)=O1(J)+H*O5(J)/6.0D0
S1      0004200.5       O2(J)=O2(J)+H*O6(J)/6.0D0
S1      0004300.6       CONTINUE
S1      0004400.        RETURN
S1      0004500.        END
S2      0000100.        SUBROUTINE MATINV (X,MMM)
S2      0000200.        DOUBLE PRECISION X,Y,W,A,TEST,R,DELT
S2      0000300.        DIMENSION X(MMM,MMM), Y(13,13), K(13), W(13,13)
S2      0000400.        DATA DELT,LOOPS/1.0D-8,2/
S2      0000500.        KSIG=0
S2      0000600.        M=MMM
S2      0000700.        MM=M-1
S2      0000800.        DO 1 J=1,M
S2      0000900.        K(J)=J
S2      0001000.        DO 1 I=1,M
S2      0001100.1       Y(I,J)=X(I,J)
S2      0001200.        DO 6 I=1,M
S2      0001300.        A=0.D0
S2      0001400.        DO 2 J=I,M
```

```
S2    0001500.      IF (DABS(Y(J,1)).LE.A) GO TO 2
S2    0001600.      A=DABS(Y(J,1))
S2    0001700.      L=J
S2    0001800.2     CONTINUE
S2    0001900.      IF (A.EQ.0.DO) GO TO 18
S2    0002000.      N=K(L)
S2    0002100.      K(L)=K(I)
S2    0002200.      K(I)=N
S2    0002300.      DO 3 J=1,M
S2    0002400.      A=Y(I,J)
S2    0002500.      Y(I,J)=Y(L,J)
S2    0002600.3     Y(L,J)=A
S2    0002700.      A=Y(I,1)
S2    0002800.      DO 4 J=1,MM
S2    0002900.4     Y(I,J)=Y(I,J+1)/A
S2    0003000.      Y(I,M)=1.DO/A
S2    0003100.      DO 6 L=1,M
S2    0003200.      IF (L.EQ.I) GO TO 6
S2    0003300.      A=Y(L,1)
S2    0003400.      DO 5 N=1,MM
S2    0003500.5     Y(L,N)=Y(L,N+1)-A*Y(I,N)
S2    0003600.      Y(L,M)=-A*Y(I,M)
S2    0003700.6     CONTINUE
S2    0003800.      DO 10 I=1,M
S2    0003900.      IF (K(I).EQ.I) GO TO 10
S2    0004000.      DO 7 J=I,M
S2    0004100.      IF (K(J).EQ.I) GO TO 8
S2    0004200.7     CONTINUE
S2    0004300.      GO TO 19
S2    0004400.8     DO 9 L=1,M
S2    0004500.      A=Y(L,I)
S2    0004600.      Y(L,I)=Y(L,J)
S2    0004700.9     Y(L,J)=A
S2    0004800.      K(J)=K(I)
S2    0004900.10    CONTINUE
S2    0005000.      DO 15 N=1,LOOPS
S2    0005100.      TEST=0.DO
S2    0005200.      DO 12 J=1,M
S2    0005300.      DO 12 I=1,M
S2    0005400.      R=0.DO
S2    0005500.      DO 11 L=1,M
S2    0005600.11    R=R-X(I,L)*Y(L,J)
S2    0005700.      IF (I.EQ.J) R=R+1.ODO
S2    0005800.      TEST=DMAX1(TEST,DABS(R))
S2    0005900.12    W(I,J)=R
S2    0006000.      IF (TEST.LE.DELT) GO TO 16
S2    0006100.      DO 14 J=1,M
S2    0006200.      DO 14 I=1,M
S2    0006300.      A=0.DO
S2    0006400.      DO 13 L=1,M
S2    0006500.13    A=A+Y(I,L)*W(L,J)
S2    0006600.14    Y(I,J)=Y(I,J)+A
S2    0006700.15    CONTINUE
S2    0006800.      KSIG=3
```

```
S2      0006900.        PRINT 21, TEST
S2      0007000.16      DO 17 J=1,M
S2      0007100.        DO 17 I=1,M
S2      0007200.17      X(I,J)=Y(I,J)
S2      0007300.        GO TO 20
S2      0007400.18      KSIG=1
S2      0007500.        GO TO 20
S2      0007600.19      KSIG=2
S2      0007700.20      IF (KSIG.NE.0) PRINT 22, KSIG
S2      0007800.        RETURN
S2      0007900.C
S2      0008000.21      FORMAT (' ERROR = ',D14.6)
S2      0008100.22      FORMAT (' ERROR INDICATOR NUMBER ',I1)
S2      0008200.        END
S3      0000100.        SUBROUTINE FCT(X,IFL)
S3      0000200.        IMPLICIT REAL*8 (A-H,O-Z)
S3      0000300.        COMMON /RUNGE/ O3(13),O4(13),O7(13),O8(13)
S3      0000400.        COMMON /EIGEN/ XL(13)
S3      0000500.        DATA NR /13/
S3      0000700.        IF(IFL.EQ.2) GO TO 20
S3      0000750.        IF(X.NE.0.D0) GO TO 50
S3      0000800.     10 DO 15 I=1,NR
S3      0000900.        O3(I)=0.D0
S3      0001000.     15 O4(I)=XL(I)
S3      0001100.        RETURN
S3      0001120.     50 DO 55 I=1,NR
S3      0001140.        O3(I)=O8(I)
S3      0001160.     55 O4(I)=XL(I)*O7(I)+O8(I)/X
S3      0001180.        RETURN
S3      0001200.     20 IF(X.NE.0.D0) GO TO 40
S3      0001250.        DO 25 I=1,NR
S3      0001300.        O3(I)=0.D0
S3      0001400.     25 O4(I)=0.D0
S3      0001500.        RETURN
S3      0001600.     40 DO 45 I=1,NR
S3      0001700.        O3(I)=X*O8(I)
S3      0001800.     45 O4(I)=XL(I)*O7(I)/X
S3      0001900.        RETURN
S3      0002000.        END
S4      0000100.        SUBROUTINE MAPOWR
S4      0000200.        IMPLICIT REAL*8 (A-H,O-Z)
S4      0000300.        DOUBLE PRECISION K,MAX
S4      0000400.        DIMENSION XK(13,13), YK(13,13), W(13,13), T(13,13), DXX(13,13), DYK(13,13,35)
S4      0000500.        COMMON /SERIES/ D1(13,13),D2(13,13),D3(13,13)
S4      0000600.        DATA NZ,K,MSER /13,3.6D1,35/
S4      0000700.        NZM1=NZ-1
S4      0000800.        DO 1 M=1,NZ
S4      0000900.        DO 1 L=1,NZ
S4      0001000.1       XK(L,M)=0.D0
S4      0001100.        XK(1,1)=2.0D0*K
S4      0001200.        XK(1,2)=-2.0D0*K
S4      0001300.        XK(NZ,NZM1)=-2.0D0*K
S4      0001400.        XK(NZ,NZ)=2.0D0*K
S4      0001500.        DO 2 L=2,NZM1
```

```
S4      0001600.         Y1=2*L
S4      0001700.         XK(L,L-1)=-K*(Y1-3.0D0)/(Y1-2.0D0)
S4      0001800.         XK(L,L)=2.0D0*K
S4      0001900.       2 XK(L,L+1)=-K*(Y1-1.0D0)/(Y1-2.0D0)
S4      0002000.         DO 4 M=1,NZ
S4      0002100.         DO 3 L=1,NZ
S4      0002200.3        YK(L,M)=0.D0
S4      0002300.4        YK(M,M)=1.0D0
S4      0002400.         DO 5 N=1,NZ
S4      0002500.         DO 5 M=1,NZ
S4      0002600.         T(M,N)=XK(M,N)
S4      0002700.5        XK(M,N)=XK(M,N)/T(1,1)
S4      0002800.         DO 9 J=1,MSER
S4      0002900.         DO 6 N=1,NZ
S4      0003000.         DO 6 M=1,NZ
S4      0003100.         W(M,N)=0.D0
S4      0003200.         DO 6 L=1,NZ
S4      0003300.6        W(M,N)=W(M,N)+YK(M,L)*XK(L,N)
S4      0003400.         DO 7 N=1,NZ
S4      0003500.         DO 7 M=1,NZ
S4      0003600.7        YK(M,N)=W(M,N)
S4      0003700.         DO 8 N=1,NZ
S4      0003800.         DO 8 M=1,NZ
S4      0003900.8        DYK(M,N,J)=YK(M,N)
S4      0004000.9        CONTINUE
S4      0004100.         RETURN
S4      0004200.         ENTRY MASER(X)
S4      0004300.         DO 11 N=1,NZ
S4      0004400.         DO 10 M=1,NZ
S4      0004500.         D1(M,N)=0.D0
S4      0004600.10       D2(M,N)=0.D0
S4      0004700.         D1(N,N)=1.0D0
S4      0004800.11       D2(N,N)=X
S4      0004900.         C4=1.0D0
S4      0005000.         C5=X
S4      0005100.         DO 13 J=1,MSER
S4      0005200.         Y2=2*J*(2*J-1)
S4      0005300.         Y3=2*J*(2*J+1)
S4      0005400.         C4=C4*T(1,1)*X*X/Y2
S4      0005500.         C5=C5*T(1,1)*X*X/Y3
S4      0005600.         IFLAG=0
S4      0005700.         DO 12 N=1,NZ
S4      0005800.         DO 12 M=1,NZ
S4      0005900.         TEMP=C4*DYK(M,N,J)
S4      0006000.         IF (TEMP.GT.1.0D-24) IFLAG=1
S4      0006100.         D1(M,N)=D1(M,N)+TEMP
S4      0006200.12       D2(M,N)=D2(M,N)+C5*DYK(M,N,J)
S4      0006300.         IF (IFLAG.EQ.0) GO TO 14
S4      0006400.13       CONTINUE
S4      0006500.         PRINT 19
S4      0006600.14       PRINT 20, X,J
S4      0006700.         ENTRY MXTRA
S4      0006800.         DO 15 N=1,NZ
S4      0006900.         DO 15 M=1,NZ
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S4    0007000.      D3(M,N)=0.D0
S4    0007100.      DO 15 L=1,NZ
S4    0007200.15    D3(M,N)=D3(M,N)+T(M,L)*D2(L,N)
S4    0007300.      DO 17 N=1,NZ
S4    0007400.      DO 16 M=1,NZ
S4    0007500.      DXX(M,N)=0.D0
S4    0007600.      DO 16 L=1,NZ
S4    0007700.16    DXX(M,N)=DXX(M,N)+D1(M,L)*D1(L,N)-D3(M,L)*D2(L,N)
S4    0007800.17    DXX(N,N)=DXX(N,N)-1.0D0
S4    0007900.      MAX=0.D0
S4    0008000.      DO 18 N=1,NZ
S4    0008100.      DO 18 M=1,NZ
S4    0008200.18    MAX=DMAX1(DXX(M,N),MAX)
S4    0008300.      IF (MAX.GT.1.0D-9) PRINT 21, X,MAX
S4    0008400.      RETURN
S4    0008500.C
S4    0008600.19    FORMAT (' WARNING -- NOT ENOUGH TERMS TAKEN ')
S4    0008700.20    FORMAT (' FOR ',F8.4,' TERMS TAKEN = ',I2)
S4    0008800.21    FORMAT (' FOR ',F8.4,' ERROR = ',F15.9)
S4    0008900.      END
S5    0000100.      IMPLICIT REAL*8 (A-H,O-Z)
S5    0000200.      COMMON /EIGEN/ XL(13)
S5    0000300.      COMMON /SERIES/ D1(13,13),D2(13,13),D3(13,13)
S5    0000350.      DIMENSION P(13,13),PIV(13,13),PO1(13),PO2(13),PO3(13),PO4(13),PT1(13,13),PT2(13,13),PT3(13,13),PT4(13,13)
S5    0000400.      DIMENSION D14(13,13,13), WDOT(13,13), UDOT(13,13), O14(13,13,13), O15(13,13,13)-
S5    0000500.     1,D4(13,13),D5(13,13),W(13,13),O5(13,13),-
S5    0000600.     2 O6(13,13),R(13,13),F1(13),F2(13),F4(13),UX(13,13),B1(-
S5    0000700.     313,13), F7(13), B2(13,13), B3(13,13), B4(13,13), D10(13,13,13), D11(13,13,13), -
S5    0000800.     4F6(13), O10(13,13,13), O11(13,13,13), -
S5    0000900.     5 F5(13), WX(13,13), UY(13,13), WY(13,13), T(13,13), SIGR(13,13), SI-
S5    0001000.     6GT(13,13), SIGZ(13,13), G(13,13), DX2(9,4), DX3(9,9), DX4(9,4), DX6(4,9)-
S5    0001100.     7, DX7(4,4), DX8(4,9), DW2(9,4), F3(13), WAL(4), WBE(9), DA(9,9), DB-
S5    0001200.     8(9,9), DC(9,4), D6(13,13), DX1(4,4), F8(13)
S5    0001300.      DATA C1,C2,C3,C3IV,XLAM/5.0D-1,5.0D-1,6.666666666666667D-1,1.5D0,7.5D-1/
S5    0001400.      DATA NR,NZ,NIC,HR,HZ/13,13,4,8.333333333333333D-2,8.333333333333333D-2/
S5    0001450.      DATA XKON,PI,IF1,IF2 /3.6D1,3.141592653589793D0,1,2/
S5    0001500.      NZM1=NZ-1
S5    0001600.      NRM1=NR-1
S5    0001700.      NOC=NZ-NIC
S5    0001800.      X1=7.5D-1/HR
S5    0001900.      X2=X1/2.0D0
S5    0002000.      X3=X2/2.0D0
S5    0002100.      X4=7.5D-1/HZ
S5    0002200.      X5=X4/2.0D0
S5    0002300.      X6=X5/2.0D0
S5    0002400.      X7=1.0D0/(HR*HZ)
S5    0002500.      X8=X7/2.0D0
S5    0002600.      X9=X8/2.0D0
S5    0002700.      X10=3.0D0*X9
S5    0002800.      X11=X10/2.0D0
S5    0002900.      X12=X8-5.0D-1/HZ
S5    0003000.      X13=X12/2.0D0
S5    0003100.      X14=1.25D0/HZ
S5    0003200.      X15=X14/2.0D0
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S5      0003203.       YP2=2.0D0*XKON
S5      0003206.       YP4=DFLOAT(NRM1)
S5      0003209.       YP1=PI/YP4
S5      0003212.       DO 201 I=1,NR
S5      0003215.   201 XL(I)=YP2*(1.0D0-DCOS(YP1*DFLOAT(I-1)))
S5      0003218.       DO 202 J=1,NR
S5      0003221.       DO 202 I=1,NR
S5      0003224.       YP3=YP1*DFLOAT((I-1)*(J-1))
S5      0003227.       P(I,J)=DCOS(YP3)
S5      0003230.   202 PIV(I,J)=P(I,J)/YP4
S5      0003233.       PIV(1,1)=PIV(1,1)/2.0D0
S5      0003236.       PIV(1,NR)=PIV(1,NR)/2.0D0
S5      0003239.       PIV(NR,1)=PIV(NR,1)/2.0D0
S5      0003242.       PIV(NR,NR)=PIV(NR,NR)/2.0D0
S5      0003245.       DO 203 J=2,NRM1
S5      0003248.       DO 203 I=2,NRM1
S5      0003251.   203 PIV(I,J)=PIV(I,J)*2.0D0
S5      0003254.       DO 204 I=1,NR
S5      0003257.       PO1(I)=1.0D0
S5      0003260.       PO2(I)=0.D0
S5      0003263.       PO3(I)=0.D0
S5      0003266.   204 PO4(I)=1.0D0
S5      0003300.       X=0.D0
S5      0003400.       XX=0.D0
S5      0003500.       DO 2 J=1,NR
S5      0003600.       DO 1 I=1,NR
S5      0004000.       O10(I,J,1)=0.D0
S5      0004100.       O11(I,J,1)=0.D0
S5      0004200.       O14(I,J,1)=0.D0
S5      0004500.     1 O15(I,J,1)=0.D0
S5      0004600.       O10(J,J,1)=1.0D0
S5      0004800.     2 O15(J,J,1)=1.0D0
S5      0004900.       DO 11 II=2,NZ
S5      0005000.       XF=X+HR
S5      0005100.       XXF=XX+HR
S5      0005200.       CALL RKGS(PO1,PO2,X,IF1)
S5      0005300.       CALL RKGS(PO3,PO4,XX,IF2)
S5      0005400.       X=XF
S5      0005500.       XX=XXF
S5      0005600.       DO 205 I=1,NR
S5      0005610.       DO 205 J=1,NR
S5      0005620.       PT1(J,I)=P(J,I)*PO1(I)
S5      0005630.       PT2(J,I)=P(J,I)*PO2(I)
S5      0005640.       PT3(J,I)=P(J,I)*PO3(I)
S5      0005650.   205 PT4(J,I)=P(J,I)*PO4(I)
S5      0005660.       DO 206 J=1,NR
S5      0005670.       DO 206 I=1,NR
S5      0005680.       SUM1=0.D0
S5      0005690.       SUM2=0.D0
S5      0005700.       SUM3=0.D0
S5      0005710.       SUM4=0.D0
S5      0005720.       DO 207 L=1,NR
S5      0005730.       SUM1=SUM1+PT1(I,L)*PIV(L,J)
S5      0005740.       SUM2=SUM2+PT2(I,L)*PIV(L,J)
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S5      0005750.        SUM3=SUM3+PT3(I,L)*PIV(L,J)
S5      0005760.    207 SUM4=SUM4+PT4(I,L)*PIV(L,J)
S5      0005770.        D10(I,J,II)=SUM1
S5      0005780.        D14(I,J,II)=SUM2/X
S5      0005790.        D11(I,J,II)=SUM3
S5      0005800.    206 D15(I,J,II)=SUM4
S5      0006500.11      CONTINUE
S5      0006600.        DO 12 N=1,NR
S5      0006700.        DO 12 M=1,NR
S5      0006800.        D5(M,N)=C2*D11(M,N,NZ)-D15(M,N,NZ)
S5      0006900.     12 D6(M,N)=C2*D10(M,N,NZ)-D14(M,N,NZ)
S5      0007000.        CALL MATINV (D5,NR)
S5      0007100.        DO 111 N=1,NR
S5      0007200.        DO 111 M=1,NR
S5      0007300.        W(M,N)=0.D0
S5      0007400.        DO 111 J=1,NR
S5      0007500.    111 W(M,N)=W(M,N)+D5(M,J)*D6(J,N)
S5      0007600.        DO 112 N=1,NR
S5      0007700.        DO 112 M=1,NR
S5      0007800.    112 D5(M,N)=C1*D5(M,N)
S5      0007900.        CALL MAPOWR
S5      0008000.        DO 23 L=1,NZ
S5      0008100.        DO 22 J=1,NZ
S5      0008200.        D10(J,L,1)=0.D0
S5      0008300.        D11(J,L,1)=0.D0
S5      0008400. 22     D14(J,L,1)=0.D0
S5      0008500. 23     D10(L,L,1)=1.0D0
S5      0008600.        DO 32 I=1,NRM1
S5      0008700.        X=HZ*DFLOAT(I)
S5      0008800.        IF(I.LE.11) GO TO 101
S5      0008900.        DO 103 N=1,NZ
S5      0009000.        DO 103 M=1,NZ
S5      0009100.        D1(M,N)=0.D0
S5      0009200.        D2(M,N)=0.D0
S5      0009300.        DO 103 L=1,NZ
S5      0009400.        D1(M,N)=D1(M,N)+D10(M,L,12)*D10(L,N,I-10)+D14(M,L,12)*D11(L,N,I-10)
S5      0009500.    103 D2(M,N)=D2(M,N)+D11(M,L,12)*D10(L,N,I-10)+D10(M,L,12)*D11(L,N,I-10)
S5      0009600.        CALL MXTRA
S5      0009700.        GO TO 102
S5      0009800.    101 CALL MASER(X)
S5      0009900.    102 DO 24 N=1,NZ
S5      0010000.        DO 24 M=1,NZ
S5      0010100.        D10(M,N,I+1)=D1(M,N)
S5      0010200.        D14(M,N,I+1)=D3(M,N)
S5      0010300.24      D11(M,N,I+1)=D2(M,N)
S5      0010400.32      CONTINUE
S5      0010500.        DO 33 N=1,NIC
S5      0010600.        DO 33 M=1,NIC
S5      0010700.33      DX1(M,N)=D3(M,N)
S5      0010800.        CALL MATINV (DX1,NIC)
S5      0010900.        DO 34 N=1,NOC
S5      0011000.        NIX=N+NIC
S5      0011100.        DO 34 M=1,NIC
S5      0011200.        DX6(M,N)=0.D0
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S5      0011300.       DX8(M,N)=0.DO
S5      0011400.       DO 34 J=1,NIC
S5      0011500.       DX8(M,N)=DX8(M,N)+DX1(M,J)*D1(J,NIX)
S5      0011600.34     DX6(M,N)=DX6(M,N)+DX1(M,J)*D3(J,NIX)
S5      0011700.       DO 35 N=1,NIC
S5      0011800.       DO 35 M=1,NIC
S5      0011900.       DX7(M,N)=0.DO
S5      0012000.       DO 35 J=1,NIC
S5      0012100.35     DX7(M,N)=DX7(M,N)+DX1(M,J)*D1(J,N)
S5      0012200.       DO 37 N=1,NOC
S5      0012300.       NIX=N+NIC
S5      0012400.       DO 37 M=1,NOC
S5      0012500.       MIX=M+NIC
S5      0012600.       DA(M,N)=0.DO
S5      0012700.       DB(M,N)=0.DO
S5      0012800.       DO 36 J=1,NIC
S5      0012900.       DA(M,N)=DA(M,N)+D3(MIX,J)*DX8(J,N)
S5      0013000.36     DB(M,N)=DB(M,N)+D3(MIX,J)*DX6(J,N)
S5      0013100.       DA(M,N)=D1(MIX,NIX)-DA(M,N)
S5      0013200.37     DB(M,N)=D3(MIX,NIX)-DB(M,N)
S5      0013300.       DO 39 N=1,NIC
S5      0013400.       DO 39 M=1,NOC
S5      0013500.       MIX=M+NIC
S5      0013600.       DC(M,N)=0.DO
S5      0013700.       DW2(M,N)=0.DO
S5      0013800.       DO 38 J=1,NIC
S5      0013900.       DW2(M,N)=DW2(M,N)+D3(MIX,J)*DX1(J,N)
S5      0014000.38     DC(M,N)=DC(M,N)+D3(MIX,J)*DX7(J,N)
S5      0014100.39     DC(M,N)=D1(MIX,N)-DC(M,N)
S5      0014200.       CALL MATINV (DA,NOC)
S5      0014300.       DO 40 N=1,NOC
S5      0014400.       DO 40 M=1,NOC
S5      0014500.       DX3(M,N)=0.DO
S5      0014600.       DO 40 J=1,NOC
S5      0014700.40     DX3(M,N)=DX3(M,N)+DA(M,J)*DB(J,N)
S5      0014800.       DO 41 N=1,NIC
S5      0014900.       DO 41 M=1,NOC
S5      0015000.       DX2(M,N)=0.DO
S5      0015100.       DX4(M,N)=0.DO
S5      0015200.       DO 41 J=1,NOC
S5      0015300.       DX2(M,N)=DX2(M,N)+DA(M,J)*DW2(J,N)
S5      0015400.41     DX4(M,N)=DX4(M,N)+DA(M,J)*DC(J,N)
S5      0015500.       KTR=0
S5      0015600.       DO 42 M=1,NR
S5      0015700.       DO 42 N=1,NZ
S5      0015800.42     WDOT(N,M)=0.DO
S5      0015900.       DO 43 N=1,NZ
S5      0016000.       DO 43 M=1,NR
S5      0016100.43     R(M,N)=0.DO
S5      0016200.44     KTR=KTR+1
S5      0016300.       DO 45 I=1,NR
S5      0016400.       DO 45 J=1,NZ
S5      0016500.45     WY(J,I)=WX(J,I)
S5      0016600.       DO 46 J=1,NZ
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S5      0016700.        DO 46 I=1,NR
S5      0016800.46      UY(I,J)=UX(I,J)
S5      0016900.        DO 47 I=1,NZ
S5      0017000.        DO 47 J=1,NR
S5      0017100.        UX(J,I)=0.D0
S5      0017200.        UDOT(J,I)=0.D0
S5      0017300.        B1(J,I)=0.D0
S5      0017400.47      B2(J,I)=0.D0
S5      0017500.        DO 48 I=1,NR
S5      0017600.        F1(I)=0.D0
S5      0017700.48      F2(I)=0.D0
S5      0017800.        DO 50 M=2,NZ
S5      0017900.        DO 50 J=1,NR
S5      0018000.        SUM1=0.D0
S5      0018100.        SUM2=0.D0
S5      0018200.        DO 49 L=1,NR
S5      0018300.        SUM1=SUM1-O11(J,L,M-1)*R(L,M-1)-O11(J,L,M)*R(L,M)
S5      0018400.49      SUM2=SUM2+O10(J,L,M-1)*R(L,M-1)+O10(J,L,M)*R(L,M)
S5      0018500.        B1(J,M)=B1(J,M-1)+HR*SUM1/2.0D0
S5      0018600.50      B2(J,M)=B2(J,M-1)+HR*SUM2/2.0D0
S5      0018700.        DO 51 L=1,NR
S5      0018800.        DO 51 J=1,NR
S5      0018900.     51 F2(J)=F2(J)+O5(J,L)*WDOT(NZ,L)-W(J,L)*B1(L,NZ)
S5      0019000.        DO 510 J=1,NR
S5      0019100.    510 F2(J)=F2(J)-B2(J,NZ)
S5      0019200.        X=HR
S5      0019300.        XX=X*X
S5      0019400.        DO 540 I=1,NR
S5      0019500.        UX(I,1)=0.D0
S5      0019600.    540 UDOT(I,1)=F2(I)/2.0D0
S5      0019700.        DO 54 I=2,NZ
S5      0019800.        DO 52 J=1,NR
S5      0019900.        F5(J)=F1(J)+B1(J,I)
S5      0020000.52      F6(J)=F2(J)+B2(J,I)
S5      0020100.        DO 53 N=1,NR
S5      0020200.        DO 53 M=1,NR
S5      0020300.        UDOT(M,I)=UDOT(M,I)+(O14(M,N,I)-O10(M,N,I)/XX)*F5(N)+(O15(M,N,I)-O11(M,N,I)/XX)*F6(N)
S5      0020400.53      UX(M,I)=UX(M,I)+(O10(M,N,I)*F5(N)+O11(M,N,I)*F6(N))/X
S5      0020500.        X=X+HR
S5      0020600.54      XX=X*X
S5      0020700.        WRITE (6,87)
S5      0020800.        DO 55 KA=1,NR
S5      0020900.55      WRITE (6,93) (UX(KA,K),K=1,NZ)
S5      0021000.        WRITE (6,88)
S5      0021100.        DO 56 KA=1,NR
S5      0021200.56      WRITE (6,93) (UDOT(KA,K),K=1,NZ)
S5      0021300.        T(1,1)=X14*(UDOT(1,1)-UDOT(2,1))
S5      0021400.        DO 57 I=2,NZ
S5      0021500.        T(I,1)=0.D0
S5      0021600.57      T(I,NR)=X4*(UDOT(NRM1,I)-UDOT(NR,I))+X10*(UX(NRM1,I)-UX(NR,I))/DFLOAT(I-1)
S5      0021700.        DO 58 I=2,NIC
S5      0021800.58      T(I,1)=X4*(UDOT(1,I)-UDOT(2,I))+X10*(UX(1,I)-UX(2,I))/DFLOAT(I-1)
S5      0021900.        T(1,NR)=X14*(UDOT(NR-1,1)-UDOT(NR,1))
S5      0022000.        T(NZ,NR)=X4*(UDOT(NRM1,NZ)-UDOT(NR,NZ))+X12*(UX(NR,NZ)-UX(NRM1,NZ))
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S5     0022100.       DO 59 J=2,NRM1
S5     0022200.       T(1,J)=X15*(UDOT(J-1,1)-UDOT(J+1,1))
S5     0022300.       T(NZ,J)=X5*(UDOT(J-1,NZ)-UDOT(J+1,NZ))+X13*(UX(J+1,NZ)-UX(J-1,NZ))
S5     0022400.       DO 59 I=2,NZM1
S5     0022500.59     T(I,J)=X5*(UDOT(J-1,I)-UDOT(J+1,I))+X11*(UX(J-1,I)-UX(J+1,I))/DFLOAT(I-1)
S5     0022600.       DO 60 I=1,NR
S5     0022700.       DO 60 J=1,NZ
S5     0022800.       WX(J,I)=0.D0
S5     0022900.       WDOT(J,I)=0.D0
S5     0023000.       B3(J,I)=0.D0
S5     0023100.60     B4(J,I)=0.D0
S5     0023200.       DO 61 I=1,NZ
S5     0023300.       F3(I)=0.D0
S5     0023400.61     F4(I)=0.D0
S5     0023500.       DO 63 M=2,NR
S5     0023600.       DO 63 J=1,NZ
S5     0023700.       SUM1=0.D0
S5     0023800.       SUM2=0.D0
S5     0023900.       DO 62 L=1,NZ
S5     0024000.       SUM1=SUM1-D11(J,L,M-1)*T(L,M-1)-D11(J,L,M)*T(L,M)
S5     0024100.62     SUM2=SUM2+D10(J,L,M-1)*T(L,M-1)+D10(J,L,M)*T(L,M)
S5     0024200.       B3(J,M)=B3(J,M-1)+HZ*SUM1/2.0D0
S5     0024300.63     B4(J,M)=B4(J,M-1)+HZ*SUM2/2.0D0
S5     0024400.       WAL(1)=C3-C1*F2(NR)
S5     0024500.       F4(1)=-C1*F2(1)
S5     0024600.       DO 64 I=2,NIC
S5     0024700.       WAL(I)=C3-C1*(UDOT(NR,I)+UX(NR,I)/(HR*DFLOAT(I-1)))
S5     0024800.64     F4(I)=-C1*(UDOT(1,I)+UX(1,I)/(HR*DFLOAT(I-1)))
S5     0024900.       DO 65 I=1,NOC
S5     0025000.       NIX=I+NIC
S5     0025100.65     WBE(I)=C3-C1*(UDOT(NR,NIX)+UX(NR,NIX)/(HR*DFLOAT(NIX-1)))
S5     0025200.       DO 68 M=1,NOC
S5     0025300.       MIX=M+NIC
S5     0025400.       SUM1=0.D0
S5     0025500.       SUM2=0.D0
S5     0025600.       DO 66 N=1,NOC
S5     0025700.       NIX=N+NIC
S5     0025800.66     SUM1=SUM1+DA(M,N)*WBE(N)-DX3(M,N)*B3(NIX,NR)
S5     0025900.       DO 67 N=1,NIC
S5     0026000.       NIX=N+NIC
S5     0026100.67     SUM2=SUM2+DX2(M,N)*WAL(N)+DX4(M,N)*(F4(N)+B4(N,NR))
S5     0026200.68     F4(MIX)=SUM1-SUM2-B4(MIX,NR)
S5     0026300.       DO 71 M=1,NIC
S5     0026400.       SUM1=0.D0
S5     0026500.       SUM2=0.D0
S5     0026600.       DO 69 N=1,NIC
S5     0026700.69     SUM1=SUM1+DX1(M,N)*WAL(N)-DX7(M,N)*(F4(N)+B4(N,NR))
S5     0026800.       DO 70 N=1,NOC
S5     0026900.       NIX=N+NIC
S5     0027000.70     SUM2=SUM2+DX6(M,N)*B3(NIX,NR)+DX8(M,N)*(F4(NIX)+B4(NIX,NR))
S5     0027100.71     F3(M)=SUM1-SUM2-B3(M,NR)
S5     0027200.       DO 73 I=1,NR
S5     0027300.       DO 72 J=1,NZ
S5     0027400.       F7(J)=F3(J)+B3(J,I)
```

```
S5      0027500.72     F8(J)=F4(J)+B4(J,I)
S5      0027600.       DO 73 N=1,NZ
S5      0027700.       DO 73 M=1,NZ
S5      0027800.       WDOT(M,I)=WDOT(M,I)+D10(M,N,I)*F8(N)+D14(M,N,I)*F7(N)
S5      0027900.73     WX(M,I)=WX(M,I)+D11(M,N,I)*F8(N)+D10(M,N,I)*F7(N)
S5      0028000.       WRITE (6,89)
S5      0028100.       DO 74 KA=1,NZ
S5      0028200.74     WRITE (6,93) (WX(KA,K),K=1,NR)
S5      0028300.       WRITE (6,90)
S5      0028400.       DO 75 KA=1,NZ
S5      0028500.75     WRITE (6,93) (WDOT(KA,K),K=1,NR)
S5      0028600.       R(1,1)=X1*(WDOT(1,1)-WDOT(2,1))
S5      0028700.       R(1,NZ)=X1*(WDOT(NZM1,1)-WDOT(NZ,1))+X8*(WX(NZM1,1)-WX(NZ,1))
S5      0028800.       R(NR,1)=X1*(WDOT(1,NR)-WDOT(2,NR))
S5      0028900.       R(NR,NZ)=X1*(WDOT(NZM1,NR)-WDOT(NZ,NR))+X8*(WX(NZ,NR)-WX(NZM1,NR))
S5      0029000.       DO 76 I=2,NRM1
S5      0029100.       R(I,1)=X1*(WDOT(1,I)-WDOT(2,I))
S5      0029200.76     R(I,NZ)=X1*(WDOT(NZM1,I)-WDOT(NZ,I))
S5      0029300.       DO 77 J=2,NZM1
S5      0029400.       R(1,J)=X2*(WDOT(J-1,1)-WDOT(J+1,1))+X9*(WX(J-1,1)-WX(J+1,1))
S5      0029500.       R(NR,J)=X2*(WDOT(J-1,NR)-WDOT(J+1,NR))+X9*(WX(J+1,NR)-WX(J-1,NR))
S5      0029600.       DO 77 I=2,NRM1
S5      0029700.77     R(I,J)=X2*(WDOT(J-1,I)-WDOT(J+1,I))
S5      0029800.       PRINT 91, KTR
S5      0029900.       IF (DABS(WX(1,1)).GT.5.0D0) STOP
S5      0030000.       DO 78 I=1,NR
S5      0030100.       DO 78 J=1,NZ
S5      0030200.       IF (DABS(WX(J,I)-WY(J,I)).GT.1.0D-6) GO TO 44
S5      0030300.78     CONTINUE
S5      0030400.       DO 79 J=1,NZ
S5      0030500.       DO 79 I=1,NR
S5      0030600.       IF (DABS(UX(I,J)-UY(I,J)).GT.1.0D-6) GO TO 44
S5      0030700.79     CONTINUE
S5      0030800.       WRITE (6,92) KTR
S5      0030900.       DO 800 I=1,NR
S5      0031000.       SIGR(I,1)=C3IV*UDOT(I,1)+XLAM*(UDOT(I,1)+WDOT(1,I))
S5      0031100.       SIGT(I,1)=XLAM*(UDOT(I,1)+WDOT(1,I))+C3IV*UDOT(I,1)
S5      0031200.   800 SIGZ(I,1)=C3IV*WDOT(1,I)+2.0D0*XLAM*UDOT(I,1)
S5      0031300.       DO 80 J=2,NZ
S5      0031400.       YJ=HR*DFLOAT(J-1)
S5      0031500.       DO 80 I=1,NR
S5      0031600.       SIGR(I,J)=C3IV*UDOT(I,J)+XLAM*(UX(I,J)/YJ+WDOT(J,I))
S5      0031700.       SIGT(I,J)=C3IV*UX(I,J)/YJ+XLAM*(UDOT(I,J)+WDOT(J,I))
S5      0031800.80     SIGZ(I,J)=C3IV*WDOT(J,I)+XLAM*(UDOT(I,J)+UX(I,J)/YJ)
S5      0031900.       G(1,1)=X2*(WX(2,1)-WX(1,1))+X5*(UX(2,1)-UX(1,1))
S5      0032000.       G(1,NZ)=X2*(WX(NZ,1)-WX(NZM1,1))+X5*(UX(2,NZ)-UX(1,NZ))
S5      0032100.       G(NR,1)=X2*(WX(2,NR)-WX(1,NR))+X5*(UX(NR,1)-UX(NRM1,1))
S5      0032200.       G(NR,NZ)=X2*(WX(NZ,NR)-WX(NZ-1,NR))+X5*(UX(NR,NZ)-UX(NRM1,NZ))
S5      0032300.       DO 81 J=2,NZM1
S5      0032400.       G(1,J)=X3*(WX(J+1,1)-WX(J-1,1))+X5*(UX(2,J)-UX(1,J))
S5      0032500.       G(NR,J)=X3*(WX(J+1,NR)-WX(J-1,NR))+X5*(UX(NR,J)-UX(NRM1,J))
S5      0032600.       DO 81 I=2,NRM1
S5      0032700.81     G(I,J)=X3*(WX(J+1,I)-WX(J-1,I))+X6*(UX(I+1,J)-UX(I-1,J))
S5      0032800.       DO 82 I=2,NRM1
```

```
S5   0032900.      G(I,1)=X2*(WX(2,I)-WX(1,I))+X6*(UX(I+1,1)-UX(I-1,1))
S5   0033000.82    G(I,NZ)=X2*(WX(NZ,I)-WX(NZM1,I))+X6*(UX(I+1,NZ)-UX(I-1,NZ))
S5   0033100.      WRITE (6,94)
S5   0033200.      DO 83 I=1,NR
S5   0033300.83    WRITE (6,93) (SIGR(I,J),J=1,NZ)
S5   0033400.      WRITE (6,95)
S5   0033500.      DO 84 I=1,NR
S5   0033600.84    WRITE (6,93) (SIGT(I,J),J=1,NZ)
S5   0033700.      WRITE (6,96)
S5   0033800.      DO 85 I=1,NR
S5   0033900.85    WRITE (6,93) (SIGZ(I,J),J=1,NZ)
S5   0034000.      WRITE (6,97)
S5   0034100.      DO 86 I=1,NR
S5   0034200.86    WRITE (6,93) (G(I,J),J=1,NZ)
S5   0034300.      STOP
S5   0034400.C
S5   0034500.87    FORMAT (///' U-MATRIX ')
S5   0034600.88    FORMAT (///' U-DOT MATRIX ')
S5   0034700.89    FORMAT (///' W-MATRIX ')
S5   0034800.90    FORMAT (///' W-DOT MATRIX ')
S5   0034900.91    FORMAT (' KTR = ',I2)
S5   0035000.92    FORMAT (///' ITERATION NUMBER ',I2,///)
S5   0035100.93    FORMAT (9(2X,F10.6))
S5   0035200.94    FORMAT (///' SIGMA-R MATRIX ')
S5   0035300.95    FORMAT (///' SIGMA-THETA MATRIX ')
S5   0035400.96    FORMAT (///' SIGMA-Z MATRIX ')
S5   0035500.97    FORMAT (///' SIGMA-RZ MATRIX ')
S5   0035600.      END
```

C.3 Annular Plate With Internal Surface Cracks

The main subroutines for this general three-dimensional cylindrical coordinate problem are shown in Figure 36. This program is similar to the previously presented two programs, except that two additional subroutines are used. Note that the matrix function routines in the circumferential and axial directions are entered at different locations which are denoted as MASER, MXTRA and MASES respectively. In addition, the input discussed for the axisymmetric problems, this program includes as input the indices of the coupling vector variables. These are denoted as arrays LR, LS and LT in the program. The integers in DATA IN are used to rearrange the elements of the circumferential matrix functions at θ_o according to the form needed in the partitioning procedure for treating mixed boundary conditions.

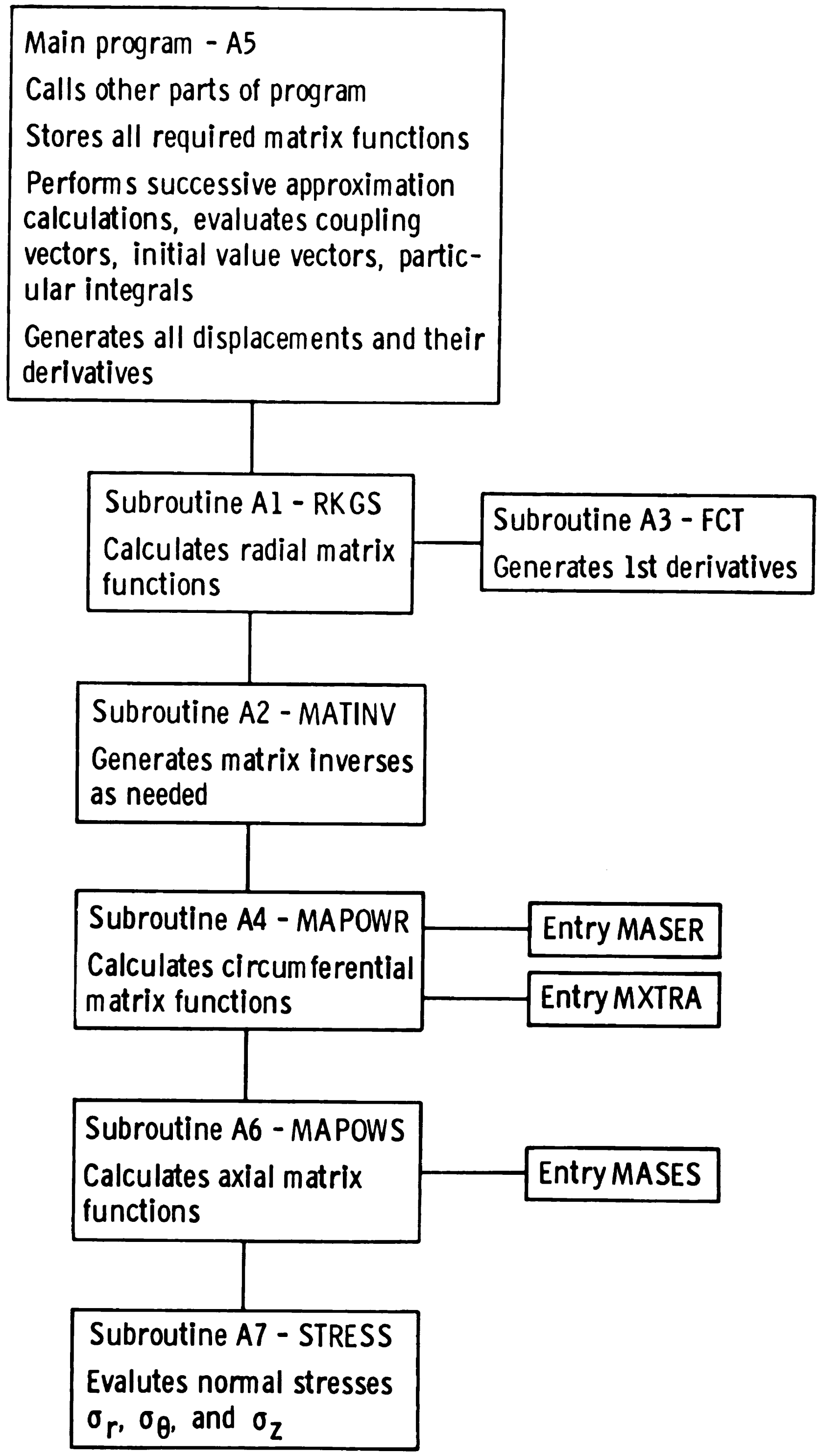

Figure 36. - Schematic representation of the annular plate with internal surface cracks computer program.

```
A1   0000100.     SUBROUTINE RKGS (O1,O2,XO)
A1   0000200.     IMPLICIT REAL*8 (A-H,O-Z)
A1   0000300.     DATA NR,HR,JJ,XJ /16,2.5D-1,64,6.4D1/
A1   0000400.     COMMON /RUNGE/ O3(16,16),O4(16,16),O7(16,16),O8(16,16)
A1   0000500.     DIMENSION O5(16,16),O6(16,16),O1(16,16),O2(16,16)
A1   0000600.     H=HR/XJ
A1   0000700.     HH=H/2.0D0
A1   0000800.     DO 6 I=1,JJ
A1   0000900.     DO 1 K=1,NR
A1   0001000.     DO 1 J=1,NR
A1   0001100.     O7(J,K)=O1(J,K)
A1   0001200.1    O8(J,K)=O2(J,K)
A1   0001300.     CALL FCT (XO)
A1   0001400.     DO 2 K=1,NR
A1   0001500.     DO 2 J=1,NR
A1   0001600.     O5(J,K)=O3(J,K)
A1   0001700.     O6(J,K)=O4(J,K)
A1   0001800.     O7(J,K)=O1(J,K)+HH*O3(J,K)
A1   0001900.2    O8(J,K)=O2(J,K)+HH*O4(J,K)
A1   0002000.     XO=XO+HH
A1   0002100.     CALL FCT (XO)
A1   0002200.     DO 3 K=1,NR
A1   0002300.     DO 3 J=1,NR
A1   0002400.     O7(J,K)=O1(J,K)+HH*O3(J,K)
A1   0002500.     O8(J,K)=O2(J,K)+HH*O4(J,K)
A1   0002600.     O5(J,K)=O5(J,K)+2.0D0*O3(J,K)
A1   0002700. 3   O6(J,K)=O6(J,K)+2.0D0*O4(J,K)
A1   0002800.     CALL FCT (XO)
A1   0002900.     DO 4 K=1,NR
A1   0003000.     DO 4 J=1,NR
A1   0003100.     O5(J,K)=O5(J,K)+2.0D0*O3(J,K)
A1   0003200.     O6(J,K)=O6(J,K)+2.0D0*O4(J,K)
A1   0003300.     O7(J,K)=O1(J,K)+H*O3(J,K)
A1   0003400. 4   O8(J,K)=O2(J,K)+H*O4(J,K)
A1   0003500.     XO=XO+HH
A1   0003600.     CALL FCT (XO)
A1   0003700.     DO 5 K=1,NR
A1   0003800.     DO 5 J=1,NR
A1   0003900.     O5(J,K)=O5(J,K)+O3(J,K)
A1   0004000.     O6(J,K)=O6(J,K)+O4(J,K)
A1   0004100.     O1(J,K)=O1(J,K)+H*O5(J,K)/6.0D0
A1   0004200.5    O2(J,K)=O2(J,K)+H*O6(J,K)/6.0D0
A1   0004300.6    CONTINUE
A1   0004400.     RETURN
A1   0004500.     END
A2   0000100.     SUBROUTINE MATINV (X,MMM)
A2   0000200.     DOUBLE PRECISION X,Y,W,A,TEST,R,DELT
A2   0000300.     DIMENSION X(MMM,MMM), Y(16,16), K(16), W(16,16)
A2   0000400.     DATA DELT,LOOPS/1.0D-8,2/
A2   0000500.     KSIG=0
A2   0000600.     M=MMM
A2   0000700.     MM=M-1
A2   0000800.     DO 1 J=1,M
A2   0000900.     K(J)=J
```

```
A2   0001000.     DO 1 I=1,M
A2   0001100.1    Y(I,J)=X(I,J)
A2   0001200.     DO 6 I=1,M
A2   0001300.     A=0.DO
A2   0001400.     DO 2 J=I,M
A2   0001500.     IF (DABS(Y(J,1)).LE.A) GO TO 2
A2   0001600.     A=DABS(Y(J,1))
A2   0001700.     L=J
A2   0001800.2    CONTINUE
A2   0001900.     IF (A.EQ.0.DO) GO TO 18
A2   0002000.     N=K(L)
A2   0002100.     K(L)=K(I)
A2   0002200.     K(I)=N
A2   0002300.     DO 3 J=1,M
A2   0002400.     A=Y(I,J)
A2   0002500.     Y(I,J)=Y(L,J)
A2   0002600.3    Y(L,J)=A
A2   0002700.     A=Y(I,1)
A2   0002800.     DO 4 J=1,MM
A2   0002900.4    Y(I,J)=Y(I,J+1)/A
A2   0003000.     Y(I,M)=1.DO/A
A2   0003100.     DO 6 L=1,M
A2   0003200.     IF (L.EQ.I) GO TO 6
A2   0003300.     A=Y(L,1)
A2   0003400.     DO 5 N=1,MM
A2   0003500.5    Y(L,N)=Y(L,N+1)-A*Y(I,N)
A2   0003600.     Y(L,M)=-A*Y(I,M)
A2   0003700.6    CONTINUE
A2   0003800.     DO 10 I=1,M
A2   0003900.     IF (K(I).EQ.I) GO TO 10
A2   0004000.     DO 7 J=I,M
A2   0004100.     IF (K(J).EQ.I) GO TO 8
A2   0004200.7    CONTINUE
A2   0004300.     GO TO 19
A2   0004400.8    DO 9 L=1,M
A2   0004500.     A=Y(L,I)
A2   0004600.     Y(L,I)=Y(L,J)
A2   0004700.9    Y(L,J)=A
A2   0004800.     K(J)=K(I)
A2   0004900.10   CONTINUE
A2   0005000.     DO 15 N=1,LOOPS
A2   0005100.     TEST=0.DO
A2   0005200.     DO 12 J=1,M
A2   0005300.     DO 12 I=1,M
A2   0005400.     R=0.DO
A2   0005500.     DO 11 L=1,M
A2   0005600.11   R=R-X(I,L)*Y(L,J)
A2   0005700.     IF (I.EQ.J) R=R+1.0DO
A2   0005800.     TEST=DMAX1(TEST,DABS(R))
A2   0005900.12   W(I,J)=R
A2   0006000.     IF (TEST.LE.DELT) GO TO 16
A2   0006100.     DO 14 J=1,M
A2   0006200.     DO 14 I=1,M
A2   0006300.     A=0.DO
```

```
A2      0006400.        DO 13 L=1,M
A2      0006500.13      A=A+Y(I,L)*W(L,J)
A2      0006600.14      Y(I,J)=Y(I,J)+A
A2      0006700.15      CONTINUE
A2      0006800.        KSIG=3
A2      0006900.        PRINT 21, TEST
A2      0007000.16      DO 17 J=1,M
A2      0007100.        DO 17 I=1,M
A2      0007200.17      X(I,J)=Y(I,J)
A2      0007300.        GO TO 20
A2      0007400.18      KSIG=1
A2      0007500.        GO TO 20
A2      0007600.19      KSIG=2
A2      0007700.20      IF (KSIG.NE.0) PRINT 22, KSIG
A2      0007800.        RETURN
A2      0007900.C
A2      0008000.21      FORMAT (' ERROR = ',D14.6)
A2      0008100.22      FORMAT (' ERROR INDICATOR NUMBER ',I1)
A2      0008200.        END
A3      0000100.        SUBROUTINE FCT(X)
A3      0000200.        IMPLICIT REAL*8 (A-H,O-Z)
A3      0000300.        COMMON/RUNGE/ O3(16,16),O4(16,16),O7(16,16),O8(16,16)
A3      0000400.        DATA Z1,Z2,Z9,Z8  /2.5D1,5.0D1,1.82378130550D0, 9.1189065275D-1 /
A3      0000500.        Z4=4.0D0*Z9
A3      0000600.        Z3=4.0D0*Z8
A3      0000700.        Z5= Z3/X**2
A3      0000800.        Z6 = 2.0D0 * Z5
A3      0000900.        Z7 = Z6 + Z2
A3      0001000.        DO 2  J=1,16
A3      0001100.        DO 2 I=1,16
A3      0001200.      2 O3(I,J)= X*O8(I,J)
A3      0001300.        DO 1  I=1,16
A3      0001400.        O4(I,1)=( Z7*O7(I,1)-Z5*O7(I,2)-Z1*O7(I,5))/X
A3      0001500.        O4(I,2)=( Z7*O7(I,2)-Z6*O7(I,1)-Z5*O7(I,3)-Z1*O7(I,6))/X
A3      0001600.        O4(I,3)=( Z7*O7(I,3)-Z5*O7(I,2)-Z6*O7(I,4)-Z1*O7(I,7))/X
A3      0001700.        O4(I,4)=( Z7*O7(I,4)-Z5*O7(I,3)-Z1*O7(I,8))/X
A3      0001800.        O4(I,5)=( Z7*O7(I,5)-Z2*O7(I,1)-Z5*O7(I,6)-Z1*O7(I,9))/X
A3      0001900.        O4(I,6)=( Z7*O7(I,6)-Z2*O7(I,2)-Z6*O7(I,5)-Z5*O7(I,7)-Z1*O7(I,10))/X
A3      0002000.        O4(I,7)=( Z7*O7(I,7)-Z2*O7(I,3)-Z5*O7(I,6)-Z6*O7(I,8)-Z1*O7(I,11))/X
A3      0002100.        O4(I,8)=( Z7*O7(I,8)-Z2*O7(I,4)-Z5*O7(I,7)-Z1*O7(I,12))/X
A3      0002200.        O4(I,9)=( Z7*O7(I,9)-Z1*O7(I,5)-Z5*O7(I,10)-Z2*O7(I,13))/X
A3      0002300.        O4(I,10)=( Z7*O7(I,10)-Z1*O7(I,6)-Z6*O7(I,9)-Z5*O7(I,11)-Z2*O7(I, -
A3      0002400.       114))/X
A3      0002500.        O4(I,11)=( Z7*O7(I,11)-Z1*O7(I,7)-Z5*O7(I,10)-Z6*O7(I,12)-Z2*O7(I, -
A3      0002600.       115))/X
A3      0002700.        O4(I,12)=( Z7*O7(I,12)-Z1*O7(I,8)-Z5*O7(I,11)-Z2*O7(I,16))/X
A3      0002800.        O4(I,13)=( Z7*O7(I,13)-Z1*O7(I,9)- Z5*O7(I,14))/X
A3      0002900.        O4(I,14)=( Z7*O7(I,14)-Z1*O7(I,10)-Z6*O7(I,13)-Z5*O7(I,15))/X
A3      0003000.        O4(I,15)=( Z7*O7(I,15)-Z1*O7(I,11)-Z5*O7(I,14)-Z6*O7(I,16))/X
A3      0003100.      1 O4(I,16)=( Z7*O7(I,16)-Z1*O7(I,12)-Z5*O7(I,15))/X
A3      0003200.        RETURN
A3      0003300.        END
A4      0000100.        SUBROUTINE MAPOWR
A4      0000200.        IMPLICIT REAL*8 (A-H,O-Z)
```

```
      DOUBLE PRECISION K,MAX
      DIMENSION XK(16,16), YK(16,16), W(16,16), T(16,16), DXX(16,16), DYK(16,16,40)
      COMMON /SER1/ D1(16,16),D2(16,16),D3(16,16)
      DATA NZ,K,MSER /16,2.5D-1,40 /
      DATA A,B,A1,A2,HR,HZ /2.5D-1,1.0D0,5.0D-1,7.5D-1,2.5D-1,1.0D-1/
      DO 1 M=1,NZ
      DO 1 L=1,NZ
    1 XK(L,M)=0.D0
      E9 = -2.0D0*A*A*K/HR**2
      E10= -A*A*K/HZ**2
      E11= 2.0D0*E10
      E12= -K*A1**2/HZ**2
      E13= 2.0D0*E12
      E14= -K*A2**2/HZ**2
      E15= 2.0D0*E14
      E16= -K*B**2/HZ**2
      E17= 2.0D0 * E16
      E18= -2.0D0*B*B*K/HR**2
      E1=2.0D0*K*(A/HR)-E9-E11
      E19= -K*A1**2/HR**2
      E21= -K*A2**2/HR**2
      E20=2.0D0*E19
      E22=2.0D0*E21
      E23= -K*A1/(2.0D0*HR)
      E24= -K*A2/(2.0D0*HR)
      E2= K-E20-E13
      E3= K-E22-E15
      E4=2.0D0*K*(-B/HR)-E18-E17
      E5 = E19+E23
      E6 = E19-E23
      E7= E21+E24
      E8= E21-E24
      XK(1,1) = E1
      XK(2,1) = E10
      XK(5,1) = E6
      XK(1,2) = E11
      XK(2,2) = E1
      XK(3,2) = E10
      XK(6,2) = E6
      XK(2,3) = E10
      XK(3,3) = E1
      XK(4,3) = E11
      XK(7,3) = E6
      XK(3,4) = E10
      XK(4,4) = E1
      XK(8,4) = E6
      XK(1,5) = E9
      XK(5,5) = E2
      XK(6,5) = E12
      XK(9,5) = E8
      XK(2,6) = E9
      XK(5,6) = E13
      XK(6,6) = E2
      XK(7,6) = E12
```

```
A4      0005700.       XK(10,6)= E8
A4      0005800.       XK(3,7) = E9
A4      0005900.       XK(6,7) = E12
A4      0006000.       XK(7,7) = E2
A4      0006100.       XK(8,7) = E13
A4      0006200.       XK(11,7)= E8
A4      0006300.       XK(4,8) = E9
A4      0006400.       XK(7,8) = E12
A4      0006500.       XK(8,8) = E2
A4      0006600.       XK(12,8)= E8
A4      0006700.       XK(5,9) = E5
A4      0006800.       XK(9,9) = E3
A4      0006900.       XK(10,9)=   E14
A4      0007000.       XK(13,9)=   E18
A4      0007100.       XK(6,10)=   E5
A4      0007200.       XK(9,10)=   E15
A4      0007300.       XK(10,10) = E3
A4      0007400.       XK(11,10) = E14
A4      0007500.       XK(14,10) = E18
A4      0007600.       XK(7,11)=E5
A4      0007700.       XK(10,11)=E14
A4      0007800.       XK(11,11)=E3
A4      0007900.       XK(12,11)=E15
A4      0008000.       XK(15,11)=E18
A4      0008100.       XK(8,12)=E5
A4      0008200.       XK(11,12)=E14
A4      0008300.       XK(12,12)=E3
A4      0008400.       XK(16,12)=E18
A4      0008500.       XK(9,13)=E7
A4      0008600.       XK(13,13)=E4
A4      0008700.       XK(14,13)=E16
A4      0008800.       XK(10,14)=E7
A4      0008900.       XK(13,14)=E17
A4      0009000.       XK(14,14)=E4
A4      0009100.       XK(15,14)=E16
A4      0009200.       XK(11,15)=E7
A4      0009300.       XK(14,15)=E16
A4      0009400.       XK(15,15)=E4
A4      0009500.       XK(16,15)=E17
A4      0009600.       XK(12,16)=E7
A4      0009700.       XK(15,16)=E16
A4      0009800.       XK(16,16)=E4
A4      0009900.       DO 4 M=1,NZ
A4      0010000.       DO 3 L=1,NZ
A4      0010100.3      YK(L,M)=0.D0
A4      0010200.4      YK(M,M)=1.0D0
A4      0010300.       TAX=0.D0
A4      0010400.       DO 30 N=1,16
A4      0010500.       DO 30 M=1,16
A4      0010600.    30 TAX=DMAX1(TAX,DABS(XK(M,N)))
A4      0010700.       DO 5 N=1,NZ
A4      0010800.       DO 5 M=1,NZ
A4      0010900.       T(M,N)=XK(M,N)
A4      0011000.5      XK(M,N)=XK(M,N)/TAX
```

```
A4      0011100.        DO 9 J=1,MSER
A4      0011200.        DO 6 N=1,NZ
A4      0011300.        DO 6 M=1,NZ
A4      0011400.        W(M,N)=0.DO
A4      0011500.        DO 6 L=1,NZ
A4      0011600.6       W(M,N)=W(M,N)+YK(M,L)*XK(L,N)
A4      0011700.        DO 7 N=1,NZ
A4      0011800.        DO 7 M=1,NZ
A4      0011900.7       YK(M,N)=W(M,N)
A4      0012000.        DO 8 N=1,NZ
A4      0012100.        DO 8 M=1,NZ
A4      0012200.8       DYK(M,N,J)=YK(M,N)
A4      0012300.9       CONTINUE
A4      0012400.        RETURN
A4      0012500.        ENTRY MASER(X)
A4      0012600.        DO 11 N=1,NZ
A4      0012700.        DO 10 M=1,NZ
A4      0012800.        D1(M,N)=0.DO
A4      0012900.10      D2(M,N)=0.DO
A4      0013000.        D1(N,N)=1.0DO
A4      0013100.11      D2(N,N)=X
A4      0013200.        C4=1.0DO
A4      0013300.        C5=X
A4      0013400.        DO 13 J=1,MSER
A4      0013500.        Y2=2*J*(2*J-1)
A4      0013600.        Y3=2*J*(2*J+1)
A4      0013700.        C4=C4*TAX*X*X/Y2
A4      0013800.        C5=C5*TAX*X*X/Y3
A4      0013900.        IFLAG=0
A4      0014000.        DO 12 N=1,NZ
A4      0014100.        DO 12 M=1,NZ
A4      0014200.        TEMP=C4*DYK(M,N,J)
A4      0014300.        IF (TEMP.GT.1.0D-24) IFLAG=1
A4      0014400.        D1(M,N)=D1(M,N)+TEMP
A4      0014500.12      D2(M,N)=D2(M,N)+C5*DYK(M,N,J)
A4      0014600.        IF (IFLAG.EQ.0) GO TO 14
A4      0014700.13      CONTINUE
A4      0014800.        PRINT 19
A4      0014900.14      PRINT 20, X,J
A4      0015000.        ENTRY MXTRA
A4      0015100.        DO 15 N=1,NZ
A4      0015200.        DO 15 M=1,NZ
A4      0015300.        D3(M,N)=0.DO
A4      0015400.        DO 15 L=1,NZ
A4      0015500.15      D3(M,N)=D3(M,N)+T(M,L)*D2(L,N)
A4      0015600.        DO 17 N=1,NZ
A4      0015700.        DO 16 M=1,NZ
A4      0015800.        DXX(M,N)=0.DO
A4      0015900.        DO 16 L=1,NZ
A4      0016000.16      DXX(M,N)=DXX(M,N)+D1(M,L)*D1(L,N)-D3(M,L)*D2(L,N)
A4      0016100.17      DXX(N,N)=DXX(N,N)-1.0DO
A4      0016200.        MAX=0.DO
A4      0016300.        DO 18 N=1,NZ
A4      0016400.        DO 18 M=1,NZ
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```
A4      0016500.18     MAX=DMAX1(DXX(M,N),MAX)
A4      0016600.       IF (MAX.GT.1.0D-9) PRINT 21, X,MAX
A4      0016700.       RETURN
A4      0016800.C
A4      0016900.19     FORMAT (' WARNING -- NOT ENOUGH TERMS TAKEN ')
A4      0017000.20     FORMAT (' FOR ',F8.4,' TERMS TAKEN = ',I2)
A4      0017100.21     FORMAT (' FOR ',F8.4,' ERROR = ',F15.9)
A4      0017200.       END
A5      0000100.       IMPLICIT REAL*8 (A-H,O-Z)
A5      0000200.       INTEGER*2  LR,LS,LT,IN
A5      0000300.       COMMON /FINAL/U,UDOT,V,VDOT,W,WDOT,AB
A5      0000400.       COMMON /SER1/ D1(16,16),D2(16,16),D3(16,16)
A5      0000500.       COMMON /SER2/ G1(16,16),G2(16,16),G3(16,16)
A5      0000600.       DIMENSION  O1(16,16),OA1(16,16,4),O2(16,16),OA2(16,16,4),O3(16,16)-
A5      0000700.      1,OA3(16,16,4),O4(16,16),OA4(16,16,4),OB1(16,16,4),OB2(16,16,4),   -
A5      0000800.      2 DB3(16,16,4),DB4(16,16,4),GB5(16,16,4),GB6(16,16,4),DA1(16,16,4),-
A5      0000900.      3 DA2(16,16,4),DA3(16,16,4),GA1(16,16,4),GA2(16,16,4),GA3(16,16,4),-
A5      0001000.      4 O5(16,16),O6(16,16),O7(16,16),OA(16,16),OB(16,16),OC(16,16),      -
A5      0001100.      5 OCA(16,16),OCAB(16,16),D4(16,16),D5(16,16),D6(16,16),DA(12,12),  -
A5      0001200.      6 DB(12,12),DX1(4,4),DX2(4,12),DX3(4,4),DX4(4,12),DC(12,4-
A5      0001300.      7), DX5(12,12),DX6(12,4),G4(16,16),G5(16,16),G6(16,16),U(16,4),V(16-
A5      0001400.      8,4),W(16,4),UDOT(16,4),VDOT(16,4),WDOT(16,4),UX(16,4),VX(16,4),   -
A5      0001500.      9 WX(16,4),B1(16,4),B2(16,4),B3(16,4),B4(16,4),B5(16,4),B6(16,4),  -
A5      0001600.      $ F1(16),F2(16),F3(16),F4(16),F5(16),F6(16),H1(16),H2(16),H3(16),H4-
A5      0001700.      $(16),H5(16),H6(16),R(16,4),S(16,4),T(16,4),REQ(64),SEQ(64),TEQ(64)-
A5      0001800.      $, LR(16,64),LS(12,64),LT(12,64),VAW(16),VBW(16),E3(16),E4(16),IN(1-
A5      0001900.      $6),FG(16),AB(4),X(82),OV1(16),OV2(16),OV3(16),OV4(16),E1(16,16),E2(16,16)
A5      0002000.       EQUIVALENCE  (R,REQ),(S,SEQ), (T,TEQ)
A5      0002100.       DATA C1,C2,C5,HR,HZ,HT,A,A1,A2,B / 5.0D-1,5.0D-1,6.666666666667   -
A5      0002200.      1 D-1, 2.5D-1,1.0D-1,5.235987756050-1,2.5D-1,5.0D-1,7.5D-1,1.0D0 /
A5      0002300.       DATA IN  /1,2,5,6,3,4,7,8,9,10,11,12,13,14,15,16 /
A5      0002400.       HT=HT/2.0D0
A5      0002500.       C4=0.D0
A5      0002600.       C3=C5
A5      0002700.       AB(1)=A
A5      0002800.       AB(2)=A1
A5      0002900.       AB(3)=A2
A5      0003000.       AB(4)=B
A5      0003100.       X(1)=1.25D0/(A*A)
A5      0003200.       X(2)=7.5D-1/(A*HR)
A5      0003300.       X(3)=7.5D-1/HR
A5      0003400.       X(4)=5.0D-1/(A*HT*HR)
A5      0003500.       X(5)=5.0D-1/(A*A*HT)
A5      0003600.       X(6)=5.0D-1/(HZ*HR)
A5      0003700.       X(7)=0.D0
A5      0003800.       X(8)= -X(5)
A5      0003900.       X(9)= 1.25D0/(A1*A1)
A5      0004000.       X(10)=3.75D-1/(A1*HR)
A5      0004100.       X(11)= 3.75D-1/HR
A5      0004200.       X(12)= 2.5D-1/(A1*HR*HT)
A5      0004300.       X(13)= 5.0D-1/(A1*A1*HT)
A5      0004400.       X(14)= 2.5D-1/(HZ*HR)
A5      0004500.       X(15)= -X(13)
A5      0004600.       X(16)= 1.25D0/(A2*A2)
```

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A5      0004700.      X(17)= 3.75D-1/(A2*HR)
A5      0004800.      X(18)= 2.5D-1/(A2*HT*HR)
A5      0004900.      X(19)= 5.0D-1/(A2*A2*HT)
A5      0005000.      X(20)= -X(19)
A5      0005100.      X(21)= 1.25D0/(B*B)
A5      0005200.      X(22)= 7.5D-1/(B*HR)
A5      0005300.      X(23)= 5.0D-1/(B*HR*HT)
A5      0005400.      X(24)= 5.0D-1/(B*B*HT)
A5      0005500.      X(25)= -X(24)
A5      0005600.      X(26)= -X(4)
A5      0005700.      X(27)= -X(12)
A5      0005800.      X(28)= -X(18)
A5      0005900.      X(29)= -X(23)
A5      0006000.      X(30)= -X(6)
A5      0006100.      X(31)= -X(14)
A5      0006200.      X(32)= 1.25D0/HT
A5      0006300.      X(33)= 2.5D-1*(2.0D0*A-HR)/(HR*HT)
A5      0006400.      X(34)= 7.5D-1*A/HT
A5      0006500.      X(35)= 5.0D-1*A/(HT*HZ)
A5      0006600.      X(36)= 7.5D-1*A1/HT
A5      0006700.      X(37)= 5.0D-1*A1/(HT*HZ)
A5      0006800.      X(38)= 6.25D-1/HT
A5      0006900.      X(39)= 1.25D-1*(2.0D0*A-HR)/(HR*HT)
A5      0007000.      X(40)= 3.75D-1*A/HT
A5      0007100.      X(41)= 2.5D-1*A/(HZ*HT)
A5      0007200.      X(42)= 3.75D-1*A1/HT
A5      0007300.      X(43)= 2.5D-1*A1/(HZ*HT)
A5      0007400.      X(44)= 3.75D-1*A2/HT
A5      0007500.      X(45)= 2.5D-1*A2/(HZ*HT)
A5      0007600.      X(46)= -1.25D-1*(2.0D0*B+HR)/(HR*HT)
A5      0007700.      X(47)= 3.75D-1*B/HT
A5      0007800.      X(48)= 2.5D-1*B/(HZ*HT)
A5      0007900.      X(49)= 7.5D-1/HZ
A5      0008000.      X(50)= 2.5D-1*(2.0D0*A-HR)/(A*HR*HZ)
A5      0008100.      X(51)= 7.5D-1/(A*HZ)
A5      0008200.      X(52)= 5.0D-1/(A*HZ*HT)
A5      0008300.      X(53)= 7.5D-1/(A1*HZ)
A5      0008400.      X(54)= 5.0D-1/(A1*HT*HZ)
A5      0008500.      X(55)= 7.5D-1/(A2*HZ)
A5      0008600.      X(56)= 5.0D-1/(A2*HZ*HT)
A5      0008700.      X(57)= -2.5D-1*(2.0D0*B+HR)/(HR*B*HZ)
A5      0008800.      X(58)= 7.5D-1/(B*HZ)
A5      0008900.      X(59)= 5.0D-1/(B*HZ*HT)
A5      0009000.      X(60)= 3.75D-1/HZ
A5      0009100.      X(61)= 3.75D-1/(A*HZ)
A5      0009200.      X(62)= 3.75D-1/(A1*HZ)
A5      0009300.      X(63)= 3.75D-1/(A2*HZ)
A5      0009400.      X(64)= 3.75D-1/(B*HZ)
A5      0009500.      X(65)= 2.5D-1/(A*HT*HZ)
A5      0009600.      X(66)= 2.5D-1/(A1*HZ*HT)
A5      0009700.      X(67)= 2.5D-1/(A2*HZ*HT)
A5      0009800.      X(68)= 2.5D-1/(B*HZ*HT)
A5      0009900.      X(69)= 1.25D-1*(2.0D0*A-HR)/(A*HR*HZ)
A5      0010000.      X(70)= -1.25D-1*(2.0D0*B+HR)/(B*HR*HZ)
```

```
A5      0010100.       X(71)=-X(41)
A5      0010200.       X(72)=-X(43)
A5      0010300.       X(73)=-X(45)
A5      0010400.       X(74)=-X(48)
A5      0010500.       X(75)=-X(52)
A5      0010600.       X(76)=-X(54)
A5      0010700.       X(77)=-X(56)
A5      0010800.       X(78)=-X(59)
A5      0010900.       X(79)=-X(65)
A5      0011000.       X(80)=-X(66)
A5      0011100.       X(81)=-X(67)
A5      0011200.       X(82)=-X(68)
A5      0011300.       READ (5,146) LR
A5      0011400.       READ (5,146) LS
A5      0011500.       READ (5,146) LT
A5      0011600.   146 FORMAT (40I2)
A5      0011700.       Y=HR
A5      0011800.       YY=HR
A5      0011900.       DO 2 J=1,16
A5      0012000.       DO 1 I=1,16
A5      0012100.       O1(I,J)=0.D0
A5      0012200.       O2(I,J)=0.D0
A5      0012300.       O3(I,J)=0.D0
A5      0012400.       O4(I,J)=0.D0
A5      0012500.       OA1(I,J,1)=0.D0
A5      0012600.       OA2(I,J,1)=0.D0
A5      0012700.       OA3(I,J,1)=0.D0
A5      0012800.       OA4(I,J,1)=0.D0
A5      0012900.       OB1(I,J,1)=0.D0
A5      0013000.     1 OB2(I,J,1)=0.D0
A5      0013100.       O1(J,J)=1.0D0
A5      0013200.       O4(J,J)=1.0D0
A5      0013300.       OA1(J,J,1)=1.D0
A5      0013400.       OA4(J,J,1)=1.D0
A5      0013500.     2 OB2(J,J,1)=1.D0
A5      0013600.       DO 11 I=2,4
A5      0013700.       YF= Y+HR
A5      0013800.       YYF=YY+HR
A5      0013900.       CALL RKGS(O1,O2,Y)
A5      0014000.       CALL RKGS(O3,O4,YY)
A5      0014100.       Y=YF
A5      0014200.       YY=YYF
A5      0014300.       DO 3 N=1,16
A5      0014400.       DO 3 M=1,16
A5      0014500.       OA1(M,N,I)=O1(M,N)
A5      0014600.       OA2(M,N,I)=O2(M,N)
A5      0014700.       OA3(M,N,I)=O3(M,N)
A5      0014800.       OA4(M,N,I)=O4(M,N)
A5      0014900.       O5(M,N)=O1(M,N)
A5      0015000.       O6(M,N)=0.D0
A5      0015100.     3 O7(M,N)=0.D0
A5      0015200.       CALL MATINV (O5,16)
A5      0015300.       DO 7 N=1,16
A5      0015400.       DO 7 M=1,16
```

```
      DO 6 J=1,16
      SUM=0.D0
      DO 5 L=1,16
    5 SUM= SUM+O5(J,L)*O3(L,N)
    6 O7(M,N)=O7(M,N)+O2(M,J)*SUM
    7 O7(M,N)=O4(M,N)-O7(M,N)
      CALL MATINV (O7,16)
      DO 9  N=1,16
      DO 9  M=1,16
      DO 9  J=1,16
      SUM=0.D0
      DO 8  L=1,16
    8 SUM=SUM+O3(J,L)*O7(L,N)
    9 O6(M,N)=O6(M,N)-O5(M,J)*SUM
      DO 10 N=1,16
      DO 10 M=1,16
      OB1(M,N,I)= O6(M,N)
   10 OB2(M,N,I)= O7(M,N)
   11 CONTINUE
      DO 12 N=1,16
      DO 12 M=1,16
      OA(M,N)= C1*O1(M,N)/B**2 -O2(M,N)
   12 OB(M,N)= C1*O3(M,N)/B**2 -O4(M,N)
      CALL MATINV (OA,16)
      DO 15 N=1,16
      DO 14 M=1,16
      OC(M,N)=0.D0
      DO 13 J=1,16
   13 OC(M,N)=OC(M,N)+OA(M,J)*OB(J,N)
   14 OC(M,N)= C1*OC(M,N)/A**2
   15 OC(N,N)= OC(N,N)+1.0D0
      CALL MATINV (OC,16)
      DO 16  N=1,16
      DO 16  M=1,16
      OCA(M,N)=0.D0
      DO 16  J=1,16
   16 OCA(M,N)=OCA(M,N)+ OC(M,J)*OA(J,N)
      DO 17  N=1,16
      DO 17  M=1,16
      OCAB(M,N)=0.D0
      DO 17  J=1,16
   17 OCAB(M,N)=OCAB(M,N)+OCA(M,J)*OB(J,N)
      DO 18  N=1,16
      DO 19  M=1,16
      O1(M,N)= C2*OCA(M,N)
      O2(M,N)= C2*OCAB(M,N)
      O5(M,N)= C1*OC(M,N)/A**2
      O6(M,N)= C1*OCAB(M,N)/A**2
      O7(M,N)=O6(M,N)
      O4(M,N)= C1*O1(M,N)/A**2
   19 O3(M,N)= C1*O2(M,N)/A**2
      OV3(N)=0.D0
      OV4(N)=0.D0
      OV1(N)=0.D0
```

```
   18 O3(N,N)=O3(N,N)-C2
      DO 20 N=1,16
      DO 20 M=1,16
      OV3(M)=OV3(M)+OCAB(M,N)*C4
      OV4(M)=OV4(M)+O7(M,N)*C4
   20 OV1(M)=OV1(M)+OCA(M,N)*C3
      DO 300 M=1,16
  300 OV4(M)=C4-OV4(M)
      DO 21 M=1,16
   21 OV2(M)= C1    *OV1(M)/A**2
      CALL MAPOWR
      DO 23 L=1,16
      DO 22 J=1,16
      DA1(J,L,1)=0.D0
      DA2(J,L,1)=0.D0
      DA3(J,L,1)=0.D0
      DB3(J,L,1)=0.D0
   22 DB4(J,L,1)=0.D0
      DA1(L,L,1)=1.D0
   23 DB4(L,L,1)=1.D0
      DO 32 I=1,3
      Y=HT* DFLOAT(I)
      IF (I.GT.1) GO TO 401
      CALL MASER(Y)
      GO TO 402
  401 DO 403 N=1,16
      DO 403 M=1,16
      D1(M,N)=0.D0
      D2(M,N)=0.D0
      DO 403 L=1,16
      D1(M,N)=D1(M,N)+DA1(M,L,2)*DA1(L,N,I)+DA3(M,L,2)*DA2(L,N,I)
  403 D2(M,N)=D2(M,N)+DA2(M,L,2)*DA1(L,N,I)+DA1(M,L,2)*DA2(L,N,I)
      CALL MXTRA
  402 DO 24 N=1,16
      DO 24 M=1,16
      DA1(M,N,I+1)= D1(M,N)
      DA2(M,N,I+1)= D2(M,N)
   24 DA3(M,N,I+1)= D3(M,N)
      DO 25 N=1,16
      DO 25 M=1,16
      D5(M,N)=0.D0
   25 D4(M,N)=D1(M,N)
      CALL MATINV(D4,16)
      DO 28 N=1,16
      DO 28 M=1,16
      DO 27 J=1,16
      SUM=0.D0
      DO 26 L=1,16
   26 SUM= SUM+ D4(J,L)*D2(L,N)
   27 D5(M,N)= D5(M,N)+ D3(M,J)*SUM
   28 D5(M,N)= D1(M,N)-D5(M,N)
      CALL MATINV (D5,16)
      DO 30 N=1,16
      DO 30 M=1,16
```

```
      D6(M,N)=0.D0
      DO 30 J=1,16
      SUM=0.D0
      DO 29 L=1,16
   29 SUM=SUM+D2(J,L)*D5(L,N)
   30 D6(M,N)=D6(M,N)-D4(M,J)*SUM
      DO 31 N=1,16
      DO 31 M=1,16
      DB3(M,N,I+1)= D6(M,N)
   31 DB4(M,N,I+1)= D5(M,N)
   32 CONTINUE
      DO 600 N=1,16
      N1=IN(N)
      DO 600 M=1,16
      M1=IN(M)
      E1(M,N)=D1(M1,N1)
  600 E2(M,N)=D2(M1,N1)
      DO 33 N=1,4
      DO 33 M=1,4
   33 DX1(M,N)= E1(M,N)
      CALL MATINV(DX1,4)
      DO 34 N=1,12
      DO 34 M=1,4
      DX4(M,N)= 0.D0
      DX2(M,N)=0.D0
      DO 34 L=1,4
      DX4(M,N)= DX4(M,N)+ DX1(M,L)*E2(L,N+4)
   34 DX2(M,N)= DX2(M,N)+ DX1(M,L)*E1(L,N+4)
      DO 35 N=1,4
      DO 35 M=1,4
      DX3(M,N)=0.D0
      DO 35 L=1,4
   35 DX3(M,N)= DX3(M,N)+DX1(M,L)* E2(L,N)
      DO 36 N=1,12
      DO 36 M=1,12
      DA(M,N)=0.D0
      DB(M,N)=0.D0
      DO 37  L=1,4
      DA(M,N)= DA(M,N)+E1(M+4,L)*DX4(L,N)
   37 DB(M,N)= DB(M,N)+E1(M+4,L)*DX2(L,N)
      DA(M,N)= E2(M+4,N+4)-DA(M,N)
   36 DB(M,N)= E1(M+4,N+4)-DB(M,N)
      CALL MATINV(DA,12)
      DO 38  N=1,4
      DO 38  M=1,12
      DC(M,N)= 0.D0
      DO 39  L=1,4
   39 DC(M,N)= DC(M,N)+E1(M+4,L)* DX3(L,N)
   38 DC(M,N)= E2(M+4,N)-DC(M,N)
      DO 40  N=1,12
      DO 40  M=1,12
      DX5(M,N)= 0.D0
      DO 40  L=1,12
   40 DX5(M,N)= DX5(M,N)+DA(M,L)*DB(L,N)
```

```
A5      0031700.        DO 41  N=1,4
A5      0031800.        DO 41  M=1,12
A5      0031900.        DX6(M,N)= 0.DO
A5      0032000.        DO 41  L=1,12
A5      0032100.     41 DX6(M,N)= DX6(M,N) + DA(M,L)*DC(L,N)
A5      0032200.        CALL MAPOWS
A5      0032300.        DO 43  L=1,16
A5      0032400.        DO 42  J=1,16
A5      0032500.        GA1(J,L,1)= 0.DO
A5      0032600.        GA2(J,L,1)= 0.DO
A5      0032700.        GA3(J,L,1)= 0.DO
A5      0032800.        GB5(J,L,1)= 0.DO
A5      0032900.     42 GB6(J,L,1)= 0.DO
A5      0033000.        GA1(L,L,1)= 1.DO
A5      0033100.     43 GB6(L,L,1)= 1.DO
A5      0033200.        DO 52  I=1,3
A5      0033300.        Y = HZ*DFLOAT(I)
A5      0033400.        CALL MASES(Y)
A5      0033500.        DO 44  N=1,16
A5      0033600.        DO 44  M=1,16
A5      0033700.        GA1(M,N,I+1)= G1(M,N)
A5      0033800.        GA2(M,N,I+1)=G2(M,N)
A5      0033900.     44 GA3(M,N,I+1)= G3(M,N)
A5      0034000.        DO 45  N=1,16
A5      0034100.        DO 45  M=1,16
A5      0034200.        G5(M,N)= 0.DO
A5      0034300.     45 G4(M,N)= G1(M,N)
A5      0034400.        CALL MATINV(G4,16)
A5      0034500.        DO 48  N=1,16
A5      0034600.        DO 48  M=1,16
A5      0034700.        DO 47  J=1,16
A5      0034800.        SUM= 0.DO
A5      0034900.        DO 46  L=1,16
A5      0035000.     46 SUM= SUM+G4(J,L)*G2(L,N)
A5      0035100.     47 G5(M,N)= G5(M,N)+G3(M,J)*SUM
A5      0035200.     48 G5(M,N)= G1(M,N)-G5(M,N)
A5      0035300.        CALL MATINV(G5,16)
A5      0035400.        DO 50  N=1,16
A5      0035500.        DO 50  M=1,16
A5      0035600.        G6(M,N)= 0.DO
A5      0035700.        DO 50  J=1,16
A5      0035800.        SUM= 0.DO
A5      0035900.        DO 49  L=1,16
A5      0036000.     49 SUM= SUM+G2(J,L)*G5(L,N)
A5      0036100.     50 G6(M,N)= G6(M,N)-G4(M,J)*SUM
A5      0036200.        DO 51  N=1,16
A5      0036300.        DO 51  M=1,16
A5      0036400.        GB5(M,N,I+1)= G6(M,N)
A5      0036500.     51 GB6(M,N,I+1)= G5(M,N)
A5      0036600.     52 CONTINUE
A5      0036700.        CALL MATINV(G1,16)
A5      0036800.        DO 53 N=1,16
A5      0036900.        DO 53 M=1,16
A5      0037000.        G2(M,N)=0.DO
```

```
A5        DO 53 L=1,16
A5     53 G2(M,N)= G2(M,N)+G1(M,L)*G3(L,N)
A5        KTR=0
A5        DO 100  N=1,4
A5        DO 100  M=1,16
A5        WDOT(M,N)= 0.D0
A5        VDOT(M,N)= 0.D0
A5    100 R(M,N)= 0.D0
A5    101 KTR= KTR+1
A5        DO 102  N=1,4
A5        DO 102  M=1,16
A5        WX(M,N)= W(M,N)
A5        VX(M,N)= V(M,N)
A5        UX(M,N)= U(M,N)
A5        U(M,N)= 0.D0
A5        UDOT(M,N)= 0.D0
A5        B1(M,N)= 0.D0
A5    102 B2(M,N)= 0.D0
A5        DO 103  M=1,16
A5        F1(M)=0.D0
A5    103 F2(M)=0.D0
A5        DO 104  M=2,4
A5        DO 104  J=1,16
A5        SUM1= 0.D0
A5        SUM2= 0.D0
A5        DO 105  L=1,16
A5        SUM1 = SUM1+OB1(J,L,M-1)*R(L,M-1)+OB1(J,L,M)*R(L,M)
A5    105 SUM2 = SUM2+ OB2(J,L,M-1)*R(L,M-1)+OB2(J,L,M)*R(L,M)
A5        B1(J,M)= B1(J,M-1)+HR*SUM1/2.0D0
A5    104 B2(J,M)= B2(J,M-1)+HR*SUM2/2.0D0
A5        KNT=0
A5        DO 106  M=1,4
A5        DO 106  N=1,4
A5        KNT= KNT+1
A5        VAW(KNT)= VDOT(M,N)/A + WDOT(4*N-3,M)
A5    106 VBW(KNT)= VDOT(M+12,N)/B + WDOT(4*N,M)
A5        DO 107  L=1,16
A5        DO 107  J=1,16
A5        F1(J)= F1(J)+O1(J,L)*VBW(L)+ O2(J,L)*VAW(L)-OC(J,L)*B1(L,4)
A5       1 -OCAB(J,L) * B2(L,4)
A5    107 F2(J)= F2(J) + O3(J,L)*VAW(L) + O4(J,L) *VBW(L)-O5(J,L)*B1(L,4)
A5       1 - O6(J,L)*B2(L,4)
A5        DO 108  L=1,16
A5        F1(L)= F1(L) - OV1(L)-OV3(L)
A5    108 F2(L)= F2(L) - OV2(L)+OV4(L)
A5        Y=HR
A5        YY=Y*Y
A5        DO 109  I=1,4
A5        DO 110  J=1,16
A5        H1(J)= F1(J) + B1(J,I)
A5    110 H2(J)= F2(J) + B2(J,I)
A5        DO 111  N=1,16
A5        DO 111  M=1,16
A5        UDOT(M,I)= UDOT(M,I) + (OA2(M,N,I)-OA1(M,N,I)/YY)*H1(N) +
```

```
      1 (OA4(M,N,I)  - OA3(M,N,I)/YY) *H2(N)
  111 U(M,I)= U(M,I) + (OA1(M,N,I) * H1(N) + OA3(M,N,I)*H2(N))/Y
      Y=Y+HR
  109 YY=Y*Y
      WRITE(6,112)
  112 FORMAT( ///' U-MATRIX ')
      DO 901 I=1,16
  901 WRITE(6,114) (U(I,J),J=1,4)
  114 FORMAT (4(1X,F15.8))
      WRITE(6,115)
  115 FORMAT (/// ' UDOT-MATRIX ')
      DO 902 I=1,16
  902 WRITE (6,114) (UDOT(I,J),J=1,4)
      DO 113  I=1,64
      L1 = LS(1,I)
      L2 = LS(2,I)
      L3 = LS(3,I)
      L4 = LS(4,I)
      L5 = LS(5,I)
      L6 = LS(6,I)
      L7 = LS(7,I)
      L8 = LS(8,I)
      L9 = LS(9,I)
      L10  = LS(10,I)
      L11  = LS(11,I)
      L12  = LS(12,I)
  113 SEQ(I)= (X(L1)+X(L2))*(U(L3,L4)-U(L5,L6)) + X(L7)*(UDOT(L3,L4)-    -
     1 UDOT(L5,L6))+X(L7)*(WDOT(L8,L9)-WDOT(L10,L11)) +                   -
     2 X(L12) * (W(L8,L9)-W(L10,L11))
      DO 116  I=1,4
      DO 116  J=1,16
      V(J,I)= 0.D0
      VDOT(J,I)= 0.D0
      B3(J,I)= 0.D0
  116 B4(J,I)= 0.D0
      DO 117  I=1,16
      F3(I)= 0.D0
  117 F4(I)= 0.D0
      DO 118  M=2,4
      DO 118  J=1,16
      SUM1 = 0.D0
      SUM2 = 0.D0
      DO 119  L=1,16
      SUM1 = SUM1 + DB3(J,L,M-1)*S(L,M-1) + DB3(J,L,M)*S(L,M)
  119 SUM2 = SUM2 + DB4(J,L,M-1)*S(L,M-1)+DB4(J,L,M)*S(L,M)
      B3(J,M)= B3(J,M-1)+HT*SUM1/2.0D0
  118 B4(J,M)= B4(J,M-1)+HT*SUM2/2.0D0
      F4(1)=C2*(UDOT(1,1)+WDOT(1,1))/4.0D0 +U(1,1)-2.5D-1*C4
      F4(2)=  C2*(UDOT(5,1) + WDOT(1,2))/4.0D0 +U(5,1)-2.5D-1*C4
      F4(3)=  C2*(UDOT(1,2) + WDOT(2,1))/2.0D0 +U(1,2)-5.0D-1*C4
      F4(4)=  C2*(UDOT(5,2) + WDOT(2,2))/2.0D0 +U(5,2)-5.0D-1*C4
      DO 700 I=1,16
      J=IN(I)
      E3(I)=B3(J,4)
```

```
A5    700 E4(I)=B4(J,4)
A5        DO 120  I=1,12
A5        SUM1 = 0.D0
A5        DO 121  J=1,12
A5    121 SUM1 = SUM1 + DX5(I,J) * E3(J+4)
A5        SUM2 = 0.D0
A5        DO 122 J=1,4
A5    122 SUM2 = SUM2 + DX6(I,J) * (-F4(J) + E4(J))
A5    120 F4(I+4)= SUM1 + SUM2+E4(I+4)
A5        DO 123 I=1,4
A5        SUM1 = 0.D0
A5        SUM2 = 0.D0
A5        DO 124  J=1,12
A5    124 SUM1 = SUM1 +DX2(I,J)*E3(J+4)+DX4(I,J)*(-F4(J+4)+E4(J+4))
A5        DO 125 J=1,4
A5    125 SUM2 = SUM2 + DX3(I,J) * (-F4(J)+E4(J))
A5    123 F3(I)= SUM1 + SUM2 + E3(I)
A5        DO 128  I=1,16
A5        J= IN(I)
A5        E3(I)= F3(J)
A5    128 E4(I)= F4(J)
A5        DO 126  I=1,4
A5        DO 127  J=1,16
A5        H3(J)= B3(J,I) - E3(J)
A5    127 H4(J)= B4(J,I) - E4(J)
A5        DO 126  N=1,16
A5        DO 126  M=1,16
A5        VDOT(M,I)= VDOT(M,I) + DA3(M,N,I)*H3(N)+DA1(M,N,I)*H4(N)
A5    126 V(M,I)= V(M,I) + DA1(M,N,I) *H3(N) + DA2(M,N,I)*H4(N)
A5        DO 200 I=1,16
A5    200 V(I,4)=0.D0
A5        WRITE(6,129)
A5    129 FORMAT (///' V-MATRIX ')
A5        DO 903 I=1,16
A5    903 WRITE (6,114) (V(I,J),J=1,4)
A5        WRITE(6,130)
A5    130 FORMAT (///' V-DOT MATRIX ')
A5       DO 904 I=1,16
A5    904 WRITE (6,114) (VDOT(I,J),J=1,4)
A5        DO 131  I=1,64
A5        L1 = LT(1,I)
A5        L2 = LT(2,I)
A5        L3 = LT(3,I)
A5        L4 = LT(4,I)
A5        L5 = LT(5,I)
A5        L6 = LT(6,I)
A5        L7 = LT(7,I)
A5        L8 = LT(8,I)
A5        L9 = LT(9,I)
A5        L10 =LT(10,I)
A5        L11 =LT(11,I)
A5        L12 =LT(12,I)
A5    131 TEQ(I)= (X(L7)+X(L2)) *(U(L3,L4)-U(L5,L6)) + X(L1)*          -
A5       1 (UDOT(L3,L4) -UDOT(L5,L6))+X(L7)*(VDOT(L8,L9)-VDOT(L10,L11))   -
```

```
A5        2 + X(L12)*(V(L8,L9)-V(L10,L11))
A5          DO 907 M=1,16
A5      907 TEQ(M)=0.D0
A5          DO 132  I=1,4
A5          DO 132  J=1,16
A5          W(J,I)= 0.D0
A5          WDOT(J,I)= 0.D0
A5          B5(J,I)= 0.D0
A5      132 B6(J,I)= 0.D0
A5          DO 133 I=1,16
A5          F5(I)= 0.D0
A5      133 F6(I)= 0.D0
A5          DO 134  M=2,4
A5          DO 134  J=1,16
A5          SUM1 = 0.D0
A5          SUM2 = 0.D0
A5          DO 135  L=1,16
A5          SUM1 = SUM1 + GB5(J,L,M-1) *T(L,M-1)+GB5(J,L,M)*T(L,M)
A5      135 SUM2 = SUM2+GB6(J,L,M-1)*T(L,M-1) + GB6(J,L,M)*T(L,M)
A5          B5(J,M)= B5(J,M-1) + HZ*SUM1/2.0D0
A5      134 B6(J,M)= B6(J,M-1)+HZ*SUM2/2.0D0
A5          KNT=0
A5          DO 136 M=1,4
A5          DO 136 N=1,4
A5          KNT=KNT+1
A5      136 FG(KNT)=-C2*(UDOT(M+12,N)+(U(M+12,N)+ VDOT(4*N,M))/AB(N))
A5          DO 137  M=1,16
A5          SUM=0.D0
A5          DO 138  L=1,16
A5      138 SUM= SUM+ G1(M,L)*FG(L)- G2(M,L)*B5(L,4)
A5      137 F6(M)= SUM-B6(M,4)
A5          DO 139  I=1,4
A5          DO 140  J=1,16
A5          H5(J)= F5(J)+ B5(J,I)
A5      140 H6(J)= F6(J)+ B6(J,I)
A5          DO 139  N=1,16
A5          DO 139  M=1,16
A5          WDOT (M,I)= WDOT(M,I)+ GA3(M,N,I)*H5(N)+GA1(M,N,I)*H6(N)
A5      139 W(M,I)= W(M,I)+ GA1(M,N,I)*H5(N)+ GA2(M,N,I)*H6(N)
A5          WRITE (6,141)
A5      141 FORMAT (/// ' W-MATRIX')
A5          DO 905 I=1,16
A5      905 WRITE (6,114) (W(I,J),J=1,4)
A5          WRITE (6,142)
A5      142 FORMAT(///' WDOT-MATRIX ')
A5          DO 906 I=1,16
A5      906 WRITE (6,114) (WDOT(I,J),J=1,4)
A5          DO 143 I=1,64
A5          L1=LR(1,I)
A5          L2=LR(2,I)
A5          L3=LR(3,I)
A5          L4=LR(4,I)
A5          L5=LR(5,I)
A5          L6=LR(6,I)
```

```
A5      0058700.         L7=LR(7,I)
A5      0058800.         L8=LR(8,I)
A5      0058900.         L9=LR(9,I)
A5      0059000.         L10=LR(10,I)
A5      0059100.         L11=LR(11,I)
A5      0059200.         L12=LR(12,I)
A5      0059300.         L13=LR(13,I)
A5      0059400.         L14=LR(14,I)
A5      0059500.         L15=LR(15,I)
A5      0059600.         L16=LR(16,I)
A5      0059700.     143 REQ(I)= X(L1)*VDOT(L2,L3)+X(L4)*(VDOT(L5,L6)-VDOT(L7,L8))         -
A5      0059800.        1 +X(L9)*(WDOT(L10,L11)-WDOT(L12,L13))+X(L14)* (V(L5,L6)-V(L7,L8)) -
A5      0059900.        2 +X(L15)*V(L2,L3)+X(L16)*(W(L10,L11)-W(L12,L13))
A5      0060000.         PRINT  91,KTR
A5      0060100.      91 FORMAT(1X,I3)
A5      0060200.         IF(DABS(U(1,1)).GT.5.0D0 ) STOP
A5      0060300.         DO 144 I=1,4
A5      0060400.         DO 144 J=1,16
A5      0060500.         IF(DABS(W(J,I)-WX(J,I)).GT.1.0D-6) GO TO 101
A5      0060600.         IF(DABS(V(J,I)-VX(J,I)).GT.1.0D-6) GO TO 101
A5      0060700.         IF(DABS(U(J,I)-UX(J,I)).GT.1.0D-6) GO TO 101
A5      0060800.     144 CONTINUE
A5      0060900.         WRITE (6,145) KTR
A5      0061000.     145 FORMAT (///' ITERATION NUMBER  ',I2,///)
A5      0061100.         CALL STRESS
A5      0061200.         STOP
A5      0061300.         END
A6      0000100.         SUBROUTINE MAPOWS
A6      0000200.         IMPLICIT REAL*8 (A-H,O-Z)
A6      0000300.         DOUBLE PRECISION K,MAX
A6      0000400.         DIMENSION XK(16,16), YK(16,16), W(16,16), T(16,16), DXX(16,16), DYK(16,16,40)
A6      0000500.         COMMON /SER2/ G1(16,16),G2(16,16),G3(16,16)
A6      0000600.         DATA NZ,K,MSER /16,2.5D-1,40 /
A6      0000700.         DATA A,B,A1,A2,HT,HR /2.5D-1,1.0D0,5.0D-1,7.5D-1,5.23598775605D-1,-
A6      0000800.        1 2.5D-1/
A6      0000900.         HT=HT/2.0D0
A6      0001000.         DO 1 M=1,NZ
A6      0001100.         DO 1 L=1,NZ
A6      0001200.1        XK(L,M)=0.D0
A6      0001300.         E9= K/HR**2
A6      0001400.         E10= 2.0D0*E9
A6      0001500.         E11= K/(A**2*HT**2)
A6      0001600.         E12=2.0D0*E11
A6      0001700.         E13=K/(A1**2*HT**2)
A6      0001800.         E14=2.0D0*E13
A6      0001900.         E15=K/(A2**2*HT**2)
A6      0002000.         E16=2.0D0*E15
A6      0002100.         E17=K/(B**2*HT**2)
A6      0002200.         E18= 2.0D0*E17
A6      0002300.         E1= E10+E12
A6      0002400.         E2=E10+E14
A6      0002500.         E3=E10+E16
A6      0002600.         E4=E10+E18
A6      0002700.         E19=K/(2.0D0*HR*A1)
```

```
A6      E20=K/(2.0D0*HR*A2)
A6      E5= E9+E19
A6      E6=E9-E19
A6      E7=E9+E20
A6      E8=E9-E20
A6      XK(1,1)=E1
A6      XK(2,1)= -E6
A6      XK(5,1)= -E11
A6      XK(1,2)=-E10
A6      XK(2,2)= E2
A6      XK(3,2)= -E8
A6      XK(6,2)= -E13
A6      XK(2,3)= -E5
A6      XK(3,3)= E3
A6      XK(4,3)= -E10
A6      XK(7,3)= -E15
A6      XK(3,4)= -E7
A6      XK(4,4)= E4
A6      XK(8,4)= -E17
A6      XK(1,5)= -E12
A6      XK(5,5)= E1
A6      XK(6,5)= -E6
A6      XK(9,5)= -E11
A6      XK(2,6)= -E14
A6      XK(5,6)= -E10
A6      XK(6,6)= E2
A6      XK(7,6)= -E8
A6      XK(10,6)= -E13
A6      XK(3,7)= -E16
A6      XK(6,7)= -E5
A6      XK(7,7)= E3
A6      XK(8,7)= -E10
A6      XK(11,7)= -E15
A6      XK(4,8)= -E18
A6      XK(7,8)= -E7
A6      XK(8,8)= E4
A6      XK(12,8)= -E17
A6      XK(5,9)= -E11
A6      XK(9,9)= E1
A6      XK(10,9)= -E6
A6      XK(13,9)= -E12
A6      XK(6,10)= -E13
A6      XK(9,10)= -E10
A6      XK(10,10)= E2
A6      XK(11,10)=-E8
A6      XK(14,10)= -E14
A6      XK(7,11)= -E15
A6      XK(10,11)= -E5
A6      XK(11,11)= E3
A6      XK(12,11)= -E10
A6      XK(15,11)= -E16
A6      XK(8,12)= -E17
A6      XK(11,12)= -E7
A6      XK(12,12)= E4
```

```
A6      0008200.       XK(16,12)= -E18
A6      0008300.       XK(9,13)= -E11
A6      0008400.       XK(13,13)= E1
A6      0008500.       XK(14,13)= -E6
A6      0008600.       XK(10,14) =-E13
A6      0008700.       XK(13,14) =-E10
A6      0008800.       XK(14,14) = E2
A6      0008900.       XK(15,14) =-E8
A6      0009000.       XK(11,15) =-E15
A6      0009100.       XK(14,15) =-E5
A6      0009200.       XK(15,15) = E3
A6      0009300.       XK(16,15) =-E10
A6      0009400.       XK(12,16) =-E17
A6      0009500.       XK(15,16) =-E7
A6      0009600.       XK(16,16) = E4
A6      0009700.       DO 4 M=1,NZ
A6      0009800.       DO 3 L=1,NZ
A6      0009900.3      YK(L,M)=0.D0
A6      0010000.4      YK(M,M)=1.0D0
A6      0010100.       TAX=0.D0
A6      0010200.       DO 30 N=1,16
A6      0010300.       DO 30 M=1,16
A6      0010400.    30 TAX=DMAX1(TAX,DABS(XK(M,N)))
A6      0010500.       DO 5 N=1,NZ
A6      0010600.       DO 5 M=1,NZ
A6      0010700.       T(M,N)=XK(M,N)
A6      0010800.5      XK(M,N)=XK(M,N)/TAX
A6      0010900.       DO 9 J=1,MSER
A6      0011000.       DO 6 N=1,NZ
A6      0011100.       DO 6 M=1,NZ
A6      0011200.       W(M,N)=0.D0
A6      0011300.       DO 6 L=1,NZ
A6      0011400.6      W(M,N)=W(M,N)+YK(M,L)*XK(L,N)
A6      0011500.       DO 7 N=1,NZ
A6      0011600.       DO 7 M=1,NZ
A6      0011700.7      YK(M,N)=W(M,N)
A6      0011800.       DO 8 N=1,NZ
A6      0011900.       DO 8 M=1,NZ
A6      0012000.8      DYK(M,N,J)=YK(M,N)
A6      0012100.9      CONTINUE
A6      0012200.       RETURN
A6      0012300.       ENTRY MASES(X)
A6      0012400.       DO 11 N=1,NZ
A6      0012500.       DO 10 M=1,NZ
A6      0012600.       G1(M,N)=0.D0
A6      0012700.10     G2(M,N)=0.D0
A6      0012800.       G1(N,N)=1.0D0
A6      0012900.11     G2(N,N)=X
A6      0013000.       C4=1.0D0
A6      0013100.       C5=X
A6      0013200.       DO 13 J=1,MSER
A6      0013300.       Y2=2*J*(2*J-1)
A6      0013400.       Y3=2*J*(2*J+1)
A6      0013500.       C4=C4*TAX*X*X/Y2
```

```
A6      0013600.        C5=C5*TAX*X*X/Y3
A6      0013700.        IFLAG=0
A6      0013800.        DO 12 N=1,NZ
A6      0013900.        DO 12 M=1,NZ
A6      0014000.        TEMP=C4*DYK(M,N,J)
A6      0014100.        IF (TEMP.GT.1.0D-16) IFLAG=1
A6      0014200.        G1(M,N)=G1(M,N)+TEMP
A6      0014300.12      G2(M,N)=G2(M,N)+C5*DYK(M,N,J)
A6      0014400.        IF (IFLAG.EQ.0) GO TO 14
A6      0014500.13      CONTINUE
A6      0014600.        PRINT 19
A6      0014700.14      PRINT 20, X,J
A6      0014800.        DO 15 N=1,NZ
A6      0014900.        DO 15 M=1,NZ
A6      0015000.        G3(M,N)=0.D0
A6      0015100.        DO 15 L=1,NZ
A6      0015200.15      G3(M,N)=G3(M,N)+T(M,L)*G2(L,N)
A6      0015300.        DO 17 N=1,NZ
A6      0015400.        DO 16 M=1,NZ
A6      0015500.        DXX(M,N)=0.D0
A6      0015600.        DO 16 L=1,NZ
A6      0015700.16      DXX(M,N)=DXX(M,N)+G1(M,L)*G1(L,N)-G3(M,L)*G2(L,N)
A6      0015800.17      DXX(N,N)=DXX(N,N)-1.0D0
A6      0015900.        MAX=0.D0
A6      0016000.        DO 18 N=1,NZ
A6      0016100.        DO 18 M=1,NZ
A6      0016200.18      MAX=DMAX1(DXX(M,N),MAX)
A6      0016300.        IF (MAX.GT.1.0D-9) PRINT 21, X,MAX
A6      0016400.        RETURN
A6      0016500.C
A6      0016600.19      FORMAT (' WARNING -- NOT ENOUGH TERMS TAKEN ')
A6      0016700.20      FORMAT (' FOR ',F8.4,' TERMS TAKEN = ',I2)
A6      0016800.21      FORMAT (' FOR ',F8.4,' ERROR = ',F15.9)
A6      0016900.        END
A7      0000100.        SUBROUTINE STRESS
A7      0000200.        IMPLICIT REAL*8 (A-H,O-Z)
A7      0000300.        COMMON /FINAL/U,UDOT,V,VDOT,W,WDOT,AB
A7      0000400.        DIMENSION U(16,4),AB(4),UDOT(16,4),V(16,4),VDOT(16,4),W(16,4),WDOT(16,4)
A7      0000500.        DIMENSION ST(16,4),SR(16,4),SZ(16,4)
A7      0000600.        DATA CI,XL /1.5D0,7.5D-1/
A7      0000700.    114 FORMAT(4(1X,F15.8))
A7      0000800.        DO 950 J=1,4
A7      0000900.        SR(1,J)=CI*UDOT(1,J)+XL*((U(1,J)+VDOT(4*J-3,1))/AB(J)+WDOT(J,1))
A7      0001000.        SR(2,J)=CI*UDOT(2,J)+XL*((U(2,J)+VDOT(4*J-3,2))/AB(J)+WDOT(J+4,1))
A7      0001100.        SR(3,J)=CI*UDOT(3,J)+XL*((U(3,J)+VDOT(4*J-3,3))/AB(J)+WDOT(J+8,1))
A7      0001200.        SR(4,J)=CI*UDOT(4,J)+XL*((U(4,J)+VDOT(4*J-3,4))/AB(J)+WDOT(J+12,1))
A7      0001300.        SR(5,J)=CI*UDOT(5,J)+XL*((U(5,J)+VDOT(4*J-2,1))/AB(J)+WDOT(J,2))
A7      0001400.        SR(6,J)=CI*UDOT(6,J)+XL*((U(6,J)+VDOT(4*J-2,2))/AB(J)+WDOT(J+4,2))
A7      0001500.        SR(7,J)=CI*UDOT(7,J)+XL*((U(7,J)+VDOT(4*J-2,3))/AB(J)+WDOT(J+8,2))
A7      0001600.        SR(8,J)=CI*UDOT(8,J)+XL*((U(8,J)+VDOT(4*J-2,4))/AB(J)+WDOT(J+12,2))
A7      0001700.        SR(9,J)=CI*UDOT(9,J)+XL*((U(9,J)+VDOT(4*J-1,1))/AB(J)+WDOT(J,3))
A7      0001800.        SR(10,J)=CI*UDOT(10,J)+XL*((U(10,J)+VDOT(4*J-1,2))/AB(J)+WDOT(J+4,3))
A7      0001900.        SR(11,J)=CI*UDOT(11,J)+XL*((U(11,J)+VDOT(4*J-1,3))/AB(J)+WDOT(J+8,3))
A7      0002000.        SR(12,J)=CI*UDOT(12,J)+XL*((U(12,J)+VDOT(4*J-1,4))/AB(J)+WDOT(J+12,3))
```

```
      SR(13,J)=CI*UDOT(13,J)+XL*((U(13,J)+VDOT(4*J,1))/AB(J)+WDOT(J,4))
      SR(14,J)=CI*UDOT(14,J)+XL*((U(14,J)+VDOT(4*J,2))/AB(J)+WDOT(J+4,4))
      SR(15,J)=CI*UDOT(15,J)+XL*((U(15,J)+VDOT(4*J,3))/AB(J)+WDOT(J+8,4))
      SR(16,J)=CI*UDOT(16,J)+XL*((U(16,J)+VDOT(4*J,4))/AB(J)+WDOT(J+12,4))
      ST(1,J)=CI*(VDOT(4*J-3,1)+U(1,J))/AB(J)+XL*(UDOT(1,J)+WDOT(J,1))
      ST(2,J)=CI*(VDOT(4*J-3,2)+U(2,J))/AB(J)+XL*(UDOT(2,J)+WDOT(J+4,1))
      ST(3,J)=CI*(VDOT(4*J-3,3)+U(3,J))/AB(J)+XL*(UDOT(3,J)+WDOT(J+8,1))
      ST(4,J)=CI*(VDOT(4*J-3,4)+U(4,J))/AB(J)+XL*(UDOT(4,J)+WDOT(J+12,1))
      ST(5,J)=CI*(VDOT(4*J-2,1)+U(5,J))/AB(J)+XL*(UDOT(5,J)+WDOT(J,2))
      ST(6,J)=CI*(VDOT(4*J-2,2)+U(6,J))/AB(J)+XL*(UDOT(6,J)+WDOT(J+4,2))
      ST(7,J)=CI*(VDOT(4*J-2,3)+U(7,J))/AB(J)+XL*(UDOT(7,J)+WDOT(J+8,2))
      ST(8,J)=CI*(VDOT(4*J-2,4)+U(8,J))/AB(J)+XL*(UDOT(8,J)+WDOT(J+12,2))
      ST(9,J)=CI*(VDOT(4*J-1,1)+U(9,J))/AB(J)+XL*(UDOT(9,J)+WDOT(J,3))
      ST(10,J)=CI*(VDOT(4*J-1,2)+U(10,J))/AB(J)+XL*(UDOT(10,J)+WDOT(J+4,3))
      ST(11,J)=CI*(VDOT(4*J-1,3)+U(11,J))/AB(J)+XL*(UDOT(11,J)+WDOT(J+8,3))
      ST(12,J)=CI*(VDOT(4*J-1,4)+U(12,J))/AB(J)+XL*(UDOT(12,J)+WDOT(J+12,3))
      ST(13,J)=CI*(VDOT(4*J,1)+U(13,J))/AB(J)+XL*(UDOT(13,J)+WDOT(J,4))
      ST(14,J)=CI*(VDOT(4*J,2)+U(14,J))/AB(J)+XL*(UDOT(14,J)+WDOT(J+4,4))
      ST(15,J)=CI*(VDOT(4*J,3)+U(15,J))/AB(J)+XL*(UDOT(15,J)+WDOT(J+8,4))
      ST(16,J)=CI*(VDOT(4*J,4)+U(16,J))/AB(J)+XL*(UDOT(16,J)+WDOT(J+12,4))
      SZ(1,J)=CI*WDOT(J,1)+XL*(UDOT(1,J)+(U(1,J)+VDOT(4*J-3,1))/AB(J))
      SZ(2,J)=CI*WDOT(J+4,1)+XL*(UDOT(2,J)+(U(2,J)+VDOT(4*J-3,2))/AB(J))
      SZ(3,J)=CI*WDOT(J+8,1)+XL*(UDOT(3,J)+(U(3,J)+VDOT(4*J-3,3))/AB(J))
      SZ(4,J)=CI*WDOT(J+12,1)+XL*(UDOT(4,J)+(U(4,J)+VDOT(4*J-3,4))/AB(J))
      SZ(5,J)=CI*WDOT(J,2)+XL*(UDOT(5,J)+(U(5,J)+VDOT(4*J-2,1))/AB(J))
      SZ(6,J)=CI*WDOT(J+4,2)+XL*(UDOT(6,J)+(U(6,J)+VDOT(4*J-2,2))/AB(J))
      SZ(7,J)=CI*WDOT(J+8,2)+XL*(UDOT(7,J)+(U(7,J)+VDOT(4*J-2,3))/AB(J))
      SZ(8,J)=CI*WDOT(J+12,2)+XL*(UDOT(8,J)+(U(8,J)+VDOT(4*J-2,4))/AB(J))
      SZ(9,J)=CI*WDOT(J,3)+XL*(UDOT(9,J)+(U(9,J)+VDOT(4*J-1,1))/AB(J))
      SZ(10,J)=CI*WDOT(J+4,3)*XL*(UDOT(10,J)+(U(10,J)+VDOT(4*J-1,2))/AB(J))
      SZ(11,J)=CI*WDOT(J+8,3)+XL*(UDOT(11,J)+(U(11,J)+VDOT(4*J-1,3))/AB(J))
      SZ(12,J)=CI*WDOT(J+12,3)+XL*(UDOT(12,J)+(U(12,J)+VDOT(4*J-1,4))/AB(J))
      SZ(13,J)=CI*WDOT(J,4)+XL*(UDOT(13,J)+(U(13,J)+VDOT(4*J,1))/AB(J))
      SZ(14,J)=CI*WDOT(J+4,4)+XL*(UDOT(14,J)+(U(14,J)+VDOT(4*J,2))/AB(J))
      SZ(15,J)=CI*WDOT(J+8,4)+XL*(UDOT(15,J)+(U(15,J)+VDOT(4*J,3))/AB(J))
  950 SZ(16,J)=CI*WDOT(J+12,4)+XL*(UDOT(16,J)+(U(16,J)+VDOT(4*J,4))/AB(J))
      WRITE (6,951)
  951 FORMAT(///' SIGMA-R MATRIX ')
      DO 952 I=1,16
  952 WRITE (6,114) (SR(I,J),J=1,4)
      WRITE (6,953)
  953 FORMAT (///' SIGMA-THETA MATRIX ')
      DO 954 I=1,16
  954 WRITE (6,114) (ST(I,J),J=1,4)
      WRITE (6,955)
  955 FORMAT(///' SIGMA-Z MATRIX ')
      DO 956 I=1,16
  956 WRITE (6,114) (SZ(I,J),J=1,4)
      RETURN
      END
```

C.4 Bar With Through-Thickness Central Crack

The computer program for the solution of problems in Cartesian coordinates can be conveniently divided into four parts. The schematic representation of these four parts for the bar with a through-thickness central crack computer program is shown in Figure 37. The main program is denoted as CCRAG while the subroutines are designated as CBITT, CMINV and CVECT. Subroutine CBITT-EIGEN calculates the eigenvalues, eigenvectors, modal matrices and the matrix functions required in the x-y-z directional differential equation solutions. Subroutine CMINV-MATINV uses the Gauss-Jordan maximum pivot strategy to numerically generate all the required matrix inverses. Subroutine CVECT-VECTOR is used to evaluate the coupling vectors in the x-y-z directions respectively.

The input for each routine is typed in at the beginning of each part using the DATA statements. Note that the subroutines CBITT-EIGEN and CVECT-VECTOR are each called three times, with the arguments of the call vector defining the coordinate direction along which the variables are evaluated. The program is completely general in that the dimensions of the bar or the selected grid increments can be changed by suitable changes in the DIMENSION, DATA and COMMON statements only. The program includes a large number of PRINT statements which are used to follow the execution of the program at the conversation terminal. Note that

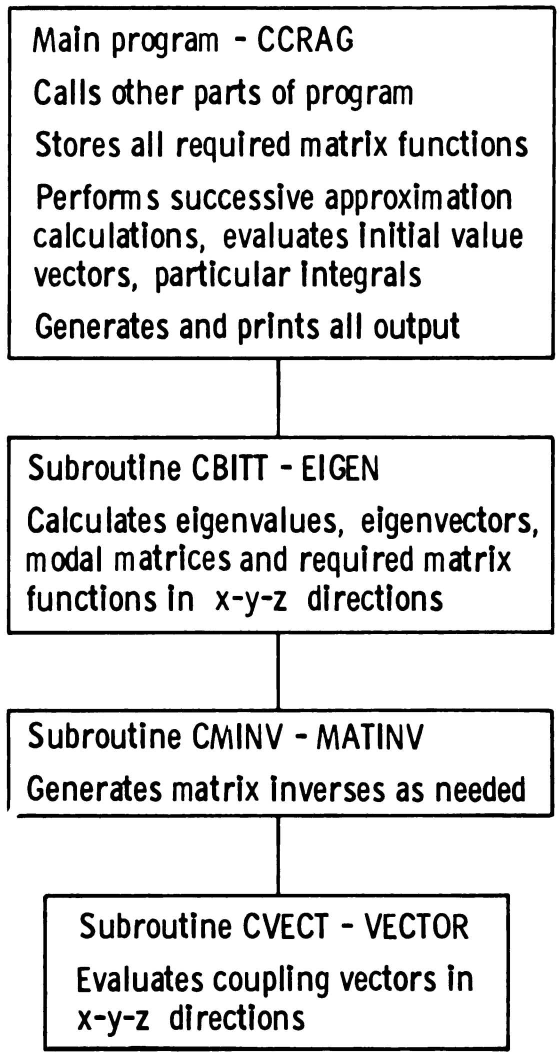

Figure 37. - Schematic representation of the bar with throug-thickness central crack computer program.

most matrix multiplications in this program are performed in a column by column operation which is done solely to minimize the paging in the IBM-360 computer.

The output for this problem includes the displacements after each iteration and their derivatives $\{\mathring{u}\}$, $\{\mathring{v}\}$, $\{\mathring{w}\}$, and the normal stresses $\{\sigma_x\}$, $\{\sigma_y\}$ and $\{\sigma_z\}$ after convergence of the successive approximation procedure has been obtained. Information about matrix function error checks, singularity of matrices and number of iterations is also displayed at the terminal.

```
CBITT        SUBROUTINE EIGEN(XA1,XA2,XA3,XI,HH,A1,A2,J1,J2,J3,J4,IFL)
CBITT        IMPLICIT REAL*8 (A-H,O-Z)
CBITT        DIMENSION DXX(98,98),XA1(J4,J4,J1),XA2(J4,J4,J1),XA3(J4,J4,J1),-
CBITT       1 D1(98,98),D2(98,98),D3(98,98),D4(98),D5(98),D6(98),P(98,98),-
CBITT       2 PIV(98,98),PT1(98,98),PT2(98,98),PT3(98,98),EL(98),FL(98)
CBITT        DIMENSION XI(J4,J4),PT4(98,98)
CBITT        DIMENSION  D1T(98,98),D2T(98,98)
CBITT        DATA PI /3.141592653589793D0/
CBITT        NX=J1
CBITT        NY=J2
CBITT        NZ=J3
CBITT        MX=J4
CBITT        H=HH
CBITT        C1=A1
CBITT        C2=A2
CBITT        NXM1=NX-1
CBITT        NYM1=NY-1
CBITT        NZM1=NZ-1
CBITT        FX=DFLOAT(NYM1*NZM1)
CBITT        FY=PI/DFLOAT(NYM1)
CBITT        FZ=PI/DFLOAT(NZM1)
CBITT        DO 2  I=1,MX
CBITT        DO 2  J=1,MX
CBITT      2 XA1(J,I,1)= 0.D0
CBITT        DO 3  I=1,MX
CBITT        DO 3  J=1,MX
CBITT      3 XA2(J,I,1)= 0.D0
CBITT        DO 4  I=1,MX
CBITT        DO 4  J=1,MX
CBITT      4 XA3(J,I,1)= 0.D0
CBITT        DO 7  I=1,MX
CBITT      7 XA1(I,I,1)= 1.D0
CBITT        KNT=0
CBITT        DO 40 I=1,NZ
CBITT        S1=FZ*DFLOAT(I-1)
CBITT        DO 40 J=1,NY
CBITT        S2=FY*DFLOAT(J-1)
CBITT        KNT=KNT+1
CBITT        EL(KNT)=C1*(1.0D0-DCOS(S1))+C2*(1.0D0-DCOS(S2))
CBITT        EL(KNT)=DSQRT(2.0D0*EL(KNT))
CBITT        JNT=0
CBITT        DO 40 M=1,NZ
CBITT        S3=S1*DFLOAT(M-1)
CBITT        DO 40 N=1,NY
CBITT        S4=S2*DFLOAT(N-1)
CBITT        JNT=JNT+1
CBITT     40 P(JNT,KNT)=DCOS(S3)*DCOS(S4)
CBITT        PRINT 2001
CBITT   2001 FORMAT(' END OF LOOP 40')
CBITT        DO 404 I=1,MX
CBITT        DO 404 J=1,MX
CBITT    404 PIV(J,I)=P(J,I)/FX
CBITT        DO 405 I=1,MX
CBITT        DO 405 J=1,MX
```

```
CBITT          M=MOD(I,NY)
CBITT          N=MOD(J,NY)
CBITT          IF((M.EQ.0.OR.M.EQ.1).AND.(N.EQ.0.OR.N.EQ.1)) GO TO 406
CBITT          IF(M.NE.0.AND.M.NE.1.AND.N.NE.0.AND.N.NE.1) GO TO 407
CBITT          GO TO 405
CBITT      406 PIV(J,I)=PIV(J,I)/2.0D0
CBITT          GO TO 405
CBITT      407 PIV(J,I)=PIV(J,I)*2.0D0
CBITT      405 CONTINUE
CBITT          PRINT 2002
CBITT     2002 FORMAT(' END OF LOOP 405')
CBITT          DO 408 I=1,NY
CBITT          DO 408 J=1,NY
CBITT      408 PIV(J,I)=PIV(J,I)/2.0D0
CBITT          M=MX-NY
CBITT          DO 409 I=1,NY
CBITT          DO 409 J=1,NY
CBITT          MPJ=M+J
CBITT      409 PIV(MPJ,I)=PIV(MPJ,I)/2.0D0
CBITT          DO 410 I=1,NY
CBITT          MPI=M+I
CBITT          DO 410 J=1,NY
CBITT      410 PIV(J,MPI)=PIV(J,MPI)/2.0D0
CBITT          DO 411 I=1,NY
CBITT          MPI=M+I
CBITT          DO 411 J=1,NY
CBITT          MPJ=M+J
CBITT      411 PIV(MPJ,MPI)=PIV(MPJ,MPI)/2.0D0
CBITT          MXY=MX-2*NY
CBITT          DO 412 I=1,MXY
CBITT          MPI=NY+I
CBITT          DO 412 J=1,MXY
CBITT          MPJ=NY+J
CBITT      412 PIV(MPJ,MPI)=PIV(MPJ,MPI)*2.0D0
CBITT          PRINT 2003
CBITT     2003 FORMAT(' END OF LOOP 412')
CBITT          DO 10 II=1,NXM1
CBITT          Y=H*DFLOAT(II)
CBITT          DO 41 J=1,MX
CBITT       41 FL(J)=Y*EL(J)
CBITT          DO 42 J=1,MX
CBITT          D4(J)=DCOSH(FL(J))
CBITT          TEMP=DSINH(FL(J))
CBITT          D6(J)=EL(J)*TEMP
CBITT          IF (J.EQ.1) GO TO 43
CBITT          D5(J)=TEMP/EL(J)
CBITT          GO TO 42
CBITT       43 D5(J)=Y
CBITT       42 CONTINUE
CBITT          PRINT 2004
CBITT     2004 FORMAT(' END OF LOOP 42')
CBITT          DO 48 J=1,MX
CBITT          DO 48 I=1,MX
CBITT       48 PT1(I,J)=PIV(I,J)*D4(I)
```

```
CBITT          DO 148 J=1,MX
CBITT          DO 148 I=1,MX
CBITT      148 PT2(I,J)=PIV(I,J)*D5(I)
CBITT          DO 248 J=1,MX
CBITT          DO 248 I=1,MX
CBITT      248 PT3(I,J)=PIV(I,J)*D6(I)
CBITT          PRINT 2005
CBITT     2005 FORMAT(' END OF LOOP 248')
CBITT          DO 49 J=1,MX
CBITT          DO 49 I=1,MX
CBITT          SUM=0.D0
CBITT          DO 491 L=1,MX
CBITT      491 SUM=SUM+PT1(L,I)*P(L,J)
CBITT       49 D1(I,J)=SUM
CBITT          PRINT 2006
CBITT     2006 FORMAT(' END OF LOOP 49')
CBITT          DO 149 J=1,MX
CBITT          DO 149 I=1,MX
CBITT          SUM=0.D0
CBITT          DO 492 L=1,MX
CBITT      492 SUM=SUM+PT2(L,I)*P(L,J)
CBITT     149 D2(I,J)=SUM
CBITT          PRINT 2007
CBITT     2007 FORMAT(' END OF LOOP 149')
CBITT          DO 249 J=1,MX
CBITT          DO 249 I=1,MX
CBITT          SUM=0.D0
CBITT          DO 493 L=1,MX
CBITT      493 SUM=SUM+PT3(L,I)*P(L,J)
CBITT      249 D3(I,J)=SUM
CBITT          PRINT 2009
CBITT     2009 FORMAT(' END OF LOOP 249')
CBITT          IF (II.NE.NXM1) GO TO 496
CBITT          DO 1901 N=1,MX
CBITT          DO 1901 M=1,MX
CBITT     1901 D1T(M,N)=D1(N,M)
CBITT          PRINT 1501
CBITT     1501 FORMAT(' END OF LOOP 1901 ')
CBITT          DO 1902 N=1,MX
CBITT          DO 1902 M=1,MX
CBITT     1902 D2T(M,N)=D2(N,M)
CBITT          PRINT 1502
CBITT     1502 FORMAT(' END OF LOOP 1902 ')
CBITT          DO 1913 N=1,MX
CBITT          DO 1903 M=1,MX
CBITT          SUM2=0.D0
CBITT          SUM1=0.D0
CBITT          DO 1904 L=1,MX
CBITT     1904 SUM1=SUM1+D1T(L,M)*D1(L,N)
CBITT          DO 1905 L=1,MX
CBITT     1905 SUM2=SUM2+D2T(L,M)*D3(L,N)
CBITT     1903 DXX(M,N)=SUM1-SUM2
CBITT     1913 DXX(N,N)=DXX(N,N)-1.0D0
CBITT          PRINT 1503
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CBITT   0016300. 1503 FORMAT(' END OF LOOP 1913 ')
CBITT   0016400.      SAX=0.D0
CBITT   0016500.      DO 28 N=1,MX
CBITT   0016600.      DO 28 M=1,MX
CBITT   0016700.   28 SAX=DMAX1(DABS(DXX(M,N)),SAX)
CBITT   0016800.      PRINT 29,Y,SAX
CBITT   0016900.   29 FORMAT(' FOR',F8.4,' ERROR = ',F15.9)
CBITT   0017000.      IF(IFL.EQ.0) GO TO 496
CBITT   0017100.      DO 497 J=1,MX
CBITT   0017200.      DO 497 I=1,MX
CBITT   0017300.  497 PT4(I,J)=PIV(I,J)/D4(I)
CBITT   0017400.      DO 498 J=1,MX
CBITT   0017500.      DO 498 I=1,MX
CBITT   0017600.      SUM=0.D0
CBITT   0017700.      DO 499 L=1,MX
CBITT   0017800.  499 SUM=SUM+PT4(L,I)*P(L,J)
CBITT   0017900.  498 XI(I,J)=SUM
CBITT   0018000.  496 DO 35 N=1,MX
CBITT   0018100.      DO 35  M=1,MX
CBITT   0018200.   35 XA1(M,N,II+1)= D1(M,N)
CBITT   0018300.      DO 39 N=1,MX
CBITT   0018400.      DO 39 M=1,MX
CBITT   0018500.   39 XA2(M,N,II+1)=D2(M,N)
CBITT   0018600.      DO 36  N=1,MX
CBITT   0018700.      DO 36  M=1,MX
CBITT   0018800.   36 XA3(M,N,II+1)= D3(M,N)
CBITT   0018900.   10 CONTINUE
CBITT   0019000.      PRINT 1014
CBITT   0019100. 1014 FORMAT(' END OF D GENERATION ')
CBITT   0019200.      RETURN
CBITT   0019300.      END
CCRAG   0000100.      IMPLICIT REAL*8 (A-H,O-Z)
CCRAG   0000200.      DOUBLE PRECISION K1,K2,K3,K4,K6,K7
CCRAG   0000300.      DIMENSION OA1(35,35,14),OA2(35,35,14),OA3(35,35,14),DA1(70,70,7),-
CCRAG   0000400.     2 GA3(98,98,5),O22(35,35),O21(35,35),DX1(35,35),DX2(35,35),-
CCRAG   0000500.     3 DA2(70,70,7),DA3(70,70,7),GA1(98,98,5),GA2(98,98,5),-
CCRAG   0000600.     5 DX3(35,35),DX4(35,35),DA(35,35),DB(35,35),DC(35,35),DX5(35,35),        -
CCRAG   0000700.     6 DX6(35,35),DAB(35,35),DAC(35,35),G22(98,98),G21(98,98)
CCRAG   0000800.      DIMENSION U(35,14),UX(35,14),UDOT(35,14),R(35,14),B1(35,14),B2(35,14),F2(35),H2(35),VAW(35)-
CCRAG   0000900.     1,V(70,7),VX(70,7),VDOT(70,7),S(70,7),B3(70,7),B4(70,7),F3(70),F4(70),H3(70),H4(70),VAL(35),VBE(35),VBW(70)-
CCRAG   0001000.     2,W(98,5),WX(98,5),WDOT(98,5),T(98,5),B5(98,5),B6(98,5),F6(98),H6(98),VCW(98)
CCRAG   0001100.      DIMENSION SIGX(35,14),SIGY(70,7),SIGZ(98,5)
CCRAG   0001200.      DIMENSION DA1T(70,70),DA3T(70,70),D22(70,70)
CCRAG   0001300.      EQUIVALENCE (VBW(1),VAL(1)),(VBW(36),VBE(1))
CCRAG   0001400.      DATA XK,HX,HY,HZ /2.5D-1,1.538461538461554D-1,2.916666666666666D-1,3.75D-1/
CCRAG   0001500.      DATA  C1,C2,CI,XL /5.0D-1,6.666666666666667D-1,1.5D0,7.5D-1/
CCRAG   0001600.      DATA NX,NY,NZ /14,7,5/
CCRAG   0001700.      DATA KOC,IF1,IF2,IF3 /7,1,0,1/
CCRAG   0001800.      NIC=KOC*NZ
CCRAG   0001900.      M1=(NX-1)*NZ
CCRAG   0002000.      M2=(NY-1)*NX
CCRAG   0002100.      M3=(NZ-1)*NY
CCRAG   0002200.      MX= NY*NZ
CCRAG   0002300.      MY= NX*NZ
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CCRAG   0002400.       MZ= NX*NY
CCRAG   0002500.       NOC= MY-NIC
CCRAG   0002600.       K2 = XK/(HY*HY)
CCRAG   0002700.       K3 = XK/(HZ*HZ)
CCRAG   0002800.       K6 = XK/(HX*HX)
CCRAG   0002900.       CALL EIGEN(GA1,GA2,GA3,G22,HZ,K2,K6,NZ,NX,NY,MZ,IF3)
CCRAG   0002950.       DO 18 N=1,MZ
CCRAG   0003000.       DO 18 M=1,MZ
CCRAG   0003050.       SUM=0.D0
CCRAG   0003100.       DO 118 L=1,MZ
CCRAG   0003150.   118 SUM=SUM+GA3(M,L,NZ)*G22(L,N)
CCRAG   0003200.    18 G21(M,N)=SUM
CCRAG   0003250.       PRINT 2999
CCRAG   0003300.  2999 FORMAT(' END OF LOOP 18')
CCRAG   0003600.       CALL EIGEN(DA1,DA2,DA3,D22,HY,K6,K3,NY,NZ,NX,MY,IF2)
CCRAG   0003700.       DO 201 I=1,MY
CCRAG   0003800.       DO 201 J=1,MY
CCRAG   0003900.   201 DA1T(I,J)=DA1(J,I,NY)
CCRAG   0004000.       DO 202 I=1,MY
CCRAG   0004100.       DO 202 J=1,MY
CCRAG   0004200.   202 DA3T(I,J)=DA3(J,I,NY)
CCRAG   0004300.       DO 3 I=1,NIC
CCRAG   0004400.       DO 3 J=1,NIC
CCRAG   0004500.     3 DX1(J,I)=DA3T(J,I)
CCRAG   0004600.       CALL MATINV (DX1,NIC)
CCRAG   0004700.       DO  4  N=1,NOC
CCRAG   0004800.       NIX= NIC+N
CCRAG   0004900.       DO  4  M=1,NIC
CCRAG   0005000.       SUM=0.D0
CCRAG   0005100.       DO 104 L=1,NIC
CCRAG   0005200.     104 SUM=SUM+DX1(M,L)*DA3T(L,NIX)
CCRAG   0005300.     4 DX2(M,N)=SUM
CCRAG   0005340.       PRINT 2001
CCRAG   0005380.  2001 FORMAT(' END OF LOOP 4 ')
CCRAG   0005400.       DO 5 N=1,NIC
CCRAG   0005500.       DO 5 M=1,NIC
CCRAG   0005600.       SUM=0.D0
CCRAG   0005700.       DO 105 L=1,NIC
CCRAG   0005800.     105 SUM=SUM+DX1(M,L)*DA1T(L,N)
CCRAG   0005900.     5 DX3(M,N)=SUM
CCRAG   0006000.       DO 6 N=1,NOC
CCRAG   0006100.       NIX= NIC+N
CCRAG   0006200.       DO 6  M=1,NIC
CCRAG   0006300.       SUM=0.D0
CCRAG   0006400.       DO 106 L=1,NIC
CCRAG   0006500.      106 SUM=SUM+DX1(M,L)*DA1T(L,NIX)
CCRAG   0006600.     6 DX4(M,N)=SUM
CCRAG   0006700.       DO 7 N=1,NOC
CCRAG   0006800.       NIX=N+NIC
CCRAG   0006900.       DO 7 M=1,NOC
CCRAG   0007000.       MIX= M+NIC
CCRAG   0007100.       SUM=0.D0
CCRAG   0007200.       DO 8 L=1,NIC
CCRAG   0007300.     8 SUM= SUM+DA3(L,MIX,NY)*DX4(L,N)
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CCRAG    0007400.      7 DA(M,N)=DA1T(MIX,NIX)-SUM
CCRAG    0007500.        DO 9 N=1,NOC
CCRAG    0007600.        NIX = N+NIC
CCRAG    0007700.        DO 9 M=1,NOC
CCRAG    0007800.        SUM= 0.D0
CCRAG    0007900.        MIX=M+NIC
CCRAG    0008000.        DO 10 L=1,NIC
CCRAG    0008100.     10 SUM=SUM+DA3(L,MIX,NY)*DX2(L,N)
CCRAG    0008200.      9 DB(M,N)=DA3T(MIX,NIX)-SUM
CCRAG    0008240.        PRINT 2004
CCRAG    0008280.   2004 FORMAT(' END OF LOOP 9')
CCRAG    0008300.        DO 11 N=1,NIC
CCRAG    0008400.        DO 11 M=1,NOC
CCRAG    0008500.        SUM= 0.D0
CCRAG    0008600.        MIX= M+NIC
CCRAG    0008700.        DO 12 L=1,NIC
CCRAG    0008800.     12 SUM=SUM+DA3(L,MIX,NY)*DX3(L,N)
CCRAG    0008900.     11 DC(M,N)=DA1T(MIX,N)-SUM
CCRAG    0009000.        CALL MATINV(DA,NOC)
CCRAG    0009100.        DO 13  N=1,NIC
CCRAG    0009200.        DO 13  M=1,NOC
CCRAG    0009300.        SUM= 0.D0
CCRAG    0009400.        MIX= M + NIC
CCRAG    0009500.        DO 113  L=1,NIC
CCRAG    0009600.    113 SUM=SUM+DA3(L,MIX,NY)*DX1(L,N)
CCRAG    0009700.     13 DX5(M,N)=SUM
CCRAG    0009800.        DO 14  N=1,NIC
CCRAG    0009900.        DO 14  M=1,NOC
CCRAG    0010000.        SUM= 0.D0
CCRAG    0010100.        DO 114  L=1,NOC
CCRAG    0010200.    114 SUM= SUM + DA(M,L)*DX5(L,N)
CCRAG    0010300.     14 DX6(M,N)=SUM
CCRAG    0010400.        DO 15  N=1,NOC
CCRAG    0010500.        DO 15  M=1,NOC
CCRAG    0010600.        SUM= 0.D0
CCRAG    0010700.        DO 115  L=1,NOC
CCRAG    0010800.    115 SUM= SUM + DA(M,L)*DB(L,N)
CCRAG    0010900.     15 DAB(M,N)=SUM
CCRAG    0011000.        DO 16  N=1,NIC
CCRAG    0011100.        DO 16  M=1,NOC
CCRAG    0011200.        SUM= 0.D0
CCRAG    0011300.        DO 116  L=1,NOC
CCRAG    0011400.    116 SUM= SUM+DA(M,L)*DC(L,N)
CCRAG    0011500.     16 DAC(M,N)=SUM
CCRAG    0011540.        PRINT 2007
CCRAG    0011580.   2007 FORMAT(' END OF LOOP 16')
CCRAG    0011600.        CALL EIGEN(DA1,DA2,DA3,D22,HX,K3,K2,NX,NY,NZ,MX,IF1)
CCRAG    0011650.        DO 2 N=1,MX
CCRAG    0011700.        DO 2 M=1,MX
CCRAG    0011750.        SUM=0.D0
CCRAG    0011800.        DO 102 L=1,MX
CCRAG    0011850.    102 SUM=SUM+DA3(M,L,NX)*D22(L,N)
CCRAG    0011900.      2 D21(M,N)=SUM
CCRAG    0011950.        PRINT 3000
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CCRAG    0012000. 3000 FORMAT(' END OF LOOP 2')
CCRAG    0012300.      KTR=0
CCRAG    0012400.      DO 20 N=1,NX
CCRAG    0012500.      DO 20 M=1,MX
CCRAG    0012600.   20 R(M,N)=0.D0
CCRAG    0012700.      DO 21 N=1,NY
CCRAG    0012800.      DO 21 M=1,MY
CCRAG    0012900.   21 VDOT(M,N)=0.D0
CCRAG    0013000.      DO 22 N=1,NZ
CCRAG    0013100.      DO 22 M=1,MZ
CCRAG    0013200.   22 WDOT(M,N)=0.D0
CCRAG    0013300.   23 KTR=KTR+1
CCRAG    0013340.      PRINT 3001
CCRAG    0013380. 3001 FORMAT(' END OF LOOP 23')
CCRAG    0013400.      DO 24 N=1,NX
CCRAG    0013500.      DO 24 M=1,MX
CCRAG    0013600.   24 UX(M,N)=U(M,N)
CCRAG    0013700.      DO 124 N=1,NX
CCRAG    0013800.      DO 124 M=1,MX
CCRAG    0013900.      B1(M,N)=0.D0
CCRAG    0014000.  124 B2(M,N)=0.D0
CCRAG    0014100.      DO 38 M=1,MX
CCRAG    0014200.   38 F2(M)=0.D0
CCRAG    0014300.      DO 25 M=2,NX
CCRAG    0014320.      DO 25 J=1,MX
CCRAG    0014340.      SUM2=0.D0
CCRAG    0014360.      DO 26 L=1,MX
CCRAG    0014380.  26  SUM2=SUM2+OA1(L,J,M-1)*R(L,M-1)+OA1(L,J,M)*R(L,M)
CCRAG    0014400.   25 B2(J,M)=B2(J,M-1)+HX*SUM2/2.0D0
CCRAG    0014420.      PRINT 2008
CCRAG    0014440. 2008 FORMAT(' END OF LOOP 25')
CCRAG    0014460.      DO 4025 M=2,NX
CCRAG    0014480.      DO 4025 J=1,MX
CCRAG    0014500.      SUM1=0.D0
CCRAG    0014520.      DO 4026 L=1,MX
CCRAG    0014540. 4026 SUM1=SUM1-OA2(L,J,M-1)*R(L,M-1)-OA2(L,J,M)*R(L,M)
CCRAG    0014560. 4025 B1(J,M)=B1(J,M-1)+HX*SUM1/2.0D0
CCRAG    0014580.      PRINT 3004
CCRAG    0014600. 3004 FORMAT(' END OF LOOP 4025')
CCRAG    0015200.      KNT=0
CCRAG    0015300.      DO 27 M=1,NZ
CCRAG    0015400.      J=M1+M
CCRAG    0015500.      DO 27 N=1,NY
CCRAG    0015600.      K=NX*N
CCRAG    0015700.      KNT=KNT+1
CCRAG    0015800.   27 VAW(KNT)=-C1*(VDOT(J,N)+WDOT(K,M))
CCRAG    0015900.      DO 28 J=1,MX
CCRAG    0016000.      DO 28 L=1,MX
CCRAG    0016100.   28 F2(J)=F2(J)+O22(L,J)*VAW(L)-O21(L,J)*B1(L,NX)
CCRAG    0016200.      DO 29 J=1,MX
CCRAG    0016300.   29 F2(J)=F2(J)-B2(J,NX)
CCRAG    0016400.      DO 30 I=1,NX
CCRAG    0016420.      DO 31 J=1,MX
CCRAG    0016440.   31 H2(J)=F2(J)+B2(J,I)
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CCRAG   0016460.      DO 30 M=1,MX
CCRAG   0016480.      SUM2=0.D0
CCRAG   0016500.      DO 130 N=1,MX
CCRAG   0016520.  130 SUM2=SUM2+DA1(N,M,I)*B1(N,I)+DA2(N,M,I)*H2(N)
CCRAG   0016540.   30 U(M,I)=SUM2
CCRAG   0016560.      PRINT 3010
CCRAG   0016580. 3010 FORMAT(' END OF LOOP 30 ')
CCRAG   0016600.      DO 4030 I=1,NX
CCRAG   0016620.      DO 4031 J=1,MX
CCRAG   0016640. 4031 H2(J)=F2(J)+B2(J,I)
CCRAG   0016660.      DO 4030 M=1,MX
CCRAG   0016680.      SUM1=0.D0
CCRAG   0016700.      DO 4130 N=1,MX
CCRAG   0016720. 4130 SUM1=SUM1+DA3(N,M,I)*B1(N,I)+DA1(N,M,I)*H2(N)
CCRAG   0016740. 4030 UDOT(M,I)=SUM1
CCRAG   0016760.      PRINT 3011
CCRAG   0016780. 3011 FORMAT(' END OF LOOP 4030 ')
CCRAG   0016800.      WRITE (6,32)
CCRAG   0016820.   32 FORMAT(///' U-MATRIX')
CCRAG   0016840.      DO 33 I=1,MX
CCRAG   0016860.   33 WRITE (6,34) (U(I,J),J=1,NX)
CCRAG   0016880.   34 FORMAT (6(1X,F15.8))
CCRAG   0018400.      CALL VECTOR(W,WDOT,U,UDOT,NY,NZ,NX,HY,HZ,HX,MY,MZ,MX,1,S)
CCRAG   0018500.      DO 37 N=1,NY
CCRAG   0018600.      DO 37 M=1,MY
CCRAG   0018700.   37 VX(M,N)=V(M,N)
CCRAG   0018800.      DO 137 N=1,NY
CCRAG   0018900.      DO 137 M=1,MY
CCRAG   0019000.      B3(M,N)=0.D0
CCRAG   0019100.  137 B4(M,N)=0.D0
CCRAG   0019200.      DO 39 M=1,MY
CCRAG   0019300.      F3(M)=0.D0
CCRAG   0019400.   39 F4(M)=0.D0
CCRAG   0019500.      DO 40 M=2,NY
CCRAG   0019520.      DO 40 J=1,MY
CCRAG   0019540.      SUM2=0.D0
CCRAG   0019560.      DO 41 L=1,MY
CCRAG   0019580.   41 SUM2=SUM2+DA1(L,J,M-1)*S(L,M-1)+DA1(L,J,M)*S(L,M)
CCRAG   0019600.   40 B4(J,M)=B4(J,M-1)+HY*SUM2/2.0D0
CCRAG   0019620.      PRINT 2009
CCRAG   0019640. 2009 FORMAT(' END OF LOOP 40 ')
CCRAG   0019660.      DO 4040 M=2,NY
CCRAG   0019680.      DO 4040 J=1,MY
CCRAG   0019700.      SUM1=0.D0
CCRAG   0019720.      DO 4041 L=1,MY
CCRAG   0019740. 4041 SUM1=SUM1-DA2(L,J,M-1)*S(L,M-1)-DA2(L,J,M)*S(L,M)
CCRAG   0019760. 4040 B3(J,M)=B3(J,M-1)+HY*SUM1/2.0D0
CCRAG   0019780.      PRINT 3012
CCRAG   0019800. 3012 FORMAT(' END OF LOOP 4040')
CCRAG   0020400.      KNT=0
CCRAG   0020500.      DO 42 M=1,KOC
CCRAG   0020600.      DO 42 N=1,NZ
CCRAG   0020700.      K=NY*(N-1)+1
CCRAG   0020800.      KNT=KNT+1
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CCRAG    0020900.    42 F4(KNT)=-C1*(UDOT(K,M)+WDOT(M,N))
CCRAG    0021000.       KNT=0
CCRAG    0021100.       DO 43 M=1,NX
CCRAG    0021200.       L=M2+M
CCRAG    0021300.       DO 43 N=1,NZ
CCRAG    0021400.       K=NY*N
CCRAG    0021500.       KNT=KNT+1
CCRAG    0021600.    43 VBW(KNT)=C2-C1*(UDOT(K,M)+WDOT(L,N))
CCRAG    0021700.       DO 44 I=1,NOC
CCRAG    0021800.       MIX=I+NIC
CCRAG    0021900.       SUM1=0.D0
CCRAG    0022000.       SUM2=0.D0
CCRAG    0022100.       DO 45 J=1,NOC
CCRAG    0022200.       NIX=J+NIC
CCRAG    0022300.    45 SUM1=SUM1+DA(I,J)*VBE(J)-DAB(I,J)*B3(NIX,NY)
CCRAG    0022400.       DO 46 J=1,NIC
CCRAG    0022500.    46 SUM2=SUM2+DX6(I,J)*VAL(J)+DAC(I,J)*(F4(J)+B4(J,NY))
CCRAG    0022600.    44 F4(MIX)=SUM1-SUM2-B4(MIX,NY)
CCRAG    0022700.       DO 47 I=1,NIC
CCRAG    0022800.       SUM1=0.D0
CCRAG    0022900.       SUM2=0.D0
CCRAG    0023000.       DO 48 J=1,NIC
CCRAG    0023100.    48 SUM1=SUM1+DX1(I,J)*VAL(J)-DX3(I,J)*(F4(J)+B4(J,NY))
CCRAG    0023200.       DO 49 J=1,NOC
CCRAG    0023300.       NIX=NIC+J
CCRAG    0023400.    49 SUM2=SUM2+DX2(I,J)*B3(NIX,NY)+DX4(I,J)*(F4(NIX)+B4(NIX,NY))
CCRAG    0023500.    47 F3(I)=SUM1-SUM2-B3(I,NY)
CCRAG    0023600.       DO 50 I=1,NY
CCRAG    0023620.       DO 51 J=1,MY
CCRAG    0023640.       H3(J)=F3(J)+B3(J,I)
CCRAG    0023660.    51 H4(J)=F4(J)+B4(J,I)
CCRAG    0023680.       DO 50 M=1,MY
CCRAG    0023700.       SUM2=0.D0
CCRAG    0023720.       DO 150 N=1,MY
CCRAG    0023740.   150 SUM2=SUM2+DA1(N,M,I)*H3(N)+DA2(N,M,I)*H4(N)
CCRAG    0023760.    50 V(M,I)=SUM2
CCRAG    0023780.        PRINT 3015
CCRAG    0023800. 3015 FORMAT(' END OF LOOP 50')
CCRAG    0023820.       DO 4050 I=1,NY
CCRAG    0023840.       DO 4051 J=1,MY
CCRAG    0023860.       H3(J)=F3(J)+B3(J,I)
CCRAG    0023880. 4051 H4(J)=F4(J)+B4(J,I)
CCRAG    0023900.       DO 4050 M=1,MY
CCRAG    0023920.       SUM1=0.D0
CCRAG    0023940.       DO 4150 N=1,MY
CCRAG    0023960. 4150 SUM1=SUM1+DA3(N,M,I)*H3(N)+DA1(N,M,I)*H4(N)
CCRAG    0023980. 4050 VDOT(M,I)=SUM1
CCRAG    0024000.       PRINT 3016
CCRAG    0024020. 3016 FORMAT(' END OF LOOP 4050 ')
CCRAG    0024040.       WRITE (6,52)
CCRAG    0024060.    52 FORMAT (///' V-MATRIX')
CCRAG    0024080.       DO 53 I=1,MY
CCRAG    0024100.    53 WRITE (6,34) (V(I,J),J=1,NY)
CCRAG    0025600.       CALL VECTOR(U,UDOT,V,VDOT,NZ,NX,NY,HZ,HX,HY,MZ,MX,MY,O,T)
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CCRAG   0025700.       DO 56 N=1,NZ
CCRAG   0025800.       DO 56 M=1,MZ
CCRAG   0025900.    56 WX(M,N)=W(M,N)
CCRAG   0026000.       DO 156 N=1,NZ
CCRAG   0026100.       DO 156 M=1,MZ
CCRAG   0026200.       B5(M,N)=0.D0
CCRAG   0026300.   156 B6(M,N)=0.D0
CCRAG   0026400.       DO 57 M=1,MZ
CCRAG   0026500.    57 F6(M)=0.D0
CCRAG   0026600.       DO 58 M=2,NZ
CCRAG   0026620.       DO 58 J=1,MZ
CCRAG   0026640.       SUM2=0.D0
CCRAG   0026660.      DO 59 L=1,MZ
CCRAG   0026680.    59 SUM2=SUM2+GA1(L,J,M-1)*T(L,M-1)+GA1(L,J,M)*T(L,M)
CCRAG   0026700.    58 B6(J,M)=B6(J,M-1)+HZ*SUM2/2.0D0
CCRAG   0026720.       PRINT 2010
CCRAG   0026740.  2010 FORMAT(' END OF LOOP 58 ')
CCRAG   0026760.       DO 4058 M=2,NZ
CCRAG   0026780.       DO 4058 J=1,MZ
CCRAG   0026800.       SUM1=0.D0
CCRAG   0026820.       DO 4059 L=1,MZ
CCRAG   0026840.  4059 SUM1=SUM1-GA2(L,J,M-1)*T(L,M-1)-GA2(L,J,M)*T(L,M)
CCRAG   0026860.  4058 B5(J,M)=B5(J,M-1)+HZ*SUM1/2.0D0
CCRAG   0026880.       PRINT 3017
CCRAG   0026900.  3017 FORMAT(' END OF LOOP 4058')
CCRAG   0027500.       KNT=0
CCRAG   0027600.       DO 60 M=1,NY
CCRAG   0027700.       L=M3+M
CCRAG   0027800.       DO 60 N=1,NX
CCRAG   0027900.       K=NZ*N
CCRAG   0028000.       KNT=KNT+1
CCRAG   0028100.    60 VCW(KNT)=-C1*(UDOT(L,N)+VDOT(K,M))
CCRAG   0028200.       DO 61 J=1,MZ
CCRAG   0028300.       DO 61 L=1,MZ
CCRAG   0028400.    61 F6(J)=F6(J)+G22(L,J)*VCW(L)-G21(L,J)*B5(L,NZ)
CCRAG   0028500.       DO 62 J=1,MZ
CCRAG   0028600.    62 F6(J)=F6(J)-B6(J,NZ)
CCRAG   0028700.       DO 63 I=1,NZ
CCRAG   0028720.       DO 64 J=1,MZ
CCRAG   0028740.    64 H6(J)=F6(J)+B6(J,I)
CCRAG   0028760.       DO 63 M=1,MZ
CCRAG   0028780.       SUM2=0.D0
CCRAG   0028800.       DO 163 N=1,MZ
CCRAG   0028820.   163 SUM2=SUM2+GA1(N,M,I)*B5(N,I)+GA2(N,M,I)*H6(N)
CCRAG   0028840.    63 W(M,I)=SUM2
CCRAG   0028860.       PRINT 3018
CCRAG   0028880.  3018 FORMAT(' END OF LOOP 63')
CCRAG   0028900.       DO 4063 I=1,NZ
CCRAG   0028920.       DO 4064 J=1,MZ
CCRAG   0028940.  4064 H6(J)=F6(J)+B6(J,I)
CCRAG   0028960.       DO 4063 M=1,MZ
CCRAG   0028980.       SUM1=0.D0
CCRAG   0029000.       DO 4163 N=1,MZ
CCRAG   0029020.  4163 SUM1=SUM1+GA3(N,M,I)*B5(N,I)+GA1(N,M,I)*H6(N)
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CCRAG    0029040. 4063 WDOT(M,I)=SUM1
CCRAG    0029060.      PRINT 3020
CCRAG    0029080. 3020 FORMAT(' END OF LOOP 4063')
CCRAG    0029100.      WRITE (6,65)
CCRAG    0029120.   65 FORMAT(///' W-MATRIX')
CCRAG    0029140.      DO 66 I=1,MZ
CCRAG    0029160.   66 WRITE (6,34) (W(I,J),J=1,NZ)
CCRAG    0030600.      CALL VECTOR(V,VDOT,W,WDOT,NX,NY,NZ,HX,HY,HZ,MX,MY,MZ,O,R)
CCRAG    0030700.      PRINT 69,KTR
CCRAG    0030800.   69 FORMAT(1X,I3)
CCRAG    0030900.      IF(DABS(U(1,1)).GT.5.0D0) STOP
CCRAG    0031000.      DO 70 I=1,NX
CCRAG    0031100.      DO 70 J=1,MX
CCRAG    0031200.      IF(DABS(U(J,I)-UX(J,I)).GT.1.0D-6) GO TO 23
CCRAG    0031300.   70 CONTINUE
CCRAG    0031400.      DO 71 I=1,NY
CCRAG    0031500.      DO 71 J=1,MY
CCRAG    0031600.      IF(DABS(V(J,I)-VX(J,I)).GT.1.0D-6) GO TO 23
CCRAG    0031700.   71 CONTINUE
CCRAG    0031800.      DO 72 I=1,NZ
CCRAG    0031900.      DO 72 J=1,MZ
CCRAG    0032000.      IF(DABS(W(J,I)-WX(J,I)).GT.1.0D-6) GO TO 23
CCRAG    0032100.   72 CONTINUE
CCRAG    0032200.      WRITE (6,73) KTR
CCRAG    0032300.   73  FORMAT(///' .ITERATION NUMBER ',I2,///)
CCRAG    0032400.      KNT=0
CCRAG    0032500.      DO 74 K=1,NZ
CCRAG    0032600.      DO 74 I=1,NY
CCRAG    0032700.      KNT=KNT+1
CCRAG    0032800.      DO 74 J=1,NX
CCRAG    0032900.      L1=K+NZ*(J-1)
CCRAG    0033000.      L2=J+NX*(I-1)
CCRAG    0033100.   74 SIGX(KNT,J)=CI*UDOT(KNT,J)+XL*(VDOT(L1,I)+WDOT(L2,K))
CCRAG    0033200.      KNT=0
CCRAG    0033300.      DO 77 K=1,NX
CCRAG    0033400.      DO 77 I=1,NZ
CCRAG    0033500.      KNT=KNT+1
CCRAG    0033600.      DO 77 J=1,NY
CCRAG    0033700.      L1=K+NX*(J-1)
CCRAG    0033800.      L2=J+NY*(I-1)
CCRAG    0033900.   77 SIGY(KNT,J)=CI*VDOT(KNT,J)+XL*(WDOT(L1,I)+UDOT(L2,K))
CCRAG    0034000.      KNT=0
CCRAG    0034100.      DO 80 K=1,NY
CCRAG    0034200.      DO 80 I=1,NX
CCRAG    0034300.      KNT=KNT+1
CCRAG    0034400.      DO 80 J=1,NZ
CCRAG    0034500.      L1=K+NY*(J-1)
CCRAG    0034600.      L2=J+NZ*(I-1)
CCRAG    0034700.   80 SIGZ(KNT,J)=CI*WDOT(KNT,J)+XL*(UDOT(L1,I)+VDOT(L2,K))
CCRAG    0034705.      PRINT 3021
CCRAG    0034710. 3021 FORMAT(' END OF JOB ')
CCRAG    0034715.      WRITE (6,35)
CCRAG    0034720.   35 FORMAT(///' UDOT-MATRIX')
CCRAG    0034725.      DO 36 I=1,MX
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CCRAG   0034730.   36 WRITE (6,34) (UDOT(I,J),J=1,NX)
CCRAG   0034735.      WRITE (6,54)
CCRAG   0034740.   54 FORMAT(///' VDOT-MATRIX ')
CCRAG   0034745.      DO 55 I=1,MY
CCRAG   0034750.   55 WRITE (6,34) (VDOT(I,J),J=1,NY)
CCRAG   0034755.      WRITE (6,67)
CCRAG   0034760.   67 FORMAT(///' WDOT-MATRIX ')
CCRAG   0034765.      DO 68 I=1,MZ
CCRAG   0034770.   68 WRITE (6,34) (WDOT(I,J),J=1,NZ)
CCRAG   0034800.      WRITE (6,75)
CCRAG   0034900.   75 FORMAT(///' SIGMA-X MATRIX ')
CCRAG   0035000.      DO 76 I=1,MX
CCRAG   0035100.   76 WRITE (6,34) (SIGX(I,J),J=1,NX)
CCRAG   0035200.      WRITE (6,78)
CCRAG   0035300.   78 FORMAT(///' SIGMA-Y MATRIX ')
CCRAG   0035400.      DO 79 I=1,MY
CCRAG   0035500.   79 WRITE (6,34) (SIGY(I,J),J=1,NY)
CCRAG   0035600.      WRITE (6,81)
CCRAG   0035700.   81 FORMAT(///' SIGMA-Z MATRIX ')
CCRAG   0035800.      DO 82 I=1,MZ
CCRAG   0035900.   82 WRITE (6,34) (SIGZ(I,J),J=1,NZ)
CCRAG   0036000.      STOP
CCRAG   0036100.      END
CMINV   0000100.      SUBROUTINE MATINV (X,MMM)
CMINV   0000200.      DOUBLE PRECISION X,Y,W,A,TEST,R,DELT
CMINV   0000300.      DIMENSION X(MMM,MMM), Y(35,35), K(35), W(35,35)
CMINV   0000400.      DATA DELT,LOOPS/1.0D-8,2/
CMINV   0000500.      KSIG=0
CMINV   0000600.      M=MMM
CMINV   0000700.      MM=M-1
CMINV   0000800.      DO 1 J=1,M
CMINV   0000900.      K(J)=J
CMINV   0001000.      DO 1 I=1,M
CMINV   0001100.1     Y(I,J)=X(I,J)
CMINV   0001200.      DO 6 I=1,M
CMINV   0001300.      A=0.D0
CMINV   0001400.      DO 2 J=I,M
CMINV   0001500.      IF (DABS(Y(J,1)).LE.A) GO TO 2
CMINV   0001600.      A=DABS(Y(J,1))
CMINV   0001700.      L=J
CMINV   0001800.2     CONTINUE
CMINV   0001900.      IF (A.EQ.0.D0) GO TO 18
CMINV   0002000.      N=K(L)
CMINV   0002100.      K(L)=K(I)
CMINV   0002200.      K(I)=N
CMINV   0002300.      DO 3 J=1,M
CMINV   0002400.      A=Y(I,J)
CMINV   0002500.      Y(I,J)=Y(L,J)
CMINV   0002600.3     Y(L,J)=A
CMINV   0002700.      A=Y(I,1)
CMINV   0002800.      DO 4 J=1,MM
CMINV   0002900.4     Y(I,J)=Y(I,J+1)/A
CMINV   0003000.      Y(I,M)=1.D0/A
CMINV   0003100.      DO 6 L=1,M
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CMINV    0003200.       IF (L.EQ.I) GO TO 6
CMINV    0003300.       A=Y(L,1)
CMINV    0003400.       DO 5 N=1,MM
CMINV    0003500.5      Y(L,N)=Y(L,N+1)-A*Y(I,N)
CMINV    0003600.       Y(L,M)=-A*Y(I,M)
CMINV    0003700.6      CONTINUE
CMINV    0003800.       DO 10 I=1,M
CMINV    0003900.       IF (K(I).EQ.I) GO TO 10
CMINV    0004000.       DO 7 J=I,M
CMINV    0004100.       IF (K(J).EQ.I) GO TO 8
CMINV    0004200.7      CONTINUE
CMINV    0004300.       GO TO 19
CMINV    0004400.8      DO 9 L=1,M
CMINV    0004500.       A=Y(L,I)
CMINV    0004600.       Y(L,I)=Y(L,J)
CMINV    0004700.9      Y(L,J)=A
CMINV    0004800.       K(J)=K(I)
CMINV    0004900.10     CONTINUE
CMINV    0005000.       DO 15 N=1,LOOPS
CMINV    0005100.       TEST=0.D0
CMINV    0005200.       DO 12 J=1,M
CMINV    0005300.       DO 12 I=1,M
CMINV    0005400.       R=0.D0
CMINV    0005500.       DO 11 L=1,M
CMINV    0005600.11     R=R-X(I,L)*Y(L,J)
CMINV    0005700.       IF (I.EQ.J) R=R+1.0D0
CMINV    0005800.       TEST=DMAX1(TEST,DABS(R))
CMINV    0005900.12     W(I,J)=R
CMINV    0006000.       IF (TEST.LE.DELT) GO TO 16
CMINV    0006100.       DO 14 J=1,M
CMINV    0006200.       DO 14 I=1,M
CMINV    0006300.       A=0.D0
CMINV    0006400.       DO 13 L=1,M
CMINV    0006500.13     A=A+Y(I,L)*W(L,J)
CMINV    0006600.14     Y(I,J)=Y(I,J)+A
CMINV    0006700.15     CONTINUE
CMINV    0006800.       KSIG=3
CMINV    0006900.       PRINT 21, TEST
CMINV    0007000.16     DO 17 J=1,M
CMINV    0007100.       DO 17 I=1,M
CMINV    0007200.17     X(I,J)=Y(I,J)
CMINV    0007300.       GO TO 20
CMINV    0007400.18     KSIG=1
CMINV    0007500.       GO TO 20
CMINV    0007600.19     KSIG=2
CMINV    0007700.20     IF (KSIG.NE.0) PRINT 22, KSIG
CMINV    0007800.       RETURN
CMINV    0007900.C
CMINV    0008000.21     FORMAT (' ERROR = ',D14.6)
CMINV    0008100.22     FORMAT (' ERROR INDICATOR NUMBER ',I1)
CMINV    0008200.       END
CVECT    0000100.       SUBROUTINE VECTOR(V1,V2,W1,W2,LX,LY,LZ,HX,HY,HZ,LD1,LD2,LD3,IFG,R1)
CVECT    0000200.       IMPLICIT REAL*8 (A-H,O-Z)
CVECT    0000300.       DIMENSION V1(LD2,LY),V2(LD2,LY),W1(LD3,LZ),W2(LD3,LZ),V(98,14),-
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```
CVECT         1 VDOT(98,14),W(98,14),WDOT(98,14),R1(LD1,LX),R(98,14)
CVECT          M6=LD1
CVECT          M5=LD2
CVECT          M4=LD3
CVECT          NX=LX
CVECT          NY=LY
CVECT          NZ=LZ
CVECT          NXM1=NX-1
CVECT          NYM1=NY-1
CVECT          NZM1=NZ-1
CVECT          M1=NXM1*NZ
CVECT          M3=NZM1*NY
CVECT          M7=NYM1*NX
CVECT          M2=(NX-2)*NZ
CVECT          M9=(NZ-2)*NY
CVECT          Q1=7.5D-1/HX
CVECT          Q2=5.0D-1/(HX*HY)
CVECT          Q3=5.0D-1/(HX*HZ)
CVECT          Q4=Q1/2.0D0
CVECT          Q5=Q2/2.0D0
CVECT          Q6=Q3/2.0D0
CVECT          DO 201 J=1,NY
CVECT          DO 201 I=1,M5
CVECT      201 V(I,J)=V1(I,J)
CVECT          DO 202 J=1,NY
CVECT          DO 202 I=1,M5
CVECT      202 VDOT(I,J)=V2(I,J)
CVECT          DO 203 J=1,NZ
CVECT          DO 203 I=1,M4
CVECT      203 W(I,J)=W1(I,J)
CVECT          DO 204 J=1,NZ
CVECT          DO 204 I=1,M4
CVECT      204 WDOT(I,J)=W2(I,J)
CVECT          DO 101 I=1,MX
CVECT      101 R(I,1)=0.D0
CVECT          R(1,NX)=Q1*(VDOT(M2+1,1)-VDOT(M1+1,1))+Q1*(WDOT(NX-1,1)-WDOT(NX,1))-
CVECT         1 +Q2*(V(M2+1,1)-V(M1+1,1))+Q3*(W(NX-1,1)-W(NX,1))
CVECT          R(NY,NX)=Q1*(VDOT(M2+1,NY)-VDOT(M1+1,NY))+Q1*(WDOT(M4-1,1)-WDOT(M4,1))-
CVECT         1 -Q2*(V(M2+1,NY)-V(M1+1,NY))+Q3*(W(M4-1,1)-W(M4,1))
CVECT          R(M3+1,NX)=Q1*(VDOT(M1,1)-VDOT(M5,1))+Q1*(WDOT(NX-1,NZ)-WDOT(NX,NZ))-
CVECT         1 +Q2*(V(M1,1)-V(M5,1))-Q3*(W(NX-1,NZ)-W(NX,NZ))
CVECT          R(M6,NX)=Q1*(VDOT(M1,NY)-VDOT(M5,NY))+Q1*(WDOT(M4-1,NZ)-WDOT(M4,NZ))-
CVECT         1 -Q2*(V(M1,NY)-V(M5,NY))-Q3*(W(M4-1,NZ)-W(M4,NZ))
CVECT          DO 102 J=2,NXM1
CVECT          K=M7+J
CVECT          L2=J*NZ
CVECT          L1=L2-NZ
CVECT          I2=L2+1
CVECT          I1=I2-2*NZ
CVECT          L3=L2+NZ
CVECT          R(1,J)=Q4*(VDOT(I1,1)-VDOT(I2,1))+Q4*(WDOT(J-1,1)-WDOT(J+1,1))-
CVECT         1 +Q5*(V(I1,1)-V(I2,1))+Q6*(W(J-1,1)-W(J+1,1))
CVECT          R(NY,J)=Q4*(VDOT(I1,NY)-VDOT(I2,NY))+Q4*(WDOT(K-1,1)-WDOT(K+1,1))-
CVECT         1 -Q5*(V(I1,NY)-V(I2,NY))+Q6*(W(K-1,1)-W(K+1,1))
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CVECT         R(M3+1,J)=Q4*(VDOT(L1,1)-VDOT(L3,1))+Q4*(WDOT(J-1,NZ)-WDOT(J+1,NZ))-
CVECT        1 +Q5*(V(L1,1)-V(L3,1))-Q6*(W(J-1,NZ)-W(J+1,NZ))
CVECT     102 R(M6,J)=Q4*(VDOT(L1,NY)-VDOT(L3,NY))+Q4*(WDOT(K-1,NZ)-WDOT(K+1,NZ))-
CVECT        1 -Q5*(V(L1,NY)-V(L3,NY))-Q6*(W(K-1,NZ)-W(K+1,NZ))
CVECT         DO 103 I=2,NYM1
CVECT         L=NX*I
CVECT         K=NX*(I-1)
CVECT         I3=I+M3
CVECT         R(I,NX)=Q1*(VDOT(M2+1,I)-VDOT(M1+1,I))+Q1*(WDOT(L-1,1)-WDOT(L,1))-
CVECT        1 +Q3*(W(L-1,1)-W(L,1))
CVECT         R(I3,NX)=Q1*(VDOT(M1,I)-VDOT(M5,I))+Q1*(WDOT(L-1,NZ)-WDOT(L,NZ))-Q3*(W(L-1,NZ)-W(L,NZ))
CVECT         DO 103 J=2,NXM1
CVECT         K1=K+J
CVECT         L2=J*NZ
CVECT         L1=L2-NZ
CVECT         I2=L2+1
CVECT         I1=I2-2*NZ
CVECT         L3=L2+NZ
CVECT         R(I,J)=Q4*(VDOT(I1,I)-VDOT(I2,I))+Q4*(WDOT(K1-1,1)-WDOT(K1+1,1))+Q6*(W(K1-1,1)-W(K1+1,1))
CVECT     103 R(I3,J)=Q4*(VDOT(L1,I)-VDOT(L3,I))+Q4*(WDOT(K1-1,NZ)-WDOT(K1+1,NZ))-Q6*(W(K1-1,NZ)-W(K1+1,NZ))
CVECT         DO 104 K=2,NZM1
CVECT         I1=M2+K
CVECT         I2=M1+K
CVECT         DO 105 I=NY,M9,NY
CVECT         I3=I+NY
CVECT         R(I+1,NX)=Q1*(VDOT(I1,1)-VDOT(I2,1))+Q1*(WDOT(NX-1,K)-WDOT(NX,K))+Q2*(V(I1,1)-V(I2,1))
CVECT         R(I3,NX)=Q1*(VDOT(I1,NY)-VDOT(I2,NY))+Q1*(WDOT(M4-1,K)-WDOT(M4,K))-Q2*(V(I1,NY)-V(I2,NY))
CVECT         DO 105 J=2,NXM1
CVECT         L2=K+J*NZ
CVECT         L1=L2-2*NZ
CVECT         L=M7+J
CVECT         R(I+1,J)=Q4*(VDOT(L1,1)-VDOT(L2,1))+Q4*(WDOT(J-1,K)-WDOT(J+1,K))+Q5*(V(L1,1)-V(L2,1))
CVECT     105 R(I3,J)=Q4*(VDOT(L1,NY)-VDOT(L2,NY))+Q4*(WDOT(L-1,K)-WDOT(L+1,K))-Q5*(V(L1,NY)-V(L2,NY))
CVECT          DO 104 L=2,NYM1
CVECT         I=NY*(K-1)+L
CVECT         I3=L*NX
CVECT         I4=I3-NX
CVECT         R(I,NX)=Q1*(VDOT(I1,L)-VDOT(I2,L))+Q1*(WDOT(I3-1,K)-WDOT(I3,K))
CVECT         DO 104 J=2,NXM1
CVECT         K1=I4+J
CVECT         L2=K+J*NZ
CVECT         L1=L2-2*NZ
CVECT     104 R(I,J)=Q4*(VDOT(L1,L)-VDOT(L2,L))+Q4*(WDOT(K1-1,K)-WDOT(K1+1,K))
CVECT         IF (IFG.NE.1) GO TO 107
CVECT         R(1,1)=Q1*(VDOT(1,1)-VDOT(NZ+1,1))+Q1*(WDOT(1,1)-WDOT(2,1))-
CVECT        1 +Q2*(V(1,1)-V(NZ+1,1))+Q3*(W(1,1)-W(2,1))
CVECT         R(NY,1)=Q1*(VDOT(1,NY)-VDOT(NZ+1,NY))+Q1*(WDOT(M7+1,1)-WDOT(M7+2,1))-
CVECT        1 -Q2*(V(1,NY)-V(NZ+1,NY))+Q3*(W(M7+1,1)-W(M7+2,1))
CVECT         R(NY+1,1)=Q1*(VDOT(2,1)-VDOT(NZ+2,1))+Q1*(WDOT(1,2)-WDOT(2,2))+Q2*(V(2,1)-V(NZ+2,1))
CVECT         R(2*NY,1)=Q1*(VDOT(2,NY)-VDOT(NZ+2,NY))+Q1*(WDOT(M7+1,2)-WDOT(M7+2,2))-Q2*(V(2,NY)-V(NZ+2,NY))
CVECT         R(2*NY+1,1)=Q1*(VDOT(3,1)-VDOT(NZ+3,1))+Q1*(WDOT(1,3)-WDOT(2,3))+Q2*(V(3,1)-V(NZ+3,1))
CVECT         R(3*NY,1)=Q1*(VDOT(3,NY)-VDOT(NZ+3,NY))+Q1*(WDOT(M7+1,3)-WDOT(M7+2,3))-Q2*(V(3,NY)-V(NZ+3,NY))
CVECT         R(3*NY+1,1)=Q1*(VDOT(4,1)-VDOT(NZ+4,1))+Q1*(WDOT(1,4)-WDOT(2,4))+Q2*(V(4,1)-V(NZ+4,1))
CVECT         R(4*NY,1)=Q1*(VDOT(4,NY)-VDOT(NZ+4,NY))+Q1*(WDOT(M7+1,4)-WDOT(M7+2,4))-Q2*(V(4,NY)-V(NZ+4,NY))
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CVECT   0011200.      R(4*NY+1,1)=Q1*(VDOT(5,1)-VDOT(NZ+5,1))+Q1*(WDOT(1,5)-WDOT(2,5))+Q2*(V(5,1)-V(NZ+5,1))
CVECT   0011300.      R(5*NY,1)=Q1*(VDOT(5,NY)-VDOT(NZ+5,NY))+Q1*(WDOT(M7+1,5)-WDOT(M7+2,5))-Q2*(V(5,NY)-V(NZ+5,NY))
CVECT   0011310.      R(5*NY+1,1)=Q1*(VDOT(6,1)-VDOT(NZ+6,1))+Q1*(WDOT(1,6)-WDOT(2,6))+Q2*(V(6,1)-V(NZ+6,1))
CVECT   0011320.      R(6*NY,1)=Q1*(VDOT(6,NY)-VDOT(NZ+6,NY))+Q1*(WDOT(M7+1,6)-WDOT(M7+2,6))-Q2*(V(6,NY)-V(NZ+6,NY))
CVECT   0011330.      R(6*NY+1,1)=Q1*(VDOT(7,1)-VDOT(NZ+7,1))+Q1*(WDOT(1,7)-WDOT(2,7))+Q2*(V(7,1)-V(NZ+7,1))
CVECT   0011340.      R(7*NY,1)=Q1*(VDOT(7,NY)-VDOT(NZ+7,NY))+Q1*(WDOT(M7+1,7)-WDOT(M7+2,7))-Q2*(V(7,NY)-V(NZ+7,NY))
CVECT   0011400.      DO 106 I=2,NYM1
CVECT   0011500.      L=NX*(I-1)+1
CVECT   0011600.      I3=NY+I
CVECT   0011700.      I4=2*NY+I
CVECT   0011800.      I5=3*NY+I
CVECT   0011900.      I6=4*NY+I
CVECT   0011910.      I7=5*NY+I
CVECT   0011920.      I8=6*NY+I
CVECT   0011930.      R(I7,1)=Q1*(VDOT(6,I)-VDOT(NZ+6,I))+Q1*(WDOT(L,6)-WDOT(L+1,6))
CVECT   0011940.      R(I8,1)=Q1*(VDOT(7,I)-VDOT(NZ+7,I))+Q1*(WDOT(L,7)-WDOT(L+1,7))
CVECT   0012000.      R(I6,1)=Q1*(VDOT(5,I)-VDOT(NZ+5,I))+Q1*(WDOT(L,5)-WDOT(L+1,5))
CVECT   0012100.      R(I5,1)=Q1*(VDOT(4,I)-VDOT(NZ+4,I))+Q1*(WDOT(L,4)-WDOT(L+1,4))
CVECT   0012200.      R(I4,1)=Q1*(VDOT(3,I)-VDOT(NZ+3,I))+Q1*(WDOT(L,3)-WDOT(L+1,3))
CVECT   0012300.      R(I,1)=Q1*(VDOT(1,I)-VDOT(NZ+1,I))+Q1*(WDOT(L,1)-WDOT(L+1,1))+Q3*(W(L,1)-W(L+1,1))
CVECT   0012400.  106 R(I3,1)=Q1*(VDOT(2,I)-VDOT(NZ+2,I))+Q1*(WDOT(L,2)-WDOT(L+1,2))
CVECT   0012500.  107 DO 205 J=1,NX
CVECT   0012600.      DO 205 I=1,M6
CVECT   0012700.  205 R1(I,J)=R(I,J)
CVECT   0012800.      RETURN
CVECT   0012900.      END
```

C.5 Bar With Through-Thickness Double Edge Cracks

Since the solution of this problem is very similar to that discussed in Appendix C.4, the enclosed computer program also follows the schematic diagram of Figure 37. The input and output data are handled identically to that of the central crack problem and only the initial value vectors are appreciably modified. Minor changes in the generation of the coupling vectors are also required.

```
CBITT   0000100.      SUBROUTINE EIGEN(XA1,XA2,XA3,XI,HH,A1,A2,J1,J2,J3,J4,IFL)
CBITT   0000200.      IMPLICIT REAL*8 (A-H,O-Z)
CBITT   0000300.      DIMENSION DXX(98,98),XA1(J4,J4,J1),XA2(J4,J4,J1),XA3(J4,J4,J1),-
CBITT   0000400.     1 D1(98,98),D2(98,98),D3(98,98),D4(98),D5(98),D6(98),P(98,98),-
CBITT   0000500.     2 PIV(98,98),PT1(98,98),PT2(98,98),PT3(98,98),EL(98),FL(98)
CBITT   0000600.      DIMENSION XI(J4,J4),PT4(98,98)
CBITT   0000650.      DIMENSION  D1T(98,98),D2T(98,98)
CBITT   0000700.      DATA PI /3.141592653589793D0/
CBITT   0000800.      NX=J1
CBITT   0000900.      NY=J2
CBITT   0001000.      NZ=J3
CBITT   0001100.      MX=J4
CBITT   0001200.      H=HH
CBITT   0001300.      C1=A1
CBITT   0001400.      C2=A2
CBITT   0001500.      NXM1=NX-1
CBITT   0001600.      NYM1=NY-1
CBITT   0001700.      NZM1=NZ-1
CBITT   0001800.      FX=DFLOAT(NYM1*NZM1)
CBITT   0001900.      FY=PI/DFLOAT(NYM1)
CBITT   0001950.      FZ=PI/DFLOAT(NZM1)
CBITT   0002000.      DO 2  I=1,MX
CBITT   0002050.      DO 2  J=1,MX
CBITT   0002100.    2 XA1(J,I,1)= 0.D0
CBITT   0002150.      DO 3  I=1,MX
CBITT   0002200.      DO 3  J=1,MX
CBITT   0002250.    3 XA2(J,I,1)= 0.D0
CBITT   0002300.      DO 4  I=1,MX
CBITT   0002350.      DO 4  J=1,MX
CBITT   0002400.    4 XA3(J,I,1)= 0.D0
CBITT   0002450.      DO 7  I=1,MX
CBITT   0002500.    7 XA1(I,I,1)= 1.D0
CBITT   0002550.      KNT=0
CBITT   0002600.      DO 40 I=1,NZ
CBITT   0002650.      S1=FZ*DFLOAT(I-1)
CBITT   0002700.      DO 40 J=1,NY
CBITT   0002750.      S2=FY*DFLOAT(J-1)
CBITT   0002800.      KNT=KNT+1
CBITT   0002850.      EL(KNT)=C1*(1.0D0-DCOS(S1))+C2*(1.0D0-DCOS(S2))
CBITT   0002900.      EL(KNT)=DSQRT(2.0D0*EL(KNT))
CBITT   0002950.      JNT=0
CBITT   0003000.      DO 40 M=1,NZ
CBITT   0003050.      S3=S1*DFLOAT(M-1)
CBITT   0003100.      DO 40 N=1,NY
CBITT   0003150.      S4=S2*DFLOAT(N-1)
CBITT   0003200.      JNT=JNT+1
CBITT   0003250.   40 P(JNT,KNT)=DCOS(S3)*DCOS(S4)
CBITT   0003270.      PRINT 2001
CBITT   0003290. 2001 FORMAT(' END OF LOOP 40')
CBITT   0003300.      DO 404 I=1,MX
CBITT   0003350.      DO 404 J=1,MX
CBITT   0003400.  404 PIV(J,I)=P(J,I)/FX
CBITT   0003450.      DO 405 I=1,MX
CBITT   0003500.      DO 405 J=1,MX
```

```
CBITT   0003550.       M=MOD(I,NY)
CBITT   0003600.       N=MOD(J,NY)
CBITT   0003650.       IF((M.EQ.0.OR.M.EQ.1).AND.(N.EQ.0.OR.N.EQ.1)) GO TO 406
CBITT   0003700.       IF(M.NE.0.AND.M.NE.1.AND.N.NE.0.AND.N.NE.1) GO TO 407
CBITT   0003750.       GO TO 405
CBITT   0003800.   406 PIV(J,I)=PIV(J,I)/2.0D0
CBITT   0003850.       GO TO 405
CBITT   0003900.   407 PIV(J,I)=PIV(J,I)*2.0D0
CBITT   0003950.   405 CONTINUE
CBITT   0003970.       PRINT 2002
CBITT   0003990.  2002 FORMAT(' END OF LOOP 405')
CBITT   0004000.       DO 408 I=1,NY
CBITT   0004050.       DO 408 J=1,NY
CBITT   0004100.   408 PIV(J,I)=PIV(J,I)/2.0D0
CBITT   0004150.       M=MX-NY
CBITT   0004200.       DO 409 I=1,NY
CBITT   0004250.       DO 409 J=1,NY
CBITT   0004300.       MPJ=M+J
CBITT   0004350.   409 PIV(MPJ,I)=PIV(MPJ,I)/2.0D0
CBITT   0004400.       DO 410 I=1,NY
CBITT   0004450.       MPI=M+I
CBITT   0004500.       DO 410 J=1,NY
CBITT   0004550.   410 PIV(J,MPI)=PIV(J,MPI)/2.0D0
CBITT   0004600.       DO 411 I=1,NY
CBITT   0004650.       MPI=M+I
CBITT   0004700.       DO 411 J=1,NY
CBITT   0004750.       MPJ=M+J
CBITT   0004800.   411 PIV(MPJ,MPI)=PIV(MPJ,MPI)/2.0D0
CBITT   0004850.       MXY=MX-2*NY
CBITT   0004900.       DO 412 I=1,MXY
CBITT   0004950.       MPI=NY+I
CBITT   0005000.       DO 412 J=1,MXY
CBITT   0005050.       MPJ=NY+J
CBITT   0005100.   412 PIV(MPJ,MPI)=PIV(MPJ,MPI)*2.0D0
CBITT   0005120.       PRINT 2003
CBITT   0005140.  2003 FORMAT(' END OF LOOP 412')
CBITT   0005150.       DO 10 II=1,NXM1
CBITT   0005200.       Y=H*DFLOAT(II)
CBITT   0005250.       DO 41 J=1,MX
CBITT   0005300.    41 FL(J)=Y*EL(J)
CBITT   0005350.       DO 42 J=1,MX
CBITT   0005400.       D4(J)=DCOSH(FL(J))
CBITT   0005450.       TEMP=DSINH(FL(J))
CBITT   0005500.       D6(J)=EL(J)*TEMP
CBITT   0005550.       IF (J.EQ.1) GO TO 43
CBITT   0005600.       D5(J)=TEMP/EL(J)
CBITT   0005650.       GO TO 42
CBITT   0005700.    43 D5(J)=Y
CBITT   0005750.    42 CONTINUE
CBITT   0005770.       PRINT 2004
CBITT   0005790.  2004 FORMAT(' END OF LOOP 42')
CBITT   0005800.       DO 48 J=1,MX
CBITT   0005850.       DO 48 I=1,MX
CBITT   0005900.    48 PT1(I,J)=PIV(I,J)*D4(I)
```

```
CBITT   0005950.       DO 148 J=1,MX
CBITT   0006000.       DO 148 I=1,MX
CBITT   0006050.   148 PT2(I,J)=PIV(I,J)*D5(I)
CBITT   0006100.       DO 248 J=1,MX
CBITT   0006150.       DO 248 I=1,MX
CBITT   0006200.   248 PT3(I,J)=PIV(I,J)*D6(I)
CBITT   0006220.       PRINT 2005
CBITT   0006240.  2005 FORMAT(' END OF LOOP 248')
CBITT   0006250.       DO 49 J=1,MX
CBITT   0006300.       DO 49 I=1,MX
CBITT   0006350.       SUM=0.D0
CBITT   0006400.       DO 491 L=1,MX
CBITT   0006450.   491 SUM=SUM+PT1(L,I)*P(L,J)
CBITT   0006500.    49 D1(I,J)=SUM
CBITT   0006520.       PRINT 2006
CBITT   0006540.  2006 FORMAT(' END OF LOOP 49')
CBITT   0006550.       DO 149 J=1,MX
CBITT   0006600.       DO 149 I=1,MX
CBITT   0006650.       SUM=0.D0
CBITT   0006700.       DO 492 L=1,MX
CBITT   0006750.   492 SUM=SUM+PT2(L,I)*P(L,J)
CBITT   0006800.  149 D2(I,J)=SUM
CBITT   0006820.       PRINT 2007
CBITT   0006840.  2007 FORMAT(' END OF LOOP 149')
CBITT   0006850.       DO 249 J=1,MX
CBITT   0006900.       DO 249 I=1,MX
CBITT   0006950.       SUM=0.D0
CBITT   0007000.       DO 493 L=1,MX
CBITT   0007050.   493 SUM=SUM+PT3(L,I)*P(L,J)
CBITT   0007100.   249 D3(I,J)=SUM
CBITT   0007120.       PRINT 2009
CBITT   0007140.  2009 FORMAT(' END OF LOOP 249')
CBITT   0007200.       IF (II.NE.NXM1) GO TO 496
CBITT   0007250.       DO 1901 N=1,MX
CBITT   0007300.       DO 1901 M=1,MX
CBITT   0007350.  1901 D1T(M,N)=D1(N,M)
CBITT   0007400.       PRINT 1501
CBITT   0007450.  1501 FORMAT(' END OF LOOP 1901 ')
CBITT   0007500.       DO 1902 N=1,MX
CBITT   0007550.       DO 1902 M=1,MX
CBITT   0007600.  1902 D2T(M,N)=D2(N,M)
CBITT   0007650.       PRINT 1502
CBITT   0007700.  1502 FORMAT(' END OF LOOP 1902 ')
CBITT   0007750.       DO 1913 N=1,MX
CBITT   0007800.       DO 1903 M=1,MX
CBITT   0007850.       SUM2=0.D0
CBITT   0007900.       SUM1=0.D0
CBITT   0007950.       DO 1904 L=1,MX
CBITT   0008000.  1904 SUM1=SUM1+D1T(L,M)*D1(L,N)
CBITT   0008050.       DO 1905 L=1,MX
CBITT   0008100.  1905 SUM2=SUM2+D2T(L,M)*D3(L,N)
CBITT   0008150.  1903 DXX(M,N)=SUM1-SUM2
CBITT   0008200.  1913 DXX(N,N)=DXX(N,N)-1.0D0
CBITT   0008250.       PRINT 1503
```

```
CBITT    0008300. 1503 FORMAT(' END OF LOOP 1913 ')
CBITT    0008350.      SAX=0.D0
CBITT    0008400.      DO 28 N=1,MX
CBITT    0008450.      DO 28 M=1,MX
CBITT    0008500.  28  SAX=DMAX1(DABS(DXX(M,N)),SAX)
CBITT    0008550.      PRINT 29,Y,SAX
CBITT    0008600.   29 FORMAT(' FOR',F8.4,' ERROR = ',F15.9)
CBITT    0013300.      IF(IFL.EQ.0) GO TO 496
CBITT    0013400.      DO 497 J=1,MX
CBITT    0013500.      DO 497 I=1,MX
CBITT    0013600.  497 PT4(I,J)=PIV(I,J)/D4(I)
CBITT    0013700.      DO 498 J=1,MX
CBITT    0013800.      DO 498 I=1,MX
CBITT    0013900.      SUM=0.D0
CBITT    0014000.      DO 499 L=1,MX
CBITT    0014100.  499 SUM=SUM+PT4(L,I)*P(L,J)
CBITT    0014200.  498 XI(I,J)=SUM
CBITT    0014300.  496 DO 35 N=1,MX
CBITT    0014400.      DO 35  M=1,MX
CBITT    0014500.   35 XA1(M,N,II+1)= D1(M,N)
CBITT    0014600.      DO 39 N=1,MX
CBITT    0014700.      DO 39 M=1,MX
CBITT    0014800.   39 XA2(M,N,II+1)=D2(M,N)
CBITT    0014900.      DO 36  N=1,MX
CBITT    0015000.      DO 36  M=1,MX
CBITT    0015100.   36 XA3(M,N,II+1)= D3(M,N)
CBITT    0015200.   10 CONTINUE
CBITT    0015230.      PRINT 1014
CBITT    0015250. 1014 FORMAT(' END OF D GENERATION ')
CBITT    0015300.      RETURN
CBITT    0015400.      END
CCRAG    0000100.      IMPLICIT REAL*8 (A-H,O-Z)
CCRAG    0000200.      DOUBLE PRECISION K1,K2,K3,K4,K6,K7
CCRAG    0000300.      DIMENSION OA1(35,35,14),OA2(35,35,14),OA3(35,35,14),DA1(70,70,7),-
CCRAG    0000400.     2 GA3(98,98,5),O22(35,35),O21(35,35),DX1(35,35),DX2(35,35),-
CCRAG    0000500.     3 DA2(70,70,7),DA3(70,70,7),GA1(98,98,5),GA2(98,98,5),-
CCRAG    0000600.     5DX3(35,35),DX4(35,35),DA(35,35),DB(35,35),DC(35,35),DX5(35,35),-
CCRAG    0000700.     6DX6(35,35),DAB(35,35),DAC(35,35),G22(98,98),G21(98,98)
CCRAG    0000800.      DIMENSION U(35,14),UX(35,14),UDOT(35,14),R(35,14),B1(35,14),B2(35,14),F2(35),H2(35),VAW(35)-
CCRAG    0000900.     1,V(70,7),VX(70,7),VDOT(70,7),S(70,7),B3(70,7),B4(70,7),F3(70),F4(70),H3(70),H4(70),VAL(35),VBE(35),VBW(70)-
CCRAG    0001000.     2,W(98,5),WX(98,5),WDOT(98,5),T(98,5),B5(98,5),B6(98,5),F6(98),H6(98),VCW(98)
CCRAG    0001100.      DIMENSION SIGX(35,14),SIGY(70,7),SIGZ(98,5)
CCRAG    0001200.      DIMENSION DA1T(70,70),DA3T(70,70),D22(70,70)
CCRAG    0001300.      EQUIVALENCE (VBW(1),VAL(1)),(VBW(36),VBE(1))
CCRAG    0001400.      DATA XK,HX,HY,HZ /2.5D-1,1.53846153846154D-1,2.91666666666666D-1,3.75D-1/
CCRAG    0001500.      DATA  C1,C2,CI,XL /5.0D-1,6.66666666666667D-1,1.5D0,7.5D-1/
CCRAG    0001600.      DATA NX,NY,NZ /14,7,5/
CCRAG    0001700.      DATA KOC,IF1,IF2,IF3 /7,1,0,1/
CCRAG    0001800.      NOC=KOC*NZ
CCRAG    0001850.      KIC=NX-KOC
CCRAG    0001900.      M1=(NX-1)*NZ
CCRAG    0002000.      M2=(NY-1)*NX
CCRAG    0002100.      M3=(NZ-1)*NY
CCRAG    0002200.      MX= NY*NZ
```

```
CCRAG    0002300.         MY= NX*NZ
CCRAG    0002400.         MZ= NX*NY
CCRAG    0002450.         NIC=MY-NOC
CCRAG    0002500.         K2=XK/(HY*HY)
CCRAG    0002550.         K3=XK/(HZ*HZ)
CCRAG    0002600.         K6=XK/(HX*HX)
CCRAG    0002650.         CALL EIGEN(GA1,GA2,GA3,G22,HZ,K2,K6,NZ,NX,NY,MZ,IF3)
CCRAG    0002700.         DO 18 N=1,MZ
CCRAG    0002750.         DO 18 M=1,MZ
CCRAG    0002800.         SUM=0.D0
CCRAG    0002850.         DO 118 L=1,MZ
CCRAG    0002900.     118 SUM=SUM+GA3(M,L,NZ)*G22(L,N)
CCRAG    0002950.      18 G21(M,N)=SUM
CCRAG    0003000.         PRINT 2999
CCRAG    0003050.    2999 FORMAT(' END OF LOOP 18')
CCRAG    0003600.         CALL EIGEN(DA1,DA2,DA3,D22,HY,K6,K3,NY,NZ,NX,MY,IF2)
CCRAG    0003700.         DO 201 I=1,MY
CCRAG    0003800.         DO 201 J=1,MY
CCRAG    0003900.     201 DA1T(I,J)=DA1(J,I,NY)
CCRAG    0004000.         DO 202 I=1,MY
CCRAG    0004100.         DO 202 J=1,MY
CCRAG    0004200.     202 DA3T(I,J)=DA3(J,I,NY)
CCRAG    0004300.         DO 3 I=1,NIC
CCRAG    0004350.         MIX=NOC+I
CCRAG    0004400.         DO 3 J=1,NIC
CCRAG    0004450.         NIX=NOC+J
CCRAG    0004500.       3 DX1(J,I)=DA3T(NIX,MIX)
CCRAG    0004550.         CALL MATINV (DX1,NIC)
CCRAG    0004600.         DO  4  N=1,NOC
CCRAG    0004650.         DO  4  M=1,NIC
CCRAG    0004700.         SUM=0.D0
CCRAG    0004750.         DO 104 L=1,NIC
CCRAG    0004800.         NIX=NOC+L
CCRAG    0004850.       104 SUM=SUM+DX1(M,L)*DA3T(NIX,N)
CCRAG    0004900.       4 DX2(M,N)=SUM
CCRAG    0004910.         PRINT 2001
CCRAG    0004920.    2001 FORMAT(' END OF LOOP 4 ')
CCRAG    0004950.         DO 5 N=1,NIC
CCRAG    0005000.         NIX=NOC+N
CCRAG    0005050.         DO 5 M=1,NIC
CCRAG    0005100.         SUM=0.D0
CCRAG    0005150.         DO 105 L=1,NIC
CCRAG    0005200.         MIX=NOC+L
CCRAG    0005250.       105 SUM=SUM+DX1(M,L)*DA1T(MIX,NIX)
CCRAG    0005300.       5 DX3(M,N)=SUM
CCRAG    0005350.         DO 6 N=1,NOC
CCRAG    0005400.         DO 6  M=1,NIC
CCRAG    0005450.         SUM=0.D0
CCRAG    0005500.         DO 106 L=1,NIC
CCRAG    0005550.         NIX=NOC+L
CCRAG    0005600.        106 SUM=SUM+DX1(M,L)*DA1T(NIX,N)
CCRAG    0005650.       6 DX4(M,N)=SUM
CCRAG    0005700.         DO 7 N=1,NOC
CCRAG    0005750.         DO 7 M=1,NOC
```

```
CCRAG    0005800.        SUM=0.D0
CCRAG    0005850.        DO 8 L=1,NIC
CCRAG    0005900.        MIX=NOC+L
CCRAG    0005950.      8 SUM= SUM+DA3(MIX,M,NY)*DX4(L,N)
CCRAG    0006000.      7 DA(M,N)=DA1T(M,N)-SUM
CCRAG    0006050.        DO 9 N=1,NOC
CCRAG    0006100.        DO 9 M=1,NOC
CCRAG    0006150.        SUM= 0.D0
CCRAG    0006200.        DO 10 L=1,NIC
CCRAG    0006250.        MIX=NOC+L
CCRAG    0006300.     10 SUM=SUM+DA3(MIX,M,NY)*DX2(L,N)
CCRAG    0006350.      9 DB(M,N)=DA3T(M,N)-SUM
CCRAG    0006360.        PRINT 2004
CCRAG    0006370.   2004 FORMAT(' END OF LOOP 9 ')
CCRAG    0006400.        DO 11 N=1,NIC
CCRAG    0006450.        NIX=NOC+N
CCRAG    0006500.        DO 11 M=1,NOC
CCRAG    0006550.        SUM= 0.D0
CCRAG    0006600.        DO 12 L=1,NIC
CCRAG    0006650.        MIX=NOC+L
CCRAG    0006700.     12 SUM=SUM+DA3(MIX,M,NY)*DX3(L,N)
CCRAG    0006750.     11 DC(M,N)=DA1T(M,NIX)-SUM
CCRAG    0006800.        CALL MATINV(DA,NOC)
CCRAG    0006850.        DO 13  N=1,NIC
CCRAG    0006900.        DO 13  M=1,NOC
CCRAG    0006950.        SUM= 0.D0
CCRAG    0007000.        DO 113  L=1,NIC
CCRAG    0007050.        MIX=NOC+L
CCRAG    0007100.    113 SUM=SUM+DA3(MIX,M,NY)*DX1(L,N)
CCRAG    0007150.     13 DX5(M,N)=SUM
CCRAG    0007200.        DO 14  N=1,NIC
CCRAG    0007250.        DO 14  M=1,NOC
CCRAG    0007300.        SUM= 0.D0
CCRAG    0007350.        DO 114  L=1,NOC
CCRAG    0007400.    114 SUM= SUM + DA(M,L)*DX5(L,N)
CCRAG    0007450.     14 DX6(M,N)=SUM
CCRAG    0007500.        DO 15  N=1,NOC
CCRAG    0007550.        DO 15  M=1,NOC
CCRAG    0007600.        SUM= 0.D0
CCRAG    0007650.        DO 115  L=1,NOC
CCRAG    0007700.    115 SUM= SUM + DA(M,L)*DB(L,N)
CCRAG    0007750.     15 DAB(M,N)=SUM
CCRAG    0007800.        DO 16  N=1,NIC
CCRAG    0007850.        DO 16  M=1,NOC
CCRAG    0007900.        SUM= 0.D0
CCRAG    0007950.        DO 116  L=1,NOC
CCRAG    0008000.    116 SUM= SUM+DA(M,L)*DC(L,N)
CCRAG    0008050.     16 DAC(M,N)=SUM
CCRAG    0008060.        PRINT 2007
CCRAG    0008070.   2007 FORMAT(' END OF LOOP 16 ')
CCRAG    0011600.        CALL EIGEN(DA1,DA2,DA3,D22,HX,K3,K2,NX,NY,NZ,MX,IF1)
CCRAG    0011650.        DO 2 N=1,MX
CCRAG    0011700.        DO 2 M=1,MX
CCRAG    0011750.        SUM=0.D0
```

```
CCRAG   0011800.       DO 102 L=1,MX
CCRAG   0011850.   102 SUM=SUM+OA3(M,L,NX)*O22(L,N)
CCRAG   0011900.     2 O21(M,N)=SUM
CCRAG   0011950.       PRINT 3000
CCRAG   0012000.  3000 FORMAT(' END OF LOOP 2')
CCRAG   0012300.       KTR=0
CCRAG   0012400.       DO 20 N=1,NX
CCRAG   0012500.       DO 20 M=1,MX
CCRAG   0012600.    20 R(M,N)=0.D0
CCRAG   0012700.       DO 21 N=1,NY
CCRAG   0012800.       DO 21 M=1,MY
CCRAG   0012900.    21 VDOT(M,N)=0.D0
CCRAG   0013000.       DO 22 N=1,NZ
CCRAG   0013100.       DO 22 M=1,MZ
CCRAG   0013200.    22 WDOT(M,N)=0.D0
CCRAG   0013300.    23 KTR=KTR+1
CCRAG   0013340.       PRINT 3001
CCRAG   0013380.  3001 FORMAT(' END OF LOOP 23 ')
CCRAG   0013400.       DO 24 N=1,NX
CCRAG   0013500.       DO 24 M=1,MX
CCRAG   0013600.    24 UX(M,N)=U(M,N)
CCRAG   0013700.       DO 124 N=1,NX
CCRAG   0013800.       DO 124 M=1,MX
CCRAG   0013900.       B1(M,N)=0.D0
CCRAG   0014000.   124 B2(M,N)=0.D0
CCRAG   0014100.       DO 38 M=1,MX
CCRAG   0014200.    38 F2(M)=0.D0
CCRAG   0014300.       DO 25 M=2,NX
CCRAG   0014400.       DO 25 J=1,MX
CCRAG   0014600.       SUM2=0.D0
CCRAG   0014700.       DO 26 L=1,MX
CCRAG   0014900.    26  SUM2=SUM2+OA1(L,J,M-1)*R(L,M-1)+OA1(L,J,M)*R(L,M)
CCRAG   0015100.    25 B2(J,M)=B2(J,M-1)+HX*SUM2/2.0D0
CCRAG   0015105.       PRINT 2008
CCRAG   0015110.  2008 FORMAT(' END OF LOOP 25')
CCRAG   0015115.       DO 4025 M=2,NX
CCRAG   0015120.       DO 4025 J=1,MX
CCRAG   0015125.       SUM1=0.D0
CCRAG   0015130.       DO 4026 L=1,MX
CCRAG   0015135.  4026 SUM1=SUM1-OA2(L,J,M-1)*R(L,M-1)-OA2(L,J,M)*R(L,M)
CCRAG   0015140.  4025 B1(J,M)=B1(J,M-1)+HX*SUM1/2.0D0
CCRAG   0015145.       PRINT 3004
CCRAG   0015150.  3004 FORMAT(' END OF LOOP 4025')
CCRAG   0015200.       KNT=0
CCRAG   0015300.       DO 27 M=1,NZ
CCRAG   0015400.       J=M1+M
CCRAG   0015500.       DO 27 N=1,NY
CCRAG   0015600.       K=NX*N
CCRAG   0015700.       KNT=KNT+1
CCRAG   0015800.    27 VAW(KNT)=-C1*(VDOT(J,N)+WDOT(K,M))
CCRAG   0015900.       DO 28 J=1,MX
CCRAG   0016000.       DO 28 L=1,MX
CCRAG   0016100.    28 F2(J)=F2(J)+O22(L,J)*VAW(L)-O21(L,J)*B1(L,NX)
CCRAG   0016200.       DO 29 J=1,MX
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CCRAG    0016300.    29 F2(J)=F2(J)-B2(J,NX)
CCRAG    0016400.       DO 30 I=1,NX
CCRAG    0016500.       DO 31 J=1,MX
CCRAG    0016600.    31 H2(J)=F2(J)+B2(J,I)
CCRAG    0016700.       DO 30 M=1,MX
CCRAG    0016900.       SUM2=0.D0
CCRAG    0017000.       DO 130 N=1,MX
CCRAG    0017200.   130 SUM2=SUM2+OA1(N,M,I)*B1(N,I)+OA2(N,M,I)*H2(N)
CCRAG    0017400.    30 U(M,I)=SUM2
CCRAG    0017404.       PRINT 3010
CCRAG    0017408.  3010 FORMAT(' END OF LOOP 30 ')
CCRAG    0017412.       DO 4030 I=1,NX
CCRAG    0017416.       DO 4031 J=1,MX
CCRAG    0017420.  4031 H2(J)=F2(J)+B2(J,I)
CCRAG    0017422.       DO 4030 M=1,MX
CCRAG    0017424.       SUM1=0.D0
CCRAG    0017428.       DO 4130 N=1,MX
CCRAG    0017432.  4130 SUM1=SUM1+OA3(N,M,I)*B1(N,I)+OA1(N,M,I)*H2(N)
CCRAG    0017436.  4030 UDOT(M,I)=SUM1
CCRAG    0017440.       PRINT 3011
CCRAG    0017444.  3011 FORMAT(' END OF LOOP 4030 ')
CCRAG    0017500.       WRITE (6,32)
CCRAG    0017600.    32 FORMAT(///' U-MATRIX')
CCRAG    0017700.       DO 33 I=1,MX
CCRAG    0017800.    33 WRITE (6,34) (U(I,J),J=1,NX)
CCRAG    0017900.    34 FORMAT (6(1X,F15.8))
CCRAG    0018400.       CALL VECTOR(W,WDOT,U,UDOT,NY,NZ,NX,HY,HZ,HX,MY,MZ,MX,1,S)
CCRAG    0018500.       DO 37 N=1,NY
CCRAG    0018600.       DO 37 M=1,MY
CCRAG    0018700.    37 VX(M,N)=V(M,N)
CCRAG    0018800.       DO 137 N=1,NY
CCRAG    0018900.       DO 137 M=1,MY
CCRAG    0019000.       B3(M,N)=0.D0
CCRAG    0019100.   137 B4(M,N)=0.D0
CCRAG    0019200.       DO 39 M=1,MY
CCRAG    0019300.       F3(M)=0.D0
CCRAG    0019400.    39 F4(M)=0.D0
CCRAG    0019500.       DO 40 M=2,NY
CCRAG    0019600.       DO 40 J=1,MY
CCRAG    0019800.       SUM2=0.D0
CCRAG    0019900.       DO 41 L=1,MY
CCRAG    0020100.    41 SUM2=SUM2+DA1(L,J,M-1)*S(L,M-1)+DA1(L,J,M)*S(L,M)
CCRAG    0020300.    40 B4(J,M)=B4(J,M-1)+HY*SUM2/2.0D0
CCRAG    0020330.       PRINT 2009
CCRAG    0020360.  2009 FORMAT(' END OF LOOP 40 ')
CCRAG    0020363.       DO 4040 M=2,NY
CCRAG    0020366.       DO 4040 J=1,MY
CCRAG    0020369.       SUM1=0.D0
CCRAG    0020372.       DO 4041 L=1,MY
CCRAG    0020375.  4041 SUM1=SUM1-DA2(L,J,M-1)*S(L,M-1)-DA2(L,J,M)*S(L,M)
CCRAG    0020378.  4040 B3(J,M)=B3(J,M-1)+HY*SUM1/2.0D0
CCRAG    0020381.       PRINT 3012
CCRAG    0020384.  3012 FORMAT(' END OF LOOP 4040')
CCRAG    0020400.       KNT=NOC
```

```
CCRAG    0020450.        DO 42 M=1,KIC
CCRAG    0020500.        MIX=M+KOC
CCRAG    0020550.        DO 42 N=1,NZ
CCRAG    0020600.        K=NY*(N-1)+1
CCRAG    0020650.        KNT=KNT+1
CCRAG    0020700.     42 F4(KNT)=-C1*(UDOT(K,MIX)+WDOT(MIX,N))
CCRAG    0020750.        KNT=0
CCRAG    0020800.        DO 43 M=1,NX
CCRAG    0020850.        L=M2+M
CCRAG    0020900.        DO 43 N=1,NZ
CCRAG    0020950.        K=NY*N
CCRAG    0021000.        KNT=KNT+1
CCRAG    0021050.     43 VBW(KNT)=C2-C1*(UDOT(K,M)+WDOT(L,N))
CCRAG    0021100.        DO 44 I=1,NOC
CCRAG    0021150.        SUM1=0.D0
CCRAG    0021200.        SUM2=0.D0
CCRAG    0021250.        DO 45 J=1,NOC
CCRAG    0021300.     45 SUM1=SUM1+DA(I,J)*VAL(J)-DAB(I,J)*B3(J,NY)
CCRAG    0021350.        DO 46 J=1,NIC
CCRAG    0021400.        MIX=NOC+J
CCRAG    0021450.     46 SUM2=SUM2+DX6(I,J)*VBE(J)+DAC(I,J)*(F4(MIX)+B4(MIX,NY))
CCRAG    0021500.     44 F4(I)=SUM1-SUM2-B4(I,NY)
CCRAG    0021550.        DO 47 I=1,NIC
CCRAG    0021600.        NIX=NOC+I
CCRAG    0021650.        SUM1=0.D0
CCRAG    0021700.        SUM2=0.D0
CCRAG    0021750.        DO 48 J=1,NIC
CCRAG    0021800.        MIX=NOC+J
CCRAG    0021850.     48 SUM1=SUM1+DX1(I,J)*VBE(J)-DX3(I,J)*(F4(MIX)+B4(MIX,NY))
CCRAG    0021900.        DO 49 J=1,NOC
CCRAG    0021950.     49 SUM2=SUM2+DX2(I,J)*B3(J,NY)+DX4(I,J)*(F4(J)+B4(J,NY))
CCRAG    0022000.     47 F3(NIX)=SUM1-SUM2-B3(NIX,NY)
CCRAG    0023600.        DO 50 I=1,NY
CCRAG    0023700.        DO 51 J=1,MY
CCRAG    0023800.        H3(J)=F3(J)+B3(J,I)
CCRAG    0023900.     51 H4(J)=F4(J)+B4(J,I)
CCRAG    0024000.        DO 50 M=1,MY
CCRAG    0024200.        SUM2=0.D0
CCRAG    0024300.        DO 150 N=1,MY
CCRAG    0024500.    150 SUM2=SUM2+DA1(N,M,I)*H3(N)+DA2(N,M,I)*H4(N)
CCRAG    0024700.     50 V(M,I)=SUM2
CCRAG    0024704.         PRINT 3015
CCRAG    0024708.   3015 FORMAT(' END OF LOOP 50')
CCRAG    0024712.        DO 4050 I=1,NY
CCRAG    0024716.        DO 4051 J=1,MY
CCRAG    0024720.        H3(J)=F3(J)+B3(J,I)
CCRAG    0024724.   4051 H4(J)=F4(J)+B4(J,I)
CCRAG    0024728.        DO 4050 M=1,MY
CCRAG    0024732.        SUM1=0.D0
CCRAG    0024736.        DO 4150 N=1,MY
CCRAG    0024740.   4150 SUM1=SUM1+DA3(N,M,I)*H3(N)+DA1(N,M,I)*H4(N)
CCRAG    0024744.   4050 VDOT(M,I)=SUM1
CCRAG    0024748.        PRINT 3016
CCRAG    0024752.   3016 FORMAT(' END OF LOOP 4050 ')
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CCRAG   0024800.       WRITE (6,52)
CCRAG   0024900.    52 FORMAT (///' V-MATRIX')
CCRAG   0025000.       DO 53 I=1,MY
CCRAG   0025100.    53 WRITE (6,34) (V(I,J),J=1,NY)
CCRAG   0025600.       CALL VECTOR(U,UDOT,V,VDOT,NZ,NX,NY,HZ,HX,HY,MZ,MX,MY,O,T)
CCRAG   0025700.       DO 56 N=1,NZ
CCRAG   0025800.       DO 56 M=1,MZ
CCRAG   0025900.    56 WX(M,N)=W(M,N)
CCRAG   0026000.       DO 156 N=1,NZ
CCRAG   0026100.       DO 156 M=1,MZ
CCRAG   0026200.       B5(M,N)=0.D0
CCRAG   0026300.   156 B6(M,N)=0.D0
CCRAG   0026400.       DO 57 M=1,MZ
CCRAG   0026500.    57 F6(M)=0.D0
CCRAG   0026600.       DO 58 M=2,NZ
CCRAG   0026700.       DO 58 J=1,MZ
CCRAG   0026900.       SUM2=0.D0
CCRAG   0027000.      DO 59 L=1,MZ
CCRAG   0027200.    59 SUM2=SUM2+GA1(L,J,M-1)*T(L,M-1)+GA1(L,J,M)*T(L,M)
CCRAG   0027400.    58 B6(J,M)=B6(J,M-1)+HZ*SUM2/2.0D0
CCRAG   0027430.       PRINT 2010
CCRAG   0027460.  2010 FORMAT(' END OF LOOP 58 ')
CCRAG   0027463.       DO 4058 M=2,NZ
CCRAG   0027466.       DO 4058 J=1,MZ
CCRAG   0027469.       SUM1=0.D0
CCRAG   0027472.       DO 4059 L=1,MZ
CCRAG   0027475.  4059 SUM1=SUM1-GA2(L,J,M-1)*T(L,M-1)-GA2(L,J,M)*T(L,M)
CCRAG   0027478.  4058 B5(J,M)=B5(J,M-1)+HZ*SUM1/2.0D0
CCRAG   0027481.       PRINT 3017
CCRAG   0027484.  3017 FORMAT(' END OF LOOP 4058')
CCRAG   0027500.       KNT=0
CCRAG   0027600.       DO 60 M=1,NY
CCRAG   0027700.       L=M3+M
CCRAG   0027800.       DO 60 N=1,NX
CCRAG   0027900.       K=NZ*N
CCRAG   0028000.       KNT=KNT+1
CCRAG   0028100.    60 VCW(KNT)=-C1*(UDOT(L,N)+VDOT(K,M))
CCRAG   0028200.       DO 61 J=1,MZ
CCRAG   0028300.       DO 61 L=1,MZ
CCRAG   0028400.    61 F6(J)=F6(J)+G22(L,J)*VCW(L)-G21(L,J)*B5(L,NZ)
CCRAG   0028500.       DO 62 J=1,MZ
CCRAG   0028600.    62 F6(J)=F6(J)-B6(J,NZ)
CCRAG   0028700.       DO 63 I=1,NZ
CCRAG   0028800.       DO 64 J=1,MZ
CCRAG   0028900.    64 H6(J)=F6(J)+B6(J,I)
CCRAG   0029000.       DO 63 M=1,MZ
CCRAG   0029200.       SUM2=0.D0
CCRAG   0029300.       DO 163 N=1,MZ
CCRAG   0029500.   163 SUM2=SUM2+GA1(N,M,I)*B5(N,I)+GA2(N,M,I)*H6(N)
CCRAG   0029700.    63 W(M,I)=SUM2
CCRAG   0029704.       PRINT 3018
CCRAG   0029708.  3018 FORMAT(' END OF LOOP 63')
CCRAG   0029712.       DO 4063 I=1,NZ
CCRAG   0029716.       DO 4064 J=1,MZ
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CCRAG   0029720. 4064 H6(J)=F6(J)+B6(J,I)
CCRAG   0029722.      DO 4063 M=1,MZ
CCRAG   0029724.      SUM1=0.D0
CCRAG   0029728.      DO 4163 N=1,MZ
CCRAG   0029732. 4163 SUM1=SUM1+GA3(N,M,I)*B5(N,I)+GA1(N,M,I)*H6(N)
CCRAG   0029736. 4063 WDOT(M,I)=SUM1
CCRAG   0029740.      PRINT 3020
CCRAG   0029744. 3020 FORMAT(' END OF LOOP 4063')
CCRAG   0029800.      WRITE (6,65)
CCRAG   0029900.   65 FORMAT(///' W-MATRIX')
CCRAG   0030000.      DO 66 I=1,MZ
CCRAG   0030100.   66 WRITE (6,34) (W(I,J),J=1,NZ)
CCRAG   0030600.      CALL VECTOR(V,VDOT,W,WDOT,NX,NY,NZ,HX,HY,HZ,MX,MY,MZ,O,R)
CCRAG   0030700.      PRINT 69,KTR
CCRAG   0030800.   69 FORMAT(1X,I3)
CCRAG   0030900.      IF(DABS(U(1,NX)).GT.5.0D0) STOP
CCRAG   0031000.      DO 70 I=1,NX
CCRAG   0031100.      DO 70 J=1,MX
CCRAG   0031200.      IF(DABS(U(J,I)-UX(J,I)).GT.1.0D-6) GO TO 23
CCRAG   0031300.   70 CONTINUE
CCRAG   0031400.      DO 71 I=1,NY
CCRAG   0031500.      DO 71 J=1,MY
CCRAG   0031600.      IF(DABS(V(J,I)-VX(J,I)).GT.1.0D-6) GO TO 23
CCRAG   0031700.   71 CONTINUE
CCRAG   0031800.      DO 72 I=1,NZ
CCRAG   0031900.      DO 72 J=1,MZ
CCRAG   0032000.      IF(DABS(W(J,I)-WX(J,I)).GT.1.0D-6) GO TO 23
CCRAG   0032100.   72 CONTINUE
CCRAG   0032200.      WRITE (6,73) KTR
CCRAG   0032300.   73  FORMAT(///' ITERATION NUMBER ',I2,///)
CCRAG   0032400.      KNT=0
CCRAG   0032500.      DO 74 K=1,NZ
CCRAG   0032600.      DO 74 I=1,NY
CCRAG   0032700.      KNT=KNT+1
CCRAG   0032800.      DO 74 J=1,NX
CCRAG   0032900.      L1=K+NZ*(J-1)
CCRAG   0033000.      L2=J+NX*(I-1)
CCRAG   0033100.   74 SIGX(KNT,J)=CI*UDOT(KNT,J)+XL*(VDOT(L1,I)+WDOT(L2,K))
CCRAG   0033200.      KNT=0
CCRAG   0033300.      DO 77 K=1,NX
CCRAG   0033400.      DO 77 I=1,NZ
CCRAG   0033500.      KNT=KNT+1
CCRAG   0033600.      DO 77 J=1,NY
CCRAG   0033700.      L1=K+NX*(J-1)
CCRAG   0033800.      L2=J+NY*(I-1)
CCRAG   0033900.   77 SIGY(KNT,J)=CI*VDOT(KNT,J)+XL*(WDOT(L1,I)+UDOT(L2,K))
CCRAG   0034000.      KNT=0
CCRAG   0034100.      DO 80 K=1,NY
CCRAG   0034200.      DO 80 I=1,NX
CCRAG   0034300.      KNT=KNT+1
CCRAG   0034400.      DO 80 J=1,NZ
CCRAG   0034500.      L1=K+NY*(J-1)
CCRAG   0034600.      L2=J+NZ*(I-1)
CCRAG   0034700.   80 SIGZ(KNT,J)=CI*WDOT(KNT,J)+XL*(UDOT(L1,I)+VDOT(L2,K))
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CCRAG   0034705.      PRINT 3021
CCRAG   0034710. 3021 FORMAT(' END OF JOB ')
CCRAG   0034715.      WRITE (6,35)
CCRAG   0034720.   35 FORMAT(///' UDOT-MATRIX')
CCRAG   0034725.      DO 36 I=1,MX
CCRAG   0034730.   36 WRITE (6,34) (UDOT(I,J),J=1,NX)
CCRAG   0034735.      WRITE (6,54)
CCRAG   0034740.   54 FORMAT(///' VDOT-MATRIX ')
CCRAG   0034745.      DO 55 I=1,MY
CCRAG   0034750.   55 WRITE (6,34) (VDOT(I,J),J=1,NY)
CCRAG   0034755.      WRITE (6,67)
CCRAG   0034760.   67 FORMAT(///' WDOT-MATRIX ')
CCRAG   0034765.      DO 68 I=1,MZ
CCRAG   0034770.   68 WRITE (6,34) (WDOT(I,J),J=1,NZ)
CCRAG   0034800.      WRITE (6,75)
CCRAG   0034900.   75 FORMAT(///' SIGMA-X MATRIX ')
CCRAG   0035000.      DO 76 I=1,MX
CCRAG   0035100.   76 WRITE (6,34) (SIGX(I,J),J=1,NX)
CCRAG   0035200.      WRITE (6,78)
CCRAG   0035300.   78 FORMAT(///' SIGMA-Y MATRIX ')
CCRAG   0035400.      DO 79 I=1,MY
CCRAG   0035500.   79 WRITE (6,34) (SIGY(I,J),J=1,NY)
CCRAG   0035600.      WRITE (6,81)
CCRAG   0035700.   81 FORMAT(///' SIGMA-Z MATRIX ')
CCRAG   0035800.      DO 82 I=1,MZ
CCRAG   0035900.   82 WRITE (6,34) (SIGZ(I,J),J=1,NZ)
CCRAG   0036000.      STOP
CCRAG   0036100.      END
CMINV   0000100.      SUBROUTINE MATINV (X,MMM)
CMINV   0000200.      DOUBLE PRECISION X,Y,W,A,TEST,R,DELT
CMINV   0000300.      DIMENSION X(MMM,MMM), Y(35,35), K(35), W(35,35)
CMINV   0000400.      DATA DELT,LOOPS/1.0D-8,2/
CMINV   0000500.      KSIG=0
CMINV   0000600.      M=MMM
CMINV   0000700.      MM=M-1
CMINV   0000800.      DO 1 J=1,M
CMINV   0000900.      K(J)=J
CMINV   0001000.      DO 1 I=1,M
CMINV   0001100.1     Y(I,J)=X(I,J)
CMINV   0001200.      DO 6 I=1,M
CMINV   0001300.      A=0.D0
CMINV   0001400.      DO 2 J=I,M
CMINV   0001500.      IF (DABS(Y(J,1)).LE.A) GO TO 2
CMINV   0001600.      A=DABS(Y(J,1))
CMINV   0001700.      L=J
CMINV   0001800.2     CONTINUE
CMINV   0001900.      IF (A.EQ.0.D0) GO TO 18
CMINV   0002000.      N=K(L)
CMINV   0002100.      K(L)=K(I)
CMINV   0002200.      K(I)=N
CMINV   0002300.      DO 3 J=1,M
CMINV   0002400.      A=Y(I,J)
CMINV   0002500.      Y(I,J)=Y(L,J)
CMINV   0002600.3     Y(L,J)=A
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CMINV   0002700.      A=Y(I,1)
CMINV   0002800.      DO 4 J=1,MM
CMINV   0002900.4     Y(I,J)=Y(I,J+1)/A
CMINV   0003000.      Y(I,M)=1.DO/A
CMINV   0003100.      DO 6 L=1,M
CMINV   0003200.      IF (L.EQ.I) GO TO 6
CMINV   0003300.      A=Y(L,1)
CMINV   0003400.      DO 5 N=1,MM
CMINV   0003500.5     Y(L,N)=Y(L,N+1)-A*Y(I,N)
CMINV   0003600.      Y(L,M)=-A*Y(I,M)
CMINV   0003700.6     CONTINUE
CMINV   0003800.      DO 10 I=1,M
CMINV   0003900.      IF (K(I).EQ.I) GO TO 10
CMINV   0004000.      DO 7 J=I,M
CMINV   0004100.      IF (K(J).EQ.I) GO TO 8
CMINV   0004200.7     CONTINUE
CMINV   0004300.      GO TO 19
CMINV   0004400.8     DO 9 L=1,M
CMINV   0004500.      A=Y(L,I)
CMINV   0004600.      Y(L,I)=Y(L,J)
CMINV   0004700.9     Y(L,J)=A
CMINV   0004800.      K(J)=K(I)
CMINV   0004900.10    CONTINUE
CMINV   0005000.      DO 15 N=1,LOOPS
CMINV   0005100.      TEST=0.DO
CMINV   0005200.      DO 12 J=1,M
CMINV   0005300.      DO 12 I=1,M
CMINV   0005400.      R=0.DO
CMINV   0005500.      DO 11 L=1,M
CMINV   0005600.11    R=R-X(I,L)*Y(L,J)
CMINV   0005700.      IF (I.EQ.J) R=R+1.0DO
CMINV   0005800.      TEST=DMAX1(TEST,DABS(R))
CMINV   0005900.12    W(I,J)=R
CMINV   0006000.      IF (TEST.LE.DELT) GO TO 16
CMINV   0006100.      DO 14 J=1,M
CMINV   0006200.      DO 14 I=1,M
CMINV   0006300.      A=0.DO
CMINV   0006400.      DO 13 L=1,M
CMINV   0006500.13    A=A+Y(I,L)*W(L,J)
CMINV   0006600.14    Y(I,J)=Y(I,J)+A
CMINV   0006700.15    CONTINUE
CMINV   0006800.      KSIG=3
CMINV   0006900.      PRINT 21, TEST
CMINV   0007000.16    DO 17 J=1,M
CMINV   0007100.      DO 17 I=1,M
CMINV   0007200.17    X(I,J)=Y(I,J)
CMINV   0007300.      GO TO 20
CMINV   0007400.18    KSIG=1
CMINV   0007500.      GO TO 20
CMINV   0007600.19    KSIG=2
CMINV   0007700.20    IF (KSIG.NE.0) PRINT 22, KSIG
CMINV   0007800.      RETURN
CMINV   0007900.C
CMINV   0008000.21    FORMAT (' ERROR = ',D14.6)
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CMINV   0008100.22      FORMAT (' ERROR INDICATOR NUMBER ',I1)
CMINV   0008200.        END
CVECT   0000100.        SUBROUTINE VECTOR(V1,V2,W1,W2,LX,LY,LZ,HX,HY,HZ,LD1,LD2,LD3,IFG,R1)
CVECT   0000200.        IMPLICIT REAL*8 (A-H,O-Z)
CVECT   0000300.        DIMENSION V1(LD2,LY),V2(LD2,LY),W1(LD3,LZ),W2(LD3,LZ),V(98,14),-
CVECT   0000400.       1 VDOT(98,14),W(98,14),WDOT(98,14),R1(LD1,LX),R(98,14)
CVECT   0000500.        M6=LD1
CVECT   0000600.        M5=LD2
CVECT   0000700.        M4=LD3
CVECT   0000800.        NX=LX
CVECT   0000900.        NY=LY
CVECT   0001000.        NZ=LZ
CVECT   0001100.        NXM1=NX-1
CVECT   0001200.        NYM1=NY-1
CVECT   0001300.        NZM1=NZ-1
CVECT   0001400.        M1=NXM1*NZ
CVECT   0001500.        M3=NZM1*NY
CVECT   0001600.        M7=NYM1*NX
CVECT   0001700.        M2=(NX-2)*NZ
CVECT   0001800.        M9=(NZ-2)*NY
CVECT   0001900.        Q1=7.5D-1/HX
CVECT   0002000.        Q2=5.0D-1/(HX*HY)
CVECT   0002100.        Q3=5.0D-1/(HX*HZ)
CVECT   0002200.        Q4=Q1/2.0D0
CVECT   0002300.        Q5=Q2/2.0D0
CVECT   0002400.        Q6=Q3/2.0D0
CVECT   0002500.        DO 201 J=1,NY
CVECT   0002600.        DO 201 I=1,M5
CVECT   0002700.    201 V(I,J)=V1(I,J)
CVECT   0002800.        DO 202 J=1,NY
CVECT   0002900.        DO 202 I=1,M5
CVECT   0003000.    202 VDOT(I,J)=V2(I,J)
CVECT   0003100.        DO 203 J=1,NZ
CVECT   0003200.        DO 203 I=1,M4
CVECT   0003300.    203 W(I,J)=W1(I,J)
CVECT   0003400.        DO 204 J=1,NZ
CVECT   0003500.        DO 204 I=1,M4
CVECT   0003600.    204 WDOT(I,J)=W2(I,J)
CVECT   0003700.        DO 101 I=1,MX
CVECT   0003800.    101 R(I,1)=0.D0
CVECT   0003900.        R(1,NX)=Q1*(VDOT(M2+1,1)-VDOT(M1+1,1))+Q1*(WDOT(NX-1,1)-WDOT(NX,1))-
CVECT   0004000.       1 +Q2*(V(M2+1,1)-V(M1+1,1))+Q3*(W(NX-1,1)-W(NX,1))
CVECT   0004100.        R(NY,NX)=Q1*(VDOT(M2+1,NY)-VDOT(M1+1,NY))+Q1*(WDOT(M4-1,1)-WDOT(M4,1))-
CVECT   0004200.       1 -Q2*(V(M2+1,NY)-V(M1+1,NY))+Q3*(W(M4-1,1)-W(M4,1))
CVECT   0004300.        R(M3+1,NX)=Q1*(VDOT(M1,1)-VDOT(M5,1))+Q1*(WDOT(NX-1,NZ)-WDOT(NX,NZ))-
CVECT   0004400.       1 +Q2*(V(M1,1)-V(M5,1))-Q3*(W(NX-1,NZ)-W(NX,NZ))
CVECT   0004500.        R(M6,NX)=Q1*(VDOT(M1,NY)-VDOT(M5,NY))+Q1*(WDOT(M4-1,NZ)-WDOT(M4,NZ))-
CVECT   0004600.       1 -Q2*(V(M1,NY)-V(M5,NY))-Q3*(W(M4-1,NZ)-W(M4,NZ))
CVECT   0004700.        DO 102 J=2,NXM1
CVECT   0004800.        K=M7+J
CVECT   0004900.        L2=J*NZ
CVECT   0005000.        L1=L2-NZ
CVECT   0005100.        I2=L2+1
CVECT   0005200.        I1=I2-2*NZ
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CVECT   0005300.       L3=L2+NZ
CVECT   0005400.       R(1,J)=Q4*(VDOT(I1,1)-VDOT(I2,1))+Q4*(WDOT(J-1,1)-WDOT(J+1,1))-
CVECT   0005500.      1 +Q5*(V(I1,1)-V(I2,1))+Q6*(W(J-1,1)-W(J+1,1))
CVECT   0005600.       R(NY,J)=Q4*(VDOT(I1,NY)-VDOT(I2,NY))+Q4*(WDOT(K-1,1)-WDOT(K+1,1))-
CVECT   0005700.      1 -Q5*(V(I1,NY)-V(I2,NY))+Q6*(W(K-1,1)-W(K+1,1))
CVECT   0005800.       R(M3+1,J)=Q4*(VDOT(L1,1)-VDOT(L3,1))+Q4*(WDOT(J-1,NZ)-WDOT(J+1,NZ))-
CVECT   0005900.      1 +Q5*(V(L1,1)-V(L3,1))-Q6*(W(J-1,NZ)-W(J+1,NZ))
CVECT   0006000.   102 R(M6,J)=Q4*(VDOT(L1,NY)-VDOT(L3,NY))+Q4*(WDOT(K-1,NZ)-WDOT(K+1,NZ))-
CVECT   0006100.      1 -Q5*(V(L1,NY)-V(L3,NY))-Q6*(W(K-1,NZ)-W(K+1,NZ))
CVECT   0006200.       DO 103 I=2,NYM1
CVECT   0006300.       L=NX*I
CVECT   0006400.       K=NX*(I-1)
CVECT   0006500.       I3=I+M3
CVECT   0006600.       R(I,NX)=Q1*(VDOT(M2+1,I)-VDOT(M1+1,I))+Q1*(WDOT(L-1,1)-WDOT(L,1))-
CVECT   0006700.      1 +Q3*(W(L-1,1)-W(L,1))
CVECT   0006800.       R(I3,NX)=Q1*(VDOT(M1,I)-VDOT(M5,I))+Q1*(WDOT(L-1,NZ)-WDOT(L,NZ))-Q3*(W(L-1,NZ)-W(L,NZ))
CVECT   0006900.       DO 103 J=2,NXM1
CVECT   0007000.       K1=K+J
CVECT   0007100.       L2=J*NZ
CVECT   0007200.       L1=L2-NZ
CVECT   0007300.       I2=L2+1
CVECT   0007400.       I1=I2-2*NZ
CVECT   0007500.       L3=L2+NZ
CVECT   0007600.       R(I,J)=Q4*(VDOT(I1,I)-VDOT(I2,I))+Q4*(WDOT(K1-1,1)-WDOT(K1+1,1))+Q6*(W(K1-1,1)-W(K1+1,1))
CVECT   0007700.   103 R(I3,J)=Q4*(VDOT(L1,I)-VDOT(L3,I))+Q4*(WDOT(K1-1,NZ)-WDOT(K1+1,NZ))-Q6*(W(K1-1,NZ)-W(K1+1,NZ))
CVECT   0007800.       DO 104 K=2,NZM1
CVECT   0007900.       I1=M2+K
CVECT   0008000.       I2=M1+K
CVECT   0008100.       DO 105 I=NY,M9,NY
CVECT   0008200.       I3=I+NY
CVECT   0008300.       R(I+1,NX)=Q1*(VDOT(I1,1)-VDOT(I2,1))+Q1*(WDOT(NX-1,K)-WDOT(NX,K))+Q2*(V(I1,1)-V(I2,1))
CVECT   0008400.       R(I3,NX)=Q1*(VDOT(I1,NY)-VDOT(I2,NY))+Q1*(WDOT(M4-1,K)-WDOT(M4,K))-Q2*(V(I1,NY)-V(I2,NY))
CVECT   0008500.       DO 105 J=2,NXM1
CVECT   0008600.       L2=K+J*NZ
CVECT   0008700.       L1=L2-2*NZ
CVECT   0008800.       L=M7+J
CVECT   0008900.       R(I+1,J)=Q4*(VDOT(L1,1)-VDOT(L2,1))+Q4*(WDOT(J-1,K)-WDOT(J+1,K))+Q5*(V(L1,1)-V(L2,1))
CVECT   0009000.   105 R(I3,J)=Q4*(VDOT(L1,NY)-VDOT(L2,NY))+Q4*(WDOT(L-1,K)-WDOT(L+1,K))-Q5*(V(L1,NY)-V(L2,NY))
CVECT   0009100.        DO 104 L=2,NYM1
CVECT   0009200.       I=NY*(K-1)+L
CVECT   0009300.       I3=L*NX
CVECT   0009400.       I4=I3-NX
CVECT   0009500.       R(I,NX)=Q1*(VDOT(I1,L)-VDOT(I2,L))+Q1*(WDOT(I3-1,K)-WDOT(I3,K))
CVECT   0009600.       DO 104 J=2,NXM1
CVECT   0009700.       K1=I4+J
CVECT   0009800.       L2=K+J*NZ
CVECT   0009900.       L1=L2-2*NZ
CVECT   0010000.   104 R(I,J)=Q4*(VDOT(L1,L)-VDOT(L2,L))+Q4*(WDOT(K1-1,K)-WDOT(K1+1,K))
CVECT   0010100.       IF (IFG.NE.1) GO TO 107
CVECT   0010104.       R(7*NY+1,1)=Q1*(VDOT(8,1)-VDOT(NZ+8,1))+Q1*(WDOT(1,8)-WDOT(2,8))+Q2*(V(8,1)-V(NZ+8,1))
CVECT   0010106.       R(8*NY,1)=Q1*(VDOT(8,NY)-VDOT(NZ+8,NY))+Q1*(WDOT(M7+1,8)-WDOT(M7+2,8))-Q2*(V(8,NY)-V(NZ+8,NY))
CVECT   0010110.       R(8*NY+1,1)=Q1*(VDOT(9,1)-VDOT(NZ+9,1))+Q1*(WDOT(1,9)-WDOT(2,9))+Q2*(V(9,1)-V(NZ+9,1))
CVECT   0010120.       R(9*NY,1)=Q1*(VDOT(9,NY)-VDOT(NZ+9,NY))+Q1*(WDOT(M7+1,9)-WDOT(M7+2,9))-Q2*(V(9,NY)-V(NZ+9,NY))
CVECT   0010130.       R(9*NY+1,1)=Q1*(VDOT(10,1)-VDOT(NZ+10,1))+Q1*(WDOT(1,10)-WDOT(2,10))+Q2*(V(10,1)-V(NZ+10,1))
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CVECT  0010140.       R(10*NY,1)=Q1*(VDOT(10,NY)-VDOT(NZ+10,NY))+Q1*(WDOT(M7+1,10)-WDOT(M7+2,10))-Q2*(V(10,NY)-V(NZ+10,NY))
CVECT  0010200.       R(10*NY+1,1)=Q1*(VDOT(11,1)-VDOT(NZ+11,1))+Q1*(WDOT(1,11)-WDOT(2,11))+Q2*(V(11,1)-V(NZ+11,1))
CVECT  0010250.       R(11*NY,1)=Q1*(VDOT(11,NY)-VDOT(NZ+11,NY))+Q1*(WDOT(M7+1,11)-WDOT(M7+2,11))-Q2*(V(11,NY)-V(NZ+11,NY))
CVECT  0010300.       R(11*NY+1,1)=Q1*(VDOT(12,1)-VDOT(NZ+12,1))+Q1*(WDOT(1,12)-WDOT(2,12))+Q2*(V(12,1)-V(NZ+12,1))
CVECT  0010350.       R(12*NY,1)=Q1*(VDOT(12,NY)-VDOT(NZ+12,NY))+Q1*(WDOT(M7+1,12)-WDOT(M7+2,12))-Q2*(V(12,NY)-V(NZ+12,NY))
CVECT  0010400.       R(12*NY+1,1)=Q1*(VDOT(13,1)-VDOT(NZ+13,1))+Q1*(WDOT(1,13)-WDOT(2,13))+Q2*(V(13,1)-V(NZ+13,1))
CVECT  0010450.       R(13*NY,1)=Q1*(VDOT(13,NY)-VDOT(NZ+13,NY))+Q1*(WDOT(M7+1,13)-WDOT(M7+2,13))-Q2*(V(13,NY)-V(NZ+13,NY))
CVECT  0010600.       R(13*NY+1,1)=Q1*(VDOT(14,1)-VDOT(NZ+14,1))+Q1*(WDOT(1,14)-WDOT(2,14))-Q3*(W(1,14)-W(2,14))-
CVECT  0010650.      1+Q2*(V(14,1)-V(NZ+14,1))
CVECT  0010700.       R(14*NY,1)=Q1*(VDOT(14,NY)-VDOT(NZ+14,NY))+Q1*(WDOT(M7+1,14)-WDOT(M7+2,14))-Q3*(W(M7+1,14)-W(M7+2,14))-
CVECT  0010750.      1-Q2*(V(14,NY)-V(NZ+14,NY))
CVECT  0010800.       DO 106 I=2,NYM1
CVECT  0010850.       L=NX*(I-1)+1
CVECT  0010855.       I7=7*NY+I
CVECT  0010860.       I8=8*NY+I
CVECT  0010870.       I9=9*NY+I
CVECT  0010900.       I10=10*NY+I
CVECT  0010960.       I11=11*NY+I
CVECT  0011020.       I12=12*NY+I
CVECT  0011080.       I13=13*NY+I
CVECT  0011140.       R(I7,1)=Q1*(VDOT(8,I)-VDOT(NZ+8,I))+Q1*(WDOT(L,8)-WDOT(L+1,8))
CVECT  0011146.       R(I8,1)=Q1*(VDOT(9,I)-VDOT(NZ+9,I))+Q1*(WDOT(L,9)-WDOT(L+1,9))
CVECT  0011148.       R(I9,1)=Q1*(VDOT(10,I)-VDOT(NZ+10,I))+Q1*(WDOT(L,10)-WDOT(L+1,10))
CVECT  0011150.       R(I10,1)=Q1*(VDOT(11,I)-VDOT(NZ+11,I))+Q1*(WDOT(L,11)-WDOT(L+1,11))
CVECT  0011200.       R(I11,1)=Q1*(VDOT(12,I)-VDOT(NZ+12,I))+Q1*(WDOT(L,12)-WDOT(L+1,12))
CVECT  0011250.       R(I12,1)=Q1*(VDOT(13,I)-VDOT(NZ+13,I))+Q1*(WDOT(L,13)-WDOT(L+1,13))
CVECT  0011350.   106 R(I13,1)=Q1*(VDOT(14,I)-VDOT(NZ+14,I))+Q1*(WDOT(L,14)-WDOT(L+1,14))-Q3*(W(L,14)-W(L+1,14))
CVECT  0012500.   107 DO 205 J=1,NX
CVECT  0012600.       DO 205 I=1,M6
CVECT  0012700.   205 R1(I,J)=R(I,J)
CVECT  0012800.       RETURN
CVECT  0012900.       END
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