A CLASS OF TIME DOMAIN MODELS FOR THE NUMERICAL SOLUTION OF TRANSMISSION LINE PROBLEMS

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ABSTRACT

A CLASS OF TIME DOMAIN MODELS
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by Thomas Lynn Drake

The requirements of modern technology necessitate extremely large and complex systems which contain transmission lines as components. These systems no longer are composed of components which are described by linear equations. Therefore, time domain techniques must be used to analyze these systems. The equations which describe these large and complex systems are most easily solved by numerical techniques with large scale digital computers.

The specific subject of this thesis is the derivation of three general classes of time domain models, using difference equations, which give the numerical solution to the lossless transmission line of finite length. In addition to giving the numerical solution along the line, any model relates the voltage and current variables, corresponding to the transmission line linear graph representation, in a manner such that each graph element which represents each port of the transmission line can be formulated in the system graph as either a branch or a chord. The system for which the transmission line is a component may be nonlinear as well as linear.

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One of the main results of this thesis other than deriving the three classes of time domain models is the approach which is used in performing these derivations. Instead of directly approaching the general transmission line problem, three transmission line problems, each having certain identifying characteristics, are first treated by standard numerical methods. The superposition principle, even though the mathematical description of the boundaries may be nonlinear, is then applied to combine the results for these specific transmission line problems to obtain these three classes of time domain models. This general approach by-passes a number of the difficulties which are normally encountered by convential techniques.

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Thomas Lynn Drake

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I. INTRODUCTION

1.0 Introduction

A lossless parallel transmission line of length L is described mathematically by the telegrapher's equations

$$-\frac{\partial V(x,t)}{\partial x} = l\,\frac{\partial I(x,t)}{\partial t}$$

$$-\frac{\partial I(x,t)}{\partial x} = c\,\frac{\partial V(x,t)}{\partial t}$$

where l and c are the inductance and capacitance parameters for the line. For a given set of initial conditions and an appropriate boundary condition for each end of the line, there are a number of techniques, both analytical and numerical, discussed in the literature for finding the solution to these equations.

Most analytical techniques for finding the solution to these equations are applicable only to a restricted class of problems, generally those for which the mathematical descriptions of the boundaries are linear. Numerical methods on the other hand, are by no means so restricted. In fact, most numerical methods of solution allow the mathematical descriptions of the boundaries to be nonlinear as well as linear. Hence, the subject of the thesis is concerned with numerical methods of solution for these equations.

Any numerical technique consists of a finite set of equations, obtained by some method, for which the numerical solution to these equations approximates $V(x_j,t_k)$ and $I(\bar{x}_p,t_k)$ respectively on the two sets of points in the x,t-plane, (x_j,t_k) and $(\bar{x}_p,t_k)$, where the two sets of points, (x_j) and $(\bar{x}_p)$, are finite. The two sets of points, (x_j,t_k) and $(\bar{x}_p,t_k)$ are respectively called the voltage and current net points. In addition, the two sets of points, (x_j) and $(\bar{x}_p)$, are respectively called the voltage and current node points. Normally, the sets of points, (t_k), (x_j), and $(\bar{x}_p)$, are chosen such that the points belonging to any given set are equally spaced. Hence, if Δx and Δt are the increments of the variables x and t, the sets of voltage and current net points are given by $x_j=x_0+j\Delta x$, $t_k=k\Delta t$, where $j=0,1,2,\ldots,J$ and $k=0,1,2,\ldots,$ and $\bar{x}_p=\bar{x}_0+p\Delta x$, $t_k=k\Delta t$, where $p=0,1,2,\ldots,P$ and $k=0,1,2,\ldots,$. Generally, x_0 and $\bar{x}_0$ are related either as $x_0=\bar{x}_0$ or $x_0=\bar{x}_0\pm\Delta x/2$.

The numerical techniques which are discussed in the literature can be placed in one of three general categories. The first such category contains numerical techniques which employ a Fourier series. This technique simply uses numerical techniques to find the Fourier coefficients of the series instead of the normal analytical techniques. This method of solution generally can only be used in cases where the mathematical descriptions of the boundaries are linear.

The second category contains the methods which approximate the transmission line by a passive lumped parameter electrical equivalent network. The resulting network equations can then be solved by some numerical technique. This approximate equivalent network approach is probably the most widely used by the engineering profession. The biggest shortcomings of this approach are that all physically realizable passive networks either require a large number of elements to be a good approximation, an exceedingly small Δt to solve the network equations numerically, or both.

The last category contains finite-difference methods for the approximate numerical solution for problems of this kind. This is the general approach which has been studied extensively by applied mathematicians. The partial derivatives are first approximated by difference equations at the net points and the resulting difference equations are then solved. Instead of attempting to solve the telegrapher's equations, the literature is primarily concerned with solving the wave equation by finite-difference methods. Summaries of these methods are given by Richtmyer (1), Kunz (2), and Fox (3). These methods are derived on the basis that each end of the line is terminated in a voltage (current) source which has a zero source impedance (admittance). If the line is terminated differently, most of the properties which are derived for these methods are no longer valid. Therefore, their biggest shortcoming is that most practical problems encountered in electrical engineering do not have boundary conditions for which these methods were intended.

The specific subject of the thesis is the derivation of three general classes of numerical time domain models, using difference equations, which give the approximate numerical solution to the lossless transmission line of finite length. The equations, corresponding to any given time domain model, are generally not realizable as a passive lumped parameter electrical network. In addition, these three classes of models are applicable for the practical problems encountered in electrical engineering. The mathematical descriptions of the transmission line boundaries are allowed to be nonlinear.

The mathematical development of the thesis treats a normalized set of telegrapher's equations which are obtained by introducing a change of variable. By letting

$$V(x,t) = \sqrt{l/c}\, E(x,t),$$

the normalized telegrapher's equations can be written as

$$-\frac{\partial E(x,t)}{\partial x} = \sqrt{lc}\,\frac{\partial I(x,t)}{\partial t}$$

$$-\frac{\partial I(x,t)}{\partial x} = \sqrt{lc}\,\frac{\partial E(x,t)}{\partial t}.$$

The resulting set of equations has the properties that the characteristic impedance has been transformed to unity and the phase velocity v_p remains unchanged. The quantity $\sqrt{lc}$ is easily recognized to be $1/v_p$.

Each numerical time domain model is derived on the basis that the transmission line, having finite length, is to be a two port component in a given system. For any given model, the set of equations which define this model are such that the transmission line terminals can be represented as a two part linear graph with one element in each part. In addition to giving the approximate solution at the node points, these equations relate the voltage and current variables, corresponding to the transmission line linear graph representation, in a manner such that each graph element can be formulated in the system graph as either a branch or a chord.

Unless otherwise stated, the mathematical development assumes that both linear graph elements are chords. Chapter 5 shows that the remaining three cases, one by symmetry, immediately follow from this case.

For this case, the increment Δx is chosen as $\Delta x = L/k$, where k is a positive integer. The set of voltage node points, (x_j), are defined as $x_j = j\Delta x$, where $j=0,1,2,\ldots,k$. In addition, the set of current node points, $(\bar{x}_p)$, are defined as $\bar{x}_p = p\Delta x + \Delta x/2$, where $p=0,1,2,\ldots,k-1$. These sets of node points are defined differently for the cases where either one or both graph elements are not in the cotree of the system graph.

The node points in a given set are equally spaced. Hence, standard numerical methods which use equally spaced data points are applicable for the derivation of the time domain models. On the other hand, these two sets of node points are purposely defined so that the intersection of the two sets is the null set. The basic reason for defining the node points in this manner is that the derivations and the construction of any time domain model is greatly simplified. This will become evident during the derivation of these methods.

1.1 Superposition Principle.

All three classes of time domain models are derived on the basis of certain mathematical properties of the theoretical solution to the normalized telegrapher's equations. One important property on which the derivations are based is the superposition principle. Hence, certain specific applications of this principle must be discussed first in order to indicate the development of the three classes of time domain models.

For the purposes of this discussion, let us assume that the transmission line is a component in a given system. In addition, let us assume that the entire system can be represented as a two part linear graph containing two elements in each part for which the linear graph which represents the transmission line is the cotree of the system graph.

As far as the solution of normalized telegrapher's equations is concerned, the initial and boundary conditions are given as

$$E(x,0) = h_1(x) \qquad 0 < x < L$$

$$I(x,0) = h_2(x) \qquad 0 \leq x \leq L$$

$$E(0,t) = E_{s1}(I(0,t),t) \qquad 0 \leq t < \infty$$

$$E(L,t) = E_{s2}(I(L,t),t) \qquad 0 \leq t < \infty$$

where $h_1(x)$ and $h_2(x)$ are specified functions of x. The functions $E_{s1}(I(0,t),t)$ and $E_{s2}(I(L,t),t)$ are determined in some manner for $t \geq 0$ by the mathematical description of the system components which correspond to the branches of the tree.

Because of the linearity of the telegrapher's equations the superposition principle can be applied at $t=t_0$ as long as the boundary conditions for the telegrapher's equations are properly defined. This property is used to subdivide the transmission line problem at $t=t_0$ into three problems, each having certain identifying characteristics.

Let us consider the three transmission line problems which are described by the following sets of initial and boundary conditions.

$$E_1(x,t_0) = 0 \qquad 0 < x < L \qquad 1.1.0$$

$$I_1(x,t_0) = 0 \qquad 0 < x \leq L$$

$$E_1(L,t) = 0 \qquad t_0 \le t < \infty$$

$$E_1(0,t) = E_{s1}(I(0,t),t) \qquad t_0 \le t < \infty$$

$$I_1(0,t_0) = E_1(0,t_0)$$

$$E_2(x,t_0) = 0 \qquad 0 < x < L \qquad 1.1.1$$

$$E_2(L,t) = E_{s2}(I(L,t),t) \qquad t_0 \le t < \infty$$

$$I_2(x,t_0) = 0 \qquad 0 \le x < L$$

$$E_2(0,t) = 0 \qquad t_0 \le t < \infty$$

$$I_2(L,t_0) = E_2(L,t_0)$$

$$E_3(x,t_0) = E(x,t_0) \qquad 0 < x < L \qquad 1.1.2$$

$$I_3(x,t_0) = I(x,t_0) \qquad 0 < x < L$$

$$E_3(0,t) = E_3(L,t) = 0 \qquad t_0 \le t < \infty$$

$$I_3(0,t_0) = I(0,t_0) - E(0,t_0)$$

$$I_3(L,t_0) = I(L,t_0) + E(L,t_0)$$

It is quite clear that the solution for $E(x,t)$ and $I(x,t)$ for $t \ge t_0$ is given as

$$E(x,t) = E_1(x,t) + E_2(x,t) + E_3(x,t)$$

$$I(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t)$$

where all current orientations are from $x=0$ to $x=L$.

Upon examining these three transmission line problems, one finds that there are two transmission line problems which have zero initial conditions and one identically zero boundary condition. The remaining problem has the boundary conditions specified as zero but has initial conditions which may be nonzero. For the purposes of simplifying the discussion throughout the thesis, the transmission line problems which have the identifying characteristics the same as the problems which are described by Eqs. 1.1.0, 1.1.1, or 1.1.2 are classified respectively as problems of type 1, type 2, or type 3.

At $t=t_0\Delta t$, the same subdivision process can be applied to the solution of each of the two transmission line problems which are described by Eqs. 1.1.0 and 1.1.1. These two solutions at $t=t_0+\Delta t$ define two problems of type 3, a problem of type 1, and a problem of type 2. This subdivision process in effect states that the solution for $E(x,t)$ and $I(x,t)$ for $t \geq t_0+\Delta t$ can be considered the sum of the solutions of a problem of type 1, a problem of type 2, and three problems of type 3.

Let us consider the two problems of type 3 which were defined at $t=t_0+\Delta t$. Both of these problems are determined respectively by some process which converts the boundary conditions $E(0,t)$ and $E(L,t)$ for the interval $t_0 \leq t < t_0+\Delta t$ into initial conditions for these two problems of type 3 which were defined at $t=t_0+\Delta t$.

At $t=t_0+\Delta t$, the three problems of type 3 can be combined by superposition to define one problem of type 3. As a result, the solution for $E(x,t)$ and $I(x,t)$ for $t \geq t_0+\Delta t$ can also be considered as the sum of the solutions of a problem of type 1, type 2, and type 3.

The specific approach of the thesis is then first to derive two general classes of time domain models which are applicable for obtaining the approximate numerical solution for $E_3(x,t)$ and $I_3(x,t)$ at the defined node points. Next, a method for converting the boundary conditions $E(0,t)$ and $E(L,t)$ for the interval $t_0 \leq t < t_0+\Delta t$ into initial conditions in order to define two problems of type 3 at $t=t_0+\Delta t$ is derived. It is quite clear that a time domain model for the problem of type 3 and the method for converting the boundary conditions into initial conditions can be combined by superposition to obtain a time domain model which is valid for the original problem. This combination process introduces three classes of time domain models.

1.2 Thesis Outline.

According to the discussion as given in section 1.1, it is necessary to derive a time domain model which is applicable for the problem of type 3. In order to obtain these models, the fact is used that a problem of type 3 has the same solution as an infinite line with certain periodic initial conditions for the interval $0 \leq x \leq L$. Therefore, instead of directly approaching the problem of type 3, the infinite transmission line is treated extensively in Chapter 2.

In Chapter 2, two classes of time domain models are derived for the infinite line. One such class of models is derived on the basis of a basic numerical solution which is defined in this chapter. The other such class of models is derived by obtaining a set of ordinary differential equations in t by approximating the partial derivatives with respect to x at certain defined node points by difference methods.

Chapter 3 shows that both classes of time domain models which were derived for the infinite line can be transformed into the finite line problem of type 3. As a result, all the properties which are valid for the infinite line, having periodic initial conditions, are also valid for the problem of type 3. One important result of this chapter is the derivation of a matrix which is called the transformation matrix which transforms all the properties of the infinite line to this finite line problem of type 3.

In Chapter 4, a class of methods is derived which transforms the boundary condition, $E(0,t)$, for a problem of type 1 for the interval $t_0 \leq t < t_0+\Delta t$ into an initial condition at $t=t_0+\Delta t$. Each method derives a set of launching numbers, one for each node, such that the initial condition at each node point at $t=t_0+\Delta t$ is the product of $E(0,n\Delta t)$ and a launching number. Because of the similarity existing between a problem of type 1 and type 2, this class of methods for obtaining these launching numbers is also valid for the problem of type 2. The specific approach of this chapter is the use of the superposition principle in order to show that for a given problem of type 1, its solution at $t=p\Delta t$ is the sum of the solutions of p problems of type 3 which are staggered in time for $0 < x \leq L$. This class of methods is then derived on the basis that the sum of these p solutions for the p problems of type 3 must approximate the solution to the problem of type 1 at $t=p\Delta t$ in a certain sense.

Since the results of Chapters 3 and 4 are valid only for the case where both graph elements which represent the transmission line are formulated as chords, Chapter 5 shows that these results are also valid for the other three cases which arise in the formulation. In order to obtain this result, the node points must be redefined such that the node point at either $x=0$ or $x=L$ is a voltage (current) node point when the linear graph element which represents this port is formulated as a chord (branch).

Since there is no current node point at either $x=0$ or $x=L$, there is no information present in these equations about the approximate solution to $I(x,t)$ at these points. In order to provide this information, Chapter 6 derives some properties which are valuable for determining an interpolation formula which is applicable for determining the solution for the currents at these points. In addition, these special properties are also applicable in certain situations for determining the solution of either $E(x,t)$ or $I(x,t)$ at other values of x than the defined node points.

The principle objective of Chapter 7 is to apply the superposition principle to combine the results of Chapters 2, 3, 4, 5, and 6 to obtain three classes of time domain models which are applicable to the general transmission line problems which occur in electrical engineering. Since there are two classes of numerical time domain models defined for the problem of type 3, the three classes of models are created by the method for which the results of Chapter 3 are integrated with the results of Chapter 4. In addition, a transmission line problem is worked to illustrate the concepts which are presented in this chapter.

II. INFINITE TRANSMISSION LINE

2.0 Introduction.

An infinite lossless parallel transmission line is described mathematically by the normalized telegrapher's equations. In order to distinguish the infinite line case from the finite transmission line case, the variables, $V(x,t)$ and $S(x,t)$, are used in place of $E(x,t)$ and $I(x,t)$ to represent the solution to these partial differential equations.

The specific subject of this chapter is the definition of two general classes of time domain models, which use difference methods, for the purpose of obtaining an approximate numerical solution to the infinite line. One such class of models is derived on the basis of a basic numerical solution while the other class of models is obtained by approximating the partial derivatives with respect to x at certain node points and obtaining a set of ordinary linear differential equations. In addition, certain properties of the approximate solution such as periodicity will be investigated.

Once the initial condition functions, $V(x,0)$ and $S(x,0)$, are specified for $-\infty < x < \infty$, the analytical solution for $V(x,t)$ and $S(x,t)$ can be immediately written for all x and t as

$$V(x,t) = \frac{V(x-v_pt,0) + S(x-v_pt,0)}{2} + \frac{V(x+v_pt,0) - S(x+v_pt,0)}{2}$$

$$S(x,t) = \frac{V(x-v_pt,0) + S(x-v_pt,0)}{2} + \frac{S(x+v_pt,0) - V(x+v_pt,0)}{2} .$$

Since the analytical solution for infinite line problem can be immediately written from the specified initial conditions, it may now seem of limited value to pursue the definition of the two general classes of time domain models. The only reason for treating the infinite line time domain models is that all of these models and their properties are also valid for the finite line problem of type 3. The main advantage of treating the infinite line over the finite line is that there are no boundaries which must be taken into consideration. Hence, standard numerical methods which use more accurate central differences can be used for the purpose of deriving these models.

The voltage and current node points are defined respectively as $x_j=j\Delta x$, $j=0,\pm 1,\pm 2,\pm 3,\ldots,$ and $\bar{x}_p=p\Delta x-\Delta x/2$, $p=0,\pm 1,\pm 2,\pm 3,\ldots,$. In order to simplify the notation, the functions $f(2n,t)$ and $f(2n-1,t)$ are respectively the approximate solution for $V(n\Delta x,t)$ and $S(n\Delta x-\Delta x/2,t)$ which are obtained by some time domain model.

Each initial condition is allowed to be periodic in x with period 2L or aperiodic in x. If an initial condition is aperiodic in x, where either V(x,0) or S(x,0) is given by h(x), for $-\infty < x < \infty$, h(x) is required to satisfy the Fourier integral theorem and to be differentiable almost everywhere. When V(x,0) or S(x,0) is periodic, this initial condition is required to be differentiable almost everywhere.

2.1 Basic Numerical Solution.

The theoretical solution for f(n,t) for all t and $n=0,\pm1,\pm2,\pm3,\ldots,$ in terms of the specified initial conditions is

$$f(n,t) = \frac{V(n\Delta x/2-v_p t,0) + (-1)^n V(n\Delta x/2+v_p t,0)}{2} + \frac{S(n\Delta x/2-v_p t,0) + (-1)^{n+1} S(n\Delta x/2+v_p t,0)}{2} . \qquad 2.1.0$$

Let us assume that the specified initial condition V(x,0) is known only at the voltage node points. In addition, let us assume that S(x,0) is only known at the current node points. The values of V(x,0) and S(x,0) at values of x other than at their respective node points can be obtained by means of interpolation. One method of interpolation is to define two infinite series, $P_v(x)$ and $P_s(x)$, which pass through f(2n,0) and f(2n-1,0) respectively for $n=0\pm1,\pm2,\pm3,\ldots,$. It is quite evident from the definition of f(n,0) that $P_v(x)$ and $P_s(x)$ are the infinite series approximations to $V(x,0)$ and $S(x,0)$. Hence, an approximate solution for $f(n,t)$ can be obtained by replacing $V(x,0)$ and $S(x,0)$ in Eq. 2.1.0 by $P_v(x)$ and $P_s(x)$. This approximate solution for $f(n,t)$ is then given by

$$f(n,t) = \frac{P_v(n\Delta x/2-v_pt) + (-1)^n P_v(n\Delta x/2+v_pt)}{2} + \frac{P_s(n\Delta x/2-v_pt) + (-1)^{n+1} P_s(n\Delta x/2+v_pt)}{2} . \qquad 2.1.1$$

It is quite clear that there are a number of infinite series which pass through the prescribed points but are not equal at the remaining values of x. For example, $P_v(x)$ is given be the two infinite series

$$P_v(x) = \sum_{n=-\infty}^{\infty} f(2n,0) \frac{\sin^2 \pi(x/\Delta x-n)}{\pi^2(x/\Delta x-n)^2}$$

$$P_v(x) = \sum_{n=-\infty}^{\infty} f(2n,0) \frac{\sin \pi(x/\Delta x-n)}{\pi(x/\Delta x-n)}$$

where both series equal $f(2n,0)$ at $x=n\Delta x$. In order to correct this ambiguity, the exact form of $P_v(x)$ and $P_s(x)$ must be defined. Once these two infinite series are uniquely defined, Eq. 2.1.1 will be defined as the basic numerical solution for $f(n,t)$.

Let the operator δ^k for $k=0,1,2,\ldots$, be defined as

$$\delta^k f(n,t) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(n-k+2i,t) \qquad 2.1.2$$

where $\binom{k}{i}$ is a binomial coefficient. Then the operator δ^{k+j} has the property

$$\delta^{j+k} f(n,t) = \delta^j(\delta^k f(n,t)) = \delta^k(\delta^j f(n,t)). \qquad 2.1.3$$

If Δ^k corresponds to the standard central difference operator (2) for fixed t, this operator, Δ^k, is related to the operator δ^k by

$$\Delta^k f(n,t) = (-1)^k \delta^k f(n,t).$$

The most accurate interpolation polynomials for a given interval are those which lead to interpolation at or near the middle of the interval. These types of formulas are called central-difference formulas. One such formula is Stirling's interpolation formula (2) which represents an interpolation polynomial that is based on tabulated values symmetrically placed with respect to x_0. Hence if $x_0 = n\Delta x$ and $\bar{x} = (x-x_0)/\Delta x$, one method for finding $P_v(x)$ is by using an infinite degree Sterling's polynomial interpolation formula. This formula for $P_v(\bar{x})$ can be written as

$$P_v(\bar{x}) = f(2n,0) - \bar{x}\mu\delta f(2n,0) + \qquad 2.1.4$$

$$\sum_{k=1}^{\infty} \frac{\prod_{j=0}^{k-1}(\bar{x}^2-j^2)}{(2k)!}\delta^{2k}f(2n,0) -$$

$$\sum_{k=2}^{\infty} \bar{x}\frac{\prod_{j=1}^{k-1}(\bar{x}^2-j^2)}{(2k-1)!}\mu\delta^{2k-1}f(2n,0)$$

where

$$\mu\delta^{2k-1}f(2n,0) = \frac{\delta^{2k-1}f(2n+1,0) + \delta^{2k-1}f(2n-1,0)}{2} .$$

It is quite clear that $P_v(\bar{x})=f(2n+2i,0)$ for $\bar{x}=i$.

Another type of central difference formula is a Bessel's interpolation formula (2). This interpolation formula is also based on tabulated values symmetrically placed with respect to $x_1=x_0+\Delta x/2$. By letting $v=(x-x_1)/\Delta x$, $P_v(v)$ can be written in terms of a Bessel's interpolation formula as

$$P_v(v) = \mu\delta^0 f(2n+1,0) - v\delta f(2n+1,0) + \qquad 2.1.5$$

$$\sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k}\left[v^2-(2j-1)^2/4\right]}{(2k)!}\mu\delta^{2k}f(2n+1,0) -$$

$$\sum_{k=1}^{\infty} v \frac{\prod_{j=1}^{k}\left[v^2-(2j-1)^2/4\right]}{(2k+1)!} \delta^{2k+1} f(2n+1,0)$$

where

$$\mu\delta^{2k} f(2n+1,0) = \frac{\delta^{2k} f(2n+2,0) + \delta^{2k} f(2n,0)}{2} .$$

For $v=i-1/2$, $P_v(v) = f(2n+2i,0)$.

Consider the even function with respect to t which is given by

$$\frac{P_v(n\Delta x-v_p t) + P_v(n\Delta x+v_p t)}{2} .$$

By using Eq. 2.1.4, this function is given by

$$\frac{P_v(n\Delta x-v_p t) + P_v(n\Delta x+v_p t)}{2} = f(2n,0) +$$

$$\sum_{k=1}^{\infty} \frac{\prod_{j=0}^{k-1}\left[(v_p t/\Delta x)^2-j^2\right]}{(2k)!} \delta^{2k} f(2n,0)$$

In the same manner, the odd function with respect to t, given by

$$\frac{P_v(n\Delta x+\Delta x/2-v_p t) - P_v(n\Delta x+\Delta x/2+v_p t)}{2}$$

can be written terms of Eq. 2.1.5 as

$$\frac{P_v(n\Delta x+\Delta x/2-v_pt) - P_v(n\Delta x+\Delta x/2+v_pt)}{2} = \frac{v_pt}{\Delta x} \quad f(2n+1,0) +$$

$$\sum_{k=1}^{\infty} \frac{v_pt}{\Delta x} \frac{\prod_{j=1}^{k}\left[(v_pt/\Delta x)^2-(2j-1)^2/4\right]}{(2k+1)!} \delta^{2k+1}f(2n+1,0) .$$

If the same interpolation formulas are used to find $P_s(x)$, then an approximate solution for $f(n,t)$ is given by

$$f(n,t) = f(n,0) + v\delta f(n,0) + \frac{v^2}{2!}\delta^2 f(n,0) + \qquad 2.1.6$$

$$\frac{v(v^2-1/4)}{3!}\delta^3 f(n,0) + \frac{v^2(v^2-1)}{4!}\delta^4 f(n,0) +\ldots+\ldots$$

where $v=v_pt/\Delta x$. It will later become evident that both Stirling's and Bessel's formula give the same infinite series for $P_v(x)$. Since Eq. 2.1.6 is obtained directly from Eq. 2.1.1 by replacing $P_v(x)$ and $P_s(x)$ by either their Stirling's or Bessel's formula representation, Eq. 2.1.6 must be the basic numerical solution for $f(n,t)$. When Bessel's and Sterling's formulas are used to obtain interpolation polynomials of finite degree, Bessel's and Stirling's interpolating polynomials respectively pass through an even and an odd number of points. Hence, these polynomials are generally different in the finite degree case. For the infinite degree case, these infinite series as given by Eqs. 2.1.4 and 2.1.5 give the same function. In order to show this result, it is necessary to write an alternate form for Eqs. 2.1.4 and 2.1.5.

Consider the polynomial $p_j^{2n}(x)$ of degree 2n which is given by

$$p_j^{2n}(x) = \frac{(-1)^{j-1} n!\ n!}{(n+j)!\ (n-j)!} \times \frac{\prod_{p=1}^{n} \left[1 - \frac{x^2}{p^2 \Delta x^2} \right]}{j\Delta x - x} .$$

This polynomial has the property that

$$p_j^{2n}(x) = 0 \qquad x = p\Delta x,\ p = n, -n+1, \ldots, j-1$$

$$p_j^{2n}(x) = 0 \qquad x = p\Delta x,\ p = j+1, j+2, \ldots, n$$

$$p_j^{2n}(x) = 1 \qquad x = j\Delta x$$

By letting $\bar{x} = x - \bar{n}\Delta x$, an interpolating polynomial which passes through $f(2\bar{n}+2i, 0)$ for $i = 0, \pm 1, \pm 2, \ldots, \pm n$, can be written as

$$p^{2n}(\bar{x}) = f(2\bar{n}, 0)\ p_0^{2n}(\bar{x}) +$$

$$\sum_{j=1}^{n} \left[f(2\bar{n}+2j, 0)\ p_j^{2n}(\bar{x}) + f(2\bar{n}-2j, 0)\ p_{-j}^{2n}(\bar{x}) \right] .$$

The polynomial $p^{2n}(\bar{x})$ is identical to all interpolation polynomials of degree 2n which pass through the prescribed points. Hence, $p^{2n}(\bar{x})$ is identical to a Stirling's approximation which is centered about $x = \bar{n}\Delta x$. In the limit as $n \rightarrow \infty$, $p^{2n}(\bar{x})$ is identical to the Stirling's approximation for $P_v(\bar{x})$ as given by Eq. 2.1.4.

Let us consider the limit as $n \rightarrow \infty$ of

$$f(2\bar{n}+2j,0)\ p_j^{2n}(\bar{x})$$

for fixed j. According to Knopp (4), this limit is defined and is given by

$$\lim_{n\to\infty} f(2\bar{n}+2j,0)\ p_j^{2n}(\bar{x}) = f(2\bar{n}+2j,0)\ \frac{\sin \pi(\bar{x}-j\Delta x)/\Delta x}{\pi(\bar{x}-j\Delta x)/\Delta x} .$$

Hence, the limit as $n\to\infty$ of $p^{2n}(\bar{x})$ must be given as

$$\lim_{n\to\infty} p^{2n}(\bar{x}) = f(2\bar{n},0)\ \frac{\sin \pi\bar{x}/\Delta x}{\pi\bar{x}/\Delta} +$$

$$\sum_{j=1}^{\infty}\left[f(2\bar{n}+2j,0)\ \frac{\sin \pi(\bar{x}-j\Delta x)/\Delta x}{\pi(\bar{x}-j\Delta x)/\Delta x} + \right.$$

$$\left. f(2\bar{n}-2j,0)\ \frac{\sin \pi(\bar{x}+j\Delta x)/\Delta x}{\pi(\bar{x}+j\Delta x)/\Delta x}\right] .$$

An interpolation polynomial which passes through $f(2\bar{n}+2i,0)$ for $i=-n,-n+1,\ldots,-1,0,+1,\ldots,n,n+1$, can also be written as

$$p^{2n+1}(\bar{x}) = f(2\bar{n},0)\ \frac{(n+1)\ p_0^{2n+2}(\bar{x})}{n+1+\bar{x}/\Delta x} +$$

$$\sum_{j=1}^{n}\left[f(2\bar{n}+2j,0)\ \frac{(n+j+1)\ p_j^{2n+2}(\bar{x})}{n+1+\bar{x}/\Delta x} + \right.$$

$$\left. f(2\bar{n}-2j,0)\ \frac{(n-j+1)\ p_{-j}^{2n+2}(\bar{x})}{n+1+\bar{x}/\Delta x}\right] +$$

$$f(2\bar{n}+2n+2,0)\ \frac{(2n+2)\ p_{n+1}^{2n+2}(\bar{x})}{n+1+\bar{x}/\Delta x} .$$

This polynomial, $p^{2n+1}(\bar{x})$, of degree $2n+1$ is identical to a Bessel's approximation which is centered about $x=\bar{n}\Delta x+\Delta x/2$.

The limit as $n \to \infty$ is

$$\lim_{n \to \infty} p^{2n+1}(\bar{x}) = f(2\bar{n},0)\, \frac{\sin \pi \bar{x}/\Delta x}{\pi \bar{x}/\Delta x} +$$

$$\sum_{j=1}^{\infty} \left[f(2\bar{n}+2j,0)\, \frac{\sin \pi(\bar{x}-j\Delta x)/\Delta x}{\pi(\bar{x}-j\Delta x)/\Delta x} + \right.$$

$$\left. f(2\bar{n}-2j,0)\, \frac{\sin \pi(\bar{x}+j\Delta x)/\Delta x}{\pi(\bar{x}+j\Delta x)/\Delta x} \right] .$$

It can be concluded that Eqs. 2.1.4 and 2.1.5 are equivalent definitions of $P_v(x)$.

Using this form of the interpolation formulas, $P_v(x)$ and $P_s(x)$ respectively can be written as shown in Eqs. 2.1.7

$$P_v(x) = f(0,0)\, \frac{\sin \pi x/\Delta x}{\pi x/\Delta x} + \tag{2.1.7}$$

$$\sum_{j=1}^{\infty} \left[f(2j,0)\, \frac{\sin \pi(x-j\Delta x)/\Delta x}{\pi(x-j\Delta x)/\Delta x} + f(-2j,0)\, \frac{\sin \pi(x+j\Delta x)/\Delta x}{\pi(x+j\Delta x)/\Delta x} \right]$$

$$P_s(x) = f(1,0)\, \frac{\sin \pi(x-\Delta x/2)/\Delta x}{\pi(x-\Delta x/2)/\Delta x} +$$

$$\sum_{j=1}^{\infty} \left[f(2j+1,0)\, \frac{\sin \pi(x-j\Delta x-\Delta x/2)/\Delta x}{\pi(x-j\Delta x-\Delta x/2)/\Delta x} + \right.$$

$$\left. f(-2j+1,0)\, \frac{\sin \pi(x+j\Delta x-\Delta x/2)/\Delta x}{\pi(x+j\Delta x-\Delta x/2)/\Delta x} \right]$$

where $-\infty < x < \infty$.

An alternate form of the basic numerical solution for $f(n,t)$ which is identical to Eq. 2.1.6 can now be obtained by substituting Eqs. 2.1.7 into Eq. 2.1.1.

Sections 2.2 and 2.3 will use the expression for $P_v(x)$ and $P_s(x)$ as given by Eqs. 2.1.7 to show that these approximating functions are defined for all x. Therefore, the basic numerical solution must also be defined for all x.

2.2 Periodic Initial Conditions.

In section 2.1, $P_v(x)$ and $P_s(x)$ are given by Eqs. 2.1.7. If an initial condition function is periodic in x with period 2L, it will be shown that the approximating function is defined for all x and that the approximating function is a Fourier series on 2k sample points.

If the initial condition V(x,0) is periodic in x with period 2L, Δx is defined to be $\Delta x = L/k$, where k is a positive integer. For this choice of Δx, the functions f(2j,0) are given as

$$f(2j+2nk,0) = P_v(j\Delta x,0)$$

for $j=0,1,2,\ldots,2k-1$, and $n=0,\pm 1,\pm 2,\ldots,$. Hence, $P_v(x)$ can be written in terms of f(2j,0) for $j=0,1,2,\ldots,2k-1$, as

$$P_v(x) = \sum_{j=0}^{2k-1} f(2j,0)\,\frac{\sin \pi(x/\Delta x - j)}{2\pi k}\left\{\frac{1}{x/2k\Delta x - j/2k} + \sum_{i=1}^{\infty}\left[\frac{1}{x/2k\Delta x - j/2k - i} + \frac{1}{x/2k\Delta x - j/2k + i}\right]\right\}.$$

According to Knopp (4), the infinite series portion of the expression is given by

$$\frac{1}{x/2k\Delta x - j/2k} + \sum_{i=1}^{\infty}\left[\frac{1}{x/2k\Delta x - j/2k - i} + \frac{1}{x/2k\Delta x - j/2k + i}\right] =$$

$$\cot \pi(x/2k\Delta x - j/2k).$$

Therefore,

$$P_v(x) = \sum_{j=0}^{2k-1} \frac{f(2j,0)\ \sin \pi(x/\Delta x - j)}{2k} \cot \pi(x/2k\Delta x - j/2k) .$$

In order to simplify this expression for $P_v(x)$, let

$$z = (x/2k\Delta x) - (j/2k)$$

$$y(z) = \frac{\sin(2k\pi z)\ \cos \pi z}{\sin \pi z} .$$

Upon substituting,

$$P_v(x) = \sum_{j=0}^{2k-1} \frac{f(2j,0)\ y(z)}{2k} .$$

But y(z) can also be written as

$$y(z) = \frac{\cos^2 \pi z\ \ \sin \pi(2k-1)z}{\sin \pi z} + \cos \pi z \cos \pi(2k-1)z .$$

This expression for y(z) can be simplified by applying Lagrange's identity (5). Hence,

$$y(z) = \cos^2 \pi z \left[1 + 2\sum_{m=1}^{k-1} \cos 2m\pi z\right] +$$

$$\cos \pi z \cos \pi(2k-1)z .$$

By means of trigonometric identities,

$$y(z) = 1 + 2 \sum_{m=1}^{k-1} \left[\cos 2\pi mz + \cos 2\pi kz\right]$$

$$= 1 + 2 \sum_{m=1}^{k-1} \left[\cos m\pi x/L \cos m\pi j\Delta x/L\right] +$$

$$2 \sum_{m=1}^{k-1} \left[\sin m\pi x/L \sin m\pi j\Delta x/L\right] + \cos k\pi x/L \cos \pi j \ .$$

$p_v(x)$ can now be written as

$$P_v(x) = \frac{A_0}{2} + \sum_{m=1}^{k-1} A_m \cos m\pi x/L + \sum_{m=1}^{k-1} B_m \sin m\pi x/L +$$

$$\frac{A_k}{2} \cos k\pi x/L$$

where

$$A_m = \frac{1}{k} \sum_{j=0}^{2k-1} f(2j,0) \cos m\pi j\Delta x/L$$

$$B_m = \frac{1}{k} \sum_{j=0}^{2k-1} f(2j,0) \sin m\pi j\Delta x/L \ .$$

This is a Fourier series on 2k sample points (6).

If the initial condition $S(x,0)$ is periodic in x with period $2L$ and $\Delta x = L/k$, the same argument can be applied to show that the approximating function $P_s(x)$ can also be written as a Fourier series on $2k$ sample points.

Since the initial conditions are differentiable almost everywhere, in the limit as $\Delta x \longrightarrow 0$, the approximating functions will approach the actual initial conditions almost everywhere.

2.3 Aperiodic Initial Conditions.

The purpose of this section is to show that the approximating functions for the aperiodic case are defined and that each approximating function approximates a Fourier integral of the initial condition.

In section 2.0, an initial condition was required to satisfy the hypothesis of the Fourier integral theorem (5) for the aperiodic case. Hence at every point $x(-\infty < x < \infty)$, where $V(x,0)$ has a right and left-hand derivative, $V(x,0)$ is represented by its Fourier integral as follows:

$$\frac{V(x+0,0) + V(x-0,0)}{2} =$$

$$\lim_{\alpha \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} V(x',0) \frac{\sin \alpha(x'-x)}{x'-x} dx'$$

Consider the function $\bar{V}(x,0)$ which is given by

$$\bar{V}(x,0) = \frac{1}{\pi} \int_{-\infty}^{\infty} V(x',0) \frac{\sin \pi(x'-x)/\Delta x}{x'-x} dx' .$$

It is clear that $\bar{V}(x,0)$ and $V(x,0)$ have the same frequency spectrum for $0 \leq \alpha < \pi/\Delta x$. For $\alpha > \pi/\Delta x$, the function $\bar{V}(x,0)$ contains no spectral components. In addition, if $\pi/\Delta x$ is sufficiently large, $\bar{V}(x,0)$ is a good approximation to $V(x,0)$.

If the expression for $\bar{V}(x,0)$ is numerically integrated using the trapezoidal rule (6), an approximate expression for $\bar{V}(x,0)$ is given by

$$\bar{V}(x,0) \doteq \sum_{j=-\infty}^{\infty} V(j\Delta x,0) \frac{\sin \pi(x-j\Delta x)/\Delta x}{\pi(x-j\Delta x)/\Delta x} .$$

At $x=j\Delta x$ for $j=0,\pm 1,\pm 2,\pm 3,\ldots$, $\bar{V}(x,0)$ is identically equal to $V(x,0)$.

Since $V(x',0)$ satisfies the hypothesis of the Fourier integral theorem, the approximate expression for $\bar{V}(x,0)$ must be an absolutely convergent series. Upon rearranging this series, it can be seen that the approximate expression for $\bar{V}(x,0)$ and the expression for $P_V(x)$ which is given by Eq. 2.1.7 are identical. Hence, $P_v(x)$ is an approximate Fourier integral of $V(x,0)$. In the limit as $\Delta x \longrightarrow 0$, $P_v(x)$ becomes the exact Fourier integral of $V(x,0)$.

The same procedure can be applied to $S(x,0)$ to show the validity of the approximating function $P_s(x)$.

2.4 Numerical Differentiation.

The standard approach for finding a set of difference equations which give the approximate numerical solution to an equation containing partial derivatives is to approximate the partial derivatives with some numerical differentiation scheme. A technique similar to this will be used to solve the telegrapher's equations numerically. Hence, the purpose of this section is to define a method of numerical differentiation.

Consider the function $g(x,t)$ which is given by

$$g(x,t) = \frac{V(x+j\Delta x+\Delta x/2,t) - V(-x+j\Delta x+\Delta x/2,t)}{2} .$$

At $t=t_0$, $g(x,t_0)$ is an odd function with respect to x. In addition, at x=0, the partial derivative with respect to x of $g(x,t)$ is identical to the partial derivative with respect to x of $V(x+j\Delta x+/2,t)$. Therefore, the specific approach is to derive a numerical differentiation formula which approximates the partial derivative of $g(x,t)$ with respect to x at $x=0$.

Once the functional values of $g(x,t_0)$ at $x=\pm\Delta x/2, \pm 3\Delta x/2, \pm 5\Delta x/2, \ldots, \pm(2n+1)\,\Delta x/2$, are given, $g(x,t_0)$ can be approximated with a Bessel's interpolation polynomial of degree 2n+1 which passes through the prescribed points. Hence, $g(x,t_0) \doteq p_{2n+1}(x)$, where $p_{2n+1}(x)$ is interpolating polynomial. This polynomial in terms of the known values of $V(x+m\Delta x+\Delta x/2,t_0)$ can be written as

$$p_{2n+1}(x) = -\frac{x}{\Delta x}\delta f(2m+1,t_0) - \sum_{k=1}^{n} \frac{x}{\Delta x} \frac{\prod_{j=1}^{k}\left[(x/\Delta x)^2-(2j-1)^2/4\right]}{(2k+1)!}\,\delta^{2k+1} f(2m+1,t_0) \;..$$

It is evident that the interpolating polynomial is an odd function of x.

The derivative of $p_{2n+1}(x)$ evaluated at $x=0$ is defined as a 2n+1 degree approximation for the partial derivative of $V(j\Delta x+\Delta x/2,t_0)$ with respect to x. This approximation is given by

$$p'_{2n+1}(0) = -\sum_{m=1}^{n+1} \frac{\prod_{i=1}^{m}(2i-3)^2}{\Delta x\,(-2)^{m-1}\,(2m-1)!}\,\delta^{2m-1} f(2j+1,t_0) \qquad 2.4.0$$

$$= \frac{\partial E(j\Delta x+\Delta x/2,t_0)}{\partial x}$$

In order to get an idea of the error which results by using this numerical differentiation formula, a remainder term will be derived. For the derivation, it will be assumed that the (2n+3)th partial derivative of $g(x,t_0)$ with respect to x exists in the interval of interpolation, $-(2n+1)\Delta x/2 \leq x \leq (2n+1)\Delta x/2$. The function $g(x,t_0)$ is given exactly for all x in this interval by

$$g(x,t_0) = p_{2n+1}(x) + h(x)$$

where h(x) is unknown function of x. Since $p_{2n+1}(x)$ is equal to $g(x,t_0)$ at $x=0,\pm\Delta x/2,\pm 3\Delta x/2,\ldots,\pm(2n+1)\Delta x/2$, h(x) must have real roots at these points. Therefore, h(x) must be a function which is identically zero or an odd function which has at least 2n+3 real roots in the interval of interpolation.

The partial derivative of $g(x,t_0)$ with respect to x in this interval is given exactly by

$$\frac{\partial g(x,t_0)}{\partial x} = p'_{2n+1}(x) + h'(x)$$

where $p'_{2n+1}(x)$ is a polynomial of degree 2n. By Rolle's theorem (7), h'(x) has at least 2n+2 real roots (z_i) such that $|z_i| < (2n+1)\Delta x/2$. Therefore, the partial derivative of $g(x,t_0)$ with respect to x can now be written as

$$\frac{\partial g(x,t_0)}{\partial x} = p'_{2n+1}(x) + k(x)\,\frac{\prod_{i=1}^{2n+2}(x-z_i)}{(2n+2)!}\,. \qquad 2.4.0$$

Consider the function w(y) which is defined by

$$w(y) = \frac{\partial g(y,t_0)}{\partial y} - p'_{2n+1}(y) - k(x)\,\frac{\prod_{i=1}^{2n+2}(y-z_i)}{(2n+2)!}\,.$$

By virtue of Eq. 2.4.0, w(y) has at least 2n+3 real roots $x, z_1, \ldots, z_{2n+2}$. Upon choosing x in the interval of interpolation such that these roots are distinct, Rolle's theorem can be applied to show that w'(y) has at least 2n+2 distinct real roots $(\overline{z}_i)$ such that $|\overline{z}_i| < (2n+1)\Delta x/2$. Therefore by successive applications of Rolle's theorem, k(x) can be determined. Hence,

$$k(x) = \frac{\partial^{(2n+3)} g(\xi, t_0)}{\partial y^{(2n+3)}}$$

where $-(2n+1)\Delta x/2 < \xi < (2n+1)\Delta x/2$.

The partial derivative of $g(x,t_0)$ can now be written as

$$\frac{\partial g(x,t_0)}{\partial x} = p'_{2n+1}(x) + \frac{\partial^{(2n+3)} g(\xi, t_0)}{\partial x^{(2n+3)}}\,\frac{\prod_{i=1}^{2n+2}(x-z_i)}{(2n+2)!}\,.$$

Since our main interest is the partial derivative of $g(x,t_0)$ with respect to x at x=0, this partial derivative is given by

$$\frac{\partial g(0,t_0)}{\partial x} = p'_{2n+2}(0) + \frac{\partial^{(2n+3)} g(\xi,t_0)}{\partial x^{(2n+3)}} \; \frac{\prod_{i=1}^{2n+2} (-z_i)}{(2n+2)!} \;.$$

But

$$\left| \frac{\prod_{i=1}^{2n+2} z_i}{(2n+2)!} \right| \le \frac{(\Delta x)^{2n+2}}{(2)^{2n+2}} \; \frac{(1)\ (3)\ (5)\cdots(2n+1)}{(2)\ (4)\ (6)\cdots(2n+2)} \le \frac{(\Delta x)^{2n+2}}{(2)^{2n+3}} \;.$$

It can now be concluded that

$$\left| \frac{\partial g(0,t_0)}{\partial x} - p'_{2n+1}(0) \right| \le \left| \frac{\partial^{(2n+3)} g(\xi,t_0)}{\partial x^{(2n+3)}} \right| \; \frac{(\Delta x)^{2n+2}}{2^{2n+3}}$$

whenever the (2n+3)th partial derivative of $g(x,t_0)$ exists.

The same argument can be used to obtain a numerical differentiation formula for

$$\frac{\partial s(x+j\Delta x,t)}{\partial x}$$

at $x=0$ and $t=t_0$.

2.5 Time Domain Models.

The preliminary material which was derived in the previous sections can now be applied to obtain two general classes of time domain models which represent the infinite line. A time domain model is a set of equations for which the numerical solution is $f(n,j\Delta t)$, where $n=0,\pm 1,\pm 2,$ $\pm 3,\ldots,$ and $j+0,1,2,3,\ldots,$ and for which the solution $f(n,j\Delta t)$ approximates the solution of the infinite line at the defined net points for certain choices of Δx and Δt.

One class of models is derived on the basis of obtaining a set of ordinary differential equations in t by replacing the partials with respect to x with difference equations. The second class of models is derived on the basis of the previously derived basic numerical solution.

Definition: Class 1-Numerical Time Domain Models: A time domain model belongs to class 1 if and only if the time domain model can be obtained by the following steps.

1. Write the appropriate telegrapher's equations at the defined voltage and current node points.
2. Approximate both the partials for $j=0,\pm 1,\pm 2,\ldots,$ $\frac{\partial v(j\Delta x+\Delta x/2,t)}{\partial x}$ and $\frac{\partial s(j\Delta x,t)}{\partial x}$, by means of a 2n+1 degree numerical differentiation formula as given by Eq. 2.4.0. For any given model, n is an integer equal to or greater than zero which is fixed for all j.
3. Replace the partial derivatives with respect to x with their respective difference equation approximations.
4. The resulting set of equations is a set of ordinary differential equations in time.

It is quite clear that the initial conditions on the line at t=0 provide the necessary initial conditions to solve these equations.

The definition of the class 1-numerical time domain model gives a set of rules for the construction of a class 1 model. These rules in effect say that any model which belongs to class 1 can be written as

$$\frac{df(j,t)}{dt} = v_p \sum_{m=1}^{m+1} \frac{\prod_{i=1}^{m} (2i-3)^2}{\Delta x \ (-2)^{m-1} (2m-1)!} \delta^{2m-1} f(j,t) \qquad 2.5.0$$

for all j and t where 2n+1 is the degree of the numerical differentiation formula. If these resulting differential equations were solved by some analytical means, the error of the solution would be of the order of $(\Delta x)^{2n+2}$.

In order to illustrate class 1, let us look at some particular examples of models from this class.

Example 1: n=0

$$\frac{df(j,t)}{dt} = \frac{v_p}{\Delta x} \delta f(j,t) \qquad j=0,\pm 1,\pm 2,\pm 3,\ldots,$$

Example 2: n=1

$$\frac{df(j,t)}{df} = \frac{v_p}{\Delta x} \left[\delta f(j,t) - \frac{\delta^3 f(j,t)}{24} \right] \qquad j=0,\pm 1,\pm 2,\ldots,$$

These examples respectively are obtained by employing a 1st and 3rd order numerical differentiation formula.

<u>Definition</u>: Class 2-Numerical Time Domain Models: A numerical time domain model belongs to class 2 if and only if the model can be obtained by the following steps.

1. Given the basic numerical solution for $f(j,t)$ which is given by Eq. 2.1.6 and is written in terms of $f(j,0)$.
2. With this expression for $f(j,t)$, write the solutions for $f(j,\Delta t)$, $f(j,0)$, $f(j,-\Delta t),\ldots,$ and $f(j,-j_2\Delta t)$.
3. Derive a relationship for finding $f(j,\Delta t)$ in terms of $f(j,0)$, $f(j,-\Delta t),\ldots,$ and $f(j,-j_2\Delta t)$.
4. Truncate this relationship. If $\delta^k f(j,0)\, p(v)$, where $v=v_p\Delta t/\Delta x$ and $p(v)$ is a polynomial in v, would be the next nonzero term of this expression if the series were extended, the remainder term can be obtained by replacing $\delta^k f(j,0)$ by $(\Delta x)^k \dfrac{\partial^{(k)} v(\xi,0)}{\partial x^{(k)}}$ when $k+j$ is even or by $(\Delta x)^k \dfrac{\partial^{(k)} s(\xi,0)}{\partial x^{(k)}}$ when $k+j$ is odd where $j\Delta x-k\Delta x/2 < \xi < j\Delta x+k\Delta x/2$.
5. Use the same truncated expression to find $f(j,\Delta t)$ for all j.

For $j_2=0$, a model belonging to class 2 is just an expression for $f(j,\Delta t)$ obtained by truncating the basic numerical solution for $f(j,\Delta t)$ after n terms. For $j_2\neq 0$, the model is still a truncated version of the basic numerical solution for $f(j,\Delta t)$ except that certain terms of the best numerical solution have been replaced by known values of $f(j,-\Delta t), \ldots,$ and $f(j,-j_2\Delta t)$.

Some examples of models from class 2 are:

Example 3:

$$f(j,\Delta t) = f(j,0) + \frac{v_p\Delta t}{\Delta x}\,\delta f(j,0) \qquad j=0,\pm 1,\pm 2,\ldots,$$

Example 4:

$$f(j,\Delta t) = f(j,0) + \frac{v_p\Delta t}{\Delta x}\,\delta f(j,0) + \ldots + \ldots$$

$$f(j,-\Delta t) = f(j,0) - \frac{v_p\Delta t}{\Delta x}\,\delta f(j,0) + \ldots + \ldots$$

By subtracting these two expressions which eliminates all even terms,

$$f(j,\Delta t) = f(j,-\Delta t) + \frac{2v_p\Delta t}{\Delta x}\,\delta f(j,0) + \ldots + \ldots$$

Truncating,

$$f(j,\Delta t) = f(j,-\Delta t) + \frac{2v_p\Delta t}{\Delta x}\,\delta f(j,0) \ .$$

Example 5:

$$f(j,\Delta t) = f(j,0) + \frac{v_p\Delta t}{\Delta x}\,\delta f(j,0) + \frac{(v_p\Delta t)^2}{2(\Delta x)^2}\,\delta^2 f(j,0) + \ldots$$

$$f(j,-\Delta t) = f(j,0) - \frac{v_p \Delta t}{\Delta x} \delta f(j,0) + \frac{(v_p \Delta t)^2}{2(\Delta x)^2} \delta^2 f(j,0) + \ldots$$

By adding the two expressions which eliminates all odd terms, $f(j,\Delta t)$ is now given by

$$f(j,\Delta t) = 2\, f(j,0) - f(j,-\Delta t) + \frac{(v_p \Delta t)^2}{2(\Delta x)^2} \delta^2 f(j,0) + \ldots + \ldots$$

Truncating,

$$f(j,\Delta t) = 2\, f(j,0) - f(j,-\Delta t) + \frac{(v_p \Delta t)^2}{2(\Delta x)^2} \delta^2 f(j,0) \; .$$

This example from class 2 is the model which is given in most numerical analysis books (1,2,3) as an approximation to the wave equation.

For a model belonging to either class 1 or class 2, the defining expression for $f(j,t)$ for all j is independent of j for fixed Δx and Δt. If the nodes were divided into mutually disjoint sets, different models from a given class could be used to represent each set. From a mathematical point of view, this is a perfectly valid way of obtaining a time domain model for the line. In addition, two much larger classes of models could be obtained. The only problem of defining models in this manner is that the resulting models would be far more complex in form and not nearly as practical for computer solution. There is no apparent advantage in doing this. Hence, this thesis only considers models belonging to the presently defined classes even though the results of the thesis are generally valid for the more complicated models.

2.6 Methods of Numerical Solution for the Time Domain Models.

Now that two general classes of time domain models are defined, this section will discuss certain numerical techniques for finding their solutions.

First of all, let us consider the models belonging to class 1. It is quite clear that any numerical technique for solving linear ordinary differential equations can be used to solve the differential equations which are defined by any model belonging to this class. Because of the linearity of the equations and the special properties of the operators δ^p, given by Eq. 2.1.3, these equations can be converted to difference equations by treating each equation individually once the method of solution is stated.

Let us now consider one differential equation belonging to a given model for this class. This equation can be written as

$$\frac{df(j,t)}{dt} = \frac{v_p}{\Delta x}\left[d_1\delta + d_3\ \delta^3 + \ldots + d_{2m+1}\ \delta^{2m+1} \right] f(j,t)$$

$$= H(\Delta x, \Delta t)\ f(j,t)$$

where d_{2i+1} is a constant for $i=0,1,2,\ldots,m$, and $H(\Delta x,\Delta t)$ is an operator which is a function of Δx and Δt. If $H(\Delta x,\Delta t)$ is considered a constant, then any numerical technique of solution that uses the values of $f(j,n\Delta t)$, $f(j,(n-1)\Delta t),\ldots,$ and $f(j,(n-j_2)\Delta t)$ to find $f(j,(n+1)\Delta t)$ can be written as

$$f(j,(n+1)\Delta t) = \sum_{l=0}^{j_2}\sum_{p=0}^{m_i} b_{i,p}(\Delta t)\; H^p(\Delta x,\Delta t)\; f(j,(n-i)\Delta t)$$

where $b_{i,p}(\Delta t)$ for $i=0,1,2,3,\ldots,j_2$, and $p=0,1,2,\ldots,m_i$, is a function of Δt. By treating $H^p(\Delta x,\Delta t)$ as a polynomial in δ, $f(j,(n+1)\Delta t)$ is given by

$$f(j,(n+1)\Delta t) = \sum_{l=0}^{j_2} a_i(\Delta x,\Delta t,\delta)\; f(j,(n-i)\Delta t) \qquad 2.6.0$$

where $a_i(\Delta x,\Delta t,\delta)$ for $i=0,1,2,\ldots,j_2$, is an operator which is a function of $\Delta x, \Delta t$, and δ. Equation 2.6.0 is simply the resulting difference equation which is obtained from this differential equation once the method of numerical solution is specified.

If the same numerical technique is used to solve each differential equation belonging to this model, then the difference equation, Eq. 2.6.0, is valid for all j. In fact, there is no apparent reason for using different methods on different equations belonging to this model.

A first order model belonging to class 1 is given by

$$\frac{df(j,t)}{dt} = \frac{v_p}{\Delta x} \delta f(j,t)$$

for $j=0,\pm 1,\pm 2,\ldots,$. Some examples that use the Runge-Kutta methods (2) to solve this model are shown to illustrate the construction process for obtaining Eq. 2.6.0.

Example 1: Runge Kutta first order:

$$H(\Delta x,\Delta t) = \frac{v_p}{\Delta x} \delta$$

For this technique of solution,

$$f(j,(n+1)\Delta t) = \left[1 + \Delta t\ H(\Delta x,\Delta t)\right] \quad f(j,n\Delta t)$$

$$= f(j,n\Delta t) + \frac{v_p \Delta t}{\Delta x} \delta f(j,n\Delta t).$$

Example 2: Runge Kutta second order:

$$H(\Delta x,\Delta t) = \frac{v_p \Delta t}{\Delta x} \delta$$

$$f(j,(n+1)\Delta t) = \left[1 + \Delta t\ H(\Delta x,\Delta t) + \frac{\Delta t^2\ H(\Delta x,\Delta t)}{2}\right] f(j,n\Delta t)$$

$$= \left[1 + \frac{v_p \Delta t}{\Delta x} + \frac{(v_p \Delta t)^2}{2\Delta x^2} \delta^2\right] \quad f(j,n\Delta t)$$

The definition of a class 2 model gives a difference equation of the form

$$f(j,\Delta t) = \sum_{i=0}^{j_2} a_i(\Delta x,\Delta t,\delta)\ f(j,-i\Delta t) \qquad 2.6.1$$

where $a_i(\Delta x, \Delta t, \delta)$ is an operator which is a function of $\Delta x, \Delta t$, and δ. Since $f(j,\Delta t)$ for $j=0,\pm 1,\pm 2,\ldots,$ determines a new initial value problem at $t=\Delta t$, this formula may be used to find $f(j,2\Delta t)$. Hence, Eq. 2.6.1 can be used recursively to find $f(j,n\Delta t)$ for $n=0,1,2,\ldots,$.

For certain models from class 1 and certain numerical methods of solution for these models, there are models from class 2 which give the same sets of difference equations. But there are also certain cases where this is not true. For example, consider the model from class 1 given by a third degree approximation to the partials.

$$\frac{df(j,t)}{dt} = \frac{v_p}{\Delta x}\left[\delta f(j,t) - \delta^3 f(j,t)/24\right]$$

A Runge Kutta first order method for solving these resulting differential equations gives the following difference equations.

$$f(j,(n+1)\Delta t) =$$
$$f(j,n\ t) + \frac{v_p \Delta t}{\Delta x}\left[\delta f(j,n\Delta t) - \delta^3 f(j,n\Delta t)/24\right]$$

Since this difference equation gives $f(j,(n+1)\Delta t)$ in terms of $f(j,n\Delta t)$, any model belonging to class 2 which would give the same difference equation must be obtained by truncating the basic numerical solution, Eq. 2.1.6. It is quite clear that Eq. 2.1.6 can not be truncated to give the same difference equation.

A number of these methods derive a difference equation expression for $f(n,(p+1)\Delta t)$ in terms of $f(j,p\Delta t),\ldots,$ and $f(j,(p-j_2)\Delta t)$ for $n=0,\pm1,\pm2,\ldots,$ and $j=0,\pm1,\pm2,\ldots,$. Such difference methods are called difference methods with (j_2+2) levels (1) in t. For $p=0$ and $j_2\neq0$, where are the values of $f(j,-\Delta t),\ldots,$ and $f(j,-j_2\Delta t)$ obtained in the case where only $f(j,0)$ is specified? This question can only be answered by saying that there is insufficient information present to find $f(j,\Delta t)$.

Let us consider a difference method which gives $f(j,(p+1)\Delta t)$ in terms of $f(j,p\Delta t)$ for all j. If $f(j,0)$ are specified for all j, then the values of $f(j,t)$ for $t=\Delta t,\ldots,$ $(p+1)\Delta t$, can be calculated for all j by this difference method. Since the values of $f(j,(p+1)\Delta t)$ are known for $m=0,1,2,\ldots,p+1$, a different difference method can be used to find $f(j,m\Delta t)$ for $m=p+2,p+3,\ldots,$. This new difference method can be multi-level $(j_2\neq0)$ difference method in t.

2.7 Special Properties of Difference Equation Solution.

When the specified initial conditions have special properties, the solution to any model belonging to class 1 or class 2 must also have special properties. This section will derive some of these properties which will be used in later chapters.

First, let the initial condition functions, $V(x,0)$ and $S(x,0)$, be respectively an odd and an even function with respect to $x=m\Delta x$. For $p=0,1,2,\ldots,$ the relationships

$$\delta^p f(2j+2m,0) = \delta^p f(-2j+2m,0)$$

$$\delta^p f(2j+1+2m,0) = -\delta^p f(-2j-1+2m,0)$$

must be true for all j. Hence for $j_2=0$, the solution any difference equation for $f(j,\Delta t)$ which is given by either Eq. 2.6.0 or 2.6.1 must have the property

$$\delta^p f(2j+2m,\Delta t) = -\delta^p f(-2j+2m,\Delta t)$$

$$\delta^p f(2j+1+2m,\Delta t) = \delta^p f(-2j-1+2m,\Delta t)$$

for all j. By means of induction, it can be seen that Eqs. 2.7.0 must be true for all j and n=0,1,2,....,.

$$\delta^p f(2j+2m,n\Delta t) = \delta^p f(-2j+2m,n\Delta t) \qquad 2.7.0$$

$$\delta^p f(2j+1+2m,n\Delta t) = -\delta^p f(-2j-1+2m,n\Delta t)$$

For the multilevel difference equations, given by Eqs. 2.6.0 and 2.6.1, where $f(j,n\Delta t)$, $f(j,(n-1)\Delta t)$,....., and $f(j,(n-j_2)\Delta t)$ satisfy Eqs. 2.7.0 for all j, the solution for $f(j,(n+1)\Delta t)$ must satisfy Eqs. 2.7.0.

This same argument can be applied to the case where $V(x,0)$ and $S(x,0)$ are respectively an even and an odd function with respect to $x=m\Delta x+\Delta x/2$. The result of this argument is given by Eqs. 2.7.1.

$$\delta^p f(2j+2m+1,n\Delta t) = -\delta^p f(-2j+2m+1,n\Delta t) \qquad 2.7.1$$

$$\delta^p f(2j+2m+2,n\Delta t) = \delta^p f(-2j+2m,n\Delta t)$$

Now let us consider the case where $V(x,0)$ and $S(x,0)$ are periodic in x with period L. In addition, let Δx be chosen such that $\Delta x = L/k$, where k is a positive integer. Because of this periodicity, the functions $f(j,0)$ are given by

$$f(2j+2nk,0) = V(j\Delta x,0)$$

$$f(2j+1+2nk,0) = S(j\Delta x+\Delta x/2,0)$$

where $n=0,\pm1,\pm2,\ldots,$ and $j=0,1,2,\ldots,k-1$. These equations also imply that the following is also valid

$$\delta^p f(j+2nk,0) = \delta^p f(j,0)$$

for $j=0,1,2,\ldots,2k-1$, $p=0,1,2,\ldots,$ and $n=0,\pm1,\pm2,\ldots,$. Hence for $j_2=0$, the solution to either Eq. 2.6.0 or 2.6.1 must have the property that

$$\delta^p f(j+2nk,\Delta t) = \delta^p f(j,\Delta t).$$

By induction, this argument can be used to show that Eqs. 2.7.2 are valid for $j=0,1,2,\ldots,2k-1$, $p=0,1,2,\ldots,$ and all n.

$$\delta^p f(j+2nk,m\Delta t) = \delta^p f(j,m\Delta t) \qquad 2.7.2$$

Equation 2.7.2 is also valid when multilevel difference equations are used to find $f(j,p\ t)$.

2.8 Conclusion.

Two classes of infinite line time domain models have been derived. Any model which belongs to class 1 is obtained by approximating the partial derivatives with respect to x at certain node points and obtaining a set of ordinary linear differential equations. On the other hand, any model which belongs to class 2 was derived on the basis of a basic numerical solution which was defined in section 2.1.

In addition to deriving these two classes of time domain models, sections 2.2 and 2.3 showed that the basic numerical solution is defined and can be written as either a Fourier series on 2k sample points, an approximate Fourier integral, or a combination of both. Whereas, section 2.7 showed that whenever the specified initial conditions had special properties, the solution to any model by the methods of numerical solution which are given in section 2.6 must also have certain special properties.

The findings of this chapter are extensively used in the remaining chapters of this thesis.

III. TRANSMISSION LINE OF TYPE 3

3.0 Introduction.

A problem of type 3 is a lossless parallel transmission line of length L for which a boundary condition at each end of the line is defined to be zero for $t \geq 0$ but has specified initial conditions at $t=0$. The line is defined to start at $x=0$ and to end at $x=L$. This chapter will specifically treat the case where the boundary conditions $E(0,t)$ and $E(L,t)$ are specified to be identically zero for $t \geq 0$.

If $E(x,0)$ and $I(x,0)$ are the specified initial conditions for this problem of type 3, an infinite line for which the initial conditions are

$$V(x+2nL,0) = E(x,0) \qquad 0 \leq x \leq 1$$
$$V(-x+2nL,0) = -E(x,0) \qquad 0 \leq x \leq L$$
$$S(x+2nL,0) = I(x,0) \qquad 0 \leq x \leq L$$
$$S(-x+2nL,0) = I(x,0) \qquad 0 \leq x \leq L$$

where $n=0,\pm 1,\pm 2,\pm 3,\ldots,$ has the same solution for $0 \leq x \leq L$ as the problem of type 3 for all t. Hence, an approximate numerical solution can be obtained for this problem of type 3 by solving the infinite line problem for certain specified initial conditions.

Consider the functions $V(x,0)$ and $S(x,0)$. Both of these functions are periodic in x with period 2L. If $f(j,n\Delta t)$ is the approximate numerical solution to this infinite line by any difference method defined in Chapter 2, then according to Eqs. 2.7.2, when $\Delta x = L/k$, the relationship

$$f(i+4pk,n\Delta t) = f(i,n\Delta t)$$

where $i=0,1,2,\ldots,4k-1$, and $p=0,\pm 1, \pm 2,\ldots$, is true.

Since $V(x,0)$ and $S(x,0)$ are also an odd and an even function of x respectively with respect to $x=0$, Eqs. 2.7.0 state that this relationship further reduces to

$$\begin{aligned} f(2i+4pk,j\Delta t) &= f(2i,j\Delta t) \\ f(-2i+4pk,j\Delta t) &= -f(2i,j\Delta t) \\ f(2i+1+4pk,j\Delta t) &= f(2i+1,j\Delta t) \\ f(-2i-1+4pk,j\Delta t) &= f(2i+1,j\Delta t) \end{aligned} \tag{3.0.0}$$

where $p=0,\pm 1,\pm,\ldots,$ and $i=0,1,2,\ldots,k-1$. This relationship also states that $f(2pk,j\Delta t)$ for all p is identically zero independent of the difference method.

Because of these special infinite line properties, any model or difference method derived in Chapter 2 can be reduced to a set of 2k equations for which the numerical solution is $f(p,j\Delta t)$, where $p=0,1,2,\ldots,2k-1$. Since $f(0,t)$ is identically zero, the actual equations which need be solved can be further reduced to $(2k-1)$ equations.

If $f(i,j\Delta t)$ or $f(i,t)$ is replaced by $g(i,j\Delta t)$ or $g(i,t)$ for $i=0,1,2...,2k$, these resulting $(2k+1)$ equations are defined as a time domain model for the given problem of type 3. The functions $g(2i,t)$ for $i=0,1,...,k$, and $g(2i-1,t)$ for $i=1,2,...,k$, are defined as the approximate numerical solutions for $E(i\Delta x,t)$ and $I(i\Delta x-\Delta x/2,t)$ respectively. Since the functions, $g(0,t)$ and $g(2k,t)$, must be identically zero, only the remaining $(2k-1)$ equations need by solved. Hence, this chapter will investigate methods for deriving these $(2k-1)$ equations and their methods of solution.

3.1 Transformation Matrix C.

The time domain models and difference methods derived in Chapter 2 are all in terms of the operator δ^m for $m=0,1,2,....$. Because of the special infinite line properties stated in sections 3.0 and 2.7, $\delta^m f(j,t)$ for $j=0,\pm,\pm 2,...$, can be written in terms of $f(p,t)$ for $p=0,1,2,...,2k$. This section will derive a transformation matrix C such that if $c^m_{p,j}$ is a typical entry of C^m, then

$$\delta^m f(p,t) = \sum_{j=1}^{2k-1} c^m_{p,j}\ f(j,t)$$

for $p=1,2,...,2k-1$.

If $f(i,t)$ is substituted for $f(i,j\Delta t)$ in Eqs. 3.0.0, then the relationship

$$\delta^m f(2i+4pk,t) = \delta^m f(2i,t) \qquad 3.1.0$$

$$\delta^m f(-2i+4pk,t) = -\delta^m f(2i,t)$$

$$\delta^m f(2i+1+4pk,t) = \delta^m f(2i+1,t)$$

$$\delta^m f(-2i-1+4pk,t) = \delta^m f(2i+1,t)$$

where $i=0,1,2,...,k$, and $p=0,\pm1,\pm2,...,$ is valid. Since

$$\delta^m f(2pk,t) = -\delta^m f(0,t)$$

$$\delta^m f(-2k+4pk,t) = -\delta^m f(2k,t)$$

for $p=0,\pm1,\pm2,...,$ $\delta^m f(2pk,t)$ must be identically zero for all t.

Let us consider $\delta^{m+1} f(i,t)$ for $i=1,2,...,2k-1$. According to Eqs. 2.1.3, $\delta^{m+1} f(i,t)$ can be written as

$$\delta^{m+1} f(i,t) = \delta^m f(i-1,t) - \delta^m f(i+1,t).$$

But

$$\delta^m f(0,t) = \delta^m f(2k,t)=0.$$

Hence $\delta^{m+1} f(i,t)$ is given as

$$\delta^{m+1} f(1,t) = \delta^m f(2,t)$$

$$\delta^{m+1} f(i,t) = \delta^m f(i-1,t) - \delta^m f(i+1,t) \qquad i=2,3,...,2k-2$$

$$\delta^{m+1} f(2k-1,t) = \delta^m f(2k-2,t).$$

This gives a direct linear relationship between $\delta^{m+1}f(i,t)$ and $\delta^m f(i,t)$ for $i=1,2,\ldots,2k-1$, which is independent of m. This linear relationship can be expressed by means of a $(2k-1)\times(2k-1)$ matrix C. It is quite clear that a typical entry $c_{i,j}$ of C is

$$c_{i,i+1} = -1 \qquad i=1,2,\ldots,2k-1 \tag{3.1.1}$$
$$c_{i+1,i} = 1 \qquad i=1,2,\ldots,2k-1$$
$$c_{i,j} = 0 \qquad i\neq j+1, \quad j\neq i+1.$$

This resulting matrix is called the transformation matrix.

By definition,

$$\begin{bmatrix} \delta^{m+1}f(1,t) \\ \delta^{m+1}f(2,t) \\ \cdot \\ \cdot \\ \delta^{m+1}f(2k-2,t) \\ \delta^{m+1}f(2k-1,t) \end{bmatrix} = C \begin{bmatrix} \delta^{m}f(1,t) \\ \delta^{m}f(2,t) \\ \cdot \\ \cdot \\ \delta^{m}f(2k-2,t) \\ \delta^{m}f(2k-1,t) \end{bmatrix} \tag{3.1.2}$$

Hence $\delta^m f(i,t)$ for $i=1,2,\ldots,2k-1$, can be written in terms of $\delta^0 f(i,t)=f(i,t)$ as

$$\begin{bmatrix} \delta^m f(1,t) \\ \delta^m f(2,t) \\ \cdot \\ \cdot \\ \delta^m f(2k-2,t) \\ \delta^m f(2k-1,t) \end{bmatrix} = C^m \begin{bmatrix} f(1,t) \\ f(2,t) \\ \cdot \\ \cdot \\ f(2k-2,t) \\ f(2k-1,t) \end{bmatrix} . \qquad 3.1.3$$

It is quite clear that this is a very convenient means of expressing $\delta^m f(i,t)$.

The matrix C has at least one nonzero entry and at most two nonzero entries in each row or column. This property indicates that special methods can be used to calculate C^{m+1}. If C_i^m is the ith row of C^m, then it is quite clear that

$$C_i^{m+1} = C_{i-1}^m - C_{i+1}^m \qquad i=2,3,\ldots,2k-2 \qquad 3.1.4$$

$$C_1^{m+1} = -C_2^m$$

$$C_{2k-1}^{m+1} = C_{2k-2}^m .$$

In effect, this says that if this recursion method is employed to calculate C^{m+1} from C^m and C, then $2(4k^2-6k+2)$ additions and no multiplications would be required. If the product was taken in the ordinary sense, then $(2k-1)(4k^2-4k+1)$ additions and $(4k^2-4k+1)$ multiplications are required. If the time to perform a multiplication is equal to or greater than the time required to perform an addition, the matrix product in the ordinary sense takes at least k times longer to execute than the recursion formula.

3.2 Transformation of Time Domain Models Belonging to Class 1.

Given a time domain model, belonging to class 1, for the infinite line. There are two approaches for obtaining a set of (2k-1) equations which give the approximate numerical solution to the problem of type 3. The first approach is to directly transform the infinite line time domain model, given by Eqs. 2.5.0, to the finite line problem of type 3. This process would give (2k-1) ordinary linear differential equations for which the solution approximates $E(x,t)$ and $I(x,t)$ at the defined nodes. A numerical method of solution then can be defined to solve these resulting differential equations.

The second approach is to define a method of numerical solution for the infinite line model. The resulting set of difference equations, given by Eqs. 2.6.0, can then be transformed to the finite line problem. As far as the end result is concerned, both approaches are identical. This section treats the first approach.

A time domain model for the infinite line belonging to class 1 which is obtained by using a (2n+1)th degree difference equation approximation for the partial derivatives with respect to x can be written as

$$\frac{df(j,t)}{dt} = \sum_{m=1}^{n+1} \frac{v_p \left[\prod_{i=1}^{m} (2i-3)^2 \right]}{\Delta x \ (-2)^{m-1} (2m-1)!} \ \delta^{2m-1} f(j,t)$$

where $j=0,\pm 1,\pm 2,\ldots,$. Because of the special infinite line properties given by Eqs. 3.1.0, the solution to this infinite set of differential equations can be obtained by solving the finite set of (2k-1) differential equations which are

$$\frac{df(j,t)}{dt} = \sum_{m=1}^{n+1} \frac{v_p \left[\prod_{i=1}^{m} (2i-3)^2 \right]}{\Delta x \ (-2)^{m-1} \ (2m-1)!} \ \delta^{2m-1} f(j,t)$$

where $j=1,2,\ldots,2k-1$.

If F(t) is a (2k-1)x1 matrix

$$F(t) = \begin{bmatrix} f(1,t) \\ \cdot \\ \cdot \\ f(2k-1,t) \end{bmatrix} \qquad 3.2.0$$

with a typical entry of f(i,t), then these (2k-1) differential equations can be written in matrix form as

$$\frac{dF(t)}{dt} = \sum_{m=1}^{n+1} \frac{v_p \left[\prod_{i=1}^{m} (2i-3)^2 \right]}{\Delta x \ (-2)^{m-1} (2m-1)!} \quad C^{2m-1} F(t).$$

Let the (2k-1)x1 matrix G(t) be defined as

$$G(t) = \begin{bmatrix} g(1,t) \\ g(2,t) \\ \cdot \\ \cdot \\ g(2k-2,t) \\ g(2k-1,t) \end{bmatrix} . \qquad 3.2.1$$

By replacing F(t) by G(t), a set of (2k-1) differential equations are obtained for which g(i,t), where i=1,2,...,2k-1, is the solution. Since g(i,t) is the approximate solution for E(x,t) and I(x,t) at the defined node points, this resulting set of equations is a time domain model for the problem of type 3 obtained by transforming the infinite line time domain models belonging to class 1.

Since this time domain model is obtained directly from the infinite line time domain model, the mathematical properties are identical to those which were derived for the infinite line. Probably the most important advantage of transforming the infinite line model or difference method is that all the approximations for the partial derivatives are obtained by means of central difference formulas of degree (2n+1). It is not obvious by approaching the problem of type 3 directly that central difference approximations can be used at all nodes.

Any numerical technique can be used to find the numerical solution to these equations. One convenient procedure for converting these differential equations to difference equations is as follows. If the (2k-1)x(2k-1) matrix $\mathbf{A}_n$ is given by

$$\mathbf{A}_n = \sum_{m=1}^{n+1} \frac{v_p \left[\prod_{i=1}^{m} (2i-3)^2 \right]}{\Delta x \ (-2)^{m-1} (2m-1)!} C^{2m-1} \qquad 3.2.2$$

then this model can be expressed in matrix notation as

$$\frac{dG(t)}{dt} = \mathbf{A}_n \ G(t). \qquad 3.2.3$$

If G(t) and $\mathbf{A}_n$ are treated respectively as a single variable and a constant, then any numerical technique of solution that uses the values of $G(j\Delta t)$, $G((j-1)\Delta t), \ldots,$ and $G((j-j_2\Delta t)$, to find $G((n+1)\Delta t)$ can be written as

$$G((j+1)\Delta t) = \sum_{i=0}^{j_2} \sum_{p=0}^{m_i} b_{i,p}(\Delta t) \ \mathbf{A}_n^p \ G((j-i)\Delta t) \qquad 3.2.4$$

where $b_{i,j}(\Delta t)$ for $i=0,1,2,\ldots,j_2$, and $p=0,1,2,\ldots,m_i$, is a function of the increment Δt. By treating G(t) and $\mathbf{A}_n$ as matrices, Eqs. 3.2.4. give us the desired difference equations.

Some examples of difference equations derived from a first order model belonging to class 1 are:

Example 1: Runge Kutta first order:

$$G((j+1)\Delta t) = \left[U + \frac{v_p \Delta t}{\Delta x} C \right] G(j\Delta t)$$

Example 2: Runge Kutta second order:

$$G((j+1)\Delta t) = \left[U + \frac{v_p \Delta t}{\Delta x} C + \frac{(v_p \Delta t)^2}{2\ \Delta x^2} C^2 \right] G(j\Delta t)$$

3.3 Transformation of Class 2 Time Domain Models.

It was shown in section 2.1 by Eqs. 2.1.6 that the basic numerical solution for the infinite line is

$$f(j,t) = f(j,0) + v\,\delta f(j,0) + v^2 \frac{\delta^2 f(j,0)}{2!} + v(v^2-1/4) \frac{\delta^3 f(j,0)}{3!} + v^2(v^2-1) \frac{\delta^4 f(j,0)}{4!} + \ldots$$

where $j=0,\pm1,\pm2,\ldots,$ and $v=v_p t/\Delta x$. Because of the special infinite line relations given by Eqs. 3.1.0, it is only necessary to consider the basic numerical solution for $f(i,t)$ for $i=1,2,\ldots,2k-1$.

If the transformation matrix C and the matrix F(t) are used, it is quite evident that this basic numerical solution can be written in matrix notation as

$$F(t) = F(0) + v\,C\,F(0) + v^2\,\frac{C^2F(0)}{2!} + \qquad 3.3.0$$
$$v(v^2-1/4)\,\frac{C^3F(0)}{3!} + \ldots$$

If $F(t)$ is replaced by $G(t)$, the result is an alternative form of the basic numerical solution at the defined nodes for the problem of type 3.

The definition of a class 2 time domain model which is given in section 2.5 can be applied to this form of the basic numerical solution. The resulting equations are identically the transformed class 2 infinite line time domain model.

3.4 Transformation of Difference Methods.

Instead of transforming the time domain models as shown in sections 3.2 and 3.3, the difference methods which are given in section 2.6 by Eqs. 2.6.0 and 2.6.1 can be transformed to the finite line case.

A difference method, given by either Eqs. 2.6.0 or 2.6.1, can be written as

$$f(j,(n+1)\Delta t) = \sum_{i=0}^{j_2} a_i(\Delta x,\Delta t,\delta)\; f(j,(n-i)\Delta t)$$

where $a_i(\Delta x, \Delta t, \delta)$ is an operator. This operator can be expressed as

$$a_i(\Delta x, \Delta t, \delta) = \sum_{p=0}^{m_i} \bar{d}_{i,p}(\Delta x,\Delta t)\,\delta^p$$

where $\bar{d}_{i,p}(\Delta x, \Delta t)$ is a constant for fixed Δx and Δt.

If the same procedure is used as was given in sections 3.2 and 3.3, this transformed difference can obviously be written as

$$G(j\Delta t) = \sum_{i=0}^{j_2} \sum_{p=0}^{m_i} \bar{d}_{i,p}(\Delta x, \Delta t)\ C^p\ G((j-i)\Delta t).$$

3.5 Conclusion.

The time domain models which are presented in this section are obtained directly from the infinite line time domain models by means of a finite-infinite line transformation. Therefore, the mathematical properties of these models are identical to those derived for the infinite line models. The most important advantage of obtaining the models in this manner is that all the approximations for the partial derivatives with respect to x are obtained by more accurate central difference formulas.

IV. LAUNCHING NUMBERS

4.0 Introduction.

A problem of type 1 is a transmission line problem which has zero initial conditions and one identically zero boundary condition. One such problem of type 1 has the initial and boundary conditions specified as

$$
\begin{aligned}
&E(x,0) = I(x,0) = 0 && 0 < x \leq L && 4.0.0\\
&E(L,t) = 0 && 0 \leq t < \infty\\
&I(0,0) = E_{s1}(0)\\
&E(0,t) = E_{s1}(t) && 0 \leq t < \infty
\end{aligned}
$$

where $E_{s1}(t)$ is a specified function of t.

In order to indicate the development of this chapter, let us consider the problem of type 1 for which the boundary and initial conditions are specified as follows:

$$
\begin{aligned}
&E_{n,1}(x,0) = I_{n,1}(x,0) = 0 && 0 < x \leq L\\
&E_{n,1}(L,t) = 0 && 0 \leq t < \infty\\
&I_{n,1}(0,0) = E_{s1}(n\Delta t)\\
&E_{n,1}(0,t) = 0 && t < 0\\
&\qquad\quad = E_{s1}(t+n\Delta t) && 0 \leq t < \Delta t\\
&\qquad\quad = 0 && t \geq \Delta t
\end{aligned}
$$

Since $E_{n,1}(0,t)$ is identically zero for $t \geq \Delta t$, it is possible to define a problem of type 3 at $t=\Delta t$ for which the initial and boundary conditions are

$$E_n(x,0) = E_{n,1}(x,\Delta t) \qquad 0 \leq x \leq L \qquad 4.0.1$$

$$I_n(x,0) = I_{n,1}(x,\Delta t) \qquad 0 \leq x \leq L$$

$$E_n(0,t) = E_n(L,t) = 0 \qquad 0 \leq t < \infty$$

such that

$$E_{n,1}(x,t+\Delta t) = E_n(x,t) \qquad 4.0.2$$

$$I_{n,1}(x,t+\Delta t) = I_n(x,t)$$

for $0 \leq t < \infty$ and $0 \leq x \leq L$.

Since the lossless parallel transmission line is a linear problem, the superposition principle may be applied at any time. Hence, it is quite evident that the relationship

$$E(x,t) = \sum_{n=0}^{p} E_{n,1}(x,t-n\Delta t) \qquad 4.0.3$$

$$I(x,t) = \sum_{n=0}^{p} I_{n,1}(x,t-n\Delta t)$$

is valid for $p\Delta t \leq t < (p+1)\Delta t$. If Eqs. 4.0.2 are substituted into Eqs. 4.0.3, then the solution for $E(x,t)$ and $I(x,t)$ can be written as

$$E(x,t) = \sum_{n=0}^{p-1} E_n(x,t-(n+1)\Delta t) \quad + E_{p,1}(x,t-p\Delta t)$$

$$I(x,t) = \sum_{n=0}^{p-1} I_n(x,t-(n+1)\Delta t) \quad + I_{p,1}(x,t-p\Delta t)$$

where $p\Delta t \leq t < (p+1)\Delta t$. At $t=p\Delta t$, the solutions for $E(x,p\Delta t)$ and $I(x,p\Delta t)$ are given as

$$E(x,p\Delta t) = \sum_{n=0}^{p-1} E_n(x,(p-n-1)\Delta t) \qquad 0 < x \leq L \qquad 4.0.4$$

$$I(x,p\Delta t) = \sum_{n=0}^{p-1} I_n(x,(p-n-1)\Delta t) \qquad 0 < x \leq L$$

$$E(0,p\Delta t) = E_{s1}(p\Delta t)$$

$$I(0,p\Delta t) = \sum_{n=0}^{p-1} I_n(0,(p-n-1)\Delta t) \quad + E_{s1}(p\Delta t).$$

Equations 4.0.4 simply state that the solutions for $E(x,p\Delta t)$ and $I(x,p\Delta t)$, where $0 < x \leq L$, can be obtained by summing the solutions for p problems of type 3.

The general method of numerical solution can now be stated for the problem of type 1 which is specified by the initial and boundary conditions given by Eqs. 4.0.0. The nonzero boundary condition, $E_{s1}(t)$, at $t=n\Delta t$ is used to define a problem of type 3 at $t=(n+1)\Delta t$. The numerical method of solution for a problem of type 3 was covered in detail in Chapter 3. According to Eqs. 4.0.4, the numerical solution at the interior node points at $t=p\Delta t$ is the superposition of the numerical solutions of p problems of type 3. If $g_n(j,0)$, $j=0,1,2,\ldots,2k$, where $\Delta x=L/k$, are the initial conditions to the problem of type 3 which was defined at $t=(n+1)\Delta t$, this chapter specifically derives a class of transformations for which

$$g_n(j,0) = b_{j,1}(\Delta x,\Delta t)\ E_{s1}(n\Delta t) \qquad j=0,1,\ldots,2k \qquad 4.0.5$$

such that

$$g(j,p\Delta t) = \sum_{n=0}^{p-1} g_n(j,(p-n-1)\Delta t) \qquad j=1,2,\ldots,2k \qquad 4.0.6$$

$$g(0,p\Delta t) = E_{s1}(p\Delta t)$$

are the approximate numerical solutions to $E(x,p\Delta t)$ and $I(x,p\Delta t)$ at the defined net points. Any transformation from this class of transformations defines the $b_{j,1}(\Delta x,\Delta t)$, where $j=0,1,2,\ldots,2k$, which are employed in Eqs. 4.0.5. These quantities, $b_{j,1}(\Delta x,\Delta t)$, are defined as launching numbers.

The specific approach for deriving this class of transformations is to use the basic numerical solution, given by Eqs. 3.3.0, to find $g_n(j,(p-n-1)\Delta t)$, and then compare the functions $g(j,p\Delta t)$, given by Eqs. 4.0.6, for all p, to an approximate Fourier integral, an approximate Foureir series, and an analytical solution for this problem of type 1.

4.1 Approximate Fourier Integral.

Part of this chapter is based on an approximate Fourier integral. Hence, this section is devoted to the derivation of the approximate Fourier integral.

The Fourier integral transform pairs are

$$g(\alpha) = \int_{-\infty}^{\infty} f(x') \cos(\alpha(x'-x))\, dx'$$

$$f(x) = \lim_{\alpha \to \infty} (1/\pi) \int_{0}^{\alpha} g(\alpha')\, d\alpha'$$

where $g(\alpha)$ and $f(x)$ at present bear no relationship to the functions $g(n,t)$ or $f(n,t)$.

Let us approximate $g(\alpha)$ by $\bar{g}(\alpha)$ where $\bar{g}(\alpha)$ is given as

$$\bar{g}(\alpha) = g(\alpha) \qquad 0 \leq \alpha \leq \pi/\Delta x$$

$$\bar{g}(\alpha) = 0 \qquad \alpha > \pi/\Delta x$$

It is quite clear that this is a good approximation if Δx is chosen sufficiently small. If $\bar{g}(\alpha)$ is used in place of $g(\alpha)$, the transform pairs are

$$\bar{g}(\alpha) = \int_{-\infty}^{\infty} f(x') \cos(\alpha(x'-x))\, dx' \qquad 0 \leq \alpha \leq \pi/\Delta x$$

$$\bar{g}(\alpha) = 0 \qquad\qquad \alpha > \pi/\Delta x$$

$$\bar{f}(x) = (1/\pi)\int_0^{\pi/\Delta x} \bar{g}(\alpha)\, d\alpha$$

where $\bar{f}(x)$ is an approximation to $f(x)$. If $\bar{g}(\alpha)$ is eliminated from these equations, $\bar{f}(x)$ can be written as

$$\bar{f}(x) = (1/\pi)\int_{-\infty}^{\infty} f(x')\, dx' \int_0^{\pi/\Delta x} \cos(\alpha'(x'-x))\, d\alpha'$$

$$= \int_{-\infty}^{\infty} f(x') \frac{\sin \pi(x'-x)/\Delta x}{\pi(x'-x)/\Delta x}\, dx' .$$

If this integral is numerically integrated by means of the trapezoidal rule, the function $\bar{f}(x)$ is then approximated by

$$\bar{f}(x) = \frac{\Delta x'}{\Delta x}\sum_{-\infty}^{\infty} f(n\Delta x') \frac{\sin \pi(n\Delta x'-x)/\Delta x}{\pi(n\Delta x'-x)/\Delta x} \qquad 4.1.0$$

where $\Delta x'$ may or may not equal Δx. Equation 4.1.0 is defined as the approximate Fourier integral of $f(x)$.

If $\bar{f}(x)$ is given by Eqs. 4.1.0, it is quite clear that $\bar{f}(x)$ approximates $f(x)$ for Δx and $\Delta x'$ (generally $\Delta x' \leq \Delta x$) sufficiently small. Since

$$\frac{\sin \pi(n\Delta x'-x)/\Delta x}{\pi(n\Delta x'-x)/\Delta x}$$

contains no spectral components for $\alpha > \pi/\Delta x$, $\bar{f}(x)$ can have no spectral components for $\alpha > \pi/\Delta x$. In addition, when $\Delta x' = \Delta x$, this is the same Fourier integral approximation as was given in section 2.3.

4.2 Approximate Fourier Integral and Series Solution.

The main purpose of this section is to derive an approximate Fourier integral and series solution for $E(x,p\Delta t)$ and $I(x,p\Delta t)$. In fact, special forms of an approximate Fourier integral and series solution for $E(x,p\Delta t)$ and $I(x,p\Delta t)$ are derived which will be used in later sections of this chapter. The reasons for developing these special forms will become evident in later sections.

First of all, let us consider the problem of type 1 for which the nonzero boundary condition is given by the function $E^p_{s1}(t)$ where

$$\begin{aligned} E^p_{s1}(t) &= E_{s1}(t) && 0 \le t < p\Delta t \\ &= 0. && t \ge p\Delta t \end{aligned} \qquad 4.2.0$$

If the analytical solution for this problem is given by $E^p(x,t)$ and $I^p(x,t)$, it is quite clear that

$$\begin{aligned} E^p(x,t) &= E(x,t) && 0 \le t < p\Delta t, \quad 0 \le x \le L \\ I^p(x,t) &= I(x,t) && 0 \le t < p\Delta t, \quad 0 \le x \le L \\ E^p(x,p\Delta t) &= E(x,p\Delta t) && 0 < x \le L \\ I^p(x,p\Delta t) &= I(x,p\Delta t) && 0 < x \le L \end{aligned}$$

is valid. In addition for $t \geq p\Delta t$, this problem corresponds to a problem of type 3 where $E^p(0,t)$ and $E^p(L,t)$ are identically zero.

For $t \geq p\Delta t$, the analytical solution for $E^p(x,t)$ and $I^p(x,t)$ can be written as

$$E^p(x,t) = \sum_{n=-\infty}^{\infty} E^p_{s1}(t-x/v_p-2nL/v_p) - \sum_{n=-\infty}^{\infty} E^p_{s1}(t+x/v_p-2nL/v_p)$$

$$I^p(x,t) = \sum_{n=-\infty}^{\infty} E^p_{s1}(t-x/v_p-2nL/v_p) + \sum_{n=-\infty}^{\infty} E^p_{s1}(t+x/v_p-2nL/v_p).$$

Both the functions $E^p(x,t)$ and $I^p(x,t)$ are periodic in both $x, (-\infty < x < \infty)$ and $t, (-\infty < t < \infty)$ even though the solutions are valid for $0 \leq x \leq L$ and $p\Delta t \leq t < \infty$. This forces $E^p(x,t)$ and $I^p(x,t)$ to satisfy the finite-infinite line transformation as discussed in Chapter 3.

If we let $t' = t - p\Delta t$ and define the function $M^p(-v_p t')$ as

$$M^p(-v_p t') = E^p_{s1}(t'+p\Delta t), \qquad 4.2.1$$

then the solution for $E^p(x,t')$ and $I^p(x,t')$ for $t' \geq 0$ can be written as

$$E^p(x,t') = \sum_{n=-\infty}^{\infty} M^p(-v_p t'+x+2nL) - \sum_{n=-\infty}^{\infty} M^p(-v_p t'-x+2nL) \qquad 4.2.2$$

$$I^p(x,t') = \sum_{n=-\infty}^{\infty} M^p(-v_p t'+x+2nL) + \sum_{n=-\infty}^{\infty} M^p(-v_p t'-x+2nL).$$

By letting

$$V^p(x) = \sum_{n=-\infty}^{\infty} M^p(x+2nL), \qquad 4.2.3$$

$E^p(x,t')$ and $I^p(x,t')$ can now be expressed as

$$E^p(x,t') = V^p(-v_p t'+x) - V^p(-v_p t'-x) \qquad 4.2.4$$

$$I^p(x,t') = V^p(-v_p t'+x) + V^p(-v_p t'-x)$$

where $V^p(x)$ is periodic in x with period 2L.

Since $V^p(x)$ is periodic in x with period 2L, this function can be expanded in a Fourier series as

$$V^p(x) = \bar{a}_0^p/2 + \sum_{n=1}^{\infty} \left[\bar{a}_n^p \cos(n\pi x/L) + \bar{b}_n^p \sin(n\pi x/L)\right] \qquad 4.2.5$$

where

$$\bar{a}_n^p = (1/L)\int_0^{2L} V^p(x) \quad \cos(n\pi x/L)\ dx$$

$$\bar{b}_n^p = (1/L)\int_0^{2L} V^p(x)\ \sin(n\pi x/L)\ dx.$$

If $x=v_p t$ and the trapezoidal rule is used to numerically integrate the integrals to find the coefficients, it can easily be shown that

$$\bar{a}_n^p = \frac{v_p \Delta t}{L} \sum_{j=1}^{p} M^p(jv_p\Delta t)\ \cos(n\pi jv_p\Delta t/L) \qquad 4.2.6$$

$$\bar{b}_n^p = \frac{v_p \Delta t}{L} \sum_{j=1}^{p} M^p(jv_p\Delta t)\ \sin(n\pi jv_p\Delta t/L).$$

The function $N^p(\Delta x',-v_p t')$ is defined as the approximate Fourier integral of $M^p(-v_p t')$. According to Eqs. 4.1.0, the approximate Fourier integral of $M^p(-v_p t')$ is given as

$$N^p(\Delta x',-v_p t') = \qquad 4.2.7$$

$$\frac{v_p \Delta t}{\Delta x'} \sum_{n=1}^{p} M^p(nv_p\Delta t)\ \frac{\sin \pi(nv_p\Delta t+v_p t')/\Delta x'}{\pi(nv_p\Delta t+v_p t')/\Delta x'}$$

where $\Delta x'k'=L$. The increment $\Delta x'$ may or may not be equal to the increment Δx used to solve the transmission line problems of type 3. Hence an approximate Fourier integral solution can be obtained by first rearranging the absolute convergent series which are given in Eqs. 4.2.2 and then be replacing $M^p(-v_p t')$ by its approximate Fourier integral

$$
\begin{aligned}
E_f^p(x,t') = {} & N^p(\Delta x',-v_p t'+x) - N^p(\Delta x',-v_p t'-x) + \\
& \sum_{n=1}^{\infty}\left[N^p(\Delta x',-v_p t'+x+2nL) + N^p(\Delta x',-v_p t'+x-2nL)\right] - \\
& \sum_{n=1}^{\infty}\left[N^p(\Delta x',-v_p t'-x+2nL) + N^p(\Delta x',-v_p t'-x-2nL)\right]
\end{aligned}
\qquad 4.2.8
$$

$$
\begin{aligned}
I_f^p(x,t') = {} & N^p(\Delta x',-v_p t'+x) + N^p(\Delta x',-v_p t'-x) + \\
& \sum_{n=1}^{\infty}\left[N^p(\Delta x',-v_p t'+x+2nL) + N^p(\Delta x',-v_p t'+x-2nL)\right] + \\
& \sum_{n=1}^{\infty}\left[N^p(\Delta x',-v_p t'-x+2nL) + N^p(\Delta x',-v_p t'-x-2nL)\right]
\end{aligned}
$$

where $E_f^p(x,t')$ and $I_f^p(x,t')$ are respectively the approximate Fourier integral solutions for $E^p(x,t')$ and $I^p(x,t')$ when $t' \geq 0$.

By letting the function $V_a^p(x)$ be defined as

$$
V_a^p(x) = N^p(\Delta x',x) + \sum_{n=1}^{\infty}\left[N^p(\Delta x',x+2nL) + N^p(\Delta x',x-2nL)\right] \qquad 4.2.9
$$

the approximate solution for $E_f^p(x,t')$ and $I_f^p(x,t')$ can be written as

$$E_f^p(x,t') = V_a^p(-v_p t'+x) - V_a^p(-v_p t'-x) \qquad 4.2.10$$

$$I_f^p(x,t') = V_a^p(-v_p t'+x) + V_a^p(-v_p t'-x).$$

This function is a periodic function in x with period 2L. Therefore, it can be represented as a Fourier series. If $N^p(\Delta x', x+2nL)$ is replaced by its defining equation, Eq. 4.2.7, then $V_a^p(x)$ can be written as

$$V_a^p(x) = \frac{v_p\Delta t}{\Delta x'} \sum_{n=1}^{p} M^p(nv_p\Delta t)\left\{\frac{\sin \pi(nv_p\Delta t-x)/\Delta x'}{\pi(nv_p\Delta t-x)/\Delta x'} + \right.$$

$$\left. \sum_{i=1}^{\infty} \left[\frac{\sin \pi(nv_p\Delta t-x-2iL)/\Delta x'}{\pi(nv_p\Delta t-x-2iL)/\Delta x'} + \frac{\sin \pi(nv_p\Delta t-x+2iL)/\Delta x'}{\pi(nv_p\Delta t-x+2iL)/\Delta x'}\right]\right\}.$$

But according to section 2.2, $V_a^p(x)$ can now be written as

$$V_a^p(x) = a_0^p/2 + \sum_{m=1}^{k'-1} \left[a_n^p \cos(n\pi x/L) + \right. \qquad 4.2.11$$

$$\left. b_n^p \sin(n\pi x/L)\right] + \left[a_{k'}^p \cos(k'\pi x/L) + b_{k'}^p \sin(k'\pi x/L)\right]/2.$$

where

$$a_m^p = \frac{v_p\Delta t}{L} \sum_{m=1}^{p} M^p(nv_p\Delta t)\ \cos(m\pi nv_p\Delta t/L). \qquad 4.2.12$$

$$b_m^p = \frac{v_p\Delta t}{L} \sum_{m=1}^{p} M^p(nv_p\Delta t)\ \sin(m\pi nv_p\Delta t/L).$$

If the Fourier coefficients which are given by Eqs. 4.2.12 are compared to the coefficients as given by Eqs. 4.2.6, it can be concluded that these are the same coefficients. Therefore, by truncating $V^p(x)$ after k' terms and obtaining the coefficients by numerical integration, the result is the function $V_a^p(x)$.

Let us examine briefly the solutions for $E_f^p(x,t')$ and $I_f^p(x,t')$ at t'=0. If Eqs. 4.2.11 is substituted into Eqs. 4.2.10, the solutions at t'=0 are

$$E_f^p(x,0) = \sum_{n=1}^{k'-1} b_n^p \sin(n\pi x/L) + b_{k'}^p \sin(k'\pi x/L) \qquad 4.2.13$$

$$I_f^p(x,0) = a_0^p + 2\sum_{n=1}^{k'-1} a_n^p \cos(n\pi x/L) + a_{k'}^p \cos(k'\pi x/L)$$

At the defined node points, the k'th term is zero and need not be considered. If $E_{s1}^p(t)$ is discontinuous at t=pΔt, it is quite likely that $V^p(x)$ is discontinuous at x=2nL, where $n=0,\pm 1,\pm 2,\ldots,$. As a result, these approximations as given by Eqs. 4.2.13 can tend to oscillate about the actual solutions. These oscillations decreases slowly as we move away from the discontinuity. This property can not be overlooked and is taken into consideration in this and a later section.

One method which helps to correct this situation is the introduction of Lanczos' σ factors (6). If $E_f^p(x,t')$ and $I_f^p(x,t')$ for t'=0 are truncated after the n'th term, the introduction of these factors in effect replace these truncated approximations by smoothed function which are given as

$$\bar{E}_f^p(x,0) = n'/L \int_{x-L/2n'}^{x+L/2n'} E_{f,n'}^p(x',0)\, dx'$$

$$\bar{I}_f^p(x,0) = n'/L \int_{x-L/2n'}^{x+L/2n'} I_{f,n'}^p(x',0)\, dx'$$

where

$$E_{f,n'}^p(x',0) = 2\sum_{n=1}^{n'} b_n^p \sin(n\pi x/L)$$

$$I_{f,n'}^p(x',0) = a_0^p + 2\sum_{n=1}^{n'} a_n^p \cos(n\pi x/L)$$

and $n' < k'$. Once this integration or averaging process has been performed, $\bar{E}_f^p(x,0)$ and $\bar{I}_f^p(x,0)$ are given by

$$\bar{E}_f^p(x,0) = 2\sum_{n=1}^{n'} \sigma_n b_n^p \sin(n\pi x/L) \qquad 4.2.14$$

$$\bar{I}_f^p(x,0) = a_0^p + 2\sum_{n=1}^{n'} \sigma_n a_n^p \cos(n\pi x/L)$$

where

$$\sigma_n = \frac{\sin(\pi n/2n')}{(\pi n/2n')} . \qquad 4.2.15$$

4.3 Concept of Consistent Initial Values.

First of all, let us consider the transmission line problem of type 3 for which the initial conditions at the defined nodes, $g_n(j,0)$, where $j=0,1,2,\ldots,2k$, are specified. If the solution for $g_n(j,(p-n-1)\Delta t)$ is obtained by some method described in Chapter 3 from these specified initial values, this chapter assumes that the increments Δx and Δt are chosen such that $g_n(j,(p-n-1)\Delta t)$ approximates the basic numerical solution as given in section 3.3 for all p such that $p \geq n+1$. Hence, the basic numerical solution is employed to express $g_n(j,(p-n-1)\Delta t)$ in this chapter.

If Δx and Δt are not chosen sufficiently small, regardless of how the launching numbers, $b_{j,1}(\Delta x, \Delta t)$ are chosen, it is not likely that $g(j, p\Delta t)$ is an approximate solution to $E(x, p\Delta t)$ and $I(x, p\Delta t)$ at the node points. In order to avoid this situation when comparing the functions $g(j,t)$ to either the approximate Fourier solutions or the analytical solutions for $E(x,t)$ and $I(x,t)$ which are given in section 4.2, the assumption is always made that these increments have been properly chosen.

The set of initial values $g_n(j,0)$ for $n=0,1,2,\ldots,p-1$, and $j=0,1,2,\ldots,2k$, is a set of consistent initial values if and only if the following conditions are satisfied:

1. Let $g_n(j,t)$ be obtained by using the basic numerical solution as defined in section 3.3.
2. There exists a $\overline{\Delta x}$ such that if $\Delta x < \overline{\Delta x}$, there corresponds a $\overline{\Delta t}$ which depends on Δx such that if $\Delta t < \overline{\Delta t}$, then $g(j, p\Delta t)$ approximates $E^p(x,t')$ and $I^p(x,t')$ for all p at the defined node points when $t'=0$.

It is clear that the transformation defined by the launching numbers $b_{j,1}(\Delta x, \Delta t)$ defines a valid transformation only if the set of initial values $g_n(j,0)$ are a set of consistent initial values.

4.4 Derivation of $g(j, p\Delta t)$.

Before the function $g(j, p\Delta t)$ can be compared to the function $E^p(x,t)$ and $I^p(x,t)$ as given by Eqs. 4.2.2 or the

Fourier approximations to these functions at $t'=0$, $g(j,p\Delta t)$ must be placed in the proper form so that a comparison can be made. This section develops such a form.

Given the initial values $g_n(j,0)$ for $j=0,1,2,\ldots,2k$, and $n=0,1,2,\ldots$, whether these values are consistent initial values or not. Since the functions $g_n(j,t)$, $j=0,1,\ldots,2k$, are the approximate solutions to a problem of type 3 at certain node points which are obtained from the initial values $g_n(j,0)$, $j=0,1,2,\ldots,2k$, the functions $g_n(0,t)$ and $g_n(2k,t)$ are required to be identically zero for all t. If we let $f_n(j,t)=g_n(j,t)$ for $j=0,1,2,\ldots,2k$, and use the relationships given by Eqs. 3.1.0, the problem of type 3 has been transformed to the infinite line situation which was described in Chapter 2. The functions $f_n(j,t)$ are used in place of $f(j,t)$ for identification purposes.

Consider the function $z^p(j,t)$ which is defined as

$$z^p(j,t) = \sum_{n=0}^{p-1} f_n(j,t-(n+1)\Delta t) \qquad 4.4.0$$

where $j=0,\pm 1,\pm 2,\ldots,$. If we have a set of consistent initial values, the functions $z^p(j+4mk,t)$ for $j=0,1,2,\ldots,2k$, and $m=0,\pm 1,\pm 2,\ldots$, would give us the desired approximate numerical solution at $t=p\Delta t$ to either $E^p(j\Delta x,t)$ or $I^p(j\Delta x+\Delta x/2,t)$, depending on whether j is odd or even.

The basic numerical solution for $f_n(m,t)$ can be obtained by substituting Eqs. 2.1.7 into Eq. 2.1.1. If this basic numerical solution is used to represent $f_n(m,t)$ and upon introducing the change of variable $t'=t-p\Delta t$ and $i=p-n-1$, the function $z^p(m,t')$ can be written as follows:

$$
\begin{aligned}
z^p(m,t') = \sum_{i=0}^{p-1} \Bigg\{ & f_{p-i-1}(m,0) \, \frac{\sin \pi(iv_p\Delta t+v_pt')/\Delta x}{\pi(iv_p\Delta t+v_pt')/\Delta x} + \\
& \frac{f_{p-i-1}(m+1,0)}{2} \, \frac{\sin \pi(iv_p\Delta t+\Delta x/2+v_pt')/\Delta x}{\pi(iv_p\Delta t+\Delta x/2+v_pt')/\Delta x} - \\
& \frac{f_{p-i-1}(m+1,0)}{2} \, \frac{\sin \pi(iv_p\Delta t-\Delta x/2+v_pt')/\Delta x}{\pi(iv_p\Delta t-\Delta x/2+v_pt')/\Delta x} + \\
& \sum_{j=1}^{\infty} \Bigg[\frac{f_{p-i-1}(m+2j,0)}{2} \, \frac{\sin \pi(iv_p\Delta t-j\Delta x+v_pt')/\Delta x}{\pi(iv_p\Delta t+j\Delta x+v_pt')/\Delta x} + \\
& \frac{f_{p-i-1}(m-2j,0)}{2} \, \frac{\sin \pi(iv_p\Delta t+j\Delta x+v_pt')/\Delta x}{\pi(iv_p\Delta t+j\Delta x+v_pt')/\Delta x} \Bigg] + \\
& \sum_{j=1}^{\infty} \Bigg[\frac{f_{p-i-1}(m+2j,0)}{2} \, \frac{\sin \pi(iv_p\Delta t+j\Delta x+v_pt')/\Delta x}{\pi(iv_p\Delta t+j\Delta x+v_pt')/\Delta x} + \\
& \frac{f_{p-i-1}(m-2j,0)}{2} \, \frac{\sin \pi(iv_p\Delta t-j\Delta x+v_pt')/\Delta x}{\pi(iv_p\Delta t-j\Delta x+v_pt')/\Delta x} \Bigg] + \\
& \sum_{j=1}^{\infty} \Bigg[\frac{f_{p-i-1}(m+2j+1,0)}{2} \, \frac{\sin \pi(iv_p\Delta t+j\Delta x+\Delta x/2+v_pt')/\Delta x}{\pi(iv_p\Delta t+j\Delta x+\Delta x/2+v_pt')/\Delta x} + \\
& \frac{f_{p-i-1}(m-2j+1,0)}{2} \, \frac{\sin \pi(iv_p\Delta t-j\Delta x+\Delta x/2+v_pt')/\Delta x}{\pi(iv_p\Delta t-j\Delta x+\Delta x/2+v_pt')/\Delta x} \Bigg] -
\end{aligned}
$$

$$\sum_{j=1}^{\infty}\left[\frac{f_{p-i-1}(m+2j+1,0)}{2}\;\frac{\sin\pi(iv_p\Delta t-j\Delta x-\Delta x/2+v_pt')/\Delta x}{\pi(iv_p\Delta t-j\Delta x-\Delta x/2+v_pt')/\Delta x}+\right.$$

$$\left.\frac{f_{p-i-1}(m-2j+1,0)}{2}\;\frac{\sin\pi(iv_p\Delta t+j\Delta x-\Delta x/2+v_pt')/\Delta x}{\pi(iv_p\Delta t+j\Delta x-\Delta x/2+v_pt')/\Delta x}\right]\Bigg\}$$

This expression for $z^p(m,t')$ as given by Eqs. 4.4.1 is quite complicated. The expression can be simplified greatly if two new functions, $J^p(j,x)$ and $H^p(j,x)$, and their properties are introducted. Let $J^p(j,x)$ be defined as

$$J^p(j,x) = 0 \qquad\qquad x \le -v_p\Delta t/2$$

$$J^p(j,x) = 0 \qquad\qquad v_p\Delta t(n-1/2) \le x < v_p\Delta t(n+1/2)$$

where $j=0,\pm1,\pm2,\ldots,$. It is quite clear that

$$J^p(j,x)\Big|_{x=nv_p\Delta t} = f_{p-n-1}(j,0).$$

The function $H^p(j,x)$ is the approximate Fourier integral, defined by Eqs. 4.1.0, of $J^p(j,x)$. Therefore,

$$H^p(j,x) = \frac{v_p\Delta t}{\Delta x}\sum_{n=0}^{p-1} f_{p-n-1}(j,0)\;\frac{\sin\pi(nv_p\Delta t-x)/\Delta x}{\pi(nv_p\Delta t-x)/\Delta x}. \qquad 4.4.2$$

Because of the relationships derived in Chapter 3, given by Eqs. 3.1.0, the following relationships for $H^p(j,x)$ are valid for $j=0,\pm1,\pm2,\ldots,$.

$$H^p(j+4ik,x) = H^p(j,x) \qquad i=0,\pm 1,\pm 2,\ldots, \qquad 4.4.3$$

$$H^p(2j+1,x) = -H^p(-2j-1,x)$$

$$H^p(2j,x) = H^p(-2j,x)$$

Let us now employ Eqs. 4.4.2 and 4.4.3 to simplify Eqs. 4.4.1. It is quite clear that the expression for $z^p(m,t')$ as given by Eqs. 4.4.4 is valid.

$$z^p(m,t') = \frac{\Delta x}{2v_p\Delta t}\sum_{m=1}^{2k-1}\Bigg\{ H^p(n,-v_pt'+(m-n)\Delta x/2) + \qquad 4.4.4$$

$$(-1)^{n+m}\, H^p(n,-v_pt'+(n-m)\Delta x/2) +$$

$$(-1)^{n+1}\, H^p(n,-v_pt'+(m+n)\Delta x/2) +$$

$$(-1)^{m+1}\, H^p(n,-v_pt'-(n+m)\Delta x/2) +$$

$$\sum_{i=1}^{\infty}\Big[H^p(n,-v_pt'+(m-n)\Delta x/2+2iL) +$$

$$H^p(n,-v_pt'+(m-n)\Delta x/2-2iL)\Big] +$$

$$(-1)^{n+m}\sum_{i=1}^{\infty}\Big[H^p(n,-v_pt'+(n-m)\Delta x/2+2iL) +$$

$$H^p(n,-v_pt'+(n-m)\Delta x/2-2iL)\Big] +$$

$$(-1)^{n+1}\sum_{i=1}^{\infty}\Big[H^p(n,-v_pt'+(m+n)\Delta x/2+2iL) +$$

$$H^p(n,-v_pt'+(m+n)\Delta x/2-2iL)\Big] +$$

$$(-1)^{m+1} \sum_{i=1}^{\infty} \left[H^p(n,-v_p t'-(n+m)\Delta x/2+2iL) + H^p(n,-v_p t'-(n+m)\Delta x/2-2iL) \right] \Bigg\}$$

This expression places $z^p(m,t')$ directly in terms of $H^p(n,x)$ for $n=1,2,\ldots,2k-1$.

For simplification purposes, let us introduce two new functions $z_j^p(m,-v_p t'+2iL)$ for $j=1,2$, and $i=0,\pm 1,\pm 2,\ldots,$. These functions are given as

$$z_1^p(m,-v_p t+2iL) = \frac{\Delta x}{2v_p \Delta t} \sum_{n=1}^{2k-1} H^p(n,-v_p t'+(m-n)\Delta x/2+2iL) \qquad 4.4.5$$

$$z_2^p(m,-v_p t'+2iL) = \frac{\Delta x}{2v_p \Delta t} \sum_{n=1}^{2k-1} (-1)^{n+1} H^p(n,v_p t'+(m+n)\Delta x/2+2iL). \qquad 4.4.6$$

Therefore, Eqs. 4.4.7 is obtained by substituting these functions into Eqs. 4.4.4.

$$\begin{aligned} z^p(m,t') = {} & z_1^p(m,-v_p t') + z_2^p(m,-v_p t') + \\ & (-1)^{m+1} \left[z_1^p(-m,-v_p t') + z_2^p(-m,-v_p t') \right] + \\ & \sum_{i=1}^{\infty} \left[z_1^p(m,-v_p t'+2iL) + z_1^p(m,-v_p t'-2iL) \right] + \\ & \sum_{i=1}^{\infty} \left[z_2^p(m,-v_p t'+2iL) + z_2^p(m,-v_p t'-2iL) \right] + \end{aligned} \qquad 4.4.7$$

$$(-1)^{m+1} \sum_{i=1}^{\infty} \left[z_1^p -m,-v_p t'+2iL) + z_1^p(-m,-v_p t'-2iL) \right] +$$

$$(-1)^{m+1} \sum_{i=1}^{\infty} \left[z_2^p(-m,-v_p t'+2iL) + z_2^p(-m,-v_p t'-2iL) \right]$$

Before this relationship, given by Eqs. 4.4.7, can be investigated, the initial values, $g_n(j,0)$ or $f_n(j,0)$, must be defined in some manner so that a comparison can be made. Let $g_n(j,0)$ be defined by Eqs. 4.0.5. That is,

$$g_n(j,0) = b_{j,1}(\Delta x,\Delta t)\ E_{s1}(n\Delta t).$$

If this expression is substituted into Eqs. 4.4.2, the function $H^p(j,x)$ is now given as

$$H^p(j,x) =$$

$$b_{j,1}(\Delta x,\Delta t)\ \frac{v_p\Delta t}{\Delta x} \sum_{n=0}^{p-1} E_{s1}((p-n-1)\Delta t)\ \frac{\sin \pi(nv_p\Delta t-x)/\Delta x}{\pi(nv_p\Delta t-x)/\Delta x}\ .$$

If we let $x=-v_p t'+v_p\Delta t$ and compare the resulting expression to Eqs. 4.2.7, it is evident that Eq. 4.4.8 is valid.

$$H^p(j,-v_p t'-v_p\ t) = b_{j,1}(\Delta x,\Delta t)\ N^p(\Delta x,-v_p t') \qquad 4.4.8$$

If Eqs. 4.4.8 is substituted into Eqs. 4.4.5 and 4.4.6, then $z_1^p(m,-v_p t'+2iL)$ and $z_2^p(m,-v_p t'+2iL)$ are now given as

$$z_1^p(m,-v_p t'+2iL) = \qquad 4.4.9$$

$$\frac{\Delta x}{2v_p\Delta t} \sum_{n=1}^{2k-1} b_{n,1}(\Delta x,\Delta t)\ N^p(\Delta x,-v_p t'+(m-n)\Delta x/2+2iL+v_p\Delta t)$$

$$z_2^p(m,-v_pt'+2iL) = \qquad 4.4.10$$

$$\frac{\Delta x}{2v_p\Delta t}\sum_{n=1}^{2k-1}(-1)^{n+1}b_n(\Delta x,\Delta t)\ N^p(\Delta x,-v_pt'+(m+n)\Delta x/2+2iL+v_p\Delta t).$$

Therefore, $z_1^p(m,-v_pt'+2iL)$ and $z_2^p(m,-v_pt'+2iL)$ can be placed directly in terms of the approximate Fourier integral of $M^p(-v_pt')$ which is given by Eq. 4.2.7.

The function $z^p(m,t')$ can also be expressed in terms of the quantity $V_a^p(x)$ which was defined by Eq. 4.2.9. In order to show this result, let us define the functions $y_1^p(m,-v_pt')$ and $y_2^p(m,-v_pt')$ as

$$y_i^p(m,-v_pt') = z_i^p(m,-v_pt') + \sum_{i=1}^{\infty}\left[z_i^p(m,-v_pt'+2iL) + z_i^p(m,-v_pt'-2iL)\right]. \qquad 4.4.11$$

By means of Eqs. 4.2.9, 4.4.9, and 4.4.10, it can easily be shown that $y_1^p(m,-v_pt')$ and $y_2^p(m,-v_pt')$ can be written as

$$y_1^p(m,-v_pt') = \qquad 4.4.12$$

$$\frac{\Delta x}{2v_p\Delta t}\sum_{n=1}^{2k-1}b_{n,1}(\Delta x,\Delta t)\ V_a^p(-v_pt'+(m-n)\Delta x/2+v_p\Delta t)$$

$$y_2^p(m,-v_pt') = \qquad 4.4.13$$

$$\frac{\Delta x}{2v_p\Delta t}\sum_{n=1}^{2k-1} b_{n,1}(\Delta x,\Delta t)\ V_a^p(-v_pt'+(m+n)\Delta x/2+v_p\Delta t).$$

Therefore,

$$z^p(m,t') = y_1^p(m,-v_pt') + y_2^p(m,-v_pt') + (-1)^{m+1}\left[y_1^p(-m,-v_pt') + y_2^p(-m,-v_pt')\right]. \qquad 4.4.14$$

The desired expressions for $z^p(m,t')$ have now been derived. In section 4.5, these expressions will be interpreted. In sections 4.6 and 4.7, these expressions for $z^p(m,t')$ will be compared to both the analytical solution and the approximate Fourier solutions to obtain the launching numbers.

4.5 Interpretation of $z^p(m,-v_pt')$.

In section 4.2, an analytical solution for $E^p(x,t')$ and $I^p(x,t')$, given by Eqs. 4.2.2, was derived. In section 4.4, a number of special forms for $z^p(m,t')$ were derived. The objective of this section is to compare these two solutions for the purposes of interpreting the results of section 4.4.

First of all, the function $w^p(m,t')$ is introduced for notation purposes. Let this function be defined as follows:

$$w^p(2m,t') = E^p(m\Delta x,t') \qquad m=0,1,2,\ldots,k \qquad 4.5.0$$

$$w^p(2m+1,t') = I^p(m\Delta x+\Delta x/2,t') \quad m=0,1,2,\ldots,2k-1$$

Therefore,

$$w^p(m,t') = \sum_{n=-\infty}^{\infty} M^p(-v_p t'+m\Delta x/2+2nL) + (-1)^{m+1}\sum_{n=-\infty}^{\infty} M^p(-v_p t'-m\Delta x/2+2nL). \qquad 4.5.1$$

If this problem was a semi-infinite transmission line which was defined to start at $x=0$ and to extend to infinity in the positive x direction, the solution for $w^p(m,t')$ would then be given as

$$w^p(m,t') = M^p(-v_p t'+m\Delta x/2)$$

where $m=0,1,2,\ldots,$. Therefore, for the finite line, the remaining terms which appear in Eq. 4.5.1 for n positive results from reflections which occur at $x=0$ and $x=L$. The terms which appear in Eq. 4.5.1 for negative n are always zero for $0 \le x \le L$ and were introduced to obtain periodicity in x and t'.

According to Eq. 4.4.7, $z^p(m,t')$ can be written as

$$
\begin{aligned}
z^p(m,t') = {} & z_1^p(m,-v_pt') + z_2^p(m,-v_pt') + \\
& (-1)^{m+1}\left[z_1^p(-m,-v_pt') + z_2^p(-m,-v_pt')\right] + \\
& \sum_{n=1}^{\infty}\left[z_1^p(m,-v_pt'+2nL) + z_1^p(m,-v_pt'-2nL)\right] + \\
& \sum_{n=1}^{\infty}\left[z_2^p(m,-v_pt'+2nL) + z_2^p(m,-v_pt'-2nL)\right] + \\
(-1)^{m+1} & \sum_{n=1}^{\infty}\left[z_1^p(-m,-v_pt'+2nL) + z_1^p(-m,-v_pt'-2nL)\right] + \\
(-1)^{m+1} & \sum_{n=1}^{\infty}\left[z_2^p(-m,-v_pt'+2nL) + z_2^p(-m,-v_pt'-2nL)\right] .
\end{aligned}
$$

In order to better interpret the functions $z_1^p(m,-v_pt'+2nL)$ and $z_2^p(m,-v_pt'+2nL)$, let us look at $z^p(m,t')$ for the semi-infinite transmission line case. For this case, $z^p(m,t')$ is given as

$$
\begin{aligned}
z^p(m,t') = {} & \left[z_1^p(m,-v_pt') + z_2^p(m,-v_pt')\right] + \\
& (-1)^{m+1}\left[z_1^p(-m,-v_pt') + z_2^p(-m,-v_pt')\right] .
\end{aligned} \tag{4.5.2}
$$

If $z_1^p(m,-v_pt')$ and $z_2^p(m,-v_pt')$ are respectively defined by Eqs. 4.4.9 and 4.4.10, then for m=0,1,2,..., and $t' \geq 0$, it can be shown that $z_1^p(-m,-v_pt')$ is approximately zero. For the finite line case, it can then be concluded that the remaining terms in the expression for n positive result from the reflections at x=0 and x=L. For n negative, the terms have no effect on the solution for $0 \leq x \leq L$ and are added to obtain periodicity in x.

Let us analyze the result which is given by Eq. 4.5.2. First of all, $z_1^p(m,-v_p t')$ and $z_2^p(m,-v_p t')$ for m=0,1,2,3,..., are incident waves traveling on this semi-infinite transmission line. The remaining terms are reflected waves. Therefore, $z_2^p(-m,-v_p t')$ for m=0,1,2,..., can be interpreted as a false reflected wave which is created by the definition of the p semi-infinite line problems. By looking at Eq. 4.4.10, it can be seen that the incident wave which is given by $z_2^p(m,-v_p t')$ results from the reflection of $z_2^p(-m,-v_p t')$ at x=0. The function $z_1^p(-m,-v_p t')$ was introduced in the process of transforming the semi-infinite line to the infinite line situation.

The main objective of this chapter is to find a set of launching numbers, $b_{j,1}(\Delta x,\Delta t)$, j=1,2,...,2k-1, such that

$$z^p(m,0) \doteq w^p(m,0) \qquad 4.5.3$$

for all p and m=0,1,2,...,2k. Since both these functions represent the solutions to problems of type 3, this relationship is satisfied at m=0 and m=2k independent of the launching numbers.

If Eq. 4.5.3 is satisfied, then the two semi-infinite line situations which have been discussed must also be satisfied. Therefore, it can be concluded that the relationship

$$M^p(m\Delta x/2) \doteq z_1^p(m,0) + z_2^p(m,0) + (-1)^{m+1}\left[z_1^p(-m,0) + z_2^p(-m,0)\right] \qquad 4.5.4$$

for m=1,2,...,2k-1, and p=0,1,2,..., must be satisfied. Conversely, if this relationship is satisfied, then Eq. 4.5.3 must also be satisfied for the finite line case.

In a later section, this relationship, Eq. 4.5.4, will be used to obtain the launching numbers.

4.6 Fourier Integral Comparison.

The object of this section is to find a set of launching numbers such that Eq. 4.5.4 is satisfied. Since $M^p(x)$ is generally discontinuous at x=0, then it is quite clear that $N^p(\Delta x,x)$ is the approximate Fourier integral of a discontinuous function. Therefore, in the vicinity of x=0, $N^p(\Delta x,x)$ could be a very poor approximation to $M^p(x)$ and probably the assumption that

$$N^p(\Delta x,x) \doteq M^p(x)$$

is invalid in the vicinity of this discontinuity. It will be shown in this section that if this comparison is correctly made that it is possible to avoid this problem.

Normally, the transmission line is found to be a two port component in a larger system. For this system, only the terminal variables, E(0,t), I(0,t), E(L,t), and I(L,t), are generally of interest. For example, when $E(0,t)$ is specified in some manner, the approximate numerical solution for $I(0,t)$ is generally required. At present there is no information regarding this variable. In Chapter 6, it will be shown that this variable is obtained by interpolation from the values of current which are known at the defined current node points located near $x=0$. Therefore, it is necessary that in deriving these launching numbers that $I(x,t)$ must be determined very accurately at the defined current node points which are used for interpolation purposes. In addition, the launching numbers must be chosen such that the terminal variables at $x=L$ can be determined quite accurately. As long as these terminal characteristics are determined accurately, the accuracy is not as important at the interior voltage nodes.

Before we can proceed with the comparison which is indicated in Eq. 4.5.4, it is necessary to consider the function $y(x)$. This function is defined as

$$y(x) = N^p(\Delta x, x+v_p\Delta t) + N^p(\Delta x, -x+v_p\Delta t). \qquad 4.6.0$$

If Eq. 4.2.7 is substituted into this equation, $y(x)$ can be expressed now as

$$y(x) = \frac{v_p\Delta t}{\Delta x}\sum_{n=0}^{p-1} M^p((n+1)v_p\Delta t)\ \frac{\sin \pi(nv_p\Delta t-x)/\Delta x}{\pi(nv_p\Delta t-x)/\Delta x} +$$

$$\frac{v_p\Delta t}{\Delta x}\sum_{n=0}^{p-1} M^p((n+1)v_p\Delta t)\ \frac{\sin \pi(nv_p\Delta t+x)/\Delta x}{\pi(nv_p\Delta t+x)/\Delta x}\ .$$

By defining $\bar{M}^p(x)$ as

$$\begin{aligned} \bar{M}^p(x) &= M^p(v_p\Delta t+x) && 0 \leq x < \infty && 4.6.1 \\ &= M^p(v_p\Delta t-x) && -\infty < x \leq 0 \end{aligned}$$

and $\bar{N}^p(\ x,x)$ as the approximate Fourier transform of $\bar{M}^p(x)$, $y(x)$ can now be expressed as

$$y(x) = \bar{N}^p(\Delta x,x) + \frac{v_p\Delta t}{\Delta x}\ M^p(v_p\Delta t)\ \frac{\sin \pi x/\Delta x}{\pi x/\Delta x}\ . \qquad 4.6.2$$

The function $\bar{M}^p(x)$ does not have a discontinuity at $x=0$.

Before we proceed to derive the launching numbers, let us introduce the quantities $\bar{b}_{j,1}$ which are defined as

$$\bar{b}_{j,1} = \frac{\Delta x}{v_p\Delta t}\ b_{j,1}(\Delta x,\Delta t). \qquad 4.6.3$$

These quantities are called transformation constants. Therefore, instead of deriving the launching numbers $b_{j,1}(\Delta x,\Delta t)$, the quantities $\bar{b}_{j,1}$ will be derived. It will become evident that the discussion is greatly simplified as a result of the introduction of these transformation constants.

Let us now assume that

$$\bar{b}_{2j,1} = 0$$

for j=0,1,2,...,k. By substituting Eqs. 4.4.9 and 4.4.10 into Eq. 4.5.4, then for m=1,2,3,...,k, it can be shown that this relationship reduces to

$$M^p((2m-1)\Delta x/2) = \sum_{n=1}^{k} \frac{\bar{b}_{2n-1,1}}{2} y((m-n)\Delta x) + \sum_{n=1}^{k} \frac{\bar{b}_{2n-1,1}}{2} y((m+n-1)\Delta x) \quad . \qquad 4.6.4$$

In effect, Eq. 4.5.4 is being studied at the defined current nodes.

If the increments Δx and Δt are chosen sufficiently small, then the approximation

$$\bar{N}^p(\Delta x, x) \doteq \bar{M}^p(x)$$

is a valid approximation. In addition, if

$$v_p \Delta t/\Delta x << 1$$

then the approximation

$$y(x) \doteq \bar{M}^p(x)$$

is also valid. Therefore, for m=1,2,3,...,k, the relationship as given by Eq. 4.6.4 reduces to

$$M^p((2m-1)\Delta x/2) \doteq \sum_{n=1}^{k} \frac{\bar{b}_{2n-1,1}}{2} \bar{M}^p((m-n)\Delta x) + \sum_{n=1}^{k} \frac{\bar{b}_{2n-1,1}}{2} \bar{M}^p((m+n-1)\Delta x). \qquad 4.6.5$$

For discussion purposes, let us further assume that

$$\bar{b}_{2j-1,1} = 0$$

for $j > j'$ where j' is a positive integer such that

$$1 \le j' \le k.$$

For this assumption, let us first look at the relation which is given by Eq. 4.6.5 for $m \ge j'$. Since the arguments of $\bar{M}^p(x)$ are always positive, the function $\bar{M}^p(x)$ can now be replaced by $M^p(x+v_p\Delta t)$. By introducing the change of variable

$$x' = (2m-1)\Delta x/2 + v_p\Delta t,$$

this relation now reduces to

$$M^p(x'-v_p\Delta t) \doteq \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(x'-(2n-1)\Delta x/2) + \qquad 4.6.6$$

$$\sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(x'+(2n-1)\Delta x/2).$$

This relationship states that the transformation constants are certain constants of an interpolation formula which gives $M^p(x'-v_p\Delta t)$ in terms of $M^p(x'+(2n-1)\Delta x/2)$ and $M^p(x'-(2n-1)\Delta x/2)$ for $n=1,2,\ldots,j'$.

Let $P^{j'}(z)$ be an interpolation polynomial of degree $(2j'-1)$ which passes through the following points

$$P^{j'}(z)\Big|_{z=(2n-1)\Delta x/2} = M^p(x'+z)\Big|_{z=(2n-1)\Delta x/2}$$

$$P^{j'}(z)\Big|_{z=-(2n-1)\Delta x/2} = M^p(x'+z)\Big|_{z=-(2n-1)\Delta x/2}$$

where $n=1,2,3,\ldots,j'$. This polynomial can be written by means of a Bessels' central difference interpolation formula as

$$P^{j'}(z) = \mu y_0 + v\delta y_0 + \frac{(v^2-1/4)}{2!}\mu\delta^2 y_0 + \cdots + \cdot\ \cdot$$

where

$$v = -z/\Delta x$$

$$\delta^{2p+1}y_0 = \sum_{n=0}^{2p+1} -1^n \binom{2p+1}{n} M^p(x'-(2p+1)\Delta x/2+n\Delta x)$$

$$\mu\delta^{2p}y_0 = \sum_{n=0}^{2p} -1^n \binom{2p}{n} M^p(x'+\Delta x/2-p\Delta x+n\Delta x)/2 + \sum_{n=0}^{2p} -1^n \binom{2p}{n} M^p(x'-\Delta x/2-p\Delta x+n\Delta x)/2 \ .$$

This polynomial is an interpolating polynomial of degree (2j'-1) which has the property that

$$M^p(x'+z) \doteq P^{j'}(z).$$

In addition, the coefficients are in terms of $M^p(x'+n\Delta x/2)$ for $n=\pm1,\pm2,\pm5,...,\pm(2j'-1)$.

Since it has already been assumed that $v_p\Delta t << \Delta x$, the assumption can be made that

$$M^p(x'-v_p\Delta t) \doteq M^p(x').$$

Therefore, the transformation constants can be obtained by replacing $M^p(x'-v_p\Delta t)$ with $P^{j'}(0)$ in Eq. 4.6.6 and then equating coefficients. Therefore, these constants can be found by means of the expression

$$P^{j'}(0) = \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(x'-(2n-1)\Delta x/2) + \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(x'+(2n-1)\Delta x/2) \qquad 4.6.7$$

where

$$P^{j'}(0) = \sum_{n=1}^{j'} \frac{\prod_{i=1}^{n} (2i-3)^2}{(-4)^{n-1}(2(n-1))!} \mu\delta^{2n-1}y_0 \ . \qquad 4.6.8$$

Some examples which use the results of this derivation are as follows.

<u>Example 1</u>: $j' = 1$

$$\bar{b}_{1,1}\left[M^p(x'+\Delta x/2) + M^p(x'-\Delta x/2)\right]/2 = P^1(0)$$

where

$$P^1(0) = \left[M^p(x'+\Delta x/2) + M^p(x'-\Delta x/2)\right]/2$$

Therefore,

$$\bar{b}_{1,1} = 1$$

and

$$b_{1,1}(\Delta x, \Delta t) = v_p\Delta t/\Delta x.$$

<u>Example 2</u>: $j'=2$

$$\bar{b}_{1,1}\left[M^p(x'+\Delta x/2) + M^p(x'-\Delta x/2)\right]/2 +$$

$$\bar{b}_{3,1}\left[M^p(x'+3\Delta x/2) + M^p(x'-3\Delta x/2)\right]/2 = P^2(0)$$

where

$$P^2(0) = \frac{5\left[M^p(x'+\Delta x/2) + M^p(x'-\Delta x/2)\right]}{8} - \frac{M^p(x'+3\Delta x/2) + M^p(x'-3\Delta x/2)}{8} .$$

Therefore,

$$\bar{b}_{1,1} = (5/4)$$

$$\bar{b}_{3,1} = -(1/4)$$

and

$$b_{1,1}(\Delta x, \Delta t) = \frac{5v_p\Delta t}{4\ \Delta x}$$

$$b_{3,1}(\Delta x,\Delta t) = -\frac{v_p\Delta t}{4\Delta x} .$$

If the transformation constants are determined by Eqs. 4.6.7 and 4.6.8, then Eq. 4.6.5 is satisfied for $m=j', j'+1, j'+2, \ldots, k$. Let us now look at this relationship, Eq. 4.6.5, for $m=1,2,\ldots,j'-1$, when the transformation constants have been determined by Eqs. 4.6.7 and 4.6.8. It can be shown that the relationship now becomes

$$M^p((2m-1)\Delta x/2) \doteq \sum_{n=1}^{m} \frac{\bar{b}_{2n-1,1}}{2} M^p((m-n)\Delta x+v_p\Delta t) + \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p((m+n-1)\Delta x+v_p\Delta t) + \sum_{n=m+1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p((n-m)\Delta x+v_p\Delta t) . \qquad 4.6.9$$

Since the transformation constants have already been determined, this relationship can no longer be interpreted as an interpolation formula as was the case of Eq. 4.6.6. But interpreting this formula in a different manner, it can be shown that Eqs. 4.6.7 and 4.6.8 are still valid for these constants.

In order to show that these transformation constants also give the desired results at $x=(2m-1)\Delta x/2$ where $m=1,2,\ldots,j'$, it is necessary to briefly discuss the subject of linear filters (6). Such linear filters are used primarily in data smoothing applications. For discussion purposes, let us suppose that we have a time-varying function $h(t)$ and decide to smooth this function. By introducing a linear filter, the smoothed function $\bar{h}(t)$ is given as

$$\bar{h}(t) = \sum_{j=-p'}^{p'} d_j \cdot h(t+j\Delta x)$$

where

$$\sum_{j=-p'}^{p'} d_j = 1. \qquad 4.6.10$$

It can be seen that this function, $\bar{h}(t)$, is a weighted average of the functional values $h(t+j\Delta x)$ for $j=0,\pm 1,\pm 2,\ldots,\pm p'$.

It can be seen that the transformation constants, determined by Eqs. 4.6.7 and 4.6.8, are independent of the functional values of $M^p(x)$. If we look at these equations for the case when $M^p(x)$ is a constant, it can be concluded that these transformation constants must also have the property that

$$\sum_{j=1}^{j'} \bar{b}_{2j-1,1} = 1. \qquad 4.6.11$$

Therefore, the requirement which is given by Eq. 4.6.10 is satisfied. It can be concluded that for $m=1,2,\ldots,j'-1$, the approximation, given by Eq. 4.6.4, is satisfied.

So far in this discussion it has been shown that if

$$\bar{b}_{2j,1} = 0 \qquad j=1,2,\ldots,k-1$$

$$\bar{b}_{2j-1,1} = 0 \qquad j > j'$$

$$(v_p \Delta t)/\Delta x << 1$$

and if the remaining transformation constants are determined by eqs. 4.6.7 and 4.6.8, then Eq. 4.5.4 is satisfied whenever $m=1,3,5,\ldots,2k-1$. In other words, Eq. 4.5.4 is satisfied at the current nodes. Let us keep the transformation constants defined in the same way and let us investigate Eq. 4.5.4 at the voltage node points.

The functions $z_1^p(m,-v_p t')$ and $z_2^p(m,-v_p t')$ are both linear combinations of discontinuous functions. But at the current nodes, these discontinuous functions can be combined to produce continuous functions. At the voltage node points, this is no longer the case. Even though this creates some minor problems, the relationship, given by Eq. 4.5.4, can still be investigated.

For the time being, let us assume that the assumption

$$M^p(x) \doteq N^p(\Delta x, x)$$

is valid for all x. For this assumption, it can be shown that

$$z_1^p(-m,0) \doteq 0 \qquad m=2,3,\ldots,2k-2$$

$$z_2^p(-m,0) \doteq 0. \qquad m=2j',2j'+2,\ldots,2k-2$$

Therefore, by introducing the change of variable

$$\bar{x} = m\Delta x + v_p \Delta t,$$

Eq. 4.5.4 reduces to

$$M^p(\bar{x}-v_p\Delta t) \doteq \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(\bar{x}-(2n-1)\Delta x/2) + \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(\bar{x}+(2n-1)\Delta x/2).$$

for m=2',2j'+2,...,2k-2. If we compare this result with Eq. 4.6.6, it can be seen that these equations are the same equations except that the variable x' has been replaced by $\bar{x}$. Since the transformation constants satisfy Eq. 4.6.6, they must also satisfy this result. For m=2j',2j'+2,...,2k-2, the discontinuity presents no problem since we are not evaluating

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$N^p(\Delta x, x)$ in the vicinity of the discontinuity. It can now be concluded that these transformation constants satisfy Eq. 4.5.4 at the voltage nodes which are located at $x=n\Delta x$, where $n=j', j'+1, \ldots, k-1$. In fact if $j'=1$, this equation is satisfied at all the voltage node points.

If we investigate Eq. 4.5.4 at the voltage nodes which are located at $x=n\Delta x$, where $n=1,2,\ldots j'-1$, it can be seen that $z_2^p(-2n,0)$ is no longer zero. In fact the sign which appears in front of $z_2^p(-2n,0)$ is now a negative sign where in the current case, the sign was positive. It can then be concluded that Eq. 4.5.4 is not satisfied at these nodes. For the assumption that

$$N^p(x) \doteq N^p(\Delta x, x),$$

let us investigate this error. At $x=n\Delta x$, $z_2^p(-2n,0)$ can be written as

$$z_2^p(-2n,0) = \sum_{j=n+1}^{k} \frac{\bar{b}_{2j-1,1}}{2} M^p((j-n)\Delta x - \Delta x/2 + v_p \Delta t).$$

If $g(2n,t')$ and $M^p(n\Delta x)$ are respectively the actual solution and the desired solution at $x=n\Delta x$, where $n=1,2,\ldots,k-1$, then it can be shown that

$$g(2n,0) \doteq M^p(n\Delta x) - 2\, z_2^p(-2n,0).$$

By means of linear filter theory, it can be shown that

$$g(2n,0) \doteq M^p(n\Delta x)\left[1-\sum_{j=m+1}^{k}\bar{b}_{2j-1,1}\right] \qquad 4.5.12$$

where $M^p(n\Delta x)$ is the desired solution. This says the actual solution is the true solution multiplied by a constant.

So far in this section, the discussion has been based on the assumption that

$$\bar{b}_{2j,1} = 0 \qquad j=1,2,\ldots,k-1$$

For discussion purposes, let us assume that

$$\bar{b}_{2j-1,1} = 0 \qquad j=j'+1,j'+2,\ldots,k$$

$$\bar{b}_{2j,1} = 0 \qquad j=n'+1,n'+2,\ldots,k-1$$

$$M^p(x) \doteq N^p(\Delta x,x)$$

and that the remaining constants have not been determined. Let us look at Eq. 4.5.4 for the case where m is greater than both j' and n'. For this choice of m,

$$z_1^p(-m,0) \doteq 0$$

$$z_2^p(-m,0) \doteq 0$$

Upon making the change of variable

$$x' = m\Delta x/2 + v_p \Delta t,$$

Eq. 4.5.4 now reduces to

$$\begin{aligned} M^p(x'-v_p\Delta t) \doteq & \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(x'-(2n-1)\Delta x/2) \quad + \\ & \sum_{n=1}^{j'} \frac{\bar{b}_{2n-1,1}}{2} M^p(x'+(2n-1)\Delta x/2) \quad + \\ & \sum_{n=1}^{n'} \frac{\bar{b}_{2n,1}}{2} M^p(x'-n\Delta x) \quad - \\ & \sum_{n=1}^{n'} \frac{\bar{b}_{2n,1}}{2} M^p(x'+n\Delta x). \end{aligned}$$

The method for obtaining these constants will be to consider separately the terms which contain the constants $\bar{b}_{2j-1,1}$.

Let us first look at the terms which contain the constants $\bar{b}_{2j,1}$. If we look at these terms from the viewpoint of linear filter theory, the values of these terms should be approximately zero independent of the value of the constants. An alternate way of looking at these terms is to define a polynomial $P_{n'}(z)$ of degree 2n' which passes through the points

$$P_{n'}(z)\Big|_{z=n\Delta x} = n\Delta x \; M^p(x'+n\Delta x)$$

where $n=0,\pm 1,\pm 2,\ldots,\pm n'$. Let us look at the function $P_{n'}(z)/z$. It is quite clear that this function passes through the following points.

$$P_{n'}(z)/z \Big|_{z=n\Delta x} = M^p(x'+n\Delta x) \qquad n=1,2,\ldots,n'$$

$$P_{n'}(z)/z \Big|_{z=-n\Delta x} = M^p(x'-n\Delta x) \qquad n=1,2,\ldots,n'$$

Since the limit of this function as $z \to 0$ is defined, these constants can be obtained by equating the limit as $z \to 0$ of this interpolation polynomial to the terms which contain the constants $\bar{b}_{2j,1}$. This function has the property

$$\lim_{z \to 0} P_{n'}(z)/z = \lim_{z \to 0} \left[P_{n'}(z) - P_{n'}(-z)\right]/2z$$

where

$$\left[P_{n'}(z)-P_{n'}(-z)\right]/2z = M^p(x'+z) - M^p(x'-z).$$

Therefore, the limit as $z \to 0$ must be approximately zero. Hence, the same conclusion is obtained as was the conclusion obtained by looking at these terms from the filter theory point of view.

Since the sum of the terms which contain the $\bar{b}_{2j,1}$'s is approximately zero, the terms which contain the constants $\bar{b}_{2j-1,1}$ must be defined in the same manner as was the case when the constants, $\bar{b}_{2j,1}$, were zero.

If we look at Eq. 4.5.4 for m less than both j' and n', we have the same terms as we had in the case where $\bar{b}_{2j,1}$ but with some addition terms added since the constants $\bar{b}_{2j,1}$ are no longer zero. Let the function $h((2j-1)\Delta x/2)$, $2j-1 \leq m$, be the sum of these addition terms at the current nodes. Hence,

$$h((2j-1)\Delta x/2) \doteq \sum_{n=1}^{n'} \frac{\bar{b}_{2n,1}}{2} M^p((2j-1+2n)\Delta x/2) + \sum_{n=1}^{j-1} \frac{\bar{b}_{2n,1}}{2} M^p((2j-1-2n)\Delta x/2) - \sum_{n=1}^{j-1} \frac{\bar{b}_{2n,1}}{2} M^p((2n-2j+1)\Delta x/2).$$

This result states that Eq. 4.5.4 can only be satisfied at these current nodes when the $\bar{b}_{2j,1}$'s are identically zero. Therefore, these constants must be zero in order to correctly determine I(0,t).

4.7 Fourier Series Comparison.

In section 4.2, an approximate Fourier series solution for $E^p(x,t')$ and $I^p(x,t')$, given by Eqs. 4.2.13, was derived. In section 4.4, some special forms for $z^p(m,t')$ were derived. The object of this section is to compare the approximate Fourier series solution as given by Eq. 4.2.13 to

$z^p(m,t')$ which is given by Eq. 4.4.15. Once this comparison is performed, a set of launching numbers can be obtained.

First, let us consider the functions $E_3^p(x,t')$ and $I_3^p(x,t')$ which are given as

$$E_3^p(x,t') = y_1^p(2x/\Delta x,-v_pt') + y_2^p(2x/\Delta x,-v_pt') - y_1^p(-2x/\Delta x,-v_pt') - y_2^p(-2x/\Delta x,-v_pt') \qquad 4.7.0$$

$$I_3^p(x,t') = y_1^p(2x/\Delta x,-v_pt') + y_2^p(2x/\Delta x,-v_pt') + y_1^p(-2x/\Delta x,-v_pt') + y_2^p(-2x/\Delta x,-v_pt')$$

where $y_1^p(2x/\Delta x,-v_pt')$ and $y_2^p(2x/\Delta x,-v_pt')$ are respectively given by Eqs. 4.4.12 and 4.4.13. It is quite clear that

$$E_3^p(m\Delta x/2,t') = z^p(m,t') \qquad m=0,2,4,6,\ldots,2k$$

$$I_3^p(m\Delta x/2,t') = z^p(m,t') \qquad m=1,3,5,\ldots,2k-1$$

is true.

By substituting Eqs. 4.2.12, 4.4.12 and 4.4.13 into Eqs. 4.7.0 and letting $t'=0$, $E_3^p(x,t')$ and $I_3^p(x,t')$ can now be written as

$$E_3^p(x,0) = \sum_{n=1}^{k-1} \Big[B_n\ a_n^p \sin(n\pi v_p \Delta t/L) \sin(n\pi x/L) + \qquad 4.7.1$$

$$B_n\ b_n^p \cos(n\pi v_p \Delta t/L) \sin(n\pi x/L) +$$

$$A_n\ b_n^p \cos(n\pi v_p \Delta t/L) \sin(n\pi x/L) -$$

$$A_n\ a_n^p \sin(n\pi v_p \Delta t/L) \sin(n\pi x/L) \Big] +$$

$$(A_k/2)\ b_k^p \cos(k\pi v_p \Delta t/L) \sin(k\pi x/L) -$$

$$(A_k/2)\ a_k^p \sin(k\pi v_p \Delta t/L) \sin(k\pi x/L)$$

$$I_3^p(x,0) = (A_0\ a_0^p)/2 + \sum_{n=1}^{k-1} \Big[A_n\ a_n^p \cos(n\pi v_p \Delta t/L) \cos(n\pi x/L) +$$

$$A_n\ b_n^p \sin(n\pi v_p \Delta t/L) \cos(n\pi x/L) -$$

$$B_n\ b_n^p \cos(n\pi v_p \Delta t/L) \cos(n\pi x/L) +$$

$$B_n\ a_n^p \sin(n\pi v_p \Delta t/L) \cos(n\pi x/L) \Big] +$$

$$(A_k/2)\ a_k^p \cos(k\pi v_p \Delta t/L) \cos(k\pi x/L) +$$

$$(A_k/2)\ b_k^p \sin(k\pi v_p \Delta t/L) \cos(k\pi x/L)$$

where

$$B_n = \sum_{j=1}^{k-1} \bar{b}_{2j,1} \sin(nj\pi \Delta x/L) \qquad 4.7.2$$

$$A_n = \sum_{j=1}^{k} \bar{b}_{2j-1,1} \cos(n\pi(2j-1)\Delta x/2L).$$

Since the kth term is zero at the defined node points, these terms can be disregarded in the comparison.

The transformation constants would be best chosen if

$$E^p(x,0) = E_3^p(x,0)$$

$$I^p(x,0) = I_3^p(x,0)$$

at the defined node points. This requirement is satisfied only when the following relationship is satisfied for $n=0,1,2,\ldots,k-1$.

$$\bar{b}_n^p + \sum_{j=1}^{\infty}\left[-\bar{b}^p_{2kj-n} + \bar{b}_{2kj+n}\right] = \tag{4.7.3}$$

$$B_n\left[a_n^p \sin(n\pi v_p \Delta t/L) + b_n^p \cos(n\pi v_p \Delta t/L)\right] +$$

$$A_n\left[b_n^p \cos(n\pi v_p \Delta t/L) - a_n^p \sin(n\pi v_p \Delta t/L)\right]$$

$$\bar{a}_n^p + \sum_{j=1}^{\infty}\left[a^p_{2kj-n} + a^p_{2kj+n}\right] =$$

$$A_n\left[a_n^p \cos(n\pi v_p \Delta t/L) + b_n^p \sin(n\pi v_p \Delta t/L)\right] -$$

$$B_n\left[b_n^p \cos(n\pi v_p \Delta t/L) - a_n^p \sin(n\pi v_p \Delta t/L)\right]$$

For a given Δx and Δt, there are no constants A_n and B_n such that this relationship is satisfied for all p. Hence, this relationship as it stands does not provide a means for obtaining the transformation constants.

An alternate approach is to use the approximate Fourier series expressions for $E^p(x,t')$ and $I^p(x,t')$ which are given by Eqs. 4.2.13. If we truncate these expressions after the (k-1)th term, then it is possible to obtain an approximate Fourier series expression for $E^p(x,t')$ and $I^p(x,t')$ which contains the same number of terms as does $E_3^p(x,t')$ and $I_3^p(x,t')$. Since $E^p(x,t')$ and $I^p(x,t')$ for $t'=0$ are generally discontinuous functions at $x=0$, these truncated series may tend to oscillate about the actual solution. In order to correct this situation, let us introduce the Lanczos' σ factors which are discussed in section 4.2.

By performing a term by term comparison of the respective series, it is possible to obtain the following relationship between the coefficients.

$$\begin{aligned} \sigma_n b_n^p &= B_n \left[a_n^p \sin(n\pi v_p \Delta t/L) + b_n^p \sin(n\pi v_p \Delta t/L)\right] + \\ &\quad A_n \left[b_n^p \cos(n\pi v_p \Delta t/L) - a_n^p \sin(n\pi v_p \Delta t/L)\right] \\ \sigma_n a_n^p &= A_n \left[a_n^p \cos(n\pi v_p \Delta t/L) + b_n^p \sin(n\pi v_p \Delta t/L)\right] + \\ &\quad B_n \left[b_n^p \cos(n\pi v_p \Delta t/L) - a_n^p \sin(n\pi v_p \Delta t/L)\right] \end{aligned} \qquad 4.7.4$$

Upon performing the indicated mathematics, the quantities B_n and A_n are given as

$$B_n = \sigma_n \sin(n\pi v_p \Delta t/L) \qquad n=1,2,\ldots,k-1 \qquad 4.7.5$$

$$A_n = \sigma_n \cos(n\pi v_p \Delta t/L) \qquad n=0,1,2,\ldots,k-1$$

By substituting the defining equations for B_n and A_n, Eqs. 4.7.2, into these relationships, it is possible to solve for the transformation constants $\bar{b}_{j,1}$. Once these transformation constants have been determined, the launching numbers, $b_{j,1}(\Delta x, \Delta t)$, can be determined by Eq. 4.6.3.

Let us investigate Eqs. 4.7.5. If the assumption that

$$v_p \Delta t/L << 1$$

is made, then the assumptions that

$$\sin(n\pi v_p \Delta t/L) \doteq 0 \qquad n=1,2,\ldots,k-1$$

$$\cos(n\pi v_p \Delta t/L) \doteq 1 \qquad n=0,1,2,\ldots,k-1$$

are valid, Therefore,

$$B_n \doteq 0 \qquad n=1,2,\ldots,k-1$$

$$A_n = \sigma n. \qquad n=0,1,2,\ldots,k-1$$

According to Eqs. 4.7.2, the quantity B_n is a linear combination of the transformation constants $\bar{b}_{2j,1}$. Upon solving for $\bar{b}_{2j,1}$, we find that

$$\bar{b}_{2j,1} = 0. \qquad j=1,2,\ldots,k-1$$

This is the same result as was obtained in section 4.6.

In order to find the remaining transformation constants, it is necessary to take the inverse of a (k)x(k) matrix for which the quantity A_n corresponds to a row of this matrix. By investigating this matrix, it can be seen that these constants depend on Δx or the size of the matrix. For this reason, the transformation constants or launching numbers as determined by section 4.6 are more useful.

4.8 Problem of Type 2.

A problem of type 2 is a transmission line problem which has zero initial conditions and one identically zero boundary condition at x=0. One such problem of type 2 has the initial and boundary conditions specified as

$$E_2(x,0) = I_2(x,0) = 0 \qquad 0 \leq x < L \qquad 4.8.0$$

$$E_2(0,t) = 0 \qquad 0 \leq t < \infty$$

$$-I_2(L,0) = E_{s2}(0)$$

$$E_2(L,t) = E_{s2}(t) \qquad 0 \leq t < \infty$$

where $E_{s2}(t)$ is a specified function of t. The current orientations for this problem are from x=0 to x=L. Because of the similarity existing between this problem and a problem of type 1, it is possible to use methods which have been derived to obtain a set of launching numbers which are valid for transforming the boundary condition for the interval $n\Delta t \leq t < (n+1)\Delta t$ to an initial condition.

By letting

$$E_1(L-x,t) = E_2(x,t)$$

$$I_1(L-x,t) = -I_2(x,t)$$

this problem of type 2 has been transformed to a problem of type 1 for which the solutions are $E_1(x,t)$ and $I_1(x,t)$. Therefore, if $b_{j,2}(\Delta x,\Delta t)$, $j=1,2,...,2k-1$, are the launching numbers for the problem of type 2 which is described by Eqs. 4.8.0, these launching numbers are given by

$$b_{j,2}(\Delta x,\Delta t) = (-1)^j \; b_{2k-j,1}(\Delta x,\Delta t) \qquad 4.8.1$$

where $j=1,2,...,2k-1$.

4.9 Conclusion.

Two classes of methods are derived which transform the boundary condition, $E(0,t)$, for a problem of type 1 for the interval $t_0 \le t < t_0+\Delta t$ into an initial condition at $t=t_0+\Delta t$. Each method derives a set of launching numbers, one for each node, such that the initial condition at each node point at $t=t_0+\Delta t$ is the product of $E(0,n\Delta t)$ and a launching number. This class of methods is also valid for the problem of type 2.

Of the two classes of methods which are given in sections 4.6 and 4.7, the general method as given by section 4.6 is the preferred method. There are two basic reasons for preferring this general method. First of all, any launching number can be placed in terms of a transformation constant by Eq. 4.6.3. For any given method from section 4.6, these transformation constants are independent of Δx where as this is not the case in section 4.7. Secondly, the sum of the solutions of the problems of type 3 is compared to the analytical solution whereas in section 4.7, this comparison is made to an approximate Fourier series solution.

V. REMAINING CASES

5.0 Introduction.

In Chapters 3 and 4, the derivations are performed with the assumption that both boundary conditions are specified voltages. The main object of this chapter is to demonstrate that if the functions g(j,t) and f(j,t) are defined in a slightly different manner, the derivations are equally valid for cases for which the boundary conditions are either both specified currents or a specified voltage and a specified current.

Let us consider the case where both boundary conditions are specified currents. In this situation, $I_{s1}(t)$ and $I_{s2}(t)$ are now specified boundary conditions instead of $E_{s1}(t)$ and $E_{s2}(t)$. Let $\Delta x=L/k$, where k is a positive integer equal to or greater than 1. In addition, let the functions g(2n,t) for n=0,1,2,...,k, and g(2n+1,t) for n=0,1,2,...k-1, be respectively the approximate numerical solutions for $I(n\Delta x,t)$ and $E(n\Delta x+\Delta x/2,t)$. For the function g(j,t) defined in this manner, this case is simply the dual of the case already presented. That is, the derivations for this case are identical to the results derived in Chapters 3 and 4 except that all voltages have been replaced by currents and all currents have been replaced by voltages. Therefore, the results are valid for the case where both boundary conditions are specified currents.

The remaining portion of this chapter specifically treats the case where one boundary condition is a specified voltage and the other is a specified current.

5.1 Chapter 3 Validity.

This section will specifically show that for the case where the boundary conditions $E(0,t)$ and $I(L,t)$ are specified to be identically zero, the results derived in Chapter 3 are still valid if the function $g(j,t)$ is defined in a slightly different manner. This case corresponds to a lossless parallel transmission line for which a short is located at $x=0$ and an open is located at $x=L$.

An infinite line for which the initial conditions are

$$
\begin{aligned}
V(x+4nL,0) &= E(x,0) \qquad\qquad 5.1.0\\
V(-x+4nL,0) &= -E(x,0)\\
V(-x+2nL,0) &= E(x,0)\\
V(x+2nL,0) &= -E(x,0)\\
S(x+4nL,0) &= I(x,0)\\
S(-x+4nL,0) &= I(x,0)\\
S(-x+2nL,0) &= -I(x,0)\\
S(x+2nL,0) &= -I(x,0)
\end{aligned}
$$

where $0 \le x \le L$ and $n=0,\pm1,\pm2,\pm3,\ldots,$ has the same solution for $0 \le x \le L$ as this problem. Hence, an approximate numerical solution can be obtained for this problem by solving the infinite line problem for the initial conditions given by Eqs. 5.1.0.

Both the functions V(x,0) and S(x,0) have special properties as was the case in Chapter 3. First of all, both functions are periodic in x with period 4L. Secondly, V(x,0) and S(x,0) are respectively an odd and an even function of x with respect to x=0. In addition to these two properties, V(x,0) and S(x,0) are respectively an even and an odd function of x with respect to x=L.

Let the increment Δx be given as $\Delta x = 2L/k'$, where k' is a positive odd integer. Because of these special properties, the functions f(n,t) have some very special relationships. By means of Eqs. 2.7.0 and 2.7.1, one of these special relationships can be written as

$$\delta^m f(-2i,t) = -\delta^m f(2i,t) \qquad 5.1.1$$

$$\delta^m f(2i+1,t) = \delta^m f(-2i-1,t)$$

$$\delta^m f(2i+k',t) = -\delta^m f(-2i+k',t)$$

$$\delta^m f(2i+1+k',t) = \delta^m f(-2i-1+k',t)$$

where $i=0,\pm1,\pm2,\ldots,$ and $m=0,1,2,\ldots,$. Eqs. 5.1.1 in effect state that Eqs. 5.1.2 are also valid.

$$\delta^m f(0,t) = 0 \qquad m=0,1,2,\ldots, \qquad 5.1.2$$

$$\delta^m f(k',t) = 0$$

Let us now define the functions $g(2n,t)$, $n=0,1,2,\ldots,(k'-1)/2$, and $g(2n+1,t)$, $n=0,1,2,\ldots,(k'-1)/2$, respectively as the approximate numerical solutions for $E(n\Delta x,t)$ and $I(n\Delta x+\Delta x/2,t)$.

Since

$$\delta^m g(i,t) = \delta^m f(i,t) \qquad i=0,1,2,\ldots,k',$$

it is quite clear that if the quantity 2k is replaced by k', the proof for C^m and its properties as given in Chapter 3 are also valid for this case. Hence, the result is restated as follows:

$$C^m \begin{bmatrix} g(1,t) \\ g(2,t) \\ g(3,t) \\ \cdot \\ \cdot \\ g(k'-1,t) \end{bmatrix} = \begin{bmatrix} \delta^m g(1,t) \\ \delta^m g(2,t) \\ \delta^m g(3,t) \\ \cdot \\ \cdot \\ \delta^m g(k'-1,t) \end{bmatrix} \qquad 5.1.3$$

In addition, the remaining properties except possibly those given in section 3.0 are also valid.

5.2 <u>Chapter 4 Validity</u>.

This section specifically treats the case where the boundary and initial conditions are given as

$$E(x,0) = I(x,0) = 0 \qquad 0 < x \leq L \qquad 5.2.0$$

$$I(L,t) = 0 \qquad 0 \leq t < \infty$$

$$E(0,t) = E_{s1}(t) \qquad 0 \leq t < \infty$$

$$I(0,0) = E_{s1}(0).$$

This case corresponds to a lossless parallel transmission line with a voltage driver at x=0 and an open circuit at x=L. The object of this section is to show that the results of Chapter 4 are valid for this case.

Let us first consider the lossless parallel transmission line of length 2L for which the initial and boundary conditions are given as

$$E^*(x,0) = I^*(2L,0) = 0 \qquad 0 < x < 2L \qquad 5.2.1$$

$$I^*(0,0) = I^*(2L,0) = E_{s1}(0)$$

$$E^*(0,t) = E^*(2L,t) = E_{s1}(t). \qquad 0 \leq t < \infty$$

It can be easily verified that the solutions for the transmission line problem which is described by Eqs. 5.2.0 are identical to the solutions for this problem for $0 \leq x \leq L$.

Since this problem, defined by the boundary and initial conditions as given by Eqs. 5.2.1, is a linear problem, the superposition principle can be applied. Hence, it is only necessary to solve the problem of type 1, having a length of 2L, for which the boundary and initial conditions are given by Eqs. 5.2.2.

$$E_1^*(x,0) = I_1^*(x,0) = 0 \qquad 0 < x \leq 2L \qquad 5.2.2$$

$$E_1^*(2L,t) = 0 \qquad 0 \leq t < \infty$$

$$I_1^*(0,0) = E_{s1}(0)$$

$$E_1^*(0,t) = E_{s1}(t) \qquad 0 \le t < \infty$$

It is quite clear that the solutions for $E^*(x,t)$ and $I^*(x,t)$ are given as

$$E^*(x,t) = E_1^*(x,t) + E_1^*(2L-x,t) \tag{5.2.3}$$

$$I^*(x,t) = I_1^*(x,t) - I_1^*(2L-x,t).$$

The problem of type 1 for which the boundary and initial conditions are given by Eqs. 5.2.2 satisfies all the requirements of Chapter 4. If $\Delta x=2L/k'$, where k' is a positive integer, the launching numbers $b_{j,1}(\Delta x,\Delta t)$, $j=1,2,\ldots,2k'-1$, are valid for this problem.

Let $g_{n,1}^*(j,0)$, $j=0,1,2,\ldots,2k'$, be the initial conditions to a problem of type 3 which was defined at $t=(n+1)\Delta t$. Chapter 4 showed that if

$$g_{n,1}^*(j,0) = b_{j,1}(\Delta x,\Delta t)\ E_{s1}(n\Delta t) \qquad j=1,2,\ldots,2k'-1$$

$$g_{n,1}^*(0,0) = 0$$

$$g_{n,1}^*(2k',0) = 0,$$

then

$$g_1^*(j,p\Delta t) = \sum_{n=0}^{p-1} g_{n,1}^*(j,(p-n-1)\Delta t) \qquad j=1,2,\ldots,2k'$$

$$g_1^*(0,p\Delta t) = E_{s1}(p\Delta t)$$

are the approximate numerical solutions to $E_1^*(x,p\Delta t)$ and $I_1^*(x,p\Delta t)$ at the defined node points. Therefore, if $g^*(j,t)$, $j=0,1,2,\ldots,2k'$, are the approximate numerical solutions for $E^*(x,t)$ and $I^*(x,t)$ at the same node points, then the solution for $g^*(j,t)$, $j=0,1,2,\ldots,2k'$, can be written as

$$g^*(j,t) = g_1^*(j,t) + (-1)^j g_1^*(2k'-j,t). \qquad 5.2.5$$

The result, given by Eq. 5.2.5, states that the numerical solution for $E^*(x,p\Delta t)$ and $I^*(x,p\Delta t)$ at the interior node points is the sum of 2p problems of type 3. By applying the superposition principle, the two problems of type 3 which were defined at $t=(n+1)\Delta t$ can be combined to form one problem of type 3 such that if

$$b^*_{j,1}(\Delta x,\Delta t) = b_{j,1}(\Delta x,\Delta t) + (-1)^j b_{2k'-j}(\Delta x,\Delta t) \qquad 5.2.6$$

and

$$\bar{g}_n(j,0) = b^*_{j,1}(\Delta x,\Delta t)\ E_{s1}(n\Delta t) \qquad j=1,2,\ldots,2k'-1$$

$$\bar{g}_n(0,0) = 0$$

$$\bar{g}_n(2k',0) = 0$$

then

$$g^*(j,p\Delta t) = \sum_{n=0}^{p-1} \bar{g}_n(j,(p-n-1)\Delta t) \qquad j=1,2,\ldots,2k'-1$$

$$g^*(0,p\Delta t) = E_{s1}(p\Delta t)$$

$$g^*(2k',p\Delta t) = E_{s1}(p\Delta t)$$

gives the same solution for $g^*(j,t)$ as does Eq. 5.2.6.

Let us now transform the problem of type 3 for which the initial conditions are $\bar{g}_n(j,0)$ to the infinite line case. If $f_n(m,0)$, $m=0,\pm 1,\pm 2,\ldots,$ are the initial condition for this infinite line problem, when k' is odd, it can be easily shown that the solutions $f_n(m,t)$ satisfy Eqs. 5.1.1. Therefore, if

$$g_n(j,0) = b^*_{j,1}(\Delta x,\Delta t)\ E_{s1}(n\Delta t) \qquad j=1,2,\ldots,k'-1$$

$$g_n(0,0) = 0$$

$$g_n(k',0) = 0,$$

then

$$g(j,p\Delta t) = \sum_{n=0}^{p-1} g_n(j,(p-n-1)\Delta t) \qquad j=1,2,\ldots,k'$$

$$g(0,p\Delta t) = E_{s1}(p\Delta t)$$

are the numerical solutions for $E(j\Delta x/2,p\Delta t)$, $j=0,2,4,\ldots,(k'-1)$, and $I(j\Delta x/2,p\Delta t)$, $j=1,3,\ldots,k'$. In addition, if the same time domain model is used to obtain $g_n(j,(p-n-1)\Delta t)$ as is used to obtain $\bar{g}_n(j,(p-n-1)\Delta t)$, then Eq. 5.2.7 must be true.

$$g(j,p\Delta t) = g^*(j,p\Delta t) \qquad j=0,1,2,\ldots,k' \qquad 5.2.7$$

For the case where

$$b_{j,1}(\Delta x,\Delta t) = 0$$

for $j=k'+1,k'+2,...,2k'$, the launching numbers as given by Eq. 5.2.6 are the same as those given in Chapter 4.

If the duality principle is applied to this result, it can be easily seen that this result is equally valid for the case where $E(L,t)$ is specified to be identically zero and $I(0,t)$ is a specified function of t.

5.3 Conclusion.

It can be concluded that if the functions $g(j,t)$ are defined in a slightly different manner, the results of Chapters 3 and 4 are equally valid for the three cases which arise when both boundary conditions are not specified voltages.

For the case when both boundary conditions are specified voltages (currents), the increment Δx is chosen as $\Delta x=L/k$, where k is a positive integer. The functions $g(2j,t)$, $j=0,1,2,...,k$, and $g(2j+1,t)$, $j=0,1,2,...,k-1$, are defined respectively as the approximate solutions for $E(j\Delta x,t)$ $(I(j\Delta x,t))$ and $I(j\Delta x+\Delta x/2,t)$ $(E(j\Delta x+\Delta x/2,t)$.

When the boundary conditions at $x=0$ and $x=L$ are respectively a specified voltage (current) and a specified current (voltage), the increment Δx is chosen as $\Delta x=2L/k'$, where k' is a positive odd integer. The functions $g(2j,t)$,

$j=0,1,2,\ldots,(k'-1)/2$, and $g(2j+1,t)$, $j=0,1,2,\ldots,(k'-1)/2$, are defined respectively as the approximate numerical solutions for $E(j\Delta x,t)$ $(I(j\Delta x,t))$ and $I(j\Delta x+\Delta x/2,t)$ $(E(j\Delta x+\Delta x/2,t))$. In addition, the launching numbers for this case are defined by Eq. 5.2.6.

VI. INTERPOLATION FORMULAS

6.0 Introduction.

In Chapters 3 and 4, there has been no mention of I(0,t) or I(L,t). For a practical problem, the approximate numerical solution for these two variables are generally of particular interest. In addition to these two variables, a given problem might require the knowledge of the approximate numerical solution for E(x,t) or I(x,t) at some value of x other than at a node point. This chapter specifically derives methods for obtaining these results.

If the knowledge of E(x,t) or I(x,t) are required at $t=t_0$ and $x=x_0$, it is assumed that there is no information present concerning the specified boundary conditions for $t > t_0$ except in the case where a boundary condition is specified to be identically zero for all t.

There are two general methods which can be employed to find E(x,t) and I(x,t) at some value of x other than at a node point. One such method is the use of interpolation or extrapolation formulas. The other method is to numerically solve the appropriate telegrapher equation at the point of interest. The use of an interpolation or extrapolation formula is by far more convenient to apply. Hence, this chapter will only treat this method.

The most obvious approach is simply to apply any standard interpolation or extrapolation formula for which g(j,t) are known quantities. For values of x near the ends of the line, it is quite clear that this approach does not result in central difference formulas. Hence, this chapter derives some methods for which central difference formulas can by employed for every value of x for certain special cases.

6.1 At Least One Boundary Condition Specified Zero.

For the case for which at least one boundary condition is specified as zero, the solutions for problems of this type have certain special properties. These special properties can be employed to obtain central difference interpolation formulas for values of x near the end of the line for which the boundary condition is specified as zero.

Let us first consider the transmission line of length L for which the initial and boundary conditions are given as follows:

$$\begin{aligned}
&E(0,t) = E_{s1}(t) && 0 \leq t < \infty && 6.1.0\\
&E(x,0) = h_1(x) && 0 < x \leq L &&\\
&I(x,0) = h_2(x) && 0 \leq x < L &&\\
&I(L,t) = 0 && 0 \leq t < \infty &&
\end{aligned}$$

It is quite obvious that the transmission line of length 2L for which the boundary and initial conditions are given by

Eqs. 6.1.1 has the same solution for $0 \le x \le L$. Because of the

$$E^*(2L-x,0) = E^*(x,0) = E(x,0) \qquad 0<x<L \qquad 6.1.1$$
$$-I^*(2L-x,0) = I^*(x,0) = I(x,0) \qquad 0 \le x \le L$$
$$E^*(2L,t) = E^*(0,t) = E_{s1}(t) \qquad 0 \le t < \infty$$

symmetry of this equivalent problem, the result as given by Eqs. 6.1.2 must be valid. This result in effect states that

$$E(x,t) = E^*(x,t) = E^*(2L-x,t) \qquad 0 \le x \le L \qquad 6.1.2$$
$$I(x,t) = I^*(x,t) = -I^*(2L-x,t) \qquad 0 \le x \le L$$

the solution E(x,t) and I(x,t) can be considered respectively as an even and an odd function of x with respect to x=L.

Now, let us consider the transmission line of length L for which the initial and boundary conditions are given as follows:

$$E(0,t) = E_{s1}(t) \qquad 0 \le t < \infty \qquad 6.1.3$$
$$E(x,0) = h_1(x) \qquad 0<x<L$$
$$I(x,0) = h_2(x) \qquad 0 \le x \le L$$
$$E(L,t) = 0 \qquad 0 \le t < \infty$$

An equivalent problem which gives the same solution for $0 \le x \le L$ is given as follows:

$$-E^*(2L-x,0) = E^*(x,0) = E(x,0) \qquad 0<x<L \qquad 6.1.4$$
$$I^*(2L-x,0) = I^*(x,0) = I(x,0) \qquad 0 \le x \le L$$
$$-E^*(2L,t) = E(0,t) = E_{s1}(t) \qquad 0<t<\infty$$

Because of the symmetry of this problem with respect to x=L, the result as given by Eqs. 6.1.5. must be valid. For this

$$E(x,t) = E^*(x,t) = E^*(2L-x,t) \qquad 0 \le x \le L \qquad 6.1.5$$
$$I(x,t) = I^*(x,t) = I^*(2L-x,t) \qquad 0 \le x \le L$$

case, the solutions E(x,t) and I(x,t) can be considered respectively an odd and an even function of x with respect to x=L.

It can be concluded that when E(x,t) is specified to be identically zero at a boundary, then E(x,t) and I(x,t) at $t=t_0$ can be considered respectively an odd and an even function of x with respect to this boundary. Likewise, when I(x,t) is specified to be zero at a boundary, E(x,t) and I(x,t) at $t=t_0$ can be considered respectively an even and an odd function of x with respect to the boundary. These properties are independent of the boundary condition at the other end of the line.

6.2 Both Boundary Conditions Not Identically Zero.

This section derives some properties about the solution for the case for which the conditions of section 6.1 are not satisfied.

Let us consider the transmission line of length L for which the initial and boundary conditions are given as

$$E(0,t) = E_{s1}(I(0,t),t) \qquad 0 \le t < \infty \qquad 6.2.0$$
$$E(x,0) = h_1(x) \qquad 0 < x < L$$

$$I(x,0) = h_2(x) \qquad 0 \le x \le L$$

$$E(L,t) = E_{s2}(I(L,t),t) \qquad 0 \le t < \infty$$

where the boundary condition on each end of the line is allowed to be a function of the other boundary variable. It is quite clear that if either $E_{s1}(I(0,t),t)$ or $E_{s2}(I(L,t),t)$ are not identically zero, this problem does not satisfy the conditions as stated in section 6.1. If the specified boundary conditions are a function of $I(0,t)$ and $I(L,t)$, it is absolutely necessary that the numerical solution for these two variables be known. Hence, it is necessary to derive some properties about $I(x,t)$ which can be used for interpolation purposes.

The transmission line itself is linear. Hence, the superposition principle may be applied at $t=t_0$ in order to divide this problem into two problems for which the sum of the solutions of the two problems gives the solution to the original problem. The initial and boundary conditions for the two resulting problems are given as follows:

$$E_1(0,t) = E_{s1}(I(0,t),t) \qquad t_0 \le t < \infty \qquad 6.2.1$$

$$E_1(x,t_0) = 0 \qquad 0 < x < L$$

$$I_1(x,t_0) = 0 \qquad 0 \le x \le L$$

$$E_1(L,t) = 0 \qquad t_0 \le t < \infty$$

$$E_2(0,t) = 0 \qquad t_0 \le t < \infty \qquad 6.2.2$$

$$I_2(x,t_0) = I(x,t_0) \qquad 0 \le x \le L$$

$$E_2(x,t_0) = E(x,t_0) \qquad 0 < x < L$$

$$E_2(L,t) = E_{s2}(I(L,t),t) \qquad t_0 \le t < \infty$$

The initial condition $I_2(x,t_0)$ for $0 \le x \le L$ contains all the information about $I(x,t_0)$. Since the problem for which the initial and boundary conditions are given by Eqs. 6.2.2 satisfies the conditions of section 6.1, the function $I(x,t_0)$ can be considered an even function of x with respect to x=0. A similar argument can be used to show that $I(x,t_0)$ can be considered an even function of x with respect to x=L as far as interpolation is concerned.

Let us now consider the case where I(0,t) and I(L,t) are specified functions of t. The duality principle can be applied to show that E(x,t) can be considered for interpolation purposes as an even function of x with respect to both x=0 and x=L. For the remaining case when E(0,t) and I(L,t) are both specified in some manner, I(x,t) and E(x,t) can be considered even functions respectively with respect to x=0 and x=L.

6.3 Conclusion.

The results presented in this chapter have indicated that if one variable (either E(x,t) or I(x,t)) is specified at a boundary in some manner, the remaining variable can be considered an even function of x with respect to this boundary as far as interpolation is concerned. This specified variable can be a function of the remaining variable. In addition, the specified variable has no special property as far as being an even or an odd function of x with respect to this boundary except when it is specified to be identically zero.

VII. GENERAL TRANSMISSION LINE PROBLEM

7.0 Introduction

The main objective of this chapter is to demonstrate how the concepts of Chapters 3, 4, 5, and 6 can be combined to derive three classes of time domain models which can be applied to a very general transmission line problem. For demonstration purposes, let us consider the transmission line problem for which the initial and boundary conditions are given as follows.

$$
\begin{aligned}
I(x,0) &= h_1(x) & & 0 \le x \le L & & 7.0.0 \\
E(x,0) &= h_2(x) & & 0 < x < L & & \\
E(0,t) &= E_{s1}(I(0,t),t) & & 0 \le t < \infty & & \\
E(L,t) &= E_{s2}(I(L,t),t) & & 0 \le t < \infty & &
\end{aligned}
$$

This problem allows E(0,t) and E(L,t) to respectively depend on I(0,t) and I(L,t) in some prescribed manner. This dependence may be either linear or nonlinear.

Before any attempt is made to apply the results of the previous chapters, let us discuss certain preliminary aspects of the numerical solution. First of all, the increment Δx is chosen as $\Delta x = L/k$, where k is a positive integer. In addition, g(2n,t), n=0,1,2,...,k, and g(2n+1,t), n=0,1,2,...,k-1, are respectively the approximate numerical solutions for $E(n\Delta x,t)$ and $I(n\Delta x+\Delta x/2,t)$.

Since $E(0,t)$ and $E(L,t)$ are dependent on $I(0,t)$ and $I(L,t)$, it is necessary to employ an interpolation formula to calculate these variables. The results of Chapter 6 can be directly applied in this case for the purposes of obtaining such an interpolation formula. Hence, $I(0,t)$ and $I(L,t)$ can be written as shown in Eqs. 7.0.1. The same formula need not be used

$$\begin{bmatrix} I(0,t) \\ I(L,t) \end{bmatrix} = \begin{bmatrix} d_{i,1} & \cdot & \cdot & d_{i,i} & \cdot & \cdot & d_{i,k} \\ d_{2,1} & \cdot & \cdot & d_{2,i} & \cdot & \cdot & d_{2,k} \end{bmatrix} \begin{bmatrix} g(1,t) \\ g(3,t) \\ \cdot \\ \cdot \\ g(2i-1,t) \\ \cdot \\ g(2k-1,t) \end{bmatrix} \qquad 7.0.1$$

for both $I(0,t)$ and $I(L,t)$. It is quite clear that this interpolation formula is independent of the difference method employed to find the solution for $g(j,t)$.

The functions $g(0,t)$ and $g(2k,t)$ must be specified as

$$g(0,t) = E_{s1}(I(0,t),t) \qquad 0 \le t < \infty \qquad 7.0.2$$

$$g(2k,t) = E_{s2}(I(L,t),t) \qquad 0 \le t < \infty$$

independent of the method of solution for $g(j,t)$,

$j=1,2,\ldots,2k-1$. At $t=0$, the specified values of $I(0,t)$ and $I(L,t)$ should be used to calculate these boundary conditions. At all other values of t, the interpolation formula as given by Eqs. 7.0.1 must be used to determine the values of $I(0,t)$ and $I(L,t)$ in Eqs. 7.0.2.

In order to indicate how the results of the previous chapters can be applied to this problem to obtain the solution for $g(j,t)$, the superposition principle will be used. At $t=n\Delta x$, this problem may be subdivided into three separate problems for which the initial and boundary conditions are given as follows.

$$
\begin{array}{lll}
E_n(x,n\Delta t) = E(x,n\Delta t) & 0<x<L & 7.0.3 \\
I_n(x,n\Delta t) = I(x,n\Delta t) & 0\le x\le L & \\
E_n(0,t) = 0 & n\Delta t\le t<\infty & \\
E_n(L,t) = 0 & n\Delta t\le t<\infty &
\end{array}
$$

$$
\begin{array}{lll}
E_{n,1}(x,n\Delta t) = 0 & 0<x\le L & 7.0.4 \\
I_{n,1}(x,n\Delta t) = 0 & 0\le x\le L & \\
E_{n,1}(L,t) = 0 & n\Delta t\le t<\infty & \\
E_{n,1}(0,t) = E_{s1}(I(0,t),t) & n\Delta t\le t<\infty &
\end{array}
$$

$$
\begin{array}{lll}
E_{n,2}(x,0) = 0 & 0\le x<L & 7.0.5 \\
I_{n,2}(x,0) = 0 & 0\le x<L &
\end{array}
$$

$$E_{n,2}(0,t) = 0 \qquad\qquad n\Delta t \leq t < \infty$$

$$E_{n,2}(L,t) = E_{s2}(I(L,t),t) \qquad\qquad n\Delta t \leq t < \infty$$

The solution to the original problem for $t \geq n\Delta t$ is simply the sum of the solutions to these three problems. That is, $E(x,t)$ and $I(x,t)$ are given by

$$E(x,t) = E_n(x,t) + E_{n,1}(x,t) + E_{n,2}(x,t) \qquad 7.0.6$$

$$I(x,t) = I_n(x,t) + I_{n,1}(x,t) + I_{n,2}(x,t)$$

for $t \geq n\Delta t$. The orientation of the three currents in Eqs. 7.0.6 are assumed to be from $x=0$ to $x=L$. The problem, described by Eqs. 7.0.3, satisfies the conditions of Chapter 3. The remaining two problems, described by Eqs. 7.0.4 and 7.0.6, each satisfy the conditions of Chapter 4 as long as the boundary conditions, $E_{n,1}(0,t)$ and $E_{n,2}(L,t)$, for $t \geq n\Delta t$ are properly defined.

Let the functions $g_n(j,t)$, $g_{n,1}(j,t)$, and $g_{n,2}(j,t)$ be respectively the numerical solutions to these three problems for which the initial and boundary conditions are given by Eqs. 7.0.3, 7.0.4, and 7.0.5. It is quite clear that the following relationships must be true, independent of the method of numerical solution.

$$g(j,t) = g_n(j,t) + g_{n,1}(j,t) + g_{n,2}(j,t) \qquad 7.0.7$$

$$g_n(j, n\Delta t) = g(j, n\Delta t) \qquad j=1,2,\ldots,2k-1 \qquad 7.0.8$$

$$g_n(0,t) = g_n(2k,t) = 0 \qquad n\Delta t \le t < \infty$$

$$g_{n,1}(j, n\Delta t) = 0 \qquad j=1,2,\ldots,2k-1 \qquad 7.0.9$$

$$g_{n,1}(0,t) = g(0,t) \qquad n\Delta t \le t < \infty$$

$$g_{n,1}(2k,t) = 0 \qquad n\Delta t \le t < \infty$$

$$g_{n,2}(j, n\Delta t) = 0 \qquad j=1,2,\ldots,2k-1 \qquad 7.0.10$$

$$g_{n,2}(0,t) = 0 \qquad n\Delta t \le t < \infty$$

$$g_{n,2}(2k,t) = g(2k,t) \qquad n\Delta t \le t < \infty$$

The numerical procedure for finding $g(j,(n+1)\Delta t)$, $j=0,1,2,\ldots,2k-1$, is essentially finding $g_n(p,(n+1)\Delta t)$, $g_{n,1}(p,(n+1)\Delta t)$, and $g_{n,2}(p,(n+1)\Delta t)$ for $p=1,2,\ldots,2k-1$, by the methods indicated in the previous chapters. According to the results of Chapter 4, the quantities $g_{n,1}(p,(n+1)\Delta t)$ and $g_{n,2}(p,(n+1)\Delta t)$, $p=1,2,\ldots,2k-1$, respectively depend only on the values of $g_{n,1}(0,n\Delta t)$ and $g_{n,2}(2k,n\Delta t)$. Once all of these values are determined, Eqs. 7.0.7 can be used to find $g(j,(n+1)\Delta t)$ for $j=1,2,\ldots,2k-1$. By means of the interpolation formula, Eqs. 7.0.1, and Eqs. 7.0.2, $g(0,(n+1)\Delta t)$ and $g(j(n+1)\Delta t)$ can be determined. Since the functions $g(j,(n+1)\Delta t)$ corresponds to the approximate numerical solution for $E(x,(n+1)\Delta t)$ and $I(x,(n+1)\Delta t)$ at certain prescribed node points, the whole process of superposition can be repeated to find $g(j,(n+2)\Delta t)$.

The next sections of this chapter indicate in detail how the results of Chapters 3 and 4 can be employed to obtain three classes time domain models for which $g(j,t)$, $j=0,1,...,2k$, is the solutions.

7.1 Classes A and B.

This section indicates how the results of Chapters 3, 4, and 6 can be employed to obtain two classes of time domain models which use difference methods in both the x and t dimensions. In order to obtain these models, it is first necessary to find the solution for $g_n(j,(n+1)\Delta t)$, $g_{n,1}(j,(n+1)\Delta t)$, and $g_{n,2}(j,(n+1)\Delta t)$. Once this result is shown, this section demonstrates that the difference methods for each of these three transmission line problems can be combined to produce two classes of time domain models which represent the transmission line.

Since the transmission line problem, described by Eqs. 7.0.8, is a problem of type 3, the results of Chapter 3 are valid for obtaining $g_n(j,(n+1)\Delta t)$ for $j=1,2,...,2k-1$, from the specified initial conditions which are given at $t=n\Delta t$. There are two possible techniques for obtaining this result. One technique uses any difference method, given by Eqs. 3.2.4, which is derived for a class 1 time domain model. The other technique uses any class 2 time domain model. Regardless of the technique, the matrix equation, Eqs. 7.1.0, must be valid for obtaining $g_n(j,(n+1)\Delta t)$. The matrix A_0^* is

$$\begin{bmatrix} g_n(1,(n+1)\Delta t) \\ g_n(2,(n+1)\Delta t) \\ \cdot \\ \cdot \\ g_n(2k-1,(n+1)\Delta t) \end{bmatrix} = A_0^* \begin{bmatrix} g_n(1,n\Delta t) \\ g_n(2,n\Delta t) \\ \cdot \\ \cdot \\ g_n(2k-1,n\Delta t) \end{bmatrix} \qquad 7.1.0$$

determined by a given two level difference method which uses either of the two indicated techniques. The entries of A_0^* are functions of both Δx and Δt.

The results of Chapter 4 can be directly applied to the remaining two transmission line problems, described by Eqs. 7.0.9 and 7.0.10, in order to obtain $g_{n,1}(j,(n+1)\Delta t)$ and $g_{n,2}(j,(n+1)\Delta t)$ for $j=1,2,\ldots,2k-1$. The results of Chapter 4 states that there are two sets of launching numbers, $(b_{j,1}(\Delta x,\Delta t))$ and $(b_{j,2}(\Delta x,\Delta t))$, such that Eqs. 7.1.1 are true for $j=1,\ldots,2k-1$.

$$g_{n,1}(j,(n+1)\Delta t) = b_{j,1}(\Delta x,\Delta t)\ g_{n,1}(0,n\Delta t) \qquad 7.1.1$$

$$g_{n,2}(j,(n+1)\Delta t) = b_{j,2}(\Delta x,\Delta t)\ g_{n,2}(2k,n\Delta t)$$

In Chapter 4, there is a general method for determining these launching numbers. These launching numbers may be chosen by the same specific method or by a different method. If the same specific method is employed, then these two sets are related by

$$b_{j,2}(\Delta x,\Delta t) = (-1)^j b_{2k-j,1}(\Delta x,\Delta t) \quad j=1,2,\ldots,2k-1.$$

Before $g_{n,1}(0,(n+1)\Delta t)$ and $g_{n,2}(2k,(n+1)\Delta t)$ can be determined, $I(0,(n+1)\Delta t)$ and $I(L,(n+1)\Delta t)$ must be known. By means of Eqs. 7.0.7 and 7.0.1, $I(0,(n+1)\Delta t)$ and $I(L,(n+1)\Delta t)$ are given as follows.

$$\begin{bmatrix} I(0,(n+1)\Delta t) \\ I(L,(n+1)\Delta t) \end{bmatrix} = \begin{bmatrix} d_{1,1} & \cdot & \cdot & d_{1,k} \\ d_{2,1} & \cdot & \cdot & d_{2,k} \end{bmatrix} \left\{ \begin{bmatrix} g_n(1,(n+1)\Delta t) \\ g_n(3,(n+1)\Delta t) \\ \cdot \\ \cdot \\ g_n(2k-1,(n+1)\Delta t) \end{bmatrix} + \begin{bmatrix} g_{n,1}(1,(n+1)\Delta t) \\ g_{n,1}(3,(n+1)\Delta t) \\ \cdot \\ \cdot \\ g_{n,1}(2k-1,(n+1)\Delta t) \end{bmatrix} + \begin{bmatrix} g_{n,2}(1,(n+1)\Delta t) \\ g_{n,2}(3,(n+1)\Delta t) \\ \cdot \\ \cdot \\ g_{n,2}(2k-1,(n+1)\Delta t) \end{bmatrix} \right\} \quad 7.1.2$$

Once these variables are known, $g_{n,1}(0,(n+1)\Delta t)$ and $g_{n,2}(2k,(n+1)\Delta t)$ can be determined by Eqs. 7.0.2.

Now that the functions $g_n(j,(n+1)\Delta t)$, $g_{n,1}(j,(n+1)\Delta t)$, and $g_{n,2}(j,(n+1)\Delta t)$ are known for $j=0,1,2,\ldots,2k$, $g(j,(n+1)\Delta t)$, is given by Eqs. 7.0.7. With this knowledge of $g(j,(n+1)\Delta t)$, three new transmission line problems can be created at $t=(n+1)\Delta t$. Therefore, the entire process can then be repeated for finding $g(j,(n+2)\Delta t)$.

The discussion has indicated how the results of Chapters 3, 4, and 6 are applied to determine $g(j,(n+1)\Delta t)$ from $g(j,n\Delta t)$. It is clear that a number of the indicated steps can be combined. Hence, the general form of the equations required to obtain a difference equation solution can be written as follows.

$$\begin{bmatrix} g(1,(n+1)\Delta t) \\ g(2,(n+1)\Delta t) \\ \cdot \\ \cdot \\ g(2k-1,(n+1)\Delta t) \end{bmatrix} = A_0^* \begin{bmatrix} g(1,n\Delta t) \\ g(2,n\Delta t) \\ \cdot \\ \cdot \\ g(2k-1,n\Delta t) \end{bmatrix} + \qquad 7.1.3$$

$$\begin{bmatrix} b_{1,1}(\Delta x,\Delta t) & b_{1,2}(\Delta x,\Delta t) \\ b_{2,1}(\Delta x,\Delta t) & g_{2,2}(\Delta x,\Delta t) \\ \cdot & \cdot \\ \cdot & \cdot \\ b_{2k-1}(\Delta x,\Delta t) & b_{2k-1,2}(\Delta x,\Delta t) \end{bmatrix} \begin{bmatrix} g(0,n\Delta t) \\ g(2k,n\Delta t) \end{bmatrix}$$

$$\begin{bmatrix} I(0,(n+1)\Delta t) \\ I(L,(n+1)\Delta t) \end{bmatrix} = \begin{bmatrix} d_{1,1} & \cdot & \cdot & d_{1,k} \\ d_{2,1} & \cdot & \cdot & d_{2,k} \end{bmatrix} \begin{bmatrix} g(1,(n+1)\Delta t) \\ g(3,(n+1)\Delta t) \\ \cdot \\ \cdot \\ g(2k-1,(n+1)\Delta t) \end{bmatrix} \qquad 7.1.4$$

$$g(0,(n+1)\Delta t) = E_{s1}(I(0,(n+1)\Delta t),(n+1)\Delta t) \qquad 7.1.5$$

$$g(2k,(n+1)\Delta t) = E_{s2}(I(L,(n+1)\Delta t),(n+1)\Delta t)$$

Equations 7.1.3 and 7.1.4 are simply a restatement of Eqs. 7.0.1 and 7.0.2.

The same argument can be directly applied for difference methods are greater than two level. The end result can be obtained by replacing Eqs. 7.1.3 by Eqs. 7.1.6 where the matrix G(t) is defined by Eqs. 3.2.1. The other sets of

$$G((n+1)\Delta t) = \sum_{k=0}^{j_2} A_i^* \, G((n-i)\Delta t) \; + \qquad 7.1.6$$

$$\begin{bmatrix} b_{1,1}(\Delta x,\Delta t) & b_{1,2}(\Delta x,\Delta t) \\ b_{2,1}(\Delta x,\Delta t) & b_{2,2}(\Delta x,\Delta t) \\ \cdot & \\ \cdot & \\ b_{2k-1,1}(\Delta x,\Delta t) & b_{2k-1,1}(\Delta x,\Delta t) \end{bmatrix} \begin{bmatrix} g(0,n\Delta t) \\ g(2k,n\Delta t) \end{bmatrix}$$

equations remain the same.

Now that certain results of the previous chapters have combined, it is possible to define two classes of time domain models which represent the transmission line.

<u>Definition</u>: Class A-Time Domain Model:

A time domain model belongs to class A if and only if the model can be obtained by the following steps.

1. The model is described by Eqs. 7.1.3 or 7.1.6, and 7.1.4.
2. The matrices A_0^*, $A_1^*, \ldots,$ and A_{j2}^* are obtained by a difference method in both x and t which is derived from a class 1 time domain model.

<u>Definition</u>: Class B-Time Domain Models:

A time domain model belongs to class B if and only if the model can be obtained by the following steps.

1. The model is described by Eqs. 7.1.3 or 7.1.6, and 7.1.4.
2. The matrices A_0^*, A_1^*,..., and $A_{j_2}^*$ correspond to a time domain model belonging to class 2.

The only difference between the two classes of models is the technique for which the matrices A_0^*, A_1^*,..., and $A_{j_2}^*$ are obtained. For a model belonging to class A, these matrices have a class 1 time domain model as a basis. Class B on the other hand have the class 2 time domain models as its basis. Since these two classes are not mutually disjoint, it is possible for a model to belong to both classes.

7.2 <u>Class C</u>.

This section combines the results of Chapters 3, 4, and 6 to derive a set of (2k-1) linear ordinary differential equations for the solutions are $g(j,t)$ for $j=1,2,\ldots,2k-1$.

Let us first consider the problem of type 3 which is described by Eqs. 7.0.8. The approximate solution, $g_n(j,t)$, to this problem can be found by solving a set of (2k-1) ordinary differential equations which are given by Eqs. 3.2.3. These equations for a (2p+1) degree approximation to the partial derivations can be written as

$$\frac{dG_n(t)}{dt} = A_p\, G_n(t)$$

where the matrices $G_n(t)$ and A_p are respectively defined by Eqs. 3.2.1 and 3.2.2.

If Eqs. 7.1.1 are restated, these equations state that

$$g_{n,1}(j,(n+1)\Delta t) = b_{j,1}(\Delta x,\Delta t)\; g_{n,1}(0,n\Delta t) \qquad j=1,\ldots,2k-1$$

$$g_{n,2}(j,(n+1)\Delta t) = b_{j,2}(\Delta x,\Delta t)\; g_{n,2}(2k,n\Delta t) \qquad j=1,\ldots,2k-1$$

are true for the remaining two problems. This relationship uses the fact that $g_{n,1}(j,n\Delta t)$ and $g_{n,2}(j,n\Delta t)$ for $j=1,2,\ldots,2k-1$ are zero. For any given method for finding the launching numbers, $b_{j,1}(\Delta x,\Delta t)$ and $b_{j,2}(\Delta x,\Delta t)$, Chapter 4 showed that these launching numbers can be written as

$$b_{j,1}(\Delta x,\Delta t) = (v_p\Delta t/\Delta x)\; \bar{b}_{j,1} \qquad j=1,2,\ldots,2k-1 \qquad 7.2.1$$

$$b_{j,2}(\Delta x,\Delta t) = (v_p\Delta t/\Delta x)\; \bar{b}_{j,2} \qquad j-1,2,\ldots,2k-1$$

where $\bar{b}_{j,1}$, and $\bar{b}_{j,2}$ are constants independent of Δx or Δt.

If Eqs. 7.2.1 are substituted into Eqs. 7.1.1, the following relationships can be obtained.

$$\frac{g_{n,1}(j,(n+1)\Delta t) - g_{n,1}(j,n\Delta t)}{\Delta t} = \frac{v_p}{\Delta x}\,\bar{b}_{j,1}\; g_{n,1}(0,n\Delta t)$$

$$\frac{g_{n,2}(j,(n+1)\Delta t) - g_{n,2}(j,n\Delta t)}{\Delta t} = \frac{v_p}{\Delta x}\,\bar{b}_{j,2}\; g_{n,2}(2k,n\Delta t)$$

But

$$\frac{dg_{n,1}(j,n\Delta t)}{dt} \doteq \frac{g_{n,1}(j,(n+1)\Delta t) - g_{n,1}(j,n\Delta t)}{\Delta t}$$

$$\frac{dg_{n,2}(j,n\Delta t)}{dt} \doteq \frac{g_{n,2}(j,(n+1)\Delta t) - g_{n,2}(j,n\Delta t)}{\Delta t} .$$

Therefore,

$$\frac{dg_{n,1}(j,n\Delta t)}{dt} \doteq \frac{v_p}{\Delta x} \bar{b}_{j,1}\ g_{n,1}(0,n\Delta t) \qquad 7.2.2$$

$$\frac{dg_{n,1}(j,n\Delta t)}{dt} \doteq \frac{v_p}{\Delta x} \bar{b}_{j,2}\ g_{n,2}(2k,n\Delta t).$$

If this relationship, given in Eq. 7.2.2, is combined with Eqs. 7.2.0, the solution for $g(j,t)$, $j=1,2,\ldots,2k-1$, can be found by solving the following $(2k-1)$ ordinary differential equations.

$$\frac{dG(t)}{dt} = A_p\ G(t) + \frac{v_p}{\Delta x} \begin{bmatrix} \bar{b}_{1,1} & \bar{b}_{1,2} \\ \bar{b}_{2,1} & \bar{b}_{2,2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \bar{b}_{2k-1,1} & \bar{b}_{2k-1,2} \end{bmatrix} \begin{bmatrix} g(0,t) \\ g(2k,t) \end{bmatrix} \qquad 7.2.3$$

In addition to Eqs. 7.2.3, it is necessary to use Eqs. 7.0.1 and 7.0.2 to properly define the quantities $g(0,t)$, $g(2k,t)$, $I(0,t)$ and $I(L,t)$. The third class of time domain models can now be defined.

Definition: Class C-Time domain Models:

A time domain model belongs to class C if and only if the model is described by Eqs. 7.0.1 and 7.2.3.

7.3 Discussion of Models.

Let us assume that this parallel line of length L is a single component in a given system. Each port can be represented by a linear graph of one part which contains exactly one element. The variables $e_1(t)$ and $i_1(t)$ can be associated with the graph representing the port at x=0. For the port x=L, the variables $e_2(t)$ and $i_2(t)$ are associated with its linear graph representation. These graph variables are associated with the transmission line variables as shown in Eqs. 7.3.0 where the minus sign is required because of the

$$\begin{aligned} e_1(t) &= E(0,t) \\ i_1(t) &= I(0,t) \\ e_2(t) &= E(L,t) \\ i_2(t) &= -I(L,t) \end{aligned} \qquad 7.3.0$$

assumed current orientations.

Let us assume that a time domain model which belongs to either class A or class B is employed to relate these graph variables. If we assume that the increments Δx and Δt are properly chosen for this transmission line model, it is quite clear that this model defines the relationship

$$i_1((n+1)\Delta t) = h_3(e_1(n\Delta t), e_2(n\Delta t), n\Delta t) \qquad 7.3.1$$

$$i_2((n+1)\Delta t) = h_4(e_1(n\Delta t), e_2(n\Delta t), n\Delta t)$$

where the functions $h_3(e_1(t), e_2(t), t)$ and $h_4(e_1(t), e_2(t), t)$ are given by Eqs. 7.3.1, may or may not be the desired relationship to represent the transmission line as a two port component. The answer to this question is determined by the methods of numerical solution which is to be used to solve the system for which the transmission line is a two port component.

For discussion purposes, let us assume that Eqs. 7.3.1 are the desired relationships. The most difficult problem which occurs when using the results of section 7.1 is the problem of choosing the increment Δt once Δx has been chosen. For a given model for the transmission line, the increment which is required is a function of this model and a function of the mathematical description of the entire system. For example, if Δt is chosen such that the transmission line model is stable for the case when $e_1(t)$ and $e_2(t)$ are specified functions of t and are independent of $i_1(t)$ and $i_2(t)$, there is a strong possibility that the equations which describe the transmission line are unstable for the case when $e_1(t)$ and $e_2(t)$ depend on $i_1(t)$ and $i_2(t)$ in some manner.

The transmission line time domain model which belongs to class C relates these graph variables in an entirely different manner. In fact, this class of models gives the relationship

$$\frac{di_1(t)}{dt} = \bar{h}_3(e_1(t),e_2(t),t) \qquad 7.3.2$$

$$\frac{di_2(t)}{dt} = \bar{h}_4(e_1(t),e_2(t),t)$$

where the functions $\bar{h}_3(e_1(t),e_2(t),t)$ and $\bar{h}_4(e_1(t),e_2(t),t)$ are determined by Eqs. 7.2.3 and 7.0.1. The method of numerical solution which is used to solve the differential equations, given by Eqs. 7.2.3, can be chosen so that it is the same as the method of numerical solution which is empolyed to solve the entire system. For example, if the other components in the system are described by a set of differential equations, then numerical method for solving these differential equations can also be employed to solve Eqs. 7.2.3.

One advantage of using a class C model is that some technique of numerical solution can be used which automatically decides on the increment Δt.

7.4 Both Graph Elements are not Chords.

All of the discussion so far presented in this chapter is applicable to transmission line problems for which the transmission line linear graph representation has been formulated as chords. The findings of Chapters 5 and 6 can be directly applied to the results of sections 7.0, 7.1, 7.2, and 7.3 to show that by modifying the equations which describe the models, the models are also applicable to the other cases which arise in formulation.

Upon applying the findings of Chapters 5 and 6, the results are stated as follows:

Both Graph elements are branches:

1. $\Delta x = L/k$, where k is a positive integer.
2. The functions $g(2j,t)$, $j=0,1,2,...,k$, and $g(2j+1,t)$, $j=0,1,2,...,k-1$, are respectively defined as the approximate numerical solutions for $I(j\Delta x,t)$ and $E(j\Delta x+\Delta x/2,t)$.
3. In Eqs. 7.0.1, replace $I(0,t)$ and $I(L,t)$ respectively by $E(0,t)$ and $E(L,t)$.
4. Upon performing these modifications, the three classes of time domain models are valid for this case.

One graph element is a chord and the other is a branch:

1. $\Delta x=2L/k'$, where k' is a positive odd integer.
2. When the port at x=0 is formulated as a chord (branch), the functions $g(2j,t)$, $j=0,1,2,...,(k'-1)/2$, and $g(2j+1,t)$, $j=0,1,2,...,(k'-1)/2$, are defined respectively as the approximate numerical solutions for $E(j\Delta x,t)(I(j\Delta x,t))$ and $I(j\Delta x+\Delta x/2,t)(E(j\Delta x+\Delta x/2,t))$.
3. Replace the quantity 2k by k' in all equations in sections 7.0, 7.1, and 7.2.

4. In Eqs. 7.0.1, replace I(L,t)(E(0,t)) by E(L,t)(I(0,t)).
5. Use Eqs. 5.2.6 to recalculate the launching numbers.
6. Upon performing these modifications, the three classes of time domain models are valid for this case.

7.5 Example Problem.

In order to illustrate the concepts which are presented in this chapter, let us work a simple problem. This problem consists of a lossless transmission line of length L which is terminated respectively at x=0 and x=L with a specified voltage source and a matched load. In addition, it will be assumed that the initial conditions on the line are specified to be zero.

For simplification purposes, the parameters of the transmission line are specified as

$$\sqrt{l/c} = 1 \text{ ohm}$$

$$v_p = 1 \text{ meter/second}$$

$$L = 1 \text{ meter.}$$

This choice of parameters forces the telegrapher's equations to be in normalized form. In addition, the computations required to obtain any time domain model is simplified.

The linear graph for this problem can be drawn as shown in Fig. 1 where elements 2 and 3 are associated respectively

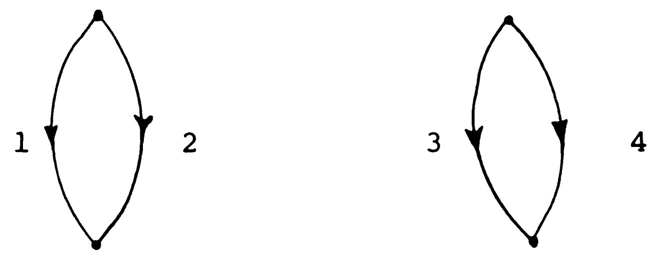

Fig. 1 Oriented linear graph for example problem.

with the transmission line ports at x=0 and x=L. Elements 1 and 4 are associated respectively with the specified voltage source and the matched load. If we let elements 1 and 4 be the tree for this graph, the f-cut set, f-circuit, and element equations are respectively:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_4(t) \\ i_2(t) \\ i_3(t) \end{bmatrix} = 0 \qquad 7.5.0$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_4(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = 0 \qquad 7.5.1$$

$$\begin{aligned} e_1(t) &= 2t & 0 \le t \le 0.5 \qquad 7.5.2 \\ &= 1 & 0.5 \le t \le 1.5 \end{aligned}$$

$$= -2t + 4 \qquad\qquad 1.5 \leq t \leq 2.0$$

$$= 0 \qquad\qquad 2.0 \leq t \leq \infty$$

$$e_2(t) = E(0,t)$$

$$i_2(t) = I(0,t)$$

$$e_3(t) = E(L,t)$$

$$i_3(t) = -I(L,t)$$

$$e_4(t) = i_4(t)$$

Before any attempt is made to solve these equations, the time domain model which is used to represent the transmission line must be specified.

Since there are an infinite number of time domain models which belong to each class of models, the transmission line portion of the problem will be represented by a model which belongs to class C because it is described by differential equations. To further illustrate some of the concepts, this problem will be worked four times using two different models and 2 choices of Δx for each model.

For this problem, the increment Δx must be chosen as $\Delta x = 1/L$, where k is a positive integer, independent of the method of numerical solution. In addition, the functions $g(2j,t)$, $j=0,1,2,\ldots,k$, and $g(2j+1,t)$, $j=0,1,2,\ldots,k-1$, are defined respectively as the numerical solution for $E(j\Delta x,t)$ and $I(j\Delta x + x/2,t)$.

According to section 7.2, any model which belongs to class C can be written in general terms as

$$\frac{dG(t)}{dt} = \mathbf{A}_p\, G(t) + (v_p/\Delta x) \begin{bmatrix} \bar{b}_{1,1} & \bar{b}_{1,2} \\ \bar{b}_{2,1} & \bar{b}_{2,2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \bar{b}_{2k-1,1} & b_{2k-1,2} \end{bmatrix} \begin{bmatrix} g(0,t) \\ g(2k,t) \end{bmatrix} \qquad 7.5.3$$

$$\begin{bmatrix} I(0,t) \\ I(L,t) \end{bmatrix} = \begin{bmatrix} d_{1,1} & \cdot & \cdot & d_{1,i} & \cdot & \cdot & d_{1,k} \\ d_{2,1} & \cdot & \cdot & d_{2,i} & \cdot & \cdot & d_{2,k} \end{bmatrix} \begin{bmatrix} g(1,t) \\ g(2i-1,t) \\ \cdot \\ g(2k-1,t) \end{bmatrix} \qquad 7.5.4$$

Once the matrix $\mathbf{A}_p$, the interpolation formulas, and the launching numbers or transformation constants are chosen, the time domain model has been defined. In order to solve this resulting time domain model, it is only necessary to specify Δx, Δt, the method of numerical solution, and the boundary conditions.

In order to find the terminal variable $I(0,t)$, let us define an interpolation formula in terms of the functions $g(1,t)$ and $g(3,t)$. According to Chapter 6, the function $I(x,t)$ can be assumed to be an even function of x with respect to $x=0$. Therefore, if

$$I(x,t) = a_0 + a_2\ x^2$$

such that

$$g(1,t) = a_0 + a_2\ (\Delta x/2)^2$$

$$g(3,t) = a_0 + a_2\ (3\Delta x/2)^2$$

then

$$I(0,t) = 9\ g(1,t)/8 - g(3,t)/8. \qquad 7.5.5$$

By using the same interpolation formula to find $I(L,t)$ in terms of $g(2k-1,t)$ and $g(2k-3,t)$, $I(L,t)$ is then given as

$$I(L,t) = 9\ g(2k-1,t)/9 - g(2k-3,t)/8. \qquad 7.5.6$$

The interpolation formulas will remain the same when defining both time domain models.

The matrices A_p, $p=0,1,2,3,\ldots$, are defined by Eqs. 3.2.2. Therefore, once the value of p is chosen, this portion of the domain model has been specified. For the two specific models which will be solved numerically, p is chosen as zero. Therefore,

$$A_0 = k\ C \qquad 7.5.7$$

where C is the transformation matrix which is defined by Eqs. 3.1.1 and k is a scalar.

Finally, two sets of transformation constants must be specified in order to completely specify these two time domain models. One class of methods for obtaining these transformation constants is derived in Section 4.6. If the results are used from the two examples which are given in section 4.6 and the results of section 4.8 are applied, one such set is

$$\bar{b}_{1,1} = 1 \qquad 7.5.8$$

$$\bar{b}_{j,1} = 0 \qquad j=2,3,\ldots,2k-1$$

$$\bar{b}_{2k-1,2} = -1$$

$$\bar{b}_{j,2} = 0 \qquad j=1,2,\ldots,2k-2$$

and the other set is

$$\bar{b}_{1,1} = 5/4 \qquad 7.5.9$$

$$\bar{b}_{3,1} = -1/4$$

$$b_{j,1} = 0 \qquad j=4,5,\ldots,2k-1$$

$$b_{2,1} = 0$$

$$\bar{b}_{2k-1,2} = -5/4$$

$$\bar{b}_{2k-3,2} = 1/4$$

$$\bar{b}_{2k-2,2} = 0$$

$$\bar{b}_{j,2} = 0. \qquad j=1,2,\ldots,2k-4$$

Let us now solve Eqs. 7.5.0, 7.5.1, and 7.5.2 for $i_2(t)$ and $i_4(t)$. The numerical solution for $i_2(t)$ and $i_4(t)$ along with the analytical solutions for these variables are shown for the interval $0 \leq t \leq 4$ in Figures 2, 3, 4, and 5 where the information concerning each technique of solution is given as follows:

Figure 2:

1. $\Delta t = 1/80$ seconds.
2. $k = 4$.
3. Transformation constants specified by Eqs. 7.5.8.
4. Runge Kutta fourth order method of solution.

Figure 3:

1. $\Delta t = 1/80$ seconds.
2. $k = 8$.
3. Transformation constants specified by Eqs. 7.5.8.
4. Runge Kutta fourth order method of solution.

Figure 4:

1. $\Delta t = 1/80$ seconds.
2. $k = 4$.
3. Transformation constants specified by Eqs. 7.5.9.
4. Runge Kutta fourth order method of solution.

Figure 5:

1. $\Delta t = 1/80$ seconds.
2. $k = 8$.

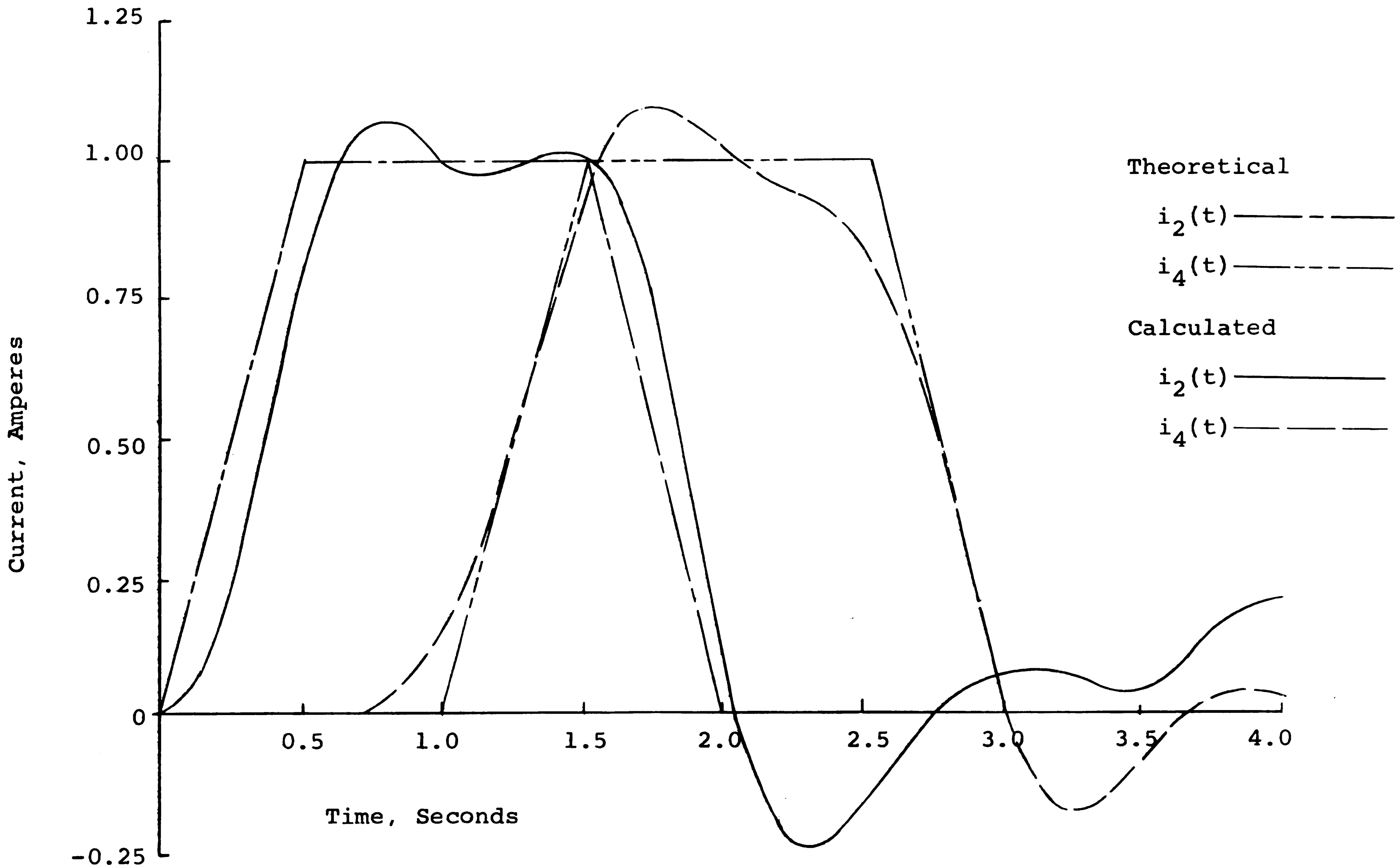

Figure 2. Graph of solutions for k=4 and transformation constants specified by Eqs. 7.5.8.

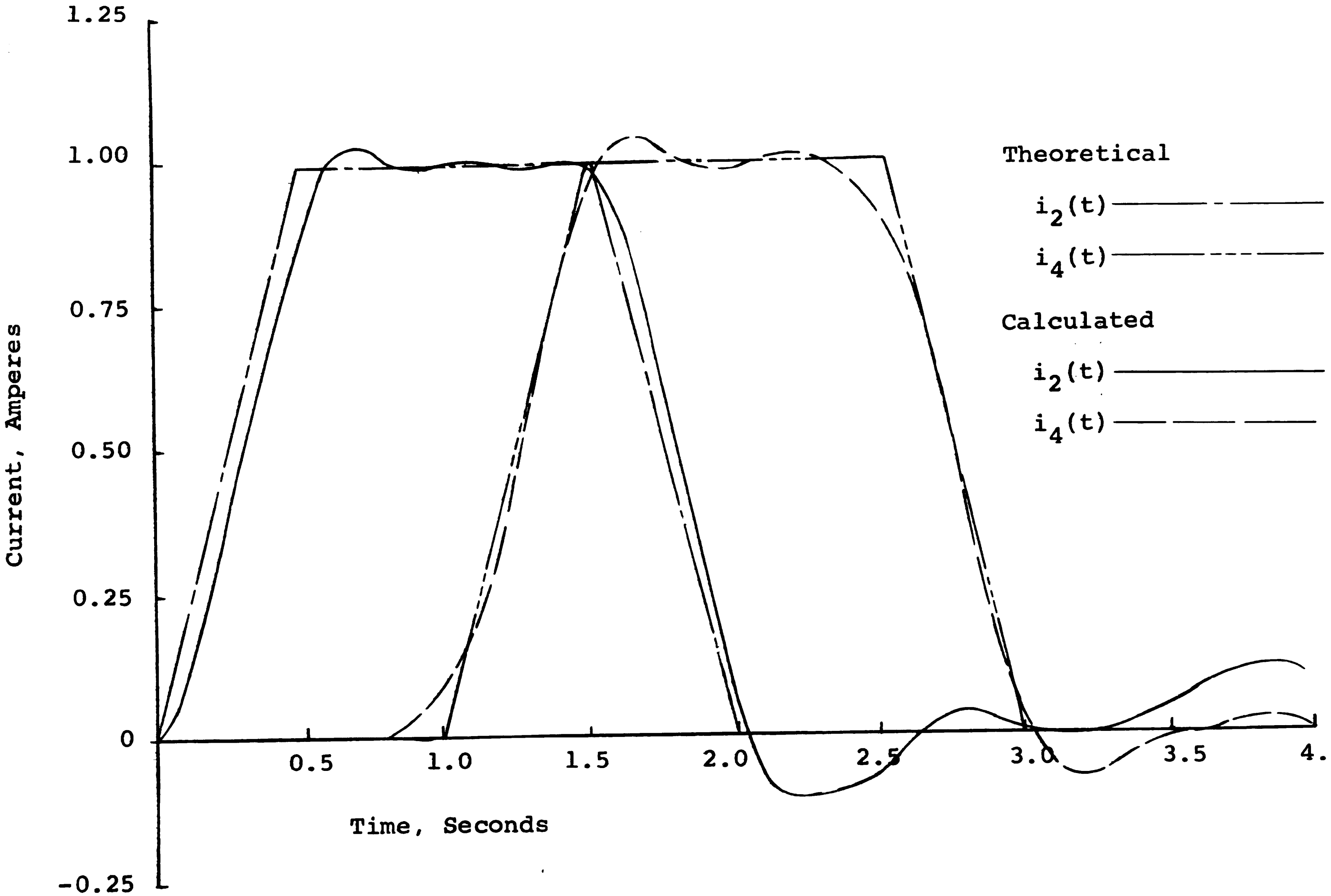

Figure 3. Graph of solutions for k=8 and transformation constants specified by Eqs. 7.5.8.

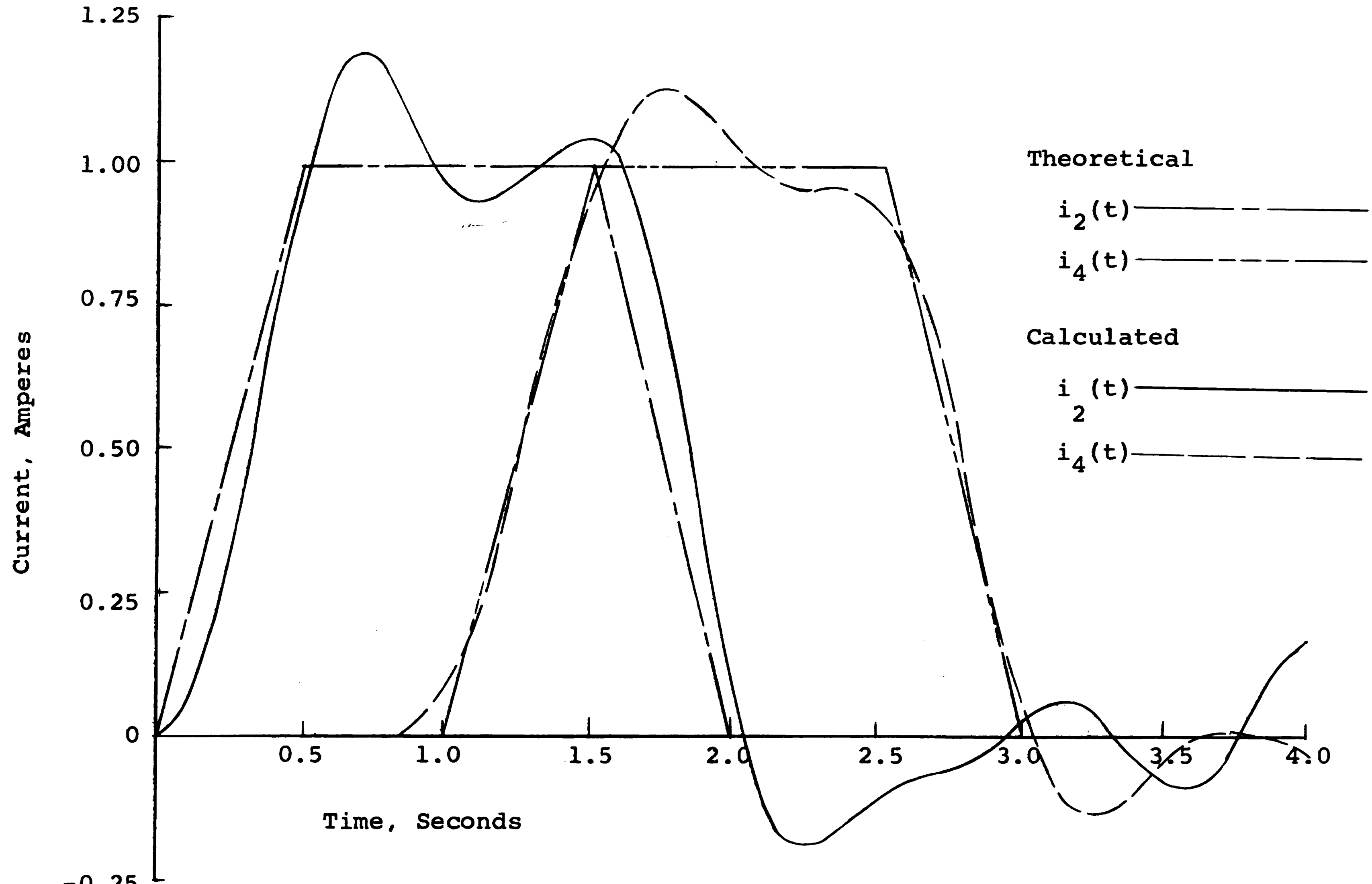

Figure 4. Graph of solutions for k=4 and transformation constants specified by Eqs. 7.5.9.

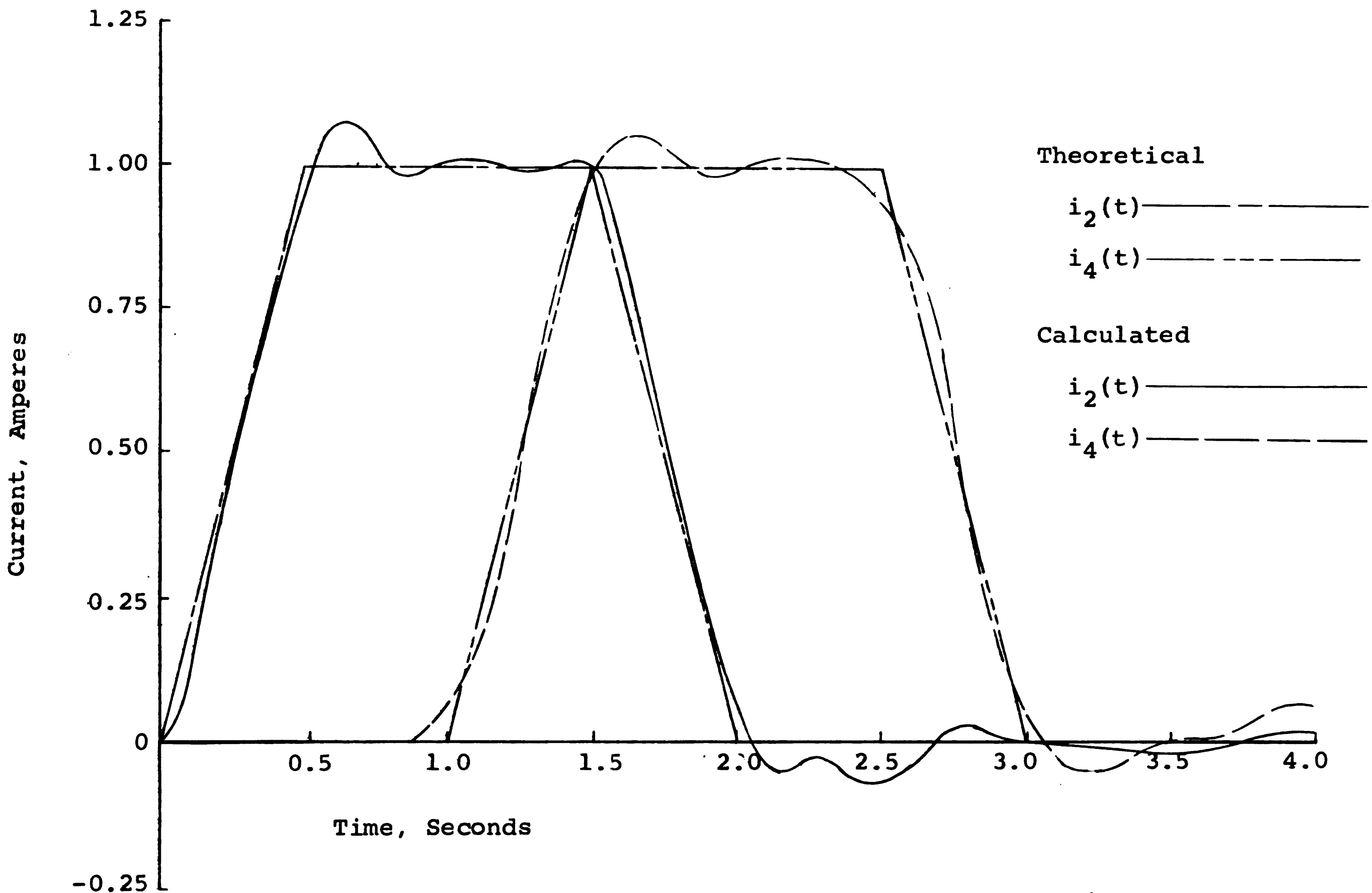

Figure 5. Graph of solutions for k=8 and transformation constants specified by Eqs. 7.5.9.

3. Transformation constants specified by Eqs. 7.5.9.

4. Runge Kutta fourth order method of solution.

Let us now discuss certain aspects of the numerical solutions for $i_2(t)$ and $i_4(t)$ which are given by Figures 2, 3, 4, and 5. If Figure 2 is compared to Figure 3 and Figure 4 is compared to Figure 5, the effect of the increment Δx on the numerical solutions is clearly shown. Hence, if Δx was chosen even smaller, the accuracy of the approximation would be improved.

Since the input current, $i_2(t)$, for all of the models is obtained by interpolation from the calculated values of current at certain interior node points, the time delay property of the equations causes the calculated input current to lag the theoretical solution. This time delay property is corrected at the expense of overshoot to some extent by using the transformation constants which are specified by Eqs. 7.5.9. This effect is also reduced by a smaller Δx since the time delay between the input and the first current node is reduced.

For $t \geq 2$, the numerical solution for $i_2(t)$ is not zero where theoretically this current is zero. The reason for this result is that the time domain models do not truly have a characteristic impedance equal to unity. As a result, reflections occur at $x=0$ and $x=L$. This effect is also observed in the solution for $i_4(t)$. The reduction of this effect by decreasing Δx is shown by comparing the results for k=4 to k=8. This phenomena is generally present in all transmission line approximations whether they are those derived in the thesis or not.

The variable $i_4(t)$ corresponds to the output current. Upon observing $i_4(t)$, we see that the model gives us the correct time delay. In addition, we see that $i_4(t)$ begins to rise before the theoretical solution begins to rise. This property can generally be expected from all transmission line approximations.

It can then be concluded that for sufficiently small Δx these models provide the desired representation.

7.6 Conclusion.

The findings of the previous chapters have been combined in this chapter by superposition to obtain the three classes of time domain models which are applicable to the transmission line problems encountered in electrical engineering. In addition, a numerical example was worked to show certain aspects of the numerical solution which are obtained by the use of these models.

VIII. CONCLUSION

8.0 Conclusion

Three classes of time domain models have been derived which are directly applicable to the practical problems which are encountered in electrical engineering. Any model which belongs to either class A or class B is defined by a finite set of difference equations in both the x and t dimension. On the other hand, a model which belongs to class C is described by a finite set of ordinary differential equations which are in normal form.

For the case where the transmission line linear graph representation is formulated as chords, the general form of the equation for any model which belongs to class A or class B are given by Eqs. 7.1.3 or 7.1.6, and 7.1.4. The general method for determining the matrices $\mathbf{A}_i^*$, $i=0,1,\ldots,j_2$, which appear in these equations is one of the principle results of Chapter 3. The result of Chapter 4 is the derivation of a general method for obtaining the launching numbers which appear in these equations. The remaining terms which appear in these equations correspond to the interpolation formulas used to find the numerical solution for $I(0,t)$ and $I(L,t)$. Chapter 6 derives certain properties which are useful in determining these interpolation formulas.

The general form of the equations which describe any model belonging to class C are given by Eqs. 7.0.1 and 7.2.3. The terms appearing in these equations are found by applying the results of Chapter 2, 3, 4, and 6 for the case when the transmission line linear graph representation is formulated in the cotree.

For the other three cases which result when both of the linear graph elements which represent the transmission line are not formulated as chords, the results of Chapter 5 and section 7.4 must be applied to slightly modify the equations which represent any model even though the general form of the equations remains essentially the same.

One of the main results of this thesis other than deriving these classes of time domain models is the approach which was used in the derivations. This approach consisted of the specific use of the superposition principle to subdivide the general transmission problem into 3 transmission line problems which are easily treated individually. One of these resulting problems was treated by a finite-infinite line transformation which provided an effective means for approximating all of the partial derivatives with respect to x by more accurate central difference formulas. The remaining problems are treated by a transformation which transforms the boundary conditions for a given interval to initial conditions. This transformation is defined by a set of launching numbers.

8.1 Limitations.

It must be pointed out that the telegrapher's equations or the wave equation describe a two conductor transmission line only when the algebraic sum of the currents in any cross-section is zero. This condition must be satisfied before any of the models can be expected to be a valid representation. This requirement generally places some restrictions on the system for which one or more transmission lines are components.

8.2 Additional Problems.

This thesis derives an infinite number of time domain models. One subject which warrants further study is the choice of a best model or models. It would first be necessary to formulate a criteria for making this choice.

One additional problem would be to extend the findings of this thesis to the lossy transmission line. By making the appropriate change of variable, it is clear that the distortionless line immediately follows. For the lossy lines which are not distortionless, some of the findings of this thesis can be easily extended while others are not.

First of all, the derivations in Chapters 2 and 3 which treat the time domain models, belonging to class 1, can be readily modified to include lossy lines by the introduction of the appropriate dissapation terms which appear in the telegrapher's equations. On the other hand, modification of the derivations associated with time domain models which belong to class 2 can be quite complex. The reason for this is that the extention of the basic numerical solution to include lossy lines results in a very complicated expression.

Since the derivations in Chapter 4 are based on certain forms of this basic numerical solution, there is no easy way to modify the launching numbers to include lossy lines. The transformation constants which are given by

$$\bar{b}_{1,1} = 1$$

$$\bar{b}_{j,1} = 0 \qquad j=2,3,\ldots,2k-1$$

$$\bar{b}_{2k-1,2} = 1$$

$$\bar{b}_{j,2} = 0 \qquad j=1,2,\ldots,2k-2$$

are valid for this lossy case.

Finally, the results in Chapters 5, 6, and 7 are valid whenever the best numerical solution does not enter in any way into the derivations. It can be concluded in order to extend a large portion of the thesis, it would be necessary to first investigate the basic numerical solution which is applicable or the lossy infinite line and to secondly obtain the Fourier approximations, used in Chapter 4, which are valid for the lossy line. Once these items are determined, the same approach as was used in the thesis can be applied.

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