SOME PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT

SOME PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY

By

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This thesis is primarily concerned with some partial differential equations with delay, for example

$$\left(\frac{\partial u}{\partial t}\right)(x,t) = c\Delta u(x,t) - \int_{-\infty}^{t} g(t-s)\,\Delta u(s)\,ds$$

where Δ denotes the Laplacian. In the first chapter, several physical models are discussed, which lead to equations of this type.

In the second chapter, a more general equation

$$\left(\frac{d}{dt}u\right)(t) = cAu(t) - \int_{-\infty}^{t} g(t-s)Au(s)\,ds$$

where A is an infinitesimal generator and is considered at first. This equation is formally transformed into a second equation.

This second equation is shown to have a unique solution for appropriate initial conditions. A semigroup is defined using these solutions. The infinitesimal generator for this

semigroup is found, and its spectrum computed. In the last section it is shown that the solution of the modified equation gives solutions of the original equation when $A = \Delta$ and the solution is in a weak sense.

The third chapter discusses existence and uniqueness for the nonhomogeneous linear equation and the perturbed equation.

The last chapter involves a saddle point property for the perturbed equation.

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To my parents

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CHAPTER I

INTRODUCTION

Functional differential equations have been well studied by various writers. Such equations arise in physical models where the rate of change of a system depends not only on the present state of the system, but also on the past state or "history" of the system. A natural generalization of functional differential equations, which are ordinary differential equations where the derivative depends on the history of the system [11], are equations involving partial derivatives and the history of the system. These equations occur as models in some physical and biological problems. In this introduction models of 1. Gene frequency, 2. Heat conduction, 3. The "dangling spider," and 4. Viscoelasticity will be considered.

1. Gene frequency.

Fleming [9] has used the equation

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \alpha p - \beta$$

to describe the frequency of a selectively-neutral gene, in a habitat consisting of a number of colonies of animals arranged in a row. Here $p(t,x)$ is the frequency of the gene type at time t and position x, α is a positive constant depending on the rate of reproduction and death, and β is a positive constant taking into account other variables. The migration from colony to colony is

$$\frac{p(t,x+h) + p(t,x-h) - 2p(t,x)}{2h}$$

which is approximated by $\frac{\partial^2 p}{\partial x^2}$. Note that this model assumes that the rate of change of the gene frequency depends on the migration at the present moment only. To relax this assumption a more realistic model might be

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \int_{-\infty}^{t} g(t-s)\left(\frac{\partial^2 p}{\partial x^2}\right)(s,x)\,ds + \alpha p - \beta,$$

where $g(s) \to 0$ as $s \to \infty$, $g(s) \geq 0$. This integral involves the past history, and thus takes into account the rate of migration in past times. Since $g(s) \to 0$ as $s \to \infty$, the system is of fading memory type. That is, although the value of p in the distance past does affect the integral, it does so less than the value of p in the immediate past.

This model also assumes that the colonies are arranged in a row. It is more valid to consider colonies arranged in a plane. By reasoning similar to that in the preceding model, this can be represented by

$$\frac{\partial p}{\partial t} = \Delta p + \int_{-\infty}^{t} g(t-s)\,\Delta p(s,x)\,ds + \alpha p - \beta,$$

where all the variables have the same meaning as before, except x is a two vector.

2. Heat conduction.

Jace W. Nunziato [21] has studied a similar equation in regard to heat conduction in materials with memory. Let M be a homogeneous heat conductor with memory. For each $x \in B$, there are three response functionals: the free energy E, the entropy N, and the heat flux Q. Each of these depend on the temperature T and the temperature gradient ∇T, both at the present time and the past history.

To be more specific, let $T_T(s) = T(s-t)$ and $\nabla T_t(s) = \nabla T(s-t)$, considered as elements of a function space. Let $H = \{g:(-\infty,0) \to R$ such that $\int_0^\infty g(s)h^2(s)ds < \infty\}$. (Identifying functions equal except on sets of measure 0.) $h(s)$ is a positive function decreasing to 0. $T(x,t)$ is the temperature at position x, time t. $T(t)$ is the same function considered as a function from the real numbers into a Hilbert space H. For each t, $T(\cdot,t) \in H$. H is a Hilbert space. Let $<\ ,\ >$ be the inner product in H. Then Q (similarly E and N) is a function of

$$(T,\nabla T,T_t,\nabla T_t) \in R \times H \times R^3 \times H^3.$$

E is assumed twice Fréchet differentiable, N and Q each once Fréchet differentiable.

Define $G = E + TN$, where G the internal energy. Then G and Q must satisfy the energy balance equation

$$G = - \nabla Q + r$$

where r is the heat supply from the body's surroundings.

The second law of thermodynamics gives the Clausius-Duhem inequality [3]

$$\dot{N} \geq \frac{r}{T} - \nabla\left(\frac{1}{T} Q\right).$$

To obtain constitutive equations these equations are linearized. This gives after considerable computation

$$Q = -K(0)\nabla T - \int_0^\infty K'(s)\nabla T(t-s)\,ds.$$

In the same way

$$G = G_0 + \alpha(0)T + \int_0^\infty \alpha'(s)T(t-s)\,ds.$$

Here K is the heat conduction relaxation function and Q is the energy temperature relaxation function.

Combining these equations and using the energy balance equation results in

$$\alpha(0)\dot{T}(x,t) + \int_0^\infty \alpha'(s)\dot{T}(x,t-s)\,ds$$

$$= K(0)\nabla^2 T(x,t) + \int_0^\infty K'(s)\nabla^2 T(x,t-s)\,ds + r(x,t).$$

If $\alpha'(s) = 0$ (in other words if the energy temperature relaxation function is constant) then the equation becomes

$$\dot{T}(x,t) = \alpha\Delta T(x,t) + \int_0^\infty g(s)\Delta T(x,t-s)\,ds + r(x,t)$$

an equation similar to the previous one. Paul Davis has studied this equation [7], [8].

3. The "dangling spider."

Another example of a model where such an equation appears is the "dangling spider," studied by B. Coleman and D. Owen [5] among others. A mass M hangs from a ceiling by a massless but strechable filament of length z. The forces acting on the ball are the tension T in the filament and a body force F in the z-direction. It can in many cases be assumed $F = -\frac{dh(z)}{dz}$, where h is independent of time, that is, F is a potential.

Assume T at time t is given by a function g of the history of z up to t: $T(t) = g(z_t)$. Using Newton's second law, the equation of motion for the ball is $M\ddot{z} = F(z) - g(z_t)$.

If

$$T(t) = G(0)z_t(0) + \int_0^\infty G'(s)z_t(-s)ds$$

(this form appears in the linear theory of viscoelasticity [4]) where G is the relaxation function, and $G(0)$ the equilibrium modulus, then the equation becomes

$$M\ddot{z} = F(z) - G(0)z_t(0) - \int_0^\infty G'(s)z_t(-s)ds.$$

For F, the simplest nontrivial case, and also a case with physical importance, is when $F = gM$, the gravitational case. In more general cases, F does depend on z.

4. Viscoelasticity.

Another example is linear viscoelasticity of the Boltz-Mann type, considered by Dafermos [6] and Coleman [4]. Consider a one dimensional homogeneous body with density $\rho > 0$, stress at position x at time $t = \sigma(x,t)$, displacement from position x at time $t = u(x,t)$, satisfying the constitutive equation $\sigma(x,t) = c\,u_x(x,t) - \int_{-\infty}^{t} g(t-\tau)u_x(x,\tau)\,d\tau$ where $g(\xi) \geq 0$, $g'(\xi) \leq 0$, and $c - \int_0^{\infty} g(\xi)\,d\xi > 0$.

The body has endpoints b and d, which remain fixed, so $u(b,t) = u(d,t) = 0$, for all $t \in R$.

The state of the body at time t is described by the displacement $u(x,t)$, the momentum $v(x,t)$, and the history of displacement $u_t(x,s)$.

The equation of motion is

$$\rho\,\ddot{u}(x,t) = c\,u_{xx}(x,t) - \int_{-\infty}^{t} g(t-\tau)u_{xx}(x,\tau)\,d\tau .$$

In the following chapters, a generalization of the equations which appear in the first two models will be studied. Chapter 2 is concerned with the homogeneous linear equation. Existence and uniqueness is shown for a related equation. A semigroup is constructed using solutions to this equation, and its infinitesimal generator is computed. The spectrum is of the infinitesimal is computed. In the last section of chapter two, the solution of the modified equation is shown to give solutions of the original equation in a weak sense.

The third chapter is concerned with the nonhomogeneous equation and the perturbed linear equation. Basic existence and uniqueness theorems are proven for these.

In the fourth chapter a saddle point property is established for the perturbed equation. This involves more accurate determination of the spectrum of the semigroup.

CHAPTER II

EXISTENCE, UNIQUENESS AND THE INFINITESIMAL GENERATOR

§1. Introduction.

In this chapter we consider some partial differential equations with delay, for example

$$(2.1)\quad \frac{\partial u}{\partial t}(x,t) = c\Delta u(x,t) - \int_{-\infty}^{0} g(t-s)\,\Delta u(x,s)\,ds, \quad t \geq 0$$

where u is a scalar function of $x \in R^n$ and $t \in R$. "Δ" denotes the Laplacian with respect to x and g is a scalar function defined on $(-\infty, 0]$ and c is a positive real number.

Let Ω be an open, bounded, connected subset of R^n with boundary $\partial\Omega$. Let $\bar{\Omega}$ be the closure of Ω. Let $\partial\Omega$ be locally Lipschitz, that is, if $x \in \partial\Omega$, there is a neighborhood U_x of x in R^n such that $U_x \cap \partial\Omega$ is the graph of a Lipschitz function. Together with (2.1) we impose the following initial-boundary conditions:

$$u(x,t) = 0 \qquad x \in \partial\Omega, \quad t \geq 0$$

$$u(x,t) = w(x,t) \qquad x \in \bar{\Omega}, \quad t \leq 0$$

where $w(x,t)$ is a given function.

The solutions to (2.1) found here will be solutions in the sense of distributions, or a "weak solution" [15], with the solution $u(x,t)$ defined on a Sobolev space [1].

Let $C_0^\infty(\Omega)$ be the infinitely differentiable functions into R with compact support in Ω, and define $|u|_{H_0^1} = \sqrt{\int_\Omega |\text{grad } u|^2}$. Let $H_0^1(\Omega)$ be the completion of C_0^∞ in the norm $|\cdot|_{H_0^1}$. Then $H_0^1(\Omega)$ is a Hilbert space [15]. The Laplacian is defined on C_0^∞ and can be extended to a closed self-adjoint operator on a dense subset of $H_0^1(\Omega)$. [15]

Now the solution $u(x,t)$ will be considered as an element in a Hilbert space. Let $a > 0$ be a positive number such that $\int_0^\infty e^{2at}|g'(t)|^2 dt < \infty$.

Let $X = L_{2,e^{at}}[-\infty,0;H_0^1(\Omega)]$, be the Hilbert space of functions from $(-\infty,0]$ into $H_0^1(\Omega)$ with inner product

$$\langle f,g\rangle_X = \int_{-\infty}^0 e^{2at}\langle f(t),g(t)\rangle_{H_0^1}\, dt$$

where $\langle\ ,\ \rangle_{H_0^1}$ is the inner product in $H_0^1(\Omega)$.

The space $H_0^1(\Omega)$ is used to correspond to the boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0.$$

Elements in $C_0^\infty(\Omega)$ clearly satisfy this condition, so the condition $u(\cdot,t) \in H_0^1(\Omega)$ corresponds to $u(x,t) = 0$, by the trace theorem [15]. Also let $V = X \times H_0^1(\Omega)$.

W will be used as the space from which the initial condition will be taken. If $(w,p) \in w$, $W \in X$, $p \in H_0^1(\Omega)$, then $u(x,t) = w(x,t)$ for $t < 0$, and $u(x,0) = p(x)$. Since $w \in X$ does not imply w is continuous, both $w(x,t)$ for $t < 0$ and $w(x,0)$ must be specified.

The first question concerning (2.1) and the related initial boundary condition is naturally, "Does a solution exist and is the solution unique?" To show (2.1) has a solution in the sense of distribution under the previously mentioned restrictions, we will use semigroup theory.

A semigroup is a one parameter family of continuous linear operators $T(t)$ on a Banach space E such that [13]

(1) $T(t+s) = T(t)T(s)$ for $t,s \geq 0$.

(2) For any fixed x, $T(t)x$ is a continuous function in t for $t > 0$, in norm on the space E.

(3) $T(0) = I$, the identity operator on E.

The infinitesimal generator of the semigroup $T(t)$ is defined by

$$A(x) = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad x \in D(A),$$

where the domain of $A, D(A)$, is given by

$$D(A) = \{x \mid \lim_{x \to 0^+} \frac{T(t)x - x}{t} \text{ exists}\}.$$

It can be shown that A is a closed linear operator with dense domain and that the semigroup is uinquely determined by the infinitesimal generator [13].

In the next sections a semigroup will be found using solutions of an equation related to (2.1). The infinitesimal generator of this semigroup will be found and its spectrum computed. Then using a special case of the general equation, it will be shown weak solutions of (2.1) are obtained.

§2. The modified equation.

Let E be a Banach space and A the infinitesimal generator of a semigroup on E. The semigroup generated by A will be denoted by e^{At}. Let $a \geq 0$, set $K = L_{2,e^{at}}(-\infty,0;E)$. The Banach space E in applications can be chosen to correspond to boundary conditions, for instance $H_0^1(\Omega)$ will correspond to the boundary condition $u(x) = 0$, $x \in \partial\Omega$.

Now consider the equation

$$\frac{\partial u}{\partial t} = cAu - \int_{-\infty}^{t} g(t-s)Au(s)ds \tag{2.1}$$

with g differentiable

$$\int_0^\infty |g(s)|^2 e^{2as} ds < \infty, \quad \text{and}$$

$$\int_0^\infty |g'(s)|^2 e^{2as} ds < \infty$$

for some $a > 0$. Suppose this equation has a unique solution for any initial condition (w,p), $w \in K$, $p \in E$. Then the variation of constants formula [12] gives

$$u(t) = e^{cAt}p + \frac{1}{c}\int_0^t e^{cA(t-s)}\left(\int_{-\infty}^{s} g(s-s')Au(s')ds'\right)ds$$

for $t > 0$. Integration by parts gives

$$(2.2)\quad u(t) = e^{cAt}p + \frac{1}{c}\int_{-\infty}^{t} g(t-s)u(s)ds$$

$$-\frac{1}{c}e^{cAt}\left(\int_{-\infty}^{0} g(-s')u(s')ds'\right) - \frac{1}{c}\left(\int_0^t e^{cA(t-s)}u(s)ds\right)g(0)$$

$$-\frac{1}{c}\int_0^t e^{cA(t-s)}\left(\int_{-\infty}^{s} g'(s-s')u(s')ds'\right)ds\ .$$

This is the equation that will be studied in the next two sections.

§3. Existence and uniqueness.

First, it will be shown that equation (2.2) has a unique solution for (w,p), $w \in K$, $p \in E$.

Lemma 2.1. Let $w \in L_{2,e^{at}}(-\infty,0;E)$. Then $G(t) = \int_{-\infty}^{0} g(t-s)w(s)ds$ is a continuous function of t for $t \geq 0$.

Proof: For any $t_1, t_2 \geq 0$, $|G(t_1) - G(t_2)|$
$\leq \left|\int_{-\infty}^{0}(g(t_1-s)-g(t_2-s))w(s)ds\right| \leq \left(\int_{-\infty}^{0}|g(t_1-s)-g(t_2-s)|^2 e^{-2as}ds\right)^{\frac{1}{2}}$
$\left(\int_{-\infty}^{0} e^{2as}|w(s)|_E^2\, ds\right)^{\frac{1}{2}}$.

Since $\int_{-\infty}^{0} e^{-2as}(g(t_1-s)-g(t_2-s))^2 ds \to 0$ as $t_1 \to t_2$, $G(t)$ is continuous.

<u>Theorem 2.1</u>. Let E be a Banach space, let A be the infinitesimal generator of a semigroup on E and let $g[0,\infty) \to R$ satisfy

(i) $\int_0^{\infty} [g(t)]^2 e^{2at} dt < \infty$;

(ii) $g'(t)$ exists for $0 \leq t < \infty$; and

(iii) $\int_0^{\infty} [g'(t)]^2 dt < \infty$.

Then the equation

$$u(t) = e^{cAt}p + \frac{1}{c}\int_{-\infty}^{t} g(t-s)u(s)ds \tag{2.2}$$

$$- e^{cAt}\frac{1}{c}\Big[\int_{-\infty}^{0} g(-s)u(s)ds - \frac{1}{c}\int_0^t e^{cA(t-s)}\int_{-\infty}^{0} g'(s-s')$$

$$u(s')ds')ds - \frac{1}{c}\Big[\int_0^t e^{cA(t-s)}u(s)ds\Big]g(0)$$

has a unique solution on $[0,t_1]$ where $t_1 > 0$ is arbitrary.

<u>Proof</u>: Let $C[0,t_1;E]$ be the Banach space of continuous functions from $[0,t_1]$ to E, $0 < t_1 < \infty$ with norm

$$|u|_c = \max_{t\in[0,t_1]} |e^{-kt}u(t)|_E$$

where k is some positive number. Define

$$[M(w,p)v](t) = e^{cAt}p + \frac{1}{c}\int_{-\infty}^{0} g(t-s)w(s)ds$$

$$+ \frac{1}{c}\int_0^t g(t-s)v(s)ds - \frac{1}{c}e^{cAt}\int_{-\infty}^{0} g(-s)w(s)ds$$

$$- \frac{1}{c}\int_0^t e^{cA(t-s)}\left(\int_{-\infty}^{0} g(s-s')w(s')ds\right)ds$$

$$- \frac{1}{c}\int_0^t e^{cA(t-s)}\int_0^s g(s-s')v(s')ds'$$

$$- \frac{1}{c}\left[\int_0^t e^{cA(t-s)}v(s)ds\right]g(0).$$

It will be shown $M(w,p)$ is a contraction for k chosen correctly.

Note that $e^{cAt}(p+\int_{-\infty}^{0} g(-s)w(s)ds)$ is continuous in t and $\int_0^t g(t-s)v(s)ds$ is continuous (since both g and v are continuous). By Lemma 2.1, $\frac{1}{c}\int_{-\infty}^{0} g(t-s)w(s)ds$ is continuous. For the other terms in $M(w,p)$ continuity is shown using the theorem that $\int_0^t e^{cA(t-s)}f(s)ds$ is continuous if f is continuous [12]. Hence, $M(w,p)\colon C[0,t;E] \to C[0,t;E]$.

Next, we have for any $v(t), r(t)$ in $C[0,t;E]$

$$\sup_{t\in[0,t_1]} |e^{-kt}[M(w,p)v(t) - M(w,p)r(t)]$$

$$\leq \sup_{t\in[0,t_1]} e^{-kt}\frac{1}{c}\int_0^t |g(t-s)|\,|v(s) - r(s)|ds$$

$$+ \sup_{t\in[0,t_1]} e^{-kt}\frac{1}{c}\int_0^t |e^{cA(t-s)}|\left(\int_0^s |g(s-s')|\,|v(s')-v(s')|ds'\right)ds$$

$$+ \sup_{t\in[0,t_1]} e^{-kt} \frac{1}{c}|g(0)| \int_0^t |e^{cA(t-s)}| |v(s) - v(s)| ds$$

$$\leq \sup_{t\in[0,t_1]} e^{-kt} \frac{1}{c} \int_0^t e^{ks} e^{-ks} |g(t-s)| |v(s) - r(s)| ds$$

$$+ \sup_{t\in[0,t_1]} e^{-kt} \frac{1}{c} \int_0^t |e^{cA(t-s)}| (\int_0^s |g(s-s')| |v(s')-r(s')| ds') ds$$

$$+ \sup_{t\in[0,t_1]} |g(0)| e^{-kt} \frac{1}{c} \int_0^t e^{ks} e^{-ks} |v(s) - r(s)| ds.$$

Now, $|e^{cAt}| \leq Re^{\alpha t}$ for some $R > 0$ and some α. [13] Using this

$$\sup_{t\in[0,t_1]} e^{-kt} \frac{1}{c} \int_0^t e^{is} e^{-ks} |g(t-s)| |v(s) - r(s)| ds$$

$$+ \sup_{t\in[0,t_1]} e^{-kt} \frac{1}{c} \int_0^t |e^{cA(t-s)}| (\int_0^s |g(s-s')| |v(s')-r(s')| ds') ds$$

$$+ \sup_{t\in[0,t_1]} |g(0)| e^{-kt} \frac{1}{c} \int_0^t e^{ks} |v(s) - r(s)| ds$$

$$\leq \frac{1}{c} e^{-kt} (\int_0^t e^{ks} |v - r|_c ds) \max_{s\in[0,t_1]} |g(s)|$$

$$+ \frac{R}{c} e^{-kt} \int_0^t e^{\alpha s} (\int_0^s \text{Max } |g(s)| e^{ks'} |v - r|_c ds') ds$$

$$+ \sup_{t\in[0,t_1]} (|g(0)| e^{-kt} \frac{1}{c} \int_0^t e^{ks} |v - r|_c ds)$$

$$\leq \left(\frac{\text{Max}|g(s)|, s\in[0,t_1]}{kc} + \frac{\text{Max}|g(s)| s\in[0,t_1]}{c(k+c)} Re^{\alpha t_1} + \frac{1}{k}|g(0)| \frac{1}{c}\right) |v-r|_c.$$

Choosing k large enough so that

$$\max_{s\in[0,t_1]} |g(s)|\left(\frac{1}{c} + \frac{R}{(k+\alpha)c} e^{\alpha t_1}\right) + |g(0)|\frac{1}{c} < k$$

makes $M(w,p)$ a contraction. Therefore $M(w,p)$ has a unique fixed point. Since t_1 can be any positive number, this gives a unique solution to (2.2), for all $t > 0$.

Theorem 2.2. Let $w \in K$, and $p \in E$, K and E the same Banach spaces as in Theorem 2.1. Let $u(t)$ be the solution of (2.2) with initial value (w,p). Then $|u(t)|_E \leq C(t)|(w,p)|_{K\times E}$ where $C(t)$ depends on t but not on (w,p).

Proof: By (2.2) after taking norms

$$|u(t)|_E \leq Re^{\alpha t}|p|_E + \frac{1}{c}\int_{-\infty}^{0} |g(t-s)||w(s)|_E ds$$

$$+ \frac{1}{c}\int_0^t |g(t-s)||u(s)|_E ds + Re^{\alpha t}G|w|_K$$

$$+ \frac{R}{c}\int_0^t e^{\alpha(t-s)}\left(\int_0^s |g'(s-s')||u(s')|_E ds'\right)ds$$

$$+ \frac{R|g(0)|}{c}\int_0^t e^{\alpha(t-s)}(u(s))ds$$

$$\leq Re^{\alpha t}[|p|_E + G|w|_K] + \frac{1}{c}G(t)$$

$$+ \frac{1}{c}\max_{[0,t]}(g(s)|\int_0^t |u(s)|_E ds$$

$$+ \frac{R}{c} \frac{1}{\sqrt{2a}} e^{(\alpha+a)t} G \int_0^t u(s')e^{-(\alpha+a)s'} ds'$$

$$+ \frac{R|g(0)|}{c} \int_0^t e^{\alpha(t-s)} |u(s)|_E ds.$$

Now fix t_1, $0 < t_1 < \infty$. Then on $[0,t_1]$, $e^{\alpha t} \leq e^{\alpha t_1}$. So using Gronwall's inequality on the above equation,

$$\|u(t_1)\|_E \leq Re^{\alpha t}[\|p\|_E + G\|w\|_K] + \frac{1}{c} G(t)$$

$$+ (\exp \int_0^{t_1} [\operatorname*{Max}_{[0,t_1]} g(s)) \frac{1}{c} + \frac{R}{c} \frac{1}{\sqrt{2a}} e^{(\alpha+a)t_1} e^{-(\alpha+a)t}$$

$$+ \frac{R(g(0))}{c} e^{\alpha(t_1-t)} dt]) |p|_E$$

$$\leq c(t) |(w,p)|_{K\times E} .$$

Now define $T(t)$ for $t \geq 0$ by

$$T(t)(w,p) = (u_t,u(t)) \tag{2.3}$$

where u is the solution of (2.2) for the given (w,p) and

$$u_t(s) = w(t-s) \qquad t < -s$$

$$u_t(s) = u(t-s) \qquad t > -s$$

<u>Theorem 2.3.</u> $T(t)$ defined by (2.3) is a strongly continuous semigroup of linear operators on $K \times E$.

<u>Proof:</u> For each $t, T(t)$ is linear since equation (2.2) is linear. $T(t)$ is bounded by Theorem 2.2. $T(t)(w,p)$ is continuous by Theorem 2.1.

So to show $T(t)$ is a strongly continuous semigroup, only $T(t_1)T(t_2) = T(t_1+t_2)$ must be checked. Let $w \in K$, $p \in E$, then

$$u(t_1+t_2) = e^{cA(t_1+t_2)}p + \frac{1}{c}\int_{-\infty}^{t_1+t_2} g(t_1+t_2-s)u(s)ds$$

$$-\frac{1}{c}e^{cA(t_1+t_2-s)}\int_{-\infty}^{0} g\ -s)u(s)ds$$

$$-\frac{1}{c}\int_0^t e^{cA(t-s)}(\int_{-\infty}^{s} g'(s-s')u(s')ds')ds$$

$$-\frac{1}{c}g(0)\int_0^{t_1+t_2} e^{cA(t_1+t_2-s)}u(s)ds$$

$$= e^{cAt_1}(e^{cAt_2}p + \frac{1}{c}\int_{-\infty}^{t_2} g(t_2-s)u(s)ds$$

$$-\frac{1}{c}e^{cA(t_2-s)}\int_{-\infty}^{0} g(-s)u(s)ds$$

$$-\frac{1}{c}(\int_0^{t_2} e^{cA(t_2-s)}(\int_{-\infty}^{s} g'(s-s')u(s')ds')ds$$

$$-\frac{1}{c}[\int_0^{t_2} e^{cA(t_2-s)}u(s)ds]g(0)) + \frac{1}{c}\int_{-\infty}^{t_1+t_2} g(t_1+t_2-s)u(s)ds$$

$$+ -\frac{1}{c}\int_{t_2}^{t_1+t_2} e^{cA(t_1+t_2-s)}(\int_{-\infty}^{s} g'(s-s')u(s')ds')ds$$

$$-\frac{1}{c}\int_{t_2}^{t_1+t_2} e^{cA(t_1+t_2-s)}u(s)ds\,g(0) - e^{cAt_1}\frac{1}{c}\int_{-\infty}^{t_2} g(t_2-s)u(s)ds$$

$$= e^{cAt_1}u(t_2) + \frac{1}{c}\int_{-\infty}^{t_1} g(t_1-s)u_{t_2}(s)ds$$

$$- \frac{1}{c}e^{cAt_1}\int_{-\infty}^{0} g(-s)u_{t_2}(s)ds$$

$$- \frac{1}{c}\int_0^{t_1} e^{cA(t_1-s)}\left(\int_{-\infty}^{s} g'(s-s'')u_{t_2}(s'')ds''\right)ds$$

$$- \frac{1}{c}\left(\int_0^{t_1} e^{cA(t_1-s)}u_{t_2}(s)ds\right)g(0)$$

making the change of variables $-t_2 + s \to s$, and $t_2 + s' \to s'$. So $T(t_1 + t_2)(w,p) = T(t_1)(T(t_2)(w,p))$, and so $T(t)$ is a semigroup.

§4. The infinitesimal generator.

Now the infinitesimal generator of $T(t)$ will be found.

<u>Theorem 2.4</u>. Let $T(t)$ be the semigroup given in Theorem 2.3. Then the infinitesimal generator of $T(t)$ is

$$\mathcal{A}(w,p) = (\dot{w}, A(cp - \int_{-\infty}^{0} g(-s)w(s)ds)) \tag{2.4}$$

where the domain $D(A)$ of $\mathcal{A}$ is given by

$D(\mathcal{A}) = \{(w,p) \mid w, \dot{w} \in K,\ p \in E,\ \lim_{t\to 0} w(t) = p,$ and $(cp - \int_{-\infty}^{0} g(-s)w(s)ds) \in D(A)\}$.

<u>Proof</u>: Let $\mathcal{A}(w,p)$ be the infinitesimal generator of $T(t)$. Then for $(w,p) \in D(\mathcal{A})$

$$\mathcal{A}(w,p) = \lim_{t\to 0}\frac{T(t)(w,p) - (w,p)}{t} = \lim_{t\to 0}\left(\frac{u_t - w}{t}, \frac{u(t) - p}{t}\right).$$

By definition $\lim\limits_{t\to 0} \frac{w_t - w}{t} = \dot{w}$ where $\dot{w}$ is the weak derivative of w [15]. For $\dot{w}$ to exist, $\lim\limits_{t\to 0} w(t)$ must exist. [20]

Since

$$\lim_{t\to 0^+} u(t) = \lim_{t\to 0} [e^{cAt} p + \frac{1}{c}\int_{-\infty}^{t} g(t-s)u(s)ds$$

$$-\frac{1}{c} e^{cAt} \int_{-\infty}^{0} g(-s)u(s)ds - \frac{1}{c} (\int_{0}^{t} e^{cA(t-s)}u(s)ds)g(0)$$

$$-\frac{1}{c}\int_{0}^{t} e^{cA(t-s)} (\int_{-\infty}^{s} g'(s-s')u(s')ds')]ds = p, \quad \lim_{t\to 0} u(t)$$

$$= \lim_{t\to 0} w(t) = p .$$

For the second component of $\frac{1}{t}(T(t)(w,p) - (w,p))$,

$$\frac{1}{t}(u(t) - p) = (e^{cAt}[p + - \frac{1}{c}\int_{-\infty}^{0} g(-s)w(s)ds]$$

$$- (p + - \frac{1}{c}\int_{-\infty}^{0} g(t-s)w(s)ds) + \frac{1}{c}\int_{-\infty}^{0} g(t-s)w(s)ds$$

$$+ \frac{1}{c}\int_{0}^{t} g(t-s)u(s)ds - \frac{1}{c} (\int_{0}^{t} e^{cA(t-s)}u(s)ds)g(0)$$

$$- \frac{1}{c}\int_{0}^{t} e^{cA(t-s)} (\int_{-\infty}^{0} g'(s-s')w(s')ds')ds$$

$$- \frac{1}{c}\int_{0}^{t} e^{cA(t-s)} (\int_{0}^{s} (s-s')w(s')ds')ds) \frac{1}{t}$$

$$= \frac{1}{t}[e^{cA(t-s)} (p - \frac{1}{c}\int_{-\infty}^{0} g(-s)w(s)ds) - (p - \frac{1}{c}\int_{-\infty}^{0} g(-s)w(s)ds)]$$

$$\frac{1}{ct}(\int_{-\infty}^{0} [g(t-s) - g(-s)]w(s)ds - \frac{1}{c}\int_{0}^{t} e^{cA(t-s)} (\int_{-\infty}^{0} g'(s-s')w(s')ds')$$

$$+ \frac{1}{c} \left[\int_0^t (g(t-s) - g(0))u(s)\,ds\right]\frac{1}{t}$$

$$+ \frac{1}{c} \left(\int_0^t [1 - e^{cA(t-s)}]g(0)p\ ds\right)\frac{1}{t}$$

$$+ \frac{1}{c} \left[\int_0^t e^{cA(t-s)}\ g(0)[p\,[\ u(s)\ ]ds\right]\frac{1}{t}$$

$$+ - \frac{1}{c} \left(\int_0^t e^{cA(t-s)} \left(\int_0^s g'(s-s')u(s')\,ds'\right)ds\right)\frac{1}{t}$$

We will denote these terms by

(i) $\frac{1}{t} \left(e^{cAt}(p - \frac{1}{c} \int_{-\infty}^{0} g(-s)w(s)\,ds) - (p - \frac{1}{c} \int_{-\infty}^{0} g(-s)w(s)\,ds\right)$

(ii) $\frac{1}{ct} \int_{-\infty}^{0} [g(t-s) - g(-s)]w(s)\,ds$

$$- \frac{1}{c} \int_0^t e^{cA(t-s)} \int_{-\infty}^{0} g'(s-s')w(s')\,ds$$

(iii) $\frac{1}{c} \left[\int_0^t (g(t-s) - g(0))u(s)\,ds\right]\frac{1}{t}$

(iv) $\frac{1}{c} \left(\int_0^t [1 - e^{cA(t-s)}]g(0)p\ ds\right)\frac{1}{t}$

(v) $\int_0^t e^{cA(t-s)}\ g(0)[p - u(s)\,ds]\frac{1}{t}$

(vi) $\frac{1}{c} \left[\int_0^t e^{cA(t-s)} \left(\int_0^s g'(s-s')u(s')\,ds'\right)ds\right]\frac{1}{t}$

each of these terms will be considered separately.

As $t \to 0$, term (vi)

$$\frac{1}{ct} \left|\int_0^t e^{cA(t-s)} \left(\int_0^s g'(s-s')u(s')\,ds'\right)ds\right|$$

$$\leq \frac{1}{ct}\ t\ Re^{\alpha t}\left(\int_0^t g'(s)^2\ ds\right)^{\frac{1}{2}}\left(\int_0^t u(s)^2\ ds\right)^{\frac{1}{2}}$$

$$= \frac{r}{c}\ e^{\alpha t}\left(\int_0^t g'(s)^2 ds\right)^{\frac{1}{2}}\left(\int_0^t u(s)^2 ds\right)^{\frac{1}{2}} \to 0.$$

For term (v), since $u(s)$ is continuous for $s \geq 0$, for any $\epsilon > 0$, there exist δ such that

$$0 \leq s < \delta \Rightarrow |p - u(s)| < \epsilon.$$

And so

$$\frac{1}{t}\left|\frac{1}{c}\left[\left(\int_0^t e^{cA(t-s)}[p-u(s)]ds\right) \cdot g(0)\right| \leq t \cdot \frac{1}{c} Re^{\alpha t} g(0)\epsilon\right.$$

$$\leq Re^{\alpha\delta} g(0)\epsilon \to 0 \quad \text{as} \quad t \to 0, \quad \text{since} \quad \epsilon \quad \text{is arbitrary.}$$

For the term (vi)

$$\frac{1}{ct}\left(\int_0^t [1 - e^{cA^{\circ}t-s)}]p\, ds\right)g(0)$$

$$= \frac{1}{t} \cdot \frac{1}{c} g(0) \int_0^t (1 - e^{cAs})p\, ds \left|\frac{1}{t} \cdot \frac{1}{c} \cdot g(0) \int_0^t (1 - e^{cAs})p\, ds\right|$$

$$\leq \frac{1}{t} \cdot c \cdot g(0) \cdot t \sup_{0 \leq s \leq t} |e^{cAs} p - p|.$$

Since $e^{cAs}p$ is a continuous function of s, this $\to 0$ as $t \to 0$.

For the term (iii)

$$\left|\frac{1}{t} \int_0^t (g(t-s) - g(0))u(s)ds\right|$$

$$\leq \max_{s\in[0,t]} (|g(s) - g(0)|\,|u(s)|)\, \frac{1}{t} \cdot t \to 0$$

as $s \to 0$, since g and u are continuous for $s \geq 0$.

Next, for the term (ii),

$$\left[\frac{1}{c} \int_{-\infty}^0 [g(t-s) - g(-s)]w(s)ds\right.$$

$$\left. - \frac{1}{c} \int_0^t e^{cA(t-s)} \left(\int_{-\infty}^0 g'(s-s')w(s')ds'\right)ds\right]\frac{1}{t}$$

$$= \frac{1}{c} \int_{-\infty}^0 [g(t-s) - g(-s)]\frac{1}{t}\, w(s)ds - \frac{1}{c} \int_{-\infty}^0 g'(-s')w(s')ds'$$

$$- \frac{1}{c}\int_0^t\int_{-\infty}^0 (g'(s-s') - g(-s')w(s')ds')ds\ \frac{1}{t}$$

$$- \frac{1}{c}\int_0^t (e^{cA(t-s)} - 1)(\int_{-\infty}^0 g'(s-s')w(s')ds'ds$$

$$= \frac{1}{c}\int_{-\infty}^0 [g(t-s) - g(-s)]\frac{1}{t}\ w(s)ds - \frac{1}{c}\int_{-\infty}^0 g'(-s')w(s')ds$$

$$- \frac{1}{ct}\int_0^t (G(s) - g(0)|ds$$

$$- \frac{1}{ct}\int_0^t (e^{cA(t-s)} - 1)G(s)ds.$$

Now,

$$\lim_{t\to 0} \frac{1}{c}\int_{-\infty}^0 \frac{g(t-s) - g(-s)}{t}\ w(s)ds = \frac{1}{c}\int_{-\infty}^0 g'(-s)w(s)ds$$

so

$$\lim_{t\to 0} \frac{1}{c}\int_{-\infty}^0 \frac{g(t-s) - g(-s)}{t}\ w(s)ds - \frac{1}{c}\int_{-\infty}^0 g'(-s)w(s)ds = 0.$$

Also,

$$\lim_{t\to 0} \frac{1}{ct}\int_0^t (G(s) - G(0))ds = 0,$$

and

$$\lim_{t\to 0} \frac{1}{ct}\int_0^t (e^{cA(t-s)} - 1)G(s)ds = 0,$$

since G is continuous. Therefore

$$\lim_{t\to 0} \frac{u_t - p}{t} = \lim_{t\to 0} (e^{cAt}[p - \frac{1}{c}\int_{-\infty}^0 g(-s)w(s)ds]$$
$$- p - \frac{1}{c}\int_{-\infty}^0 g(-s)u(s)ds)\frac{1}{t}$$

and the existence of either limit implies the existence of the other. By semigroup theory, the right hand side equals

$$A(cp - \int_{-\infty}^0 g(-s)w(s)ds),$$

and so $(w,p) \in D(A)$ iff $w,\dot{w} \in K$, $\lim_{t\to 0} w(t) = p$, and

$$A(cp - \int_{-\infty}^0 g(-s)w(s)ds) \in E.$$

In this case

$$\mathcal{A}(w,p) = (\dot{w}, A(cp - \int_{-\infty}^{0} w(s)g(-s)ds)).$$

Now the spectrum of $\mathcal{A}$ will be found. First a few definitions are needed.

Let $R(\lambda,V) = (V - \lambda I)^{-1}$, where λ is a complex number, and V is a linear, not necessarily continuous operator on E. Let $\rho(V) = \{\lambda \in \mathbb{C} \mid R(\lambda,V)$ exists, is continuous, and domain = $E\}$ and let $\sigma(V)$ = complement in $\mathbb{C}$ of $\rho(V)$. $\sigma(V)$ is divided into three subsets.

$P\sigma(V) = \{\lambda \in \mathbb{C} \mid R(\lambda,V)$ does not exist$\}$.

$C\sigma(V) = \{\lambda \in \mathbb{C} \mid R(\lambda,V)$ exists with dense domain, but is not continuous$\}$.

$R\sigma(V) = \{\lambda \in \mathbb{C} \mid R(\lambda,V)$ exists, but its domain is not dense$\}$.

$\rho(V)$ is called the resolvent of V, $\sigma(V)$ the spectrum of V and $P\sigma(V)$, $\mathbb{C}\sigma(V)$ and $R\sigma(V)$ the point, continuous, and residual spectrum respectively.

We have the following lemma

<u>Lemma 2.2</u>. For $\mathrm{Re}(\lambda) > -a$, define

$$M_{\lambda}(u)(t) = \int_{0}^{t} e^{\lambda(t-s)} u(s)ds.$$

Then $M_\lambda : K \to K$ is a bounded linear operator and $|M_\lambda| \le \frac{1}{\lambda + a}$.

Proof: Using the Cauchy-Schwarts inequality and Fubini's theorem, we have

$$\left| \int_{-\infty}^{0} e^{2at} \left(\int_{0}^{t} e^{\lambda(t-s)} u(s)\,ds \right)^2 dt \right|_E$$

$$\le \int_{-\infty}^{0} e^{2at} \left(\int_{t}^{0} e^{\lambda(t-s)} |u(s)|_E ds \right)^2 dt$$

$$\le \int_{-\infty}^{0} \left(\int_{t}^{0} e^{(\lambda+a)(t-s)} e^{as} |u(s)|_E ds \right)^2 dt$$

$$\le \int_{-\infty}^{0} \left(\int_{t}^{0} e^{(\lambda+a)(t-s)} ds \right) \cdot \left(\int_{t}^{0} e^{(\lambda+a)(t-s)} e^{2as} |u(s)|_E^2 ds \right) dt$$

$$\le \frac{1}{\lambda + a} \int_{-\infty}^{0} \left(\int_{t}^{0} e^{(\lambda+a)(t-s)} e^{2as} |u(s)|_E^2 ds \right) dt$$

$$= \frac{1}{\lambda + a} \int_{-\infty}^{0} \left(\int_{-\infty}^{t} e^{(\lambda+a)(t-s)} e^{2as} |u(s)|_E^2 dt \right) ds$$

$$= \frac{1}{\lambda + a)^2} \int_{-\infty}^{0} e^{2as} |u(s)|^2 ds = \frac{1}{(\lambda + a)^2} |u|_K^2 .$$

So $|M_\lambda| \le \frac{1}{\lambda + a}$.

The next theorem describes the spectrum of $\mathcal{Q}$ in terms of the spectrum of A.

Theorem 2.5. Let $\mathcal{Q}$ be the operator in Theorem 2.4. Let

$$M_\lambda u(t) = \int_{0}^{t} e^{\lambda(t-s)} u(s)\,ds$$

$$\tilde{g}(t) = \int_{0}^{\infty} e^{-\lambda} g(s)\,ds$$

$$\mu = \frac{\lambda}{c - \tilde{g}(\lambda)}, \quad J_\lambda = R(\lambda, A)A$$

$$p = \frac{\mu}{\lambda} R(\mu, A)v + \frac{\mu}{\lambda} \int_{-\infty}^{0} g(-s) M_\lambda J_\lambda u(s)\,ds$$

where $v \in E$, $u \in K$. Let

$$\sigma_1 = \{\lambda \mid \mathrm{Re}\ \lambda \leq -a\}$$

$$\sigma_2 = \{\lambda \mid \mu \in \sigma(A)\}$$

$$\sigma_3 = \{\lambda \mid c - \tilde{g}(\lambda) = 0\}.$$

Then

$$\sigma(\mathcal{A}) = \sigma_1 \cup \sigma_2 \cup \sigma_3.$$

Also, if $\lambda \in \rho(\mathcal{A})$, then $R(\lambda, \mathcal{A}) = (M_\lambda u + e^{\lambda t} p, p)$.

__Proof__: First, $\tilde{g}(\lambda)$ exists for $\mathrm{Re}(\lambda) > -a$ and

$$\left| \int_0^\infty e^{-\lambda s} g(s)\,ds \right| \leq \left(\int_0^\infty e^{(-\mathrm{Re}\,\lambda + -a)2s}\,ds \right)^{\frac{1}{2}}$$

$$\left| \int_0^\infty e^{2as} g^2(s)\,ds \right|^{\frac{1}{2}} \leq \frac{1}{\sqrt{(\mathrm{Re}\ \lambda + a)}} \cdot G$$

Now, suppose $(\mathcal{A} - \lambda I)(w, p) = (u, v)$. Then $\dot{w} - \lambda w = u$. So

$$w = \int_0^t e^{\lambda(t-s)} u(s)\,ds + e^{\lambda t} p,$$

$$\mu = \frac{\lambda}{c - \tilde{g}(\lambda)}, \quad J_\lambda = R(\lambda, A)A$$

$$p = \frac{\mu}{\lambda} R(\mu, A)v + \frac{\mu}{\lambda} \int_{-\infty}^{0} g(-s) M_\lambda J_\lambda u(s)ds$$

where $v \in E$, $u \in K$. Let

$$\sigma_1 = \{\lambda | \mathrm{Re}\ \lambda \leq -a\}$$

$$\sigma_2 = \{\lambda | \mu \in \sigma(A)\}$$

$$\sigma_3 = \{\lambda | c - \tilde{g}(\lambda) = 0\}.$$

Then

$$\sigma(\mathcal{A}) = \sigma_1 \cup \sigma_2 \cup \sigma_3.$$

Also, if $\lambda \in \rho(\mathcal{A})$, then $R(\lambda, \mathcal{A}) = (M_\lambda u + e^{\lambda t}p, p)$.

<u>Proof</u>: First, $\tilde{g}(\lambda)$ exists for $\mathrm{Re}(\lambda) > -a$ and

$$\left|\int_0^\infty e^{-\lambda s} g(s)ds\right| \leq \left(\int_0^\infty e^{(-\mathrm{Re}\lambda + -a)2s}\, ds\right)^{\frac{1}{2}}$$

$$\left|\int_0^\infty e^{2as} g^2(s)ds\right|^{\frac{1}{2}} \leq \frac{1}{\sqrt{(\mathrm{Re}\ \lambda + a)}} \cdot G$$

Now, suppose $(\mathcal{A} - \lambda I)(w, p) = (u, v)$. Then $\dot{w} - \lambda w = u$.

So

$$w = \int_0^t e^{\lambda(t-s)} u(s)ds + e^{\lambda t} p,$$

since $\lim_{t\to 0} w(t) = p$. Substituting this for w in the equation gives

$$cAp - \int_{-\infty}^{0} g(-s)\left(\int_{0}^{s} e^{\lambda(s-s')}Au(s')ds'\right)ds - \left(\int_{0}^{\infty} e^{-\lambda s}g(s)ds\right)Ap - \lambda p = v.$$

So

$$[c-\tilde{g}(\lambda)]Ap - \lambda p = v + \int_{-\infty}^{0} g(-s)\,(M_\lambda Au)(s)ds\ .$$

If $c - \tilde{g}(\lambda) \neq 0$, dividing by $c - \tilde{g}(\lambda)$ gives

$$Ap - \frac{\lambda}{c-\tilde{g}(\lambda)}\,p = \frac{v}{c-\tilde{g}(\lambda)} + \left(\int_{-\infty}^{0} g(-s)\,(M_\lambda Au)(s)ds\right)\frac{1}{c-\tilde{g}(\lambda)}\ .$$

If $\dfrac{\lambda}{c-\tilde{g}(\lambda)} = \mu \in \rho(A)$, then

$$p = (R(\mu,A)v)\,\frac{\mu}{\lambda} + \frac{\mu}{\lambda}\int_{-\infty}^{0} g(-s)M_\lambda(R(\mu,A)Au)(s)ds\ .$$

If $\dfrac{\lambda}{c-\tilde{g}(\lambda)} \in \sigma(A)$, then p will not be a continuous function of (u,v) where $(u,v) \in K \times E$. Also if $\mathrm{Re}(\lambda) \le -a$, M_λ is not a bounded linear operator, so again $\lambda \in \sigma(\mathcal{A})$.

If $c - \tilde{g}(\lambda) = 0$, then the equation for p becomes

$$-\lambda p = v + \int_{-\infty}^{0} g(-s)\,(M_\lambda Au)(s)ds\ .$$

So

$$p = -\frac{v}{\lambda} - \frac{1}{\lambda}\int_{-\infty}^{0} g(-s)\,(M_\lambda Au)(s)ds\,.$$

$$= -\frac{v}{\lambda} - \frac{1}{\lambda}\,A\int_{-\infty}^{0} g(-s)\,(M_\lambda u)(s)ds$$

Since A is an infinitesimal generator it has dense domain and so $R(\lambda,\mathcal{A})$ is defined here with dense domain, but is not continuous. So $\lambda \in C\sigma(\mathcal{A})$.

Now suppose $\mu \in P\sigma(\mathcal{A})$, and $\mathrm{Re}(\lambda) > -a$. Then if $(A-\mu I)q = 0$, $(\mathcal{A}-\lambda I)(e^{\lambda t}q, q) = (0, C(A-\mu I)q) = (0,0)$, so $\lambda \in P\sigma(\mathcal{A})$.

Finally, for $\mathrm{Re}\,\lambda \leq -a$, since functions vanishing for $s \leq t_1, t_1$ variable are dense in K, it can be shown $\lambda \in C\sigma(\mathcal{A})$. Since this is not actually used in what follows, details won't be given.

§5. Decomposing W into subspaces invariant under $\mathcal{A}$.

In this section, it will be shown W can be decomposed into subspaces invariant under $\mathcal{A}$ for the case where $A =$ the Laplacian Δ, and $H_0^1(\Omega) = E$. To do this, several definitions are needed.

Let E be a Banach space and V be a linear operator on E with domain $D(V)$. Let the null space of V be denoted by $\mathcal{N}(V)$ and the range of V by $\mathcal{R}(V)$. Also let λ be a complex number, and let the generalized eigenspace of λ, denoted by $\mathcal{M}_\lambda(V)$, be $\{x \mid x \in E,\ (V-\lambda I)^k x = 0$ for some $k \geq 1$, k an integer$\}$.

A point $\lambda \in P\sigma(V)$ is said to be normal [10] if

1. $\mathcal{M}_\lambda(V)$ is finite dimensional.
2. $\mathcal{M}_\lambda(V) = \mathcal{N}(V-\lambda I)^k$ for some integer k.
3. $E = \mathcal{M}_\lambda(V) \oplus \mathcal{N}(V-KI)^k$.

The set of all normal eigenvalues of V will be denoted by $N\sigma(V)$.

The decomposition of W will depend on the normal eigenvalues of $\mathcal{A}$, the operator defined in Theorem 2.4. So it must be shown that $\mathcal{A}$ has normal eigenvalues. This is done using the theory of Banach space valued functions of a complex variable.

Let E be a Banach space. A function $f: C \to E$ is analytic in an open subset $G \subset C$ if $\lim_{\mu \to \lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda}$ exists at all points $\lambda \in G$ [24].

A function f has a pole at a point $\lambda \in C$ if f is analytic in a deleted neighborhood of λ and $\lim_{\mu \to \lambda} (\mu - \lambda)^n f(\mu)$ exists for some integer $n \geq 1$ [24].

<u>Lemma 2.3</u>. Let A and $\mathcal{A}$ be the same operators as in Theorem 2.4. Let $\mu \in N\sigma(A)$, and a be as in Theorem 2.4. Suppose $\operatorname{Re} \lambda > -a$. Then $\lambda \in N\sigma(\mathcal{A})$.

<u>Proof</u>: By Theorem 2.5,

$$R(\lambda, \mathcal{A})(u,v) = (M_\lambda u + e^{\lambda t} p_\lambda, p_\lambda)$$

where

$$p_\lambda = \frac{\mu}{\lambda} R(\mu, A)v + \frac{\mu}{\lambda} \int_{-\infty}^{0} g(-s)(M_\lambda AR(\mu, A)u)(s)ds.$$

Now if $\mu_1 \in N\sigma(A)$, then $R(\mu,A)$ has a pole at μ_1 [24]. Since μ is an analytic function of λ and $R(\mu,A)$ has a pole at μ_1 as a function of μ, $R(\mu,A)$ as a function of λ has a pole at λ_1 [24], where λ_1 is a number such that

$$\frac{\lambda_1}{c - \tilde{g}(\lambda_1)} = \mu_1 .$$

Thus, for some n,

$$\lim_{\lambda \to \lambda_2} R(\mu,A)(\lambda - \lambda_1)^n$$

exists. Also, if we let $J_\lambda = AR(\mu,A)$, then

$$J_\lambda = AR(\mu, A) = I + \mu R(\mu, A),$$

and so

$$\lim_{\lambda \to \lambda_1} (\lambda - \lambda_1)^n J_\lambda = \lim_{\lambda \to \lambda_1} (\lambda - \lambda_1)^n [I + \mu R(\mu, A)]$$

$$= \lim_{\lambda \to \lambda_1} (\lambda - \lambda_1)^n R(\mu, A) .$$

Thus J_λ has a pole at λ_1.

Now

$$\lim_{\lambda \to \lambda_1} p_\lambda (\lambda - \lambda_1)^n = \lim_{\lambda \to \lambda_1} \frac{\mu}{\lambda} R(\mu,A)(\lambda - \lambda_1)^n$$

$$+ \frac{\mu_1}{\lambda_1} \int_{-\infty}^{0} g(-s) M_{\lambda_1} (\lim_{\lambda \to \lambda_1} (\lambda - \lambda_1)^n J_\lambda)(w)(s)\,ds$$

and since the limit on the right exists, p_λ has a pole at λ_1. Now

$$\lim_{\lambda\to\lambda_1} (\lambda - \lambda_1)^n R(\lambda, \mathcal{A})(w,p)$$

$$= \lim_{\lambda\to\lambda_1} ((\lambda - \lambda_1)^n M_\lambda w + e^{\lambda t}(\lambda - \lambda_1)^n p_\lambda, (\lambda - \lambda_1)^n p_\lambda)$$

and since the right-hand side has a limit, the limit on the left exists, and so $R(\lambda,\mathcal{A})$ has a pole at λ_1. Since $R(\lambda,\mathcal{A})$ has a pole at λ_1, $\lambda_1 \in N\sigma(\mathcal{A})$ [24].

Now consider the case where A = the Laplacian Δ, and $H_0^1(\Omega) = E$. This is done not only to correspond to the models in Chapter 1, but also to make E a Hilbert space and A self-adjoint. These properties will be used in the following theorems.

For the Laplacian, if Ω is an open, bounded region with boundary, then Δ has only normal eigenvalues, which consist of negative real numbers [16]. Thus by Theorem 2.5, if $\mathcal{A}$ is the operator defined in Theorem 2.4, for the case where $E = H_0^1(\Omega)$ and $A = \Delta$ and λ is a complex number, then $\lambda \in \sigma(\mathcal{A})$ if and only if one of the three following conditions are satisfied:

1. $\operatorname{Re} \lambda \leq -a$
2. $\dfrac{\lambda}{c - \tilde{g}(\lambda)} \in \sigma(\Delta)$
3. $c - \tilde{g}(\lambda) = 0.$

Now for the Laplacian on $H_0^1(\Omega)$, $\sigma(A) = N\sigma(\Delta)$. So for $A = \Delta$ and $E = H_0^1(\Omega)$, Lemma 2.3 gives if

$$\frac{\lambda}{c - \tilde{g}(\lambda)} \in \sigma(\Delta)$$

then $\lambda \in N\sigma(\mathcal{A})$.

Now define

$$|u|_{H^{-1}} = \sup \{ \int uv \, , \, |v|_{H_0^1} = 1\}$$

for u an infinitely differentiable function with compact support in Ω. Let H^{-1} be the completion of this space in the norm $| \ |_{H^{-1}}$. Also define

$$< a,b >_H = \int_\Omega ab \, .$$

Then H is the dual space of H_0^1 using the duality pairing $< \ , \ >_H$. Also, Δ is a continuous linear operator from H_0^1 to H^{-1} [ ]. This space is used to define a bilinear form on W and it will be shown later that if $u(t)$ is a solution of (2.2) with $E = H_0^1(\Omega)$ and $A = \Delta$, then for $t \geq 0$, $\dot{u}(t)$ is in $H^{-1}(\Omega)$.

Recall the definition of W. In this case,

$$W = L_{2,e^{at}}(-\infty, 0 ; H_0^1(\Omega)) \times H_0^1(\Omega) \, .$$

Now a symmetric bilinear form will be defined on W by

$$(2.5) \qquad < (w,p), (u,v) >_{\mathcal{A}} = C < p,v >_H$$
$$- \int_{-\infty}^0 (\int_s^0 < w(s-s'), \Delta u(s') >_H ds') g(-s) ds \, .$$

Since Δ is a continuous linear operator from H_0^1 into H^{-1}, this is defined.

This bilinear form is similar to the one defined by Jack Hale for ordinary delay differential equations [11], and is used for a similar purpose, to decompose W into subspaces which are invariant for (2.1).

First, it will be shown that $\mathcal{A}$ is "self-adjoint" with respect to the bilinear form (2.5).

<u>Lemma 2.4</u>. Let $\mathcal{A}$ be the operator defined in Theorem 2.4 for the case where $E = H_0^1(\Omega)$ and $A = \Delta$. Let $< \ , \ >_{\mathcal{A}}$ be the bilinear form defined by (2.5). Let w and u be elements of $L_{2,e^{at}}(-\infty, 0 ; H_0^1(\Omega))$, where a is as in Theorem 2.4, let p and q be elements of $H_0^1(\Omega)$, let $(w,p) \in D(\mathcal{A})$ and $(u,q) \in D(\mathcal{A})$. Then

$$< \mathcal{A}(w,p), (u,v) > = < (w,p), \mathcal{A}(u,v) > .$$

<u>**Proof:**</u>

$$< \mathcal{A}(w,p), (u,v) >_{\mathcal{A}} = < (\dot{w}, c\,\Delta p - \int_{-\infty}^{0} g(-s) w(s)\,ds), (u,v) >_{\mathcal{A}}$$

$$= < c\,\Delta p - \int_{-\infty}^{0} g(-s) A\, w(s)\,ds \ , v >_H$$

$$- \int_{-\infty}^{0} \left(\int_{s}^{0} < \dot{w}(s-s') \ , \Delta u(s') >_H (-s)\,ds' \right) ds$$

$$= < c\,\Delta p, v >_H - \int_{-\infty}^{0} g(-s) < \Delta w(s) \ , v >_H ds$$

$$- \int_{-\infty}^{0} \left(\int_{s}^{0} < \dot{w}(s-s'), \Delta u(s') >_H g(-s)\,ds' \right) ds .$$

Since the Laplacian is self-adjoint in the $\langle\ ,\ \rangle_H$ inner product, and since

$$\frac{d}{ds'}\langle \Delta w(s-s'),u(s')\rangle_H = -\langle \Delta\dot{w}(s-s'),u(s')\rangle_H + \langle \Delta w(s-s'),\dot{u}(s')\rangle_H$$

we have

$$\langle c\Delta p,v\rangle_H = \langle p,c\Delta v\rangle_H$$

$$-\int_{-\infty}^{0} f(-s)\ \langle \Delta w(s),v\rangle_H ds = \int_{-\infty}^{0} g(-s)\langle w(s),\Delta v\rangle_H ds.$$

Integration gives

$$\langle \Delta w(0),u(s)\rangle_H - \langle \Delta w(s),u(0)\rangle_H = -\int_0^s \langle \Delta\dot{w}(s-s'),u(s')\rangle_H ds'$$

$$+\int_0^s \langle \Delta w(s-s'),\dot{u}(s')\rangle_H$$

where $w(0) = \lim_{t\to 0^-} w(t)$ and $u(0) = \lim_{t\to 0^-} u(t)$. Since (w,p) and (u,q) are in $D(\mathcal{A})$, Theorem 2.5 gives $w(0) = p$ and $u(0) = q$. This gives

$$\langle \Delta p,u(s)\rangle_H - \langle \Delta w(s),q\rangle_H =$$

$$-\int_0^s \langle \Delta\dot{w}(s-s'),u(s')\rangle_H ds' + \int_0^s \langle \Delta w(s-s'),\dot{u}(s')\rangle_H.$$

Using this,

$$< c\Delta p, v >_H - \int_{-\infty}^{0} g(-s) < \Delta w(s), v >_H ds$$

$$- \int_{-\infty}^{0} \left(\int_{s}^{0} < \dot{w}(s-s'), \Delta u(s') >_H g(-s) ds' \right) ds$$

$$= < p, c\Delta v >_H - \int_{-\infty}^{0} g(-s) < \Delta w(s), v >_H ds$$

$$+ \int_{-\infty}^{0} g(-s) < \Delta w(s), v >_H ds - \int_{-\infty}^{0} g(-s) < p, \Delta u(s) >_H ds$$

$$- \int_{-\infty}^{0} \left(\int_{s}^{0} < w(s-s'), \Delta \dot{u}(s') >_H g(-s) ds' \right) ds .$$

Making the change of variables $s'' = s - s'$ in the last integral, we have

$$< \mathcal{A}(w,p), (u,v) >_{\mathcal{A}} = < p, c\Delta v >_H - \int_{-\infty}^{0} g(-s) < p, \Delta u(s) >_H ds$$

$$- \int_{-\infty}^{0} \left(\int_{s}^{0} < \dot{u}(s-s''), \Delta w(s'') > ds'' \right) g(-s) ds$$

$$= < (w,p), \mathcal{A}(u,v) >_{\mathcal{A}} .$$

This result is used in later calculations.

Lemma 2.5. Let $\mathcal{A}$ be the operator defined by (2.4) with $E = H_0^1(\Omega)$ and $A = \Delta$ and let k be an integer ≥ 1. Then a necessary and sufficient condition that the equation

$$(\mathcal{A} - \lambda I)^k (w,p) = (u,v)$$

has a solution for some given (u,v) is that

$$(2.6) \qquad < (u,v), (s,q) >_{\mathcal{A}} = 0$$

for all $(s,q) \in \mathcal{N}(\mathcal{A} - \lambda I)^k$.

Proof: It is easy to show (2.6) is necessary. If $(\mathcal{A}-\lambda I)^k(w,p) = (u,v)$, then using Lemma 2.4,

$$< \mathcal{A}-\lambda I)^k(w,p),(s,q) >_{\mathcal{A}} =$$

$$< (w,p),(\mathcal{A}-\lambda I)^k(s,q) >_{\mathcal{A}} = <(w,p),(0,0)>_{\mathcal{A}} = 0$$

for $(s,q) \in \eta((\mathcal{A}-\lambda I)^k)$.

To show the condition (2.6) is sufficient, a characterization of $\eta((\mathcal{A}-\lambda I)^k)$ is needed.
If $(\mathcal{A}-\lambda I)^k(w,p) = 0$, then $(\frac{d}{dt}-\lambda I)^k w = 0$. So for some $\lambda_{j+1} \in H_0^1(\Omega)$,

$$w = \sum_{j=0}^{k-1} \gamma_{j+1} \frac{e^{\lambda t}}{j!} t^j .$$

Also, for $0 \le i \le k$

$$(\mathcal{A}-\lambda I)^i(w,p)$$

$$= ((\frac{d}{dt}-\lambda I)^i \sum_{j=0}^{k-1} \gamma_{j+1} \frac{e^{\lambda t}}{j!} t^j , \lim_{t\to 0^-} (\frac{d}{dt}-\lambda I) \sum_{j=0}^{k-1} \gamma_{j+1} \frac{e^{\lambda t}}{j!} t^j))$$

$$= \left(\sum_{j=0}^{k-i-1} \gamma_{j+i+1} \frac{e^{\lambda t}}{j!} t^j , \gamma_{i+1} \right) .$$

Since

$$(\mathcal{A}-\lambda I)(\mathcal{A}-\lambda I)^i(w,p) = (\mathcal{A}-\lambda I)^{i+1}(w,p) ,$$

this gives

$$(\mathcal{A}-\lambda I)\left(\sum_{j=0}^{k-i-1} \gamma_{j+i+1} \frac{e^{\lambda t}}{j!} t^j , \gamma_{i+1} \right) = \left(\sum_{j=0}^{k-i-2} \gamma_{j+i+2} \frac{e^{\lambda t}}{j!} t^j , \gamma_{i+2} \right)$$

which gives, using (2.4),

<u>Proof:</u> It is easy to show (2.6) is necessary. If $(\mathcal{A}-\lambda I)^k(w,p) = (u,v)$, then using Lemma 2.4,

$$< \mathcal{A}-\lambda I)^k(w,p),(s,q) >_{\mathcal{A}} =$$

$$< (w,p),(\mathcal{A}-\lambda I)^k(s,q) >_{\mathcal{A}} = <(w,p),(0,0)>_{\mathcal{A}} = 0$$

for $(s,q) \in \mathcal{N}((\mathcal{A}-\lambda I)^k)$.

To show the condition (2.6) is sufficient, a characterization of $\mathcal{N}((\mathcal{A}-\lambda I)^k)$ is needed. If $(\mathcal{A}-\lambda I)^k(w,p) = 0$, then $(\frac{d}{dt}-\lambda I)^k w = 0$. So for some $\lambda_{j+1} \in H_0^1(\Omega)$,

$$w = \sum_{j=0}^{k-1} \gamma_{j+1} \frac{e^{\lambda t}}{j!} t^j .$$

Also, for $0 \leq i \leq k$

$$(\mathcal{A}-\lambda I)^i(w,p)$$

$$= ((\frac{d}{dt}-\lambda I)^i \sum_{j=0}^{k-1} \gamma_{j+1} \frac{e^{\lambda t}}{j!} t^j , \lim_{t \to 0^-} (\frac{d}{dt}-\lambda I) \sum_{j=0}^{k-1} \gamma_{j+1} \frac{e^{\lambda t}}{j!} t^j))$$

$$= \left(\sum_{j=0}^{k-i-1} \gamma_{j+i+1} \frac{e^{\lambda t}}{j!} t^j , \gamma_{i+1} \right) .$$

Since

$$(\mathcal{A}-\lambda I)(\mathcal{A}-\lambda I)^i(w,p) = (\mathcal{A}-\lambda I)^{i+1}(w,p) ,$$

this gives

$$(\mathcal{A}-\lambda I)\left(\sum_{j=0}^{k-i-1} \gamma_{j+i+1} \frac{e^{\lambda t}}{j!} t^j , \gamma_{i+1} \right) = \left(\sum_{j=0}^{k-i-2} \gamma_{j+i+2} \frac{e^{\lambda t}}{j!} t^j , \gamma_{i+2} \right)$$

which gives, using (2.4),

$$c\,\Delta\gamma_{i+1} - \int_{-\infty}^{0} g(-s) \sum_{j=0}^{k-i-1} \Delta\gamma_{k+i+1} \frac{e^{\lambda s}}{j!} s^j ds\; \lambda\gamma_{i+1} = \gamma_{i+2}$$

for $0 \leq i \leq k-2$, and

$$c\,\Delta\gamma_k - \left(\int_{-\infty}^{0} g(-s)e^{\lambda s}ds\right)\Delta\gamma_k - \lambda\gamma_k = 0$$

for $i = k-1$.

The last equation gives

$$\gamma_k \in \mathcal{N}(\Delta - \mu I).$$

The equation for $i = k-2$ is, after some simplification,

$$[c-\tilde{g}(\lambda)]\Delta\gamma_{k-1} - \tilde{g}'(\lambda)\Delta\gamma_k - \lambda\gamma_{k-1} = \gamma_k$$

or

$$\Delta\gamma_{k-1} - \mu\gamma_{k-1} - \frac{\tilde{g}'(\lambda)}{c-\tilde{g}(\lambda)}\Delta\gamma_k = \gamma_k \frac{1}{c-\tilde{g}(\lambda)}.$$

Applying $(\Delta - \mu I)$ to this gives

$$(\Delta - \mu I)^2 \gamma_{k-1} = 0$$

and so $\gamma_{k-1} \in \mathcal{N}(\Delta - \mu I)^2$. But since the Laplacian is self-adjoint, $\mathcal{N}(\Delta - \mu I)^2 = \mathcal{N}(\Delta - \mu I)$ and so $\gamma_{k-1} \in \mathcal{N}(\Delta - \mu I)$.

Proceeding by induction, $\gamma_k \in \mathcal{N}(\Delta - \mu I)$ for $1 \leq i \leq k$, and Also $(-1)^i \frac{\tilde{g}^{(i)}(\lambda)}{i!} = \frac{1}{\lambda}$. So if $(w,p) \in \mathcal{N}(\mathcal{A} - \lambda I)^k$, then

(2.5) $w = \sum_{j=0}^{k-1} \gamma_{j+1} e^{\lambda t} \frac{t^j}{j!}$, and $p = \gamma_1$, where γ_j is an eigenfunction of the Laplacian, with eigenvalue λ, and

$(-1)^j \dfrac{\tilde{g}^{(i)}(\lambda)}{i!} = \dfrac{1}{\lambda}$. Direct computation shows this is also sufficient.

Now if $<(u,v),(s,q)>_{\mathcal{A}} = 0$ for all $(s,q) \in \mathcal{N}(\mathcal{A}-\lambda I)^k$, then by equation (2.5)

$$c < v, \gamma_k >_H - \int_{-\infty}^{0} g(-s) \left(\int_{s}^{0} \sum_{j=0}^{k-1} < \gamma_{j+1} \frac{e^{\lambda(s-s')}}{j!} (s-s')^j , \Delta u(s') > ds \right)$$

$$= c < v, \gamma_k >_H - \sum_{j=0}^{k-1} < \gamma_{j+1}, \int_{-\infty}^{0} g(-s) \int_{s}^{0} e^{\lambda(s-s')} \frac{(s-s')^j}{j!} \Delta u(s')$$

$$> ds = 0$$

where each γ_i is any element in $\mathcal{N}(\Delta - \mu I)$. Since the γ_i's are arbitrary elements in $\mathcal{N}(\Delta - \mu I)$, this means

$$c < v, \gamma_k >_H - < \gamma_k, \int_{-\infty}^{0} g(-s) \left(\int_{s}^{0} e^{\lambda(s-s')} \Delta u(s')\, ds' \right) ds >_H = 0$$

for all $\gamma_k \in \mathcal{N}(\Delta - \mu I)$. Since $\Delta - \mu I$ satisfies Fredholm's alternative [ ], this means

$$cv - \int_{-\infty}^{0} g(-s) M_\lambda \ \Delta u(s)\, ds \in R(\Delta - \mu I)$$

and so,

$$cv - \int_{-\infty}^{0} g(-s) M_\lambda \Delta(s)\, ds = (\Delta - \mu I) L$$

for some $L \in H_0^1$.

Also, since each γ_i is arbitrary,

$$< \gamma_i, \int_{-\infty}^{0} g(-s) \left(\int_{s}^{0} e^{\lambda(s-s')} \Delta u\, (s') ds' \right) ds > = 0$$

for each $\gamma_i \in \mathcal{N}(\Delta - \mu I)$. In the same way

$$\int_{-\infty}^{0} g(-s) M_\lambda^{\,i}\,\Delta u(s)\,ds = (\Delta - \mu I)\,\psi_i$$

for some $\psi_i \in H_0^1$.

Let $p = \frac{\mu}{\lambda} L$, and let

$$w = \sum_{j=0}^{k-2} e^{\lambda t}\, t^{k-1-i}\, \psi_i + e^{\lambda t} p + (M_\lambda^{\,k} u)(t) .$$

Then direct calculation shows

$$(\mathcal{A} - \lambda I)^k (w,p) = (u,v) .$$

Thus, $(u,v) \in \mathcal{R}(\mathcal{A} - \lambda I)^k$.

<u>Theorem 2.6</u>. Let $\{\lambda_1, \ldots, \lambda_n\}$ be a finite set of eigenvalues for the operator $\mathcal{A}$, $\mathcal{A}$ the same operator as in Theorem 2.4, for the case $E = H_0^1(\Omega)$, and $A =$ the Laplacian.

Let $\mathcal{M}_{\lambda_i}(\mathcal{A})$ be the generalized eigenspace of $\mathcal{A}$ for λ_i and $\mathcal{M} = \mathcal{M}_{\lambda_1}(\mathcal{A}) \oplus \mathcal{M}_{\lambda_2}(\mathcal{A}) \cdots \oplus \mathcal{M}_{\lambda_n}(\mathcal{A})$. Also let

$$\mathcal{M}^0 = \{y \in W \mid \langle y, z \rangle_{\mathcal{A}} = 0 \quad \text{for all} \quad z \in \mathcal{M}\} .$$

Then $\mathcal{M}$ and $\mathcal{M}^0$ are invariant under the map $e^{\mathcal{A}t}$, i.e. if $y \in \mathcal{M}$, then $e^{at} y \in \mathcal{M}$, and if $z \in \mathcal{M}^0$, then $e^{\mathcal{A}t} z \in \mathcal{M}^0$ for all $t > 0$.

Let $[\varphi_1, \cdots, \varphi_m\}$ be a basis for $\mathcal{M}$. Then there is another basis $\{\psi_1, \cdots, \psi_m\}$ for $\mathcal{M}$ such that

$$\langle \varphi_i, \psi_i \rangle_{\mathcal{A}} = 1 \quad \text{and} \quad \langle \varphi_i, \psi_j \rangle = 0 \quad \text{if} \quad i \neq j .$$

Define $P: W \to W$ by

$$PZ = \sum_{i=1}^{m} \langle \psi_i, Z \rangle_{\mathcal{A}}\, \varphi_i . \tag{2.7}$$

Then P is a projection onto $\mathcal{M}$, that is, if $z \in W$, $Pz \in \mathcal{M}$, and if $z \in \mathcal{M}$, $Pz = z$. Also for any $t \geq 0$

$$e^{\mathcal{A}t}P = Pe^{\mathcal{A}t}$$

Also $I - P$ is a projection onto $\mathcal{M}^0$, and

$$e^{\mathcal{A}t}(I - P) = (I - P)e^{\mathcal{A}t} .$$

<u>Proof</u>: First, since the Laplacian has only normal eigenvalues, each λ_i is a normal eigenvalue. Thus, each of the $\mathcal{M}_{\lambda_i}(\mathcal{A})$ is finite dimensional and so is their direct sum. Also each $\mathcal{M}_{\lambda_i}(\mathcal{A}) = \mathcal{N}(\mathcal{A} - \lambda I)^{k_i}$ for some k_i. Since $\mathcal{M}_{\lambda_i}(\mathcal{A}) \cap \mathcal{M}_{\lambda_j}(\mathcal{A}) = \{0\}$, Lemma 2.3 and the definition of normal eigenvalue give

$$W = \mathcal{M}_{\lambda_1}(\mathcal{A}) \oplus \mathcal{M}_{\lambda_2}(\mathcal{A}) \oplus \cdots \oplus \mathcal{M}_{\lambda_n}(\mathcal{A}) \oplus (\bigcap_{i=1}^{n} \mathcal{N}(\mathcal{A}-\lambda_1) . \tag{2.8}$$

Now assume $\{\varphi_1, \cdots, \varphi_m\}$ is a basis for $\mathcal{M}$. If for every $\psi \in \mathcal{M}$, $\psi \neq 0$, there is a φ such that $\langle \psi, \varphi \rangle_{\mathcal{A}} \neq 0$, that is, if $\langle\ ,\ \rangle_{\mathcal{A}}$ is nondegenerate on $\mathcal{M}$ [14], then the Gram-Schmidt process will give a basis $\{\psi_1, \cdots, \psi_n\}$ such that $\langle \varphi_i , \psi_i \rangle_{\mathcal{A}} = 1$ and $\langle \varphi_i , \psi_j \rangle = 0$ if $i \neq j$.

Now suppose $\langle \psi, \varphi_j \rangle = 0$ for $j = 1, \cdots, m$. Then by Lemma 2.4, $\psi \in \mathcal{R}(\mathcal{A} - \lambda_i I)^{k_i}$ for $i = 1, \cdots, n$. But then, since $\psi \in \mathcal{M}$ and $\psi \in \bigcap_{i=1}^{n} \mathcal{R}(\mathcal{A} - \lambda_i I)^{k_i}$, $\psi = 0$. So $\langle\ ,\ \rangle_{\mathcal{A}}$ is nondegenerate on $\mathcal{M}$.

Let $P(x) = \sum_{i=1}^{n} \langle x, \psi_i \rangle_{\mathcal{A}} \varphi_i$. Then $Px \in \mathcal{M}$. Also $P(\varphi_i) = \sum_{i=1}^{n} \langle \varphi_i, \psi_i \rangle_{\mathcal{A}} \varphi_i = \varphi_i$. Since P is linear and sends each element of a basis of $\mathcal{M}$ into itself, $P|_{\mathcal{M}}$ is the identity. So $P: W \to W$ is a projection onto $\mathcal{M}$.

Now let $y \in \mathcal{M}^0$. Then since $\langle y, \varphi \rangle = 0$ for any $\varphi \in \mathcal{M}$, $\langle y, \psi_i \rangle = 0$ for all i. So $Py = 0$.

Conversely, suppose $Py = 0$. Since the φ_i form a basis, they are linearly independent. Using equation (2.7), $\langle y, \psi_i \rangle = 0$ for each ψ_i. Since the ψ_i form a basis of $\mathcal{M}$, $\langle y, z \rangle = 0$ for any $z \in \mathcal{M}$, and $y \in \mathcal{M}^0$. Thus $\mathcal{N}(P) = \mathcal{M}^0$. Since P is a projection, $\mathcal{N}(P) = \mathcal{R}(I - P)$, which shows that $I - P$ is a projection of W onto $\mathcal{M}^0$.

Now, $\mathcal{M}^0 = \bigcap_{i=1}^{n} \mathcal{R}(\mathcal{A} - \lambda I)^{k_i}$ by Lemma 2.5. Since $e^{\mathcal{A}t}\mathcal{A}(x) = \mathcal{A}e^{\mathcal{A}(t)}(x)$ for $x \in D(\mathcal{A})$ and, $e^{\mathcal{A}t}\mathcal{R}(\mathcal{A} - \lambda I)^k(x) = \mathcal{R}(\mathcal{A} - \lambda I)^k e^{\mathcal{A}t}x$, $\mathcal{R}(\mathcal{A} - \lambda I)^k$ is invariant under $e^{\mathcal{A}t}$.

Now let $x \in W$. Then by (2.8)

$$x = y + z, \quad \text{where } y \in \mathcal{M} \text{ and } z \in \mathcal{M}^0 \text{ and}$$

$$Pe^{\mathcal{A}t}x = Pe^{\mathcal{A}t}(y + z) = Pe^{\mathcal{A}t}y + Pe^{\mathcal{A}t}z$$

$$= e^{\mathcal{A}t}y = e^{\mathcal{A}t}Py = e^{\mathcal{A}t}Px.$$

In the same way

$$(I - P)e^{\mathcal{A}t}x = e^{\mathcal{A}t}(I - P)x.$$

§6. The operator $\mathcal{A}$ and solutions of (1).

Now the connection between the operator $\mathcal{A}$ and solutions of (2.1) will be shown in the special case where $A = \Delta$ and $E = H_0^1(\Omega)$.

Theorem 2.7. Let $X = L_{2,e^{at}}(-\infty,0;H_0^1(\Omega))$, $W = X \times H_0^1(\Omega)$, let $w \in X$, and $p \in H_0^1(\Omega)$.

Define $P_1: W \to X$ by $P_1(w,p) = w$, and $P_2: W \to H_0^1(\Omega)$ by $P_2(w,p) = p$.

Let $\mathcal{A}$ be the operator defined by equation (2.4), with $A = \Delta$ and $E = H_0^1(\Omega)$. Then for any given initial condition $(w,p) \in W$, the equation

$$\text{(2.9)} \qquad \dot{u}(t) = \Delta(c\,u(t) - \int_{-\infty}^{t} g(t-s)u(s)ds)$$

with $\dot{u}(t) \in H^{-1}(\Omega)$ has a unique solution given by

$$\text{(2.10)} \qquad u(t) = P_2 e^{\mathcal{A}t}(w,p), \ t \geq 0.$$

Proof: By semigroup theory [13], if $(v,q) \in D(\mathcal{A})$, then $e^{\mathcal{A}t}(v,q) \in D(\mathcal{A})$ for all $t > 0$ and

$$\frac{d}{dt}(e^{\mathcal{A}t}(v,q)) = \mathcal{A}(e^{\mathcal{A}t}(v,q)).$$

Now let

$$e^{\mathcal{A}t}(v,q) = (v(t_i),q(t)),$$

where $v(t_i) \in X$ for each t, and $q(t) \in H_0^1$ for each t.

By Theorem 2.4,

$$\frac{d}{ds} v(t,s) = \frac{d}{dt} v(t,s).$$

Thus

$$v(t,s) = v(t+s).$$

Also since

$$\lim_{s\to 0^-} v(t,s) = \lim_{s\to 0^-} v(t+s) = q(t),$$

it follows that $v(t) = q(t)$, for $t > 0$. From this, it follows that

$$\begin{aligned} q(t) &= \Delta(c\, q(t) - \int_{-\infty}^{0} g(-s)v(t,s)ds) \\ &= \Delta(c\, q(t) - \int_{-\infty}^{0} g(-s)q(t+s)ds) \\ &= \Delta(c\, q(t) - \int_{-\infty}^{t} g(t-s)q(s)ds) \end{aligned}$$

Also, by definition,

$$q(t) = P_2 e^{\mathcal{A}t}(w,p).$$

Now take any $(w,p) \in W$. $W(\mathcal{A})$ is dense in W [13], so there exist $x_n \in W$ such that

$$\lim_{n\to\infty} x_n = (w,p).$$

Then each $Pe^{\mathcal{A}t}x_n$ satisfies

$$\frac{d}{dt}(P_2 e^{\mathcal{A}t}x_n) = \Delta(c(P_2 e^{\mathcal{A}t}x_n) - \int_{-\infty}^{t} g(t-s)(P_2 e^{\mathcal{A}s}x_n)\,ds).$$

Δ is a continuous linear operator from H_0^1 to H^{-1} and $|e^{\mathcal{A}t}| \leq ke^{\alpha t}$ for some constants k and α not depending on t [13]. Thus $e^{\mathcal{A}t}x_n \to e^{\mathcal{A}t}(w,p)$ uniformly on any interval $[0,T]$, $T < \infty$, and

$$\Delta(c(P_2 e^{\mathcal{A}t}x_n) - \int_{-\infty}^{t} g(t-s)(P_2 e^{\mathcal{A}s}x_n)\,ds)$$

converges uniformly in the norm of H^{-1} on $[0,T]$. Therefore

$$\frac{d}{dt}(P_2 e^{\mathcal{A}t}x_n) \to \frac{d}{dt}(P_2 e^{\mathcal{A}t}x)$$

by the theorem on uniform convergence of derivatives [15]. So $P_2 e^{\mathcal{A}t}(w,p) = u(t)$ satisfies

$$\dot{u}(t) = \Delta(c\,u(t) - \int_{-\infty}^{t} g(t-s)u(s)\,ds)$$

$P_1 x_n \to w$, and $P_2 x_n \to p$, so $u(t) = w(t)$ for $t < 0$ and $u(0) = p$. So u satisfies the initial conditions. Also, u is unique since if u satisfies (2.9) with $\dot{u}(t) \in H_1^{-1}$ the variation of constants formula gives

$$u(t) = e^{c\Delta t}p - \int_{0}^{t} \Delta e^{c\Delta(t-s)} \left(\int_{-\infty}^{s} g(s-s')u(s')\,ds' \right) ds .$$

Integration by parts gives $u(t)$ is a solution of (2.2) which by Theorem 2.1 is unique.

CHAPTER III

NONHOMOGENEOUS LINEAR PROBLEMS AND PERTURBED NONLINEAR PROBLEMS

§1. Introduction.

This chapter will first be concerned with nonhomogeneous linear equations of the type

$$\dot{u}(t) = c\,\Delta u(t) - \int_{-\infty}^{t} g(t-s)\,\Delta u(s)\,ds + f(t). \tag{3.1}$$

A solution will be in the same sense as in section 2.6, that is, $u(t)$ will be in $H_0^1(\Omega)$ for each t, $\dot{u}(t)$ will be in $H^{-1}(\Omega)$.

Once a result has been obtained for (3.1), it will then be used to obtain a result on equations of the form

$$\dot{u}(t) = c\,\Delta u(t) - \int_{-\infty}^{t} g(t-s)\,\Delta u(s)\,ds + f(t,u_t,u(t)) \tag{3.2}$$

These results come easily from the variation of constants formula for semigroups.

Theorem 3.1. [13]. If A is an infinitesimal generator on a Banach space E, $f\colon [0,T] \to E$ a differentiable function, and $x \in D(A)$, then the equation

$$\dot{u}(t) = A\,u(t) + f(t),\ u(0) = x$$

has a unique solution

$$(3.3) \qquad u(t) = e^{At}x + \int_0^t e^{A(t-s)} f(s)\,ds\ .$$

§2. The inhomogeneous linear equation.

<u>Theorem 3.2</u>. Let Ω and $H_0^1(\Omega)$ be as in section 2,5. Let $a > 0$,

$$X = L_{2,e^{at}}(-\infty,0\ ;\ H_0^1(\Omega) \quad \text{and}$$

$$W = X \times H_0^1(\Omega)\ .$$

Let $g\colon (-\infty,0] \to R$ be differentiable and satisfy

$$\text{(i)} \quad \int_0^\infty [g(s)]^2 e^{2as} ds < \infty,$$

$$\text{(ii)} \quad \int_0^\infty [g'(s)]^2 e^{2as} ds < \infty.$$

For any $T > 0$, $x \in X$, $p \in H_0^1$ and continuous $f\colon [0,T] \to H_0^1(\Omega)$, the equation

$$(3.1) \qquad \dot{u}(t) = c\,\Delta u(t) - \int_{-\infty}^t g(t-s)\,\Delta u(s)\,ds + f(t)$$

$$u(t) = w(t), \quad t < 0$$

$$u(0) = p$$

has a unique solution on $[0,T]$, with $u(t) \in H_0^1(\Omega)$ and $\dot{u}(t) \in H^{-1}(\Omega)$ for each $t \geq 0$.

<u>Proof</u>: Let $\mathcal{A}$ be the infinitesimal generator defined by equation (2.4) in the case where $E = H_0^1(\Omega)$ and $A = \Delta$. Let $v \in X$, $q \in H_0^1$, $(v,q) \in D(\mathcal{A})$, and $r\colon [0,T] \to H_0^1$ be differentiable. Then $h\colon [0,T] \to W$ defined by $h(t) = (0,r(t))$ is differentiable so the equation

$$\dot{\varphi}(t) = \mathcal{A}\,\varphi(t) + h(t) \tag{3.4}$$

with initial condition

$$\varphi(0) = (v,q)$$

has, by the previously stated theorem, the unique solution

$$\varphi(t) = e^{\mathcal{A}t}(v,q) + \int_0^t e^{\mathcal{A}(t-s)}h(s)\,ds.$$

Now, as in section 2.6, if $w \in X$, and $p \in H_0^1$, then let

$$P_1(w,p) = w \qquad P_2(w,p) = p.$$

For each $t \geq 0$, $\varphi(t) \in W$. So $P_1\varphi(t) \in X$. This means $P_1\varphi(t)$ is a function on $(-\infty,0]$. Let $\lambda(t,s) = [P_1\varphi(t)](s)$, for $s \leq 0$. Then λ is defined for $s \leq 0$ and $t \geq 0$. Also let $P_2\varphi(t) = \mu(t)$. Then $\varphi(t) = (\lambda(t,\cdot),\ \mu(t))$.

By equation (3.4) and equation (2.4), since

$$\frac{\partial}{\partial t}(\varphi(t)) = \left(\left(\frac{\partial\lambda}{\partial t}\right)(t,\cdot)\right),\ \left(\frac{\partial\mu}{\partial t}\right)(t)\ ,$$

$$\left(\frac{\partial}{\partial t}\lambda\right)(t,s) = \left(\frac{\partial\lambda}{\partial s}\right)(t,s)$$

so

$$\lambda(t,s) = \lambda(t+s).$$

Also by Theorem 3.1, $(\lambda(t,\cdot),\mu(t)) \in D(\mathcal{A})$ for each $t \geq 0$. Thus $\lim_{s\to 0^-} \lambda(t,s) = \mu(t)$. From this $\lambda(t) = \mu(t)$ and hence $\mu(t)$ satisfies

$$\dot{\mu}(t) = c\,\Delta\mu(t) - \int_{-\infty}^{t} g(t-s)\Delta\mu(s)ds + r(t)\ ,$$

setting $\mu(t) = v(t)$ for $t < 0$. Also

$$\mu(0) = P_2 e^{\mathcal{A}(0)}(v,q) = q\ .$$

Now $D(\mathcal{A})$ is dense in W. Also, the differentiable functions from $[0,T]$ into $H_0^1(\Omega)$ are dense in $C[0,T\,;H_0^1(\Omega)]$ [15]. So let $(w,p) \in W$ and let $f(t)\colon [0,T] \to H_0^1(\Omega)$ be continuous. Choose $(v_n,q_n) \in D(\mathcal{A})$, such that $\lim_{n\to\infty} (v_n,q_n) \to (w,p)$, and let r_n be differentiable and $r_n \to f$ uniformly on $[0,T]$ as $n \to \infty$. Then

$$e^{\mathcal{A}t}(v_n,q_n) + \int_0^T e^{\mathcal{A}(t-s)}(0,v_n(s))ds \to$$

$$e^{\mathcal{A}t}(w,p) + \int_0^T e^{\mathcal{A}(t-s)}(0,f(s))ds$$

uniformly on $[0,T]$. Also, if $\mu_n(t) = P_2 e^{\mathcal{A}t}(v_n,a_n)$

$$\mu_n(t) = P_2 e^{\mathcal{A}t}(v_n,q_n) + \int_0^t e^{\mathcal{A}(t-s)}(0,f(s))ds,$$

then

$$\dot{\mu}_n(t) = \Delta((c\,\mu_n(t) - \int_{-\infty}^{t} g(t-s)\mu_n(s)ds) + v_n(t)$$

converges uniformly on $[0,T]$ in the norm on H^{-1}. Therefore

$$\dot{\mu}_n(t) \to \frac{d}{dt} P_2(e^{\mathcal{A}t}(w,p) + \int_0^t e^{\mathcal{A}(t-s)}(0,f(s))ds)$$

and thus

$$(3.5) \qquad u(t) = P_2(e^{\mathcal{A}t}(w,p) + \int_0^t e^{\mathcal{A}(t-s)}(0,f(s))ds$$

satisfies

$$u(0) = p = P_2 e^{\mathcal{A}(0)}(w,p)$$

$$u(t) = w(t), \ t < 0$$

$$\dot{u}(t) = c\,\Delta u(t) - \int_{-\infty}^t g(t-s)\,\Delta u(s)ds + f(t).$$

This not only shows a solution of (3.1) exists, it gives a formula (3.5) for obtaining the solution. Uniqueness comes from the uniqueness of the homogeneous equation.

§3. The perturbed linear equation.

In this section the equation

$$(3.2) \qquad \dot{u}(t) = c\,\Delta u(t) - \int_{-\infty}^t g(t-s)\,\Delta u(s)\,ds + f(t,u_t,u(t))$$

$$u(t) = w(t), \ t < 0$$

$$u(0) = p$$

will be studied. Again, for each $t > 0$, $u(t) \in H_0^1$ and $\dot{u}(t) \in H^{-1}$. Using the variation of constants formula in section 3.2, we have

$$(3.6) \quad u(t) = P_2(e^{\mathcal{A}t}(w,p) + \int_0^t e^{\mathcal{A}(t-s)}(0,f(s,u_s,u(s))ds).$$

Conversely, if a solution to (3.4) can be found, then by Theorem 1, (3.2) has the solution $u(t)$.

Theorem 3.3. Let $\mathcal{A}$ be the infinitesimal generator in Theorem 2.4 for the case where $A = \Delta$ and $E = H_0^1$. Let

$$X = L_{2,e^{at}}(-\infty, 0; H_0^1(\Omega)), \quad \text{and}$$

$$W = X \times H_0^1(\Omega),$$

U be an open set in W, $f: R \times U \to H_0^1$. Assume that f satisfies

1. f is continuous in t.
2. There is an $L > 0$ such that for any t_1, for any $x_1, x_2 \in U$,

$$|f(t_1,x_1) - f(t_1,x_2)|_{H_0'} \leq L\,|x_1 - x_2|_W .$$

If $(w,p) \in U$, then

$$\dot{u}(t) = c\,\Delta u(t) - \int_{-\infty}^{t} g(t-s)\,\Delta u(s)\,ds + f(t,u_t,u(t)) \tag{3.2}$$

$$u(t) = w(t),\ t < 0$$

$$u(0) = p$$

has a unique solution in $[0,t_1]$, for some $t_1 > 0$.

Proof: This is just using the variation of constants formula (3.4) and Banach's contraction mapping theorem. There are K_1, K_2 such that $|e^{\mathcal{A}t}| \leq K_1 e^{K_2 t}$ [13].

Choose $\delta > 0$ and $T > 0$ so that

$$\{(s,q) \mid |(s,q) - (w,p)|_W < \delta\} \subset U$$

and

$$|(e^{\mathcal{A}h} - I)(w,p)|_W \leq \frac{\delta}{4} \quad \text{for} \quad 0 \leq h \leq T.$$

(Since $e^{\mathcal{A}t}(w,p)$ is continuous in t, this can be done). Also let $B = \max |f(t,w,p)|_{0 \leq t \leq 1}$, $TK_1 e^{K_2 T}(B + L\delta) < \frac{\delta}{4}$, $T < 1$, $|w - w_T| < \frac{\delta}{4}$, and $Te^{aT} < \frac{1}{2}$.

Let $S = \{q \in C[0,T;W] \mid |g(t) - (w,p)| \leq \delta_1$ for all $t \in [0,T]$, $(P,g(t))(s) = P_2 g)(s)$, $0 < s < t\}$. Then S is is a complete metric space since it's a closed subspace of $C[0,T;W]$.

Define $G: S \to S$ by $G(v(t)) = (g_t, g(t))$, where

$$g(t) = P_2(e^{\mathcal{A}t}(w,p) + \int_0^t e^{\mathcal{A}(t-s)} f(s,v(s))ds).$$

It will be shown that G is a strict contraction on S. First,

$$|(G(v))(t) - (w,p)|_W \leq |(e^{\mathcal{A}t} - I)(w,p)|_W$$

$$+ \left| \int_0^t e^{\mathcal{A}(t-s)} f(s,v(s))ds \right| \leq \frac{\delta}{4} + (TK_1 e^{K_2 T})(B + L\delta) < \frac{1}{2}\delta,$$

by the choice of T. Also

$$|w - P_1(G(v)|(t)| \leq |w - w_t|_W + \left(\int_{-t}^0 e^{-2as} P_2 |Gv(s)|^2 ds \right)^{\frac{1}{2}}$$

$$\leq \frac{\delta}{4} + t\, e^{at} \delta \frac{1}{2} < \delta.$$

Clearly $(P_1(Gv)(t))s = P_2 Gv(s)$, so $G: S \to S$. Also, G

is a contraction. Let v and q be elements of S. Then

$$|Gv - G_q|_s \leq \sup_{t\in[0,T]} |P_2 \int_0^t e^{\mathcal{Q}(t-s)} (f(s,v(s)) - f(s,q(s))ds|_{H_0^1}$$

$$\leq K_1 e^{K_2 T} L |v - q|_s < \frac{1}{2} |v - q|$$

by choice of T, and thus, G is a contraction, and so has a unique fixed point. This fixed point gives a solution of (3.6), which by Theorem 3.2 gives a solution of (3.2).

CHAPTER IV

THE SADDLE POINT PROPERTY

§1. The spectrum of the semigroup.

This chapter will be concerned with the saddle point property for the equation

$$(4.1)\quad \dot{u}(t) = c\,\Delta u - \int_{-\infty}^{t} g(t-s)\,\Delta u\,(s)ds + f(t,u_t,u(t))$$

with $\lim\limits_{w\to 0} f(t,w) = o(|w|)$.

To prove the saddle point property for (1), the spectrum of $e^{\mathcal{A}t}$ must be found. By a general theorem on semigroups [13], if $\lambda \in \sigma(\mathcal{A})$, then $e^{\lambda t} \in \sigma(e^{\mathcal{A}t})$. Also, if $e^{\lambda t} \in P\sigma(e^{\mathcal{A}t})$ then $\lambda + \frac{2\pi\, in}{t} \in P\sigma(\mathcal{A})$ for some integer n. But $e^{\mathcal{A}t}$ has continuous spectrum besides 0, which for a general semigroup does not necessarily arise from the spectrum of the infinitesimal generator. Hence, something more is needed to find the spectrum of $e^{\mathcal{A}t}$. When the spectrum of $e^{\mathcal{A}t}$ is found, this can be used to determine the rate of growth of $e^{\mathcal{A}t}$ restricted to invariant subspaces.

First, a lemma is needed.

<u>Lemma 4.1</u>. Let g and A be as in Theorem 2.1 with the additional restrictions:

1. g is decreasing

2. $c - \tilde{g}(0) > 0$.

3. There exists a K such that $|R(A,\lambda)| < \frac{1}{\lambda + k}$ for $\lambda > -k$.

(This means $A + kI$ generates a contractive semigroup [13]).

Assume there is a root λ of

$$\lambda = k(\tilde{g}(\lambda) - c) + \frac{2g(0)}{c}$$

with $\operatorname{Re}\lambda > -a$ and let

$$\lambda_1 = \max\{\operatorname{Re}\lambda \,|\, \lambda = k(\tilde{g}(\lambda) - c) + \frac{2g(0)}{c}\}.$$

Then there is a constant $M \geq 1$ such that, for $w \in K$ and $p \in E$ the solution u of equation 2.2 with initial condition (w,p) satisfies

$$|u(t)| \leq M e^{\lambda_1 t} |(w,p)|_{K \times E}.$$

Proof: We have that $u(t)$ satisfies equation 2.2 so

$$\begin{aligned} u(t) = {} & e^{cAt}\left(p - \frac{1}{c}\int_{-\infty}^{0} g(-s)w(s)\,ds\right) + \frac{1}{c}\int_{-\infty}^{t} g(t-s)w(s)\,ds \\ & - \frac{1}{c}\int_{0}^{t} e^{cA(t-s)}\left(\int_{-\infty}^{0} g'(s-s')w(s')\,ds'\right)ds \\ & + \frac{1}{c}\int_{0}^{t} g(t-s)u(s)\,ds - \frac{g(0)}{c}\int_{0}^{t} e^{cA(t-s)}u(s)\,ds \end{aligned}$$

$$- \frac{1}{c}\int_0^t e^{cA(t-s)}\left(\int_0^s g'(s-s')u(s')ds'\right)ds .$$

Since $A + kI$ generates a contractive semigroup, $|e^{cAt}| \leq e^{-ckt}$ for $t \geq 0$ [13]. Also, since $g(s) \geq 0$ and $g'(s) \leq 0$, we have, by taking norms, that

$$(4.2) \qquad |u(t)|_E \leq e^{-kct}\left|p - \frac{1}{c}\int_{-\infty}^0 g(-s)w(s)ds\right|_E$$

$$+ \left|\frac{1}{c}\int_{-\infty}^0 g(t-s)w(s)ds\right|_E + \frac{1}{c}\int_0^t e^{-kc(t-s)}\left|\int_{-\infty}^0 g'(s-s')w(s')ds'\right|_E ds$$

$$+ \frac{1}{c}\int_0^t g(t-s)|u(s)|_E ds + \frac{g(0)}{c}\int_0^t e^{-kc(t-s)}|u(s)|_E ds$$

$$+ \frac{1}{c}\int_0^t e^{-kc(t-s)}\left(\int_0^s -g'(s-s')|u(s')|ds'\right)ds .$$

Using integration by parts on (4.2) gives

$$|u(t)|_E \leq e^{-kct}\left|\left(p - \frac{1}{c}\int_{-\infty}^0 g(-s)w(s)ds\right)\right|_E$$

$$+ \frac{1}{c}\left|\int_{-\infty}^0 g(t-s)w(s)ds\right|_E + \frac{1}{c}\int_0^t e^{-kc(t-s)}\left|\int_{-\infty}^0 g'(s-s')w(s')ds'\right|_E$$

$$+ k < \int_0^t e^{-kc(t-s)}\int_0^s g(t-s)|u(s)|_E ds$$

$$+ \frac{2g(0)}{c}\int_0^t e^{-kc(t-s)}|u(s)|_E ds .$$

Let

$$G = \left(\int_{-\infty}^{0} (g(-s))^2 \, e^{-2as} ds \right)^{\frac{1}{2}},$$

and

$$G_1 = \left(\int_{-\infty}^{0} [g'(-s)]^2 e^{-2as} ds \right)^{\frac{1}{2}}.$$

Then

$$\left| p - \frac{1}{c} \int_{-\infty}^{0} g(-s) w(s) \, ds \right|_E \leq (g+1) \, |(w,p)|_{K \times E} \, .$$

Also

$$\left| \int_{-\infty}^{0} g(t-s) w(s) \, ds \right| \leq \left(\int_{-\infty}^{0} g^2(t-s) e^{-2as} ds \right)^{\frac{1}{2}} |w|_K \, .$$

Since

$$\int_{-\infty}^{0} g^2(t-s) e^{-2as} ds \leq \int_{-\infty}^{t} g^2(t-s) e^{-2as} ds$$

$$= e^{-2at} \int_{-\infty}^{t} g^2(t-s) e^{2a(t-s)} ds = e^{-2at} G^2 ,$$

$$\int_{-\infty}^{0} g^2(t-s) e^{-2as} ds)^{\frac{1}{2}} \leq e^{-at} G \, .$$

Next, consider the term

$$\int_{0}^{t} e^{-kc(t-s)} \left| \int_{-\infty}^{0} g'(s-s') w(s') \, ds' \right| ds \, .$$

Now as above,

$$\left| \int_{-\infty}^{0} g'(s-s') w(s') \, ds' \right|_{H_0^1} \leq (G_1 + 1) \, |W|_K \, e^{-as}$$

and so

$$\int_0^t e^{-kc(t-s)} \left| \int_{-\infty}^0 g'(s-s')w(s')ds' \right| ds$$

$$\leq \int_0^t e^{-kc(t-s)} e^{-as} (G_1+1) \, |w|_K ds$$

$$= \int_0^t e^{-kct} e^{(kc-2a)s} (G_1+1) \, |w|_K ds$$

$$\leq \frac{1}{(a-kc)} e^{-kct} |G_1+1| \, |w|_K \, .$$

From this

$$e^{-kct} \left|p - \frac{1}{c} \int_{-\infty}^0 g(-s)w(s)ds\right|_E + \frac{1}{c}\left| \int_{-\infty}^0 g(t-s)w(s)ds\right|_E$$

$$+ \frac{1}{c} \left| \int_0^t e^{-kc(t-s)} \int_{-\infty}^0 g'(s-s')w(s')ds')ds \right|_E$$

$$\leq Q \, e^{-kct} \, | \, (w,p) \, |_{K\times E}$$

where

$$Q = (G+1) + \frac{1}{c} G + \frac{1}{a-kc} (G_1+1),$$

i.e. Q is a constant not depending on (w,p). Thus

$$|u(t)|_E \leq Q \, |(w,p)|_{K\times E} \, e^{-kct}$$

$$+ K \int_0^t e^{-kc(t-s)} \left(\int_0^s g(s-s') \, |u(s')|_E ds'\right) ds$$

$$+ \frac{2g(0)}{c} \int_0^t e^{-kc(t-s)} |u(s)|_E ds \, .$$

Let $v(f)$ be the solution of

(4.3)

$$v(t) = Q|(w,p)|_{K\times E}\, e^{-kct} + K\int_0^t e^{-kc(t-s)}\left(\int_0^s g(s-s')v(s')ds'\right)ds$$

$$+ \frac{2g(0)}{c}\int_0^t e^{-kc(t-s)}v(s)ds$$

then $|u(t)|_E \leq v(t)$ for $t \geq 0$ by a comparison theorem for Volterra integral equations [19]. But if v satisfies (4.3) for $t \geq 0$, setting $v(t) = 0$ for $t < 0$ and differentiating gives

$$(4.2) \qquad v'(t) = -kcv(t) + \frac{2g(0)}{c}v(t) + k\int_{-\infty}^t g(t-s)v(s)ds$$

with $v(t)$ satisfying the initial condition

$$v(t) = 0 \quad t < 0$$

$$v(0) = Q|(w,p)|_{K\times E}.$$

Hence, by [20], $v(t)$ increases no more rapidly than the real part of the root of the characteristic equation for (4.2) with largest real part, to be exact,

$$v(t) \leq M_1 e^{\lambda_1 t}\, Q|(w,p)|_w$$

where $M_1 > 1$ is some positive number. Since $\lambda > \lambda_1$, and $|u(t)|_E \leq v(t)$,

$$|u(t)|_E \le M_1 e^{\lambda_1 t} Q\,|(w,p)|_{K\times E} \le M\,Q\,e^{\lambda t}\,|(w,p)|_w .$$

Now a few definitions are needed. In section 2.5 the set of normal eigenvalues $N\sigma(V)$ of an operator V on a Banach space E was defined.

The essential spectrum of V, denoted by $\sigma_\epsilon(V)$, is defined as

$$\sigma_\epsilon(V) = \sigma(V) - N\sigma(V)$$

and the essential spectral radius of V by

$$r_\epsilon(V) = \sup\{|\lambda| \,\big|\, \lambda \in \sigma_\epsilon(V)\}$$

when this exists. If V is a bounded operator, then $\sigma(V)$ is bounded, so $r_\epsilon(V)$ exists.

The Kuratowski measure of noncompactness $\alpha(\beta)$ for a bounded subset B of a Banach space E is defined by $\alpha(\beta) = \inf\{r \,|\, B$ can be covered by a finite number of balls of radius $r\}$ [23].

If V is a bounded linear operator from E to E, the measure of noncompactness, $\alpha(V)$, of V is defined by

$$\alpha(V) = \inf\{r \,|\, \alpha(V(B)) \le r\,\alpha(V), \quad \text{for all } B \subset E\}.$$

Clearly $\alpha(V) \le |V|$. Also let $\mu_1, \mu_2, \cdots$ be the eigenvalue of the Laplacian on Ω, arranged in decreasing order, with multiple eigenvalues listed once for each multiplicity and $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ be the associated eigenvectors.

If $B_n = \text{span}\{\varphi_1,\cdots,\varphi_n\}$ let $V_{1,n}$ be the orthogonal projection of $H_0^1(\Omega)$ onto B_n and let $V_{2,n} = I - V_{1,n}$. Now let $P_{1,n}: W \to W$ be defined by $P_{1,n}(w,p) = (u, V_{1,n}p)$ where $u(s) = V_{1,n}\, w(s)$. Then $P_{1,n}P_{1,n}(w,p) = P_{1,n}(w,p)$, so $P_{1,n}$ is a projection. Let $P_{2,n} = I - P_{1,n}$. These definitions and projections will be used in the following theorem

<u>Theorem 4.1</u>. Let $g, \Omega, H_0^1(\Omega)$, X, W and $\mathcal{A}$ be as in Theorem 2.6. Set

$$\tilde{g}(\lambda) = \int_0^\infty e^{-\lambda s} g(s)\,ds.$$

Assume g also satisfies

(i) g is decreasing

(ii) $c - \tilde{g}(0) > 0$.

Then the equation $c - \tilde{g}(\lambda) = 0$ has a largest real root λ_1. $\mathcal{A}$ generates a semigroup $e^{\mathcal{A}t}$ and $r_\epsilon(e^{\mathcal{A}t}) = e^{\lambda_2 t}$.

Also, if $\mu > \lambda_1$, there are only finitely many elements in $\sigma(\mathcal{A}) \cap \{\lambda \,|\, \mathrm{Re}\,\lambda > \mu\}$. All these points are elements in the normal spectrum. If P is the projection defined in Chapter 2, section 5.4, then $|(I-P)e^{\mathcal{A}t}| \leq K e^{\mu t}$ for some constant $K > 0$.

<u>Proof</u>: By Theorem 2.4, $\mathcal{A}$ is an infinitesimal generator. Let $W_{1,n} = P_{1,n}(W)$ and $W_{2,n} = P_{2,n}(W)$ If $v \in H_0^1(\Omega)$ and $\Delta v \in H_0^1(\Omega)$, then

$$v = \sum_{i=1}^{\infty} \langle v, \varphi_i \rangle_{H_0^1} \varphi_i$$

and

$$\Delta v = \sum_{i=1}^{\infty} \mu_i \langle v, \varphi_i \rangle_{H_0^1} \varphi_i .$$

Hence

$$V_{1,n} \Delta v = \sum_{i=1}^{n} \mu_i \langle v, \varphi_i \rangle_{H_0^1} \varphi_i = \Delta V_{1,n} v .$$

If $(w,p) \in D(\mathcal{A})$,

$$P_{1,n}\mathcal{A}(w_1 p) = (V_{1,n}\dot{w}, V_{1,n}\Delta(c p - \int_{-\infty}^{0} g(-s) w(s) ds)$$

$$= ((V_{1,n}w), \Delta(c V_{1,n} - \int_{-\infty}^{0} g(-s) \; V_{1,n} \; w(s) ds)$$

$$= \mathcal{A} \; P_{1,n}(w,p) .$$

Also,

$$P_{2,n}\mathcal{A}(w,p) = (I - P_{1,n}) \mathcal{A} (w,p) = \mathcal{A}(I - P_{1,n})(w,p)$$

$$= \mathcal{A} P_{2,n} (w,p) .$$

Now suppose $r_\epsilon(e^{\mathcal{A}t}) > e^{\lambda t}$ and $e^{\lambda_3 t} = r_\epsilon(e^{\mathcal{A}t})$. Let $\lambda_2 = \dfrac{\lambda_1 + \lambda_3}{2}$. Since $g(s)$ is decreasing and positive, $\tilde{g}(\lambda)$ is decreasing for λ real and $|\tilde{g}(\lambda)| \le \tilde{g}(\text{Re}\,\lambda)$. Let

$$k = \frac{\lambda_2 - \frac{2g(0)}{c}}{c - \tilde{g}(\lambda_2)}$$

and z be a solution of $k(c - \tilde{g}(z)) = z - \dfrac{2g(0)}{c}$.

Then

$$k(c - \operatorname{Re}\tilde{g}(z)) = \operatorname{Re}(z) - \frac{2g(0)}{c}.$$

Now

$$c - \operatorname{Re}\tilde{g}(z) \geq c - |\tilde{g}(z)| \geq c - \tilde{g}(\operatorname{Re} z).$$

Since $c - \tilde{g}(\operatorname{Re}(z)) > 0$, and $K < 0$, we have that

$$k \geq \frac{\operatorname{Re}(z) - \frac{2g(0)}{c}}{c - \tilde{g}(\operatorname{Re} z)}.$$

Since $\dfrac{\lambda - \frac{2g(0)}{c}}{c - \tilde{g}(\lambda)}$ is increasing on $[\lambda_1, \frac{2g(0)}{c}]$, $\operatorname{Re} z < \lambda_2$.

Suppose $x \in B^{\perp}$. Then

$$< \Delta x, x > \leq + (\lambda_{n+1}) |x|^2_{H_1^0} \leq + K|x|^2_{H_1^0}.$$

and hence $\Delta + kI$ generates a contractive semigroup [13].

Also $W_{2,n}$ is invariant for $\mathcal{A}$, since $P_{2,n}\mathcal{A}x = \mathcal{A}P_{2,n}x$ for $x \in D(\mathcal{A})$. Define $B_n + = \{v \in H_0^1(\Omega) \mid <v, s>_{H_0^1} = 0$ for all $s \in B_n\}$ ($B_n^{\perp}$ is the orthogonal complement of B_n).

If $v \in B_n^{\perp}$ and $\Delta v \in H_0^1(\Omega)$, then

$$<x, \Delta x>_{H_0^1} = < \sum_{n+1}^{\infty} <x, \varphi_i> \varphi_i , \sum_{n+1}^{\infty} \mu_i <x, \varphi_i> \varphi_i >$$

$$= \sum_{n+1}^{\infty} \mu_i <x, \varphi_i>^2 \leq \mu_{n+1} |x|^2_{H_0^1(\Omega)} \leq K|x|^2_{H_0^1}.$$

So $(\Delta + kI)|_{B_n^{\perp}}$ generates a contractive semigroup [13].

Also if $x \in D(\mathcal{A}) \cap W_{2,n}$,

$$\mathcal{A}|_{W_{2,n}}(w,p) = (w, \Delta|_{B_n^\perp}(cp - \int_{-\infty}^{0} g(-s)w(s)ds)).$$

Since $\Delta|_{B_n^\perp}$ satisfies the requirements for A in Lemma

$$|e^{\mathcal{A}t}|_{W_{2,n}} = |P_2 e^{\mathcal{A}t}| \leq Q\, e^{-kt} < Q\, e^{-\lambda_2 t}$$

where Q is some constant > 0.

Recall the definition of the measure of noncompactness α of an operator. From inequality (4.4),

$$\alpha(P_{2,n} e^{\mathcal{A}t}) \leq Q\, e^{\lambda_2 t} .$$

Now consider $\mathcal{A}|_{W_{1,n}}$. On B_n, the Laplacian is a continuous linear operator and hence

$$\mathcal{A}|_{W_{1,n}} = (\dot{w}, \Delta|B_n(cp - \int_{-\infty}^{0} g(-s)w(s)ds)).$$

Thus $\mathcal{A}|_{W_{1,n}}$ is the infinitesimal generator of the solution semigroup to an ordinary delay differential equation and by a general theorem [20],

$$r_\epsilon\left(e^{\mathcal{A}|_{W_1} t}\right) = e^{-at}$$

where a is the positive number such that

$$\int_0^\infty g(s)^2 e^{2as} ds < \infty .$$

By another theorem [22], for any bounded operators V and S on a Banach E,

$$r_\epsilon(V) = \lim_{n\to\infty} \sqrt[n]{\alpha(V^n)} .$$

Also

$$\alpha(V+S) \leq \alpha(V) + \alpha(S).$$

Therefore

$$r_\epsilon(e^{\ t}) = \lim_{n\to\infty} (\alpha(e^{n\mathcal{A}t}))^{\frac{1}{n}} \leq \lim_{n\to\infty} (\alpha(P_{1,n}e^{n\mathcal{A}t}) + \alpha(P_{2,n}e^{n\mathcal{A}t}))$$

$$\leq \lim_{n\to\infty} (e^{\lambda_2 nt} + \alpha(P_{2,n}e^{n\mathcal{A}t}))^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{\lambda_2 t}(1 + e^{-\lambda_2 t}\ \alpha(P_{2,n}e^{n\mathcal{A}t}))^{\frac{1}{n}} .$$

Now

$$\lim_{n\to\infty} (e^{-\lambda_2 + n}\ \alpha(P_{2,n}e^{n\mathcal{A}t})^{\frac{1}{n}}$$

$$= e^{-\lambda_2 t} \cdot e^{at} = e^{(a-\lambda_2)t} < 1 ,$$

since $\lambda_2 > a$ and hence,

$$\lim_{n\to\infty} (e^{-\lambda_2 tn}\ \alpha(P_{2,n}e^{n\mathcal{A}t})) = 0$$

and

$$r_\epsilon(e^{\mathcal{A}t}) \leq e^{\lambda_2 t} \lim_{n\to\infty} (1 + e^{-\lambda_2 t}\ \alpha(P_{2,n}e^{n\mathcal{A}t}))^{\frac{1}{n}}$$

$$= e^{\lambda_2 t} \lim_{n\to\infty} (1+0)^{\frac{1}{n}} = e^{\lambda_2 t} .$$

This implies that $r_\epsilon(e^{\mathcal{A}t}) \leq e^{\lambda_2 t}$, contracting $r_\epsilon(e^{\mathcal{A}t}) = e^{\lambda_3 t}$. Hence, $r_\epsilon(e^{\mathcal{A}t}) \leq e^{\lambda_1 t}$. Since $e^{\lambda_1 t} \in \sigma_\epsilon(\mathcal{A})$, $r_\epsilon(e^{\mathcal{A}t}) = e^{\lambda_1 t}$.

For any $\mu > \lambda$, $e^{\mathcal{A}t}$ has only finitely many normal eigenvalues λ with $|\lambda| > e^{\mu t}$, since

$$N\sigma(e^{\mathcal{A}t}) \cap \{\lambda \,|\, |\lambda| > e^{\mu t}\} = N\sigma_{\mu}(e^{\mathcal{A}t})$$

is a compact subset of the normal eigenvalues and so has only finitely many points. If $e^{\lambda t} \in N\sigma_{\mu}(e^{\mathcal{A}t})$, then only finitely many $\lambda + \frac{2\pi\, in}{t}$ can be eigenvalues of $\mathcal{A}$, as otherwise λ would have an infinite eigenspace, contradicting $\lambda \in N\sigma(e^{\mathcal{A}t})$. Also, each such point is a normal eigenvalue of $\mathcal{A}$ [13]. Thus, $\mathcal{A}$ has only finitely many points of spectrum with real part greater than μ, all normal eigenvalues.

Now let $\lambda_1, \cdots, \lambda_m$ be all these eigenvalues, and P the projection defined in Theorem 2.6. Then $Pe^{\mathcal{A}t} = e^{\mathcal{A}t}P$, $\sigma(Pe^{\mathcal{A}t}) = \sigma(e^{\mathcal{A}t}) - \{\lambda_1, \cdots, \lambda_m\}$ [16]. So $r_{\sigma}(Pe^{\mathcal{A}t}) \leq e^{\mu' t} < e^{\mu t}$, since

$$r_{\sigma}(Pe^{\mathcal{A}t}) = \lim_{n \to \infty} \sqrt[n]{Pe^{\mathcal{A}nt}}, \text{ and } \lim_{t \to \infty} (Pe^{\mathcal{A}t})e^{-\mu t} = 0.$$

Since $\lim\limits_{t \to \infty} (Pe^{\mathcal{A}t})e^{-\mu t}$ exists,

$$|(Pe^{\mathcal{A}t})e^{-\mu t}| = e^{-\mu t}|Pe^{\mathcal{A}t}|$$

is bounded for $t \geq 0$. Therefore for some $M > 0$, $e^{-\mu t}|Pe^{\mathcal{A}t}| \leq M$ for all $t \geq 0$, that is $|Pe^{\mathcal{A}t}| \leq Me^{\mu t}$ for $t \geq 0$. Results similar to this are given in [17] and [18].

§2. The Saddle Point property.

In this section, we consider the equation

$$(4.4)\quad \dot{u}(t) = c\,\Delta u - \int_{-\infty}^{t} g(t-s)\,\Delta u(s)\,ds + h(t,(u_t,u(t)))$$

where g satisfies the hypotheses in Theorem (2.1), and $h\colon R \times W \to H_0^1$ is continuous and satisfies:

$$(4.5)\qquad |h(t,z) - h(t,q)|_{H_0^1} \leq \alpha(\zeta)\,|z-q|_W$$

for all $|z|_W$, $|q|_W < \zeta$, where α is continuous and $\lim_{\zeta\to 0} \alpha(\zeta) = 0$. Let $\mathcal{A}$ be the infinitesimal generator associated with the linearized equation (2.1), and $\{\lambda_1,\cdots,\lambda_n\}$ the first n eigenvalues of this operator (counting multiplicities), arranged in order of real part. Suppose that

$$(4.6)\ \operatorname{Re}\lambda_i > \mu > \lambda^* = \max\{\lambda \mid c - \tilde{g}(\lambda) = 0\}\ \ i = 1,\cdots,n\,.$$

The projection from W onto the subspace generated by the generalized eigenspaces corresponding to the eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$ will be denoted by P. The projection $I-P$ will be denoted by Q.

We can now state the saddle point property for equation (4.4):

<u>Theorem 4.2</u>. Consider the equation (4.4). Assume that all of the above hypotheses are satisfied. Then there are constants $\delta > 0$ and $M \geq 1$ such that

(1) Let $S(\mu)$ denote the set of initial values $(w,p) \in W$ such that

(a) $|(w,p)|_W < \frac{\rho}{2M}$

(b) If $u(t)$ is the solution of (4.4) with initial value (w,p), then $|e^{-\mu t} u(t)|_{H_0^1} < \delta$ for all $t \geq 0$.

Then $S(\mu)$ is homeomorphic under the mapping $Q|_S$ to the closed ball of radius $\frac{\rho}{2M}$ in $Q(W)$. Also S is tangent to $Q(W)$ at zero, that is

$$\lim_{|Q(W)| \to 0} \frac{\operatorname{dist}((w,p),Q(W))}{|(w,p)|_W} = 0$$

and $\lim_{t\to\infty} e^{-\mu t}|u(t)|_{H_0^1} = 0.$

(2) Let $U(\mu)$ denote the set of initial values $(w,p) \in W$ such that

(c) $|(w,p)|_W \leq \frac{\rho}{2M}$

(d) There is a solution $u(t)$ of (4.3) defined for all real numbers t, such that $w(t) = u(t)$ for $t < 0$, and $u(0) = p$.

(e) $|e^{-\mu t} u(t)| < \delta$ for all $t \leq 0$.

Then $U(\mu)$ is homeomorphic to the closed ball of radius $\frac{\rho}{2M}$ in $P(W)$, under the mapping $P|_{U(\mu)}$. U is tangent to $P(W)$ at 0 and $\lim_{t\to-\infty} |e^{\mu t} u(t)| \to 0$.

<u>Proof</u>: Let $\mathcal{A}_1 = P\mathcal{A}, \mathcal{A}_2 = Q\mathcal{A}$, $W_1 = P(W)$ and $W_2 = Q(W)$.

First, let $(w,p) \in S(\mu)$. Then for each t by Theorem 3.1,

$$(u_t, u(t)) = e^{\mathcal{A}t}(w,p) + \int_0^t e^{\mathcal{A}(t-s)}(0, h(s,(u_s, u(s)))ds.$$

Now, $(u_t, u(t)) = P(u_t, u(t)) + Q(u_t, u(t))$. Using these projections,

$$P(u_t, u(t)) = e^{\mathcal{A}_1 tp}(w,p) + \int_0^t e^{\mathcal{A}_1(t-s)} P(u_s, u(s))ds.$$

Since $P\mathcal{A} = \mathcal{A}_I$ is a bounded linear operator, $e^{\mathcal{A}_1 t}$ exists for all t, positive and negative. So

$$e^{\mathcal{A}_1 t} P(u_t, u(t)) = P(w,p) + \int_0^t e^{-\mathcal{A}_1 s} P(0, h(s,(u_s, u(s))))ds.$$

Now $\mathcal{A}_1|_{W_1}$ has spectrum $\{\lambda_1, \cdots, \lambda_n\}$. From this $|(e^{-\mathcal{A}_1 t}|_{W_1})| \leq e^{-\lambda_n t}$ for $t \geq 0$. So

$$\lim_{t\to\infty} |e^{-\mathcal{A}_1 t} P(u_t, u(t))|_W \leq \lim_{t\to\infty} e^{(-\lambda_n+\mu)t} \sup_{t\in[0,\infty)} |e^{-\mu t}(u_t, u(t))|_W = 0.$$

This gives $P(w,p) = -\int_0^\infty e^{-a,s} P(0, h(s,(u_s, u(s))))ds.$

Substituting this into the equation for $(u_t, u(t))$ gives

$$(4.7)\quad (u_t, u(t)) = e^{\mathcal{A}_2 t} R + \int_0^t e^{\mathcal{A}_2(t-s)} Q(0, h(s,(u_s, u(s))))ds$$
$$- \int_t^\infty e^{\mathcal{A}_1(t-s)} P(0, h(s,(u_s, u(s))))ds$$

where $R = Q(w,p)$.

Now suppose $R \in W_2$. Then it will be shown (4.7) has a unique solution $u(t)$ for each R with $|R|_W \leq \frac{\rho}{2M}$, for ρ small enough.

Let $C = \{z \mid z\colon [0,\infty) \to W,\ z$ continuous, $|e^{-\mu t} z(t)|_W \to 0$ as $t \to \infty\}$. It will be shown

$$(T_R z)(t) = e^{\alpha_2 t} R + \int_0^t e^{\alpha_2 (t-s)} Q(0,h(s,(u_s,u(s))))ds$$

$$- \int_t^\infty e^{\alpha_1 (t-s)} P(0,h(s,(u_s,u(s))))ds$$

is a contraction for $|R|_W$ small enough.

First, $T_R z$ is continuous. $e^{\alpha_1 t} R$ is continuous in t, and

$$\int_0^t e^{\alpha_2 (t-s)} Q(0,h(s,u_s,u(s))ds$$

is continuous in t by the same argument as in Theorem (2.1). The last term is continuous by the usual theorem on continuous dependence of an integral on a parameter. Now by Theorem 4.1, $|e^{\alpha_2 t}| \leq M e^{-\mu t}$ for all $t \geq 0$, where M is a positive constant ≥ 1. So

$$\sup_{t \geq 0} |e^{-\mu t} T_R z(t)| < \infty,$$

since

$$\sup e^{-\mu t}(T_R z(t)) \leq \sup_{t\geq 0} e^{-\mu t} e^{\lambda_{n+1}} M |R|$$

$$+ e^{-\mu t}\left(\int_0^t e^{\lambda_{n+1}(t-s)} |P| e^{-\mu s} ds\right) \alpha(\rho) |z|_C$$

$$+ e^{-\mu t}\left(\int_+^\infty e^{-\lambda_n (t-s)} |Q| e^{-\mu s} ds\right) \alpha(\rho) |z|_C$$

$$\leq |R| M + \alpha(\rho)\left[\frac{|\rho|}{M-\lambda_{n+1}} + \frac{|Q|}{\lambda_n - M}\right] |z|_C$$

So $T_R z\colon C \to C$. Also let $BC(\rho) = \{z \in C \mid |z|_c \le \rho\}$. Then if $z \in BC(\rho)$, from (4.5),

$$|R_R z|_c \le M|R|_w + \alpha(\rho)\left(\frac{|\rho|}{\lambda_n-\mu} + \frac{|Q|}{\mu-\lambda_{n+1}}\right)|z|_c \le M|R|_w + \frac{1}{2}\rho,$$

if ρ is chosen so small that $K(\rho)\left(\frac{|\rho|}{\lambda_n-\mu} + \frac{|Q|}{\mu-\lambda_{n+1}}\right) < \frac{1}{2}$. So choosing $|R|_w \le \frac{\rho}{2M}$,

$$T_R\colon BC(\rho) \to BC(\rho).$$

Also T_R is a contraction on $BC(\rho)$. Using (4.5),

$$|R_R z - T_R q|_c \le \sup_{t\ge 0} e^{-\mu t}\Big[\int_0^t e^{\lambda_{n+1}(t-s)}\alpha(\rho)|P|e^{+\mu s}|z-q|_c ds$$

$$+ |Q|\int_t^\infty e^{-\lambda_n(t-s)}\alpha(\rho)e^{\mu(s)}|z-q|_c ds$$

$$\le \alpha(\rho)\left[\frac{|Q|}{-\lambda_{n+1}+\mu}\right] \le \frac{1}{2}$$

by choice of ρ. So T_R has a unique fixed point z in $BC(\rho)$. Hence, z satisfies

$$z(t) = e^{\mathcal{A}_2}R + \int_0^t e^{\mathcal{A}_2(t-s)}Q(0,h(s,z(s)))ds$$

$$- \int_t^\infty e^{\mathcal{A}_1(t-s)}P(0,h(s,z(s)))ds$$

or

$$z(t) = e^{\mathcal{A}(t)}\left(R - \int_0^\infty e^{-\mathcal{A}_1 s}P(0,h(s,z(s)))ds\right)$$

$$+ \int_0^t e^{\mathcal{A}(t-s)}(0,h(s,z(s))ds.$$

By Theorem 3.1, $P_2 z(s)$ gives a solution of (4.3). Also

$$\lim_{t\to\infty} |e^{-\mu t} z(t)| \le e^{-\mu t} e^{\lambda_{n+1} t} M|R|$$
$$+ e^{\lambda_{n+1} - \mu t} \frac{|Q|}{\lambda_{n+1} - \mu} \alpha(\rho) + e^{(\mu - \lambda_n) t} \frac{P}{\mu - \lambda_n \alpha} \alpha(\rho) \to 0$$

as $t \to \infty$. Furthermore, if $(w,p) \in s(\mu)$, then

$$Q(w,p) - (w,p) = -P(w,p)$$
$$= \int_0^\infty e^{\alpha_1 s} P(0, g(s, us, u(s)))ds.$$

Since, if $|R| \le \frac{\rho}{2M}$, $|u(t)|_c \le \rho$ by definition of $S(\mu)$,

$$\left|\int_0^\infty e^{-\alpha_1 s} P(0, g(s, u_s, u(s)))ds\right|$$
$$\le \frac{1}{\lambda_n - \mu} k(|u_t, u(t)|_c) |u_t, u(t)|_c$$
$$\le \frac{1}{\lambda_n - \mu} k(2M|P|(w,p)) 2M|P|(w,p)$$

Hence

$$\lim_{|(w,p)|\to 0} \frac{|(w,p) - Q(w,p)|_w}{|(w,p)|_w} \le K(2M|P||(w,p)|_w) \to 0,$$

and $S(\mu)$ is tangent to W_2 at 0.

For the unstable manifold, if $(w,p) \in U(\rho)$, then since $u(t)$ exists for all $t \le 0$, if $t_1 < t$,

$$(u_t,u(t)) = e^{\mathcal{A}(t-t_1)}(u_{t_1},u(t_1))$$

$$+ \int_{t_1}^{t} e^{\mathcal{A}(t-s)}(0,h(s,(u_s,u(s))))ds.$$

Taking projections of this gives

$$Q(u_t,u(t)) = e^{\mathcal{A}_2(t-t_1)}Q(u_{t_1},u(t_1))$$

$$+ \int_{t_1}^{t} e^{\mathcal{A}_2(t-s)} Q(0,g(s,(u_s,u(s))))ds.$$

Since

$$|e^{\mathcal{A}_2(t-t_1)}Q(u_{t_1},u(t_1))| \leq \rho e^{+\mu t}e^{-\lambda_n t}M|Q|,$$

and $\lim\limits_{t\to\infty} \rho e^{\mu t}e^{-\lambda_n t}M|Q| = 0$, letting $t \to -\infty$ gives

$$Q(u_t,u(t)) = \int_{-\infty}^{t} e^{\mathcal{A}_2(t-s)} Q(0,h(s,(u_s,u(s))))ds .$$

Also

$$(w,p) = e^{-\mathcal{A}t}(u_t,u(t)) + \int_{t}^{0} e^{\mathcal{A}(-s)}(0,h(s,(u_s,u(s))))ds.$$

So

$$P(w,p) = e^{-\mathcal{A}_1 t}P(u_t,u(t)) + \int_{t}^{0} e^{\mathcal{A}_1(-s)} P(0,h(s,(u_s,u(s))))ds$$

since $e^{\mathcal{A}_1 t}$ exists for all t, this becomes

$$P(u_t,u(t)) = e^{\mathcal{A}_1 t}P(w,p) - \int_{t}^{0} e^{\mathcal{A}_1(t-s)} P(0,h(s,(u_s,u(s)))) .$$

Adding the equations for $Q(u_t,u(t))$ and $P(u_t,u(t))$ gives

$$(4.8)\qquad (u_t, u(t)) = P(u_t, u(t)) + Q(u_t, u(t))$$

$$= e^{\alpha_1 t} P(w,p) - \int_t^0 e^{\alpha_1 (t-s)} P(0, h(s,(u_s, u(s))))ds$$

$$+ \int_\infty^t e^{\alpha_2 (t-s)} Q(0), g(s,(u_s, u(s))))ds.$$

Now computations similar to the ones for (4.4) give the results for the set $U(\mu)$.

§3. **Illustration of the projections in section 4.1.**

Now an example will be given of the projections in section 1, in the case where R^n is R. This is slightly simpler than the general case.

Let $\Omega = [0, 2\pi]$. Then equation (2.1) is

$$(\frac{\partial}{\partial t} u)(x,t) = c(\frac{\partial^2}{\partial x^2} u)(x,t) - \int_{-\infty}^t g(t-s)(\frac{\partial^2}{\partial s^2} u)(x,s)ds$$

$$u(0,t) = u(2\pi,t) = 0$$

$$u(x,t) = w(x,t) \qquad t \leq 0.$$

Here $\mu_n = -n^2$ and $\varphi_n = (\sin nx) \dfrac{1}{\sqrt{\pi}}$.

Any $w(x,t) \in L_{2,e^{at}}(-\infty, 0; H_0^1(0,2\pi))$ can be written in the form

$$w(x,t) = \sum_{j=1}^{\infty} f_j(t) \sin(jx).$$

Thus

$$P_{1,n}(w(x,t)) = \sum_{j=1}^{n} f_j \sin(jx)$$

and

$$P_{2,n}(w(x,t)) = \sum_{j=n+1}^{\infty} f_j(t) \sin (j x).$$

If $u(t,x)$ is the solution of (2.1) with initial value $w(t,x)$, then

$$u(t,x) = \sum_{j=1}^{\infty} h_j(t) \sin (j x)$$

where $h_n(t)$ satisfies

$$h_n'(t) = -cn^2 h_n(t) + n^2 \int_{-\infty}^{t} g(t-s) h_n(s) ds.$$

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