RELATIVE PROJECTIVITY AND RADICALS IN MODULAR GROUP ALGEBRAS

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ABSTRACT

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By

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Let H be a subgroup of a finite group G and F a field of characteristic p. Using J.A. Green's concept of vertices, our first main result characterizes completely the class $\mathcal{P}$ of those subgroups H of G for which every FG-module is H-projective. The pair (G,H) is said to have a projective-pairing if $H \in \mathcal{P}$. We show that the class $\mathcal{P}$ consists of precisely those subgroups of G which contain a p-Sylow subgroup of G (p = characteristic F). As it turns out, two other classes - the class $\mathcal{R}$ of subgroups H of G for which $\text{Rad } FG \subseteq (\text{Rad } FH)\cdot FG$, Rad denoting the Jacobson Radical of the ring concerned, and the class $\mathcal{C}$ of subgroups H of G for which the induced FG-module $\mathfrak{N}^G = \mathfrak{N} \otimes_{FH} FG$ is completely reducible for each irreducible FH-module $\mathfrak{N}$ - are almost equivalent to the class $\mathcal{P}$. We show that for normal subgroups these three concepts coincide. Otherwise examples exist showing that these three classes are distinct in general.

A (finite) group G is called a PRC-group if $\mathcal{P} = \mathcal{R} = \mathcal{C}$. Our main results may be summarized as follows: G is a PRC-group over F of characteristic p if it satisfies any of the

following conditions: (1) $p = 0$ or $p \nmid |G|$, (2) G is a p-group, (3) G is a Frobenius group with kernel G' and complement a p-Sylow subgroup of G, (4) G is an extension of a PRC-group by a p'-group, and (5) G is an extension of a p-group by a PRC-group.

Finally, we attempt to spot out "Projective - Sensitive" subgroups of G, which are the subgroups H of G satisfying (i) Whenever (G,M) is a projective-pairing then $M \geq H$ or $(H, H \cap M)$ is a projective-pairing, and (ii) If $K \leq H$ and (H,K) is a projective-pairing then there exists an $M \leq G$ such that (G,M) is a projective-pairing and $K = M \cap H$.

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NOTATION AND TERMINOLOGY

G	=	a finite group
$H \leq G$	=	H is a subgroup of G
$H \trianglelefteq G$	=	H is a normal subgroup of G
F	=	Base field of characteristic p
FG	=	Group algebra of G over F
$[G:H]$	=	index of subgroup H in G
$\lvert G \rvert$	=	order of G
G'	=	the derived group of G
$Z(FG)$	=	Center of the group algebra FG
P	=	a p-Sylow subgroup of G (p = characteristic of F)
$H \leq_G K$	=	H,K are subgroups of G and there is an x in G such that $x^{-1}Hx \leq K$
H^x	=	$x^{-1}Hx$, $x \in G$, $H \leq G$
Rad FG	=	the Jacobson Radical of the group algebra FG
$\dim_F V$	=	dimension of the vector space V over F
$\oplus$	=	direct sum
$\otimes$	=	tensor product
$\mathfrak{N}^G$	=	$\mathfrak{N} \otimes_{FH} FG$, $\mathfrak{N}$ being an FH-module, $H \leq G$
$\mathfrak{M}_H$	=	an FG-module $\mathfrak{M}$ considered as an FH-module, $H \leq G$
$\mathfrak{A}_G(H)$	=	ideal in FG generated by $\{h-1 \mid h \in H \leq G\}$
$\mathfrak{A}_G^{-1}(I)$	=	$\{g \in G \mid g - 1 \in I$, I an ideal in $FG\}$

CHAPTER I

INTRODUCTION TO THE PROBLEM AND PRELIMINARY RESULTS

§1. Introduction

Let H be a subgroup of a finite group G and F a field of characteristic p. The H-projective FG-modules were studied by G. Hochschild and D.G. Higman [7], [6]. Green used this concept to classify indecomposable FG-modules through the theory of Vertices and Sources [4]. Using his concept of Vertices, our first main result (Theorem 3) characterizes the class $\mathcal{P} = \{H < G \mid \text{every FG-module is H-projective}\}$. For a subgroup H in $\mathcal{P}$ we call (G,H) a projective-pairing. During the course of our investigations we are led to study two more classes: the class $\mathcal{R}$ of subgroups H of G for which Rad FG $\subseteq$ (Rad FH).FG and the class $\mathcal{C}$ of subgroups H of G for which $\mathfrak{N}^G = \mathfrak{N} \otimes_{FH} FG$ is completely reducible for each irreducible FH-module $\mathfrak{N}$. We show that these three concepts coincide for normal subgroups (Theorem 7). Otherwise examples exist showing the distinctness of these classes (Theorem 8).

A (finite) group G is called a PRC-group over F if $\mathcal{P} = \mathcal{R} = \mathcal{C}$. Our results in Chapter III may be summarized as follows: G is a PRC-group over F of characteristic p if it satisfies any of the following conditions: (1) p = 0 or $p \nmid |G|$ - in this case each class being the class of all subgroups

of G, (2) G is a p-group - in this case each class consists of singleton $\{G\}$ only, (3) G is a Frobenius group with kernel G' and complement a p-Sylow subgroup of G, (4) G is an extension of a PRC-group by a p'-group, and (5) G is an extension of a p-group by a PRC-group.

As a special case it follows that if G has a normal p-Sylow subgroup then G is PRC and that if Rad FG is central in FG then G is PRC. We also prove that if G is p-nilpotent or supersolvable then $\mathcal{R}$ is a subclass of $\mathcal{P}$.

In the last chapter we attempt to spot out those subgroups H of G which are "Projective-Sensitive" in the sense that they satisfy the following two conditions: (i) Whenever (G,M) is a projective-pairing then $M \geq H$ or $(H, H \cap M)$ is a projective-pairing, (ii) If $K \leq H$ and (H,K) is a projective-pairing then there exists an $M \leq G$ such that (G,M) is a projective-pairing and $K = M \cap H$. We give a necessary and sufficient condition for a normal subgroup to be projective-sensitive. Also a necessary and sufficient condition for G to have all its subgroups projective-sensitive is given. Finally, we give an application which determines the defect groups of blocks of G when G has all its subgroups projective-sensitive.

§2. Preliminary Definitions and Results

Let F be a field of characteristic $p > 0$ and G be a finite group such that p divides the order $|G|$ of G. All modules under consideration will be right modules finitely generated over F. Let $\mathfrak{M}$ be a G-module over F or an

FG-module, where FG denotes the group algebra of the group G over the field F. If H is a subgroup of G then $\mathfrak{M}$ can always be regarded as an FH-module in a natural way by restricting the domain of operators to FH. The resulting FH-module is denoted by $\mathfrak{M}_H$. Similarly if $\mathfrak{N}$ is an FH-module, the induced FG-module $\mathfrak{N} \otimes_{FH} FG$ is denoted by $\mathfrak{N}^G$, where $\otimes$ denotes the tensor-product.

Then the following elementary facts about the induced modules can easily be verified (See, for example, [4]):

(2.1) If $\mathfrak{N}$ is FH-direct sum of two FH-modules $\mathfrak{N}_1$ and $\mathfrak{N}_2$, then $\mathfrak{N}^G = \mathfrak{N}_1^G \oplus \mathfrak{N}_2^G$: as FG-direct sum.

If R is another subgroup of G such that $H \subseteq R \subseteq G$, then we have the transitive laws:

(2.2) $$(\mathfrak{N}^R)^G \cong \mathfrak{N}^G, \quad \text{and}$$

(2.3) $$(\mathfrak{M}_R)_H = \mathfrak{M}_H \ .$$

Now let $G = Hx_1 \cup Hx_2 \cup \ldots \cup Hx_n$, $n = [G:H]$ be a coset-decomposition of G over H. We shall always take $x_1 = 1$. Then

(2.4) $$\mathfrak{N}^G = \oplus \sum_{i=1}^{n} \mathfrak{N} \otimes x_i \quad \text{as vector spaces,}$$

where, for any $x \in G$, $\mathfrak{N} \otimes x = \{m \otimes x \mid m \in \mathfrak{N}\}$. It is clear that $\mathfrak{N} \otimes x$ is $F(x^{-1}Hx)$-module and that $\dim_F(\mathfrak{N} \otimes x) = \dim_F \mathfrak{N}$. We can regard $\mathfrak{N} \otimes x$ as a submodule of $(\mathfrak{N}^G)_{x^{-1}Hx}$. From (2.4) it follows that

(2.5) $$\dim_F \mathfrak{N}^G = [G:H] \cdot \dim_F \mathfrak{N} \ .$$

The mapping $m \to m \otimes x$ $(m \in \mathfrak{N})$ although it is not in general a module-isomorphism, it does induce an isomorphism between the submodule lattices of $\mathfrak{N}$ and $\mathfrak{N} \otimes x$. Thus

(2.6) $\mathfrak{N} \otimes x$ is indecomposible if and only if $\mathfrak{N}$ is so .

Also it can easily be verified that

(2.7) $$(\mathfrak{N} \otimes x)^G \cong \mathfrak{N}^G \text{ for any } x \in G .$$

We make

<u>Definition 1</u>. If $\mathfrak{M}$ and $\mathfrak{N}$ are two FG-modules, we say that $\mathfrak{M}$ is a component of $\mathfrak{N}$ if $\mathfrak{M}$ is FG-isomorphic to a direct summand of $\mathfrak{N}$.

<u>Definition 2</u>. Let H be a subgroup of G. An FG-module $\mathfrak{M}$ is said to be H-projective if every exact sequence $0 \to \mathfrak{N} \to \mathcal{L} \to \mathfrak{M} \to 0$ of FG-modules for which the associated sequence of restrictions $0 \to \mathfrak{N}_H \to \mathcal{L}_H \to \mathfrak{M}_H \to 0$ splits over FH, is itself split over FG.

<u>Remarks</u>. (1) H-projectivity is a special case of "relative-projectivity" defined by Hochschild [7]. An H-projective module $\mathfrak{M}$ is (R,S)-projective in the terminology of [7] by taking $R = FG$ and $S = FH$. In the general theory there is also the dual notion of "relative-injective" modules.

(2) We observe that $\mathfrak{M}$ is $\{1\}$-projective if and only if it is projective in the usual sense. In this sense, therefore, H-projectivity is a generalization of (usual) projectivity.

<u>Definition 3</u>. Let $H \leq G$. We say that (G,H) is a projective-pairing over F, if every exact sequence $0 \to \mathfrak{N} \to \mathcal{L} \to \mathfrak{M} \to 0$ of

FG-modules for which the exact sequence $0 \to \mathfrak{N}_H \to \mathcal{L}_H \to \mathfrak{M}_H \to 0$ splits over FH, is itself split over FG.

It follows immediately from Definition (2) above that (G,H) is a projective-pairing if and only if every FG-module is H-projective.

Definition 4. Let $G = \bigcup_{i=1}^{n} Hx_i$, $x_1 = 1$, be a coset decomposition of G over H. Then we can treat FG as a free FH-module with basis $x_1 = 1, x_2, \ldots, x_n$ and every element of FG can be written in the form $\sum_{i=1}^{n} p_i x_i$ where p_i's are in FH. The pair (G,H) is said to have property ρ over F if $\sum_i p_i x_i \in \text{Rad FG}$ implies each $p_i \in \text{Rad FH}$, where Rad denotes the Jacobson-Radical of the ring concerned.

In [11] it is shown that property ρ is independent of the choice of coset representatives and that it is a transitive-property in the sense that if $K \leq H \leq G$ and the pairs (G,H) and (H,K) have property ρ then the same is true for the pair (G,K).

It is evident that the property ρ is equivalent to the requirement $\text{Rad FG} \subseteq (\text{Rad FH}) \cdot \text{FG}$.

Comparing the dimensions, we see that if (G,H) has property ρ, then

(2.8) $$\dim_F(\text{Rad FG}) \leq [\dim_F(\text{Rad FH})] \cdot [G:H] .$$

D.G. Higman's Criteria for H-projectivity. The following characterization of H-projective FG-modules is due to D.G. Higman [6]:

(2.9) The following statements for an FG-module $\mathfrak{M}$ are equivalent:

(i) $\mathfrak{M}$ is H-projective;

(ii) $\mathfrak{M}$ is a component of $(\mathfrak{M}_H)^G$;

(iii) There exists an FH-endomorphism η of $\mathfrak{M}$ such that

$$\sum_{i=1}^{n} x_i^{-1} \eta\, x_i = 1_{\mathfrak{M}},$$

where $\{x_i\}$ are the coset representatives of G over H and $1_{\mathfrak{M}}$ is the identity map on $\mathfrak{M}$.

Note: Here we make the convention that if $\eta: \mathfrak{M} \to \mathfrak{M}$ is a map then $x\eta$ and ηx, $x \in G$, denote the mappings which take $m \in \mathfrak{M}$ to $mx\eta$ and $m\eta x$ respectively.

In [4] Green gave the following definition of H-projectivity which can easily be seen to be equivalent to those in (2.9).

(2.10) An FG-module $\mathfrak{M}$ is H-projective if and only if there exists an FH-module $\mathfrak{N}$ such that $\mathfrak{M}$ is a component of $\mathfrak{N}^G$.

We now introduce a partial ordering $\underset{G}{\leq}$ on the class of subgroups of G by saying that $H \underset{G}{\leq} R$ if for some $x \in G$, $x^{-1}Hx \leq R$. It follows that if $H \underset{G}{\leq} R$ and $R \underset{G}{\leq} H$ then H and R are conjugate subgroups of G and we express this by writing $H \underset{G}{=} R$.

From (2.2) and (2.7) we infer:

(2.11) If an FG-module $\mathfrak{M}$ is H-projective, then it is R-projective for every subgroup R such that $H \underset{G}{\leq} R$.

Now let P be a p-Sylow subgroup of G, then the index $n = [G:P]$ is prime to p and taking $\eta = n^{-1}1_{\mathfrak{M}}$ in (2.9)(iii) we obtain another theorem of D.G. Higman:

(2.12) Let P be a p-Sylow subgroup of G (p being the characteristic of F) then every FG-module is P-projective.

Combining (2.11) and (2.12) we obtain

(2.13) If H is a subgroup of G such that $H \underset{G}{\geq} P$ then (G,H) is a projective-pairing.

In order to save space and avoid lengthy repititions we abbreviate the following three classes of subgroups of G.

Definition 5. Let G be finite and F have characteristic p. We denote by

$$\mathcal{P} = \mathcal{P}(G,F) = \{H < G \mid (G,H) \text{ is a projective-pairing over } F\}$$

$$\mathcal{R} = \mathcal{R}(G,F) = \{H < G \mid (G,H) \text{ has property } \mathfrak{p} \text{ over } F\}$$

$$\mathcal{C} = \mathcal{C}(G,F) = \{H < G \mid \text{For every irreducible FH-module } \mathfrak{N}, \text{ the induced FG-module } \mathfrak{N}^G \text{ is completely reducible}\}.$$

Note that here $H < G$ means "proper" subgroup, $\{1\}$ being considered as proper subgroup.

For the class $\mathcal{P}$ we have

Definition 6. A minimal member B of $\mathcal{P}$ under the partial ordering $\underset{G}{\leq}$ is called a projective-foot of G and a maximal element T is called a projective-top of G.

These exist, because $\mathcal{P}$ is a finite class.

In the course of our investigations the notion of Frobenius groups will be important which we define next.

Definition 7. A finite group G is called Frobenius group with kernel M and complement K if $G = KM$, $M \lhd G$, $K \cap M = 1$ and $K \cap K^x = 1$ for all $x \in G - K$.

For group-theoretic properties of Frobenius groups we refer to Scott [10].

The importance of representation theoretic properties of Frobenius groups was realized by Wallace ([14] and [15]) in connection with the centrality of the radical of the group algebra. We state below his results which are pertinent to our investigation.

(2.14) If $|G| = p^a m$, $(p,m) = 1$, and Char $F = p$ then $\dim_F(\text{Rad } FG) \geq p^a - 1$, the equality holds if and only if $G = PM$, $M \triangleleft G$, $P \cap M = 1$, is Frobenius group with kernel M and complement a p-Sylow subgroup P of G.

(2.15) Rad $(FG) \subseteq Z(FG)$ if and only if G is one of the following three types:

(i) G has order prime to p

(ii) G is abelian

(iii) If P is a p-Sylow subgroup of G then $G'P$ is a Frobenius group with kernel G', the derived group of G.

We will also use the result about the dimension of the radical of FG due to Brauer and Nesbitt [2]:

(2.16) $\dim_F(\text{Rad } FG) = |G| - \sum_{k=1}^{\ell} f_k^2$, where $f_1, f_2, \ldots, f_\ell$ are the degrees of absolutely irreducible p-modular representations occurring in the regular representation of G.

For most of the other so called "standard" results of representation theory we refer to the basic text by Curtis and Reiner [3].

CHAPTER II

GENERALITIES ABOUT THE CLASSES $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$

In this chapter we characterize the class $\mathcal{P}$ completely and show that when subgroup under consideration is normal in G, the three properties are equivalent. We also prove, by exhibiting examples, that the classes $\mathcal{P}$ and $\mathcal{R}$ and $\mathcal{P}$ and $\mathcal{C}$ are entirely independent of each other while $\mathcal{C}$ is a subclass of $\mathcal{R}$.

§3. Projective-Pairing

We devote this section to classify the class $\mathcal{P}$ of all proper subgroups H of G such that (G,H) is a projective-pairing. By (2.13) if $H \underset{G}{\geq} P$, P a p-Sylow of G, then $H \in \mathcal{P}$. It is proved in Khatri and Sinha [8] that the converse of (2.13) is also true. For non-p-group G we state the main results of [8] below and refer to the same for the proofs.

<u>Lemma 1</u>. Let $H \leq S \leq G$ such that (G,S) and (S,H) are projective pairings over F. Then (G,H) is a projective-pairing.

<u>Lemma 2</u>. If (G,H) is a projective-pairing then for every subgroup S of G such that $S \underset{G}{\geq} H$, (G,S) is a projective-pairing.

<u>Theorem 1</u>. If $H \in \mathcal{P}(G,F)$ then p divides $|H|$.

<u>Theorem 2</u>. A projective-foot B of G is a p-Sylow subgroup of G.

<u>Theorem 3</u>. $\mathcal{P}(G,F)$ consists of precisely those subgroups of G which contain a p-Sylow subgroup of G.

Theorem 4. A subgroup T of G is a projective-top if and only if it is a maximal subgroup containing a p-Sylow subgroup of G.

Clearly T is not unique up to conjugacy while B is.

The results about p-groups we will include in the next chapter along with the other results.

§4. Relation Between Projective-Pairing, Property ρ and Complete Reducibility of Induced Modules for Normal Subgroups

Having characterized the class $\mathcal{P}$ in the above section, we turn our attention to its connection with the representation theory. As was realized by Sinha [12] and as we shall see later that the property of complete reducibility of induced modules is closely related with these concepts. We state below a theorem of Sinha [12]:

Theorem 5. Let H be a subgroup of G. If for each irreducible FH-module $\mathfrak{N}$, the induced module $\mathfrak{N}^G$ is completely reducible over FG then (G,H) has property ρ. Conversely if $H \trianglelefteq G$ then (G,H) property ρ implies that $\mathfrak{N}^G$ is completely reducible for every irreducible FH-module $\mathfrak{N}$.

We now prove the equivalence of projective-pairing and property ρ for normal subgroups.

Theorem 6. If H is a normal subgroup of G then (G,H) is a projective-pairing if and only if (G,H) has property ρ.

Proof. First suppose $H \trianglelefteq G$ and (G,H) is a projective-pairing. Let $G = \bigcup_{i=1}^{n} Hx_i$, $x_1 = 1$, be a coset decomposition of G over H. We observe that for $h \in H$, $x_ih = \varphi_i(h)x_i$ for all i, where each $\varphi_i(h) = x_ihx_i^{-1}$ induces an automorphism of the normal subgroup

H. We can extend φ_i by linearity to an automorphism of FH.

Now let $\mathfrak{N}$ be an irreducible FH-module. Then $\mathfrak{N}^G = \oplus \sum_{i=1}^{n} \mathfrak{N} \otimes x_i$, where each $\mathfrak{N} \otimes x_i$ is an irreducible FH-module. Thus $\mathfrak{N}^G$ is completely reducible over FH. Hence given any exact sequence of FG-modules

$$0 \to \mathfrak{M} \to \mathfrak{N}^G \to \mathcal{L} \to 0,$$

the sequence splits over FH. Then by the projective-pairing (G,H) the sequence splits over FG as well. Thus $\mathfrak{N}^G$ is completely reducible over FG.

Now let $\sum_i p_i x_i \in \text{Rad FG}$, $p_i \in \text{FH}$. Since $\mathfrak{N}^G$ is completely reducible, we have $\mathfrak{N}^G(\sum_i p_i x_i) = 0$. In particular, we have $0 = (n \otimes 1)(\sum_i p_i x_i) = \sum_i np_i \otimes x_i$, $n \in \mathfrak{N}$. This implies that each $np_i = 0$ for all $n \in \mathfrak{N}$. That is, $\mathfrak{N}p_i = 0$ for all i. Since $\mathfrak{N}$ was arbitrary irreducible FH-module, so each $p_i \in \text{Rad FH}$. This gives property ρ for (G,H).

Conversely suppose that (G,H) has property ρ. Let $\mathfrak{M}$ be an irreducible FG-module. By Clifford's theorem we have

$$\mathfrak{M}_H = \mathfrak{N}_1 \oplus \ldots \oplus \mathfrak{N}_k,$$

where each $\mathfrak{N}_i$ is an irreducible FH-module. Consider $\mathfrak{N}_1$. We have $\text{Hom}_{FH}(\mathfrak{N}_1, \mathfrak{M}_H) \cong \text{Hom}_{FG}(\mathfrak{N}_1^G, \mathfrak{M})$. Since $\mathfrak{N}_1$ is a component of $\mathfrak{M}_H$, the left hand side is non-zero and so is the right hand side. Thus $\mathfrak{M}$ is a composition factor of $\mathfrak{N}_1^G$. Since (G,H) has property ρ, by Theorem 5, $\mathfrak{N}_1^G$ is completely reducible over FG. Hence $\mathfrak{M}$ is a component of $\mathfrak{N}_1^G$. Therefore, by Green's criteria for H-projectivity, $\mathfrak{M}$ is H-projective.

As $\mathfrak{M}$ is an arbitrary irreducible FG-module, it follows that every irreducible FG-module is H-projective. In particular, the trivial FG-module, F, is H-projective. Hence vertex of F - namely a p-Sylow subgroup P of G - is contained in H. The projective pairing for (G,H) now follows from Theorem 3.

This completes the proof of the theorem.

Corollary 1. If $H \trianglelefteq G$ such that (G,H) is a projective-pairing then the representation of G, induced from an irreducible representation of H, is completely reducible.

Corollary 2. A necessary and sufficient condition for (G,H) to have a projective-pairing is that every irreducible FG-module is H-projective.

Corollary 3. Let $H \trianglelefteq G$ and $\mathfrak{M}$ an irreducible FG-module. If $\mathfrak{N}$ is a composition factor of $\mathfrak{M}_H$ then $\mathfrak{M}$ is a composition factor of $\mathfrak{N}^G$.

Though some of the implications in the next theorem go through without requiring the subgroup H to be normal in G, we state our main theorem for normal subgroups.

Theorem 7. Let $H \trianglelefteq G$. The following statements are equivalent:

(1) H contains a p-Sylow subgroup of G.

(2) (G,H) is a projective-pairing.

(3) (G,H) has property ρ.

(4) For every irreducible FH-module $\mathfrak{N}$, the induced FG-module $\mathfrak{N}^G$ is completely reducible over FG.

Proof. It follows from (2.13) and Theorems 3, 5, and 6.

We now drop the normality condition on the subgroup H and ask ourselves the question about the relation between the three properties under consideration when H is not normal in G.

By Theorem 5, $\mathcal{C}$ is a subclass of $\mathcal{R}$. Our next theorem shows that, in general, the classes $\mathcal{P}$ and $\mathcal{R}$ are independent.

Theorem 8. In general, the property of projective-pairing and property ρ are independent of each other.

Proof. We prove this theorem by exhibiting examples of group G and subgroup H such that the pair (G,H) enjoys one property but not the other.

Our first example shows that projective-pairing does not imply property ρ.

Take $G = A_5$, $H = A_4$, $p = \text{Char } F = 3$, where A_n denotes the alternating group on n symbols. Since H contains a 3-Sylow subgroup of G, (G,H) is a projective-pairing by (2.13). We show that (G,H) cannot have property ρ. We may take F to be a splitting field for G.

It is well-known that over complex field A_5 has five irreducible representations, say, T_1,T_2,T_3,T_4,T_5 of degrees 1,3,3,4,5 respectively. Taking mod 3, it follows from the results of Brauer and Nesbitt (Theorem 1 of [2] and a remark on p. 936 together with Theorem 5 of [1]) that $\overline{T}_1,\overline{T}_2,\overline{T}_3$ and $\overline{T}_4$ remain irreducible while $\overline{T}_5 \approx \overline{T}_1 + \overline{T}_4$. Since A_5 has four 3-regular classes, these are all the irreducible representations of A_5 over the field F of characteristic 3. Now by Brauer and Nesbitt's result (2.16)

$$\dim_F(\text{Rad } FA_5) = 60 - (1^2 + 3^2 + 3^2 + 4^2) = 25 .$$

Also since $H = A_4$ is a Frobenius group with complement a 3-Sylow subgroup of H,

$\dim_F(\text{Rad } FA_4) = 3 - 1 = 2$, by Wallace's result (2.14) .

Now if (G,H) has property ρ, then we must have

$$\text{Rad } FG \subseteq (\text{Rad } FH)\cdot FG \ .$$

Considering dimensions, we must have

$$\dim_F(\text{Rad } FG) \leq (\dim_F(\text{Rad } FH))\cdot[G:H] \ .$$

That is, we must have $25 \leq 2.5 = 10$, which is impossible. Thus (G,H) does not have property ρ.

Though we are not able to find an example to show that property ρ does not imply projective-pairing in finite groups, we give an example exhibiting the same when G is an infinite group and H a subgroup of finite index.

Let Char $F = 2$ and G a group generated by elements a and b subject to the relations $b^2 = 1$, $bab^{-1} = a^{-1}$. That is, G is an extension of infinite cyclic group $\langle a\rangle$ by the group $\langle b\rangle$ of order 2. It is known that Rad $FG = (0)$ (See, for example, Passman [9]).

Let $H = \langle a\rangle$. Since Rad $FG = (0)$, (G,H) has property ρ over F. Now Gustafson [5] communicated to me that if G is an infinite group and H a subgroup of finite index then (G,H) is a projective-pairing is equivalent to the index $[G:H]$ being unit in the base field. In view of this result clearly (G,H) cannot be a projective-pairing.

This completes the proof of the theorem.

<u>Corollary </u>. In general the classes $\mathcal{P}$ and $\mathcal{C}$ are independent of each other.

Proof. By Theorem 8, $\mathcal{P}$ and $\mathcal{R}$ are independent. But $\mathcal{C} \subseteq \mathcal{R}$, therefore $\mathcal{P} \not\subseteq \mathcal{C}$. The second example in the above theorem shows that $H = \langle a \rangle \in \mathcal{C}$, by Theorem 5, but $H \notin \mathcal{P}$.

This is what we wanted.

§5. Invariance of Projective-Pairing and Property ρ

In the previous sections we characterized projective-pairing completely and showed that when H was normal subgroup of G then projective-pairing was equivalent to property ρ. Also we showed that these two properties are, in general, independent of each other. A natural question now arises that how far these properties can be carried over under different algebraic transformation; for example, under taking direct products of groups or under homomorphisms. It is this question we now turn our attention to.

(A) Field-Extensions

We show that the extension of ground field F has no effect on projective-pairing and when F is splitting field for G, on property ρ. Indeed,

Theorem 9. Let G be a finite group and F a field of characteristic $p > 0$ and $L \supset F$ be a field extension. Then if for a subgroup H of G (G,H) is a projective-pairing over F, it has projective-pairing over L as well. If F is a splitting field for G, the same is true for property ρ.

Proof. The proof for projective-pairing is immediate, since the characteristic of the field is preserved under field-extensions.

We give a proof for property ρ. For standard results over radicals under field extensions we refer to [3].

Suppose now that (G,H) has property ρ over the splitting field F and $L \supset F$. Writing $G = \bigcup_i Hx_i$, let $\sum_i p_i x_i \in \text{Rad } LG$ with each $p_i \in LH$. Since $LH = FH \otimes_F L$, we may write, for each i,

$$p_i = \sum_j \ell_{ij} q_{ij} , \qquad \ell_{ij} \in L, q_{ij} \in FH .$$

Thus we have

$$\sum_{i,j} \ell_{ij} q_{ij} x_i \in \text{Rad } LG = (\text{Rad } FG) \otimes_F L \qquad (\text{See [3], §29}) .$$

This implies that, for each j,

$$\sum_i q_{ij} x_i \in \text{Rad } FG \quad \text{with} \quad q_{ij} \in FH .$$

The property ρ for (G,H) over F yields each $q_{ij} \in \text{Rad } FH$ so that $p_i = \sum_j \ell_{ij} q_{ij} \in \text{Rad } FH \otimes_F L = \text{Rad } LH$, for each i. This is what we set out to prove.

(B) Direct-Products

We prove the following theorem:

Theorem 10. Let H_1, H_2 be subgroups of the groups G_1, G_2 respectively. Then if (G_1,H_1) and (G_2,H_2) have property ρ (projective-pairing) so does the pair $(G_1 \times G_2, H_1 \times H_2)$.

Proof. As before the proof for projective-pairing is easy; for if $H_i \supseteq P_i$, $i = 1,2$, P_i = p-Sylow subgroup of G_i then $H_1 \times H_2 \supseteq P_1 \times P_2$ = p-Sylow subgroup of $G_1 \times G_2$ giving what is required. We turn to the proof for property ρ.

Write $G_1 = \bigcup_{i=1}^{n} H_1 x_i$ and $G_2 = \bigcup_{j=1}^{m} H_2 y_j$. Then we have

$$G_1 \times G_2 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (H_1 \times H_2)(x_i,y_j).$$

Now it is well-known that $F(G_1 \times G_2) = FG_1 \otimes FG_2$ as F-algebras and that $\text{Rad } F(G_1 \times G_2) = \text{Rad } FG_1 \otimes \text{Rad } FG_2$. Suppose now that $\sum_{i,j} (p_i,q_j)(x_i,y_j) \in \text{Rad } F(G_1 \times G_2)$ where for each i,j, $(p_i,q_j) \in F(H_1 \times H_2)$. Thus $\sum_{i,j} (p_i x_i, q_j y_j) \in \text{Rad } FG_1 \otimes \text{Rad } FG_2$, which gives

$\sum_i p_i x_i \in \text{Rad } FG_1$, $\sum_j q_j y_j \in \text{Rad } FG_2$ with $p_i \in FH_1$ and $q_j \in FH_2$.

Since (G_1,H_1) and (G_2,H_2) have property ρ, so we have each $p_i \in \text{Rad } FH_1$ and each $q_j \in \text{Rad } FH_2$. Hence for each i,j,

$$(p_i,q_j) \in \text{Rad } FH_1 \otimes \text{Rad } FH_2 = \text{Rad } F(H_1 \times H_2),$$

which is what we wanted.

(C) Homomorphic images

We have

Theorem 11. Let $\phi\colon G_1 \to G_2$ be an epimorphism, H_1 a subgroup of G_1 and $H_2 = \phi(H_1)$. Then (G_1,H_1) projective-pairing implies that for (G_2,H_2). If, in addition, ϕ maps $\text{Rad } FG_1$ onto $\text{Rad } FG_2$ then property ρ for (G_1,H_1) yields that for (G_2,H_2).

Proof. Once again proof for projective-pairing is immediate; for a p-Sylow subgroup is mapped onto a p-Sylow subgroup under an epimorphism and inclusion is preserved under ϕ.

We now give a proof for property ρ. Suppose that $G_1 = \bigcup_i H_1 x_i$ and $G_2 = \bigcup_j H_2 y_j$ and that $\sum_j q_j y_j \in \text{Rad } FG_2$, where

$q_j \in FH_2$, for all j.

Since $\phi(\text{Rad } FG_1) = \text{Rad } FG_2$ by hypothesis, there exists an $a \in \text{Rad } FG_1$ such that $\phi(a) = \sum_j q_j y_j$. Write $a = \sum_i p_i x_i$ with $p_i \in FH_1$. Then, by virtue of property ρ for (G_1,H_1), each p_i is in $\text{Rad } FH_1$. Let $\phi(x_i) = h_i y_i$, $h_i \in H_2$. Then

$$\sum_j q_j y_j = \phi(\sum_i p_i x_i) = \sum_i \phi(p_i)\phi(x_i) = \sum_i \phi(p_i) h_i y_i \ .$$

This implies that for each j, $q_j = \phi(p_j)h_j$. But

$$\phi(p_j) \in \phi(\text{Rad } FH_1) \subseteq \text{Rad } FH_2.$$

Hence each $q_j = \phi(p_j)h_j \in \text{Rad } FH_2$.

This gives property ρ for (G_2,H_2) and thus completes the proof of the theorem.

CHAPTER III

EQUIVALENCE OF THE CLASSES $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$

In the previous chapter we investigated projective-pairing and its relation to property ρ and complete reducibility of induced modules when the subgroup under consideration was normal. We also saw in general property ρ and projective-pairing were independent of each other. However, as we shall see, it turns out that in many cases these three classes of subgroups do coincide with each other. We now give some sufficient conditions on G which guarantee the equivalence of $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$. One condition is trivial: $p = 0$ or $p \nmid |G|$: in this case each class is the class of all subgroups of G.

§6. p-groups

For this class of groups it turns out that these three classes are vacuously equivalent. Indeed,

Theorem 12. Let G be a p-group and F a field of characteristic p. Then each of the three classes $\mathcal{P}$, $\mathcal{R}$, $\mathcal{C}$ is empty.

Proof. (1) The Class $\mathcal{P}$. Suppose that there is a subgroup H with (G,H) projective-pairing. Let L be a finite group with order prime to p and let G^* be an extension of G by L. Then clearly G is a p-Sylow subgroup of G^* and so (G^*,G) is a projective-pairing. By transitivity (Lemma 1) we obtain the projective-pairing for (G^*,H) which contradicts the fact that G is the projective-foot of G^* and thus completes the proof.

(2) The Class $\mathcal{R}$. Suppose there is an H with (G,H) property ρ. We know that Rad FG = Ideal in FG generated by the set $\{g - 1 | g \in G\}$ is of dimension $p^n - 1$ where $|G| = p^n$. Suppose $|H| = p^k$, $k \lneq n$. We have $\text{Rad } FG \subseteq (\text{Rad } FH) \cdot FG$. This gives $\dim_F(\text{Rad } FG) \leq (\dim_F \text{Rad } FH) \cdot [G:H]$; that is, $p^n - 1 \leq (p^k - 1)(p^{n-k}) = p^n - p^{n-k}$. This implies that $p^{n-k} \leq 1$, which is impossible since H is a proper subgroup of G. Thus the only subgroup of G having property ρ with G is the group G itself.

(3) The Class $\mathcal{C}$. Let H be a subgroup of G, $H \in \mathcal{C}(G,F)$. It is well-known that a p-group G has only one irreducible module over the field F of characteristic p - namely the field F itself, made into a G-module, by trivial action by elements of G. Therefore by hypothesis, we must have

$$(F_H)^G = F_G \oplus F_G \oplus \cdots \oplus F_G \quad : [G:H] \text{ copies.}$$

Thus F_G is a component of $(F_H)^G$ and so F_G is H-projective, contrary to the fact that F_G has vertex G [4].

The proof of the theorem is now complete.

§7. Special Type of Frobenius Groups

In this section we show that if G is p-nilpotent Frobenius group with kernel the normal p-complement then the three classes $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$ coincide. For groups of this type, Wallace's results [14], [15] are important to our investigation.

The following lemma, essentially due to Wallace [15] determines the radical of such a group.

<u>Lemma 3</u>. Let $G = PM$, $M \lhd G$, $P \cap M = 1$, be Frobenius group with kernel M and complement a p-Sylow subgroup P of G. Then over a field F of characteristic p,

$$\text{Rad } FG = \Big(\sum_{x\in M} x\Big)\cdot \text{Rad } FP = \text{Rad } FP \cdot \Big(\sum_{x\in M} x\Big) .$$

<u>Proof</u>. By Wallace's result (2.14) we have $\dim_F(\text{Rad } FG) = |P| - 1$. Let I be the subspace of FG spanned by $\{(\sum_{x\in M} x)(g-1) \mid g \in P\}$. Clearly $\dim_F I = |P| - 1$. Since $M \lhd G$, we have

$$y\cdot\Big(\sum_{x\in M} x\Big) = \sum_{x\in M} x = \Big(\sum_{x\in M} x\Big)\cdot y,\quad h\Big(\sum_{x\in M} x\Big) = \Big(\sum_{x\in M} x\Big)h,\quad y \in M,\ h \in P.$$

Therefore for $h,g \in P$ and $y \in M$, we have

$$hy\Big[\Big(\sum_{x\in M} x\Big)(g-1)\Big] = \Big(\sum_{x\in M} x\Big)h(g-1) = \Big(\sum_{x\in M} x\Big)[(hg-1) - (h-1)]$$

and

$$\Big[\Big(\sum_{x\in M} x\Big)(g-1)\Big]hy = \Big[(g-1)\Big(\sum_{x\in M} x\Big)\Big]hy = (g-1)h\Big(\sum_{x\in M} x\Big)y$$

$$= [(gh-1) - (h-1)]\Big(\sum_{x\in M} x\Big) = \Big(\sum_{x\in M} x\Big)[(gh-1) - (h-1)].$$

Thus I is an ideal in FG. But it is well-known that $\text{Rad } FP = \langle g-1 \mid g \in P\rangle$, hence $I = (\sum_{x\in M} x)\text{Rad } FP$ is a nilpotent ideal of dimension $|P| - 1$. This proves the lemma.

We now prove

<u>Theorem 13</u>. If $G = PM$, $M \lhd G$, $P \cap M = 1$ is a Frobenius group with kernel M, then the three classes $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$ coincide.

<u>Proof</u>. First we observe that $G' = M$ (Wallace [15]) so that by (2.15) $\text{Rad } FG \subseteq Z(FG)$.

Now suppose that (G,H) is a projective-pairing. We wish to show $H \in \mathcal{C}$. In the factorization $G = PM$, pick a p-Sylow subgroup P which H contains so that $H = P(H \cap M)$ is Frobenius group with kernel $H \cap M$. Let

$$G = \bigcup_i Hx_i \quad \text{and} \quad M = \bigcup_j (H \cap M)y_j$$

be coset decompositions of G and M respectively. By Lemma 3, a typical element r of Rad FG can be written as

$$r = a(\sum_{x \in M} x) = a(\sum_{h \in H \cap M} h)(\sum_j y_j), \text{ where } a \in \text{Rad } FP .$$

Let $\mathfrak{N}$ be an irreducible FH-module. Since Rad FG is central, we have

$$\begin{aligned} \mathfrak{N}^G \cdot r &= (\oplus \sum_i \mathfrak{N} \otimes x_i)r = \oplus \sum_i \mathfrak{N} \otimes x_i r = \oplus \sum_i \mathfrak{N} \otimes rx_i \\ &= \oplus \sum_i \mathfrak{N} \otimes [a(\sum_{h \in H \cap M} h)(\sum_j y_j)]x_i \\ &= \oplus \sum_i \mathfrak{N} \cdot [a(\sum_{h \in H \cap M} h)] \otimes (\sum_j y_j)x_i \\ &= 0, \end{aligned}$$

since $a \cdot (\sum_{h \in H \cap M} h) \in \text{Rad } FH$, H being Frobenius with kernel $H \cap M$, and $\mathfrak{N}$ irreducible over FH. Thus $\mathfrak{N}^G$ is completely reducible over FG.

Since $\mathfrak{N}$ was arbitrary, $H \in \mathcal{C}$.

We have thus shown the inclusion $\mathcal{P} \subseteq \mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{R}$ always by Theorem 5, our proof will be complete if we prove $\mathcal{R} \subseteq \mathcal{P}$.

Let H be a subgroup of G with (G,H) property ρ. Then p divides $|H|$; for it not, then FH is semi-simple and if $\sum_i p_i x_i \in \text{Rad } FG$, $p_i \in FH$, $G = \bigcup_i Hx_i$: coset decomposition of G over H, then each $p_i \in \text{Rad } FH = (0)$, and this implies semi-simplicity of FG, contrary to our hypothesis that $p \mid |G|$.

In the factorization $G = PM$, pick a p-Sylow subgroup P such that $H \cap P$ is a p-Sylow subgroup of H. Then $H = (H \cap P)\cdot(H \cap M)$ is a Frobenius group with kernel $H \cap M$. Let

$$P = \bigcup_i (H \cap P)x_i \quad \text{and} \quad M = \bigcup_j (H \cap M)y_j$$

be coset decompositions of P and M respectively. We first show that $G = \bigcup_{i,j} Hy_jx_i$ is a coset decomposition of G over H. To see this, first observe that $[G:H] = [P: H \cap P]\cdot[M: H \cap M]$. Suppose now that $Hy_jx_i = Hy_\ell x_k$, then

$$(x_ix_k^{-1})(x_kx_i^{-1}y_jx_ix_k^{-1}y_\ell^{-1}) \in H = (H \cap P)(H \cap M).$$

Therefore, there exist $a \in H \cap P$ and $b \in H \cap M$ such that

$$(x_ix_k^{-1})(x_kx_i^{-1}y_jx_ix_k^{-1}y_\ell^{-1}) = ab$$

and this gives $a^{-1}(x_ix_k^{-1}) = b(x_kx_i^{-1}y_jx_ix_k^{-1}y_\ell^{-1})^{-1} \in P \cap M = \{1\}$. Hence $x_ix_k^{-1} \in H \cap P$ and $x_kx_i^{-1}y_jx_ix_k^{-1}y_\ell^{-1} \in H \cap M$. This gives $i = k$ and consequently $j = \ell$. Thus $\{Hy_jx_i\}$ are distinct and cardinality argument proves our assertion.

Suppose that (G,H) does not have projective-pairing. Then there exists $g \in P$ such that $g \notin H$. Write $g = h_px_o$ where $h_p \in H \cap P$ and $x_o \neq 1$ is in $\{x_i\}$. Then by Lemma 3

$$r = (g-1)\left(\sum_{x \in M} x\right) \in \text{Rad } FG .$$

We now have

$$r = (h_p x_o - 1)(\sum_{x\in M} x) = h_p x_o(\sum_{x\in M} x) - (\sum_{x\in M} x)$$

$$= h_p(\sum_{x\in M} x)x_o - (\sum_{x\in M} x)\ , \qquad \text{since} \quad M \lhd G$$

$$= h_p(\sum_{h\in H\cap M} h)(\sum_j y_j)x_o - (\sum_{h\in H\cap M} h)(\sum_j y_j)$$

$$= \sum_j [h_p(\sum_{h\in H\cap M} h)]y_j x_o + \sum_j [-(\sum_{h\in H\cap M} h)]y_j\ .$$

Since $\{y_j x_i\}$ are coset-representatives of G over H and (G,H) has property ρ, we conclude that

$$h_p(\sum_{h\in H\cap M} h) \quad \text{and} \quad \sum_{h\in H\cap M} h$$

are in Rad FH. Now let F be the trivial H-module. We then have $F\cdot(\sum_{h\in H\cap M} h) = 0$. In particular,

$$0 = 1\cdot(\sum_{h\in H\cap M} h) = \sum_{h\in H\cap M} 1 = |H \cap M|,$$

which is a contradiction, since $(|H \cap M|,p) = 1$.

This completes the proof of the theorem.

Remark. It should be observed that for the inclusion $\mathcal{R} \subseteq \mathcal{P}$ we only require G to be p-nilpotent.

§8. Extensions by Groups of Orders Prime to p

In the last two sections we showed the equivalence of $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$ for two types of groups. We now extend the class of groups for which $\mathcal{P} = \mathcal{R} = \mathcal{C}$. To save space and avoid lengthy repetitions we make

Definition 8. A finite group G is called PRC-group over F if

$$\mathcal{P}(G,F) = \mathcal{R}(G,F) = \mathcal{C}(G,F).$$

We start with

Lemma 4. If $H \trianglelefteq G$, then $\operatorname{Rad} FH \subseteq \operatorname{Rad} FG$.

Proof. Let $\mathfrak{M}$ be an irreducible FG-module. Since $H \triangleleft G$, Clifford's theorem gives $\mathfrak{M}_H = \oplus \sum_i \mathfrak{N}_i$, each $\mathfrak{N}_i$ being an irreducible FH-module. We then have

$$\mathfrak{M}\cdot(\operatorname{Rad} FH) = \mathfrak{M}_H\cdot(\operatorname{Rad} FH) = \oplus \sum_i \mathfrak{N}_i(\operatorname{Rad} FH) = 0 \ .$$

Since $\mathfrak{M}$ was arbitrary, we obtain the desired result.

We next prove

Theorem 14. An extension of a PRC-group by a group with order prime to p (p = Characteristic F) is a PRC-group.

Proof. Let G be an extension of a PRC-group L by a p'-group. We wish to show $\mathcal{P}(G,F) = \mathcal{R}(G,F) = \mathcal{C}(G,F)$.

Since $\mathcal{C} \subseteq \mathcal{R}$ always by Theorem 5, it is sufficient to show $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \mathcal{P}$.

$\mathcal{P}(G,F) \subseteq \mathcal{C}(G,F)$: Let H be a subgroup of G with (G,H) a projective-pairing and let $\mathfrak{N}$ be an irreducible FH-module. Then $H \supseteq P$, a p-Sylow subgroup of G; which is also a p-Sylow subgroup of L, since $[G:L]$ is prime to p.

Since $L \triangleleft G$, $H \cap L \triangleleft H$ and so Clifford's theorem yields

$$\mathfrak{N}_{H\cap L} = \oplus \sum_i \mathfrak{N}_i \ ,$$

each $\mathfrak{N}_i$ being irreducible $F(H\cap L)$-module. By hypothesis for L, since $H \cap L \supseteq P$ = a p-Sylow subgroup of L, each $\mathfrak{N}_i^L$ is completely reducible over FL. Now $L \triangleleft G$ and (G,L) is a projective-pairing. Therefore, by Theorem 7, each $\mathfrak{N}_i^G = (\mathfrak{N}_i^L)^G$ is completely reducible over FG. This shows $(\mathfrak{N}_{H\cap L})^G = \oplus \sum_i \mathfrak{N}_i^G$ is completely

reducible over FG.

Now $(H, H \cap L)$ is a projective-pairing. Therefore, by Higman's criteria (2.9)(ii), $\mathfrak{N}$ is a component of $(\mathfrak{N}_{H \cap L})^H$. This in turn implies that $\mathfrak{N}^G$ is a component of completely reducible FG-module $(\mathfrak{N}_{H \cap L})^G = ((\mathfrak{N}_{H \cap L})^H)^G$; thus proving its complete-reducibility over FG.

Since $\mathfrak{N}$ was arbitrary, $H \in \mathcal{C}(G,F)$.

<u>$\mathcal{R}(G,F) \subseteq \mathcal{P}(G,F)$</u>: Let $H < G$ with (G,H) property ρ. Let P be a p-Sylow subgroup of G such that $H \cap P$ is a p-Sylow of H. Since $H \cap L$ is a normal subgroup of H and $H \cap L \supseteq H \cap P =$ a p-Sylow subgroup of H, by Theorem 7, $(H, H \cap L)$ has property ρ. Transitivity of property ρ yields the same for the pair $(G, H \cap L)$.

Suppose now that $\sum_i p_i x_i \in \text{Rad } FL$ with $p_i \in F(H \cap L)$, where $L = \bigcup_i (H \cap L)x_i$ is a coset-decomposition of L over $H \cap L$. We may also take $\{x_i\}$ to be some of the coset representatives of G over $H \cap L$. Since $L \lhd G$, Rad FL $\subseteq$ Rad FG by Lemma 4 and so $\sum_i p_i x_i \in \text{Rad } FG$ with $p_i \in F(H \cap L)$. By above $(G, H \cap L)$ has property ρ, hence each $p_i \in \text{Rad } F(H \cap L)$. This gives property ρ for the pair $(L, H \cap L)$.

By hypothesis for L, $(L, H \cap L)$ is a projective-pairing. Hence $H \cap L$, and therefore H, contains a p-Sylow subgroup of L, which is also a p-Sylow subgroup of G. Thus $H \in \mathcal{P}(G,F)$.

The proof of the theorem is now complete.

<u>Corollary 1</u>. If $P \lhd G$ then G is PRC.

<u>Proof</u>. Clear from Theorem 12.

<u>Corollary 2</u>. If Rad FG $\subseteq Z(FG)$ then G is PRC.

Proof. By Wallace's result (2.15), $L = G'P$ is a Frobenius group with kernel $G' = L'$ and complement a p-Sylow subgroup P of G. But L is PRC by Theorem 13 and $([G:L], p) = 1$, so the above theorem is applicable.

Corollary 3. If $|G| = pq$, p,q being primes, then G is PRC.

Proof. If G is abelian or if G is non-abelian and $p > q$, then $P \lhd G$ and Corollary 1 above is applicable. On the other hand, if G is non-abelian and $p < q$, then G is Frobenius group with P as a complement (Scott [10]) and Theorem 13 takes care of the proof.

Corollary 4. If $|G| = pq^2$, $p \nmid q - 1$, then G is PRC.

Proof. It is easy to see that G satisfying the above hypothesis has either P normal in G or is a Frobenius group with complement P and in both cases we are done.

Corollary 5. If G is supersolvable then property ρ implies projective-pairing.

Proof. If p is the highest prime dividing the order of G then $P \lhd G$ and we are done by Corollary 1 above. On the other hand, if p is the smallest prime divisor of $|G|$, then G is p-nilpotent and property ρ implies projective-pairing for p-nilpotent groups follows from the Remark following Theorem 13. Suppose now that p is an intermediate prime divisor of $|G|$. In this case, there exists a normal p-nilpotent subgroup K of G (Sylow Tower Theorem: See Scott [10]) and G is an extension of K by a p'-group. The result now follows from the Remark to Theorem 13 and the above theorem.

§9. Extensions of p-groups

The aim of this section is to further extend the class of PRC-groups. In fact, we prove that any extension of a p-group by a PRC-group is PRC.

Let H be a subgroup of G. We denote by $\mathfrak{U}_G(H)$ the ideal in FG generated by the set $\{h-1 | h \in H\}$. Similarly if I is a two-sided ideal of FG then the set $\{g \in G | g-1 \in I\}$ is denoted by $\mathfrak{U}_G^{-1}(I)$. It is easy to check that $\mathfrak{U}_G^{-1}(I)$ is a normal subgroup of G.

It is well-known that if φ: $FG \to F(G/H)$ is the map obtained by extending the canonical map $G \to G/H$ $(H \lhd G)$ to FG by linearity, then $\operatorname{Ker} \varphi = \mathfrak{U}_G(H)$ and so $FG/\mathfrak{U}_G(H) \cong F(G/H)$.

We now prove

Lemma 5. Let P be a p-Sylow subgroup of G. Then

$$\mathfrak{U}_G^{-1}(\text{Rad } FG) = \text{Core}_G(P) .$$

Proof. Let $A = \mathfrak{U}_G^{-1}(\text{Rad } FG) = \{g \in G | g-1 \in \text{Rad } FG\}$. Then $A \lhd G$, since Rad FG is a two-sided ideal in FG. Since Core $P \lhd G$, by Lemma 4, Rad(Core P) $\subseteq$ Rad FG. But it is well-known that Rad(Core P) is ideal in F(Core P) generated by $\{h-1 | h \in \text{Core } P\}$, as Core P is a p-group. Hence for each $h \in \text{Core } P$, $h-1 \in \text{Rad } FG$ and so Core $P \subseteq A$.

For the reverse inclusion, observe that for $g \in A$, $g-1$ is nilpotent. Thus there exists a least integer k such that

$$0 = (g-1)^k = g^k - \binom{k}{1} g^{k-1} + \ldots + (-1)^{k-1} \cdot 1 .$$

Since FG is free over F, each $\binom{k}{i} \equiv 0 \pmod p$, $i = 1,2,\ldots,k-1$ and $g^k = 1$. In particular, $\binom{k}{1} = k \equiv 0 \pmod p$ and so $p | k$. Thus $g \in A$ implies $p \mid |g|$. Suppose that $|A| = p^b \cdot m$ with $(p,m) = 1$. If $m > 1$ then A has elements with orders not divisible by p, contrary to what we have just proved. Hence A is a p-group. Since A is normal p-group and $\text{Core } P \subseteq A$ we have $A = \text{Core } P$ and this proves the lemma.

We can now prove

Theorem 15. Let $H < G$ such that (G,H) has property ρ. Then $H \supseteq \text{Core } P$.

Proof. If the assertion of this theorem is false then there is a $g \in \text{Core } P$ and $g \notin H$. Write $G = \bigcup_i Hx_i$, coset decomposition, so that $g = hx_i$ for some $h \in H$ and $x_i \neq 1$. By Lemma 5 we have $g - 1 = hx_i - 1.1 \in \text{Rad } FG$. Since (G,H) has property ρ, we must have $h,1 \in \text{Rad } FH$ which is impossible.

This completes the proof.

Before we come to the main theorem, we need

Lemma 6. Let I be a nilpotent ideal in a ring R and $\varphi: R \to R/I$ the natural map. If $\text{Rad } R$ denotes the nilpotent radical of R then $\varphi(\text{Rad } R) = \text{Rad}(R/I) = (\text{Rad } R)/I$.

Proof. We always have $\varphi(\text{Rad } R) \subseteq \text{Rad}(R/I)$. Now let $a + I \in \text{Rad}(R/I)$. Then for every $b + I \in R/I$,

$$(a + I)(b + I) = ab + I$$

is nilpotent; thus there is an integer m such that $(ab + I)^m = (ab)^m + I = I$. Therefore $(ab)^m \in I$. But I is a nilpotent ideal of R, hence ab is nilpotent. Since b in

R was arbitrary, $a \in \text{Rad } R$ and so $\varphi(a) = a + I \in \text{Rad}(R/I)$. This proves the first equality. The second is clear, since $\text{Ker } \varphi = I$ and $I \subseteq \text{Rad } R$.

Remark. Actually our proof shows little more than the assertion of the lemma. In fact, we have shown that $a + I$ is in $\text{Rad}(R/I)$ if and only if a is in $\text{Rad } R$. We shall use this fact.

We now come to the main theorem of this section.

Theorem 16. An extension of a p-group by a PRC-group is PRC.

Proof. Let G be an extension of a p-group A be a PRC-group. Once again we prove the inclusions $\mathcal{P} \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \mathcal{P}$.

$\mathcal{P}(G,F) \subseteq \mathcal{C}(G,F)$: Let (G,H) be a projective-pairing. Then H contains a p-Sylow subgroup of G and hence contains A. Since $A \lhd G$, $\mathfrak{I}_A(A) \subseteq \text{Rad } FG$ by Lemma 4 and so if $\mathfrak{N}$ is an irreducible FH-module, each $a \in A$ acts trivially on $\mathfrak{N}$ and $\mathfrak{N}$ can be regarded as an irreducible $F(H/A)$-module. Now clearly $(G/A, H/A)$ is a projective-pairing; hence, since G/A is PRC by hypothesis, the induced $F(G/A)$-module

$$\mathfrak{N}^{G/A} = \mathfrak{N} \otimes_{F(H/A)} F(G/A)$$

is completely reducible over $F(G/A)$. Since $a \in A$ acts trivially on $\mathfrak{N}$, we can view $\mathfrak{N}^{G/A}$ as FG-module in a natural way by defining $(n \otimes Ag)h = (n \otimes Ag)Ah$, $n \in \mathfrak{N}$, $g,h \in G$.

Define $\varphi\colon \mathfrak{N}^{G/A} \to \mathfrak{N}^G$ by $\varphi(n \otimes Ag) = n \otimes g$ and extend by linearity. Since

$$\begin{aligned}\varphi((n \otimes Ag)h) &= \varphi((n \otimes Ag)Ah) = \varphi(n \otimes Agh) = n \otimes gh = (n \otimes g)h \\ &= \varphi(n \otimes Ag)\cdot h\end{aligned}$$

and since (obviously) φ is 1-1 and onto, φ is an FG-isomorphism.

Thus $\mathfrak{N}^G$ is completely reducible over FG implying $H \in \mathcal{C}(G,F)$.

<u>$\mathcal{R}(G,F) \subseteq \mathcal{P}(G,F)$</u>: Let H be a subgroup of G with (G,H) having property ρ. Then $H \supseteq \text{Core } P \supseteq A$ by Theorem 15. Let $\varphi\colon G \to G/A$ be the natural map. We extend φ to FG by linearity. Let $G = \bigcup_i Hx_i$ be coset decomposition of G over H. Then

$$G/A = \bigcup_i (H/A)Ax_i$$

is a coset decomposition of G/A over H/A. We now show that (G/A,H/A) has property ρ.

Suppose that $\sum_i \bar{p}_i Ax_i \in \text{Rad } F(G/A)$, $\bar{p}_i \in F(H/A)$. Extending φ to group algebra FG, there exist $p_i \in FH$ such that $\varphi(p_i) = \bar{p}_i$. Thus $\varphi(\sum_i p_i x_i) \in \text{Rad } F(G/A) = \text{Rad}(FG/\mathfrak{A}_G(A))$. Since $\text{Ker } \varphi = \mathfrak{A}_G(A) \subseteq \text{Rad } FG$, Lemma 6 is applicable and we conclude that $\sum_i p_i x_i \in \text{Rad } FG$, with $p_i \in FH$. By hypothesis (G,H) has property ρ and so each p_i is in Rad FH. Hence

$$\bar{p}_i = \varphi(p_i) \in \varphi(\text{Rad } FH) = \text{Rad } F(H/A).$$

This shows that (G/A,H/A) has property ρ.

Since G/A is PRC by hypothesis, (G/A,H/A) is a projective-pairing. Thus H/A contains a p-Sylow subgroup P/A of G/A, P being a p-Sylow of G. Hence $H \supseteq P$ and (G,H) is a projective-pairing.

This completes the proof of the theorem.

<u>Corollary 1</u>. If $P \lhd G$, then G is PRC. (Corollary 1 to Theorem 14).

<u>Corollary 2</u>. If $|G| = p^2q$, then G is PRC.

<u>Proof</u>. Let n_p denote the number of p-Sylow subgroups of G. If $p > q$ then $n_p \equiv 1(\text{mod } p)$ and $n_p | q$ yields $n_p = 1$. Thus $P \lhd G$ and we are done. Suppose that $p < q$. Then $n_q = 1$ or p^2. If $n_q = p^2$ then $n_p = 1$ and so $P \lhd G$ and we are done again. If $n_q = 1$ then $G = PQ$, $Q \lhd G$, Q the q-Sylow subgroup of G. Suppose now that G has an element x of order pq. Then $H = \langle x \rangle$ has index p, the smallest prime dividing $|G|$; hence $H \lhd G$. Since H is cyclic, $K = \langle x^q \rangle$ is a normal p-subgroup of G and G is an extension of K by a group of order pq. But by Corollary 3 to Theorem 14, a group of order pq is PRC, hence G is PRC by above theorem. In case G has no element of order pq, then every element of G is either a p- or a p'-element and so G is a Frobenius group with Q as kernel and P as complement (Scott [10]). That G is PRC follows from Theorem 13.

CHAPTER IV

PROJECTIVE-SENSITIVITY

§10. Projective-Sensitivity

We start with

Definition 9. Let H be a subgroup of G. We call H to be projective-sensitive if the following holds:

(i) Whenever (G,M) is a projective-pairing then $M \supseteq H$ or $(H, H \cap M)$ is a projective-pairing.

(ii) If $K \leq H$ and (H,K) is a projective-pairing then there exists an $M \leq G$ such that (G,M) is a projective-pairing and $K = M \cap H$.

In this chapter we attempt to characterize projective-sensitive subgroups of an arbitrary finite group and give examples to show the limits of the results proved. We also give a necessary and sufficient condition for a group to have all of its subgroups projective-sensitive.

Before coming to the main results, we prove two lemmas:

Lemma 8. If $H \trianglelefteq G$ then condition (i) of Definition of projective-sensitivity is satisfied.

Proof. Let $M \leq G$ be such that (G,M) is a projective-pairing. If $M \not\supseteq H$ then, since M contains a p-Sylow subgroup P of G and H is normal in G, $H \cap M$ contains a p-Sylow subgroup $H \cap P$ of H and so $(H, H \cap M)$ is a projective-pairing.

<u>Lemma 9</u>. Let $H < G$. If $K < H$ and is such that K contains a p-Sylow subgroup of H and PK is a group where P is a p-Sylow subgroup of G containing the p-Sylow subgroup of K (of H), then $PK \cap H = K$.

<u>Proof</u>. Clearly $PK \cap H \supseteq K$. We prove the inverse containment. Suppose $|P| = p^a$, $|H| = p^b \cdot s$, $(p,s) = 1$, $b \le a$ and $|K| = p^b \cdot \ell$, $(p,\ell) = 1$, $\ell | s$. We have

$$|PK| = \frac{|P| \cdot |K|}{|P \cap K|} = \frac{p^a \cdot p^b \ell}{p^b} = p^a \cdot \ell \ .$$

Now $|PK \cap H|$ divides both $|PK| = p^a \ell$ and $|H| = p^b \cdot s$. Since $\ell | s$ and $b \le a$, $(p^a \ell, p^b s) = p^b \ell$. Since $PK \cap H$ already has K of order $p^b \cdot \ell$ we obtain $PK \cap H = K$.

This proves the lemma.

We first give a necessary and sufficient condition for a normal subgroup to be projective-sensitive.

<u>Theorem 17</u>. Let $H \lhd G$. Then H is projective-sensitive if and only if for all subgroups K of H with $K \supseteq H_p$, a p-Sylow of H (of K), PK is a group, where P is a p-Sylow subgroup of G containing H_p; p being the characteristic of F.

<u>Proof</u>. Suppose first that PK is a group with P and K satisfying the condition of the theorem. We show that H is projective-sensitive. In view of Lemma 8 it is sufficient to verify condition (ii) of the definition of projective-sensitivity. Suppose that (H,K) is a projective-pairing. By hypothesis PK is a group, P being appropriate p-Sylow subgroup of G. Taking $M = PK$ it is clear that (G,M) is a projective-pairing and Lemma 9 verifies what is required.

Conversely suppose that $H \triangleleft G$ is projective-sensitive and $K < H$ such that (H,K) is a projective-pairing. By condition (ii) there exists $M \leq G$, $M \supseteq P$ and $M \cap H = K$. Since $H \triangleleft G$, $H \cap M = K \triangleleft M$ implying $M \subseteq N_G(K)$ = normalizer of K in G. Therefore $P \subseteq N_G(K)$ and hence PK is a group. This is what we wanted to prove.

Remark. We may take PK as a choice for M in condition (ii) of projective-sensitivity.

Corollary 1. Let $H \triangleleft G$ with H_p a p-Sylow subgroup of H. If $[H:H_p]$ is a prime then H is projective-sensitive.

Proof. In this case if (H,K) is projective-pairing then either K is a p-Sylow of H or $K = H$. In both cases PK is a group, P being appropriate p-Sylow of G, and the above theorem applies.

Corollary 2. If $H \triangleleft G$ such that every subgroup of H is normal in G then H is projective-sensitive. In particular, if H is cyclic normal then H is projective-sensitive.

Proof. Clear from above theorem.

We now drop the condition of normality on H and the next theorem gives some non-normal projective-sensitive-groups.

Theorem 18. (1) Let (G,H) be a projective-pairing over F of characteristic p. Then H is projective-sensitive if and only if H contains all p-Sylow subgroups of G (p fixed).

(2) A p-subgroup H in G is projective-sensitive if and only if $H \subseteq \mathrm{Core}_G(P)$, P being a p-Sylow subgroup of G.

Proof. (1) Suppose H contains all p-Sylow subgroups $\{p^x \mid x \in G, P \text{ fixed}\}$ of G. If $M < G$ is such that (G,M) is a projective-pairing then $M \supseteq P^x$ for some $x \in G$ and so

$H \cap M \supseteq P^x$ = a p-Sylow of H giving projective-pairing for $(H, H \cap M)$. Thus condition (i) of projective-sensitivity is satisfied. Next if (H,K) is a projective-pairing then $K \supset P^x$, some $x \in G$. Taking $M = K$ we have (G,M) projective-pairing and $M \cap H = K \cap H = K$ verifying condition (ii).

Therefore H is projective-sensitive.

Conversely suppose (G,H) is a projective-pairing and there exists an $x \in G$ such that $H \not\supseteq P^x$. We show that H is not projective-sensitive. Now $H \supseteq P^y$ some $y \in G$. If $M = P^x$ then (G,M) is a projective-pairing but neither $M \supseteq H$ nor $(H, H \cap M)$ is a projective-pairing; the first statement $M \not\supseteq H$ is clear while for the second $H \cap M \subsetneqq P^x$, and so $H \cap M$ does not contain a p-Sylow of H yielding H to be non-projective-sensitive.

(2) Suppose that $H \subseteq \text{Core } P$. Wish to show H is projective-sensitive. If (G,M) is a projective-pairing then $M \supseteq P^x \supseteq \text{Core } P \supseteq H$ and so condition (i) is true. Also (H,K) projective-pairing implies $K = H$, so we may take $M = P$. Then (G,M) is a projective-pairing and $M \cap H = H = K$, thus verifying (ii). Therefore H is projective-sensitive.

Conversely, suppose a p-subgroup H is projective-sensitive and there exists an $x \in G$ such that $H \not\subseteq P^x$. Then (G,P^x) is a porjective-pairing but neither $P^x \supseteq H$ nor $(H, P^x \cap H)$ is a projective-pairing, since for a p-group H, (H,K) projective-pairing implies $K = H$. This violates the condition (i) of projective-sensitivity.

The proof of the theorem is now complete.

Theorem 19. Let H_p be a p-Sylow subgroup of H. If $H_p \subseteq \text{Core } P$ and $[H:H_p]$ is a prime then H is projective-sensitive.

Proof. Let $M \leq G$ with (G,M) projective-pairing. Then $M \supseteq P^x$, some $x \in G$, P a fixed p-Sylow of G. We have $H \cap M \supseteq H \cap P^x \supseteq H \cap \text{Core } P \supseteq H_p$ and so $(H, H \cap M)$ is a projective-pairing. This verifies condition (i) of projective-sensitivity. For condition (ii) note that if (H,K) is a projective-pairing then either K is a p-Sylow of H or $K = H$. In the former case take M = appropriate p-Sylow of G and in the latter case take $M = G$. It is trivial to verify that M has properties required by condition (ii) of projective-sensitivity.

Thus H is projective-sensitive.

Corollary . If H is of prime order q, $q \neq p$, then H is projective-sensitive.

We are now in a position to show by means of examples that normality of H is essential for both the directions in Theorem 17.

In $G = A_4$ over field F of characteristic $p = 3$, $H = Z_2$ is projective-sensitive (but not normal) by above corollary but (H,H) is a projective-pairing and PH is not a group (P a 3-Sylow of G), as its order is 6 and A_4 has no subgroup of order 6.

For the other direction take H to be a p-group not contained in Core P and let $H \subseteq P$. Then (H,H) is a projective-pairing and $PH = P$ is a group, but H is not projective-sensitive by Theorem 18(2).

We have seen in the above theorems some projective-sensitive subgroups of G. However, it may happen that G contains many more projective-sensitive subgroups. Indeed,

Theorem 20. G has every subgroup projective-sensitive if and only if $P \lhd G$, P a p-Sylow of G, p = characteristic of F.

Proof. Suppose, first, that $P \lhd G$ and let $H \leq G$. By Theorem 18, if $H \supseteq P$ or $H \subseteq P$ then H is projective-sensitive. So suppose H is not one of these.

We first verify condition (i). If $p \nmid |H|$ then FH is semi-simple and so $(H, H \cap M)$ is a projective-pairing for all $M \subseteq G$ and so (i) is trivially true. If $p \mid |H|$, then, since $P \lhd G$, the p-Sylow subgroup H_p of H is contained in P and so if (G,M) is a projective-pairing then $M \supseteq P$ implying $H \cap M \supseteq H \cap P = H_p$. Hence $(H, H \cap M)$ is a projective-pairing.

Thus in all cases condition (i) is satisfied.

Next we check condition (ii). Suppose (H,K) is a projective-pairing, $K \leq H$. Since $P \lhd G$, PK is a group. Taking $M = PK$, clearly (G,M) is a projective-pairing and, as was the case in Lemma 9, $M \cap H = K$.

This completes the sufficiency part of the theorem.

Suppose now that G has every subgroup projective-sensitive and $P \ntriangleleft G$. Then there exists an $x \in G$ such that $P \neq P^x$. By hypothesis P is projective-sensitive. Let $M = P^x$. Then (G,M) is a projective-pairing but neither $M \supseteq P$ nor $(P, P \cap M)$ is a projective-pairing. This violates the condition (i), a contradiction.

The proof of the theorem is now complete.

Our next theorem describes the representations of a group having all its subgroups projective-sensitive.

Theorem 21. If G has all its subgroups projective-sensitive over a field F of characteristic p then every irreducible representation of G over F occurs as a component of (transitive) permutation representation induced by the trivial representation of a p-Sylow subgroup of G. Consequently, degree of each irreducible representation of G is prime to p and each block of G is of lowest kind with p-Sylow subgroup as its defect group.

Proof. If G satisfies the condition of the theorem then by Theorem 20, G has unique p-Sylow subgroup P.

Let T be an irreducible representation of G afforded by (irreducible) FG-module $\mathfrak{M}$. By Clifford's theorem we have $\mathfrak{M}_p \cong \oplus \sum_i F_i$, each $F_i = F$: the trivial P-module. This gives $(\mathfrak{M}_p)^G \cong \oplus \sum_i F_i^G$. Now since $\mathfrak{M}$ is P-projective, $\mathfrak{M}$ is a component of $(\mathfrak{M}_p)^G$ and therefore, since $\mathfrak{M}$ is irreducible, there exists an i such that $\mathfrak{M}$ is a component of $F_i^G = F^G$. But F^G is a module which defines the representation of G as permutations of the right cosets Px. Hence T occurs as a component of the (transitive) permutation representation afforded by F^G. This proves the first part of the theorem.

Since $\dim_F F^G = [G:P]$, we infer that $(\dim_F \mathfrak{M}, p) = 1$. Therefore defect of each block of FG is a, where $|G| = p^a m$, $(p,m) = 1$ (Brauer and Nesbitt [2]) and each block is of lowest kind with P as defect group.

This completes the proof of the theorem.

Corollary . If G has all its subgroups projective-sensitive then vertex of each irreducible FG-module is the p-Sylow subgroup P.

Proof. By the above theorem $(\dim_F \mathfrak{M}, p) = 1$, $\mathfrak{M}$ being an irreducible FG-module. But by a theorem of Green [4], if B is the vertex of $\mathfrak{M}$ then the index $[P:B]$ divides $\dim_F \mathfrak{M}$. This proves what is desired.

We now assume that P is not normal and in the next theorem we give a sufficient condition for G to have all its normal subgroups projective-sensitive.

Theorem 22. Let $G = PM$, where P is a p-Sylow subgroup of G and $M \lhd G$ is such that each of its subgroups is normal in G. Then every normal subgroup of G is projective-sensitive.

Proof. First we observe that M and each of its subgroup is projective-sensitive by Corollary 2 to Theorem 17. Also by Theorem 18 if the normal subgroup H is a p-group or contains P then H is projective-sensitive. We show the same for $H \lhd G$ not satisfying any of the above conditions. Note that p divides $|H|$, for otherwise $H \subseteq M$.

Now if $L \leq G$ then there exists $g \in G$ such that $L = (L \cap P^g)(L \cap M)$. For choose $g \in G$ such that a p-Sylow subgroup L_p of L is contained in P^g. If $y \in L$ we may write $y = ab$, $a \in L_p$, b a p'-element. Then $b \in L \cap M$. If, in addition, $L \lhd G$ then $L = (L \cap P^g)(L \cap M) = (L \cap P^g)^{g^{-1}}(L \cap M) = (L^{g^{-1}} \cap P)\cdot(L \cap M) = (L \cap P)(L \cap M)$.

Let $H \lhd G$, $H = (H \cap P)(H \cap M) = H_p(H \cap M)$, H_p being a p-Sylow of H. In view of Lemma 8 it is sufficient to verify condition (ii) of projective-sensitivity. Let $K \subseteq H$ and (H,K) be projective-pairing. There exists $x \in H$ such that $K = H_p^x(K \cap M)$. By hypothesis $K \cap M \lhd G$, so $S = P^x(K \cap M)$ is a group, where P^x is that p-Sylow of G which contains the p-Sylow subgroup H_p^x of K. We have

$$S = P^x(K \cap M) = P^x[H_p^x(K \cap M)] = P^xK .$$

Clearly (G,S) is a projective-pairing and by Lemma 9, $S \cap H = K$.

This proves the projective-sensitivity of H and so proves the theorem.

<u>Corollary 1</u>. If $|G| = p^a q$, q a prime then every normal subgroup of G is projective-sensitive.

<u>Corollary 2</u>. D_n, the Dihedral group of order $2n$, has every subgroup projective-sensitive if p is odd and every normal subgroup projective-sensitive if $p = 2$.

<u>Proof</u>. If p is odd then $P \lhd G$ and Theorem 20 applies. If $p = 2$ then $G = PM$ where P is a 2-Sylow subgroup of G and $M \lhd G$ satisfies the condition of the above theorem.

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