# OPTIMIZATION OF DISCRETE STATE LINEAR MULTIVARIABLE SYSTEMS

1962

MICHIGAN STATE UNIVERSITY

This is to certify that the

thesis entitled

OPTIMIZATION OF DISCRETE STATE LINEAR

MULTIVARIABLE SYSTEMS

presented by

Chan Kyu Kim

has been accepted towards fulfillment
of the requirements for

Ph. D degree in Electrical Engineering

Major professor

Date November 20, 1962

O-169

OPTIMIZATION OF DISCRETE STATE LINEAR

MULTIVARIABLE SYSTEMS

by

Chan Kyu Kim

AN ABSTRACT OF A THESIS

Submitted to the
School of Advanced Graduate Studies of
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

196[illegible]

Approved ______________________________

ABSTRACT

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by Chan Kyu Kim

The object of this dissertation is to study techniques for optimizing the response characteristics of multivariable systems with respect to input signals having discrete levels.

To characterize the system behavior, a normal-form, ordinary differential equation model is assumed to be given. The output signals obtained as a solution to this model is compared to the desired or preferred output signal to produce an error functions. The objective of the optimization procedure is to minimize this error functions in the mean-square sense.

In Section II, a discrete state system model is developed from the given normal-form model for discrete-level pulses of T second durations. From this model minimalization for the error is realized by differentiating the mean-squared error function with respect to the (nxm) discrete level input pulses, where n represents the number of inputs and m the number of intervals or stages. In Section III, the problem is

viewed as an m stage process and the dynamic programming technique introduced by Bellman is used as a mean of optimization. In Section IV, the discrete-state control for the multivariable systems is analyzed.

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ACKNOWLEDGEMENT

The author wishes to express his sincere gratitude to Dr. H. E. Koenig for his guidance in preparing this thesis. Much gratitude is also due to Dr. L. W. Von Tersch, Dr. M. B. Reed, Dr. G. P. Weeg and Dr. Y. Tokad for their encouragements.

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# OPTIMIZATION OF DISCRETE STATE LINEAR MULTIVARIABLE SYSTEMS.

## I. Introduction.

Control of the multivariable systems through the use of the system configuration shown in Fig. 1.1 has been discussed by Kavanaugh.[2] Compensation of the main plant G is attempted through diagonalization of the transfer matrix to remove the interaction between the system input and output variables.

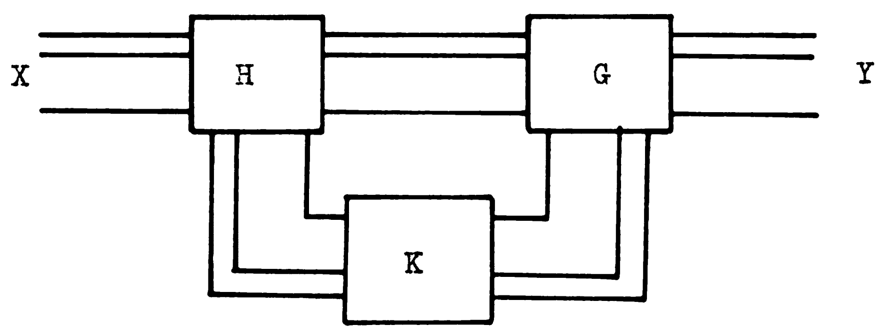


Fig. 1.1 Synthesis of the multivariable system by compensating networks

Horowitz[3] presents a method of synthesis based on the frequency response characteristics of the multivariable feedback system. In this method, the desired transfer matrix is achieved without diagonalization.

The stability problem in multivariable system is discussed in general by Bohn[4] with a discussion on the application of root locus plots in adjusting the parameters of compensating networks to achieve stability.

Amara[5] discusses control or synthesis problems in multivariable system with the non-deterministic signals along with the problem of noise reduction. A compensation network (or filter) is synthesized using the criterion of least-square. Similar system was synthesized by Hsieh and Leondes[6] for the multivariable control system in terms of correlation function between signals and noise in Wiener sense.

Mesarovic[7] analyzed the system in the interactions of the input and output variables in binary form and established the existence and uniqueness of the optimum system along with a synthesis procedure.

The objective of this thesis is to show a procedure for establishing the optimum values of discrete-level input signals to a multivariable system using the mean-square error of the output signals as a basis. In this development a mathematical model of the system in the form of a set of first-order linear equations is assumed. The conditions on the system topology and component characteristics under which such a model exists and has a solution is discussed by Hirth.[8] The optimization problem is formulated as an n-stage decision problem and uses the technique of dynamic programming in solution.

## II. Discrete-State Models of Multivariable Control Systems.

Let the mathematical model of a linear system be written in the form of set of ordinary differential equation in normal form

$$\frac{d}{dt}\begin{bmatrix} C(t) \\ V(t) \end{bmatrix} = \begin{bmatrix} F_1 \left( V(t),\ C(t),\ \overset{*}{R}(t),\ \dot{\overset{*}{R}}(t) \right) \\ F_2 \left( V(t),\ C(t),\ \overset{*}{R}(t),\ \dot{\overset{*}{R}}(t) \right) \end{bmatrix} \qquad (2.1)$$

where $C(t)^{\#}$ is a vector of cross and through variables representing the output signals and $\overset{*}{R}(t)$ is a vector representing the input signals. The vector $V(t)$ represents a set of variables internal to the system. The derivative vector in Eq. 2.1 is sometimes referred to as the state vector since the magnitude and the derivative of this vector uniquely describes the system at any time t. In general, the vector functions $F_1$ and $F_2$ may be nonlinear. However, throughout this thesis

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\# Capital letters, such as $C(t)$, are used for the notation of vector, small case letters, such as $r(t)$, are used for the scalar, and capital letter with bar, such as $\bar{R}(t)$, are used for the matrix.

only linear functions are considered with Eq. 2.1 written as

$$\frac{d}{dt}\begin{bmatrix} C(t) \\ V(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} C(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{bmatrix} \begin{bmatrix} \overset{*}{R}(t) \\ \overset{\cdot *}{R}(t) \end{bmatrix} \qquad (2.2)$$

where the submatrices $\bar{A}_{ij}$ and $\bar{B}_{ij}$ contain real, constant coefficients and the initial conditions on the state vector are given by

$$\begin{bmatrix} C(0) \\ V(0) \end{bmatrix}$$

Let P(t) be a vector representing the preferred or desired output signals with the same dimension as C(t) and let the criterion of performance be taken as the integral

$$\sigma = \int_{t_1}^{t_2} q\Big(P(t) - C(t)\Big)\, dt$$

$$= \int_{t_1}^{t_2} f\Big(P(t),\ C(t)\Big)\, dt \qquad (2.3)$$

where q and f are scalar functions of the vectors P(t) and C(t). For example, q might represent the square of the error

$$q\left[P(t) - C(t)\right] = \sum_{k=1}^{n} \left[p_k(t) - c_k(t)\right]^2$$

The basic problem is to determine the vector $R^*(t)$ such that

$$\sigma = \int_{t_1}^{t_2} f\left[C(t),\ P(t)\right] dt$$

is minimum.

The input vector $R^*(t)$ is considered as a set of discrete levels, as shown in Fig. 2.1

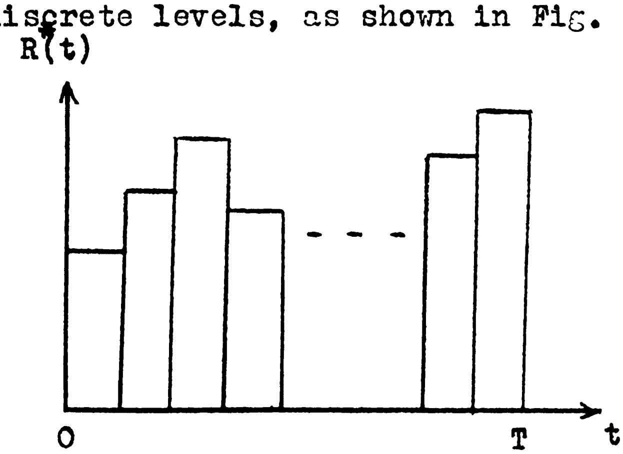


Fig. 2.1 Actuating input signals for the system input

Such an input signal might, for example, be obtained as the output of a digital controller. For continuous input signals, a discrete level approximation of the above form is assumed. In general, a finite interval (0, T) is divided into uniformly spaced sections as indicated with $\delta = T/m$ representing the width of each discrete-level interval. The magnitude of the n-inputs over interval j is represented by the vector $R^*(j\delta)$. The set of numbers representing the n input vectors over the entire interval (0, T) is represented by the nxm rectangular matrix.

$$\overline{R}^* = \left[ R^*(0) \quad R^*(\delta) \ . \ . \ . \ R^*((m-1)\delta) \right] \qquad (2.4)$$

The output signals, over the j-th interval, in general, are a function of time as well as the magnitudes of the input signals and are represented by the time-varying vector $C(j\delta + \tau)$ where

$$0 < \tau \leqq \delta = T/m$$

The set of m time-varying vectors representing the output signals over the m discrete time intervals is represented by the n x m rectangular matrix

$$\overline{C}(\tau) = \left[ C(\tau) \; C(\delta+\tau) \; . \; . \; . \; C((m-1)\delta+\tau) \right] \qquad (2.5)$$

To establish the relationship between the output or response vector $C(j\delta+\tau)$ and the input or reference vector $R^{*}(j\delta)$, consider the s-domain transfer relation obtained by taking the Laplace transform of Eq. 2.2.

$$s\begin{bmatrix} C(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} C(s) \\ V(s) \end{bmatrix} + \begin{bmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{bmatrix} \begin{bmatrix} R^{*}(s) \\ \dot{R}^{*}(s) \end{bmatrix} + \begin{bmatrix} C(0) \\ V(0) \end{bmatrix}$$

Solving for C(s) gives the form

$$C(s) = \overline{G}(s) \quad R^{*}(s) + \overline{H}(s) \quad C'(0) \qquad (2.6)$$

where

$$\overline{H}(s) = \left[ \overline{H}_{1}(s) \quad \overline{H}_{2}(s) \quad \overline{H}_{3}(s) \right]$$

and

$$C'(0) = \begin{bmatrix} C(0) \\ R^{*}(0) \\ V(0) \end{bmatrix}$$

For the first interval, i. e., $0 < t \leqq \delta$

$$R^{*}(t) = R^{*}(0) = \text{constant}$$

Therefore

$$C(s) = \frac{\overline{G}(s)}{s} \quad R^{*}(0) + \overline{H}(S) \quad C'(0) \qquad (2.7)$$

and

$$C(\tau) = \overline{G}(\tau)\, R^{*}(0) + \overline{H}(\tau)\, C'(0) \qquad (2.8)$$

where $\quad 0 < \tau \leq \delta$

C(0) represents the initial state vector and

$$\overline{G}(\tau) = L^{-1}\left[\overline{G}(s) / s\right]$$

and $\quad \overline{H}(\tau) = L^{-1}\left[\overline{H}(s)\right]$

For the second interval, consider the response C(t) to a step function of magnitude $R^{*}(0)$ applied at t = 0, i. e.,

$$C(t) = \overline{G}(t)\, R^{*}(0) + \overline{H}(t)\, C'(0) \qquad (2.9)$$

To this response add the response to a step function of magnitude $- R^{*}(0)$ applied at $t = \delta$ and a step function of magnitude $R^{*}(\delta)$ applied also at $t = \delta$. Using the shifting theorem, the result is

$$C(\delta+\tau) = \bar{G}(\delta+\tau)\, R^{*}(0) + \bar{G}(\tau)\left[R^{*}(\delta) - R^{*}(0)\right] + \bar{H}(\delta+\tau)\, C'(0)$$

where $\quad 0 < \tau \leq \delta$

In a similar manner, it can be shown that the response for interval (j + 1) is given by

$$c(j\delta+\tau) = \overline{G}(j\delta+\tau)\, R^*(0) + \overline{G}((j-1)\delta+\tau)\left[R^*(\delta) - R^*(0)\right]$$
$$+ \dots + \overline{G}(\tau)\left[R^*(j\delta) - R^*((j-1)\delta)\right]$$
$$+ \overline{H}(j\delta+\tau)\cdot c'(0) \qquad (2.10)$$

where

$$\overline{G}(j\delta+\tau) = 0 \quad \text{for} \quad 0 > \tau$$

These resulting solutions can be shown effectively in matrix form as

$$\begin{bmatrix} c(\tau) \\ \cdot \\ \cdot \\ \cdot \\ c((m-1)\delta+\tau) \end{bmatrix} = \begin{bmatrix} \overline{G}(\tau) & \cdot & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \overline{G}((m-1)\delta+\tau) & & \overline{G}(\tau) \end{bmatrix} \begin{bmatrix} R(0) \\ \cdot \\ \cdot \\ \cdot \\ R((m-1)\delta) \end{bmatrix}$$
$$+ \begin{bmatrix} \overline{H}(\tau) \\ \cdot \\ \cdot \\ \cdot \\ \overline{H}((m-1)\delta+\tau) \end{bmatrix} \begin{bmatrix} c'(0) \end{bmatrix} \qquad (2.11)$$

where

$$R(j\delta) = R^*(j\delta) - R^*((j-1)\delta)$$

Using the n x m matrices defined in Eq. 2.4 and Eq. 2.5, the above model can be written as

$$\begin{bmatrix} C(\tau) & \ldots & C((m-1)\delta+\tau) \end{bmatrix}$$
$$= \begin{bmatrix} R(0) & \ldots & R((m-1)\delta) \end{bmatrix} \begin{bmatrix} \bar{G}(\tau) & \ldots & \bar{G}((m-1)\delta+\tau) \\ & \ldots & \\ & \ldots & \\ 0 & \ldots & \bar{G}(\tau) \end{bmatrix}$$
$$+ \quad C(0) \begin{bmatrix} H(\tau) & \ldots & H((m-1)\delta+\tau) \end{bmatrix} \qquad (2.12)$$

Or

$$\bar{C}(\delta,\tau) = \bar{R}(\delta)\,\bar{G}(\tau) + C'(0)\,\bar{\bar{H}}(\tau) \qquad (2.13)$$

where

$$\bar{\bar{H}}(\tau) = \begin{bmatrix} H(\tau) & \ldots & H((m-1)\delta+\tau) \end{bmatrix}$$

Eq.2.11 and Eq. 2.12 represent the system models to be used as a basis of optimizing the input vector.

To establish a measure of the error let the preferred or desired response vector P(t) be approximated by a set of m vector functions of $\tau$ and

$\delta$, i. e.,

$$\overline{P}(\delta, \tau) = \left[ P(\tau) \; . \; . \; . \; P((m-1)\delta + \tau) \right] \qquad (2.14)$$

where $\quad 0 < \tau \leqq \delta$

In general, $\overline{P}(t)$ may or may not be continuous at the points $t = j\,\delta$. In a similar manner, let the error function defined as

$$E(t) = P(t) - C(t) \qquad (2.15)$$

be represented by an n x m matrix of functions on the m discrete intervals

$$\overline{E}(\delta, \tau) = \overline{P}(\delta, \tau) - \overline{C}(\delta, \tau) \qquad (2.16)$$

where $\overline{C}(\delta, \tau)$ is related to $\overline{R}(\delta, \tau)$ by Eq. 2.12. Thus, the error function is related to the discrete levels of the input vector by

$$\overline{E}(\delta, \tau) = \overline{P}(\delta, \tau) - \overline{R}(\delta)\,\overline{G}(\delta, \tau) - \overline{H}(\delta, \tau)c'(0) \qquad (2.17)$$

In the following development the error criteria for each of the n signals, say the k-th, is taken as

$$\sigma_k = \int_0^T \left[ e_k(t) \right]^2 dt = \int_0^T \left[ p_k(t) - c_k(t) \right]^2 dt \qquad (2.18)$$

When the error functions $e_k(t)$, where $k = 1, 2, \ldots n$, for the n output signals are represented as a vector

$$\bar{E}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \cdot \\ \cdot \\ \cdot \\ e_n(t) \end{bmatrix}$$

the error criterion in Eq. 2.19 can be written as

$$\sigma = \int_0^T \left[E(t)\right]^T \cdot \left[E(t)\right] dt \qquad (2.20)$$

Substituting

$$\begin{aligned} E(t) &= E(\tau) & 0 < t < \delta \\ &= E(\delta + \tau) & \delta < t < 2\delta \\ &\cdot & \cdot \\ &\cdot & \cdot \\ &\cdot & \cdot \\ &= E((m-1)\delta + \tau) & (j-1)\delta < t < j\delta \end{aligned} \qquad (2.21)$$

into Eq. 2.20 gives

$$\sigma = \sum_{j=0}^{m-1} \int_0^{\delta} \left[ E(j\delta + \tau) \right]^T \cdot \left[ E(j\delta + \tau) \right] \, d\tau \qquad (2.22)^{*}$$

where the $E(j\delta + \tau)$ represents the (j+1)-th column of the error matrix $\bar{E}(\delta, \tau)$. Substituting the expression for $\bar{E}(\delta, \tau)$ as given in Eq. 2.17 into Eq. 2.20 gives

$$\sigma = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} \Big\{ p_k(j\delta + \tau) - \sum_{\lambda=0}^{j-1} G_k(\lambda\delta + \tau) \cdot$$

$$R((j-1-\lambda)\delta) - h_k(j\delta + \tau) \, c_k'(0) \Big\}^2 \, d\tau \qquad (2.23)^{**}$$

Minimization of the integro-error function

$$\sigma(\bar{R})$$

with respect to the n x m discrete input levels in

$$\bar{R} = \left[ R(0) \; . \; . \; . \; R((m-1)\delta) \right]$$

can be realized by taking the partial derivatives of

---

* See Appendix B.1 for the derivation of Eq. 2.22.

** See Appendix B.2 for the derivation of Eq. 2.23.

$$\sigma(\bar{R})$$

with respect to the coefficients in $\bar{R}$. The result is the set of n x m simultaneous linear algebraic equations

$$
\begin{aligned}
\frac{\partial \sigma(\bar{R})}{\partial r_1(0)} &= 0 \\
&\vdots \\
\frac{\partial \sigma(\bar{R})}{\partial r_n(0)} &= 0 \\
&\vdots \\
\frac{\partial \sigma(\bar{R})}{\partial r_1((m-1)\delta)} &= 0 \\
&\vdots \\
\frac{\partial \sigma(\bar{R})}{\partial r_n((m-1)\delta)} &= 0
\end{aligned}
\qquad (2.24)
$$

A solution of these n x m simultaneous equations gives either a **minimization, maximization or**

the saddle point of the integro-error $\sigma(\bar{R})$.

The explicit form of the n x m simultaneous equations represented in Eq. 2.24 is

$$\frac{\partial \sigma(\bar{R})}{\partial r_1(0)} = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} 2\Big\{ p_k(j\delta+\tau) - \sum_{\lambda=0}^{j-1} G_k(\lambda\delta+\tau)\, R((j-1-\lambda)\delta) - h_k(j\delta+\tau)c_k'(0)\Big\} \cdot g_{k1}(\tau)\, d\tau = 0$$

. . . . . . . . . .

. . . . . . . . . .

$$\frac{\partial \sigma(\bar{R})}{\partial r_n((m-1)\delta)} = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} 2\Big\{ p_k(j\delta+\tau) - \sum_{\lambda=0}^{j-1} G_k(\lambda\delta+\tau)\, R((j-1-\lambda)\delta) - h_k(j\delta+\tau)c_k'(0)\Big\} g_{kn}(\tau)\, d\tau = 0 \qquad (2.25)$$

Rearranging Eq. 2.25 in matrix form, we have a set of n x m linear algebraic equation with n x m unknown, namely,

$$\left[\int_0^{\delta}\sum_{k=1}^{n} \bar{A}(\tau)\ d\tau\right] \bar{R} = \int_0^{\delta}\sum_{k=0}^{n} B(\tau)\ d\tau \qquad (2.26)$$

where $\bar{A}(\tau)$ and $B(\tau)$ are

$$\bar{A}(\tau) = \begin{bmatrix} g_{k1}(\tau)\sum_{j=0}^{m-1} G_k(j\delta+\tau) & \cdots & g_{k1}(\tau)\ G_k(\tau) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g_{kn}((m-1)\delta+\tau)\sum_{j=0}^{m-1} G_k(j\delta+\tau) & \cdots & g_{kn}((m-1)\delta+\tau)\,G_k(\tau) \end{bmatrix}$$

(2.27)

$$B(\tau) = \begin{bmatrix} g_{k1}(\tau)\left\{\sum_{j=0}^{m-1}\left[p_k(j\delta+\tau) - h_k(j\delta+\tau)\ c_k'(0)\right]\right\} \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ g_{kn}((m-1)\delta+\tau)\left\{\sum_{j=0}^{m-1}\left[p_k(j\delta+\tau) - h_k(j\delta+\tau)\ c_k'(0)\right]\right\} \end{bmatrix} \qquad (2.28)^{*}$$

* See Appendix B.3 for the derivation.

The solution to Eq. 2.26

$$\bar{R} = \left[\int_0^{\delta} \sum_{k=0}^{n} \bar{A}(\tau)\, d\tau\right]^{-1} \left[\int_0^{\delta} \sum_{k=0}^{n} B(\tau)\, d\tau\right] \qquad (2.29)$$

gives either a maximum, minimum or saddle point.

To establish a minimum, differentiate Eq. 2.23 a second time with respect to the entries of

$$\bar{R}$$

The result is

$$\frac{\partial^2 \sigma(\bar{R})}{\partial r_1(0)^2} = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} 2\left[g_{k1}(\tau)\right]^2 d\tau$$

..........

$$\frac{\partial^2 \sigma(\bar{R})}{\partial r_n(0)^2} = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} 2\left[g_{kn}(\tau)\right]^2 d\tau \qquad (2.30)$$

..........

$$\frac{\partial^2 \sigma(\bar{R})}{\partial r_1((m-1)\delta)^2} = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} 2\left[g_{k1}((m-1)\delta+\tau)\right]^2 d\tau$$

..........

$$\frac{\partial^2 \sigma(\bar{R})}{\partial r_n((m-1)\delta)^2} = \sum_{j=0}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} 2\left[g_{kn}((m-1)\delta+\tau)\right]^2 d\tau$$

Since the integrands to the right of Eq. 2.30 are positive real functions, the solution to Eq. 2.29 represents a minimum.

Example: 1

Let the s-domain relation between the input and output function for a system be given as

$$C(s) = \frac{1}{s + 1} R^{*}(s) + \frac{1}{s + 1} C(0) \quad (2.31)$$

and let

$$p(t) = t \quad (2.32)$$

be the preferred response. Determine the magnitudes of two discrete levels each of one second duration that gives minimum integro-error over the two periods with the initial condition

$$c(0) = 0.25 \quad (2.33)$$

The output signal during the interval of (0, 1) is

$$c(\tau) = (1 - e^{-\tau}) r(0) + c(0) (1 - e^{-\tau}) \quad (2.34)$$

For the interval of (1, 2) second, the output signal is

$$c(\delta+\tau) = (1 - e^{-(\delta+\tau)}) \, r(0) + (1 - e^{-\tau}) \, r(\delta) + c(0) \, (1 - e^{-(\delta+\tau)}) \qquad (2.35)$$

The integro-error function of the two intervals are

$$e(\tau) = \tau - (1 - e^{-\tau}) \, r(0) - (1 - e^{-\tau}) \, c(0)$$

$$e(\delta+\tau) = (\delta+\tau) - (1 - e^{-(\delta+\tau)}) \, r(0) - (1 - e^{-\tau}) \, r(\delta) - (1 - e^{-(\delta+\tau)}) \, c(0) \qquad (2.36)$$

The system output functions for this multi-stage process is shown in Fig. 2.1.

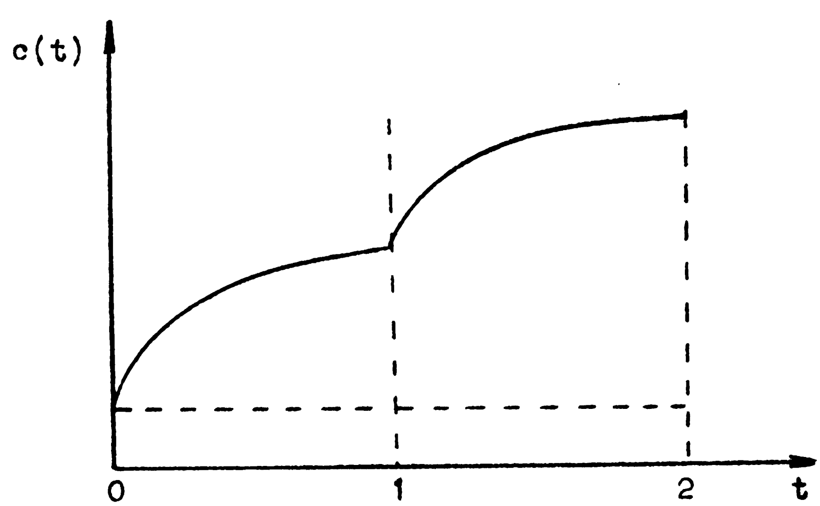


Fig. 2.1 System output variables as a function of time.

The error criteria $\sigma\left[r(0),\ r(\delta)\right]$ is

$$\sigma = \int_0^1 \left\{\tau - (1 - e^{-\tau})\ r(0) - (1 - e^{-\tau})c(0)\right\}^2 d\tau$$

$$+ \int_0^1 \left\{(\delta+\tau) - (1 - e^{-(\delta+\tau)})r(0) - (1 - e^{-\tau})r(\delta)\right.$$

$$\left. - (1 - e^{-(\delta+\tau)})\ c(0)\right\}^2 \ d\tau \qquad (2.37)$$

In order to determine the discrete input levels, take the partial of $\sigma$ with respect to $r(0)$ and $r(\tau)$ to obtain

$$\frac{\partial\sigma}{\partial r(0)} = \int_0^1 2\ (1 - e^{-\tau})\left\{\tau - (1 - e^{-\tau})r(0) - (1 - e^{-\tau})c(0)\right\} d\tau$$

$$+ \int_0^1 2\ (1 - e^{-(\delta+\tau)})\left\{(\delta+\tau) - (1 - e^{-(\delta+\tau)})r(0)\right.$$

$$\left. - (1 - e^{-\tau})\ r(\delta) - (1 - e^{-(\delta+\tau)})\ c(0)\right\} d\tau$$

$$\frac{\partial \sigma}{\partial r(\delta)} = \int_0^1 2\,(1 - e^{-\tau})\Big\{(\delta + \tau) - (1 - e^{-(\delta+\tau)})\, r(0)$$

$$- (1 - e^{-\tau})\, r(\delta) - (1 - e^{-(\delta+\tau)})\, c(0)\Big\}\, d\tau \qquad (2.38)$$

or

$$\left[\int_0^1 \bar{A}(\tau)\, d\tau\right] \begin{bmatrix} r(0) \\ r(\delta) \end{bmatrix} = \left[\int_0^1 \bar{B}(\tau)\, d\tau\right] \qquad (2.39)$$

where $\overline{A}(\tau)$ and $B(\tau)$ are

$$\bar{A}(\tau) = \begin{bmatrix} (1-e^{-\tau})+(1-e^{-(\delta+\tau)}) & (1-e^{-(\delta+\tau)})(1-e^{-\tau}) \\ (1-e^{-\tau})(1 - e^{-(\delta+\tau)}) & (1 - e^{-\tau})^2 \end{bmatrix}$$

(2.40)*

$$B(\tau)= \begin{bmatrix} (1-e^{-\tau})[\tau - c(0)(1-e^{-\tau})] + (1-e^{-(\delta+\tau)})[(\delta+\tau)-c(0)(1-e^{-\tau})] \\ (1-e^{-\tau})[(\delta+\tau) - c(0)(1-e^{-(\delta+\tau)})] \end{bmatrix}$$

(2.41)*

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* See Appendix B.4

By taking the inverse of coefficient matrix of Eq. 2.39 we have as the optimum values of $r(0)$ and $r(\delta)$

$$\begin{bmatrix} r(0) \\ r(\delta) \end{bmatrix} = \left[\int_0^1 X(\tau)\, d\tau\right]^{-1} \left[\int_0^1 B(\tau)\, d\tau\right] = \begin{bmatrix} 1.109 \\ 0.982 \end{bmatrix}$$

Example: 2

Let the s-domain system equations for a two input and two output port system be given as

$$\begin{bmatrix} c_1(s) \\ c_2(s) \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} & \frac{2}{s+1} \\ \frac{3}{s+1} & \frac{5}{s+1} \end{bmatrix} \begin{bmatrix} r_1^*(s) \\ r_2^*(s) \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} \qquad (2.42)$$

with preferred functions and intitial condition

$$\begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix} \tag{2.43}$$

$$\begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \tag{2.44}$$

The entries of $\bar{G}(t)$ are

$$\bar{G}(t) = (1 - e^{-t}) \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} \tag{2.45}$$

The discrete functions as given in Eq. 2.11 are

$$\begin{bmatrix} c_1(T) \\ c_2(T) \\ c_1(\delta + T) \\ c_2(\delta + T) \end{bmatrix} = \begin{bmatrix} 2(1-e^{-T}) & 2(1-e^{-T}) & 0 & 0 \\ 3(1-e^{-T}) & 5(1-e^{-T}) & 0 & 0 \\ 2(1-e^{-(\delta+T)}) & 2(1-e^{-(\delta+T)}) & 2(1-e^{-T}) & 2(1-e^{-T}) \\ 3(1-e^{-(\delta+T)}) & 5(1-e^{-(\delta+T)}) & 3(1-e^{-T}) & 5(1-e^{-T}) \end{bmatrix}$$

$$x \begin{bmatrix} r_1(0) \\ r_2(0) \\ r_1(\delta) \\ r_2(\delta) \end{bmatrix} + \begin{bmatrix} c_1(0)\ (1-e^{-T}) \\ c_2(0)\ (1-e^{-T}) \\ c_1(0)\ (1-c^{-(\delta+T)}) \\ c_2(0)\ (1-e^{-(\delta+T)}) \end{bmatrix} \tag{2.46}$$

The matrix of discrete error functions is, therefore,

$$E(t) = \begin{bmatrix} E(\tau) \\ E(\delta+\tau) \end{bmatrix} = \begin{bmatrix} P(\tau) \\ P(\delta+\tau) \end{bmatrix} - \begin{bmatrix} C(\tau) \\ C(\delta+\tau) \end{bmatrix}$$

$$= \begin{bmatrix} p(\tau) \\ p(\tau) \\ p(\delta+\tau) \\ p(\delta+\tau) \end{bmatrix} - \begin{bmatrix} 2(1-e^{-\tau}) & 2(1-e^{-\tau}) & 0 & 0 \\ 3(1-e^{-\tau}) & 5(1-e^{-\tau}) & 0 & 0 \\ 2(1-e^{-(\delta+\tau)}) & 2(1-e^{-(\delta+\tau)}) & 2(1-e^{-\tau}) & 2(1-e^{-\tau}) \\ 3(1-e^{-(\delta+\tau)}) & 5(1-e^{-(\delta+\tau)}) & 3(1-e^{-\tau}) & 5(1-e^{-\tau}) \end{bmatrix}$$

$$\times \begin{bmatrix} r_1(0) \\ r_2(0) \\ r_1(\delta) \\ r_2(\delta) \end{bmatrix} - \begin{bmatrix} c_1(0)\ (1-e^{-\tau}) \\ c_2(0)\ (1-e^{-\tau}) \\ c_1(0)\ (1-e^{-(\delta+\tau)}) \\ c_2(0)\ (1-e^{-(\delta+\tau)}) \end{bmatrix} \qquad (2.47)$$

The integro-error function is

$$\sigma = \int_0^1 \left\{ \left[ e_1(\tau) \right]^2 + \left[ e_2(\tau) \right]^2 \right\} d\tau$$

$$+ \int_0^1 \left\{ \left[ e_1(\delta+\tau) \right]^2 + \left[ e_2(\delta+\tau) \right]^2 \right\} d\tau \qquad (2.48)$$

where $e_1(\tau)$, $e_2(\tau)$, $e_1(\delta+\tau)$ and $e_2(\delta+\tau)$ are given in Eq. 2.47. Differentiating the integro-error function shown in Eq. 2.48 with respect to the entries of the vector $\bar{R}$ gives

$$\int_0^1 \begin{bmatrix} \bar{A}_{11}(\tau) & \bar{A}_{12}(\tau) \\ \\ \bar{A}_{21}(\tau) & \bar{A}_{22}(\tau) \end{bmatrix} d\tau \begin{bmatrix} r_1(0) \\ r_2(0) \\ r_1(\delta) \\ r_2(\delta) \end{bmatrix}$$

$$= \int_0^1 \begin{bmatrix} \left[2B_1(\tau)+3B_2(\tau)\right](1-e^{-\tau})+\left[2B_1(\delta+\tau)+3B_2(\delta+\tau)\right](1-e^{-(\delta+\tau)}) \\ \left[2B_1(\tau)+5B_2(\tau)\right](1-e^{-\tau})+\left[2B_1(\delta+\tau)+5B_2(\delta+\tau)\right](1-e^{-(\delta+\tau)}) \\ \left[2B_1(\delta+\tau)+3B_2(\delta+\tau)\right](1-e^{-\tau}) \\ \left[2B_1(\delta+\tau)+5B_2(\delta+\tau)\right](1-e^{-\tau}) \end{bmatrix} d\tau \tag{2.49}$$

where $\bar{A}_{11}(\tau)$, $\bar{A}_{12}(\tau)$, $\bar{A}_{21}(\tau)$, $\bar{A}_{22}(\tau)$, $\bar{B}_1(\tau)$, $\bar{B}_2(\tau)$, $B_1(\delta+\tau)$ and $B_2(\delta+\tau)$ are

$$\bar{A}_{11}(\tau) = \left[(1-e^{-\tau})^2 + (1-e^{-(\delta+\tau)})^2\right] \begin{bmatrix} 13 & 19 \\ 19 & 29 \end{bmatrix}$$

$$\bar{A}_{12}(\tau)=\bar{A}_{21}(\tau)=\bar{A}_{22}(\tau)=(1-e^{-\tau})(1-e^{-(\delta+\tau)}) \begin{bmatrix} 13 & 19 \\ \\ 19 & 29 \end{bmatrix}$$

$$B_1(\tau) = p_1(\tau) - c_1(0)\ (1-e^{-\tau})$$

$$B_2(\tau) = p_2(\tau) - c_2(0)\ (1-e^{-\tau})$$

$$B_1(\delta+\tau) = p_1(\delta+\tau) - c_1(0)\ (1-e^{-(\delta+\tau)})$$

$$B_2(\delta + \tau) = D_2(\delta + \tau) - c_2(0)\ (1-e^{-(\delta + \tau)}) \qquad (2.50)^{*}$$

The solution to Eq. 2.49 gives

$$\begin{bmatrix} r\ (0) \\ r\ (0) \\ r\ (\delta) \\ r\ (\delta) \end{bmatrix} = \left[ \int_0^1 \begin{matrix} \bar{A}_{11}(\tau) & \bar{A}_{12}(\tau) \\ \bar{A}_{21}(\tau) & \bar{A}_{22}(\tau) \end{matrix}\ d\tau \right]^{-1}$$

$$\int_0^1 \begin{bmatrix} [2B_1(\tau)+3B_2(\tau)\ (1-e^{-\tau})+ 2B_1(\delta+\tau)+3B_2(\ +\tau)\ (1-e^{-(\delta+\tau)}) \\ [2B_1(\tau)+5B_2(\tau)](1-e^{-\tau})+[2B_1(\delta+\tau)+5B_2(\delta+\tau)]\ (1-e^{-(\delta+\tau)}) \\ [2B_1(\delta+\tau) + 3B_2(\delta+\tau)]\ (1-e^{-\tau}) \\ [2B_1(\delta+\tau) + 5B_2(\delta+\tau)]\ (1-e^{-\tau}) \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 1.304 \\ -1.211 \\ -0.462 \\ 0.737 \end{bmatrix}$$

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* See Appendix B.5 for the derivation.

## III. Control of Multivariable System by Dynamic Programming.

The optimum magnitudes of input signals can also be realized by a process referred to by Bellman as dynamic programming.* To show this procedure and its application to the type of system under consideration here, consider the solution to the system model given in Eq. 2.11 and the error criteria

$$\sigma(\bar{R}) = \int_0^{m\delta} \left[ E(t) \right]^T \left[ E(t) \right] dt \qquad (3.1)$$

To minimize $\sigma(\bar{R})$ with respect to $\bar{R}$, let the minimization of Eq. 3.1 be written as

$$\underset{\bar{R}}{\text{Min}}\ \sigma(\bar{R}) = \underset{\bar{R}}{\text{Min}} \int_0^{m\delta} \left[ E(t) \right]^T \left[ E(t) \right] dt$$

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* Bellman, R., "Dynamic Programming," Princeton University Press, 1957

$$= \underset{R(0)}{\mathrm{Min}} \left\{ \int_0^{\delta} \left[ E(\tau) \right]^T \left[ E(\tau) \right] d\tau \right\}$$

$$+ \underset{R(\delta) \ldots R((m-1)\delta)}{\mathrm{Min}} \left\{ \int_{\delta}^{m\delta} \left[ E_2(t)^2 + \ldots + E_m(t)^2 \right] dt \right\}$$

$$= \underset{R(0)}{\mathrm{M\ i\ n}} \left\{ \left[ \int_0^{\delta} E_1(t)^2 \, dt + \left( \underset{R(\delta)}{\mathrm{Min}} \int_{\delta}^{2\delta} E_2(t)^2 \, dt + \ldots \right.\right.\right.$$

$$\left.\left.\left. + \underset{R((m-1)\delta)}{\mathrm{Min}} \int_{(m-1)\delta}^{m\delta} E_m(t)^2 \quad dt \right) \right] \right\} \qquad (3.2)$$

Applying the change of variable

$$t = (j-1)\,\delta + \tau$$

for the j-th interval, Eq. 3.2 becomes

$$\underset{\bar{R}}{\mathrm{Min}} \quad \sigma\,(\bar{R}) = \underset{R(0)}{\mathrm{Min}} \left\{ \int_0^{\delta} E_1(\tau)^2 \quad d\tau \right.$$

$$\left. + \left( \underset{R(\delta)}{\mathrm{Min}} \int_0^{\delta} E_2(\delta + \tau)^2 \, d\tau + \ldots + \underset{R((m-1)\delta)}{\mathrm{Min}} \int_0^{\delta} E_m((m-1)\delta + \tau)^2 d\tau \right) \right\}$$

$$(3.3)$$

Note that the minimization at each stage is with respect to the levels of the n input signals. The basic procedure in dynamic programming is to minimize each stage, starting from the last and working forward. At each stage the minimum is expressed as an explicit function of the signal levels of the previous interval.

To effect the last minimization with respect to $R((m-1)\delta)$, let the response function

$$C((m-1)\delta+\tau)$$

as given in Eq. 2.11 (or Eq. 2.12) be written as

$$C((m-1)\delta+\tau) = Q_{m-1}(\tau)+\overline{G}(\tau)R((m-1)\delta) \qquad (3.4)$$

where $Q_{m-1}(\tau)$ is

$$Q_{m-1}(\tau)=\sum_{j=1}^{m-1} \overline{G}(j\delta+\tau)R((j-1)\delta)+H((m-1)\delta+\tau)C(0) \qquad (3.5)$$

Note that $Q_{m-1}(\tau)$ is independent of the signal levels in the last stage. Assuming that $Q_{m-1}(\tau)$ will be optimized later, the optimum values of the signals for the last stage are obtained by differentiating $\sigma_m(\overline{R})$ with respect to $R((m-1)\delta)$. The result is a set of n linear simultaneous equation in the n variables of the vector R.

Error functions in the last interval is therefore

$$E_m((m-1)\delta+\tau)=P((m-1)\delta+\tau)-\bar{G}(\tau)R((m-1)\delta)-Q_{m-1}(\tau) \quad (3.6).$$

and for the last stage the error criteria is

$$\sigma_m(\bar{R})=\int_0^{\delta}\sum_{i=1}^{n}\left\{p_i((m-1)\delta+\tau)-G_i(\tau)R((m-1)\delta)-Q_{m-1}(\tau)\right\}^2 d\tau$$

(3.7)

Minimizing with respect to the set of discrete-level input functions gives

$$\frac{\partial\sigma_m(\bar{R})}{\partial r_1((m-1\delta)}=\int_0^{\delta}\sum_{i=1}^{n}2\left\{p_i((m-1)\delta+\tau)-G_i(\tau)R((m-1)\delta)\right.$$

$$\left.-q_{(m-1)i}(\tau)\right\}g_{i1}(\tau)\quad d\tau = 0$$

. . . .

. . . .

$$\frac{\partial\sigma_m(\bar{R})}{\partial r_n((m-1)\delta)}=\int_0^{\delta}\sum_{i=1}^{n}2\left\{p_i((m-1)\delta+\tau)-G_i(\tau)R((m-1)\delta)\right.$$

$$\left.-q_{(m-1)i}(\tau)\right\}g_{in}(\tau)\quad d\tau = 0 \quad (3.8)$$

Rearranging Eq. 3.8 in matrix form gives

$$\left[\int_0^{\delta} \bar{A}_m(\tau)\, d\tau\right] R((m-1)\delta)$$

$$= \int_0^{\delta} B_m(\tau)\, d\tau + \int_0^{\delta} D_m \left[Q_{m-1}(\tau)\right] d\tau \qquad (3.9)$$

where $\bar{A}_m(\tau)$, $B_m(\tau)$ and $D_m(\tau)$ are

$$\bar{A}_m(\tau) = \sum_{i=1}^{n} \begin{bmatrix} G_i(\tau)\ g_{i1}(\tau) \\ \cdot \\ \cdot \\ \cdot \\ G_i(\tau)\ g_{in}(\tau) \end{bmatrix} \qquad (3.10)$$

$$B_m(\tau) = \sum_{i=1}^{n} \begin{bmatrix} p_i((m-1)\delta + \tau)\ g_{i1}(\tau) \\ \cdot \\ \cdot \\ \cdot \\ p_i((m-1)\delta + \tau)\ g_{in}(\tau) \end{bmatrix} \qquad (3.11)$$

$$D_m(\tau) = \sum_{i=1}^{n} \begin{bmatrix} q_{(m-1)i}(\tau)\ g_{i1}(\tau) \\ \cdot \\ \cdot \\ \cdot \\ q_{(m-1)i}(\tau)\ g_{in}(\tau) \end{bmatrix} \qquad (3.12)$$

Taking the inverse of $A_m(\tau)$, assuming that it exists, the optimum value of the input signals in the last stage with the assumption of all signals up to that last stage are optimum, is

$$R((m-1)\delta) = \left[ \int_0^{\delta} A_m(\tau)\ d\tau \right]^{-1}$$

$$\cdot \left[ \int_0^{\delta} B_m(\tau)\ d\tau + \int_0^{\delta} D_m \left[ Q_{m-1}(\tau) \right] d\tau \right]$$

$$= \Phi_m \left[ Q_{m-1}(\tau) \right] \qquad (3.13)$$

where $\Phi_m \left[ Q_{m-1}(\tau) \right]$ is referred to as a functional since it is a function of a function $Q_{m-1}(\tau)$.

Consider next the second last stage. The function to be minimized is

$$\underset{\substack{R((m-2)\delta)\\R((m-1)\delta)}}{\text{Min}} \left\{ \int_0^{\delta} E_{m-1}^2((m-2)\delta+\tau) + E_m^2((m-1)\delta+\tau) \right\} d\tau$$

$$= \underset{R((m-2)\delta)}{\text{Min}} \left\{ \int_0^{\delta} E_{m-1}^2((m-2)\delta+\tau)\, d\tau \right.$$

$$\left. + \underset{R((m-1)\delta)}{\text{Min}} \int_0^{\delta} E_m^2((m-1)\delta+\tau) \right\} d\tau \qquad (3.14)$$

But the minimum of the last term is an explicit function of $Q_{m-1}(\tau)$ and hence as a function of $R((m-2)\delta)$, $R((m-3)\delta)$, . . . , $R(0)$ and $C(0)$. This minimum has already been established in Eq. 3.13.

To use this result in Eq. 3.14, let

$$Q_{m-1}(\tau) = Q_{m-2}(\tau) + \bar{G}(\tau)\, R((m-2)\delta) \qquad (3.15)$$

where

$$Q_{m-2}(\tau) = \sum_{j=2}^{m-1} \bar{G}((j-1)\delta+\tau)R((m-j-1)\delta) + C(0) \qquad (3.16)$$

Note that $Q_{m-2}(\tau)$ is a function only of the signal

levels up to the second last stage.

The error for the second last stage is

$$E_{m-1}((m-2)\delta+\tau)=P((m-2)\delta+\tau)-\bar{G}(\tau)R((m-2)\delta)-Q_{m-2}(\tau) \quad (3.17)$$

where $Q_{m-2}(\tau)$ has already been defined in Eq. 16.

The function to be minimized is therefore

$$\sigma_{m-1} = \int_0^{\delta}\sum_{i=1}^{n}\Big[p_i((m-2)\delta+\tau)-G_i(\tau)R((m-2)\delta) - Q_{m-2,i}(\tau)\Big]^2 d\tau \qquad (3.18)$$

The minimum of the function is established by differentiating with respect to $R((m-2)\delta)$. The result is a set of n linear algebraic equations in the n signal levels for the second last stage and are of the form

$$\left[\int_0 A_{m-1}(\tau)\, d\tau\right]^{-1} R((m-2)\delta)$$

$$= \int_0 A_{m-1}(\tau)\, d\tau + \int_0 B_{m-1}\left[Q_{m-2}(\tau)\right] d\tau \quad (3.18)$$

Again, assuming the inverse of $\overline{\Lambda}_{m-1}$ exists, the optimum value of the signal for the second last stage is

$$R((m-2)\delta) = \left[ \int_0^{\delta} \overline{\Lambda}_{m-1}(\tau)\, d\tau \right]^{-1} \left[ \int_0^{\delta} C_{m-1}(\tau)\, d\tau \right.$$

$$\left. + \int_0^{\delta} D_{m-1}\, Q_{m-2}(\tau)\, d\tau \right] = \Phi_{m-1}\left[ Q_{m-2} \right] \qquad (3.19)$$

Repeating the above process gives a set of m functionals of the form

$$\begin{aligned} R((m-1)\delta) &= \Phi_m \left[ Q_{m-1}(\tau) \right] \\ R((m-2)\delta) &= \Phi_{m-1} \left[ Q_{m-2}(\tau) \right] \\ &\;\;\vdots \\ R(0) &= \Phi_1 \left[ Q_0(\tau) \right] \end{aligned} \qquad (3.20)$$

where $Q_0(\tau) = H(\tau)\ C'(0)$.

The functions $\phi_j$ at each stage are determined by the solution of n simultaneous equations to establish the optimum magnitude of the input signals at that state as an explicit function of the arbitrary function $Q_{j-1}(\tau)$. All $Q_j(\tau)$ except $Q_0(\tau)$ are known only in terms of another arbitrary function yet unknown. Consequently, the optimum magnitudes of the signals at each stage is not actually evaluated until all m $\phi_j$ have been established. Once R(0) is known, $Q_1(\tau)$ and $R(\delta)$ are established. These, in turn, establish $R(2\delta)$ and $Q_2(\tau)$, etc.

The application of this procedure is demonstrated in the following two examples.

Example: 1

Let the s-domain relation between the input and output function for a system be given as

$$C(s) = \frac{1}{s+1} R^{*}(s) + \frac{1}{s+1} C(0) \qquad (3.21)$$

and let

$$p(t) = t \qquad (3.22)$$

be the preferred response. Determine the magnitudes of two discrete levels each of one second duration that gives minimum integro-error over the two periods with the initial condition

$$c(0) = 0.25 \qquad (3.23)$$

The output signal during the interval of (0, 1) is

$$c(\tau) = (1-e^{-\tau})\, r(0) + c(0)\, (1-e^{-\tau}) \qquad (3.24)$$

For the interval of (1, 2) second, the output signal is

$$c(\delta+\tau) = (1-e^{-(\delta+\tau)})r(0)+(1-e^{-\tau})r(\delta)$$
$$+c(0)(1-e^{-(\delta+\tau)}) \qquad (3.25)$$

The integro-error function of the two intervals are

$$e(\tau)= \quad -(1-e^{-\tau})r(0)-c(0)(1-e^{-\tau})$$

$$e(\delta+\tau)=(\delta+\tau)-(1-e^{-(\delta+\tau)})r(0)-(1-e^{-\tau})r(\delta)$$
$$-c(0)(1-e^{-(\delta+\tau)}) \qquad (3.26)$$

or

$$\begin{bmatrix} e(\tau) \\ e(\delta+\tau) \end{bmatrix} = \begin{bmatrix} \tau \\ \delta+\tau \end{bmatrix} - \begin{bmatrix} (1-e^{-\tau}) & 0 \\ (1-e^{-(\delta+\tau)}) & (1-e^{-\tau}) \end{bmatrix} \begin{bmatrix} r(0) \\ r(\delta) \end{bmatrix}$$

$$- \begin{bmatrix} c(0)(1-e^{-\tau}) \\ c(0)(1-e^{-(\delta+\tau)}) \end{bmatrix} \qquad (3.27)$$

The error function is

$$e(\delta+\tau)=(\delta+\tau)-(1-e^{-\tau})r(\delta)-Q_2(\tau) \qquad (3.28)$$

where $Q_2(\tau)$ is

$$Q_2(\tau)=(1-e^{-(\delta+\tau)})r(0)+c(0)(1-e^{-(\delta+\tau)}) \qquad (3.29)$$

For the last stage, the error criteria is

$$\sigma_2 \quad R(\delta) = \int_0^1 \left[(\delta+\tau)-(1-e^{-\tau})r(\delta)-Q_2(\tau)\right]^2 d\tau$$

(3.30)

Taking the partial derivatives with respect to $r(\delta)$ gives as the optimum for $r(\delta)$

$$r(\delta)=\left[\int_0^1 A_2(\tau)d\tau\right]^{-1}\left[\int_0^1 B_2(\tau)d\tau\right] \qquad (3.31)$$

where $A_2(\tau)$ and $B_2(\tau)$ are

$$A_2(\tau) = (1-e^{-\tau})^2 \qquad (3.32)$$

$$B_2(\tau) = \left[(\delta+\tau) - Q_2(\tau)\right](1-e^{-\tau}) \qquad (3.33)$$

or

$$r(\delta) = 3.59 - 5.95\int_0^1 Q_2(\tau)(1-e^{-\tau})\,d\tau \qquad (3.34)$$

$$= \phi_2(\tau)$$

The error criteria' $\sigma\left[\ r(0)\ \right]$ in the first stage is

$$\sigma[r(0)] = \int_0^{\delta}\left[p(\tau)-(1-e^{-\tau})r(0)-Q_1(\tau)\right]^2 d\tau \tag{3.35}$$

where $Q_1(\tau)$ is

$$Q_1(\tau) = c(0)(1-e^{-\tau}) \tag{3.36}$$

Taking the partial derivative with respect to **r(0)** gives as the **optimum for r(0)**

$$r(0) = \left[\int_0^{\delta}\bar{A}_1(\tau)\, d\tau\right]^{-1}\left[\int_0^{\delta}B_1(\tau)\, d\tau\right] \tag{3.37}$$

where $\bar{A}_1(\tau)$ and $B_1(\tau)$ are

$$\bar{A}_1(\tau) = (1-e^{-\tau})^2$$

and

$$B_1(\tau) = \left[\ p(\tau)-Q_1(\tau)\ \right](1-e^{-\tau})$$

or $\quad r(0) = 1.005$

Substituting this **r(0) into** Eq.3.34 , $r(\delta)$ **is obtained**

$$r(\delta) = 0.993$$

Example: 2

Let the s-domain system equations for a two input and two output port system be given as

$$\begin{bmatrix} c_1(s) \\ \\ c_2(s) \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} & \frac{2}{s+1} \\ \\ \frac{3}{s+1} & \frac{5}{s+1} \end{bmatrix} \begin{bmatrix} r_1(s) \\ \\ r_2(s) \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} & 0 \\ \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} c_1(0) \\ \\ c_2(0) \end{bmatrix} \qquad (3.38)$$

with the preferred function and initial condition

$$\begin{bmatrix} p_1(t) \\ \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 1.5 \\ \\ 1.0 \end{bmatrix} \qquad (3.39)$$

and

$$\begin{bmatrix} c_1(0) \\ \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ \\ 0.1 \end{bmatrix} \qquad (3.40)$$

The discrete state functions as given in Eq. 2.11 are

$$\begin{bmatrix} c_1(\tau) \\ c_2(\tau) \\ c_1(\delta+\tau) \\ c_2(\delta+\tau) \end{bmatrix} = \begin{bmatrix} (1-e^{-\tau})\,\bar{A}_{11} & \bar{A}_{12} \\ (1-e^{-(\delta+\tau)})\bar{A}_{21} & (1-e^{-\tau})\bar{A}_{22} \end{bmatrix} \begin{bmatrix} r_1(0) \\ r_2(0) \\ r_1(\delta) \\ r_2(\delta) \end{bmatrix}$$

$$+ \begin{bmatrix} c_1(0)(1-e^{-\tau}) \\ c_2(0)(1-e^{-\tau}) \\ c_1(0)(1-e^{-(\delta+\tau)}) \\ c_2(0)(1-e^{-(\delta+\tau)}) \end{bmatrix} \qquad (3.41)$$

where $\bar{A}_{11}$, $\bar{A}_{12}$, $\bar{A}_{21}$ and $\bar{A}_{22}$ are

$$\bar{A}_{11} = \bar{A}_{21} = \bar{A}_{22} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\bar{A}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix of discrete error function is therefore

$$\begin{bmatrix} e_1(\tau) \\ e_2(\tau) \\ e_1(\quad+\tau) \\ e_2(\quad+\tau) \end{bmatrix} = \begin{bmatrix} p_1(\tau) \\ p_2(\tau) \\ p_1(\delta+\tau) \\ p_2(\delta+\tau) \end{bmatrix} - \begin{bmatrix} (1-e^{-\tau})\bar{A}_{11} & \bar{A}_{12} \\ (1-e^{-(\delta+\tau)})\bar{A}_{21} & (1-e^{-\tau})\bar{A}_{22} \end{bmatrix} \begin{bmatrix} r_1(0) \\ r_2(0) \\ r_1(\delta) \\ r_2(\delta) \end{bmatrix}$$

$$- \begin{bmatrix} c\;(0)(1-e^{-\tau}) \\ c\;(0)(1-e^{-\tau}) \\ c\;(0)(1-e^{-(\delta+\tau)}) \\ c\;(0)(1-e^{-(\delta+\tau)}) \end{bmatrix} \qquad (3.42)$$

The error function in the second stage is

$$\begin{bmatrix} e_1(\delta+\tau) \\ e_2(\delta+\tau) \end{bmatrix} = \begin{bmatrix} p_1(\delta+\tau) \\ p_2(\delta+\tau) \end{bmatrix} - (1-e^{-\tau}) \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} r_1(\delta) \\ r_2(\delta) \end{bmatrix}$$

$$- \begin{bmatrix} Q_{21}(\tau) \\ Q_{22}(\tau) \end{bmatrix} \qquad (3.43)$$

where

$$\begin{bmatrix} Q_{21}(\tau) \\ Q_{22}(\tau) \end{bmatrix} = (1-e^{-(\delta+\tau)}) \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} \qquad (3.44)$$

$$+ (1-e^{-(\delta+\tau)}) \begin{bmatrix} e_1(0) \\ e_2(0) \end{bmatrix} \qquad (3.45)$$

For the second stage, the integro-error is

$$\sigma_2\left[R(\delta)\right] = \int_0^1 \left\{\left[e_1(\delta+\tau)\right]^2 + \left[e_2(\delta+\tau)\right]^2\right\} d\tau \qquad (3.46)$$

Taking the partial derivative with respect to $R(\delta)$ gives as the optimum for $R(\delta)$

$$R(\delta) = \left[\int_0^1 \bar{A}_2(\tau)d\tau\right]^{-1} \left[\int_0^1 B_2(\tau)'d\tau\right] \qquad (3.47)$$

where $\bar{A}_2(\tau)$ and $B_2(\tau)$ are

$$\bar{A}_2(\tau) = \begin{bmatrix} G_1(\tau)\, g_{11}(\tau) + G_2(\tau)\, g_{21}(\tau) \\ G_1(\tau)\, g_{12}(\tau) + G_2(\tau)\, g_{22}(\tau) \end{bmatrix}$$

$$= (1-e^{-\tau})^2 \begin{bmatrix} 13 & 19 \\ 19 & 29 \end{bmatrix} \qquad (3.48)$$

and

$$B_2(\tau) = \left[ p_1(\delta+\tau) - Q_{21}(\tau) \right] G_1(\tau)^T$$

$$+ \left[ p_2(\delta+\tau) - Q_{22}(\tau) \right] G_2(\tau)^T$$

$$= (1-e^{-\tau}) \begin{bmatrix} 6 \\ 8 \end{bmatrix} - (1-e^{-\tau}) \begin{bmatrix} 2Q_{21}+3Q_{22} \\ 2Q_{21}+5Q_{22} \end{bmatrix}$$

(3.49)

or

$$\begin{bmatrix} r_1(\delta) \\ r_2(\delta) \end{bmatrix} = \begin{bmatrix} 1.5 \\ -1.3 \end{bmatrix} - \int_0^1 (1-e^{-\tau}) \begin{bmatrix} 7.42Q_{21}+2.93Q_{22} \\ -4.44Q_{21}+2.98Q_{22} \end{bmatrix} d\tau$$

(3.50)

The error criteria $\sigma\left[R(0)\right]$ in the first stage is

$$\sigma\left[R(0)\right] = \int_0^1 \left\{ \left[e_1(\tau)\right]^2 + \left[e_2(\tau)\right]^2 \right\} d\tau \tag{3.51}$$

where the error functions $e_1(\tau)$ and $e_2(\tau)$ are

$$e_1(\tau) = p_1(\tau) - (1-e^{-\tau})\left[2r_1(0)+3r(0)\right] - Q_{11}(\tau)$$

$$e_2(\tau) = p_2(\tau) - (1-e^{-\tau})\left[3r_1(0)+5r_2(0)\right] - Q_{12}(\tau)$$

Taking the partial derivative with respect to R(0) gives as the optimum for R( 0 )

$$R(0) = \left[\int_0^{\delta} \bar{A}_1(\tau)\, d\tau\right]^{-1} \left[\int_0^{\delta} B_1(\tau)\, d\tau\right] \tag{3.52}$$

where $\bar{A}_1(\tau)$ and $B_1(\tau)$ are

$$\bar{A}_1(\tau) = \begin{bmatrix} G_1(\tau)\, g_{11}(\tau) + G_2(\tau)\, g_{21}(\tau) \\ G_1(\tau)\, g_{12}(\tau) + G_2(\tau)\, g_{22}(\tau) \end{bmatrix}$$

$$= (1-e^{-\tau})^2 \begin{bmatrix} 13 & 19 \\ 19 & 29 \end{bmatrix} \qquad (3.53)$$

and

$$B_2(\tau) = \left[p_1(\tau)-Q_{11}(\tau)\right] G_1(\tau)^T + \left[p_2(\tau)-Q_{12}(\tau)\right] G_2(\tau)^T$$

$$= (1-e^{-\tau}) \begin{bmatrix} 6 \\ 8 \end{bmatrix} - (1-e^{-\tau}) \begin{bmatrix} 2Q_{11}+3Q_{12} \\ 2Q_{11}+5Q_{12} \end{bmatrix} \qquad (3.54)$$

or

$$R(C) = \begin{bmatrix} 1.300 \\ \\ -1.211 \end{bmatrix}$$

Substituting the optimum value of step input found at $t = 0$ in Eq. 3.50 gives

$$R(\delta) = \begin{bmatrix} -0.48 \\ \\ 0.73 \end{bmatrix}$$

## IV. Discrete State Control of the Multivariable System.

In applications where the preferred function is specified only at discrete points, such as the end of each stage, the values of the input signal vector at the respective stages is uniquely determined by the solution of n simultaneous linear algebraic equations. Specifically, let the values of

$$C(0),\ C(\delta),\ \ldots,\ C((m-1)\delta)$$

be specified. These values are obtained from the system model of Eq. 2.11 when we set $\tau = 0$. Therefore, the required values of

$$R(0),\ R(\delta),\ \ldots,\ R((m-1)\delta)$$

as obtained from this models are

$$\begin{bmatrix} R(0) \\ \cdot \\ \cdot \\ \cdot \\ R((m-1)\delta) \end{bmatrix} = \begin{bmatrix} \bar{G}(0) & \cdots & 0 \\ & & \\ & & \\ & & \\ \bar{G}((m-1)\delta) & \cdots & \bar{G}(0) \end{bmatrix}^{-1} \begin{bmatrix} P(0) - H(0)C(0) \\ \\ \\ \\ P((m-1)\delta) - H((m-1)\delta)C(0) \end{bmatrix} \tag{4.1}$$

where the inverse of matrix $\bar{G}(0)$ is assumed to exist and

where the constant values

$$P(0),\ P(\delta),\ .\ .\ .\ ,P((m-1)\delta)$$

are the specified (or preferred) values at beginning of each stages.

As an example of application, consider the problem of establishing the input signals required to hold an aircraft on a specified path. The position is specified at regular intervals of time in terms of the three coordinate variables as indicated in Fig. 4.1.

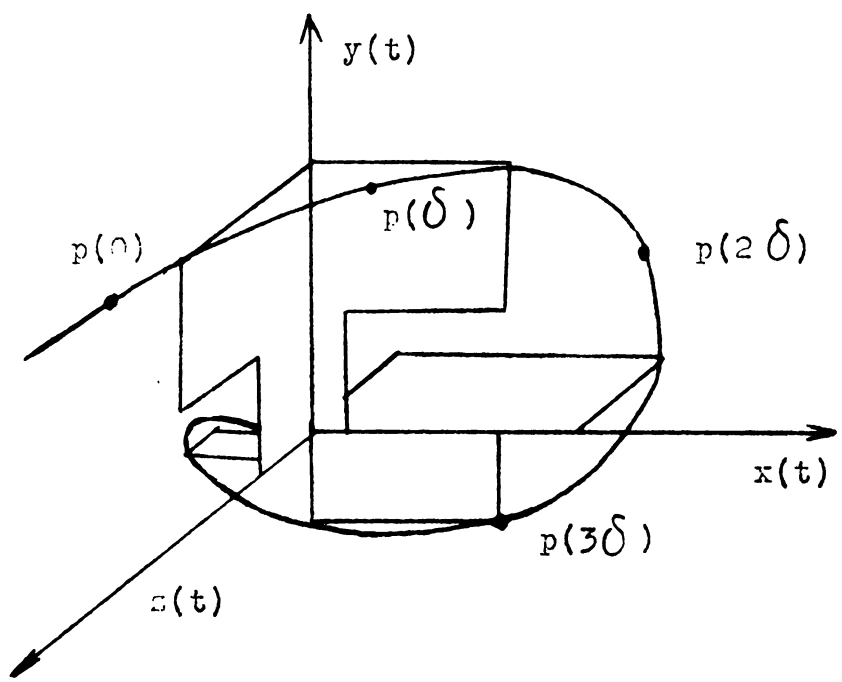


Fig. 4.1 Predetermined flight path

The s-domain model of the system, representing the inter-relationship for the system input and output variables is given as

$$\begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) & g_{13}(s) \\ g_{21}(s) & g_{22}(s) & g_{23}(s) \\ g_{31}(s) & g_{32}(s) & g_{33}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \\ w(s) \end{bmatrix}$$

$$+ \begin{bmatrix} h_{11}(s) & 0 & 0 \\ 0 & h_{22}(s) & 0 \\ 0 & 0 & h_{33}(s) \end{bmatrix} \begin{bmatrix} c_x(0) \\ c_y(0) \\ c_z(0) \end{bmatrix} \quad (4.2)$$

with the preferred function and initial condition

$$\begin{bmatrix} p_x(0) \\ p_y(0) \\ p_z(0) \end{bmatrix} \quad \cdots \quad \begin{bmatrix} p_x(3\delta) \\ p_y(3\delta) \\ p_z(3\delta) \end{bmatrix} \quad (4.3)$$

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} c_x(0) \\ c_y(0) \\ c_z(0) \end{bmatrix} \quad (4.4)$$

The discrete level input values R(0), . . .,R(3 )
are obtained from Eq. 4.1 by substituting the
given values of Eq. 4.2, Eq. 4.3 and Eq. 4.4.
The result is

$$\begin{bmatrix} u(0) \\ v(0) \\ w(0) \\ \cdot \\ \cdot \\ \cdot \\ u(3\delta) \\ v(3\delta) \\ w(3\delta) \end{bmatrix} = \begin{bmatrix} \overline{G}(0) & \cdots & 0 \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \overline{G}(3\delta) & \cdots & \overline{G}(0) \end{bmatrix}^{-1} \begin{bmatrix} p_x(0) & - c_x(0)h_{11}(0) \\ p_y(0) & - c_y(0)h_{22}(0) \\ p_z(0) & - c_z(0)h_{33}(0) \\ & \cdot \\ & \cdot \\ & \cdot \\ p_x(3\delta) & - c_x(0)h_{11}(3\delta) \\ p_y(3\delta) & - c_y(0)h_{22}(3\delta) \\ p_z(3\delta) & - c_z(0)h_{33}(3\delta) \end{bmatrix} \tag{4.5}$$

where $\overline{G}(\mathcal{T})$ is

$$\overline{G}(\mathcal{T}) = \begin{bmatrix} g_{11}(\mathcal{T}) & g_{12}(\mathcal{T}) & g_{13}(\mathcal{T}) \\ g_{21}(\mathcal{T}) & g_{22}(\mathcal{T}) & g_{23}(\mathcal{T}) \\ g_{31}(\mathcal{T}) & g_{32}(\mathcal{T}) & g_{33}(\mathcal{T}) \end{bmatrix}$$

## V. Conclusions.

The dynamic programming method for evaluating the optimum command signals for discrete level control of the multivariable systems is very effective for the integro-error criteria used in this thesis. By means of this process the optimum is found by solving successively m linear algebraic equations of order n rather than one set of equations of order (n×m).

Clearly, this procedure is most effective for the systems involving a relatively few control variables and a large number of stages, i. e., a large number of discrete control intervals. It is particularly effective, for example, for systems involving one control variable and, say, twenty stages.

As for the discrete state control of the multivariable systems, this technique can be applied only in the case of systems where the stability is assured, since the output values are optimized at each discrete time regardless of the functional behavior of the intervals.

## Appendix A Definitions

### A.1

Matrix of n x n order is

$$\bar{G}(t) = \begin{bmatrix} g_{11}(t) & \ldots & g_{1n}(t) \\ & \vdots & \\ g_{n1}(t) & \ldots & g_{nn}(t) \end{bmatrix} \qquad (a.1)$$

where each entry $g_{ij}(t)$ is inverse of Laplace transform of

$$g_{ij}(t) = L^{-1}\left[ g_{ij}(s) / s \right] \qquad (a.2)$$

of the system transfer function $\bar{G}(s)$

$$\bar{G}(s) = \begin{bmatrix} g_{11}(s) & \ldots & g_{1n}(s) \\ & \vdots & \\ g_{n1}(s) & \ldots & g_{nn}(s) \end{bmatrix} \qquad (a.3)$$

A.2

Row matrix of 1 x n order is

$$G_i(t) = \begin{bmatrix} g_{i1}(t) & . & . & . & g_{in}(t) \end{bmatrix} \qquad (a.4)$$

i-th row of Eq. a.1.

A.3

Matrix of m x n order

$$\bar{R} = \begin{bmatrix} R(0) \\ R(\delta) \\ . \\ . \\ R((m-1)\delta) \end{bmatrix} \qquad (a.5)$$

is the combination of the discrete level input functions.

A.4

Vector or column matrix

$$R(j\delta) = \begin{bmatrix} r_1(j\delta) \\ . \\ . \\ r_n(j\delta) \end{bmatrix} \qquad (a.6)$$

is the i-th column of Eq. a.5.

A.5

Any function $u(\delta+\tau)$ is defined as

$$u(\delta+\tau) = 0 \quad \left\{ \begin{array}{l} \tau < \delta \\ \tau > \delta \end{array} \right. \qquad (a.7)$$
$$= 1$$

where $0 \leq \tau < 1$

A.6

The integration of the matrix is defined as follow:

$$\int_0^{\delta} \begin{bmatrix} c_{11}(t) & \ldots & c_{1n}(t) \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ c_{n1}(t) & \ldots & c_{nn}(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} \int_0^{\delta} c_{11}(t)dt & \ldots & \int_0^{\delta} c_{1n}(t)\,dt \\ \int_0^{\delta} c_{n1}(t)dt & \ldots & \int_0^{\delta} c_{nn}(t)\,dt \end{bmatrix} \qquad (a.8)$$

A.7

Capital letters, such as $C(t)$, are used for the notation of vector, small case letters, such as $x(t)$, are used for the scalar, and capital letter with bar, such as $\bar{R}(t)$, are used for matrix.

A.8

The transpose of the vector $A(t)$ is

$$\text{Transpose of } A(t) = A(t)^{T} \qquad (A.9)$$

## Appendix B Derivations

**B.1**

The error criterion of Eq. 2.20 is

$$\sigma = \int_0^{\delta} \left[ E(t) \right]^T \left[ E(t) \right] dt + \ldots$$

$$+ \int_{(m-1)\delta}^{m\delta} \left[ E(t) \right]^T \left[ E(t) \right] dt \qquad (b.1)$$

The result of the translation of time is, then,

$$\sigma = \int_0^{\delta} \left[ E(\tau) \right]^T \left[ E(\tau) \right] d\tau + \ldots$$

$$+ \int_0^{\delta} \left[ E((m-1)\delta + \tau) \right]^T \left[ E((m-1)\delta + \tau) \right] d\tau$$

$$= \sum_{j=0}^{m-1} \int_0^{\delta} \left[ E(j\delta + \tau) \right]^T \left[ E(j\delta + \tau) \right] d\tau \qquad (b.3)$$

B.2

The error function of the (j-1)-th stage is

$$E(j\delta+\tau) = P(j\delta+\tau) - O(j\delta+\tau)$$
$$= P(j\delta+\tau) - \sum_{\lambda=0}^{j-1} \overline{G}(\lambda\delta+\tau)R((j-1-\lambda)\delta)$$
$$- H(j\delta+\tau)\, c'(0) \qquad (b.4)$$

The product of $\left[E(j\delta+\tau)\right]^T\left[E(j\delta+\tau)\right]$ is, then,

$$\left[E(j\delta+\tau)\right]^T\left[E(j\delta+\tau)\right] = \sum_{k=1}^{n}\left[e_k(j\delta+\tau)\right]^2$$

$$= \sum_{k=1}^{n}\left[p_k(j\delta+\tau) - \sum_{\lambda=0}^{j-1} G_k(\lambda\delta+\tau)R((j-1-\lambda)\delta) - h_k(j\delta+\tau)c_k(0)\right]^2$$

(b.5)

This establishes the result shown in Eq. 2.23.

B.3

By rearranging Eq. 2.25, we obtain

$$\sum_{j=1}^{m-1} \int_0^{\delta} \left[ \sum_{k=1}^{n} g_{k1}(\tau) \left[ G_k((j-1)\delta+\tau) \; . \; . \; G_k(\tau) \right] R \right] d\tau$$

$$= \sum_{j=1}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} \left[ p_k(j\delta+\tau) - c_k'(0) h_k(j\delta+\tau) \right] g_{k1}(\tau) \; d\tau$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$\sum_{j=1}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} \left[ g_{kn}((m-1)\delta+\tau) \left[ G_k((j-1)\delta+\tau) \; . \; . \; G_k(\tau) \right] R \right] d\tau$$

$$= \sum_{j=1}^{m-1} \int_0^{\delta} \sum_{k=1}^{n} \left[ p_k(j\delta+\tau) - c_k'(0) h_k(j\delta+\tau) \right] g_{kn}((m-1)+\tau) d\tau$$

(b.6)

Or, the matrix form of Eq. b.6 is

$$\int_0^{\delta}\sum_{k=1}^{n}\begin{bmatrix} g_{k1}(\tau)\sum\limits_{j=0}^{m-1} G_k(j\delta+\tau) & \cdots & g_k(\tau)G_k(\tau) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g_{kn}((m-1)\delta+\tau)\sum\limits_{j=0}^{m-1} G_k(j\delta+\tau) & \cdots & g_{kn}((m-1)\delta+\tau)G_k(\tau) \end{bmatrix} d\tau \; R$$

$$= \int_0^{\delta}\sum_{k=1}^{n}\begin{bmatrix} g_{k1}(\tau)\sum\limits_{j=0}^{m-1} p_k(j\ +\tau)-c_k'(0)h_k(j\delta+\tau) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ g_{kn}((m-1)\delta+\tau)\sum\limits_{j=0}^{m-1} p_k(j\delta+\tau)-c_k'(0)h_k(j\delta+\tau) \end{bmatrix} d\tau$$

(b.7)

[illegible].4

[illegible] integrating Eq. [illegible], we obtain

$$\int_0^1 \left[ (1-e^{-\tau})^2 + (1-e^{-(\delta+\tau)})^2 \right] r(0) \; d\tau$$

$$+ \int_0^1 (1-e^{-(\delta+\tau)})(1-e^{-\tau}) \quad r(\delta) \; d\tau$$

$$= \int_0^1 \left\{ (1-e^{-\tau}) \left[ \tau - (1-e^{-\tau}) c(0) \right] + (1-e^{-(\delta+\tau)}) \left[ (\delta+\tau) - (1-e^{-(\delta+\tau)}) c(0) \right] \right\}$$

$$x \; d\tau$$

and

$$\int_0^1 (1-e^{-\tau})(1-e^{-(\delta+\tau)}) r(0) \, d\tau + \int_0^1 (1-e^{-\tau})^2 r(\delta) \; d\tau$$

$$= \int_0^1 (1-e^{-\tau}) \left[ (\delta+\tau) - (1-e^{-(\delta+\tau)}) c(0) \right] \; d\tau$$

(b.[illegible])

Or, the matrix form of Eq. b.8 is

$$\int_0^1 \begin{bmatrix} (1-e^{-\tau})^2+(1-e^{-(\delta+\tau)})^2 & (1-e^{-(\delta+\tau)})(1-e^{-\tau}) \\ (1-e^{-\tau})(1-e^{-(\delta+\tau)}) & (1-e^{-\tau})^2 \end{bmatrix} d\tau \begin{bmatrix} r(0) \\ r(\delta) \end{bmatrix}$$

$$= \int_0^1 \begin{bmatrix} (1-e^{-\tau})\left[\tau-(1-e^{-\tau})c(0)\right]+(1-e^{-(\delta+\tau)})\left[(\delta+\tau)-(1-e^{-(\delta+\tau)})c(0)\right] \\ (1-e^{-\tau})\quad(\delta+\tau)-(1-e^{-(\delta+\tau)})\ c(0) \end{bmatrix} d\tau$$

(b.9)

B.2

From Eq. 2.48, we obtain

$$\sigma = \int_0^1 \left\{ p_1(\tau) - \begin{bmatrix} 2(1-e^{-\tau}) & 2(1-e^{-\tau}) \end{bmatrix} \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} - (1-e^{-\tau})c_1(0) \right\}^2 d\tau$$

$$+ \int_0^1 \left\{ p_2(\tau) - \begin{bmatrix} 5(1-e^{-\tau}) & 5(1-e^{-\tau}) \end{bmatrix} \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} - (1-e^{-\tau})c_2(0) \right\}^2 d\tau$$

$$+ \int_0^1 \left\{ p_1(\delta+\tau) - \begin{bmatrix} 2(1-e^{-(\delta+\tau)}) & 2(1-e^{-(\delta+\tau)}) \end{bmatrix} \begin{bmatrix} r_1(\delta) \\ r_2(\delta) \end{bmatrix} - (1-e^{-(\delta+\tau)})c_1(0) \right\}^2 d\tau$$

$$+ \int_0^1 \left\{ p_2(\delta+\tau) - \begin{bmatrix} 5(1-e^{-(\delta+\tau)}) & 5(1-e^{-(\delta+\tau)}) \end{bmatrix} \begin{bmatrix} r_1(\delta) \\ r_2(\delta) \end{bmatrix} - (1-e^{-(\delta+\tau)})c_2(0) \right\}^2 d\tau$$

(b.10)

Differentiating the error criteria of Eq. b.10 with respect to R(0) and R($\delta$) gives

$$\frac{\partial\sigma}{\partial r_1(0)} = \int_0^1 2\left[e_1(\tau)\ 2(1-e^{-\tau})+e_2(\tau)3(1-e^{-\tau})\ \right] d\tau$$

$$+\int_0^1 \ 2\left[e_1(\delta+\tau)2(1-e^{-(\delta+\tau)})+3e_2(\delta+\tau)(1-e^{-(\delta+\tau)})\right] d\tau = 0$$

$$\frac{\partial\sigma}{\partial r_2(0)} = \int_0^1 2\left[e_1(\tau)\ 2(1-e^{-\tau})+e_2(\tau)5(1-e^{-\tau})\right] d\tau$$

$$+\int_0^1 \ 2\left[e_1(\delta+\tau)2(1-e^{-(\delta+\tau)})+e_2(\delta+\tau)5(1-e^{-(\delta+\tau)})\right] d\tau = 0$$

$$\frac{\partial\sigma}{\partial r_1(\delta)} = \int_0^1 2\left[e_1(\delta+\tau)2(1-e^{-\tau})+e_2(\delta+\tau)3(1-e^{-\tau})\right] d\tau = 0$$

$$\frac{\partial\sigma}{\partial r_2(\delta)} = \int_0^1 2\left[e_1(\delta+\tau)2(1-e^{-\tau})+e_2(\delta+\tau)5(1-e^{-\tau})\right] d\tau = 0$$

(b.11)

Rearranging Eq. 3.11 in the matrix form will give the result as in Eq. 3.12.

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