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ABSTRACT

SURFACES OF EUCLIDEAN SPACES WITH PLANAR OR HELICAL GEODESICS TROUGH A POINT

By

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Compact connected surfaces in 3-dimensional Euclidean space E^3 with helical geodesics through a point are characterized as standard spheres. If the ambient manifold is a 4-dimensional Euclidean space E^4, then compact connected surfaces with helical geodesics through a point are characterized as standard spheres which lie in E^3 or pointed Blaschke surfaces which fully lie in E^4 . Such surfaces in 5-dimensional Euclidean space E^5, are characterized as standard spheres lying in E^3 or pointed Blaschke surfaces which lie fully in E^4 or E^5. We also prove that compact connected surfaces in a Euclidean space E^m $(m \geq 5)$ with helical planar geodesics through a point must lie in E^5 and these surfaces are one of model spaces above.

A surface in E^3 is locally a surface of revolution if and only if it has a point through which every geodesic is a normal section. So a complete connected surface in E^3 is a surface of revolution if and only if there is a point of the surface through which every geodesc is a normal section.

If surfaces in Euclidean space have planar geodesics through a point, then we show that those geodesics are normal sections and the Frenet curvatures of them are independent of the choice of the direction. Furthermore, a neighborhood of the point is completely determined by the Frenet curvature of a fixed geodesic.

Finally, we define a normal section of a pseudo-Riemannian submanifold in pseudo-Euclidean space in a way similar to that of the Riemannian case. Surafces in pseudo-Euclidean space with planar normal sections are classified as Veronese surfaces or flat surfaces if the surfaces do not lie in a 3-dimensional affine space.

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TABLE OF CONTENTS

INTRODUCTION

The study of surfaces in a Euclidean space is fundamental to the understanding of geometric models and provides good ideas from which to develop certain theories. So, many geometers tried to study surfaces in Euclidean space from many different points of view. The theory of pseudo-Riemannian manifolds enables us to study some surfaces in a pseudo-Euclidean space.

In this sense, surfaces in a Euclidean space or a pseudo-Euclidean space with some properties concerning geodesics and normal sections need to be examined and classified.

Helical submanifolds were first introduced in [Be.A]. Helical submanifolds in a Euclidean space or a unit sphere have been studied by K. Sakamoto, [S-1], [S-5], [S-6], since 1982. He proved that such submanifolds are either Blaschke manifolds or Euclidean planes.

On the other hand, in 1981, B.-Y. Chen and P. Verheyen, [Ch.B-V-1], [Ch.B-V-2], introduced the notion of submanifolds with geodesic normal sections and classified surfaces with geodesic normal sections in a Euclidean space. They also proved that helical submanifolds have geodesic normal sections. Later, P. Verheyen, [V], proved that a submanifold M in a Euclidean space E^m of dimension m with geodesic normal sections are helical. So the concept of submanifolds with geodesic normal sections coincides with the concept of helical submanifold if the ambient space is a Euclidean space.

In [Ho.S], S. L. Hong introduced the notion of planar geodesic immersions. Such immersions were later classified by J. A. Little, [L], and K. Sakamoto, [S-1], independently, who proved that M^n is a compact symmetric space of rank one and the second fundamental form is parallel. The Veronese surface can be considered as one of the

best examples determined by the planar geodesic immersion if the ambient space is a 5-dimensional Euclidean space E^5.

Using the theory of pseudo-Riemannian structures, C. Blomstrom, [B-2], defined planar geodesic immersions of pseudo-Riemannian submanifolds into a pseudo-Riemannian manifold and she classified complete parallel surfaces with planar geodesics in a pseudo-Euclidean space E^m_s as pseudo-Riemannian spheres, flat quadric surfaces or Veronese surfaces.

However, there has been no research on a submanifold M in a Euclidean space E^m with the property that for a fixed point p in M every geodesic passing through p is a helix of the same curvatures or every geodesic through p is planar or a normal section at the point p. From this point of view, we are going to study surfaces in a Euclidean space which have such properties and to chracterize such surfaces. Furthermore, we are going to classify surfaces of a pseudo-Euclidean space with planar normal sections.

In Chapter 0, we introduce some fundamental definitions and concepts which are the necessary background for the theory throughout this thesis.

In Chapter 1, we study compact connected surfaces in a Euclidean space with helical geodesics through a point . If the ambient space is a 3-dimensional Euclidean space E^3, then such surfaces are characterized as standard spheres. If the ambient manifold is a 4-dimensional Euclidean space E^4, then we obtain that geodesics through the point must be of rank 2, i.e., they are planar curves, and surfaces are characterized as standard spheres which lie in E^3 or pointed Blaschke surfaces which fully lie in E^4. If the ambient manifold is a 5-dimensional Euclidean space E^5, then geodesics of the given surface through the point may be of rank 4. In this case, using some fundamental equations obtained from the helices through the point, we set up a system of ordinary differential equations. By solving this system of differential equations, we have examples of pointed Blaschke surfaces which are diffeomorphic to a real projective space and lies fully in E^5. So we can characterize such surfaces as standard spheres which lie in E^3 or pointed Blaschke surfaces which lie fully in E^4 or E^5. By means of this characterization, we have a new characterization of the Veronese surface, namely, a Veronese surface is characterized as a compact connected surface with constant Gaussian curvature and a nonumbilical point through which every geodesic is a helix .

In Chapter 2, we study a surface M in a three-dimensional Euclidean space E^3 with geodesic normal sections at a point p. By adopting geodesic polar coordinates about the base point p, geodesics are proved to depend only on the arc length and thus M is characterized as locally a surface of revolution around the point p. So, if M is complete connected, then M is a surface of revolution if and only if there is a point p through which every geodesic is a normal section.

In Chapter 3, we study a surface M in a Euclidean space E^m with planar geodesics through a point p. We prove that planar geodesics through a point p are normal sections of M at p. We also prove that geodesics through p only depends on the arc length and thus Frenet curvatures are independent of the choice of the direction. So, we can precisely determine how the surface looks like in a neighborhood by means of the Frenet curvature of a fixed geodesic through p. We also observe that a surface in a Euclidean space E^m with planar geodesic through p is possible to lie fully in a considerably higher dimensional Euclidean space $E^n \subset E^m$ if p is an isolated flat point .

In Chapter 4, we define a normal section of a pseudo-Riemannian submanifold in a pseudo-Euclidean space in a way similar to that of the Riemannian case. We also define pseudo-isotropy which is a similar notion to isotropy in a Riemannian manifold and obtain that if a surface M in a pseudo-Euclidean space E^m_s with planar normal sections is not contained in a 3-dimensional affine space, then all normal sections are geodesics and hence M has planar geodesics. Using the classification theorem of C. Blomstrom, [B-2], we can classify surfaces of index r in a pseudo-Euclidean space E^m_s with planar normal sections as

flat surfaces which fully lie in $E^{2+k}_{r'k}$ (k = 2, 3) or Veronese surfaces in $E^5_{r(3-r)}$ or $E^5_{5-(r+1)(2-r)}$ if the surfaces are not contained in a 3-dimensional affine space.

CHAPTER 0

PRELIMINARIES

Let M be a smooth manifold and let $C^{\infty}(M)$ be the set of all smooth functions defined on M. Then $C^{\infty}(M)$ becomes a **R**-module, where **R** is the real number field.

Definition. A tangent vector to M at a point p is a function $X_p : C^{\infty}(M) \to \mathbf{R}$, whose value at $f \in C^{\infty}(M)$ is denoted by $X_p(f)$, such that for all f, $g \in C^{\infty}(M)$ and $r \in \mathbf{R}$,

(i) $$X_p(f + g) = X_p(f) + X_p(g),$$

(ii) $$X_p(rf) = r\, X_p(f)$$

and

(iii) $$X_p(fg) = X_p(f)\, g(p) + f(p)\, X_p(g).$$

We usually regard $X_p(f)$ as the directional derivative of f in the direction X_p at p.

The set of all tangent vectors to M at p is called the tangent space to M at p, which is denoted by $T_p(M)$. The tangent space $T_p(M)$ defines a vector space of dimension n over R. A (smooth) vector field on M is a C^{∞} mapping which assigns to each point p of M a tangent vector to M at p. The set of all tangent vectors on a manifold M of dimension n is called the tangent bundle, denoted by TM, which forms a fibre bundle over M with $T_p(M)$ as the fibre over a point p in M.

A manifold M on which there is a symmetric tensor field g of type (0,2) which is positive definite and bilinear is called a Riemannian manifold and g the Riemannian metric or the first fundamental form. If there is a nondegenerate bilinear tensor g of type (0,2) with the property that the dimension of the negative definite subbundle of TM with respect to g is constant on M, then M is called a pseudo-Riemannian or a semi-Riemannian manifold and g the pseudo-Riemannian metric. In this case, the dimension of the negative definite subbundle is called the index of the manifold M. If M is a pseudo-Riemannian manifold, we will use the notation M_r^n as M that is of dimension n and has index r. A pseudo-Riemannian manifold of index one is called Lorentzian. The Minkowski space-time E_1^4 is the best known example of a Lorentzian manifold and is largely dealt with in the special relativity. If the index is zero, then $M_0^n = M^n$ is nothing but a Riemannian manifold.

The metric tensor on a manifold defines the lengths of vector fields and angles between them. If the metric is indefinite, then a nonzero vector field may have zero or negative length. We shall say that a nonzero vector field is spacelike if it has positive length, timelike if it has negative length and lightlike or null if it has length zero. We regard a zero vector as a space like vector.

We now define a connection on a manifold.

Definition. A connection ∇ on a manifold M is a mapping of the product of the set of vector fields into the set of vector fields denoted by $\nabla(X, Y) = \nabla_X Y$ which has the linearity and derivation properties : For all $f,g \in C^\infty(M)$ and all vector fields X, Y and Z,

(i) $$\nabla_{fX+gY} Z = f\,\nabla_X Z + g\,\nabla_Y Z,$$

(ii) $$\nabla_X(fY + gY) = (Xf)Y + f\nabla_X Y + (Xg)Z + g\,\nabla_X Z,$$

$\nabla_X Y$ is called the covariant derivative of Y with respect to X for the connection ∇. Then we can define the so-called torsion T :

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where [X, Y] is the vector field defined by $[X, Y]f = X(Yf) - Y(Xf)$ for all $f \in C^{\infty}(M)$. It is known that there is a unique connection ∇ on a pseudo-Riemannian manifold satisfying

(iii) $$T = 0,$$

(iv) $$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all vector fields X, Y and Z , where $\langle\ ,\ \rangle$ is the pseudo-Riemannian metric tensor. From now on, $\langle\ ,\ \rangle$ always means a Riemannian or a pseudo-Riemannian metric tensor according to the context unless it is stated otherwise. The connection satisfying above (i) ~ (iv) is called the Levi-Civita connection.

The covariant derivative of a general tensor field S of type (r,s) is naturally defined by linearity and derivation as follows : Since S can be regarded as a multilinear mapping of $TM \times TM \times \cdots \times TM$, s copies of TM, into the space of contravariant tensors of degree r,

$$(\nabla_X S)(X_1,\ldots, X_s) = \nabla_X(S(X_1,\ldots, X_s)) - \sum_{i=1}^{s} S(X_1,\ldots, \nabla_X X_i,\ldots, X_s)$$

for any $X_1,\ldots, X_s \in TM$. By setting

$$(\nabla_X S)(X_1,\dots, X_s) = (\nabla S)(X, X_1,\dots, X_s)$$

we obtain a tensor field ∇S of type (r, s+1). We shall say that a tensor field S is parallel if $\nabla S = 0$. So the metric tensor field on a pseudo-Riemannian manifold is parallel with respect to the Levi-Civita connection.

We now define a (regular) curve γ on a manifold M.

Definition. A curve is a smooth mapping $\gamma : I \to M$, where I is an open interval. In particular, if $\frac{d\gamma}{dt} \neq 0$ for all $t \in I$, then γ is said to be regular. In the sequel, a curve means a regular curve unless it is stated otherwise. Let $s = \int_{t_0}^{t} \|\frac{d\gamma}{dt}\| dt$, where $\|\frac{d\gamma}{dt}\|^2 = <\frac{d\gamma}{dt}, \frac{d\gamma}{dt}>$. Then s is the arc length defined on the curve γ. If a curve is parametrized by arc length, then it has the unit speed.

The existence of a Riemannian metric or pseudo-Riemannian metric on a manifold provides an important tool in the study of manifolds from a geometric point of view, allowing us to introduce on such spaces many concepts of Euclidean geometry such as distances, angles between curves, areas, volumes and straight lines. Developing the idea of a straight line in a Euclidean space, we define a geodesic γ in M as a curve γ parametrized by arc length s such that γ satisfies

$$\frac{d^2x^h}{ds^2} + \sum_j \sum_i \Gamma^h_{ji} \frac{dx^j}{ds}\frac{dx^i}{ds} = 0,$$

where $\{x^h\}$ is a local coordinate system and $\nabla_{\partial/\partial x^i}\, \partial/\partial x^j = \sum_h \Gamma^h_{ji}\, \partial/\partial x^h$, $\gamma(s) = (x^h(s))$.

Thus, γ is a geodesic if and only if $\nabla_T T = 0$, where $T = \frac{d\gamma}{ds}$. In other words, a curve γ is a geodesic if and only if its velocity vector field is parallel along γ.

We now give the definition of the exponential mapping.

Definition. Let $p \in M$ and X be a unit vector in $T_p(M)$. Let $\gamma(t)$ be the geodesic emanating from p with initial velocity X with domain (a,b). Set $\exp_p tX = \gamma(t)$ for $t \in (a, b)$. $\exp_p$ is called the exponential mapping at p.

$\exp_p$ carries lines through the origin in $T_p(M)$ to geodesics through $p \in M$. Thus, the distances in M near p are approximated by distances in $T_p(M)$. As a matter of fact, $\exp_p$ gives a diffeomorphism from a neighborhood of the origin in $T_p(M)$ onto a neighborhood of p in M. For a curve γ in E^3 parametrized by arc length s, the length $\|\gamma''(s)\|$ measures how rapidly the curve pulls away from the tangent line at s in a neighborhood of s. This measurement $\kappa(s) = \|\gamma''(s)\|$ is called the curvature of γ at s. At a point where $\kappa(s) \neq 0$, a unit vector N(s) in the direction $\gamma''(s)$ is well-defined by the equation $\gamma''(s) = \kappa(s)\, N(s)$. Moreover, $\gamma''(s)$ is normal to $\gamma'(s)$. Thus, N(s) is normal to $\gamma'(s)$ and is called the principal normal vector. If $\gamma''(s) \neq 0$, the number $\tau(s)$ defined by $B'(s) = -\tau(s)N(s)$ is called the torsion of γ and $B(s) = T(s) \times N(s)$ the binormal vector at $\gamma(s)$. Then we have the following so-called Frenet formulas:

$$T'(s) = \kappa(s)N(s),$$

$$N'(s) = -\kappa(s)T(s) + \tau(s)\, B(s),$$

$$B'(s) = -\tau(s)N(s),$$

where $T(s) = \gamma'(s)$. Generalizing this idea, we obtain the Frenet frame and Frenet curvatures for a curve $\gamma : I \to M$. Let $\gamma'(s) = T_1(s)$ be the unit tangent vector and put $\kappa_1 = \| \nabla_{T_1} T_1 \|$. If κ_1 is identically zero on I, then γ is said to be of rank 1. If κ_1 is not identically zero, then one can define T_2 by $\nabla_{T_1} T_1 = \kappa_1 T_2$ on $I_1 = \{ s \in I \mid \kappa_1(s) \neq 0 \}$. Set $\kappa_2 = \| \nabla_{T_1} T_2 + \kappa_1 T_1 \|$. If κ_2 is identically zero on I_1, then γ is said to be of rank 2. If κ_2 is not identically zero on I_1, then we define T_3 by $\nabla_{T_1} T_2 = -\kappa_1 T_1 + \kappa_2 T_3$. Inductively, we can define T_d and $\kappa_d = \| \nabla_{T_1} T_d + \kappa_{d-1} T_{d-1} \|$ and if $\kappa_d = 0$ identically on $I_{d-1} = \{ s \in I \mid \kappa_{d-1}(s) \neq 0 \}$, then γ is said to be of rank d. If γ is of rank d, then we have a matrix equation

$$\nabla_{T_1}(T_1, T_2, \ldots, T_d) = (T_1, T_2, \ldots, T_d)\, \Lambda,$$

on I_{d-1}, where Λ is a $d \times d$ -matrix defined by

$$\Lambda = \begin{pmatrix} 0 & -\kappa_1 & 0 & \cdots & & \\ \kappa_1 & 0 & -\kappa_2 & 0 & & 0 \\ & \kappa_2 & 0 & & & \\ & & & \cdots & & -\kappa_{d-1} \\ 0 & & & & \kappa_{d-1} & 0 \end{pmatrix}. \tag{0.1}$$

The matrix Λ, $\{T_1, T_2, \ldots, T_d\}$ and $\kappa_1, \ldots, \kappa_d$ are called the Frenet formula, Frenet frame and Frenet curvatures of γ respectively.

We now define the curvature tensor R on a pseudo-Riemannian manifold. It is a fundamental theorem of advanced calculus that the second order partial derivatives are independent of the order of differentiation:

$$\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)$$

for all C^2 function f. For functions on manifolds the similar property $X(Yf) = Y(Xf)$ does not hold in general. As matter of fact $[X,Y]$ measures the extent by which it fails:

$$X(Yf) - Y(Xf) = [X,Y]f, \qquad f \in C^\infty(M)\,, X, Y \in TM.$$

Of course, in general $\nabla_X(\nabla_Y Z) = \nabla_Y(\nabla_X Z)$ is not true. To measure this failure of interchangeability, we define R by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in TM. \tag{0.2}$$

R is a tensor field of type (1, 3) which is called the curvature tensor. For a surface, the Gaussian curvature is intrisically defined. Generalizing the Gaussian curvature, we define the sectional curvature.

Definition. Let M^n_r be a pseudo-Riemannian manifold ($0 \le r \le n$). Suppose $p \in M^n_r$ and let Π be a nondegenerate plane spanned by X and Y in $T_p M^n_r$. The sectional curvature of the plane Π is given by

$$K_p(\Pi) = \frac{-g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

$K_p(\Pi)$ is well-defined and independent of the choice of basis of Π. It is not hard to show that the Gaussian curvature of a surface in E^3 agrees with the sectional curvature. If $K_p(\Pi)$ is independent of the choice of the nondegenerate plane Π, then we say that M^n_r has constant curvature at p. Let K be a real number. We say that M^n_r has constant curvature K if M^n_r has constant curvature K at every point of M^n_r. The famous F. Schur's Theorem tells us that if a connected pseudo-Riemannian manifold M^n_r of dimension $n \geq 3$ has constant curvature at every point, then M^n_r has constant curvature, that is, the sectional curvature is independent of the choice of the point and nondegenerate plane. Complete simply connected spaces of constant sectional curvature are called real space forms. Manifolds M and N are said to be isometric if there is a diffeomorphism φ from M to N that preserves metric tensors. If two surfaces have the same Gaussian curvature, then they are locally isometric. In [Wo], any pseudo-Riemannian space form is shown to be isometric to a pseudo-Euclidean space E^m_s, a pseudo-Riemannian sphere $S^n_s(r)$ or a hyperbolic space $H^n_s(r)$. These latter two are described as follows : Given $0 \leq s \leq n$ with $2 \leq n$ and a number $r > 0$, we define

$$(0.3) \qquad S^n_s(r) = \left\{ x \in E^{n+1}_s \mid -\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{n+1} x_i^2 = r^2 \right\}$$

which is analogous to an ordinary sphere

$$S^n(r) = \{ x \in E^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = r^2 \} = S^n_0(r).$$

$S^n_s(r)$ is called the pseudo-Riemannian sphere. We also define the so-called hyperbolic space

$$(0.4) \qquad H^n_s(r) = \left\{ x \in E^{n+1}_{s+1} \mid -\sum_{i=1}^{s+1} x_i^2 + \sum_{i=s+2}^{n+1} x_i^2 = -r^2 \right\}.$$

In the Riemannian case, we have two copies H^n_0 of the hyperbolic spaces H^n.

We denote a real space form by M(c). Then M(c) has its curvature tensor of the form

$$R(X, Y)Z = c(\langle Y, Z\rangle X - \langle Z, X\rangle Y)$$

for X, Y, $Z \in TM(c)$. So, the usual Euclidean space and the pseudo-Euclidean space have curvature tensor R = 0, and the ordinary sphere $S^n(1)$ has curvature tensor

$$R(X, Y)Z = \langle Y, Z\rangle X - \langle Z, X\rangle Y$$

and $H^n(1)$ has

$$R(X, Y)Z = \langle Z, X\rangle Y - \langle Y, Z\rangle X.$$

All of the preceding geometric concepts depend only on the pseudo-Riemannian or Riemannian metric and not on any external consideration. However, if we consider a manifold which is immersed into another, then we can observe the image of the immersion as viewed from the ambient manifold. Naturally we need some theoretical background for the submanifold theory.

Let M and $\tilde{M}$ be manifolds of dimension n and m respectively (n < m) and let $\varphi : M \to \tilde{M}$ be a differential mapping such that its differential $(d\varphi)(p) = (\varphi_*)_p$ is injective at every point p in M. Then we say that M is an <u>immersed submanifold</u> of $\tilde{M}$ and φ is an immersion from M to $\tilde{M}$. If the pullback $\varphi^*(g)$ of the metric tensor g of $\tilde{M}$ is a metric on M, then M is called a <u>pseudo-Riemannian submanifold</u> of $\tilde{M}$. The index of the pseudo-Riemannian submanifold is at most that of the ambient manifold. By observing the submanifold and the ambient manifold, we may obtain certain intrinsic properties of the submanifold which come from that of the ambient manifold. Let M be a pseudo-Riemnannian manifold of $\tilde{M}$ endowed with Levi-Civita connection $\tilde{\nabla}$. We may identify the image of M with M itself and hence we no longer distinguish vector fields on M from the images under the immersion. Let X, Y be in TM. Then we can write the covariant derivative $\tilde{\nabla}_X Y$ as

$$(0.5) \qquad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$

where $\nabla_X Y$ is a vector field tangent to M and $\sigma(X, Y)$ is a vector field normal to M. Then ∇ turns out to be the Levi-Civita connection on M and σ is a symmetric bilinear form on $TM \times TM$ which is called the <u>second fundamental form</u> of the submanifold. (0.5) is referred to as the <u>Gauss formula</u> for M in $\tilde{M}$. Let ξ be a normal vector field on M and X a vector field on M. We may then decompose $\tilde{\nabla}_X \xi$ as

$$(0.6) \qquad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $-A_\xi X$ and $D_X \xi$ are the tangential component and the normal component of $\tilde{\nabla}_X \xi$ respectively. A_ξ is called the <u>Weigarten map</u> associated to ξ and forms a self-adjoint tangent bundle endomorphism on TM satisfying $< A_\xi X, Y> = < \sigma(X, Y), \xi >$ for X, $Y \in TM$. D is a metric connection in the normal bundle $T^\perp M$ of M in $\tilde{M}$ with respect to the induced metric on $T^\perp M$. D is called the <u>normal connection</u> on M. The equation (0.6) is called the <u>Weingarten formula</u>.

A submanifold M is said to be <u>totally geodesic</u> if every geodesic in M is a geodesic in $\tilde{M}$. It is well-known that M is totally geodesic if and only if the second fundamental form σ vanishes identically.

For a normal section ξ on M, if A_ξ is everywhere proportional to the identity transformation I, that is, $A_\xi = \rho I$ for some function ρ, then ξ is called an umbilical section on M, or M is said to be umbilical with respect to ξ. If the submanifold M is umbilical with respect to every local normal section in M, then M is said to be totally umbilical.

Let $X_1, \ldots, X_n$ be an orthonormal basis of the tangent space $T_p(M)$ at the point $p \in M$ and let

$$H = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \, \sigma(X_i, X_i),$$

where $\varepsilon_i = \text{sgn} \langle X_i, X_i \rangle = \pm 1$. Then H is a normal vector which is called the mean curvature vector at p. Let $\xi_1, \ldots, \xi_{m-n}$ be an orthonormal basis of the normal space $T_p^{\perp}M$ at $p \in M$ and let $A^x = A_{\xi_x}$, then H can be written as

$$H = \frac{1}{n} \sum_{i=1}^{m-n} (\mathrm{tr} A^x) \xi_x.$$

It is easily shown that H is independent of the choice of the orthonormal basis ξ_x.

A submanifold M is called a minimal submanifold if the mean curvature vector vanishes identically. We call the submanifold a pseudo-umbilical submanifold if the Weingarten map A_H associated with the mean curvature vector H is proportional to the identity transformation.

If we use the Gauss formula and compute the curvature tensor $\tilde{R}$ of the ambient manifold, then we obtain the Gauss equation

(0.7) $$\langle R(X, Y)Z, W \rangle = \langle \tilde{R}(X, Y)Z, W \rangle + \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(Y, W), \sigma(X, Z) \rangle$$

and the Codazzi equation

$$(0.8)\qquad (\tilde{R}(X, Y)Z)^{\perp} = (\bar{\nabla}_X\sigma)(Y, Z) - (\bar{\nabla}_Y\sigma)(X, Z)$$

for all X, Y, Z, W $\in$ TM, where $^{\perp}$ denotes the normal component relative to M and $\bar{\nabla}_X\sigma$ is the covariant derivative of σ on $TM \oplus T^{\perp}M$ defined by

$$(0.9)\qquad (\tilde{\nabla}_X\sigma)(Y, Z) = D_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for X, Y, Z $\in$ TM. If the ambient manifold has constant curvature c, then the Gauss and Codazzi equations are given respectively by

$$(0.10)\qquad \begin{aligned} \langle R(X, Y)Z, W\rangle &= c(\langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle Z, X\rangle) \\ &\quad + \langle\sigma(X, W), \sigma(Y, Z)\rangle - \langle\sigma(Y, W), \sigma(X, Z)\rangle \end{aligned}$$

and

$$(0.11)\qquad (\bar{\nabla}_X\sigma)(Y, Z) - (\bar{\nabla}_Y\sigma)(X, Z) = 0.$$

Making use of the Weingarten formula, we obtain the Ricci equation : For X, Y $\in$ TM and $\xi, \eta \in T^{\perp}M$

$$(0.12)\qquad \langle\tilde{R}(X, Y)\xi, \eta\rangle = \langle R^N(X, Y)\xi, \eta\rangle - \langle[A_\xi, A_\eta] X, Y\rangle,$$

where $R^N(X, Y)\xi = D_X D_Y\xi - D_Y D_X\xi - D_{[X,Y]}\xi$ and $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$.

From these equations we see that the curvature tensors of the submanifold and the ambient maniofld are related in terms of the second fundamental form.

From time to time, a lot of geometric properties are explained with a standard sphere S^n, a real projective space RP^n, a complex projective space CP^n, a quaternion projective space HP^n and Cayley projective plane $\mathcal{O}P^2$ as examples. So we need some basic concepts and definitions concerning these spaces.

Let U be a neighborhood of the origin in $T_p(M)$ such that $\exp_p|_U$ is a diffeomorphism. We define $s : U \to U$ by $s(X) = -X$ and set $s_p = (\exp_p|_U)\circ s\circ(\exp_p|_U)^{-1}$. This s_p is called the geodesic symmetry with respect to p on $\exp_p(U)$.

The manifold M is called locally symmetric if for each $p \in M$ there exists U, a neighborhood of the origin in T_pM, such that the geodesic symmetry s_p is an isometry. Locally symmetric spaces are charaterized by $\nabla R = 0$, i.e., the curvature tensor is covariant constant.

Definition. A connected Riemannian manifold M is a symmetric space if for each $p \in M$ there exists an involutive isometry $s_p : M \to M$ such that

$$s_p \circ \exp_p = \exp_p \circ s.$$

Clearly, if M is symmetric, then M is locally symmetric.

Definition. The rank of a symmetric space is defined as the maximal dimenison of the flat submanifolds, i.e., the curvature tensor $R = 0$, which are totally geodesic in M.

S^n, RP^n, CP^n, HP^n and $\mathcal{O}P^2$ are the only examples of compact rank one symmetric spaces. These are abbreviated by CROSSes.

Let M be complete and connected. The Hopf-Rinow Theorem tells us any geodesic segment can be extended indefinitely. In general, a geodesic joining two points is not unique, for example, on a standard sphere S^2 in E^3, geodesics are great circles and so the number of geodesics joining the north pole and the south pole is infinite. However, for two points which are sufficiently close, the geodesic segment joining these two points is unique. So we need to define the following cut-map : Let X be an element of the unit tangent space $U_pM = \{X \in T_pM \mid \|X\| = 1\}$ and let γ be a geodesic emanating from p with initial velocity X, i.e., $\gamma(s) = \exp_p(sX)$, where s is the arc length. Let Seg(p,q) be the set of all geodesics from p to q which are parametrized by arc length. Then for s small enough, $Seg(p=\gamma(0),\gamma(s))$ contains only one element $\gamma|_{[0,s]}$. The set

$$A = \{\ s \in \mathbf{R}_+ \mid \ \gamma|_{[0,s]} \in \mathrm{Seg}\ (p=\gamma(0),\gamma(s))\ \}$$

is necessarily $\mathbf{R}_+$ or an interval (0, r] for some $r \in \mathbf{R}_+$, where $\mathbf{R}_+$ is the set of all positive real numbers. If $A = \mathbf{R}_+$, we say that there is no cut-point on γ; if A = (0, r], we say that $\gamma(r)$ is the <u>cut-point of p</u> and r the <u>cut-value</u> of γ. Let UM be the unit tangent bundle over M. The cut-map $\phi : UM \to \mathbf{R}_+ \cup \{\infty\}$ defined by $\phi(X) = r$ if A = (0, r] and $\phi(X) = \infty$ if $A = \mathbf{R}_+$. The cut-map is continuous. (cf. [K- N], Vol. II, p. 98).

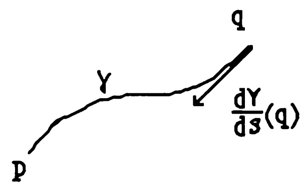
Definition. The <u>cut-locus</u> Cut(p) of a point p in M is the set of all cut-points of p, i.e,

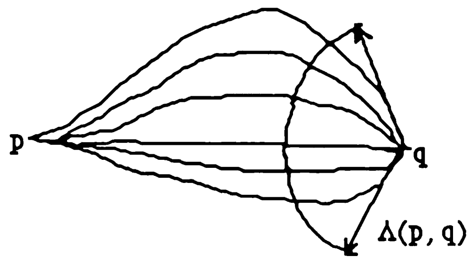
$$\mathrm{Cut}(p) = \{\ \exp_p(\phi(X)X) \mid \ X \in U_p(M)\ \}.$$

Definition. A Riemannian manifold M is said to have <u>spherical cut-locus</u> at p if for every $X \in U_p(M)$ the cut-value $\phi(X)$ is finite and does not depend on the choice of X.

For two distinct points p and q in M we define the link from p to q to be

$$\Lambda(p, q) = \{ \frac{d\gamma}{ds}(q) \in U_q(M) \mid \gamma \in Seg(p, q) \}.$$





A subset Θ of the unit sphere S of a Euclidean space V is said to be a great sphere if there exists a subspace W of V such that $\Theta = S \cap W$. By definition, the dimension of Θ is dim W - 1.

Definition. A compact Riemnannian manifold M is said to be a Blaschke manifold at the point p in M if for every q in Cut(p) the link $\Lambda(p, q)$ is a great sphere of $U_q(M)$. The manifold M is said to be a Blaschke manifold if it is a Blaschke manifold at every point in M.

The CROSSes are examples of Blaschke manifolds and the exotic spheres and exotic quaternion projective planes are examples of pointed Blaschke manifolds (cf. Besse, [Be.A], p. 143).

Blaschke conjectured that any Blaschke manifold is isometric to a CROSS. For the case that the dimension of the manifold is two, L.W. Green, [G], proved that the conjecture is true : If M is a two-dimensional Blaschke manifold, then M is isometric to a standard sphere S^2 or a real projective plane RP^2. It is still remains open when the

dimension of the manifold ≥ 3.

According to Besse, [Be. A], p. 137, a pointed Balschke manifold is characterized by the spherical cut-locus:

Theorem 0.1. For a Riemannian manifold M and a point p in M, M is a Blaschke manifold at p if and only if Cut(p) is spherical.

Throughout this dissertaton, all the manifolds and geometric quantities are C^{∞} unless it is stated otherwise.

CHAPTER 1

SURFACES OF A EUCLIDEAN SPACE WITH HELICAL GEODESICS THROUGH A POINT

§1. Some fundamental concepts.

K. Sakamoto, [S-2], [S-6], K. Tsukada, [Ts], and others studied helical immersions into a Euclidean space or a unit sphere. Here, we recall the definition of the helical immersion. Let M be a connected Riemannian manifold and $x : M \rightarrow \tilde{M}$ an isometric immersion of M into a Riemannian manifold $\tilde{M}$. If the image $x \circ \gamma$ of each geodesic γ in M has constant Frenet curvatures which are independent of the choice of the geodesic γ, then x is called a <u>helical immersion</u>. K. Sakamoto classified submanifolds of a Euclidean space or a unit sphere according to the case of even order or odd order helical immersion.

On the other hand, B.-Y. Chen and P. Verheyen, [Ch.B-V-1], [Ch.B-V-2], introduced the notion of submanifolds with geodesic normal sections in a Euclidean space and they completely classified surfaces with geodesic normal sections in a Euclidean space E^m $(3 \leq m \leq 6)$. P. Verheyen, [V], showed that submanifolds with geodesic normal sections in a Euclidean space are equivalent to the helical immersion of the submanifold into a Euclidean space. We now recall the definition of a normal section. Let M be an n-dimensional submanifold of a Euclidean space E^m of dimension m. Let p be a point of M and t be a nonzero vector tangent to M at p. Let E(p ; t) be the affine space generated by t and normal space $T_p^{\perp}M$ at p. Then the dimension of E(p ; t) is m-n+1. The intersection of M and E(p ; t) gives rise to a curve on a neighborhood of p. Such a curve is called the

normal section of M at p in the direction of t. We say that the submanifold M has geodesic normal sections if every normal section is a geodesic.

If all the geodesics in a submanifold, regarded as curves in the ambient Euclidean space, are helices of the same Frenet curvatures, then the submanifold is characterized as an n-dimensional plane or a Blaschke manifold ([S-6]). For surfaces in a Euclidean space E^m ($3 \le m \le 6$) with such property, they are planes, standard spheres or Veronese surfaces which lie in E^5 ([Ch.B-V-2]). For later use we introduce the Veronese surface in E^5 : Let (x, y, z) be the standard coordinate system of E^3, 3-dimensional Euclidean space, and (u^1, u^2, u^3, u^4, u^5) be the standard coordinate system of E^5. We consider a mapping defined by

$$u^1 = \frac{1}{\sqrt{3}}yz, \quad u^2 = \frac{1}{\sqrt{3}}zx, \quad u^3 = \frac{1}{\sqrt{3}}xy,$$

$$u^4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u^5 = \frac{1}{6}(x^2 + y^2 - 2z^2).$$

This defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points (x,y,z) and (-x, -y, -z) of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$, and this mapping defines an imbedding of the real projective plane RP^2 into $S^4(1)$. This real projective plane imbedded in $S^4(1)$ is called the Veronese surface. It is a minimal surface of $S^4(1)$.

On the other hand, it is well-known that a helical immersion is λ-isotropic. We now recall the definition of isotropy. An isometric immersion $x : M \to E^m$ is said to be λ-isotropic at a point p if $\lambda = \| \sigma(X, X) \|$ does not depend upon the choice of the unit vector X tangent to M at p. If λ is also independent of the choice of point, then x is said to be constant isotropic. It is easily seen that M is λ-isotropic at p if and only if

$$\mathfrak{S}\, A_{\sigma(X, Y)} Z = \lambda^2 \mathfrak{S} \langle X, Y \rangle Z$$

for every X, Y, Z $\in$ $T_p(M)$, where $\mathcal{S}$ denotes the cyclic sum with respect to X, Y, Z. B. O'Neill, [On], proved that M is isotropic at p if and only if

$$(1.1) \qquad <\sigma(X, X), \ \sigma(X, Y)> = 0$$

for any two orthonormal vectors X and Y in $T_p(M)$.

Using O'Neill's idea, we can prove the following.

Lemma 1.1. Let T be a symmetric tensor of type (0, r) defined on E^m. Then $\| T(u^r) \|$ does not depend on the choice of the unit vector u if and only if

$$(1.2) \qquad <T(u^r), T(u^{r-1}, u^{\perp})> = 0$$

for any vector $u^{\perp}$ which is perpendicular to u, where $T(u^r) = T(u, u, u, \ldots, u)$ and $T(u^{r-1}, u^{\perp}) = T(u, u, \ldots, u, u^{\perp})$.

Proof. Since all the unit vectors in E^m form a unit sphere, a vector $u^{\perp}$ which is perpendicular to a unit vector u is tangent to the unit sphere. So,

$$<T(u^r), T(u^r)> = C(\text{constant}) \qquad \text{on the unit sphere}$$

if and only if

$$u^{\perp}<T(u^r), T(u^r)> = 0.$$

Since u is a position vector on the unit sphere,

$$<T(u^r), T(u^{r-1}, u^{\perp})> = 0 \qquad \text{(Q. E. D.)}$$

Throughout this dissertation, $t^{\perp}$ always means a unit vector perpendicular to t for some vector t unless it is stated otherwise.

Lemma 1.2. Let M be a submanifold in a Riemannian manifold $\tilde{M}$ such that M is isotropic at a point p in M. Then we have

$$\| \sigma(e_1, e_1) \|^2 = < \sigma(e_1, e_1), \ \sigma(e_2, e_2) > + \ 2 \| \sigma(e_1, e_2) \|^2 \tag{1.3}$$

for every pair of orthonormal vectors e_1 and e_2 tangent to M at p.

Proof. Let e_1 and e_2 be orthonormal vectors tangent to M at p. Set $X = \frac{e_1+e_2}{\sqrt{2}}$, $Y = \frac{e_1-e_2}{\sqrt{2}}$. Then X and Y are orthonormal. Since M is isotropic at p,

$$< \sigma(e_1, e_1), \ \sigma(e_1, e_1) \ > \ = < \ \sigma(X, X), \ \sigma(X, X) >.$$

Using (1.1), we obtain (1.3). (Q. E. D.)

§2. Surfaces in E^m with property ($*_1$)

From now on, we assume that M is a complete connected surface in $E^m (m \geq 3)$ with Riemannain connection ∇. We also denote the normal connection by D and the Weingarten map associated to a normal vector ξ by A_ξ and the second fundamental form by σ as usual.

We now define the property ($*_1$).

($*_1$) : There is a point p in M such that every geodesic through p, which is regarded as a curve in E^m, is a helix of the same constant Frenet curvatures.

Clearly, every helical immersion satisfies the property ($*_1$).

Suppose M has the property ($*_1$). Since every geodesic has the constant curvatures, $<\sigma(t, t), \sigma(t, t)>$ does not depend on the choice of the unit vector $t \in T_pM$. So, M is isotropic at p.

Lemma 1.3. Let M be a surface in a Euclidean space E^m ($m \geq 3$). Suppose that M satisfies the property ($*_1$). Then M is isotropic at p.

We now prove

Theorem 1.4. Let M be a complete connected surface in E^3. Then M satisfies the property ($*_1$) if and only if M is a standard sphere or a plane E^2.

Proof. Suppose that M satisfies ($*_1$). By Lemma 1.3 we see that M is isotropic at p. In this case, p is an umbilical point. Choose a geodesic γ through p. Suppose γ is of rank 1. It is clear that M is a plane E^2 in E^3. Suppose that γ is of rank 2. Since every geodesic is a circle of the same radius and the same center, M is a standard sphere. Suppose that γ is of rank 3. We assume that γ is parametrized by the arc length s. Let $\gamma'(s) = T$. Then $\gamma''(s) = \sigma(T, T)$ and $\gamma'''(s) = -A_{\sigma(T,T)}T + (\bar{\nabla}_T\sigma)(T, T)$ since γ is a geodesic. Since γ is of rank 3, $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) \neq 0$, and so $T \wedge \sigma(T, T) \wedge A_{\sigma(T,T)}T \neq 0$ along γ. It follows that $T \wedge A_{\sigma(T,T)}T \neq 0$. Since M is isotropic at p, $<\sigma(t, t), \sigma(t, t^{\perp})> = 0$, where $t = T(0)$ and $\gamma(0) = p$. Accordingly, $A_{\sigma(t,t)}t \perp t^{\perp}$, i.e., $A_{\sigma(t,t)}t \wedge t = 0$. So, this case cannot occur. The converse is clear. (Q. E. D.)

We now assume that a surface M which lies in E^m ($m \geq 4$) is compact and connected.

Suppose that M satisfies the property ($*_1$). By Lemma 1.3 M is isortropic at p. The equation (1.3) implies that only two cases may occur :

Case 1) $\sigma(e_1, e_2) \equiv 0$ for any orthonormal vectors e_1 and e_2.

In this case, $\dim(\mathrm{Im}\,\sigma)_p = 1$ since M is compact, where $(\mathrm{Im}\,\sigma)_p = \{\sigma(X, Y) \mid X, Y \in T_pM\}$ is called the <u>first normal space at the point p</u>.

Case 2) $\sigma(e_1, e_2) \neq 0$ for an orthonormal basis $\{e_1, e_2\}$ of T_pM. In this case, dim $(\mathrm{Im}\,\sigma)_p \geq 2$.

Lemma 1.5. Let M be a compact connected surface in E^4 satisfying the property ($*_1$). If the dimension of the first normal space at p is one, then M is a standard sphere lying in E^3.

Proof. Suppose dim $(\mathrm{Im}\,\sigma)_p = 1$, i.e., $\sigma(e_1, e_2) = 0$ for any orthonormal basis $\{e_1, e_2\}$ of T_pM. So, p is an umbilical point of M. Choose a geodesic γ through p such that $\gamma(0) = p$ and $\gamma'(s) = T$. Then we have

(1.4) $$\gamma''(s) = \sigma(T, T),$$

(1.5) $$\gamma'''(s) = -A_{\sigma(T,T)}T + (\bar{\nabla}_T\sigma)(T, T),$$

(1.6) $$\gamma^{(4)}(s) = -(\bar{\nabla}_T A)_{\sigma(T,T)}T - \sigma(T, A_{\sigma(T,T)}T)$$

$$-2A_{(\bar{\nabla}_T\sigma)(T,\,T)}T + D_T((\bar{\nabla}_T\sigma)(T, T)).$$

where $(\bar{\nabla}_T A)_\eta Y = \nabla_X(A_\eta Y) - A_{D_X \eta} Y - A_\eta \nabla_X Y$ for $X, Y \in TM$ and $\eta \in T^\perp M$.

We are going to show that γ is of rank 2. It is enough to show that $(\bar{\nabla}_t\sigma)(t, t) = 0$, where $T(0) = t$.

Suppose that $(\bar{\nabla}_t\sigma)(t, t) \neq 0$. We may put

$$\sigma(T, T) = \kappa_1 \xi,$$

where κ_1 is the first Frenet curvature of γ and ξ a unit normal vector field along γ. As a matter of fact ξ is in the direction of the mean curvature vector H at p. Since κ_1 is constant,

$$< (\bar{\nabla}_T\sigma)(T, T), \sigma(T, T) > = 0 \qquad \text{along } \gamma$$

and since $\sigma(t, t^\perp) = 0$, we get

$$A_{(\bar{\nabla}_t\sigma)(t, t)} t = 0.$$

On the other hand, $\kappa_1 = < \sigma(T, T), \ \sigma(T,T) > = < A_{\sigma(T,T)} T, T >$. Covariant differentiation of this equation along the geodesic γ leads to

$$< (\bar{\nabla}_T A)_{\sigma(T,T)} T, T > = 0$$

because γ is a geodesic. Evaluate this at p and then we have

$$<(\bar{\nabla}_t A)_\xi t, t> = 0$$

since $\kappa_1 \neq 0$. Since this holds for any direction, linearization and the Codazzi equation (0.11) give

$$(\bar{\nabla} A)_\xi = 0.$$

So, we can obtain the following :

$$\gamma'(0) = t,$$

$$\gamma''(0) = \sigma(t, t),$$

$$\gamma'''(0) = -A_{\sigma(t,\, t)}t + (\bar{\nabla}_t \sigma)(t, t),$$

$$\gamma^{(4)}(0) = -\sigma(A_{\sigma(t,\, t)}t, t) + D_t((\bar{\nabla}_T \sigma)(T, T)).$$

Since the curvatures of γ are constant, $<\gamma''(0), \gamma'''(0)> = 0$ and $<\gamma'''(0), \gamma^{(4)}(0)> = 0$. Since $\gamma^{(4)}(0)$ is a normal vector to M, $\gamma''(0) \wedge \gamma^{(4)}(0) = 0$. Thus, γ is of rank 3. Since M is compact, this is impossible. Therefore, we have

$$(\bar{\nabla}_t \sigma)(t, t) = 0.$$

Since the curvatures are constant, $(\bar{\nabla}_T \sigma)(T, T) = 0$ along γ. So, γ is of rank 2. Thus every geodesic through p is of rank 2. Moreover, every geodesic through p is a circle

of radius $\frac{1}{\kappa_1}$ and centered at $p - \frac{1}{\kappa_1}\xi$ and so M is a standard sphere $S^2(\frac{1}{\kappa_1})$. (Q. E. D.)

Suppose dim $(\text{Im } \sigma)_p = 2$. Then there is an orthonormal basis $\{e_1, e_2\}$ of T_pM such that $\sigma(e_1, e_2) \neq 0$.

Lemma 1.6. Let M be a surface in E^m $(m \geq 4)$ such that M is isotropic at p, where dim $(\text{Im } \sigma)_p = 2$. Then $\| \sigma(e_1, e_2) \|$ does not depend on the choice of the orthonormal basis $\{e_1, e_2\}$ of T_pM.

Proof. Let $\{X, Y\}$ be an orthonormal basis of T_pM. Then there exists θ $(0 \leq \theta < 2\pi)$ such that

$$X = \cos\theta\, e_1 - \sin\theta\, e_2,$$
$$Y = \sin\theta\, e_1 + \cos\theta\, e_2,$$

for the orthormal basis $\{e_1, e_2\}$ of T_pM.

Since M is isotropic at p, $\sigma(e_1, e_1) \perp \sigma(e_1, e_2)$ and $\sigma(e_2, e_2) \perp \sigma(e_1, e_2)$. So, $\sigma(e_1, e_1) \wedge \sigma(e_2, e_2) = 0$ because dim $(\text{Im } \sigma)_p = 2$. Since $\| \sigma(e_1, e_1) \| = \| \sigma(e_2, e_2) \|$, $\sigma(e_1, e_1) = \pm\, \sigma(e_2, e_2)$. If we observe (1.3), then we obtain

$$\sigma(e_1, e_1) + \sigma(e_2, e_2) = 0, \tag{1.7}$$

that is, the mean curvature vector H vanishes at p. Therefore, we get

$$\sigma(X, Y) = \cos 2\theta\, \sigma(e_1, e_2) + \sin 2\theta\, \sigma(e_1, e_1).$$

If we compute the length of $\sigma(X, Y)$ and make use of (1.3), then we see that

$$\| \sigma(X, Y) \| = \| \sigma(e_1, e_2) \| \qquad \text{(Q. E. D.)}$$

In this case, we are also going to prove that every geodesic through p is of rank 2. Suppose $(\bar{\nabla}_t \sigma)(t, t) \neq 0$ for $t \in U_pM$.

Let $\{e_1, e_2\}$ be an orthonormal basis for T_pM. Consider a geodesic γ_1 such that $\gamma_1(0) = p$, $\gamma_1'(0) = \frac{e_1+e_2}{\sqrt{2}}$. Since γ_1 has its constant first Frenet curvature, we have

$$<(\bar{\nabla}_{T_1}\sigma)(T_1, T_1), \sigma(T_1, T_1)> = 0,$$

where $T_1 = \frac{d\gamma_1}{ds}$. Since the mean curvature vector is zero at p, we see that

$$\text{(1.8)} \qquad < (\bar{\nabla}_{e_1}\sigma)(e_1, e_1), \sigma(e_1, e_1) > + 3 < (\bar{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2) >$$

$$+ 3 < (\bar{\nabla}_{e_1}\sigma)(e_2, e_2), \sigma(e_1, e_2) > + < (\bar{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2) > = 0.$$

Consider another geodesic γ_2 such that $\gamma_2(0) = p$, $\gamma_2'(0) = \frac{e_1-e_2}{\sqrt{2}}$. Then we get

$$< (\bar{\nabla}_{T_2}\sigma)(T_2, T_2), \sigma(T_2, T_2) > = 0,$$

where $\gamma_2'(s) = T_2$. This implies

$$\text{(1.9)} \qquad - < (\bar{\nabla}_{e_1}\sigma)(e_1, e_1), \sigma(e_1, e_1) > + 3 < (\bar{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2) >$$

$$- 3 < (\bar{\nabla}_{e_1}\sigma)(e_2, e_2), \sigma(e_1, e_2) > + < (\bar{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2) > = 0.$$

Putting (1.8) and (1.9) together, we obtain

$$(1.10) \qquad 3 < (\bar{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2) > + < (\bar{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2) > = 0.$$

On the other hand, since geodesics through p have the same constant curvatures, $<(\bar{\nabla}\sigma)(X, X, X), (\bar{\nabla}\sigma)(X, X, X) >$ is independent of the choice of the unit vector X. By Lemma 1.2, we have

$$< (\bar{\nabla}\sigma)(X, X, X), (\bar{\nabla}\sigma)(X, X, X^{\perp}) > = 0$$

for every unit vector X tangent to M at p. So, $(\bar{\nabla}\sigma)(e_1, e_1, e_1) \perp (\bar{\nabla}\sigma)(e_1, e_1, e_2)$. Since $(\bar{\nabla}\sigma)(e_1, e_1, e_1) \perp \sigma(e_1, e_1)$, we get

$$(\bar{\nabla}\sigma)(e_1, e_1, e_2) \perp \sigma(e_1, e_2).$$

Therefore, (1.10) implies that

$$<(\bar{\nabla}_{e_2}\sigma)(e_2, e_2), \sigma(e_1, e_2)> = 0.$$

Since $(\bar{\nabla}_{e_2}\sigma)(e_2, e_2) \wedge \sigma(e_1, e_2) = 0$, we obtain

$$(\bar{\nabla}_{e_2}\sigma)(e_2, e_2) = 0.$$

But, this contradicts $(\bar{\nabla}_t\sigma)(t, t) \neq 0$ for $t \in U_pM$. Thus, it follows that $(\bar{\nabla}_t\sigma)(t, t) = 0$ for every $t \in U_pM$, i.e., every geodesic through p is of rank 2. By using the fundamental theorem of curves, we can write the immersion $x : M \to E^4$ with respect to the geodesic polar coordinate system (s, θ) as

(1.11) $$x(s, \theta) = C(\theta) + \frac{1}{\kappa}(\cos \kappa s)\mathbf{f}_1(\theta) + \frac{1}{\kappa}(\sin \kappa s)\mathbf{f}_2(\theta),$$

where $C(\theta)$ is a vector function depending upon θ, $\mathbf{f}_1(\theta)$ and $\mathbf{f}_2(\theta)$ are orthonormal vectors in E^4 at p depending on θ and κ is the Frenet curvature of each geodesic through p.

Without loss of generality we may assume the point p is the origin o of E^4. Then we have

(1.12) $$0 = x(0, \theta) = C(\theta) + \frac{1}{\kappa}\mathbf{f}_1(\theta) \qquad \text{for all } \theta.$$

Let e_1 and e_2 be orthonormal vectors tangent to M at o which generate the geodesic polar coordinates (s, θ).

Since $x_*(\partial/\partial s)(0, \theta) = \mathbf{f}_2(\theta) \in T_oM$,

(1.13) $$\mathbf{f}_2(\theta) = \cos\theta\, e_1 + \sin \theta\, e_2.$$

Since $(\tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s))(0, \theta) = \frac{\partial^2 x}{\partial s^2}(0,\theta) = \sigma(\mathbf{f}_2(\theta), \mathbf{f}_2(\theta))$,

(1.14) $$\sigma(\mathbf{f}_2(\theta), \mathbf{f}_2(\theta)) = -\frac{1}{\kappa}\mathbf{f}_1(\theta),$$

where $\tilde{\nabla}$ is the Riemannian connection in E^4.

Combining (1.12), (1.13) and (1.14), we obtain

$$x(s, \theta) = \frac{1}{\kappa}(\sin \kappa s)\mathbf{f}_2(\theta) + \frac{1}{\kappa^2}(1 - \cos \kappa s)\, \sigma(\mathbf{f}_2(\theta), \mathbf{f}_2(\theta))$$

$$= \frac{1}{\kappa} \sin \kappa s \cos \theta \, e_1 + \frac{1}{\kappa} \sin \kappa s \sin \theta \, e_2$$

$$+ \frac{1}{\kappa^2}(1 - \cos \kappa s)\cos 2\theta \, \sigma(e_1, e_1) + \frac{1}{\kappa^2}(1 - \cos \kappa s)\sin 2\theta \, \sigma(e_1, e_2)$$

since $\sigma(e_1, e_1) + \sigma(e_2, e_2) = 0$.

Since $\sigma(e_1, e_1) \perp \sigma(e_1, e_2)$, choose e_3 as $\dfrac{\sigma(e_1,e_1)}{\|\sigma(e_1,e_1)\|} = \dfrac{\sigma(e_1,e_1)}{\kappa}$ and e_4 as $\dfrac{\sigma(e_1,e_2)}{\|\sigma(e_1,e_2)\|} = \dfrac{\sigma(e_1,e_2)}{\kappa}$.

If we use the coordinate system with respect to e_1, e_2, e_3 and e_4, then $x(s, \theta)$ becomes

$$x(s, \theta) = \left(\frac{1}{\kappa} \sin \kappa s \cos \theta,\ \frac{1}{\kappa} \sin \kappa s \sin \theta,\ \frac{1}{\kappa}(1 - \cos \kappa s)\cos 2\theta,\ \frac{1}{\kappa}(1 - \cos \kappa s)\sin 2\theta\right). \tag{1.15}$$

We now prove

Lemma 1.7. Let M be a compact connected surface in E^4 . Suppose that M satisfies $(*_1)$ and that dim $(\mathrm{Im}\, \sigma)_p$ is maximal. Then M is a Blaschke surface at p and M is diffeomorphic to a real projective space RP^2 but not isometric to RP^2 with the standard metric.

Proof. In order to apply Theorem 0.1, it is enough to show that the cut-locus Cut(p) of the point p is spherical.

We may assume that p is the origin o of E^4. Since each geodesic through o is a circle of radius $\frac{1}{\kappa}$, we have to show that two distinct geodesics through o do not intersect on the open interval $(0, \frac{\pi}{\kappa})$. Suppose $x(s, \theta) = x(s_o, \theta_o)$ for $0< s, s_o < \frac{\pi}{\kappa}$ and $0 < |\theta - \theta_o| < \frac{\pi}{2}$. By using (1.15), we can easily derive a contradiction. Thus Cut(o) is spherical. Cut(o) is indeed the set of all antipodal points of o with respect to each geodesic through o.

According to Bott-Samelson, [Bo], [Sa],(or 7.33 Theorem of Besse, [Be.A]), we see that M is diffeomorphic to RP^2. We now prove that M is not isometric to RP^2. It is sufficient to show that the Gaussian curvature **K** cannot be constant.

Suppose that the Gaussian curvature **K** is a constant > 0. Then we can easily get $G = \frac{1}{\mathbf{K}} \sin^2(\sqrt{\mathbf{K}}\, s)$, where $G = \langle x_*(\partial/\partial\theta), x_*(\partial/\partial\theta) \rangle$. On the other hand, G can be directly computed from (1.15) as

$$G = \frac{1}{\kappa^2} \left\{ \sin^2\kappa s + 4(1 - \cos \kappa s)^2 \right\} .$$

If we compare these two equations, we have a contradiction. So, even though the surface is diffeomorphic to RP^2, it is not a standard real projective space RP^2. (Q. E. D.)

Conversely, if a compact connected surface M is a standard sphere S^2 or has the form of (1.15), then it is easily proved that M satisfies the property $(*_1)$.

Thus by combining Lemma 1.5, Lemma 1.7 and the statement above we can conclude the following

Theorem 1.8 (Classification). Let M be a compact connected surafce in E^4. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere which lies in E^3 or a Blaschke surface at a point which lies in E^4 of the form (1.15). In the second case M is

diffeomorphic to RP^2.

Remark. In the case of the Blaschke surface at the point o in the above theorem, we see that the locus of the centers of geodesics through the point o is a circle with radius κ whose points go around the circumference twice while points on a geodesic circle centered at o on M go around the geodesic circle once. On the other hand, if we compute the torsion of the cut-locus $\mathrm{Cut}(o) = x(\pi, \theta)$ of the point o, then we see that the torsion is zero and Cut(o) is indeed a circle.

Let us consider a compact connected surface M in E^5. We shall characterize surfaces in E^5 satisfying the property $(*_1)$.

We now suppose that M satisfies the property $(*_1)$. As usual, we may assume the base point p of $(*_1)$ as the origin o of E^5. Then the immersion $x : M \to E^5$ can be expressed in terms of the geodesic polar coordinates (s, θ) as

$$x(s, \theta) = r_1(\cos \beta s - 1)\mathbf{f}_1(\theta) + r_1 \sin \beta s\, \mathbf{f}_2(\theta) + r_2(\cos \delta s - 1)\mathbf{f}_3(\theta) + r_2 \sin \delta s\, \mathbf{f}_4(\theta), \tag{1.16}$$

where r_1 and r_2 are nonnegative numbers, β and δ are some positive constants and $\mathbf{f}_1(\theta)$, $\mathbf{f}_2(\theta)$, $\mathbf{f}_3(\theta)$ and $\mathbf{f}_4(\theta)$ are orthonormal vectors in T_oE^5 depending on θ.

Without loss of generality, we may assume

$$\mathbf{f}_1(0) = (1, 0, 0, 0, 0),\ \ \mathbf{f}_2(0) = (0, 1, 0, 0, 0),\ \ \mathbf{f}_3(0) = (0, 0, 1, 0, 0), \quad \mathbf{f}_4(0) = (0, 0, 0, 1, 0). \tag{1.17}$$

Let $\mathbf{f}_5(\theta)$ be a unit vector in T_oE^5 such that $\{\mathbf{f}_1(\theta), \mathbf{f}_2(\theta), \mathbf{f}_3(\theta), \mathbf{f}_4(\theta), \mathbf{f}_5(\theta)\}$ forms an orthonormal basis for T_oE^5. Automatically we may set

(1.18) $$\mathbf{f}_5(0) = (0, 0, 0, 0, 1).$$

Differentiating (1.16) with respect to s for a fixed θ, we obtain

(1.19) $$x_*(\partial/\partial s) = -r_1\beta \sin \beta s\, \mathbf{f}_1(\theta) + r_1\beta \cos \beta s\, \mathbf{f}_2(\theta) - r_2\delta \sin \delta s\, \mathbf{f}_3(\theta)$$
$$+ r_2\delta \cos \delta s\, \mathbf{f}_4(\theta).$$

Set

(1.20) $$e(\theta) = x_*(\partial/\partial s)(0, \theta) = r_1\beta\, \mathbf{f}_2(\theta) + r_2\delta\, \mathbf{f}_4(\theta),$$

which implies

(1.21) $$(r_1\beta)^2 + (r_2\delta)^2 = 1.$$

Let

$$e_1 = e(0) = r_1\beta\, \mathbf{f}_2(0) + r_2\delta\, \mathbf{f}_4(0) = (0, r_1\beta, 0, r_2\delta, 0)$$

and

$$e_2 = e(\tfrac{\pi}{2}) = r_1\beta\, \mathbf{f}_2(\tfrac{\pi}{2}) + r_2\delta\, \mathbf{f}_4(\tfrac{\pi}{2}).$$

Then $e(\theta)$ can be expressed as

(1.22) $$e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2.$$

For a fixed θ, $x(s, \theta)$ is a geodesic and thus $< \frac{\partial^2 x}{\partial s^2}, \frac{\partial x}{\partial \theta} > = 0$, which gives

(1.23)
$$\begin{aligned}
& r_1^2\beta < \mathbf{f}_1'(\theta), \mathbf{f}_2(\theta) > + r_2^2\,\delta < \mathbf{f}_3'(\theta), \mathbf{f}_4(\theta) > \\
& + (-r_1^2\beta < \mathbf{f}_1'(\theta), \mathbf{f}_2(\theta) > + r_1r_2\beta \; < \mathbf{f}_2'(\theta), \mathbf{f}_3(\theta) >) \cos \beta s \\
& - (-r_2^2\delta < \mathbf{f}_3'(\theta), \mathbf{f}_4(\theta) > + r_1r_2\delta \; < \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta) >) \cos \delta s \\
& - r_1r_2\beta < \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta) > \sin \beta s + r_1r_2\delta < \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta) > \sin \delta s \\
& - \frac{r_1r_2}{2} (\beta - \delta)(< \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta) > + < \mathbf{f}_2'(\theta), \mathbf{f}_3(\theta) >) \cos (\beta + \delta)s \\
& + \frac{r_1r_2}{2} (\beta + \delta)(< \mathbf{f}_1'(\theta), \mathbf{f}_4(\theta) > - < \mathbf{f}_2'(\theta), \mathbf{f}_3(\theta) >) \cos (\beta - \delta)s \\
& + \frac{r_1r_2}{2} (\beta - \delta)(< \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta) > - < \mathbf{f}_2'(\theta), \mathbf{f}_4(\theta) >) \sin(\beta + \delta)s \\
& + \frac{r_1r_2}{2} (\beta + \delta)(< \mathbf{f}_1'(\theta), \mathbf{f}_3(\theta) > + < \mathbf{f}_2'(\theta), \mathbf{f}_4(\theta) >) \sin(\beta - \delta)s \\
& = 0.
\end{aligned}$$

Lemma 1.9. $\beta \neq \delta$ provided the geodesics through o are of rank 4.

Proof. Suppose $\beta = \delta$. Let κ_1, κ_2 and κ_3 be the first, second and third Frenet curvatures of the geodesic $x(s, \theta)$ for a fixed θ respectively. Then

$$\kappa_1^2 = < \frac{\partial^2 x}{\partial s^2}(0, \theta), \frac{\partial^2 x}{\partial s^2}(0, \theta) > = (r_1^2 + r_2^2)\beta^4.$$

Since $< \frac{\partial x}{\partial s}(0, \theta), \frac{\partial x}{\partial s}(0, \theta) > = 1$, we obtain

$$(r_1^2 + r_2^2)\beta^2 = 1,$$

that is, $\beta^2 = \frac{1}{r_1^2 + r_2^2}$. Therefore, $\kappa_1 = \beta$.

On the other hand, the curvatures κ_i's and the frequencies β and δ have the following relations :

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \beta^2 + \delta^2 = 2\beta^2,$$

$$\kappa_1^2 \kappa_3^2 = \beta^2 \delta^2 = \beta^4 .$$

Since $\kappa_1 = \beta$, $\kappa_3^2 = \beta^2$. The first equation gives $\kappa_2 = 0$. This contradicts the fact that $x(s, \theta)$ is of rank 4. Thus, we have $\beta \neq \delta$. (Q. E. D.)

Lemma 1.10. For every θ, the geodesic $x(s, \theta)$ is periodic.

Proof. If $x(s, \theta)$ is of rank 2, then this is obvious. We now assume that $x(s, \theta)$

is of rank 4. Suppose $x(s, \theta)$ is not periodic. Then β and δ are independent over the rational numbers, that is, $\overline{x(s, \theta)} = S^1(r_1) \times S^1(r_2)$, a torus denoted by T, where $\overline{x(s, \theta)}$ is the closure of $x(s, \theta)$ in E^5. Certainly, T is contained in x(M). But T does not satisfies the property $(*_1)$. Thus $x(s, \theta)$ must be periodic for every θ. (Q. E. D.)

We now suppose that $r_1 \neq 0$ and $r_2 \neq 0$, that is, every geodesic through o is of rank 4. Combining (1.23), Lemma 1.9 and Lemma 1.10, we obtain

$$\text{(1.24)} \qquad \langle f_1'(\theta), f_2(\theta)\rangle = \langle f_1'(\theta), f_3(\theta)\rangle = \langle f_1'(\theta), f_4(\theta)\rangle$$

$$= \langle f_2'(\theta), f_3(\theta)\rangle = \langle f_2'(\theta), f_4(\theta)\rangle = \langle f_3'(\theta), f_4(\theta)\rangle = 0$$

for all θ. So we have the following system of differential equations :

$$\text{(1.25)} \qquad f_i'(\theta) = \lambda_i(\theta) f_5(\theta), \qquad f_5'(\theta) = -\sum_{i=1}^{4} \lambda_i(\theta)\, f_i(\theta)$$

for $i = 1, 2, 3$ and 4, in other words,

$$\begin{pmatrix} f_1'(\theta) \\ f_2'(\theta) \\ f_3'(\theta) \\ f_4'(\theta) \\ f_5'(\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda_1(\theta) \\ 0 & 0 & 0 & 0 & \lambda_2(\theta) \\ 0 & 0 & 0 & 0 & \lambda_3(\theta) \\ 0 & 0 & 0 & 0 & \lambda_4(\theta) \\ -\lambda_1(\theta) & -\lambda_2(\theta) & -\lambda_3(\theta) & -\lambda_4(\theta) & 0 \end{pmatrix} \begin{pmatrix} f_1(\theta) \\ f_2(\theta) \\ f_3(\theta) \\ f_4(\theta) \\ f_5(\theta) \end{pmatrix},$$

where the λ_i's are periodic functions with period 2π.

Differentiating (1.20) with respect to θ and making use of (1.22) and (1.25), we get

$$(r_1\beta\,\lambda_2(\theta) + r_2\delta\,\lambda_4(\theta))\mathbf{f}_5(\theta) = -\sin\theta\, e_1 + \cos\theta\, e_2, \tag{1.26}$$

from which, we obtain that $r_1\beta\,\lambda_2(\theta) + r_2\delta\,\lambda_4(\theta) = \pm 1$ and we may assume that $r_1\beta\,\lambda_2(\theta) + r_2\delta\,\lambda_4(\theta) = 1$. If we differentiate (1.26) twice and use (1.25), then we obtain

$$\mathbf{f}_5''(\theta) + \mathbf{f}_5(\theta) = 0$$

Since $\mathbf{f}_5(0) = (0, 0, 0, 0, 1)$ and $\mathbf{f}_5(\frac{\pi}{2}) = -e_1 = (0, -r_1\beta, 0, -r_2\delta, 0)$, we have

$$\mathbf{f}_5(\theta) = (0, -r_1\beta\sin\theta, 0, -r_2\delta\sin\theta, \cos\theta). \tag{1.27}$$

Since $\mathbf{f}_i'(\theta) = \lambda_i(\theta)\mathbf{f}_5(\theta)$ $(1 \le i \le 4)$, $\mathbf{f}_1(0) = (1, 0, 0, 0, 0)$, $\mathbf{f}_2(0) = (0, 1, 0, 0, 0)$, $\mathbf{f}_3(0) = (0, 0, 1, 0, 0)$ and $\mathbf{f}_4(0) = (0, 0, 0, 1, 0)$, we get

$$\mathbf{f}_1(\theta) = (1, -r_1\beta\int_0^\theta \lambda_1(t)\sin t\,dt, 0, -r_2\delta\int_0^\theta \lambda_1(t)\sin t\,dt, \int_0^\theta \lambda_1(t)\cos t\,dt), \tag{1.28}$$

$$\mathbf{f}_2(\theta) = (0, -r_1\beta\int_0^\theta \lambda_2(t)\sin t\,dt + 1, 0, -r_2\delta\int_0^\theta \lambda_2(t)\sin t\,dt, \int_0^\theta \lambda_2(t)\cos t\,dt), \tag{1.29}$$

$$\mathbf{f}_3(\theta) = (0, -r_1\beta\int_0^\theta \lambda_3(t)\sin t\,dt, 1, -r_2\delta\int_0^\theta \lambda_3(t)\sin t\,dt, \int_0^\theta \lambda_3(t)\cos t\,dt), \tag{1.30}$$

$$\mathbf{f}_4(\theta) = (0, -r_1\beta\int_0^\theta \lambda_4(t)\sin t\,dt, 0, -r_2\delta\int_0^\theta \lambda_4(t)\sin t\,dt + 1, \int_0^\theta \lambda_4(t)\cos t\,dt). \tag{1.31}$$

Now let us compute $\lambda_1(\theta)$. Since $< \mathbf{f}_1(\theta), \mathbf{f}_5(\theta) > = 0$ for all θ, (1.27) and (1.28) imply

$$(r_1\beta)^2 \sin\theta \int_0^\theta \lambda_1(t) \sin t\, dt + (r_2\delta)^2 \sin\theta \int_0^\theta \lambda_1(t) \sin t\, dt + \cos\theta \int_0^\theta \lambda_1(t) \cos t\, dt = 0.$$

It follows that

$$\sin\theta \int_0^\theta \lambda_1(t) \sin t\, dt + \cos\theta \int_0^\theta \lambda_1(t) \cos t\, dt = 0$$

because $(r_1\beta)^2 + (r_2\,\delta)^2 = 1$. By differentiating this, we obtain

$$\lambda_1(\theta) = -\cos\theta \int_0^\theta \lambda_1(t) \sin t\, dt + \sin\theta \int_0^\theta \lambda_1(t) \cos t\, dt,$$

which gives $\lambda_1'(\theta) = 0$ for all θ, that is λ_1 is a constant. In fact, $\lambda_1(\theta) = 0$ for all θ. Thus, $\mathbf{f}_1(\theta)$ is completely determined :

$$\mathbf{f}_1(\theta) = (1, 0, 0, 0, 0).$$

Similarly, we can compute λ_2, λ_3 and λ_4 :

$$\lambda_2(\theta) = r_1\beta, \qquad \lambda_3(\theta) = 0, \qquad \lambda_4(\theta) = r_2\delta \qquad \text{for all } \theta.$$

Consequently (1.28) ~(1.31) are precisely determined as follows :

$$\mathbf{f}_1(\theta) = (1, 0, 0, 0, 0),$$

$$\mathbf{f}_2(\theta) = (0, 1 - (r_1\beta)^2(1 - \cos\theta), 0, -r_1r_2\,\beta\delta(1 - \cos\theta), r_1\beta \sin\theta),$$

$$\mathbf{f}_3(\theta) = (0, 0, 1, 0, 0),$$

$$\mathbf{f}_4(\theta) = (1, -r_1r_2\,\beta\delta(1 - \cos\theta), 0, 1 - (r_2\delta)^2(1 - \cos\theta), r_2\delta \sin\theta).$$

These, together with (1.27), show that the immersion x the representation

(1.32) $$x(s, \theta) = (r_1 (\cos \beta s - 1), r_1 \sin \beta s - r_1\beta\,(1 - \cos\theta)(r_1^2\beta \sin \beta s + r_2^2\,\delta \sin \delta s),$$

$$r_2 (\cos \delta s - 1), r_2 \sin \delta s - r_2\delta\,(1 - \cos\theta)(r_2^2\beta \sin \beta s + r_2^2\delta \sin \delta s),$$

$$(r_1^2\beta \sin \beta s + r_2^2\delta \sin \delta s) \sin\theta).$$

In this case, each geodesic through o is periodic with period $L = \frac{2\pi p}{\beta} = \frac{2\pi q}{\delta}$ for some integers p and q. Using a similar argument to that in Lemma 1.7, we see that the cut-locus, Cut(o), of the point o is spherical and thus the surface M is a Blaschke manifold at o which is differemorphic to RP^2. Thus, we have

Proposition 1.11. Let M be a compact connected surface in E^5 with the property $(*_1)$ relative to the origin o. If every geodesic through the point o is of rank 4, then M is a Blaschke manifold at o which is diffeomorphic to RP^2 and has the form (1.32).

We now suppose that $x(s, \theta)$ is of rank 2 for every θ. Then the immersion x can

be written with respect to the geodesic polar coordinate (s, θ) as

$$(1.33) \qquad x(s, \theta) = \frac{1}{\kappa}(\cos \kappa s - 1)\, \mathbf{f}_1(\theta) + \frac{1}{\kappa} \sin \kappa s\, \mathbf{f}_2(\theta),$$

where κ is the Frenet curvature of the planar geodesic $x(s, \theta)$ for every θ and $\mathbf{f}_1(\theta)$ and $\mathbf{f}_2(\theta)$ are orthonormal vectors in E^5 at the point o. From (1.33) we obtain

$$(1.34) \qquad x_*(\partial/\partial s)(0, \theta) = \mathbf{f}_2(\theta) \in T_oM$$

and

$$(1.35) \qquad (\tilde{\nabla}_{x_*(\partial/\partial s)} x_*(\partial/\partial s))(0, \theta) = \frac{\partial^2 x}{\partial s^2}(0,\theta) = \sigma(\mathbf{f}_2(\theta), \mathbf{f}_2(\theta)) = -\kappa\, \mathbf{f}_1(\theta),$$

where $\tilde{\nabla}$ is the Riemannian connection in E^5.

Let $\{e_1, e_2\}$ be an orthonormal basis of T_oM such that

$$(1.36) \qquad \mathbf{f}_2(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2 .$$

Suppose that $\dim (\mathrm{Im}\, \sigma)_o = 1$, that is, the point o is an umbilical point of M . In this case, every geodesic through o is a circle of radius $\frac{1}{\kappa}$ and centered at $-\frac{1}{\kappa}H$. Thus, M is a standard sphere $S^2(\frac{1}{\kappa})$ which lies in E^3.

Suppose that $\dim (\mathrm{Im}\, \sigma)_o = 2$. In this case, exactly the same proof used to derive (1.15) is applied and thus the immersion x is of the form

$$(1.37) \quad x(s, \theta) = \frac{1}{\kappa}(\sin \kappa s \cos\theta, \sin \kappa s \sin\theta, (1 - \cos \kappa s)\cos 2\theta, (1 - \cos \kappa s)\sin 2\theta, 0)$$

for a suitable choice of Euclidean coordinates in E^5. Clearly, M lies in E^4.

We now assume that dim $(\mathrm{Im}\ \sigma)_o = 3$, that is, the dimension of the first normal space at the point o is maximal. Then

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) \neq 0.$$

Let

$$(1.38) \qquad e_3 = \frac{\sigma(e_1, e_1)}{\kappa}, \qquad e_4 = \frac{\sigma(e_1, e_2)}{a} \quad \text{and} \quad e_5 = \frac{\tilde{e}_5}{\|\tilde{e}_5\|},$$

where $a = \| \sigma(e_1, e_2) \|$ and $\tilde{e}_5 = \sigma(e_2, e_2) - \dfrac{g(\sigma(e_1, e_1), \sigma(e_2, e_2))}{\kappa^2}\sigma(e_1, e_1)$. Set $b = \| \tilde{e}_5 \|$. Then we have from Lemma 1.2 that

$$(1.39) \qquad b^2 + \frac{(\kappa^2 - 2a^2)^2}{\kappa^2} = \kappa^2.$$

Using (1.34), (1.35), (1.38) and (1.39), we can write the immersion x in the form

$$(1.40) \quad x(s, \theta) = \Big(\frac{1}{\kappa} \sin \kappa s \cos \theta,\ \frac{1}{\kappa} \sin \kappa s \sin \theta,\ \frac{1}{\kappa^2}(1 - \cos \kappa s)\,(\kappa - \frac{2a^2}{\kappa} \sin^2 \theta),$$

$$\frac{a}{\kappa^2}(1 - \cos \kappa s) \sin 2\theta,\ \frac{b}{\kappa^2}(1 - \cos \kappa s) \sin^2 \theta\Big)$$

for a suitable choice of the coordinates with respect to e_1, e_2, e_3, e_4 and e_5 described above.

Considering the cut-locus Cut(o) of the point o in both the cases that $\dim(\mathrm{Im}\,\sigma)_o =$ 2 and $\dim(\mathrm{Im}\,\sigma)_o = 3$, we see that M is a Blaschke surface at o.

Conversely, if the immersion x has the form (1.32), (1.37) and (1.40) or x is a standard imbedding of $S^2(\frac{1}{\kappa})$ into E^3, then it is easily checked that the surface M satisfies the property $(*_1)$.

Thus we can classify surfaces in E^5 satisfying the property $(*_1)$.

Theorem 1.12.(Classification). Let M be a compact connected surface in E^5. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere in E^3 or a Blaschke surface at a point of the form (1.37) which lies in E^4 or a Blaschke surface at a point of the form (1.40) which lies in E^5 or a Blaschke surface at a point of the form (1.32) which lies in E^5. All such pointed Blaschke surfaces are diffeomorphic to RP^2.

Remark. Let M be a compact connected surface in E^m $(m \geq 5)$. Since the dimension of the first normal space at a point is at most 3, we can conclude that M satisfies the property $(*_1)$ and the geodesics are planar if and only if M lies in E^5 and M is one of four model spaces stated in Theorem 1.12 except the case of a Blaschke surface of the form (1.32).

§3. A new characterization of the Veronese surface

The Veronese surface introduced at the beginning of this chapter certainly satisfies the property $(*_1)$. So the following question naturally arises : What is the characterization of the Veronese surface in terms of the property $(*_1)$? Since the Veronese surface is fully immersed in E^5, that is , the Veronese surface cannot lie in a hyperplane of E^5, and since every geodesic in the Veronese surface is planar, we must think of the immersion which has the form (1.40).

We are going to use the theory of submanifolds of finite type introduced and mainly developed by B.-Y. Chen, [Ch.B-3]. We recall some fundamental definitions and properties.

Let M be a compact orientable Riemannian manifold with Riemannian connection ∇ and Δ the Laplacian operator of M acting on $C^{\infty}(M)$, where

$$\Delta = -\sum_i (\nabla_{E_i} \nabla_{E_i} - \nabla_{\nabla_{E_i} E_i})$$

for an orthonormal basis $\{ E_i \}$ of TM. We define an inner product (,) on $C^{\infty}(M)$ by

$$(f, g) = \int_M fg \, dV,$$

where dV is the volume element of M. Then Δ is a self-adjoint elliptic operator with respect to (,) and it has an infinite, discrete sequence of eigenvalues :

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \uparrow +\infty.$$

Let $V_k = \{ f \in C^{\infty}(M) \mid \Delta f = \lambda_k f \}$ be the eigenspace of Δ with eigenvalue λ_k. Then $\sum_{k=0}^{\infty} V_k$ is dense in $C^{\infty}(M)$ in the L^2 - sense. Denote by $\hat{\bigoplus}_k V_k$ the completion of $\sum_{k=0}^{\infty} V_k$. We have

$$C^{\infty}(M) = \hat{\bigoplus}_k V_k.$$

For each function $f \in C^{\infty}(M)$, let f_t be the projection of f onto the subspace V_t (t =

0, 1, 2,...). Then we have the spectral decomposition

$$f = \sum_{t=0}^{\infty} f_t \qquad \text{(in the } L^2\text{-sense).}$$

Because V_0 is 1-dimensional, for any non-constant function $f \in C^{\infty}(M)$, there is a positive integer $p \geq 1$ such that $f_p \neq 0$ and

$$f - f_0 = \sum_{t \geq p} f_t,$$

where $f_0 \in V_0$ is a constant. If there are infinitely many f_t's which are nonzero, we put $q = +\infty$. Otherwise, there is an integer q, $q \geq p$, such that $f_q \neq 0$ and

$$f - f_0 = \sum_{t=p}^{q} f_t.$$

If we allow q to be $+\infty$, we have the decomposition as above for any $f \in C^{\infty}(M)$.

For an isometric immersion $x : M \to E^m$ of a compact Riemannian manifold M into E^m, we put

$$x = (x_1, x_2, \ldots, x_m),$$

where x_A is the A-th Euclidean coordinate function of M in E^m. For each x_A, we have the spectral decomposition

$$x_A - (x_A)_0 = \sum_{t=p_A}^{q_A} (x_A)_t, \quad A = 1, 2, \ldots, m.$$

For each isometric immersion $x : M \to E^m$, we put

$$p = p(x) = \inf_A \{ p_A \}, \qquad q = q(x) = \sup_A \{ q_A \}.$$

where A ranges among all A such that $x_A - (x_A)_0 \neq 0$. It is easy to see that p is an integer ≥ 1 and q is either an integer $\geq p$ or ∞. Moreover, p and q are independent of the choice of the Euclidean coordinate system in E^m. Thus p and q are well-defined. Consequently, for each isometric immersion $x : M \to E^m$ of a compact Riemannian manifold, we have a pair [p, q] associated with M. We call the pair [p, q] the <u>order of the submanifold</u> M. If we use the spectral decomposition of the coordinate functions of the immersion $x : M \to E^m$, we have

$$x = x_0 + \sum_{t=p}^{q} x_t. \tag{1.41}$$

Definition. A compact submanifold M in E^m is said to be of <u>finite type</u> if q is finite. Otherwise M is of <u>infinite type</u>.

Definition. A compact submanifold M is said to be of <u>k-type</u> (k = 1, 2, 3,...) if there are exactly k nonzero x_t's ($t \geq 1$) in the decomposition (1.41).

We can restate <u>Takahashi's Theorem</u> in terms of 1-type :

Lemma 1.13 (Takahashi [Tk] and Chen [Ch.B-3]). Let $x : M \to E^m$ be an isometric immersion of a compact Riemannian manifold M into E^m. Then x is of 1-type if and only if M is a minimal submanifold of a hypersphere of E^m.

B.-Y. Chen gave the following characterization of submanifolds of finite type.

Lemma 1.14 (Chen [Ch.B-2]). Let $x : M \to E^m$ be an isometric immersion of a compact Riemannian manifold M into E^m. Then

(i) M is of finite-type if and only if there is a non-trivial polynomial Q such that

$$Q(\Delta)\, H = 0.$$

(ii) If M is of finite type, then there is a unique monic polynomial P of least degree such that

$$P(\Delta)H = 0.$$

(iii) If M is of finite type, then M is of k-type if and only if deg P = k.

Now, coming back to the problem. We shall compute the Gaussian curvature **K** and find a condition which gives constant Gaussian curvature. Furthermore, we shall characterize the Veronese surface by examining the surface with constant Gaussian curvature and is of 1 -type.

From (1.40) we get

$$x_*(\partial/\partial s)(s, \theta) = (\cos \kappa s \cos \theta,\ \ \cos \kappa s \sin \theta, \frac{1}{\kappa} \sin \kappa s\, (\kappa - \frac{2a^2}{\kappa} \sin^2 \theta),$$

$$\frac{a}{\kappa} \sin \kappa s \sin 2\theta,\ \frac{b}{\kappa} \sin \kappa s \sin^2 \theta).$$

and

$$x_*(\partial/\partial\theta)(s, \theta) = (- \frac{1}{\kappa} \sin \kappa s \sin \theta, \frac{1}{\kappa} \sin \kappa s \cos \theta, \frac{-2a^2}{\kappa^3} (1 - \cos \kappa s) \sin 2\theta,$$

$$\frac{2a}{\kappa^2} (1 - \cos \kappa s) \cos 2\theta, \frac{b}{\kappa^2} (1 - \cos \kappa s) \sin 2\theta).$$

Then the induced first fundamental form g_{ij} is derived as

$$g_{11} = < x_*(\partial/\partial s), x_*(\partial/\partial s) > = 1, \quad g_{12} = g_{21} = < x_*(\partial/\partial s), x_*(\partial/\partial\theta) > = 0,$$

$$g_{22} = < x_*(\partial/\partial\theta), x_*(\partial/\partial\theta) > = \frac{1}{\kappa^2} \sin^2 \kappa s + \frac{4a^2}{\kappa^4} (1 - \cos \kappa s)^2.$$

Thus the line element $d\rho^2$ of M in E^5 has the form

$$d\rho^2 = ds^2 + \{ \frac{1}{\kappa^2} \sin^2 \kappa s + \frac{4a^2}{\kappa^4} (1 - \cos \kappa s)^2 \} \, d\theta^2.$$

So the Gaussian curvature **K** is given by

$$\mathbf{K} = - \frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial s^2}, \tag{1.41}$$

where $G = g_{22} = \frac{1}{\kappa^2} \sin^2 \kappa s + \frac{4a^2}{\kappa^4} (1 - \cos \kappa s)^2.$

Suppose that the Gaussian curvature **K** is a constant. Then (1.41) is equivalent to

$$2G \frac{\partial^2 \sqrt{G}}{\partial s^2} - (\frac{\partial G}{\partial s})^2 + 4 \, \mathbf{K} G^2 = 0.$$

By a straightforward and long computation, we have

$$-\frac{96a^4}{\kappa^6} - \frac{3}{2\kappa^2}(\frac{4a^2}{\kappa^2} - 1)^2 + 4\mathbf{K}\{(\frac{1}{2\kappa^2} + \frac{6a^2}{\kappa^4})^2 + \frac{32a^4}{\kappa^8} - \frac{1}{8\kappa^4}(\frac{4a^2}{\kappa^2} - 1)^2\}$$

$$+ \{ \frac{16a^2}{\kappa^2}(\frac{1}{2\kappa^2} + \frac{6a^2}{\kappa^4})(1 - \frac{4\mathbf{K}}{\kappa^2}) + \frac{4a^2}{\kappa^4}(\frac{4a^2}{\kappa^2} - 1)(7 - \frac{4\mathbf{K}}{\kappa^2}) \} \cos \kappa s$$

$$+ \{ 4(\frac{4a^2}{\kappa^2} - 1)(\frac{1}{2\kappa^2} + \frac{6a^2}{\kappa^4})(\frac{\mathbf{K}}{\kappa^2} - 1) - \frac{32a^4}{\kappa^6} + \frac{128a^4}{\kappa^8}\mathbf{K} \} \cos 2\kappa s$$

$$+ \frac{4a^2}{\kappa^4}(\frac{4a^2}{\kappa^2} - 1)(3 - \frac{4\mathbf{K}}{\kappa^2}) \cos 3\kappa s + \frac{1}{2\kappa^2}(\frac{4a^2}{\kappa^2} - 1)^2 (\frac{\mathbf{K}}{\kappa^2} - 1) \cos 4\kappa s = 0.$$

Since 1, $\cos \kappa s$, $\cos 2\kappa s$, $\cos 3\kappa s$ and $\cos 4\kappa s$ are linearly independent, we get

(a) $$-\frac{96a^4}{\kappa^6} - \frac{3}{2\kappa^2}(\frac{4a^2}{\kappa^2} - 1)^2 + 4\mathbf{K}\{(\frac{1}{2\kappa^2} + \frac{6a^2}{\kappa^4})^2 + \frac{32a^4}{\kappa^8} - \frac{1}{8\kappa^4}(\frac{4a^2}{\kappa^2} - 1)^2\}$$

$$= 0,$$

(b) $$\frac{16a^2}{\kappa^2}(\frac{1}{2\kappa^2} + \frac{6a^2}{\kappa^4})(1 - \frac{4\mathbf{K}}{\kappa^2}) + \frac{4a^2}{\kappa^4}(\frac{4a^2}{\kappa^2} - 1)(7 - \frac{4\mathbf{K}}{\kappa^2}) = 0,$$

(c) $$4(\frac{4a^2}{\kappa^2} - 1)(\frac{1}{2\kappa^2} + \frac{6a^2}{\kappa^4})(\frac{\mathbf{K}}{\kappa^2} - 1) - \frac{32a^4}{\kappa^6} + \frac{128a^4}{\kappa^8}\mathbf{K} = 0,$$

(d) $$\frac{4a^2}{\kappa^4}(\frac{4a^2}{\kappa^2} - 1)(3 - \frac{4\mathbf{K}}{\kappa^2}) = 0,$$

(e) $$\frac{1}{2\kappa^2}(\frac{4a^2}{\kappa^2} - 1)^2 (\frac{\mathbf{K}}{\kappa^2} - 1) = 0.$$

From the last two equations, we get

$$\frac{4a^2}{\kappa^2} - 1 = 0 \quad \text{or} \quad \mathbf{K} = \kappa^2 \quad \text{or} \quad \mathbf{K} = \frac{3\kappa^2}{4}.$$

If $\frac{4a^2}{\kappa^2} - 1 = 0$, then (b) implies

$$\mathbf{K} = \frac{\kappa^2}{4}. \tag{1.42}$$

If $\mathbf{K} = \kappa^2$, then (d) implies $\frac{4a^2}{\kappa^2} = 1$. It follows that $\mathbf{K} = \kappa^2$ and hence $\kappa = 0$ by (1.42). This is a contradiction. By a similar argument, $\mathbf{K} \neq \frac{3\kappa^2}{4}$.

Thus we have

Proposition 1.15. Let M be a compact connected surface in E^5 satisfying $(*_1)$ whose immersion is given by (1.40). Then the Gaussian curvature $\mathbf{K}$ is constant if and only if $\kappa = 4a^2$. In this case, the Gaussian curvature $\mathbf{K} = \frac{\kappa^2}{4} = a^2$.

In such a case, the induced metric (g_{ij}) looks like

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\kappa^2}\sin^2 \kappa s + \frac{1}{\kappa^2}(1 - \cos \kappa s)^2 \end{pmatrix} \tag{1.43}$$

Using this induced metric, we can compute the Christoffel symbols Γ^h_{ji}:

$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{11} = 0, \qquad \Gamma^1_{22} = -\frac{1}{2}\frac{\partial}{\partial s}(\log G) = -\frac{1}{\kappa}\sin \kappa s,$$

$$\Gamma^2_{22} = 0, \qquad \Gamma^1_{12} = 0, \qquad \Gamma^2_{12} = \frac{1}{2}\frac{\partial}{\partial s}(\log G) = -\frac{1}{\kappa}\sin\kappa s,$$

where $G = \frac{1}{\kappa^2}\sin^2\kappa s + \frac{1}{\kappa^2}(1 - \cos\kappa s)^2$.

Lemma 1.16. Let M be a compact connected surface in E^5 satisfying the property $(*_1)$ whose immersion has the form (1.40). If the Gaussian curvature is constant, then the Laplacian operator Δ is given by

$$\Delta = -\left(\frac{\partial^2}{\partial s^2} + \frac{1}{G}\frac{\partial^2}{\partial \theta^2}\right) - \frac{1}{2}\frac{\partial}{\partial s}(\log G)\frac{\partial}{\partial s}, \tag{1.44}$$

where $G = \frac{1}{\kappa^2}\sin^2\kappa s + \frac{1}{\kappa^2}(1 - \cos\kappa s)^2$.

It is well-known that

$$\Delta x = -2H,$$

where H is the mean curvature vector field of M. Using this equation and computing H and ΔH by means of (1.44), we obtain the following

$$\Delta H - \frac{3}{2}\kappa^2 H = 0.$$

According to Lemma 1.14, M is of 1-type and hence M is a minimal submanifold of a hypersphere of E^5.

Let us recall the Calabi's Theorem.

Theorem 1.17 (E. Calabi [C]). Let Σ be a 2-sphere with a Riemannian metric of constant Gaussian curvature **K**, and let $x : \Sigma \to S^{n-1}(r) \subset E^n$ $(n = 2m+1 \geq 3)$ be an isometric minimal immersion of Σ such that the image is not contained in any hyperplane of E^n. Then

i) The value of **K** is uniquely determined as,

$$\mathbf{K} = \frac{2}{m(m+1)r^2},$$

ii) The immersion x is uniquely determined up to a rigid motion of $S^{n-1}(r)$ and the n components of the vector x are a suitably normalized basis for the spherical harmonics of order m on Σ.

On the other hand, we can easily check that the Gaussian curvature **K** cannot be constant if the immersion has the forms (1.32) or (1.37) by computing $\mathbf{K} = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial s^2}$.

So if we apply Calabi's Theorem, we conclude

Theorem 1.18 (Characterization of a Veronese surface). Let M be a compact connected surface in E^5. Then M is a Veronese surface if and only if M has a constant Gaussian curvatutre and there is a point p which is not umbilical such that every geodesic through p is a helix of the same curvatures.

In this case, x is the first standard imbedding RP^2 into E^5 and the second standard immersion of 2-sphere into S^4.

CHAPTER 2

CHARACTERIZATION OF SURFACES OF REVOLUTION IN A 3-DIMENSIONAL EUCLIDEAN SPACE

Let M be a surface in E^3. We now define $(*_2)$.

$(*_2)$: There is a point p in M such that every geodesic through p is a normal section of M at p.

Suppose M has the property $(*_2)$. Let γ be a geodesic parametrized by the arc length s and let $\gamma(0) = p$. Then we have

$$\gamma'(s) = T,$$

$$\gamma''(s) = \sigma(T, T),$$

$$\gamma'''(s) = -A_{\sigma(T,\, T)}T + (\bar{\nabla}_T\sigma)(T, T).$$

Since γ is a normal section at p in the direction $t = T(0)$, $A_{\sigma(t,\, t)}t \wedge t = 0$, that is,

$$< \sigma(t, t), \sigma(t, t^{\perp}) > = 0.$$

Since this is true for any orthonormal vectors t and $t^{\perp}$ tangent to M at p, M is isotropic at p and p is indeed an umbilical point since the ambient manifold is E^3. Since γ is a plane curve, $\gamma(s) \wedge \gamma'(s) \wedge \gamma''(s) = 0$ for all $s \in$ Dom γ. So we can obtain

$$T \wedge A_{\sigma(T,\,T)}T \wedge \sigma(T, T) = 0,$$

which implies

$$< \sigma(T, T), \sigma(T, T^{\perp}) > = 0 \tag{2.1}$$

along γ.

Without loss of generality, we may assume p as the origin o of E^3. Since every geodesic through o is planar, we can express the immersion $x : M \to E^3$ locally on a neighborhood U of o in terms of geodesic polar coordintes (s, θ) as

$$x(s, \theta) = (h(s, \theta) \cos \theta, h(s, \theta) \sin \theta, k(s, \theta)) \tag{2.2}$$

for a suitable choice of Euclidean coordinates of E^3, where h and k are differentiable functions satisfying $h(0, \theta) = k(0, \theta) = 0$.

Differentiating (2.2) with respect to s and θ, we obtain two orthogonal vector fields tangent to M on U

$$x_*(\frac{\partial}{\partial s}) = (\frac{\partial h}{\partial s} \cos \theta, \frac{\partial h}{\partial s} \sin \theta, \frac{\partial k}{\partial s}), \tag{2.3}$$

$$x_*(\frac{\partial}{\partial \theta}) = (\frac{\partial h}{\partial \theta} \cos \theta - h \sin \theta, \frac{\partial h}{\partial \theta} \sin \theta + h \cos \theta, \frac{\partial k}{\partial \theta}), \tag{2.4}$$

where $x_*(\frac{\partial}{\partial s})(0, \theta) = (\cos\theta, \sin\theta, 0)$.

For a fixed θ, $x(s, \theta)$ is a geodesic and we thus have

$$< x_*(\frac{\partial}{\partial s}), x_*(\frac{\partial}{\partial s}) > = (\frac{\partial h}{\partial s})^2 + (\frac{\partial k}{\partial s})^2 = 1.$$

We may put

$$(2.5) \qquad \frac{\partial h}{\partial s} = \cos f(s, \theta), \qquad \frac{\partial k}{\partial s} = \sin f(s, \theta),$$

for a smooth function $f(s, \theta)$ defined on U satisfying $\cos f(0, \theta) = 1$ and $\sin f(0, \theta) = 0$ for all θ.

Lemma 2.1. $\frac{\partial}{\partial\theta}(\kappa(s, \theta))^2 = 0$ on the neighborhood U, where $\kappa(s, \theta)$ is the Frenet curvature of the geodesic $x(s, \theta)$ for a fixed θ.

Proof. Let γ be a geodesic such that $\gamma(s) = x(s, \theta)$ for some θ. Then we have

$$(\kappa(s, \theta))^2 = < \sigma(T, T), \sigma(T, T) >,$$

where $T(s) = x_*(\frac{\partial}{\partial s})(s, \theta)$. We now compute $\frac{\partial}{\partial s}(\kappa(s, \theta))^2$:

$$\frac{1}{2}\frac{\partial}{\partial\theta} < \sigma(T, T), \sigma(T, T) >$$

$$= < D_{\partial/\partial\theta}\, \sigma(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) >$$

$$= \langle (\bar{\nabla}_{\partial/\partial\theta}\sigma)(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$

$$+ 2 \langle \sigma(\nabla_{\partial/\partial\theta}\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$

$$= \langle \sigma(\nabla_{\partial/\partial s}\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$ (because of the Codazzi equation and (2.1))

$$= \langle D_{\partial/\partial s}\, \sigma(\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$

$$- \langle \sigma(\nabla_{\partial/\partial s}\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$

$$= \frac{\partial}{\partial s} \langle \sigma(\partial/\partial\theta, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$

$$- \langle \sigma(\partial/\partial\theta, \partial/\partial s), (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) \rangle$$

$$- \langle \sigma(\nabla_{\partial/\partial\theta}\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) \rangle$$

$$= - \langle \sigma(\partial/\partial\theta, \partial/\partial s), (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) \rangle$$ (because of (2.1)).

Let $\sigma(\partial/\partial\theta, \partial/\partial s) = f_1(s)\,N$ and let $\sigma(\partial/\partial s, \partial/\partial s) = g_1(s)N$, where N is the unit vector field normal to M along γ and f_1 and g_1 are some smooth functions defined along γ. Then we have

$$(\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) = g_1'(s)\, N + g_1(s)\frac{\partial N}{\partial s}.$$

(2.1) leads to $f_1(s)\, g_1(s) = 0$ for all s.

If $g_1(0) \neq 0$, then there exists an interval I contained in Dom γ such that $g_1(s) \neq 0$ for $s \in I$. So $f_1(s) = 0$ on I. Thus, we have

$$< \sigma(\partial/\partial\theta, \partial/\partial s), (\bar{\nabla}_{\partial/\partial s}\sigma)(\partial/\partial s, \partial/\partial s) > = f_1(s)\, g_1'(s) = 0 \qquad \text{on I.}$$

Suppose that $g_1(0) = 0$. Let $s_0 = \inf \{s \mid g_1(s) \neq 0 \}$. If $s_0 = 0$, then $g_1(s) \neq 0$ for $s > 0$ and thus $f_1(s) = 0$ for $s > 0$. So, $f_1(s)\, g_1'(s) = 0$ for $s > 0$. By continuity, $f_1(s)\, g_1'(s) = 0$ for $s \geq 0$. If $s_0 > 0$, then $g_1(s) = 0$ for $s < s_0$. Thus $f_1(s)\, g_1'(s) = 0$ for $s < s_0$.

If there is some $s \in$ Dom γ such that $g_1(s) = 0$, then we keep doing this argument and thus we get $\frac{\partial}{\partial\theta}(\kappa(s, \theta))^2 = 0$ for $s \in$ Dom γ. Since this is true for every θ, we have

$$\frac{\partial}{\partial\theta} < \sigma(\partial/\partial s, \partial/\partial s), \sigma(\partial/\partial s, \partial/\partial s) > = 0$$

on U. In other words, the curvature $\kappa(s, \theta)$ is independent of the choice of θ. (Q. E. D.)

Lemma 2.2. The functions h and k are independent of the choice of θ.

Proof. Differentiating (2.3) with respect to s, we get

$$\frac{\partial^2 x}{\partial s^2} = \left(\frac{\partial^2 h}{\partial s^2} \cos\theta, \frac{\partial^2 h}{\partial s^2} \sin\theta, \frac{\partial^2 k}{\partial s^2}\right). \tag{2.6}$$

Thus the curvature $\kappa(s, \theta)$ satisfies

$$(\kappa(s, \theta))^2 = (\frac{\partial f}{\partial s})^2. \tag{2.7}$$

On the other hand, (2.1) gives

$$< \frac{\partial^2 x}{\partial s^2}, \frac{\partial}{\partial \theta}(\frac{\partial x}{\partial s}) > = 0, \tag{2.8}$$

which implies

$$\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial s} = 0. \tag{2.9}$$

By Lemma 2.1, the curvatures do not depend on θ and so we choose a geodesic $x(s, \theta)$ for some θ and examine its curvature.

Suppose that $\kappa(0, \theta) = 0$. Let $s_1 = \inf \{ s \mid \kappa(s, \theta) \neq 0 \}$. If $s_1 = 0$, then $\kappa(s, \theta) \neq 0$ for $s > 0$. (2.7) and (2.9) imply $\frac{\partial f}{\partial \theta} = 0$ for $s > 0$. By continuity, $\frac{\partial f}{\partial \theta} = 0$ for $s \geq 0$. If $s_1 > 0$ (possibly $+\infty$), then $\kappa(s, \theta) = 0$ for $0 \leq s < s_1$. Then the inside of the geodesic circle S_1 centered at o with radius s_1 lies in E^2. In this case, h and k are clearly independent of the choice of θ.

Suppose that $\kappa(0, \theta) \neq 0$. Then we can choose a sufficiently small neighborhood of o where $\kappa(s, \theta) \neq 0$. Evidently $\frac{\partial f}{\partial s} \neq 0$ and thus $\frac{\partial f}{\partial \theta} = 0$ on this neighborhood. Developing this argument continuously if $\kappa(s, \theta) = 0$ for some $s \neq 0$, we see that h and k are independent of the choice of θ in the neighborhood U because $h(0, \theta) = k(0, \theta) = 0$ for all θ. (Q. E. D.)

Since the functions h and k only depend on the arc length s, (2.2) defines a surface of revolution around the point o.

Conversely, a meridian of a surface of revolution is always a geodesic and all the normal sections at the point o are geodesics through o if M is locally a surface of revolution with axis of symmetry passing through o. Thus we have

Theorem 2.3 (Local characterization). Let M be a surface in E^3. Then M is locally a surface of revolution with vertex p (around a neighborhood of p) if and only if every geodesic through p is a normal section of M at p.

Theorem 2.4 (Global characterization). Let M be a complete connected surface in E^3. Then M is a surface of revolution if and only if there is a point p in M such that every geodesic through p is a normal section of M at p.

Let M be a Riemannian manifold immersed in a Riemannian manifold $\tilde{M}$. A curve α in M which is regarded as a curve in $\tilde{M}$ is called a W-curve if the Frenet curvatures of α are constant along α. In this case, the Frenet curvatures may depend on the choice of geodesics. Together this notion and the property ($*_2$) give another characterization of submanifolds with geodesic normal sections in E^3.

Corollary 2.5. Let M be a complete connected surface in E^3. Then the following are equivalent :

(1) M satisfies the property ($*_2$) and the geodesics through the base point in the property ($*_2$) are W-curves .

(2) M satisfies the property ($*_1$).

(3) M has geodesic normal sections.

(4) M is a plane E^2 or a standard sphere S^2.

CHAPTER 3

SURFACES OF A EUCLIDEAN SPACE WITH PLANAR GEODESICS THROUGH A POINT

§ 1. Surfaces of a Euclidean space with planar geodesics through a point which is not an isolated flat point

Let M be a surface in E^m ($m \geq 3$). We define the property ($*_3$).

($*_3$) : There is a point p in M such that every geodesic through p is planar.

Lemma 3.1. Let M be a surface in E^m and let γ be a geodesic in M through p. If γ is a planar curve, then γ is a normal section of M at p.

Proof. Let us assume that γ is parametrized by the arc length s and let $\gamma(0) = p$. Then we have

$$\gamma'(s) = T,$$

$$\gamma''(s) = \sigma(T, T),$$

$$\gamma'''(s) = -A_{\sigma(T, T)}T + (\bar{\nabla}_T\sigma)(T, T).$$

Since γ is a plane curve, $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) = 0$ along γ. Thus we get

$$T \wedge \sigma(T, T) \wedge (-A_{\sigma(T, T)}T + (\bar{\nabla}_T\sigma)(T, T)) = 0,$$

which implies

$$T \wedge A_{\sigma(T, T)}T = 0 \tag{3.1}$$

and

$$\sigma(T, T) \wedge (\bar{\nabla}_T\sigma)(T, T) = 0. \tag{3.2}$$

Suppose first that $\sigma(t, t) \neq 0$, where $t = T(0)$. We can choose a neighborhood U of p such that $\sigma(u, u) \neq 0$ for any nonzero vector $u \in T_qM$, $q \in U$. So γ lies in $p + \mathrm{Span}\{t, \sigma(t, t), (\bar{\nabla}_t\sigma)(t, t)\}$ and hence γ is a normal section at p.

Suppose $\sigma(t, t) = 0$. It is enough to consider $\sigma(t, t) = 0$ and $\sigma(T, T) \neq 0$ for $s > 0$. Let N be a normal vector field to M which is parallel to $\sigma(T, T)$ along γ for $s > 0$. Then we can choose a vector field $T^\perp$ which is tangent to M along γ and perpendicular to the plane Π spanned by $\{T(s), N(s)\}$ $(s > 0)$. Extend $T^\perp(s)$ up to the point p, which will be denoted by the same notation $T^\perp$. Then $\{ t, T^\perp(0) \}$ is an orthonormal basis for T_pM and $T^\perp(0)$ is perpendicular to the plane Π. N(0) is thus a normal vector to M at p. Consequently, γ lies in $p + \mathrm{Span}\,\{t, N(0)\}$ and hence γ is a normal section at p in the direction t. (Q. E. D.)

Making use of this lemma, we see that the property $(*_2)$ is equivalent to the propertry $(*_3)$ if the ambient manifold is a 3-dimensional Euclidean space E^3.

Thus we have

Theorem 3.2. Let M be a surface in E^3. Then, M satisfies the property $(*_3)$ if and only if M is locally a surface of revolution.

Corollary 3.3. Let M be a complete connected surface in E^3. Then M satisfies the property $(*_3)$ if and only if M is a surface of revolution.

We now suppose that a surface M in E^m satisfies the property $(*_3)$. By virtue of (3.1), we get

$$< \sigma(T, T), \sigma(T, T^{\perp}) > = 0, \tag{3.3}$$

where $\gamma'(s) = T$, γ being a geodesic through p. In particular, $< \sigma(t, t), \sigma(t, t^{\perp}) > = 0$, $t = T(0)$. It is true for any unit vetor t in T_pM and thus M is isotropic at p. So we may use Lemma 1.2 later.

Let (s, θ) be the geodesic polar coordinate sytem about p. We may assume that p is the origin o of E^m. Let $\exp_o(s\, e(\theta)) = x(s, \theta)$, where $e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$ for some orthonormal basis $\{e_1, e_2\}$ for T_oM which is associated with the geodesic polar coordinates (s, θ).

Lemma 3.4. Let M be a surface in E^m with the property $(*_3)$. Then

$$\frac{\partial}{\partial\theta}(\kappa(s, \theta))^2 = 0, \tag{3.4}$$

where $\kappa(s, \theta)$ is the Frenet curvature of $x(s, \theta)$. In other words, the curvature of each geodesic through o does not depend on θ.

This lemma can be proved similarly to Lemma 2.1.

Since every geodesic through o is a plane curve and it is a normal section at o, we may represent the immersion $x : M \to E^m$ locally as

$$x(s, \theta) = h(s, \theta) \cos \theta \, e_1 + h(s, \theta) \sin \theta \, e_2 + k(s, \theta) \, N(\theta), \tag{3.5}$$

where $N(\theta)$ is a unit vector normal to M at o which may depend on θ and h and k are some smooth functions satisfying $h(0, \theta) = k(0, \theta) = 0$ for all θ.

Lemma 3.5. The functions h and k described as above do not depend on θ.

Proof. Since (s, θ) is the geodesic polar coordinate system, we have the following orthogonal vector fields tangent to M about o :

$$x_*(\frac{\partial}{\partial s}) = \frac{\partial h}{\partial s} \cos \theta \, e_1 + \frac{\partial h}{\partial s} \sin \theta \, e_2 + \frac{\partial k}{\partial s} N(\theta), \tag{3.6}$$

$$x_*(\frac{\partial}{\partial \theta}) = (\frac{\partial h}{\partial \theta} \cos \theta - h \sin \theta) e_1 + (\frac{\partial h}{\partial \theta} \sin \theta + h \cos \theta) e_2 + \frac{\partial k}{\partial \theta} N(\theta) + k N'(\theta). \tag{3.7}$$

Since $< x_*(\frac{\partial}{\partial s}), x_*(\frac{\partial}{\partial s}) > = 1$, we get

$$(\frac{\partial h}{\partial s})^2 + (\frac{\partial k}{\partial s})^2 = 1,$$

from which , we may put

$$(3.8) \qquad \frac{\partial h}{\partial s} = \cos f(s, \theta), \qquad \frac{\partial k}{\partial s} = \sin f(s, \theta),$$

where f is a smooth function defined on a neighborhood of o. Since $x_*(\frac{\partial}{\partial s})(0, \theta) = \cos\theta e_1 + \sin\theta e_2$, $\cos f(0, \theta) = 1$ and $\sin f(0, \theta) = 0$. Using (3.7) and (3.8), the curvature κ is represented as

$$(3.9) \qquad (\kappa(s, \theta))^2 = (\frac{\partial f}{\partial s})^2.$$

On the other hand, (3.3) implies

$$< \frac{\partial^2 x}{\partial s^2}, \frac{\partial}{\partial \theta}(\frac{\partial x}{\partial s}) > = 0,$$

which yields

$$(3.10) \qquad \frac{\partial f}{\partial s}\frac{\partial f}{\partial \theta} = 0.$$

The rest of the proof is exactly same as that of Lemma 2.2. (Q. E. D.)

We assume that a surface M lies in E^4 satisfying the property $(*_3)$ where the base point p in the property $(*_3)$ is not an isolated flat point. An isolated flat point p means a point such that the curvature of every geodesic through p vanishes only at p in some neighborhood of p. The curvature tensor R obviously vanishes at flat points. We also assume that the point p as the origin o of E^4. Let $\{e_1, e_2\}$ be an orthonormal basis of T_oM.

Suppose first that dim $(\mathrm{Im}\ \sigma)_o \leq 1$. In this case, by considering Lemma 1.2, we see that o is an umbilical point of M. If dim $(\mathrm{Im}\ \sigma)_o = 1$, then by choosing an appropriate Euclidean coordinate system of E^4 the immersion $x : M \to E^4$ can be locally expressed in terms of the geodesic polar coordinate system as

$$x(s, \theta) = (h(s) \cos \theta, h(s) \sin \theta, k(s), 0)$$

for some smooth functions h and k of the arc length s due toLemma 3.4, where (s, θ) is the system of geodesic polar coordinates related to $\{e_1, e_2\}$. Thus M is locally a surface of revolution about o with axis of symmetry in the direction of the mean curvature vector at o.

Suppose that dim $(\mathrm{Im}\ \sigma)_o = 0$. Since o is not an isolated flat point, there exists a neighborhood of o which is contained in a plane E^2 and this is a special case of above surface of revolution.

We now suppose dim $(\mathrm{Im}\ \sigma)_o = 2$. As we showed in Lemma 1.6, the mean curvature vector H vanishes at o and $\| \sigma(t, t^{\perp}) \|$ does not depend on the choice of orthonormal vectors t and $t^{\perp}$ tangent to M at o. Choose two unit vectors N_1 and N_2 normal to M at o such that

(3.11) $$N_1 = \frac{\sigma(e_1, e_1)}{\kappa(0)} \quad \text{and} \quad N_2 = \frac{\sigma(e_1, e_2)}{\kappa(0)},$$

where $\kappa(0)$ is the Frenet curvature at o. Since the functions h and k in (3.5) are independent of the choice of θ, (3.5) can be reduced to

(3.12) $$x(s, \theta) = h(s) \cos \theta\, e_1 + h(s) \sin \theta\, e_2 + k(s)\, N(\theta).$$

As we did before, computing the length of $x_*(\frac{\partial}{\partial s})$ by using (3.12), we may put

$$(3.13) \qquad h'(s) = \cos f(s), \qquad k'(s) = \sin f(s)$$

for some smooth function f satisfying $f(0) = 0$.

On the other hand, we obtain from (3.12)

$$k''(0) = \sigma(\partial/\partial s, \partial/\partial s)\,(0, \theta)$$

$$= \sigma(\cos\theta\, e_1 + \sin\theta\, e_2, \cos\theta\, e_1 + \sin\theta\, e_2)$$

$$= \kappa(0)\cos 2\theta\, N_1 + \kappa(0)\sin 2\theta\, N_2 .$$

Since $k''(0) = (\cos f(0))\, f'(0) = f'(0) = \pm\,\kappa(0)$,

$$(3.14) \qquad N(\theta) = \pm\,(\cos 2\theta\, N_1 + \sin 2\theta\, N_2).$$

Thus, for a suitable choice of Euclidean coordinates of E^4 associated with e_1, e_2, N_1 and N_2, the immersion x is locally determined by

$$(3.15) \qquad x(s, \theta) = \Big(\cos\theta \int_0^s \cos f(t)\, dt,\ \sin\theta \int_0^s \cos f(t)\, dt,\ \pm\cos 2\theta \int_0^s \sin f(t)\, dt,$$

$$\pm\sin 2\theta \int_0^s \sin f(t)\, dt\Big),$$

where $f(s) = \pm \int_0^s \kappa(t)\, dt$ and κ is the Frenet curvature of geodesics through o.

If a surface M has the form (3.15), it is easily checked that M satisfies $(*_3)$.

Thus we conclude

Theorem 3.6 (cf. Theorem 3.8). Let M be a surface in E^4 without isolated flat points. Then M satisfies the property ($*_3$) if and only if M is locally a surface of revolution which lies in E^3 or a surface that locally has the form (3.15).

Corollary 3.7. Let M be a complete connected surface without isolated flat points in E^4. Then M satisfies the property ($*_3$) if and only if M is a surface of revolution which lies in E^3 or a surface is globally of the form (3.15) .

We now consider a surface M which lies in E^5 satisfying the property ($*_3$). Again, we assume the base point in the property ($*_3$) is the origin of E^5, where o is not an isolated flat point.

If $\dim (\mathrm{Im}\ \sigma)_o \leq 2$, then M is locally a surface of revolution in E^3 or M is a surface with local representation about o of the form (3.15) which lies in E^4 by the exact same argument .

Suppose $\dim (\mathrm{Im}\ \sigma)_o = 3$. Then

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) \neq 0.$$

Choose three orthonormal normal vectors to M at o :

(3.16) $$N_1 = \frac{\sigma(e_1, e_1)}{\kappa(0)}, \quad N_2 = \frac{\sigma(e_1, e_2)}{a}, \quad N_3 = \frac{\tilde{N}_3}{\|\tilde{N}_3\|},$$

where $a = \|\sigma(e_1, e_2)\|$ and $\tilde{N}_3 = \sigma(e_2, e_2) - \langle \sigma(e_1, e_1), N_1 \rangle N_1$. If we compute the second fundamental form at o as we did to derive (3.14), then we obtain

$$\sigma(\partial/\partial s, \partial/\partial s)\,(0, \theta) = (\ \kappa(0)\cos^2\theta - \frac{\kappa(0)^2 - 2b^2}{\kappa(0)})\, N_1 + a \sin 2\theta\, N_2 + b \sin^2\theta\, N_3,$$

where $b = \| \tilde{N}_3 \|$. Using this equation and (3.5), we can find $N(\theta)$:

$$N(\theta) = \pm\,(\ \cos^2\theta - \frac{\kappa(0)^2 - 2b^2}{\kappa(0)^2})\, N_1 \pm \frac{a}{\kappa(0)} \sin 2\theta\, N_2 \pm \frac{b}{\kappa(0)} \sin^2\theta\, N_3.$$

Thus locally the immersion $x : M \to E^5$ may be written in terms of a suitable choice of Euclidean coordinates of E^5 as

$$\begin{aligned} x(s, \theta) = (&\cos\theta \int_0^s \cos f(t)\,dt,\quad \sin\theta \int_0^s \cos f(t)\,dt, \\ &\pm\,(\ \cos^2\theta - \frac{\kappa(0)^2 - 2b^2}{\kappa(0)^2}) \int_0^s \sin f(t)\,dt, \\ &\pm \frac{a}{\kappa(0)} \sin 2\theta \int_0^s \sin f(t)\,dt,\quad \pm \frac{b}{\kappa(0)} \sin^2\theta \int_0^s \sin f(t)\,dt\ ), \end{aligned} \tag{3.17}$$

where $f(s) = \pm \int_0^s \kappa(t)\,dt$ and κ is the Frenet curvature of geodesics through o.

Conversely, if a surface M has the form (3.17), then we can easily see that M satisfies the property ($*_3$).

Theorem 3.8 (cf. Theorem 3.6). Let M be a surafce in E^5 without isolated flat points. Then M satisfies the property ($*_3$) if and only if M is locally a surface of revolution in E^3 or a surface in E^4 which has a local representation of the form (3.15) or a surface of the form (3.17) which fully lies in E^5.

Let M be a surface in E^m. Since the dimension of the first normal space at o is at most three, we obtain the following theorem.

Theorem 3.9. Let M be a surface in E^m ($m \geq 3$) without isolated flat points. Then M satisfies the property ($*_3$) if and only if M lies locally in E^5 and M is one of the three model spaces described in Theorem 3.8.

§ 2. Surfaces of a Euclidean space with planar geodesics through a point which is an isolated flat point

In this section we study a surface M in E^m satisfying the property ($*_3$). Base point o, say the origin of E^m, is an isolated flat point.

We now assume that M is an analytic surface in E^m.

We first define the degree of an isolated flat point.

Let p be an isolated flat point of a Riemannian manifold. Then for every geodesic γ parametrized by the arc length s through $p = \gamma(0)$, its curvature $\kappa_\gamma(s)$ satisfies $\kappa_\gamma(0) = 0$ and $\kappa_\gamma(s) \neq 0$ for sufficiently small s. Let $d(p) = \inf \{ n \in \mathbf{Z}_+ \mid \kappa_\gamma^{(n)}(0) \neq 0 \}$. Then d(p) is well-defined integer ≥ 1.

Definition. d(p) is called the <u>degree</u> of the isolated flat point p.

Suppose that the analytic surface M in E^m satisfies the property ($*_3$) and that the base point in the property ($*_3$) is an isolated flat point. We also assume that the base point is the origin o of E^m.

According to (3.5) and Lemma 3.5, the immersion $x : M \to E^m$ is locally represented in terms of geodesic polar coordinates (s, θ) about o :

$$x(s, \theta) = h(s)\, e(\theta) + k(s)\, N(\theta), \tag{3.18}$$

where h and k are analytic functions of s such that $h(0) = k(0) = 0$, $h'(s) = \cos f(s)$, $k'(s) = \sin f(s)$, $f(s) = \pm \int_0^s \kappa(t)\, dt$, $e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$ and $N(\theta)$ is a unit vector normal to M at o depending on θ.

For $r \geq 2$, the r-th derivatives of h and k are :

$$h^{(r)}(s) = -(\sin f(s))\, f^{(r-1)}(s) + O_1(f', f'', \dots, f^{(r-2)})$$

and

$$k^{(r)}(s) = (\cos f(s))\, f^{(r-1)}(s) + O_2(f', f'', \dots, f^{(r-2)}),$$

where $O_i(f', f'', \dots, f^{(r-2)})$ $(i = 1, 2)$ are certain polynomials with respect to $f', f'', \dots, f^{(r-2)}$. Since o is an isolated flat point and since the curvature of each geodesic through o is independent of the choice of θ, there is an integer p (> 1) such that $\kappa(0) = \kappa'(0) = \dots = \kappa^{(p-2)}(0) = 0$ and $\kappa^{(p-1)}(0) \neq 0$, that is, the degree of o is $p - 1$. Since $\kappa(s) = \pm f'(s)$, we see that

$$h^{(r)}(0) = 0 \text{ for any } r \geq 2,$$

and

$$k^{(r)}(0) = 0 \quad (0 \leq r \leq p - 1), \qquad k^{(p)}(0) \neq 0.$$

In other words,

(3.19)
$$\frac{\partial^r x}{\partial s^r}(0, \theta) = 0 \quad (2 \le r \le p-1)$$

and

(3.20)
$$\frac{\partial^p x}{\partial s^p}(0, \theta) = k^{(p)}(0)\, N(\theta) = \kappa^{(p-1)}(0)\, N(\theta) \neq 0.$$

We now define the r-th ($r \ge 1$) covariant derivative of σ by

$$(\bar{\nabla}^r\sigma)(X_1, X_2, \ldots, X_{r+2}) = D_{X_1}((\bar{\nabla}^{r-1}\sigma)(X_2, \ldots, X_{r+2}))$$
$$- \sum_{i=2}^{r+2} (\bar{\nabla}^{r-1}\sigma)(X_2, \ldots, \nabla_{X_1}X_i, \ldots, X_{r+2}).$$

Then $\bar{\nabla}^r\sigma$ is a normal bundle valued tensor of type $(0, r+2)$. Moreover, it can be proved that

(3.21)
$$(\bar{\nabla}^r\sigma)(X_1, X_2, X_3, \ldots, X_{r+2}) - (\bar{\nabla}^r\sigma)(X_2, X_1, X_3, \ldots, X_{r+2})$$
$$= R^N(X_1, X_2)((\bar{\nabla}^{r-2}\sigma)(X_3, \ldots, X_{r+2}))$$
$$+ \sum_{i=3}^{r+2} (\bar{\nabla}^{r-2}\sigma)(X_3, \ldots, R(X_1, X_2)X_i, \ldots, X_{r+2})$$

for $r \ge 2$, where $X_1, X_2, X_3, \ldots, X_{r+2}$ are vector fields tangent to M, R^N the normal

curvature tensor, R the curvature tensor of M and $\bar{\nabla}^0\sigma = \sigma$.

On the other hand, for $r \in \mathbf{Z}_+$,

$$\frac{\partial^r x}{\partial s^r}(0, \theta) = (\bar{\nabla}^r\sigma)(t^{r+2}),$$

where $t = e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_2$ and $(\bar{\nabla}^r\sigma)(t^{r+2}) = (\bar{\nabla}^r\sigma)(t, t, t, \ldots, t)$.

Lemma 3.10. If $(\bar{\nabla}^r\sigma)(t^{r+2}) = 0$ ($0 \le r \le p-1$) and $(\bar{\nabla}^p\sigma)(t^{p+2}) \neq 0$ for all t in T_oM, then $\bar{\nabla}^p\sigma$ is symmetric and $\bar{\nabla}^r\sigma = 0$ at the point o for $0 \le r \le p-1$.

Proof. For $p = 1$, it is clear. Let $p = 2$. Since $\sigma(t, t) = 0$ for every $t \in T_oM$, o is a flat point of M. So the curvature tensor R vanishes at o. Also, $(\bar{\nabla}\sigma)(t^3) = 0$ for all $t \in T_oM$ implies $\bar{\nabla}\sigma = 0$ at the point o. Suppose it is true for $p = r - 1$ ($r > 3$). By the induction assumption and (3.21), we obtain the result. (Q. E. D.)

Thus if the point o is an isolated flat point of degree $p-1$, then we have

$$\bar{\nabla}^r\sigma = 0 \qquad (0 \le r \le p-1)$$

and

$$\bar{\nabla}^p\sigma \text{ is symmetric}$$

at the point o.

Denote by

$$(\bar{\nabla}^r\sigma)(e_1^i, e_2^j) = (\bar{\nabla}^r\sigma)(e_{11}, e_{12}, \dots, e_{1i}, e_{21}, e_{22}, \dots, e_{2j}),$$

where $i + j = r + 2$ and $e_{1h} = e_1$, $e_{2k} = e_2$ for all $h = 1, 2, \dots, i$ and $k = 1, 2, 3, \dots, j$. Since the curvature κ is independent of the choice of the geodesic through o, we get

$$\| (\bar{\nabla}^p\sigma)(e_1^{p+2}) \| = \| (\bar{\nabla}^r\sigma)(e(\theta)_1^{p+2}) \| \qquad \text{for all } \theta.$$

So we obtain the following equation.

$$-\sum_{r=1}^{p+1} \binom{p+2}{r} \cos^{2(p+2-r)}\theta \sin^{2r}\theta \, \| (\bar{\nabla}^p\sigma)(e_1^{p+2}) \|^2$$

$$-\sum_{r=1}^{p+1} \binom{p+2}{r} \cos^{2(p+2-r)}\theta \sin^{2r}\theta \, \| (\bar{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r) \|^2$$

$$+ 2\sum_{r<s} \binom{p+2}{r}\binom{p+2}{s} \cos^{2(p+2)-r-s}\theta \sin^{r+s}\theta < (\bar{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r), (\bar{\nabla}^p\sigma)(e_1^{p+2-s}, e_2^s) >$$

$$= 0,$$

which yields

(3.22) $$\binom{p+2}{q} \| (\bar{\nabla}^p\sigma)(e_1^{p+2}) \|^2$$

$$= \binom{p+2}{q} \| (\bar{\nabla}^p\sigma)(e_1^{p+2-q}, e_2^q) \|^2$$

$$+ 2 \sum_{r<s\leq 2q} \binom{p+2}{r}\binom{p+2}{s} < (\bar{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r), (\bar{\nabla}^p\sigma)(e_1^{p+2-s}, e_2^s) >$$

for $r = 2q$ $(q \geq 1)$ and

$$\sum_{r<s\leq 2q-1} \binom{p+2}{r}\binom{p+2}{s} < (\bar{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r), (\bar{\nabla}^p\sigma)(e_1^{p+2-s}, e_2^s) > = 0 \tag{3.23}$$

for $r = 2q - 1$ $(q \geq 1)$.

On the other hand, the maximal dimension of $\{ (\bar{\nabla}^p\sigma)(e_{i_1}, e_{i_2}, \ldots, e_{i_{p+2}}) \mid e_{i_j} = e_1 \text{ or } e_2 \}$ is $p+3$.

From (3.20), we see that

$$N(\theta) = \frac{1}{\| (\bar{\nabla}^p\sigma)(e_1^{p+2}) \|^2} (\bar{\nabla}^p\sigma)(e(\theta)_1^{p+2}) \tag{3.24}$$

$$= \frac{1}{\| (\bar{\nabla}^p\sigma)(e_1^{p+2}) \|^2} \sum_{r=0}^{p+2} \binom{p+2}{r} \cos^{p+2-r}\theta \sin^r\theta \cdot$$

$$(\bar{\nabla}^p\sigma)(e_1^{p+2-r}, e_2^r).$$

Thus, if o is an isolated flat point of degree p-1, then M is locally contained in at most (p + 5)-dimensional Euclidean space E^{p+5} and in this case the second furndamental form σ satisfies (3.22) and (3.23) and the immersion $x : M \to E^m$ becomes

$$x(s, \theta) = h(s) \cos \theta \, e_1 + h(s) \sin \theta \, e_2 + k(s) \, N(\theta),$$

where h and k are analytic functions such that $h^{(r)}(0) = 0$ for all $r \geq 2$, $k^{(r)}(0) = 0$ $(0 \leq r \leq p-1)$, $k^{(p)}(0) \neq 0$ and $N(\theta)$ is given by (3.24).

Thus we have

Theorem 3.11. If a surface M in E^m satisfies the property $(*_3)$ whose base point , say the origin o of E^m, is an isolated flat point of degree p - 1, then M locally lies in at most (p + 5)-dimensional Euclidean space E^{p+5} about o and is of the form :

$$(3.25) \qquad x(s, \theta) = \left(\int_0^s \cos f(t) \, dt \right) (\cos \theta \, e_1 + \sin \theta \, e_2) + \left(\int_0^s \sin f(t) \, dt \right) N(\theta),$$

where $N(\theta)$ is of the form (3.24), $f(s) = \pm \int_0^s \kappa(t) \, dt$ and $\kappa(s)$ is the Frenet curvature of geodesics through o.

Combining Theorem 3.9 and Theorem 3.11, we can classify analytic surfaces in E^m satisfying the property $(*_3)$.

Theorem 3.12 (Classification). Let M be an analytic surafce in E^m. If M satisfies the property $(*_3)$, then M is one of the following :

(1) M is locally a surface of revolution about o which lies in E^3,

(2) M is a surface of the form (3.15) about o which fully lies in E^4,

(3) M is a surface of the form (3.17) about o which fully lies in E^4,

(4) M is a surface of the form (3.25) about o which lies in E^{p+5}, where the

degree of the isolated flat point o is p -1.

Remark. According to K. Sakamoto, [S-1], and J. A. Little, [L], a surface in a Euclidean space E^m with planar geodesics must be an open portion of a plane E^2, a standard sphere S^2 or a real projective space RP^2. So M must lie in a 5-dimensional Euclidean space E^5. However, a surface M in E^m ($m \geq 3$) satisfying the property ($*_3$) may lie fully in a higher dimensional Euclidean space depending on the degree of the isolated flat point if the base point in the property ($*_3$) is an isolated flat point.

CHAPTER 4

SURFACES IN A PSEUDO-EUCLIDEAN SPACE E^m_s WITH PLANAR NORMAL SECTIONS

§1. Some fundamental concepts and definitions

We introduce some basic terminologies and definitions for later use.

If I_n denotes the unit matrix of degree n, we put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Let

$$sl(n, \mathbf{R}) = \{ \text{ all } n \times n \text{ - real matrices of trace } 0 \ \}.$$

Definition. ([B-2]). The Veronese immersions of signature (r, n-r) are defined by

$$x \in S^n_r\left(\sqrt{\frac{2(n+1)}{n}}\right) \subset E^{n+1}_r \rightarrow \frac{1}{2}\sqrt{\frac{n}{n+1}}\left(x\, {}^*x - \frac{2}{n} I_{n+1}\right) \in s_o(r, n+1-r),$$

where ${}^*x = {}^t x\, I_{r,n+1-r}$, and

$$x \in H^n_r\left(\sqrt{\frac{2(n+1)}{n}}\right) \subset E^{n+1}_{r+1} \to \frac{1}{2}\sqrt{\frac{n}{n+1}}\,(x\,{}^*x + \frac{2}{n} I_{n+1}) \in s_o(r+1, n-r),$$

where ${}^*x = {}^t x\, I_{r+1, n-r}$, and

$$s_o(p, q) = \{ A \in sl(p+q, \mathbf{R}) \mid I_{p,q}\, {}^tA\, I_{p,q} = A \}.$$

In the first case, we take $<A, B> = tr(AB)$ for the metric, so that $s_o(r, n+1-r) \cong E^N_{r(n+1-r)}$ with $N = \frac{1}{2}n(n+3)$. The image of S^n_r is contained in $S^{N-1}_{r(n+1-r)}(1)$ with zero mean curvature. In the second case we use $< A, B > = - tr(AB)$ as the metric, so that $s_o(r+1, n-r) \cong E^N_{N-(r+1)(n-r)}$, and the image of H^n_r has zero mean curvature in $H^{N-1}_{N-(r+1)(n-r)-1}(1)$. Both immersions are isometric and they are planar geodesic immersions.

Notation. $E^m_{s,t}$ denotes E^m with symmetric bilinear form $< , >$ whose signature has s negative, m - s -t positive and t zero signs, that is, t independent directions which are orthogonal to everything. When $t = 0$, we have the pseudo-Euclidean space E^m_s.

Definition. An isometric immersion $i : M^n_r \to E^m_{s,t}$ is parallel if its second fundamental form σ is covariantly constant, that is, $\bar{\nabla}\sigma = 0$. An immersion of a pseudo-Riemannian submanifold M^n_r into $E^m_{s,t}$ is full if its image is contained in no affine hyperplane of $E^m_{s,t}$.

For any function $g : E^n_r \to \mathbf{R}$, the mapping

$$x \in E^n_r \xrightarrow{i} (g(x), x, g(x)) \in E^{n+2}_{r+1}$$

is a planar geodesic immersion with $<\sigma(X, X), \sigma(X, X)> = 0$ for all unit vectors $X \in TE^n_r$, which is full in $E^{n+1}_{r,1}$ (if g is not linear). This immersion is called an expansion of E^n_r into $E^{n+1}_{r,1}$.

If the immersion i is parallel, then the geodesics are mapped to parabolas or line segments, so the function g associated with expansion must be a quadratic polynomial. Up to isometry of $E^{n+1}_{r,1}$, $g = \sum_{i=1}^{n} a_i x_i^2$, so that $i(E^n_r)$ is an elliptic or hyperbolic paraboloid or an orthogonal cylinder over one of these.

Let $g_1, \ldots, g_k$ be independent quadratic polynomials from E^n_r to $\mathbf{R}$, $k \leq \frac{1}{2}n(n+1)$. Then we can define an expansion E^n_r into $E^{n+k}_{r,k}$ by

$$x \in E^n_r \xrightarrow{i} (g_1(x), g_2(x), \ldots, g_k(x), x, g_1(x), g_2(x), \ldots, g_k(x)) \in E^{n+k}_{r,k}.$$

Let M^n_r be a pseudo-Riemannian manifold which lies in a pseudo-Euclidean space E^m_s. We define a normal section of M^n_r at a point $p \in M^n_r$ in a direction $t\ (\neq 0) \in T_pM^n_r$ in a similar way that we did in Chapter 1 : Let $E(p\ ;\ t)$ be an affine space spanned by t and $T^{\perp}_pM^n_r$. Then the intersection of M^n_r and $E(p\ ;\ t)$ gives rise to a curve in a neighborhood of p which is called the normal section of M^n_r at p in the direction t. M^n_r is said to have planar normal sections if every normal section γ is a plane curve, that is, $\gamma \wedge \gamma' \wedge \gamma'' = 0$. M^n_r is said to have pointwise planar normal sections if $\gamma(p) \wedge \gamma'(p) \wedge \gamma''(p) = 0$ for every point $p \in M^n_r$.

§ 2. Surfaces in a pseudo-Euclidean space E^m_s with planar normal sections

We now suppose that a (pseudo-Riemannian) surface M_r^2 in a pseudo-Euclidean space E_s^m has planar normal sections. Let $M = M_r^2$ (r = 0, 1, 2) in order to make the matter simple.

Let p be a point of M and let t be a nonzero vector tangent to M at a point p and let γ be the normal section of M at p in the direction t. We assume $\gamma(0) = p$. Let $T = \gamma'(s)$, where s is a parameter (which is not necessarily the arc length). Then we have

(4.1) $$\gamma''(s) = \nabla_T T + \sigma(T, T),$$

(4.2) $$\gamma'''(s) = \nabla_T \nabla_T T + \sigma(\nabla_T T, T) - A_{\sigma(T, T)} T + D_T\, \sigma(T, T).$$

$\gamma'''(0)$ is a linear combination of $\gamma'(0)$ and $\gamma''(0)$ at $\gamma(0) = p$ because γ is planar. So we have

(4.3) $$\nabla_t T = a\, t \qquad \text{for some } a \in \mathbf{R},$$

(4.4) $$(\bar{\nabla}_t \sigma)(t, t) \wedge \sigma(t, t) = 0,$$

where t = T(0). But (4.4) is true for any point p and any nonzero vector t. Thus we have

(4.5) $$(\bar{\nabla}_X \sigma)(X, X) \wedge \sigma(X, X) = 0$$

for every $X \in TM$. In particular, if t is nonnull, that is, $< t, t > \neq 0$, then we can reduce (4.3) to

(4.6) $$\nabla_t T = 0.$$

Proposition 4.1. Let M be a surface in a pseudo-Euclidean space E^m_s. Let γ be a planar normal section of M at p in the direction of a unit vector t. If $\gamma''(0)$ is null, then $\sigma(t, t)$ is a null vector.

Proof. Since γ is a plane curve which lies in the plane spanned by t and $\sigma(t, t)$, we have

$$\gamma(s) = \gamma(0) + f(s)t + g(s)\sigma(t, t)$$

for some function f and g along γ. This implies

$$\gamma''(s) = f''(s)t + g''(s)\sigma(t, t).$$

On the other hand, we see that

$$\gamma''(0) = f''(0)t + g''(0)\,\sigma(t, t) = \sigma(t, t)$$

because of (4.6). Thus, $f''(0) = 0$ and $g''(0) = 1$. Since $\gamma''(0)$ is null, $\sigma(t, t)$ is a null vector. (Q. E. D.).

Let t be any nonzero tangent vector of M at p and let γ be the normal section of M at p in the direction t. Since $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$, γ''' is a linear combination of γ' and γ''. (4.1) and (4.2) imply

(4.7) $$\nabla_T \nabla_T T - A_{\sigma(T, T)} T = b(s)\, T + c(s)\, \nabla_T T,$$

(4.8) $$\sigma(\nabla_T T, T) + D_T\, \sigma(T, T) = c(s)\, \sigma(T, T)$$

for some functions b(s) and c(s).

Suppose that γ is a geodesic. Then for an affine parameter s,

$$\nabla_T T = 0.$$

Hence we have

$$\gamma'(s) = \sigma(T, T), \qquad \gamma''(s) = -\, A_{\sigma(T,\, T)}T \;+ D_T\, \sigma(T, T).$$

Since $\gamma \wedge \gamma' \wedge \gamma'' = 0$, we obtain

(4.9) $$A_{\sigma(T,\, T)}T \;\wedge T = 0,$$

that is,

$$< \sigma(T, T), \sigma(T, T^{\perp}) > = 0$$

for all $T^{\perp} \in TM$ such that $< T, T^{\perp} > = 0$.

Suppose that γ is not a geodesic. Then $\nabla_T T \neq 0$ for $s \neq 0$. Since $(\bar{\nabla}_T \sigma)(T, T) \wedge \sigma(T, T) = 0$, we get from (4.8)

(4.10) $$\sigma(\nabla_T T, T) \wedge \sigma(T, T) = 0.$$

In particular, if t ($\neq 0$) is not null, then by the arc length parametrization, we may assume

$$< \gamma'(s), \gamma'(s) > = \varepsilon, \qquad \varepsilon = \pm 1.$$

Choose an orthonormal frame $\{ e_1, e_2\}$ on a neighborhood U of p such that e_1 is an extension of $T = \gamma'(s)$. Then $\nabla_T T = \rho(s)\, e_2$ for some function ρ since $< \nabla_T T, T > = 0$.

By using continuity, we have

Lemma 4.2. Let M be a surface of a pseudo-Euclidean space E^m_s with planar normal sections. If the normal section γ of M at p in the direction of nonnull vector t is not a geodesic on a sufficiently small neighborhood, then we have

$$\sigma(t, t) \wedge \sigma(t, t^{\perp}) = 0,$$

where $t = \gamma'(0)$ and $t^{\perp}$ is a unit vector in T_pM perpendicular to t.

Lemma 4.3. Let M be a surface in a pseudo-Euclidean space E^m_s with planar normal sections. If the normal section γ is a geodesic arc on a neighborhood of p, then we have

$$< \sigma(t, t)\, ,\ \sigma(t, t^{\perp}) > = 0$$

where $t = \gamma'(0)$ and $t^{\perp}$ is a unit vector in T_pM perpendicular to t. (In this case, $t^{\perp}$ may be t if t is a null vector.)

If M has planar normal sections, then M clearly has pointwise planar normal sections. We now prove

Lemma 4.4 ([Ch.B-5]). Let M^n_r be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space E^m_s. If M^n_r has pointwise planar normal sections, then Im σ is parallel in the normal bundle.

Proof. As we derived (4.4), we can see that M^n_r has pointwise planar normal sections if and only if

$$(4.11) \qquad (\bar{\nabla}_X \sigma)(X, X) \wedge \sigma(X, X) = 0$$

for any $X \in TM^n_r$. For any two vectors Y, $Z \in TM^n_r$, if we set $X = Y + Z$, then (4.11) gives

$$(4.12) \qquad (\bar{\nabla}_Y \sigma)(Y, Z) + (\bar{\nabla}_Z \sigma)(Z, Y) \in \operatorname{Im} \sigma.$$

Similarly, if we set $X = Y - Z$, we have

$$(4.13) \qquad (\bar{\nabla}_Y \sigma)(Y, Z) - (\bar{\nabla}_Z \sigma)(Z, Y) \in \operatorname{Im} \sigma.$$

Combining (4.12) and (4.13), we obtain

$$(\bar{\nabla}_Y \sigma)(Y, Z) \in \operatorname{Im} \sigma.$$

Together with the Codazzi equation, this implies

$$(\bar{\nabla}_X \sigma)(Y, Z) \in \operatorname{Im} \sigma.$$

for any X, Y, Z $\in TM^n_r$. Thus we conclude that the first normal space Im σ is parallel in the normal bundle. (Q. E. D.)

Using this lemma and a reduction theorem , [CH.B-5], [E], we have

Proposition 4.5. Let M^n_r be a pseudo-Riemannian submanifold of E^m_s with dim (Im σ) = k for some integer k. If M^n_r has pointwise planar normal sections, then M^n_r lies fully in an affine space of dimension $(n+k) \subset E^m_s$.

We now talk about the dimension of the first normal space of a surface in E^m_s.

Proposition 4.6. Let M be a surface in a pseudo-Euclidean space E^m_s with index r (r = 0, 1, 2). Suppose that

$$< \sigma(X, X), \sigma(X, X^{\perp}) > \not\equiv 0$$

for any orthonormal basis $\{ X, X^{\perp} \}$ of T_pM, $p \in M$. Then dim $(\mathrm{Im}\,\sigma)_p \leq 1$.

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of T_pM such that

$$< \sigma(e_1, e_1), \sigma(e_1, e_2) > \neq 0.$$

Suppose that r = 0 or 2, that is, M is spacelike or timelike. By continuity, there is an open interval (a, b) of θ such that

$$< \sigma(e(\theta), e(\theta)), \sigma(e(\theta), e(\theta)^{\perp}) > \neq 0$$

for any $\theta \in (a, b)$, where

$$\begin{pmatrix} e(\theta) \\ e(\theta)^{\perp} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

By Lemma 4.2 and Lemma 4.3 we obtain

$$\sigma(e(\theta), e(\theta)) \wedge \sigma(e(\theta), e(\theta)^{\perp}) = 0$$

for all $\theta \in (a, b)$. A straightforward computation shows that

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) = \sigma(e_1, e_1) \wedge \sigma(e_2, e_2) = \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) = 0.$$

Suppose that $r = 1$, that is, M is Lorentzian.

Let

$$\begin{pmatrix} e(\theta) \\ e(\theta)^{\perp} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

By a similar argument to that above, we get

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) = \sigma(e_1, e_1) \wedge \sigma(e_2, e_2) = \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) = 0.$$

Therefore we conclude that $\dim (\mathrm{Im}\, \sigma)_p \leq 1$. (Q. E. D.)

Lemma 4.7. Let M be a surface of a pseudo-Euclidean space E^m_s. Then

$< \sigma(X, X), \quad \sigma(X, X) >$ does not depend on the choice of the unit vector $X \in T_pM$ if and only if $< \sigma(X, X), \quad \sigma(X, X^{\perp}) > = 0$ for any orthonormal basis $\{ X, X^{\perp} \}$ of T_pM.

Proof. The proof is by the same argument as in Lemma 1.1. (Q. E. D.)

Definition. Let M^n_r be a pseudo-Riemannian submanifold of a pseudo-Riemannian manifold $\tilde{M}^m_s$. M^n_r is said to be pseudo-isotropic at $p \in M^n_r$ if the length of the second fundamental form does not depend on the choice of unit vector in $T_pM^n_r$. M^n_r is said to be pseudo-isotropic if M^n_r is pseudo-isotropic at every point of M^n_r. If M^n_r is pseudo-isotropic and the length of the second fundamental form does not depend on the choice of the point, then we say M^n_r is constant pseudo-isotropic.

Lemma 4.8. Let M be a surface in a pseudo-Euclidean space E^m_s with index r (r = 0, 1, 2). If M is pseudo-isotropic at p, then we have

$$(4.14)\quad (-1)^r < \sigma(e_1, e_1), \sigma(e_1, e_1) > = <\sigma(e_1, e_1), \sigma(e_2, e_2) > + 2 < \sigma(e_1, e_2), \sigma(e_1, e_2) >,$$

where $\{ e_1, e_2 \}$ is an orthonormal basis for T_pM.

Proof. Let $\{ e_1, e_2 \}$ be an orthonormal basis for T_pM and let $\{ X, X^{\perp} \}$ be another orthonormal basis for T_pM.

Suppose that r = 0 or 2. Then there exists $\theta \in [0, 2\pi)$ such that

$$X = \cos\theta\, e_1 + \sin\theta\, e_2, \qquad X^{\perp} = -\sin\theta\, e_1 + \cos\theta\, e_2.$$

Substitution of X and $X^{\perp}$ into $< \sigma(X, X), \sigma(X, X^{\perp}) > = 0$ gives (4.14).

Suppose that r = 1. We put

$$X = \cosh\theta\, e_1 + \sinh\theta\, e_2, \qquad X^{\perp} = \sinh\theta\, e_1 + \cosh\theta\, e_2$$

for some θ. Substitution of X and $X^{\perp}$ into $< \sigma(X, X), \sigma(X, X^{\perp}) > = 0$ gives (4.14). (Q. E. D.)

C. Blomstrom, [B-2], generalized the notion of planar geodesic immersions in the Riemannian case to that of the pseodo-Riemannian case and she proved that if $i : M_r^n \to E_s^m$ is a planar geodesic immersion, then M_r^n is constant pseudo-isotropic. From now on, if M_r^n is constant pseudo-isotropic, then we denote $< \sigma(X, X), \sigma(X, X) >$ by L, where X is a unit vector tangent to M_r^n.

Lemma 4.9 ([B-2]). Let $i : M_r^n \to E_s^m$ be a planar geodesic immersion with $L = 0$. Then M_r^n is flat if and only if the first normal space $\mathrm{Im}\,\sigma$ is composed entirely of null vectors. In particular, if M_r^n is complete, then i must be (up to rigid motion) an expansion of E_r^n into $E_{r,k}^{n+k}$, where $k = \dim(\mathrm{Im}\,\sigma)$.

Proposition 4.10. Let M be a surface which is constant pseudo-isotropic in E_{r+1}^4 with $L = 0$. If M has planar normal sections, then $\mathrm{Im}\,\sigma$ is a 1-dimensional null space, that is, M lies in $E_{r,1}^3$.

Proof. Let $p \in M$. Then $T_p^{\perp}M$ is isomorphic to E_1^2. Let $\{ e_1, e_2 \}$ be an orthonormal basis for T_pM. Then we have

$$< \sigma(e_1, e_1), \sigma(e_1, e_2) > = 0.$$

Suppose $\sigma(e_1, e_1) \wedge \sigma(e_2, e_2) \neq 0$. If $\sigma(e_1, e_2) = 0$, by (4.14) and the assumption

L = 0 we obtain

$$< \sigma(e_1, e_1), \sigma(e_2, e_2) > = 0.$$

Thus , Span { $\sigma(e_1, e_1), \sigma(e_2, e_2)$ } is isomorphic to $E^2_{0,2}$. But this is impossible in $T^\perp_p M \cong E^2_1$. Thus we get $\sigma(e_1, e_2) \neq 0$. Since $< \sigma(e_1, e_1), \sigma(e_1, e_2) > = 0$ and since $T^\perp_p M$ is a Lorentzian vector space,

$$\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) = 0 \quad \text{and} \quad \sigma(e_2, e_2) \wedge \sigma(e_1, e_2) = 0.$$

In this case, we also have Span { $\sigma(e_1, e_1), \sigma(e_2, e_2)$ } is isomorphic to $E^2_{0,2}$. Hence we have

$$\sigma(e_1, e_1) \wedge \sigma(e_2, e_2) = \sigma(e_1, e_2) \wedge \sigma(e_1, e_1) = \sigma(e_1, e_2) \wedge \sigma(e_2, e_2) = 0.$$

Consequently, Im σ is a 1-dimensional null space at every point of M. By Lemma 4.4, Im σ is parallel in the normal bundle. Thus M locally lies in $E^3_{r,1}$. (Q. E. D.)

Remark. Under the same assumptions as in Proposition 4.10, M is flat by the similar proof of Lemma 4.9. Thus, if M is complete, then $i : M \rightarrow E^m_s$ must be an expansion of E^2_r into $E^3_{r,1}$.

Blomstrom, [B-2], proved

Proposition 4.11. If $i : M \rightarrow E^5_s$ is parallel and full, then

$$<\sigma(e_1, e_1), \sigma(e_2, e_2)> - 2 < \sigma(e_1, e_2), \sigma(e_1, e_2)> = 0,$$

where $\{ e_1, e_2 \}$ is an orthonormal basis of T_pM.

Proposition 4.12. Let M be a constant pseudo-isotropic Lorentzian surface with L = 0. Then

$$<\sigma(t, t), \sigma(t, t)> = 0$$

for all null vector $t \in T_pM$.

Proof. Since M is pseudo-isotropic,

$$< \sigma(X, X), \ \sigma(X, X^{\perp})> = 0$$

for any orthonormal basis $\{ X, X^{\perp} \}$ of T_pM. Let t be a null vector of T_pM. Then

$$t = a X + a X^{\perp}$$

for some $a \ (\neq 0) \in \mathbf{R}$. By using (4.14), we have

$$<\sigma(t, t), \sigma(t, t)> \ = 2a^2 \{ < \sigma(X, X), \ \sigma(X^{\perp}, X^{\perp})> + 2 < \sigma(X, X^{\perp}), \sigma(X, X^{\perp})> \} = 0.$$

(Q. E. D.)

Proposition 4.13. Let M be a constant pseudo-isotropic surface of a pseudo-Euclidean space E^m_s with index r (r = 0, 1, 2) which has planar normal sections. If L =0, then M has planar geodesics.

Proof. Suppose $r = 1$. Let $t \neq 0$ be a nonnull vector at $p \in M$ and let γ be the normal section of M at $p = \gamma(0)$ in the direction t. Since $< t, t > \neq 0$, we may assume that γ is parametrized by the arc length s. So $< T(s), T(s) > = \varepsilon$ for sufficiently small s, where $\gamma'(s) = T(s)$ and $\gamma'(0) = t$.

Suppose $\sigma(t, t) \neq 0$. Then there exists a neighborhood U of p such that $\sigma(u, u) \neq 0$ for any unit vector $u \in T(U)$. Since γ is a plane curve which lies in a plane spanned by t and $\sigma(t, t)$, we may express

$$\gamma(s) = p + f(s)t + g(s)\sigma(t, t) \tag{4.15}$$

for some functions f and g defined on an interval I. Then we get

$$\gamma'(s) = f'(s)\, t + g'(s)\sigma(t, t),$$

$$\gamma''(s) = f''(s)\, t + g''(s)\, \sigma(t, t).$$

Since $< \gamma'(s), \gamma'(s) > = \varepsilon$ and since $L = 0$, we get

$$(f'(s))^2 = 1.$$

Thus the last equation becomes

$$\gamma''(s) = g''(s)\, \sigma(t, t).$$

Therefore, $< \gamma''(s), \gamma''(s) > = 0$ on I.

On the other hand, $\gamma''(s) = \nabla_T T + \sigma(T, T)$ and thus $< \nabla_T T, \nabla_T T > = 0$ because of

$L = 0$. Choose an orthonormal frame $\{ T, T^{\perp} \}$ along γ. Then $\nabla_T T \wedge T^{\perp} = 0$ and hence $\nabla_T T = 0$ on I. Thus γ is a geodesic arc.

Suppose $\sigma(T, T) = 0$ for some interval containing 0 , where $\gamma(0) = p$. Then γ is clearly a geodesic arc.

Suppose $\sigma(t, t) = 0$ and $\sigma(T, T) \neq 0$ for $s > 0$. Since $L = 0$, $\sigma(T, T)$ is null for $s > 0$. Extend $\sigma(T, T)$ up to $p = \gamma(0)$ and denote it by n. Then

$$\gamma(s) = p + f_1(s)t + g_1(s)\, n$$

for some functions f_1 and g_1. Using an argument similar to the one developed above, we see that γ is a geodesic arc.

Let t be a null vector tangent to M at p. Let γ be the normal section of M at p in the direction t and let $\gamma(0) = p$ and $\gamma'(0) = t$. By Proposition 4.12, we have

$$< \sigma(t, t), \sigma(t, t) > = 0.$$

We may put

$$\gamma(s) = p + f(s)t + g(s)\, \sigma(t, t)$$

for some parameter s and some functions f and g. Let $\gamma'(s) = T$. Then , we get

$$T = f'(s)t + g'(s)\, \sigma(t, t)$$

and

$$\nabla_T T + \sigma(T, T) = f''(s)t + g''(s)\, \sigma(t, t).$$

Since $< t, t > = < \sigma(t, t), \sigma(t, t) > = 0$, we have $< T, T > = 0$ and hence $< \sigma(T, T), \sigma(T, T) > = 0$ by means of Proposition 4.12. Thus we obtain

$$< \nabla_T T , \nabla_T T > = 0.$$

Suppose there is s_o such that $T(s_o) \wedge \nabla_{T(so)} T \neq 0$. Then $T(s_o)$ and $\nabla_{T(so)} T$ form a degenerate plane $E^2_{0,2}$. But this is impossible because $T_{\gamma(so)} M \cong E^2_1$. Thus we have

$$T \wedge \nabla_T T = 0.$$

So $\nabla_T T = h(s)\, T$ for some function h. γ is indeed a pregeodesic, that is, it has a parametrization as a geodesic. Thus, γ becomes a geodesic $\tilde{\gamma}$ by changing its parameter such that $\tilde{\gamma}(s) = (\gamma \circ a)(s)$ satisfying $a'' + h(a')^2 = 0$.

We now suppose that the surface M is spacelike or timelike, that is, $r = 0$ or 2.

Let p be a point of M and γ be a normal section of M at p in the direction t ($\neq 0$). Using the exact same argument as that of the first half of the case $r = 1$, we see that γ is a geodesic arc.

For a given nonzero vector t, the geodesic with initial velocity vector t is unique and hence it is the normal section. Therefore, M has planar geodesics. (Q. E. D.)

Lemma 4.14. Let M be a constant pseudo-isotropic surface in a pseudo-Euclidean space E^m_s with $L = 0$. If M has planar normal sections and if the mean curvature vector H is parallel in the normal bundle, then Im σ is spanned by null vectors.

Proof. Let $\{ e_1, e_2 \}$ be an orthonormal basis of T_pM, $p \in M$. Then the mean curvature vector H is given by

$$H = \frac{1}{2}\{ < e_1, e_1 > \sigma(e_1, e_1) + < e_2, e_2 > \sigma(e_2, e_2) \},$$

which implies

$$< H, H > = \frac{1}{2}(-1)^r < \sigma(e_1, e_1), \sigma(e_2, e_2) >.$$

Suppose that $< H, H > \neq 0$. Extend e_1 along the geodesic $\gamma_1(s) = \exp_p(s\, e_1)$ which is denoted by E_1 and extend e_2 by parallel displacement along γ_1 which is denoted by E_2. By Lemma 4.8 and $L = 0$, we get

$$< \sigma(E_1, E_1), \sigma(E_2, E_2) > = (-1)^{r+1} < H, H >.$$

Since the mean curvature vector is parallel in the normal bundle, we get

$$E_1(p) \;< \sigma(E_1, E_2), \sigma(E_1, E_2) > = 0.$$

Since $\nabla_{E_1} E_1 = 0$, we also get $\nabla_{E_1} E_2 = 0$. Thus we obtain

$$(4.16) \qquad < (\bar{\nabla}_{e_1}\sigma)(e_1, e_2), \sigma(e_1, e_2) > = 0.$$

On the other hand,

$$(4.17) \qquad 0 = E_1(p) < H, H >$$

$$= E_1(p) \;< \sigma(E_1, E_1), \sigma(E_2, E_2) >$$

$$= < (\bar{\nabla}_{e_1}\sigma)(e_1, e_1), \sigma(e_2, e_2) >$$

$$+ < \sigma(e_1, e_1), (\bar{\nabla}_{e_1}\sigma)(e_2, e_2) >.$$

Extend e_2 along the geodesic $\gamma_2(s) = \exp_p(s\, e_2)$ and denote by $\tilde{E}_2$ and extend e_1 to $\tilde{E}_1$ by parallel displacement along γ_2. Then we get

$$0 = \tilde{E}_2(p) < \sigma(\tilde{E}_1, \tilde{E}_1), \sigma(\tilde{E}_1, \tilde{E}_2) > \tag{4.18}$$

$$= < (\bar{\nabla}_{e_2}\sigma)(e_1, e_1), \sigma(e_1, e_2) >$$

$$+ < \sigma(e_1, e_1), (\bar{\nabla}_{e_2}\sigma)(e_1, e_2) >.$$

Combining (4.16), (4.17) and (4.18) and making use of (4.5), we obtain

$$(\bar{\nabla}_{e_1}\sigma)(e_1, e_1) = 0.$$

Since e_1 can be chosen arbitrarily, we have

$$\bar{\nabla}\sigma = 0,$$

that is, the immersion $i : M \to E^m_s$ is parallel.

Since $\dim(\mathrm{Im}\,\sigma) \leq 3$, $i : M \to E^m_s$ is full in at most a 5-dimesional subspace of E^m_s by Lemma 4.4. However, Proposition 4.10 implies that i cannot be full in E^4_{r+1}. If i is full in E^5_t, $< \sigma(e_1,e_1), \sigma(e_2,e_2) > = 0$ by Lemma 4.8 and Proposition 4.11. This contradicts $< H, H > \neq 0$. If $\dim(\mathrm{Im}\,\sigma) = 1$, $\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) = 0$, $\sigma(e_1, e_1) \wedge \sigma(e_2, e_2) = 0$ and $\sigma(e_1, e_2) \wedge \sigma(e_2, e_2) = 0$. Thus $\mathrm{Im}\,\sigma$ is a 1-dimensional null space. So, i cannot be full in E^3_s. Thus i is necessarily full in a degenerate 3, 4 or 5-dimensional affine space of E^m_s. Therefore, the scalar product of one of $\sigma(e_1, e_1)$, $\sigma(e_1, e_2)$ and $\sigma(e_2, e_2)$ with any vector vanishes. By considering Lemma 4.8, we get

$$< \sigma(e_1, e_1),\ \sigma(e_2, e_2) > = 0.$$

Consequently, $< H, H > = 0$ at p. Since p is arbitrary, $< H, H >$ vanishes identically on M. From this and Lemma 4.8, we conclude that $\mathrm{Im}\,\sigma$ is spanned by null vectors. (Q. E. D.)

Theorem 4.15. Let M be a constant pseudo-isotropic surface in E^m_s with planar normal sections and $L = 0$. If the mean curvature vector field H is parallel in the normal bundle, then M is flat. Moreover, if M is complete, then $i : M \to E^m_s$ is an expansion of E^2_r which is full in $E^{2+k}_{r,k}$ $(k = 1, 2, 3)$.

Proof. By Lemma 4.14, $< H, H > = 0$ and $\mathrm{Im}\,\sigma$ is generated by null vectors. We see that M is flat by considering the Gauss equation (0.7). So, if M is complete, then $i : M \to E^m_s$ is an expansion of E^2_r into $E^{2+k}_{r,k}$ $(k = 1, 2, 3)$ by Lemma 4.9. (Q. E. D.)

We now prove

Lemma 4.16. Let M^n_r be an n-dimensional pseudo-Riemannian submanifold in a pseudo-Euclidean space E^m_s with pointwise planar normal sections. If M^n_r is pseudo-

isotropic, then M_r^n is constant pseudo-isotropic.

Proof. Let $< \sigma(X, X), \sigma(X, X) > = f(p)$ for unit vectors $X \in T_pM_r^n$. Then f is a function on M_r^n. We shall prove that f is constant.

Choose a point $p \in M_r^n$ and a geodesic γ emanating from p with nonnull initial velocity vector t. We assume that γ is parametrized by the arc length s. Let $t^\perp$ be a unit vector in $T_pM_r^n$ such that $< t, t^\perp > = 0$. Extend $t^\perp$ to a vector field $T^\perp$ tangent to M_r^n which is parallel along γ and extend $t = \gamma'(0)$ on a neighborhood of $p = \gamma(0)$ such that $T = \gamma'(s)$ stays a unit vector field orthogonal to $T^\perp$. Since M_r^n is pseudo-isotropic and M has pointwise planar normal sections, we get

$$
\begin{aligned}
0 &= t < \sigma(T, T), \sigma(T, T^\perp) > \\
&= < D_t\sigma(T, T), \ \sigma(t, t^\perp) > + < \sigma(t, t), D_t\sigma(T, T^\perp) > \\
&= \ < \sigma(t, t), D_t\sigma(T, T^\perp) > \text{(because } D_t\sigma(T, T) \wedge \sigma(t, t) = 0) \\
&= \ < \sigma(t, t), (\bar{\nabla}_t\sigma)(t, t^\perp) > \quad \text{(because } \nabla_t T^\perp = 0) \\
&= < \sigma(t, t), (\bar{\nabla}_{t^\perp}\sigma)(t, t) > \\
&= \frac{1}{2}\, t^\perp < \sigma(T, T), \sigma(T, T) > \\
&= \frac{1}{2}\, t^\perp(f).
\end{aligned}
$$

Since $\dim M_r^n \geq 2$, f is a constant. So, M_r^n is constant pseudo-isotropic. (Q. E. D.)

Theorem 4.17. (Classification). Let M be a surface in a pseudo-Euclidean space E^m_s with planar normal sections. If M does not lie in a 3-dimensional affine space of E^m_s and if the mean curvature vector field H is parallel in the normal bundle, then M is an open portion of either a flat surface which locally lies in $E^{2+k}_{r,k}$ (k = 2, 3) or a Veronese surface in $E^5_{r(3-r)}$ or $E^5_{5-(r+1)(2-r)}$.

Proof. Let $M_1 = \{ p \in M \mid \dim(\mathrm{Im}\, \sigma) \leq 1 \text{ at } p \}$. Suppose $M = M_1$. Since Im σ is parallel in the normal bundle, we see that M lies in a 3-dimensional affine space of E^m_s. If $M - M_1 \neq \emptyset$, then $M - M_1$ is an open subset of M. Let U be a component of $M - M_1$. Proposition 4.6 implies that M is pseudo-isotropic. By Lemma 4.16, we see that U is constant pseudo-isotropic. Suppose L = 0 on U. Then Proposition 4.13 tells us that U has planar geodesics. Lemma 4.14 and the Gauss equation (0.7) imply that U is flat and fully lies in $E^{2+k}_{r,k}$ (k = 2, 3). Suppose $L \neq 0$. Since U has planar normal sections, $(\bar{\nabla}_X \sigma)(X, X) \wedge \sigma(X, X) = 0$ for $X \in TU$. Since L is constant on U, $(\bar{\nabla}_X \sigma)(X, X) = 0$ for all $X \in TU$. Thus the second fundamental form σ is parallel in the normal bundle, that is, $\bar{\nabla}\sigma = 0$. Let $\{ e_1, e_2 \}$ be an orthonormal basis of T_pU for some $p \in U$. If $\sigma(e_1, e_2) = 0$, then Im σ is 1-dimensional at p by Lemma 4.8. Therefore, $\sigma(e_1, e_2) \neq 0$ for any orthonormal frame $\{ e_1, e_2 \}$ on U. We also have $\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) \neq 0$. In fact, if $\sigma(e_1, e_1) \wedge \sigma(e_1, e_2) = 0$, then $\sigma(e_1, e_1)$ is a null vector. This contradicts $L \neq 0$. Making use of Lemma 4.2 and Lemma 4.3, we can conclude that U has planar geodesic normal sections. By the classification Theorem of Blomstrom, [B-2], U is an open portion of a Veronese surface in $E^5_{r(3-r)}$ or $E^5_{5-(r+1)(2-r)}$. By continuity of L, we conclude that U = M. This completes the proof. (Q. E. D.)

SUMMARY

Let M be a surface of a Euclidean space E^m. We define the property $(*_1)$, $(*_2)$ and $(*_3)$ as follows :

$(*_1)$: There is a point p in M such that every geodesic through p is a helix of the same curvatures.

$(*_2)$: There is a point p in M such that every geodesic through p is a normal section of M at p.

$(*_3)$: There is a point p in M such that every geodesic through p is planar.

Then we obtain the following results.

(1) Let M be a complete connected surface in E^3. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere or a plane.

(2) Let M be a compact connected surface in E^4. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere which lies in E^3 or a Blaschke surface at a point which lies in E^4 of the form (1.15) and is diffeomorphic to RP^2.

(3) Let M be a compact connected surface in E^5. Then M satisfies the property $(*_1)$ if and only if M is a standard sphere in E^3 or a Blaschke surface at a point of the form (1.37) which lies in E^4 or a Blaschke surface at a point of the form (1.40) which lies in E^5 or a Blaschke surface at a point of the form (1.32) which lies in E^5. All such pointed Blaschke surfaces are diffeomorphic to RP^2.

(4) Let M be a compact connected surface in E^m ($m \geq 5$). Then M satisfies the property $(*_1)$ and $(*_3)$ if and only if M lies in E^5 and M is one of four model spaces stated in (3).

(5) Let M be a compact connected surface in E^5. Then M is a Veronese surface if and only if M has a constant Gaussian curvature and satisfies the property $(*_1)$ and the base point is

not umbilical.

(6) Let M be a surface in E^3. Then M satisfies the property $(*_2)$ if and only if M is locally a surface of revolution.

(7) Let M be a complete connected surface in E^3. Then M satisfies the property $(*_2)$ if and only if M is a surface of revolution.

(8) Let M be a surface in E^3. Then the property $(*_1)$ is equivalent to $(*_3)$.

(9) Let M be a surface in E^4 without isolated flat points in E^4. Then M satisfies the property $(*_3)$ if and only if M is locally a surface of revolution which lies in E^3 or a surface that locally has the form (3.15).

(10) Let M be a surface in E^5 without isolated flat points. Then M satisfies the property $(*_3)$ if and only if M is locally a surface of revolution in E^3 or a surface in E^4 which has the form (3.15) or a surface of the form (3.17) which fully lies in E^5.

(11) Let M be a surface in E^m $(m \geq 3)$ without isolated flat points. Then M satisfies the property $(*_3)$ if and only if M lies locally in E^5 and M is one of three model spaces stated in (10).

(12) If an analytic surface M in E^m satisfies the property $(*_3)$ whose base point is an isolated flat point of degree $p\text{-}1(p > 1)$, then M is locally of the form (3.25) and M lies in a linear subspace of dimension $\leq p+5$.

(13) Let M be an analytic surface in E^m. If M satisfies the property $(*_3)$, then M is one of the following :

(a) M is locally a surface of revolution in E^3,

(b) M is a surface of the form (3.15) in E^4,

(c) M is a surface of the form (3.17) in E^4,

(d) M is a surface of the form (3.25) in E^{p+5}. In this case, the base point of the property $(*_3)$ is an isolated flat point of degree p -1.

(14) Let M be a surface which is constant pseudo-isotropic in E^4_{r+1} with $L = 0$. If M has

planar normal sections, then Im σ is a 1-dimensional null space, that is, M lies in $E^3_{r,1}$.

(15) Let M be a constant pseudo-isotropic surface of a pseudo-Euclidean space E^m_s with index r (r = 0, 1, 2) which has planar normal sections. If L =0, then M has planar geodesics.

(16) Let M be a surface in a pseudo-Euclidean space E^m_s with planar normal sections. If M does not lie in a 3-dimensional affine space of E^m_s and if the mean curvature vector field H is parallel in the normal bundle, then M is an open portion of either a flat surface which locally lies in $E^{2+k}_{r,k}$ (k = 2, 3) or a Veronese surface in $E^5_{r(3-r)}$ or $E^5_{5-(r+1)(2-r)}$.

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