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BIFURCATION OF PERIODIC ORBITS OF

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BIFURCATION OF PERIODIC ORBITS OF NONPOSITIVE DEFINITE HAMILTONIAN SYSTEMS

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ABSTRACT

BIFURCATION OF PERIODIC ORBITS OF NONPOSITIVE DEFINITE HAMILTONIAN SYSTEMS

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In this thesis, we consider the bifurcations of periodic solutions of a family of non-positive definite Hamiltonian systems of n degrees of freedom near the origin as the family passes through a semisimple resonance.

We begin with a smooth Hamiltonian H with a general semisimple quadratic part H_2 and then construct a normal form of H with respect to H_2 up to fourth order terms and make a versal deformation.

We apply the Liapunov-schmidt reduction in the presence of symmetry and further reduce the resulting bifurcation equation to a gradient system. Thus, the study of periodic solutions of the orginal system is reudced to finding critical points of a real-valued function.

As an application, we consider a system with two degrees of freedom in 1: -1 semisimple resonance by using suitable choices of the parameters to study the bifurcation as the eigenvalues split along the imaginary axis or across it and we obtain complete bifurcation patterns of periodic orbits on each energy level.

To my father and mother

who gave me mental heritage.

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TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

This thesis is mainly concerned with the study of the bifurcations of periodic solutions of a family of non-positive definite Hamiltonian systems of two degrees of freedom near an equilibrium as the family passes through the 1:-1 semisimple resonance.

We start with a smooth (C^∞) Hamiltonian function $H = H_2 + H_3 + H_4 + \ldots$ with a given normalized quadratic part

$$(0.1) \qquad H_2(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2)$$

and construct a normal form of H with respect to H_2 up to fourth order terms and make a versal deformation of H_2 to study the corresponding Hamiltonian system as parameters pass through the resonance at $\lambda = 0$, where the linearized system has two equal pairs of purely imaginary eigenvalues. In our case of 1: -1 semisimple resonance, the normal form contains nine fourth order terms and the versal deformation requires four parameters which is extremely difficult to perform complete analysis about the dynamical behavior of the system as λ varies and so we restrict ourselves to the truncated Hamiltonian containing only one fourth order term and to the codimension one bifurcations by suitably choosing one parameter so that the eigenvalues of the linearized system split <u>along</u> the imaginary axis or <u>across</u> it as λ varies across zero.

The study of non-positive definite Hamiltonian systems has been little done so far and the informations about such systems are little known.

In the case of a Hamiltonian with the normalized semi-simple quadratic part of the form

$$(0.2) \qquad H_2(x,y) = \sum_{j=1}^{n} \frac{1}{2} \lambda_j (x_j^2 + y_j^2),$$

Liapunov (1947) proved that if $\frac{\lambda_k}{\lambda_l} \neq$ integer for all $k \neq \ell$ (non-resonance condition), there exist n one-parameter familes of periodic solutions (see also Siegal and Moser [39], Hale [28]), and later Weinstein [44] removed the nonresonance condition and showed that if $\lambda_j > 0$ for all j, there exist at least n distinct periodic orbits on each energy level $H(z) = c$ for $0 < c << 1$.

The essential point in Weinstein's proof is that the condition of positive definiteness of the Hessian matrix $D^2H(0)$ implies the compactness of the energy surface $H(z) = c$ for small $c > 0$ and so one can apply either a theorem of Krasnoselski or the theory of Lyusternik-Schnirelman to obtain the desired result.

If the Hamiltonian is not positive definite, however, then the energy surface $H(z) = c$ is no longer compact and so the situation is more complicated. Moser [39] presented an example in which $D^2H(0) = \text{diag}\,(1, -1, 1, -1)$ and the Hamiltonian system possesses no nontrial periodic solutions.

More significantly, Chow and Mallet-Paret [14] proved that if H has the form

$$(0.3) \qquad H(z) = \frac{1}{2}\sum_{j=1}^{l}(x_j^2 + y_j^2) - \frac{1}{2}\sum_{j=l+1}^{n}(x_j^2 + y_j^2) + O(|z|^3),$$

and is analytic, then the corresponding Hamiltonian system

$$\dot{z} = J\nabla H(z), \text{ where } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \; I_n = n \times n \text{ identity matrix}$$

actually possesses at least $|n-2l|$ one-parameter families of periodic solutions near the origin provided that there are no 2π-periodic solutions on the zero energy level $H(z) = 0$. If $\ell = n$, then $H(z)$ is positive definite and clearly there are no 2π-periodic solutions on the surface $H(z) = 0$ and hence this result recovers a part of Weinstein's theorem. However, if $\ell = \frac{n}{2}$, e.g., $n = 2$ and $\ell = 1$ (i.e., 1: -1 resonance) then this result doesn't give any information about the existence of periodic orbits and actually Moser's example shows the nonexistence of nontrivial periodic solutions.

Recently, van der Meer [35] studied the periodic solutions of a family of Hamiltonian systems passing through the 1: -1 <u>nonsemisimple</u> resonance by examining the fibres of the normalized energy-momentum mapping by using the singularity theory of equivariant mappings.

In this thesis, we study the same 1: -1 resonance but the <u>semisimple</u> case which has four parameters in a versal deformation of H_2 and nine fourth order terms in the normal form in contrast to the <u>non-semisimple</u> case which contains two versal deformation parameters and three fourth order terms in the H_2-normal form.

Moreover, our approach to examining the periodic solutions, after normalization, is a local analysis by using the theory of Liapunov-Schmidt reduction in the presence of symmetry and reducing the resulting bifurcation equation to gradient system and studing the critical points of the reduced gradient system. We use the Lagrange multiplier method and take advantage of the equivariance symmetry of the gradiant system to solve it in a closed form.

This thesis is organized as follows. In chapter 1, we give a brief outline about the Hamiltonian systems and the theory of Hamiltonian normal forms. In Chapter 2, we introduce the theory of a versal deformation of linear systems and construct a versal deformation of H_2 given in (0.3). In Chapter 3, we use the Liapunov-schmidt reduction to examine the periodic orbits of a family of Hamiltonian systems in a normal form and obtain a real-valued function whose critical points correspond to periodic solutions of the original Hamiltonian systems. The summary of our method will be stated in Theorem 3.3.3 as a main theorem of this thesis. Finally, in Chapter 4, we apply our method in Chapter 3 to the 1: − 1 semisimple resonance problem with H_2 given by (0.1) under some restriction on the parameters and nonlinear terms and obtain explicit bifurcation results which will be summarized in Theorem 4.4.1 and Theorem 4.5.2. We conclude with a remark about the extension to a nearby nonintegrable systems and other possible methodologies to examine the periodic solutions.

CHAPTER 1: HAMILTONIAN SYSTEMS AND NORMAL FORMS

In this chapter, we will give a brief review of some basic facts about the Hamiltonian mechanics, and normal forms of Hamiltonian functions which form a background in the following chapters. Even though we are mainly working on the Euclidean space $\mathbb{R}^{2n}$, the basic structure of Hamiltonian systems will be given in the context of symplectic manifolds since the phase space of a Hamiltonian system is generally a manifold rather than Euchidean space especially when constraints are present.

Most definitions and theorem will be stated without proof. For the proofs and more detailed treatments of the above basic theories, we refer to the textbooks of Abraham and Marsden [1] and Arnold [3] and the lecture note of Cushman [22] and the thesis of van der Meer [35].

§1. Hamiltonian mechanics.

Let M be a smooth connected manifold.

Definition 1.1.1. A symplectic form ω on M is a closed, nondegenerate 2-form on M, that is, $d\omega = 0$ and for each $m \in M$, the skew-symmetric bilinear mapping $\omega(m)$: $T_mM \times T_mM \to \mathbb{R}$ is nondegenerate (i.e., $\omega(m)(v_m, w_m) = 0$ for all $w_m \in T_mM$ implies $v_m = 0$.)
The pair (M,ω) is called a symplectic manifold.

Theorem 1.1.2. Let $\omega \in \Omega^2(M)$, i.e., a 2-form on M. Then ω is nondegenerate iff M is even-dimensional, say 2n.

Definition 1.1.3. Let (M,ω) be a symplectic manifold and $H: M \rightarrow \mathbb{R}$ a given c^r function, $r \geq 1$. The vector field X_H determined by the condition

$$\omega(X_H,Y) = dH \cdot Y \tag{1.1.1}$$

is called the Hamiltonian vector field with Hamiltonian function H. We call (M,ω,X_H) a Hamiltonian system.

We will suppose H to be C^∞ in the following. Note that the nondegeneracy of ω guarantees the existence of X_H, which is a c^{r-1} vector field. Indeed, since $\omega(m)$ is nondegenerate, the linear map $\omega^\#(m)$: $T_mM \rightarrow T_m^*M$ defined by $\omega^\#(m)(v_m)\cdot w_m = \omega(m)(v_m,w_m)$ for all $w_m \in T_mM$, is invertible. Since $dH(m) \in T_m^*M$, we have

$$X_H(m) = \omega^\#(m)^{-1} \cdot dH(m) \in T_mM.$$

Let $\mathfrak{X}(M)$ be the space of smooth vector fields on M and $\mathfrak{X}^*(M)$ be its dual space, i.e., the space $\Omega^1(M)$ of one-form fields on M.
For $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^2(M)$, define $i_X\omega \in \mathfrak{X}^*(M)$ by $i_X\omega(Y) = \omega(X,Y)$. We call $i_X\omega$ the inner product of X and ω. Then, alternatively, we may define the Hamiltonian vector field by the relation

(1.1.2) $$i_{X_H}\omega = dH.$$

That is, for each $m \in M$ and each $v_m \in T_mM$,

$$dH(m) \cdot v_m = (i_{X_H}\omega)(v_m) = \omega(m)(X_H(m), v_m).$$

The following theorem shows that the definition 1.1.3 is locally equivalent to the classical one.

Theorem 1.1.4. (Darboux). (M,ω) is a symplictic manifold iff there is a chart (U,ϕ) at each $m \in M$ such that $\phi(m) = 0$, and with $\phi(u) = (x^1(u), \ldots, x^n(u), y^1(u), \ldots y^n(u))$, we have

$$\omega|U = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

The charts (U,ϕ) guaranteed by Darboux's theorem are called symplectic charts and the coordinate functions x^i, y^i are called canonical or symplectic coordinates.

Theorem 1.1.5. Let $(x_1, \ldots x_n, y_1, \ldots y_n)$ be canonical coordinates for ω, so $\omega = \Sigma\, dx_i \wedge dy_i$. Then in these coordinates

(1.1.3) $$X_H = \left(\frac{\partial H}{\partial y_i}, -\frac{\partial H}{\partial x_i}\right) = J \cdot dH$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Thus, $(x(t), y(t))$ is an integral curve of X_H iff Hamilton's equation hold:

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n.$$

<u>proof</u> $i_{X_H}\omega = \Sigma\, i_{X_H}(dx_i \wedge dy_i) = \Sigma\, (i_{X_H} dx_i) \wedge dy_i - \Sigma\, dx_i \wedge (i_{X_H} dy_i)$

$$dH = \Sigma \left(\frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial y_i} dy_i\right)$$

we have

$$i_{X_H} dx_i = \frac{\partial H}{\partial y_i}, \quad i_{X_H} dy_i = -\frac{\partial H}{\partial x_i}$$

that is,

$$X_H = \left(\frac{\partial H}{\partial y_i}, -\frac{\partial H}{\partial x_i}\right) = J \cdot dH. \qquad ///$$

Note that if $M = \mathbb{R}^{2n}$, then we have global canonical coordinates (x_i, y_i) and $\omega = \Sigma\, dx_i \wedge dy_i$, hence the Hamilton's equation in $(\mathbb{R}^{2n}, \omega = \Sigma\, dx_i \wedge dy_i)$ is globally given by (1.3), which is our classical definition of Hamiltonian system. $(\mathbb{R}^{2n}, \omega)$ is called the standard symplectic space.

The conservation of energy for the Hamiltonian system is given by the following theorem.

<u>Theorem. 1.1.6.</u> Let (M, ω, X_H) be a Hamiltonian system and $\gamma(t)$ be an integral curve for X_H, that is, $\frac{d\gamma(t)}{dt} = X_H(\gamma(t))$. Then $H(\gamma(t))$ = constant in t.

The next basic fact about the Hamiltonian systems is that their flows consist of canonical transformaitons.

Definition 1.1.7. A C^∞- map $F:(M,\omega) \to (M,\omega)$ is called a symplectic or canonical transformation if $F^*\omega = \omega$, where $F^*: \Omega^2(M) \to \Omega^2(M)$ is defined by

$$(F^*\omega)(m)(v_m, w_m) = \omega(F(m)) \cdot (dF(m)\cdot v_m, dF(m) \cdot w_m)$$

for $v_m, w_m \in T_mM$.

Theorem 1.1.8. Let (M,ω,X_H) be a Hamiltonian system and ϕ_t be the local flow of X_H. then, for each t, $\phi_t^*\omega = \omega$, that is, ϕ_t is a local one-parameter group of symplectic diffeomorphisms (on its domain). Thus ϕ_t also preserves the phase volume Ω_ω (Liouville's theorem).

Theorem 1.1.9. If ϕ is a symplectic diffeomorphism of (M,ω), then $\phi^*X_H = X_{\phi^*H}$ for every $H \in C^\infty(M)$. That is, a symplectic change of coordinates maps a Hamiltonian vector field into a Hamiltonian vector field with Hamiltonian ϕ^*H.

Definition 1.1.10. For $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, define

$$L_xf \in C^\infty(M) \text{ by } (L_xf)(m) = df(m)\cdot X(m).$$

We call L_xf the Lie derivative of f with respect to X.

Note that if $H \in C^\infty(\mathbb{R}^{2n},\omega)$ with $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$, then

(1.1.4) $$L_{X_H} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}\right).$$

Now, we introduce the definition of Poisson bracket on $C^\infty(M,\mathbb{R})$ to impose a Lie algebra structure on $C^\infty(M,\mathbb{R})$.

Definition 1.1.11: Let (M,ω) be a symplectic manifold and let $f, g \in C^\infty(M,\mathbb{R})$. The Poisson bracket $\{\ ,\ \}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is defined by

(1.1.5) $$\{f,g\}(m) = \omega(m)(X_g(m), X_f(m)) \quad \text{for each } m \in M.$$

Notice that from (1.1.1) and (1.1.4) and (1.1.5), we may write

(1.1.6) $$\{f,g\} = dg \cdot X_f = L_{X_f} g = -L_{X_g} f.$$

Since ω is skew-symmetric, so is $\{\ ,\ \}$. Thus, in canonical coordinates (x_i,y_i), $\{f,g\}$ may be written as

(1.1.7) $$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}\right).$$

From (1.1.6), it is clear that f is constant along the orbits of X_g (or g is constant along the orbits of X_f) iff $\{f,g\} = 0$. Note that $\{f,f\} = 0$ corresponds to conservation of energy for the Hamiltonian system (M,ω,f). We say that $F \in C^\infty(M,\mathbb{R})$ is an <u>integral</u> for the system (M,ω,H) if $\{H,F\} = 0$.

Definition 1.1.12: A Lie algebra is a vector space V with a bilinear operation [,] satisfying

(i) $[X,X] = 0$ for all $X \in V$ and

(ii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi identity)

for all $X,Y,Z, \in V$.

Since $\{\ ,\ \}: C^{\infty}(M) \times C^{\infty}(M) \to C^{\infty}(M)$ is a skew-symmetric bilinear form and satifies Jacobi identity, the real vector space $(C^{\infty}(M,\mathbb{R}), \{\ ,\ \})$ together with the Poisson bracket is a Lie algebra.

Theorem 1.1.13: ϕ is a symplectic diffcomorphism of (M,ω) iff ϕ preserves Poisson brackets, that is,

$$\phi^{*}\{f,g\} = \{\phi^{*}f,\ \phi^{*}g\}$$

for all $f,g \in C^{\infty}(M,\mathbb{R})$. Thus, ϕ^{*} is a Lie algebra isomorphism on $C^{\infty}(M,\mathbb{R})$.

The next fact is that Hamilton's equation may be written in Poisson bracket form.

Theorem 1.1.14: Let X_H be a Hamiltonian vector field on a symplectic manifold (M,ω) with Hamiltonian H and local flow ϕ_t. Then, for every f $\in C^{\infty}(M,\mathbb{R})$,

$$\frac{d}{dt}(f \cdot \phi_t) = \{H,\ f \cdot \phi_t\} = L_{X_H}(f \cdot \phi_t).$$

In particular, if $f = x_i$ or y_i, we have

$$(1.1.8)\qquad \dot{x}_i = \{H, x_i\} = \frac{\partial H}{\partial y_i},$$

$$\dot{y}_i = \{H, y_i\} = \frac{-\partial H}{\partial x_i}.$$

So far, we considered the Lie derivative $L_X f$ for $f \in C^\infty(M, \mathbb{R})$. We can also define the Lie derivative $L_X Y$ for $Y \in \mathfrak{X}(M)$.

Theorem 1.1.15: If $X, Y \in \mathfrak{X}(M)$, then $[L_X, L_Y] = L_X L_Y - L_Y L_X$ is an ($\mathbb{R}$ linear) derivation on $C^\infty(M, \omega)$, that is, for $f, g \in C^\infty(M, \omega)$, $[L_X, L_Y](f \cdot g) = ([L_X, L_Y]f)g + f([L_X, L_Y]g)$.

Definition 1.1.16: For $X, Y \in \mathfrak{X}(M)$, let $[X, Y] = L_X Y$ be the unique vector field such that $L_{[X,Y]} = [L_X, L_Y]$. We call $L_X Y$ the Lie derivative of Y with respect to X, or the Lie bracket of X and Y.

Notice that [,] is a skew-symmetric bilinear form on $\mathfrak{X}(M)$ and satisfies the Jacobi identity and hence the space of smooth vector fields together with the Lie bracket $(\mathfrak{X}(M), [\ ,\ ])$ forms a Lie algebra. In the local coordinates, [,] is written as

$$(1.1.9)\qquad [X, Y] = DY \cdot X - DX \cdot Y.$$

The following theorem shows the relationship between the Lie bracket of Hamiltonian vector fields and the Poisson bracket of smooth functions.

<u>Theorem 1.1.17:</u> For $f,g \in C^\infty(M)$, $[X_f, X_g] = X_{\{f,g\}}$.

Thus, the space of Hamiltonian vector fields with Lie bracket $(\mathcal{X}_H(M), [\ ])$ forms a Lie subalgebra of the Lie algebra of all smooth vector fields on M. The mapping $\rho: C^\infty(M,\mathbb{R}) \to \mathcal{X}_H(M)$ defined by $\rho(f) = X_f$ is a homomorphism of Lie algebras $(C^\infty(M), \{\ ,\ \})$ and $(\mathcal{X}_H(M), [\ ,\ ])$.

<u>Definition 1.1.18:</u> For each $F \in C^\infty(M,\mathbb{R})$, define the map

$$ad_F: C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \quad \text{by} \quad ad_F G = \{F,G\}.$$

We call the map $ad: C^\infty(M,\mathbb{R}) \to L(C^\infty(M,\mathbb{R}), C^\infty(M,\mathbb{R})): F \to ad_F$ the <u>adjoint respresentation</u> of $C^\infty(M,\mathbb{R})$.

Notice that for each $F \in C^\infty(M,\mathbb{R})$ ad_F is an inner derivation of $C^\infty(M,\mathbb{R})$ since, by the Jacobi identity, we have

$$ad_F\{G,H\} = \{ad_F G,H\} + \{G, ad_F H\}$$

for all $G,H \in C^\infty(M,\mathbb{R})$. Also, because of (1.1.7), ad_F has local expression

$$ad_F = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial y_i} = L_{X_F}.$$

<u>Definition 1.1.19:</u> For $H \in C^\infty(M,\mathbb{R})$, the Lie series is defined formally

as

$$\exp \mathrm{ad}_H = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{ad}_H^n$$

where $\mathrm{ad}_H^o = \mathrm{id}$, $\mathrm{ad}_H^n = \mathrm{ad}_H \cdot \mathrm{ad}_H^{n-1}$ for $n \geq 1$.

The Lie series is the essential tool for computing normal forms of Hamiltonian functions. In the following some basic facts about Lie series are stated.

<u>Theorem 1.1.20:</u> Let $H \in C^{\infty}(\mathbb{R}^{2n},\mathbb{R})$ with coordinates $(x,y) = (x_1, \ldots x_n, y_1, \ldots, y_n)$ and standard symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. Then,

(i) $\mathrm{ad}_H(x,y) = X_H(x,y)$, where

$$\mathrm{ad}_H(x,y) = (\mathrm{ad}_H x_1, \ldots, \mathrm{ad}_H x_n, \mathrm{ad}_H y_1, \ldots, \mathrm{ad}_H y_n).$$

(ii) $\exp(t\ \mathrm{ad}_H)\cdot(x,y)$ is the flow of X_H.

(iii) For any $F \in C^{\infty}(\mathbb{R}^{2n},\mathbb{R})$, $(F \cdot \exp \mathrm{ad}_H)(x,y) = (\exp(\mathrm{ad}_H) \cdot F)(x,y)$.

(iv) $\exp \mathrm{ad}_H$ and $\exp \mathrm{ad}_F$ commute iff $\{H,F\}$ is constant iff $[X_H,X_F] = 0$.

Notice that the space $\{\mathrm{ad}_F |\ F \in C^{\infty}(\mathbb{R}^{2n},\mathbb{R})\}$ is a Lie algebra with bracket $[\mathrm{ad}_F, \mathrm{ad}_G] = \mathrm{ad}_{\{F,G\}}$ or $[X_F,X_G] = X_{\{F,G\}}$ if we identify the vector field X with its Lie derivative L_X. Hence, the set $G = \{\exp \mathrm{ad}_F |\ F \in C^{\infty}(\mathbb{R}^{2n},\mathbb{R})\}$ forms a Lie group. Then, each one-parameter group $\{\exp t\ \mathrm{ad}_F\colon t \in \mathbb{R}\}$ forms a one-parameter subgroup of G. On the symplectic space $(\mathbb{R}^{2n},\omega)$ each one-parameter group of symplectic diffeomorphisms is the flow of a Hamiltonian vector field. Thus, we have found all one-parameter subgroups of G because each generator of G is a symplectic diffeomorphism which is the time one flow of a Hamiltonian vector field.

Definition 1.1.21: Let $(\mathbb{R}^{2n},\omega)$ be a symplectic space. A linear map $\phi:\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is symplectic iff $\omega(\phi v, \phi w) = \omega(v,w)$ for each $v,w \in \mathbb{R}^{2n}$. The set of all linear symplectic mappings of $(\mathbb{R}^{2n},\omega)$ is a Lie group $Sp(n,\mathbb{R})$ called the real symplectic group. A linear map $A: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is infinitesimally symplectic iff $\omega(Av,w) + \omega(v,Aw) = 0$ for every $v,w \in \mathbb{R}^{2n}$. The set of all infinitesmally symplectic maps is a Lie algebra $sp(n,\mathbb{R})$ under the Lie bracket $[A,B] = BA - AB$. Note that $A \in sp(n,\mathbb{R})$ iff $e^A \in Sp(n,\mathbb{R})$, which relates the Lie algebra to the corresponding Lie group.

Theorem 1.1.22: Let $\phi \in Sp(n,\mathbb{R})$ and $\lambda \in \mathbb{C}$ be an eigenvalue of ϕ of multiplicity k. Then $\frac{1}{\lambda}$, $\bar{\lambda}$, $\frac{1}{\bar{\lambda}}$ are eigenvalues of ϕ ($\bar{\lambda}$ = complex conjugate of λ) of the same multiplicity.

Theorem 1.1.23: Let $A \in sp\ (n,\mathbb{R})$ and $\lambda \in \mathbb{C}$ be an eigenvalue of A of multiplicity k. Then, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ are eigenvalues of A with the same multiplicity.

Definition 1.1.24: On $(\mathbb{R}^{2n},\omega)$, the map $\Phi: G \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a symplectic action of the Lie group G on $\mathbb{R}^{2n}$ if for each $\phi \in G$, the map

$\Phi_\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}: x \to \Phi(\phi, x)$ is symplectic.

In a natural way, the action Φ on $\mathbb{R}^{2n}$ induces an action of G on $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$

$$\Psi: G \times C^\infty(\mathbb{R}^{2n}, \mathbb{R}) \to C^\infty(\mathbb{R}^{2n}, \mathbb{R}): (\phi, H) \to H \cdot \Phi_\phi.$$

we often write $\phi \cdot H$ for $\Psi(\phi, H)$.

<u>Definition 1.1.25:</u> A Lie group G acting symplectically on $\mathbb{R}^{2n}$ is a <u>symmetry group</u> for the system $(\mathbb{R}^{2n}, \omega, H)$ if

$$\phi \cdot H = H \quad \text{for all } \phi \in G.$$

<u>Theorem 1.1.26:</u> If F is an integral for the system $(\mathbb{R}^{2n}, \omega, H)$ i.e., $\{F, H\} = 0$, then the one-parameter group $\{\exp(t\ \mathrm{ad}_F): t \in \mathbb{R}\}$ given by the flow of X_F, is a symmetry group for $(\mathbb{R}^{2n}, \omega, H)$.

The converse of the above theorem also holds in the sense that each symmetry group of a Hamiltonian system gives rise to an integral. To make this precise, we first introduce the notion of momentum mapping.

<u>Definition 1.1.27:</u> Let Φ be a symplectic action of the Lie group G on $(\mathbb{R}^{2n}, \omega)$ with the Lie algebra L. The mapping $J: \mathbb{R}^{2n} \to L^*$ is a <u>momentum mapping</u> for the action Φ if for every $\xi \in L$

$$X_{\hat{J}(\xi)}(x) = \frac{d}{dt}\Phi(\exp t\,\xi, x)\Big|_{t=0}$$

where the right-hand side is called the <u>infinitesimal generator</u> of the action corresponding to ξ and $\hat{J}(\xi) \in C^{\infty}(\mathbb{R}^{2n}, \mathbb{R})$ is defined by

$$\hat{J}(\xi)(x) = J(x) \cdot \xi.$$

<u>Theorem 1.1.28:</u> Let Φ be a symplectic action of the Lie group G on $(\mathbb{R}^{2n}, \omega)$ with the momentum mapping J. If G is a symmetry group for $(\mathbb{R}^{2n}, \omega, H)$, then $\{\hat{J}(\xi), H\} = 0$, i.e. $\hat{J}(\xi)$ is an integral for $(\mathbb{R}^{2n}, \omega, H)$.

§2. Normal forms for Hamiltonian functions

In this section, we will assume that $H \in C^{\infty}(\mathbb{R}^{2n},\mathbb{R})$ with $H(0) = 0$ and $dH(0) = 0$, that is, the origin 0 of $\mathbb{R}^{2n}$ is an equilibrium point for X_H. The goal of normal form theory is to find an origin-preserving symplectic diffeomorphism ϕ of $\mathbb{R}^{2n}$ which preserves the Hamiltonian character such that H in the new coordinates defined by ϕ, i.e., $\phi^{*}H = H \circ \phi$ is in the simplest possible form.

Let $\mathcal{P}_{\gamma}^{+}(\mathbb{R}^{2n},\mathbb{R})$ be the space of all formal power series on $\mathbb{R}^{2n}$ beginning with terms of degree $\gamma \geq 2$, and $\mathcal{P}_{j}(\mathbb{R}^{2n},\mathbb{R})$ be the space of homogeneous polynomials on $\mathbb{R}^{2n}$ of degree j. Let G be the Lie group of all origin-preserving symplectic diffeomorphisms on $\mathbb{R}^{2n}$ of the form id + $\phi^{(2)}$ where $\phi^{(2)}$ is an $\mathbb{R}^{2n}$-valued formal power series all of whose components lie in $\mathcal{P}_2^{+}(\mathbb{R}^{2n},\mathbb{R})$. The action Φ of G on $\mathbb{R}^{2n}$ induces an action $\cdot$ on $\mathcal{P}_2^{+}(\mathbb{R}^{2n},\mathbb{R})$ given by $\phi \cdot F = \Phi(\phi,F) = \phi^{*}F$ for $\phi \in G$, $F \in \mathcal{P}_2^{+}(\mathbb{R}^{2n},\mathbb{R})$. Let $Q_G(H) = \{\phi \cdot H \mid \phi \in G\}$ be the orbit of H under the action of G. Let $C(H_2)$ be a complementary space to $Q_G(H_2)$ at H_2.

<u>Definition 1.2.1</u>. Let $\overline{H}$, $H \in \mathcal{P}_2^{+}(\mathbb{R}^{2n},\mathbb{R})$ and $H = H_2 + H_3 + \ldots$ with $H_j \in \mathcal{P}_j(\mathbb{R}^{2n},\mathbb{R})$. Then we say $\overline{H}$ is a <u>H_2-normal form for H</u> if $\overline{H} \in Q_G(H) \cap C(H_2)$.

Note that the tangent space to $Q_G(H_2)$ at H_2 is $H_2 + \mathrm{ad}_{H_2}\mathcal{P}_3^{+}$ because for all $F \in \mathcal{P}_3^{+}$, $t \to \exp t\,\mathrm{ad}_F$ is a one-parameter subgroup of G which represents the tangent vector X_F to G at id. Therefore, $t \to (\exp t$

$ad_F) \cdot H_2$ is a curve in $Q(H_2)$ passing through H_2. Thus, the set of tangent vectors

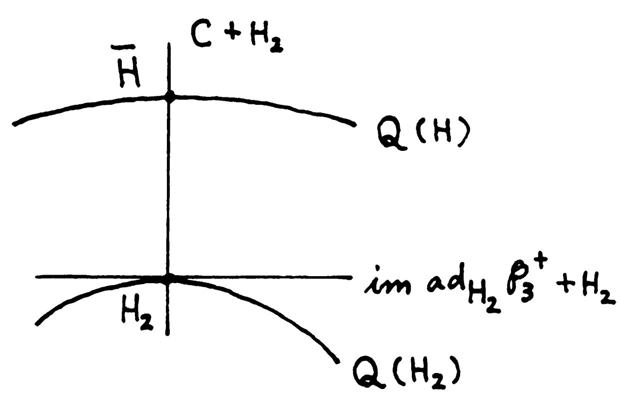
$$\frac{d}{dt}\Big|_{t=0} (\exp t\, ad_F) \cdot H_2 = \frac{d}{dt}\Big|_{t=0} \exp (t\, ad_F)\, H_2$$

$$= ad_F\, H_2$$

$$= -\, ad_{H_2} F$$

for all $F \in \mathcal{P}_3^{+}(\mathbb{R}^{2n},\mathbb{R})$ is $T_{H_2}Q(H_2)$.

If C_m is a complementary subspace to the image of the linear map $ad_{H_2}| \mathcal{P}_m : \mathcal{P}_m \to \mathcal{P}_m : f \to \{H_2, f\}$ for all $m \geq 3$, then the subspace $C = \sum_{m=3}^{\infty} C_m$ of $\mathcal{P}_3^{+}(\mathbb{R}^{2n},\mathbb{R})$ is complementary to $\mathrm{Im}\, ad_{H_2}$ of $\mathcal{P}_3^{+}$. See Figure 1.2.1.



⟨ Figure 1.2.1 ⟩

The following theorem gives an algorithm for finding a formal power series symplectic diffeomorphism ϕ_F which brings the formal power

series Hamiltonian $H = H_2 + H_3 + \ldots$ into normal form. The normalizing transformation ϕ_F is constructed by induction.

<u>Theorem 1.2.2</u>. Let $H \in \mathcal{P}_2^+(\mathbb{R}^{2n},\mathbb{R})$ and let $H_2 \neq 0$ be the quadratic part of H. Then, for each $m \in \mathbb{N}$, $m \geq 3$, there exists a $\phi \in G$ such that $\overline{H} = H \cdot \phi$ is in the H_2- normal form for H up to order m.

(proof) Suppose that $\overline{H}$ is in H_2 - normal form up to terms of degree $m-1 \geq 2$, that is, suppose there is a $F^{(m-1)} \in \mathcal{P}_3^+$ such that

$$\overline{H}^{(m)} = \phi_{F^{(m-1)}} \cdot H = \overline{H}_2 + \overline{H}_3 + \ldots + \overline{H}_{m-1} + H_m + \ldots$$

where $\overline{H}_2 = H_2$ if $m = 3$ and $\overline{H}_i \in C_i$ for $i = 3,4,5, \ldots , m-1$. For $m \geq 3$, let $F_m \in \mathcal{P}_m(\mathbb{R}^{2n},\mathbb{R})$. Then we get

$$\phi_{F_m}^{*} \overline{H}^{(m)} = \exp \mathrm{ad}_{F_m} \overline{H}^{(m)}$$

$$= \overline{H}^{(m)} + \mathrm{ad}_{F_m} \overline{H}^{(m)} + O(m+1)$$

$$= H_2 + \overline{H}_3 + \ldots + \overline{H}_{m-1} + H_m + \mathrm{ad}_{F_m} H_2 + O(m+1).$$

Therefore, the terms of degree m in the above are

$$H_m + \mathrm{ad}_{F_m} H_2 = H_m - \mathrm{ad}_{H_2} F_m.$$

Since $\mathcal{P}_m(\mathbb{R}^{2n},\mathbb{R}) = C_m \oplus \mathrm{Im}\, \mathrm{ad}_{H_2} | \mathcal{P}_m$, we may write $H_m = \overline{H}_m + \tilde{H}_m$

where $\overline{H}_m \in C_m$ and $\tilde{H}_m \in \text{im } ad_{H_2} | \mathcal{P}_m$. Since $\tilde{H}_m \in \text{im } ad_{H_2} | \mathcal{P}_m$, we can choose a $F_m \in \mathcal{P}_m$ such that $ad_{H_2} F_m = \tilde{H}_m$. With this choice of F_m,

$$\overline{H}^{(m+1)} = \phi^*_{F_m} \overline{H}^{(m)} = H_2 + \overline{H}_3 + \quad + \overline{H}_{m-1} + \overline{H}_m + O(m+1),$$

that is, ϕ_{F_m} brings $\overline{H}^{(m)}$ into H_2 - normal form up to order m. Thus we have

$$\overline{H}^{(m+1)} = \phi^*_{F_m}(\phi^*_{F(m-1)}H) = (\phi_{F(m-1)} \cdot \phi_{F_m})^* H = \phi^*_{F(m)} H.$$

This completes the inductive step of the normalization process. Repeating this step degree by degree gives a formal symplectic diffeomorphism

$$\phi = \phi_{F_3} \cdot \phi_{F_4} \cdot \ldots \cdot \phi_{F_m} \cdot \ldots = \phi_F$$

which brings H into normal form, that is,

$$\phi^*_F \cdot H = H_2 + \overline{H}_3 + \ldots + \overline{H}_m \ldots,$$

where $\overline{H}_m \in C_m$ for all $m \geq 3$. ///

Since the normal form of $H = H_2 + H_3 + \ldots + H_m + \ldots \in \mathcal{P}_2^+$ depends on the choice of complement C_m to the image of the linear map $ad_{H_2} | \mathcal{P}_m : \mathcal{P}_m \to \mathcal{P}_m$ for all $m \geq 3$, we need to know how to compute C_m in general case.

We need some basic facts about linear maps from linear algebra. (See Humphreys [29] , see also Cushman [22].)

Definition 1.2.3. Let V be a finite dimensional real vector space. A linear mapping S: V → V is semisimple if every S-invariant subspace U of V has an S-invariant complementary subspace W of V.

S being semisimple is equivalent to saying that S is diagonalizable on the complexification of V. A linear mapping N : V → V is nilpotent if there is an $m \in \mathbb{N}$ such that $N^m = 0$ but $N^{m-1} \neq 0$.

Theorem 1.2.4. Let A : V → V be a linear mapping. Then there are unique simisimple and nilpotent linear maps S and N on V such that SN = NS and A = S + N.

The maps S,N given above are called the S-N decomposition of A. The following theorem is very useful in finding C_m in the normal form.

Theorem 1.2.5. Suppose A = S + N is a S-N decomposition of a linear mapping A:V → V. Then

(a) $V = \ker S \oplus \operatorname{im} S$

(b) $\ker A = \ker S \cap \ker N$

(c) $\operatorname{im} A = \operatorname{im} S \oplus (\operatorname{im} N \cap \ker S)$.

The following shows that Theorem 1.2.4 also holds in $sp(V,\mathbb{R})$.

<u>Theorem 1.2.6</u>. Suppose that $A: (V,\omega) \to (V,\omega)$ is infinitesimally symplectic and has an S - N decomposition $A = S + N$. Then S and N are also infinitesimally symplectic.

Now, the S - N decomposition $A = S + N$ in $sp(V,\mathbb{R})$ propagates into the space $(\mathcal{P}_2, \{\ ,\ \})$ and $(gl(\mathcal{P}_m,\mathbb{R}), [\ ,\ ])$. That is, using the Theorem 1.2.6 and the isomorphism $\rho : (\mathcal{P}_2,\{\ ,\ \}) \to (sp(V,\mathbb{R}), [\ ,\ ]): f \to X_f$ of Lie algebras, every $H_2 \in \mathcal{P}_2$ has a corresponding S - N decomposition $H_2 = S_2 + N_2$ with $\{S_2,N_2\} = 0$ and $S_2,N_2 \in \mathcal{P}_2$. Further, the map $ad^{(m)}: \mathcal{P}_2 \to gl(\mathcal{P}_m,\mathbb{R}): f \to ad_f | \mathcal{P}_m = L_{X_f} | \mathcal{P}_m$ is a representation of Lie algebra $(\mathcal{P}_2, \{\ ,\ \})$ into the Lie algebra $(gl(\mathcal{P}_m,\mathbb{R}), [\ ,\ ])$ and hence if $H_2 = S_2 + N_2$ is the S - N decomposition of $H_2 \in \mathcal{P}_2$, then $ad_{H_2} | \mathcal{P}_m = ad_{S_2} | \mathcal{P}_m + ad_{N_2} | \mathcal{P}_m$ is the S - N decomposition of $ad_{H_2} | \mathcal{P}_m$. From this fact and Theorem 1.2.5, we have the following important criterion for computing a H_2- normal form for H.

<u>Theorem 1.2.7</u>: Let $H = H_2 + H_3 + \dots + H_m + \dots \in C_2^{\infty}(\mathbb{R}^{2n},\mathbb{R})$ and let $H_2 = S_2 + N_2$ be the S - N decomposition of H_2. Then, H is in <u>norm form with respect to H_2</u> iff $H_m \in C_m$ where C_m is a complement to $(\text{im } ad_{N_2}^{(m)} \cap \ker ad_{S_2}^{(m)})$ in $\ker ad_{S_2}^{(m)}$ for every $m \geq 3$, where $ad_{N_2}^{(m)} = ad_{N_2} | \mathcal{P}_m$ etc.

<u>proof</u> By the definition of C_m, and Theorem 1.2.5(C), we have

$$\mathcal{P}_m = \text{im } ad_{H_2}^{(m)} \oplus C_m$$

$$= \text{im } ad_{S_2}^{(m)} \oplus (\text{im } ad_{N_2}^{(m)} \cap \ker ad_{S_2}^{(m)}) \oplus C_m.$$

Since $ad_{S_2}^{(m)}$ is semisimple, $\mathcal{P}_m = \text{im } ad_{S_2}^{(m)} \oplus \ker ad_{S_2}^{(m)}$. Hence, $\ker ad_{S_2}^{(m)} = (\text{im } ad_{N_2}^{(m)} \cap \ker ad_{S_2}^{(m)}) \oplus C_m$. Thus, C_m is a complement of $(\text{im } ad_{N_2}^{(m)} \cap \ker ad_{S_2}^{(m)})$ in $\ker ad_{S_2}^{(m)}$. ///

Notice that if H_2 is semisimple, i.e., $H_2 = S_2$ and $N_2 = 0$, then we may take $C_m = \ker ad_{S_2}^{(m)}$. Hence, if H_2 consists of only semisimple part S_2, then we can say that the formal power series $H = S_2 + H_3 + \ldots + H_m + \ldots$ is in S_2 - normal form iff $H_m \in \ker ad_{S_2} \mid \mathcal{P}_m$ for all $m \geq 3$. Recall that we can bring an $H = H_2 + \ldots + H_m + \ldots \in C_2^\infty(\mathbb{R}^{2n},\mathbb{R})$ into S_2- normal form by a symplectic diffeomorphism $\phi_F = \exp ad_{F_3} \cdot \exp ad_{F_4} \cdot \ldots \exp ad_{F_n} \ldots$, where $F_n \in \mathcal{P}_n (n \geq 3)$. At each step, $\exp ad_{F_n}$ is determined up to terms of F_n in $\ker ad_{S_2}^{(n)}$(i.e., $F_n = \overline{F}_n + \tilde{F}_n$, $\overline{F}_n \in \ker ad_{S_2}^{(n)}$, $\tilde{F}_n \in \text{im } ad_{S_2}^{(n)}$ and $\overline{F}_n$ may be arbitrary.). Starting with $n = 3$, this freedom of choice of $\overline{F}_n$ may lead to different S_2 - normal forms up to order > 3. However, these S_2- normal forms can be transformed into one another by a symplectic diffeomorphism. Thus, the S_2- normal form is essentially unique.

Let $\mathcal{A}(S_2) = \ker ad_{S_2} \mid \mathcal{P}_2^{+}$. Then, since $\mathcal{A}(S_2)$ is closed under $\cdot$ and $\{\ ,\ \}$ and since ad_{S_2} is a derivation of $(\mathcal{P}_2^{+},\cdot)$, $(\mathcal{A}(S_2),\cdot,\{\ ,\ \})$ is a Poisson structure, that is, $(\mathcal{A}(S_2),\ \cdot\ )$ is an associative algebra with unit over $\mathbb{R}$, while $(\mathcal{A}(S_2),\ \{\ ,\ \})$ is a Lie algebra. We call $(\mathcal{A}(S_2),\ \cdot\ )$ the <u>Birkoff algebra</u> of S_2. The main goal of the semisimple case of normal form theory is to describe the Birkoff algebra, because then we know what power series appear in the normal form. The only

known general fact about the Birkoff algebra $\mathcal{A}(S_2)$ is the following.

Theorem 1.2.8. If the semisimple Hamiltonian vector field X_{S_2} corresponding to S_2 has pure imaginary eigenvalues, then $\mathcal{A}(S_2)$ is finitely generated. (For the proof, see Cuchman [22]).

Now, in order to determine the Birkoff algebra $\mathcal{A}(S_2) = \ker \mathrm{ad}_{S_2}$ for a specific semisimple quadratic Hamiltonian S_2 where X_{S_2} has purely imaginary eigenvalues, we will need to know a normal form for X_{S_2} on a symplectic vector space (V,ω).

Theorem 1.2.9 (Cushman [22]). Suppose $H_2 = S_2$ and X_{S_2} has pure imaginary eigenvalues $\pm i\alpha_j$. Then there is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of (V,ω) such that the matrix of $\omega^{\#}: V \to V^{*}$ defined by $\omega^{\#}(e) \cdot e' = \omega(e,e')$ is

$$\left[\begin{array}{c|c} \omega(e_i,e_j) & \omega(e_i,f_j) \\ \hline \omega(f_i,e_j) & \omega(f_i,f_j) \end{array}\right] = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

and

$$S_2(x,y) = \frac{1}{2} \sum_{j=1}^{n} \epsilon_j \alpha_j (x_j^2 + y_j^2),$$

where $\epsilon_j = \pm 1$, $\alpha_j > 0$ and $i\epsilon_j\alpha_j$ are the eigenvalues of X_{S_2}. Thus the matrix of X_{S_2} with respect to the above basis is

$$X_{S_2} = \left[\begin{array}{c|c} 0 & \begin{matrix} \epsilon_1\alpha_1 & & \\ & \epsilon_2\alpha_2 & \\ & & \ddots \\ & & & \epsilon_n\alpha_n \end{matrix} \\ \hline \begin{matrix} -\epsilon_1\alpha_1 & & \\ & -\epsilon_2\alpha_2 & \\ & & \ddots \\ & & & -\epsilon_n\alpha_n \end{matrix} & 0 \end{array}\right].$$

///

Now, assuming that X_{S_2} has pure imaginary eigenvalues $\pm i\alpha_j$ and is in the normal form given in Theorem 1.2.9., we can compute the Birkoff algebra ker ad_{S_2} as follows.

Let (x,y) be coordinates on $\mathbb{R}^{2n}$ corresponding to the basis given in Theorem 1.2.9. Introduce complex conjugate coordinates

$$z_j = x_j + i\epsilon_j y_j, \qquad \overline{z} = x_j - i\epsilon_j y_j$$

for $j = 1, \ldots, n$ when ϵ_j are those given in the normal form of X_{S_2}. Then the linear operator

$$L_{X_{S_2}} = ad_{S_2} = \sum_{j=1}^{n} \epsilon_j\alpha_j\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right)$$

becomes in $(z,\overline{z})$ coordinates

$$L_{\tilde{X}_{\tilde{S}_2}} = ad_{\tilde{S}_2} = -i\sum_{j=1}^{n} \alpha_j\left(z_j\frac{\partial}{\partial x_j} - \overline{z}\frac{\partial}{\partial \overline{z}_j}\right)$$

where $\tilde{S}_2 = \frac{1}{2}\sum_{j=1}^{n} \epsilon_j\alpha_j\, z_j\overline{z}_j$ and $\tilde{X}_{\tilde{S}_2}$ is the complex vector field

$$\dot{z}_j = -2i \frac{\partial \tilde{S}_2}{\partial \bar{z}_j} = - i\, \epsilon_j \alpha_j z_j$$

$$\dot{\bar{z}}_j = 2i \frac{\partial \tilde{S}_2}{\partial z_j} = i\, \epsilon_j \alpha_j \bar{z}_j \qquad \text{for } j = 1, \ldots, n.$$

The space $\mathcal{P}_m(\mathbb{R}^{2n},\mathbb{R})$ is the real space of the monomials $x^j y^k = x_1^{j_1} \ldots x_n^{j_n} y_1^{k_1} \ldots y_n^{k_n}$. In complex conjugate coordinates $\mathcal{P}_m$ corresponds to the space of Hermitian polynomials $\tilde{\mathcal{P}}_m(\mathbb{C}^{2n}, \mathbb{C})$ which is the Hermitian span of the monomials $z^j \bar{z}^k = z_1^{j_1} \ldots z_n^{j_n} \bar{z}_1^{k_1} \ldots \bar{z}_n^{k_n}$ that is, $\tilde{P}_m(z,\bar{z}) = \sum_{|j|+|k|=m} C_{jk} z^j \bar{z}^k \in \mathcal{P}_m$ if and only if $\bar{C}_{jk} = C_{kj}$.

Applying the operator $ad_{\tilde{S}_2}$ to the nomomial basis $z^j \bar{z}^k$,

$$ad_{\tilde{S}_2} \cdot z^j \bar{z}^k = - i \langle j - k, \alpha \rangle z^j \bar{z}^k$$

where $j = (j_1, \ldots, j_n)$, $k = (k_1, \ldots k_n) \in \mathbb{Z}^n_+$, $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{R}^n{+}$ and $\langle\ ,\ \rangle$ is the inner product on $\mathbb{R}^{2n}$ with norm $|\ \ |$. Therefore,

$$\ker ad_{\tilde{S}_2} = \{ \sum_{m=2}^{\infty} \sum_{|j|+|k|=m} C_{jk} z^j \bar{z}^k \mid \langle j - k, \alpha \rangle = 0 \text{ and } \bar{C}_{jk} = C_{kj} \}.$$

The relation

$$\langle j - k, \alpha \rangle = 0$$

is called the <u>resonance relation</u> corresponding to $\tilde{X}_{\tilde{S}_2}$.

The corresponding space of real formal power series in $C_2^\infty(\mathbb{R}^{2n},\mathbb{R})$ is $\ker \mathrm{ad}_{S_2}$.

The normal form theory in the case of $H_2 = N_2$ where N_2 is a nilpotent quadratic polynomial on $(\mathbb{R}^{2n}, \Sigma\, dx_j \wedge dy_j)$ is a little more complicated. By the theorem of Jacobson-Morosov, we may embed N_2 into a subalgebra of $(\mathcal{P}_2(\mathbb{R}^{2n},\mathbb{R}), \{\ ,\ \})$ which is isomorphic to $s\ell(2,\mathbb{R})$, that is, there are M_2, $T_2 \in \mathcal{P}_2\,(\mathbb{R}^{2n},\mathbb{R})$ such that $\{T_2,N_2\} = 2N_2$, $\{T_2,M_2\} = -2M_2$, $\{N_2,M_2\} = T_2$. Then the finite dimensional representation

$$\mathrm{ad}^{(m)}\colon (\mathcal{P}_2,\{\ ,\ \}) \to (g\ell(\mathcal{P}_m,\mathbb{R}), [\ ,\ ])\colon f \to \mathrm{ad}_f^{(m)}$$

of Lie algebras restricts to a finite dimensional representation of $s\ell(2,\mathbb{R})$, that is, we have the corresponding commutation reelations,

$$[\mathrm{ad}_{T_2}^{(m)}, \mathrm{ad}_{N_2}^{(m)}] = 2\,\mathrm{ad}_{N_2}^{(m)},\ [\mathrm{ad}_{T_2}^{(m)}, \mathrm{ad}_{M_2}^{(m)}] = -2\,\mathrm{ad}_{M_2}^{(m)},$$
$$[\mathrm{ad}_{N_2}^{(m)}, \mathrm{ad}_{M_2}^{(m)}] = \mathrm{ad}_{T_2}^{(m)}.$$

From the representation theory of $s\ell(2,\mathbb{R})$, we have

$$\ker \mathrm{ad}_{M_2}^{(m)} \oplus \mathrm{im}\,\mathrm{ad}_{N_2}^{(m)} = \mathcal{P}_m$$

for every $m \geq 3$. Therefore, we can say that the formal power series Hamiltonian $H = N_2 + H_3 + \ldots + H_m + \ldots$ is in N_2 - normal form iff $H_m \in \ker \mathrm{ad}_{M_2}^{(m)} \cap \ker \mathrm{ad}_{S_2}^{(m)}$ for all $m \geq 3$.

In analogy with the Birkoff algebra in the semisimple case, we call the algebra $W(N_2) = (\ker \mathrm{ad}_{M_2}, \cdot\ )$ the top weight algebra of N_2. Also, as a first step in constructing an explicit embedding of N_2 into

$s\ell\,(2,\mathbb{R})$ we need a normal form for nilpotent infinitesimally symplectic linear mapping X_{N_2}. Since in this thesis we need only the S_2 - normal form, we omit the further details about the N_2 - normal form theory. (See Notes on Normal form theory, Richard Cushman 1985, Normal Forms and Symmetry, Sanders 1985).

CHAPTER 2: VERSAL DEFORMATIONS OF QUADRATIC HAMILTONIANS

To construct a versal deformation of H_2, we need to know a versal deformation of X_{H_2} in sp $(n,\mathbb{R})$. In Section 1, we will treat briefly the theory of versal deformation of linear systems and in Section 2, we construct a versal deformation of H_2 given in (0.3).

§1 Versal deformations of Linear Systems

The reduction of a matrix in $g\ell(n,\mathbb{R})$ to its Jordan normal form or a matrix in $sp(n,\mathbb{R})$ to its normal form is an unstable process since both the normal forms themselves and conjugating transformations depend discontinuously on the elements of the original matrices. In this section we introduce the theory of versal deformtions for finding the simplest possible normal form (so called miniversal deformation) to which not only one specific matrix, but an arbitrary family of matrices close to it can be reduced by means of a mapping smoothly depending on the parameters. For further details see Arnold [2][5] and Kocak [30].

Let L be a real Lie algebra with its corresponding Lie group G, e.g., L may be $g\ell(n,\mathbb{R})$ or sp $(n,\mathbb{R})$ with $G = GL(n,\mathbb{R})$ or $Sp(n,\mathbb{R})$. Let A_o

$\in L$ and Λ^k be a small neighborhood of the origin of $\mathbb{R}^k$ for some integer k.

Definition 2.1.1 A deformation $A(\lambda)$ of A_o is a smooth mapping $A: \Lambda^k \to L$ such that $A(0) = A_o$. A deformation is also called a family, the variables λ_i parameters and the parameter space $\Lambda = \{\lambda\}$ a base of the family. Similarly, we can define a deformtion of an element of G.

Definition 2.1.2 Two deformations $A(\lambda)$ and $B(\lambda)$ of A_o are called equivalent if there exists a deformtion $C(\lambda)$ of the identity e of G with the same base such that

$$A(\lambda) = C(\lambda)\ B(\lambda)\ C^{-1}(\lambda),\ C(0) = e.$$

Let $\phi: \Lambda^\ell \to \Lambda^k$ be a smooth mapping with $\phi(0) = 0$. The mapping ϕ of the parameter space $\Lambda^\ell = \{\mu\}$ into the base of the deformation $A(\lambda)$ defines a new deformation $(\phi^* A)(\mu)$ of A_o by composition.

$$(\phi^* A)(\mu) = A(\phi(\mu)).$$

The deformation $\phi^* A$ is said to be induced by $A(\lambda)$ under the mapping ϕ.

Definition 2.1.3. A deformation $A(\lambda)$ of A_o is called versal if every other deformation $B(\mu)$ of A_o is equivalent to a deformation induced by $A(\lambda)$ under a suitable change of parameters, i.e., if there exist $C(\mu)$ and ϕ such that

$$B(\mu) = C(\mu)A(\phi(\mu))C^{-1}(\mu) \quad \text{with } C(0) = e,\ \phi(0) = 0.$$

A versal deformation $A(\lambda)$ is called universal if the inducing mapping ϕ is determined uniquely by $B(\mu)$. A versal deformation is said to be a miniversal if the dimension of the parameter space $\Lambda = \{\lambda\}$ is the smallest possible for a versal deformation.

These miniversal deformations are normal forms with the smallest possible number of parameters in the reduction to which the smooth dependence on the parameters can be preserved.

Now, in the following we introduce the important fact that a versal deformation $A(\lambda)$ of A_o is the mapping A transversal to the orbit of A_o at $\lambda = 0$.

Let Q be a smooth submanifold of a manifold L. Consider a smooth mapping $A\colon \Lambda \to L$ of another manifold Λ into L, and let λ be a point in Λ such that $A(\lambda) \in Q$.

Definition 2.1.4. The mapping $A\colon \Lambda \to L$ is called transversal to Q at λ if the tangent space to L at $A(\lambda)$ is the vector space sum of the image of tangent space to Λ at λ under A_* and the tangent space to Q at $A(\lambda)$, i.e.,

$$T_{A(\lambda)}L = A_*T_\lambda\Lambda + T_{A(\lambda)}Q,$$

where $A_*\colon T_\lambda\Lambda \to T_{A(\lambda)}L$ is the push-forward of the map A.

Now, consider a Lie algebra L with the corresponding Lie group G. The Lie group G acts on L by conjugation, called the <u>adjoint action</u> as follows.

$$Ad_g \ell = g\ell\, g^{-1} (g \in G,\ \ell \in L).$$

The orbit $Q(A_o)$ of a fixed element $A_o \in L$ under the action of G is a smooth submanifold of L defined by

$$Q(A_o) = \{Ad_j A_o = gA_o g^{-1} \mid g \in G\}.$$

<u>Theorem 2.1.5</u> A deformation $A(\lambda)$ of A_o is <u>versal</u> if and only if the mapping A is <u>transversal</u> to the orbit of A_o at $\lambda = 0$.
For the proof of this theorem, see Arnold [2].

Our next problem is to determine the mimimum number of parameters for any versal deformation of $A_o \in L$. From Theorem 2.1.5 we know that in a versal deformation of A_o the number of parameters is minimal when the vector space sum in the Definition 2.1.4 is a direct sum. Consequently, this minimum number is equal to the codimension of the orbit of A_o in L. The next argument shows that the direct sum complement of the tangent space of the orbit of A_o is the centralizer of A_o in L if ad_{A_o} is a semisimple linear mapping of L. Let us elaborate on this.

Let $A_o \in L$ where L is a Lie algebra with its Lie bracket [ ].

Definition 2.1.6. The mapping $ad_{A_o} : L \to L$ is the endomorphism of L defined by

$$ad_{A_o} X = [X,A_o] \qquad \text{for all } X \in L.$$

The kernel of this endomorphism, $\ker ad_{A_o} = \{X \in L \mid [X,A_o] = 0\}$, is called the centralizer of A_o in L.

Theorem 2.1.7. The tangent space $T_{A_o} Q(A_o)$ of the orbit of A_o at $A_o \in L$ is equal to $I_m ad_{A_o}$ in L.

proof. Consier the mapping for a fixed $A_o \in L$,

$$Ad_{A_o} : G \to L: g \to gA_o g^{-1}.$$

The image of Ad_{A_o} is the orbit of A_o in L under the action of G. Note that the derivative of Ad_{A_o} at the identity element $e \in G$ is the linear mapping $(Ad_{A_o})_*: TeG \to T_{A_o} L$. defined by

$$(Ad_{A_o})_* \cdot X = \frac{d}{dt}\Big|_{t=0} (e^{tX} A_o e^{-tX})$$

$$= XA_o - A_o X$$

$$= [X,A_o]$$

$$= \mathrm{ad}_{A_o} \cdot X \qquad \text{for all } X \in L.$$

Since $T_e G$ and $T_{A_o} L$ are isomorphic to the Lie algebra L, the above calculation shows that

$$(\mathrm{Ad}_{A_o})_* = \mathrm{ad}_{A_o}.$$

Therefore, $T_{A_o} Q(A_o) = \mathrm{Im}\ (\mathrm{Ad}_{A_o})_* = \mathrm{Im}\ \mathrm{ad}_{A_o}$. ///

Notice that since ad_{A_o} is an endomorphism of L,

$$\dim L = \dim (\mathrm{Im}\ \mathrm{ad}_{A_o}) + \dim (\mathrm{Ker}\ \mathrm{ad}_{A_o}).$$

Hence, the dimension of the centralizer of A_o is equal to the codimension of the orbit of A_o in L. Thus, the problem of constructing a miniversal deformaiton of A_o is reduced to finding a direct sum complement to $I_m\ \mathrm{ad}_{A_o}$ in L.

Now, let us consider the problem of finding a versal deformation of H_2 in the space $\mathcal{P}_2(\mathbb{R}^{2n},\mathbb{R})$ of homogeneous quadratic polynomials on $\mathbb{R}^{2n}$. We already know that the Lie algebra $(sp(n,\mathbb{R}), [\ ,\ ])$ is isomorphic to the Lie algebra $(\mathcal{P}_2, \{\ ,\ \})$ by the isomorphism ρ: $H_2 \to X_{H_2}$. Hence, if $A_o = X_{H_2} \in sp\ (n,\mathbb{R})$ and has a S - N decomposition, then H_2 has a corresponding S - N decomposition $H_2 = S_2 + N_2$ with $\{S_2,N_2\} = 0$ and S_2, $N_2 \in \mathcal{P}_2$. Furthermore, since the map $\mathrm{ad}^{(2)}: \mathcal{P}_2 \to g\ell(\mathcal{P}_2,\mathbb{R})$: H_2

$\to ad_{H_2} \big|_{\mathcal{P}_2} = L_{X_{H_2}} \big| \mathcal{P}_2$ is a representation of Lie algebra $(\mathcal{P}_2, \{\ ,\ \})$ into the Lie algebra $(g\ell(\mathcal{P}_2,\mathbb{R}), [\ ,\ ])$. $ad_{H_2} | \mathcal{P}_2$ also has a corresponding S − N decomposition $ad_{H_2} | \mathcal{P}_2 = ad_{S_2} | \mathcal{P}_2 + ad_{N_2} | \mathcal{P}_2$.

Let $Sp(2,\mathbb{R})$ act on $\mathcal{P}_2$ by composition. Then, the tangent space to the orbit of H_2 at H_2 is given by Im ad_{H_2}. Thus, to construct a miniversal deformation of H_2 we have to determine a direct sum complement to Im ad_{H_2}. If $H_2 = S_2 + N_2$, then from the Theorem 1.2.5(c) and Theorem 1.2.7 such a complement C_2 is given by the complement of (Im $ad_{N_2}^{(2)} \cap \ker ad_{S_2}^{(2)}$) in $\ker ad_{S_2}^{(2)}$. In particular , if $H_2 = S_2$ ($N_2 = 0$) then $C_2 = \ker ad_{S_2}^{(2)}$ in $\mathcal{P}_2$. The $\ker ad_{S_2}^{(2)}$ is just a Lie subalgebra of $\mathcal{P}_2$ isomorphic to the centralizer of X_{S_2} i.e., $\ker ad_{X_{S_2}}$ in $sp(2,\mathbb{R})$.

If $H_2 = S_2 + N_2$ ($N_2 \neq 0$), then to find the complement of Im ad_{N_2} in $\ker ad_{S_2}$ we embed N_2 into a subalgebra of $(\mathcal{P}_2, \{\ ,\ \})$ which is isomorphic to $s\ell(2,\mathbb{R})$ and is spanned by N_2, M_2, $T_2 \in \mathcal{P}_2$ with the commutation relations: $\{T_2,N_2\} = 2N_2$, $\{T_2,M_2\} = -2M_2$, and $\{N_2,M_2\} = T_2$. Then the finite dimensional representation

$$ad^{(2)}: (\mathcal{P}_2, \{\ ,\ \}) \to (g\ell(\mathcal{P}_2,\mathbb{R}), [\ ,\ ]): H_2 \to ad_{H_2}^{(2)}$$

of Lie algebras restricts to a finite dimensional representation of $s\ell(2,\mathbb{R})$ and has the corresponding commutation relations. From the representation theory of $s\ell(2,\mathbb{R})$, we have

$$\mathcal{P}_2 = \ker \mathrm{ad}_{M_2}{}^{(2)} \oplus \mathrm{im}\ \mathrm{ad}_{M_2}{}^{(2)}.$$

Hence, the complement C_2 of $\mathrm{im}\ \mathrm{ad}_{N_2}{}^{(2)}$ in $\ker \mathrm{ad}_{S_2}{}^{(2)}$ is given by

$$C_2 = \ker \mathrm{ad}_{M_2}{}^{(2)} \cap \ker \mathrm{ad}_{S_2}{}^{(2)}.$$

Since in this thesis we only consider the case $H_2 = S_2$ ($N_2 = 0$) we omit the further details for the nonsemisimple case. See Cushman [18,19,20,21,22] and Van der Meer [34,35].

§2. Computation of Versal deformation of H_2

Consider a Hamiltonian $H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ with $H(0) = dH(0) = 0$,

$$H(x,y) = H_2(x,y) + H_3(x,y) + H_4(x,y) + \ldots$$

where $z = (x,y) = (x_1, \ldots, x_n, y_1, \ldots y_n) \in \mathbb{R}^{2n}$

$H_j(z)$ = a homogeneous polynomial in x,y of degree j, and

$H_2(x,y)$ is the <u>nonpositive definite</u> quadratic form given by

$$(2.2.1) \qquad H_2(x,y) = \frac{1}{2}\sum_{j=1}^{\ell}(x_j^2 + y_j^2) - \frac{1}{2}\sum_{j=\ell+1}^{n}(x_j^2 + y_j^2) = \frac{1}{2} z^t Az$$

where A is 2n x 2n real diagonal matrix of the form

$$(2.2.2)\quad A = \mathrm{diag}\,(1, \ldots 1, -1, \ldots, -1, 1, \ldots, 1, -1, \ldots, -1).$$

Now, we try to find a versal deformation of H_2 given above. For each H $\in C^\infty(\mathbb{R}^{2n},\mathbb{R})$, consider the adjoint map ad_H: $C^\infty(\mathbb{R}^{2n},\mathbb{R}) \to C^\infty(\mathbb{R}^{2n},\mathbb{R})$ defined by

$$(2.2.3)\quad ad_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial y_j}\right) = \{H,\cdot\}$$

where { , } is the Poisson bracket on the Lie algebra $(C^\infty(\mathbb{R}^{2n}, \mathbb{R}), \{\ ,\ \})$. Then from (2.2.1), we have

$$(2.2.4)\quad ad_{H_2} = \sum_{j=1}^{\ell} \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}\right) - \sum_{j=\ell+1}^{n} \left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right).$$

Let $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$, $(j = 1, \ldots, n)$.
In complex conjugate coordinates $(z,\bar{z}) = (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$, we have

$$H_2(x,y) = \tilde{H}_2(z,\bar{z}) = \frac{1}{2} \sum_{j=1}^{\ell} z_j\bar{z}_j - \frac{1}{2} \sum_{j=\ell+1}^{n} z_j\bar{z}_j.$$

$$(2.2.5)\quad ad_{\tilde{H}_2}(z,\bar{z}) = -i \left[\sum_{j=1}^{\ell} \left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right) - \sum_{j=\ell+1}^{n} \left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right)\right].$$

Let $\mathcal{P}_m(z,\bar{z})$= the space of real homogeneous polynomials in $z,\bar{z}$ of degree m.

Let $P_m(z,\bar{z}) \in \mathcal{P}_m(z,\bar{z})$.

Then we may write

$$P_m(z,\bar{z}) = \sum_{|\alpha|+|\beta|=m} C_{\alpha\beta} z^{\alpha}\bar{z}^{\beta}, \qquad \bar{C}_{\alpha\beta} = C_{\beta\alpha},$$

where $z^{\alpha}\bar{z}^{\beta} = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$ and

$$|\alpha| + |\beta| = (\alpha_1 + \cdots + \alpha_n) + (\beta_1 + \cdots + \beta_n),$$

α_j, β_j = nonnegative integers $(j=1, \ldots, n)$.

Applying the operator $\operatorname{ad}_{\tilde{H}_2}$ to the basis monomial $z^{\alpha}\bar{z}^{\beta}$, we have

$$\operatorname{ad}_{\tilde{H}_2}(z^{\alpha}\bar{z}^{\beta}) = -i \left[\sum_{j=1}^{\ell} (\alpha_j - \beta_j) - \sum_{j=\ell+1}^{n} (\alpha_j - \beta_j) \right] (z^{\alpha}\bar{z}^{\beta}).$$

Hence,

$z^{\alpha}\bar{z}^{\beta} \in \ker \operatorname{ad}_{\tilde{H}_2}\big|_{\mathcal{P}_m}$ if and only if

$$(2.2.6) \quad \begin{cases} \text{(a)} \ \displaystyle\sum_{j=1}^{\ell} (\alpha_j - \beta_j) - \sum_{j=\ell+1}^{n} (\alpha_j - \beta_j) = 0 \text{ and} \\ \text{(b)} \ |\alpha| + |\beta| = m. \end{cases}$$

Now, we compute the Hilbert generators $z^{\alpha}\bar{z}^{\beta}$ for the Birkoff algebra $\ker \operatorname{ad}_{\tilde{H}_2}\big|_{\mathcal{P}_2}$ in order to construct the versal deformation of $H_2(x,y)$.

Now, the conditions (2.2.6)(a), (b) can be rewritten as

$$\begin{cases} \displaystyle\sum_{j=1}^{\ell} \alpha_j + \sum_{j=\ell+1}^{n} \beta_j = \sum_{j=1}^{\ell} \beta_j + \sum_{j=\ell+1}^{n} \alpha_j \\ \displaystyle\sum_{j=1}^{\ell} \alpha_j + \sum_{j=\ell+1}^{n} \alpha_j + \sum_{j=1}^{\ell} \beta_j + \sum_{j=\ell+1}^{n} \beta_j = m . \end{cases}$$

Let $\alpha' = (\alpha_1, \ldots, \alpha_\ell)$, $\alpha'' = (\alpha_{\ell+1}, \ldots, \alpha_n)$

$\beta' = (\beta_1, \ldots, \beta_\ell)$, $\beta'' = (\beta_{\ell+1}, \ldots, \beta_n)$

$|\alpha'| = \alpha_1 + \ldots + \alpha_\ell$, etc.

Then, the conditions (2.2.6)(a), (b) can be written as

$$(2.2.7) \quad \begin{cases} \text{(a)} & |\alpha'| + |\beta''| = |\beta'| + |\alpha''| \\ \text{(b)} & |\alpha'| + |\alpha''| + |\beta'| + |\beta''| = m. \end{cases}$$

Now, For m = 2, (2.2.7) implies

$$|\alpha'| + |\beta''| = |\beta'| + |\alpha''| = 1.$$

Hence there are 4 possible solutions for $|\alpha'|$, $|\alpha''|$, $|\beta'|$, $|\beta''|$:

(2.2.8)

$\lvert\alpha'\rvert$	$\lvert\alpha''\rvert$	$\lvert\beta'\rvert$	$\lvert\beta''\rvert$
1	1	0	0
1	0	1	0
0	1	0	1
0	0	1	1.

Let $z' = (z_1, \ldots, z_\ell)$, $z'' = (z_{\ell+1}, \ldots, z_n)$.

Then the Hilbert generators $z^\alpha \bar{z}^\beta = (z')^{\alpha'}(z'')^{\alpha''}(\bar{z}')^{\beta'}(\bar{z}'')^{\beta''}$ for ker $ad_{H_2}\big|_{\mathcal{P}_2}$ take the form:

$$(2.2.9) \quad (z')^{\alpha'}(z'')^{\alpha''},\ (z')^{\alpha'}(\bar{z}')^{\beta'},\ (z'')^{\alpha''}(\bar{z}'')^{\beta''},$$

$$(\bar{z}')^{\beta'}(\bar{z}'')^{\beta''}$$

where $|\alpha'| = |\alpha''| = |\beta'| = |\beta''| = 1$ and

$(z')^{\alpha'}(z'')^{\alpha''}$ contains $\ell\cdot(n-\ell)$ terms $z_1 z_{\ell+1}, z_1 z_{\ell+2}, \ldots, z_\ell z_n$,

$(z')^{\alpha'}(\overline{z}')^{\beta'}$ contains ℓ^2 terms $z_1 \overline{z}_1, z_1\overline{z}_2, \ldots, z_\ell\overline{z}_\ell$,

$(z'')^{\alpha''}(\overline{z}'')^{\beta''}$ contains $(n-\ell)^2$ terms $z_{\ell+1}\overline{z}_{\ell+1}, z_{\ell+1}\overline{z}_{\ell+2}, \ldots z_n\overline{z}_n$,

$(\overline{z}')^{\beta'}(\overline{z}'')^{\beta''}$ contains $\ell(n-\ell)$ terms $\overline{z}_1 \overline{z}_{\ell+1}, \overline{z}_1\overline{z}_{\ell+2}, \ldots, \overline{z}_\ell\overline{z}_n$,

Note that each term in $(\overline{z}')^{\beta'}(\overline{z}'')^{\beta''}$ is the conjugate of each term in $(z')^{\alpha'}(z'')^{\alpha''}$. Hence, the real Hilbert generators for ker ad_{H_2} are of the form

$$(2.2.10)\quad \begin{cases} \mathrm{Re}\,(z_iz_j),\ \mathrm{Im}\,(z_iz_j) & (i = 1, \ldots, \ell,\ j=\ell+1, \ldots, n) \\ \mathrm{Re}\,(z_i\overline{z}_k),\ \mathrm{Im}\,(z_i\overline{z}_k) & (i,\ k = 1, \ldots, \ell) \\ \mathrm{Re}\,(z_i\overline{z}_k),\ \mathrm{Im}\,(z_i\overline{z}_k) & (i,\ k = \ell+1, \ldots, n) \end{cases}$$

Note that each term of our quadratic form $H_2(z,\overline{z}) = \frac{1}{2}\sum_{j=1}^{\ell} z_j\overline{z}_j - \frac{1}{2}\sum_{j=\ell+1}^{n} z_j\overline{z}_j$ is in the above list.

Now, let

$$(2.2.11)\quad H_2^{\lambda}(x,y) = \sum_{i=1}^{\ell}\sum_{j=\ell+1}^{n}[a_{ij}\,\mathrm{Re}\,(z_iz_j) + b_{ij}\mathrm{Im}\,(z_iz_j)]$$

$$+ \sum_{i=1}^{\ell}\sum_{j=1}^{\ell}[C_{ij}\ \mathrm{Re}(z_i\overline{z}_j) + d_{ij}\mathrm{Im}\,(z_i\overline{z}_j)]$$

$$+ \sum_{i=\ell+1}^{n} \sum_{j=\ell+1}^{n} [e_{ij} \operatorname{Re} (z_i \bar{z}_j) + f_{ij} \operatorname{Im} (z_i \bar{z}_j)]$$

$$= \frac{1}{2} u^T B(\lambda) u \quad ,$$

where $u = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$,

and $B(\lambda) = (B_{ij}(\lambda)) \in \mathbb{R}^{2n \times 2n}$, $B_{ij}(\lambda)$ = coefficient of $x_i y_j$.

Then, the versal deformation of $H_2(x,y)$ is given by

$$(2.2.12) \qquad H_2(x,y) + H_2^{\lambda}(x,y).$$

For $m = 2k + 1$ $(k = 1, 2, \ldots)$, (2.2.7) implies

$$|\alpha'| + |\beta''| = |\beta'| + |\alpha''| = \frac{m}{2} = \frac{2k+1}{2}.$$

Since $|\alpha'| + |\alpha''| = |\beta'| + |\beta''|$ are nonnegative integers, there are <u>no</u> solutions α', α'', β', β'' satisfying the above relation. Hence, for $m = 2k + 1$ $(k = 1, 2, \ldots)$,

$$\ker \operatorname{ad}_{H_2} \Big|_{P_m} = \phi$$

i.e., there are <u>no</u> third or higher odd-powered terms in the normal for H with respect to H_2.

CHAPTER 3. LIAPUNOV - SCHMIDT REDUCTION WITH SYMMETRY

§1. Introduction

Suppose our Hamiltonian function $H(x,y)$ is in <u>normal form</u> up to a finite order m (=even integer) with respect to $H_2(x,y)$ and consider the truncated Hamiltonian function $\overline{H}(x,y)$ up to order m,

$$(3.1.1)\qquad \overline{H}(x,y) = H_2(x,y) + N(x,y), \qquad z = (x,y) \in \mathbb{R}^{2n},$$

where

$$(3.1.2)\begin{cases} H_2(x,y) = \frac{1}{2}\sum_{j=1}^{\ell}(x_j^2 + y_j^2) - \frac{1}{2}\sum_{j=\ell+1}^{n}(x_j^2 + y_j^2) = \frac{1}{2}z^t\,Az, \\ A = \mathrm{diag}\,(1, \ldots, 1, -1, \ldots, -1, 1, \ldots 1, -1, \ldots, -1) \in \mathbb{R}^{2n\times 2n}, \\ N(x,y) = H_4(x,y) + H_6(x,y) + \ldots + H_m(x,y), \\ H_k(x,y) \in \ker \mathrm{ad}_{H_2} \text{ for } k = 4, 6, \ldots, m. \end{cases}$$

Consider a linear versal deformation $\overline{H}^{\lambda}(x,y)$ of $\overline{H}(x,y)$:

$$(3.1.3)\qquad \overline{H}^{\lambda}(x,y) = H_2(x,y) + H_2^{\lambda}(x,y) + N(x,y).$$

<u>Remark</u>: Since $\overline{H}^{\lambda} \in \ker \mathrm{ad}_{H_2}$, $\{H_2, \overline{H}^{\lambda}\} = 0$. i.e. H_2 and $\overline{H}^{\lambda}$ are two integrals for the Hamiltonian system $(\mathbb{R}^{2n}, \omega, \overline{H}^{\lambda})$, where $\omega = \sum_{i=1}^{n} dx_i \wedge$

dy_i is the standard symplectic forms in $\mathbb{R}^{2n}$. Hence, in the case of 2 degrees of freedom i.e., n = 2, the system $(\mathbb{R}^4, \omega, \overline{H}^\lambda)$ is completely integrable with integrals H_2 and $\overline{H}^\lambda$ for each value of λ. However, the full nontruncated system $(\mathbb{R}^4, \omega, H^\lambda)$ where $H^\lambda(x,y) = \overline{H}^\lambda(x,y) + H_{m+1} + H_{m+2} + \ldots$ with $H_j \notin \ker ad_{H_2}$ for $j \geq m+1$ is <u>not</u> integrable. But, according to the Moser-Weinstein reduction (See [24]), for $|\lambda|$ sufficiently small, there is a C^∞ function $E^\lambda(x,y)$ depending smoothly on λ for $0 < |z| << 1$ such that E^λ is in H_2- normal and coincides with H^λ up to order m. Hence, the search for periodic solutions of a family of non-integrable systems $(\mathbb{R}^4,\omega,H^\lambda)$ can be reduced to the search for periodic solutions of a nearby family of integrable systems $(\mathbb{R}^4,\omega,E^\lambda)$. Hence, we may restrict our attention to the truncated system $(\mathbb{R}^{2n}, \omega, \overline{H}^\lambda)$. From now on, we write $H(x,y)$ for $\overline{H}(x,y)$ for the notational simplicity.

Returning to (3.1.3), consider the Hamilton's equation with Hamiltonian $H^\lambda(z) = H_2(z) + H_2^\lambda(z) + N(z)$, $z = (x,y) \in \mathbb{R}^{2n}$.

$$\begin{aligned} (3.1.4)\qquad \dot{z} &= J\nabla H^\lambda(z) \\ &= J\nabla H_2(z) + J\nabla H_2^\lambda(z) + J\nabla N(z) \\ &= JAz + JB(\lambda)z + J\nabla N(z), \end{aligned}$$

where $H_2(z) = z^T Az$, $H_2^\lambda(z) = z^T B(\lambda)z$, $B(\lambda)$ is given in (2.2.11) with $B(0) = 0$, and

$$(3.1.5) \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad I_n = \text{n x n identity matrix.}$$

Note that $(JA)^2 = JAJA = J^2A^2 = -I$

$(JA)^2 + I = 0.$

Hence, JA has eigenvalues i with multiplicity 2ℓ, (-i) with multiplicty $2(n-\ell)$. Hence, the linearized system of (3.1.4) at z = 0, that is,

$$(3.1.6) \qquad \dot{z} = JAz + JB(\lambda)z$$

passes through the 1:1:...:1: -1:-1:...:-1 resonance when $\lambda = 0$. At $\lambda = 0$, (3.1.6) becomes

$$(3.1.7) \qquad \dot{z} = JAz$$

and (3.1.7) has the solution

$$z(t) = e^{JAt}z_o$$

with the initial vector $z_o \in \mathbb{R}^{2n}$, where

$$e^{JAt} = I + JAt + \frac{1}{2!}(JAt)^2 + \frac{1}{3!}(JAt)^3 + \dots$$

$$= I + JAt - \frac{1}{2!}\,It^2 - \frac{1}{3!}\,JAt^3 + \dots$$

$$= I(1 - \frac{1}{2!}t^2 + \quad) + JA(t - \frac{1}{3!}t^3 + \dots)$$

$$= I\,(\cos t) + JA\,(\sin t).$$

That is,

$$(3.1.8) \quad e^{JAt} = I(\cos t) + JA(\sin t).$$

Therefore, the linearized equation (3.1.6) has, at $\lambda = 0$, 2n linearly independent 2π-periodic solutions, so called linear normal modes.
For $0 < |\lambda| << 1$ and $|z| << 1$, we expect that the nonlinear system (3.1.4) is close to the linear system (3.1.7) and hence may have small amplitude periodic solutions with period near 2π near the periodic solutions of the linearized system (3.1.7).
Furthermore, the equation (3.1.4)

$$\dot{z} = JAz + JB(\lambda)z + J\nabla N(z)$$

has the linear part

$$(3.1.9) \quad \dot{z} = [JA + JB(\lambda)]z$$

where the matrix $C(\lambda) = J(A + B(\lambda))$ is a smooth function of λ. Since $C(0) = JA$ has eigenvalues $+i$ with multiplicity 2ℓ and $(-i)$ with multiplicty $2(n-\ell)$, $C(\lambda)$ will have an eigenvalue of the form

$$\sigma(\lambda) \pm iw(\lambda)$$

for small $|\lambda|$, where $\sigma(0) = 0$, $\omega(0) = 1$ and σ, ω are smooth functions of λ. It may be possible to choose a particular parameter, say λ_1 with setting all the other λ's to zero so that by varying λ_1, a pair of eigenvalues of $C(\lambda)$ may vary either along the imaginary axis or across the imaginary axis. In two degrees of freedom case it turns out that the above choice is possible to examine the behavior of the periodic

orbits of the nonlinear system (3.1.4) as λ_1 varies across zero.

§2. Liapunov-Schmidt Reduction

Now, we want to study the behavior of periodic solutions of (3.1.4) as λ varies by the method of Liapunov-Schmidt Reduction in the presence of symmetry. Consider the system (3.1.4) again:

$$(3.2.1)\quad \dot{z} = J\nabla H^{\lambda}(z) = JAz + JB(\lambda)z + J\nabla N(z).$$

We introduce the time scale. Set

$$(3.2.2)\quad t = \mu\tau, \text{ for } |\mu-1| << 1.$$

Then, in the new time scale τ, (3.2.1) becomes

$$(3.2.3)\quad \frac{dz}{d\tau} = \mu\,[JAz + JB(\lambda)z + J\nabla N(z)]$$
$$= JAz + (\mu-1)JAz + \mu JB(\lambda)z + \mu J\nabla N(z).$$

Hence, a 2π-periodic solution of (3.2.3) corresponds in one to one manner to a $2\pi\mu$ - periodic solution of the original equation (3.2.1). So, henceforth, we look for 2π-periodic solutions of (3.2.3).
Set

$$(3.2.4)\quad z = e^{JA\tau}u$$

in (3.2.3), where $u \in \mathbb{R}^{2n}$.

Then, in the new coordinate u, (3.2.3) becomes

$$(3.2.5) \quad \frac{du}{d\tau} = (\mu-1)JAu + \mu e^{-Ja\tau}JB(\lambda)e^{Ja\tau}u + \mu\, e^{-JA\tau}J\nabla N(e^{JA\tau}u).$$

Now, we claim that $J\nabla \hat{H}_2^{\lambda}(e^{Ja\tau}u) = e^{Ja\tau}J\nabla \hat{H}_2^{\lambda}(u)$ and $J\nabla N(e^{JA\tau}u) = e^{JA\tau}J\nabla N(u)$. More generally, we show the following Lemma:

<u>Lemma 3.2.1</u>: Suppose $H(z) = H_2(z) + N(z)$, $z = (x,y) \in \mathbb{R}^{2n}$ is in <u>normal form</u> with respect to $H_2(z) = \frac{1}{2}\sum_{j=1}^{\ell}(x_j^2 + y_j^2) - \frac{1}{2}\sum_{j=\ell+1}^{n}(x_j^2 + y_j^2)$ $= \frac{1}{2}z^TAz$, i.e., $H \in \ker \mathrm{ad}_{H_2}$. Then, the Hamiltonian vector field $X_H(z)$ $= J\nabla H(z)$ is equivariant under the action of the one-parameter group of symplectic diffeomorphisms generated by the flow of $X_{H_2}(z)$, that is,

$$(3.2.6) \quad X_H(e^{JAt}z) = e^{JAt}X_H(z).$$

<u>Proof</u> Since H(z) is in normal form with respect to $H_2(z) = \frac{1}{2}z^TAz$, $H \in \ker \mathrm{ad}_{H_2}$ i.e., $\mathrm{ad}_{H_2}H = 0$.
Since $X_{H_2}(z) = \mathrm{ad}_{H_2}(z)$, the flow generated by $X_{H_2}(z)$ is $\exp t\mathrm{ad}_{H_2}z = (\exp tJA)z$. Also, note that $(\exp t\mathrm{ad}_{H_2})\, H(z) = H((\exp t\mathrm{ad}_{H_2})z)$ (see Theorem 1.1.20). But, $(\exp t\mathrm{ad}_{H_2})\, H(z) = H(z)$ since $\mathrm{ad}_{H_2}H = 0$. Hence,

$$H((\exp JAt)z) = H(z).$$

That is, $H(z)$ is invariant under the S^1 action of the one-parameter group $\{\exp JAt: t \in S^1\}$of symplectic diffeomorphisms. Now,

$$X_H((\exp JAt)u) = J \quad \nabla_z H(z)\Big|_{z = (\exp JAt)\cdot u}$$

$$= J \left[\frac{\partial H(\exp JAtu)}{\partial u} \frac{\partial u}{\partial z}\right]^T$$

$$= J \cdot \left[\frac{\partial H(u)}{\partial u} \cdot \exp(-JAt)\right]^T \quad (\text{since } H(\exp JAt\, u) = H(u))$$

$$= J \cdot [\exp JAt \cdot \nabla_u H(u)] \quad (\text{since} (\exp(-JAt))^T = \exp JAt)$$

$$= \exp JAt\, [J \cdot \nabla_u H(u)] \quad (\exp(-JAt) J \exp JAt = J$$

since exp JAt is symplectic)

$$= \exp JAt\, X_H(u).$$

Therefore, $X_H(\exp JAt \cdot z) = \exp JAt\, X_H(z)$. ///

Returning to equation (3.2.5), by the Lemma 3.2.1, equation (3.2.5) can be written as

$$(3.2.7) \quad \frac{du}{d\tau} = (\mu-1)JAu + \mu JB(\lambda)u + \mu J\nabla N(u)$$

$$= (\mu-1)\ J\nabla H_2(u) + \mu \quad J\nabla H_2^\lambda(u) + \mu\ J\nabla N(u)$$

$$= J\ \nabla[(\mu-1)H_2(u) + \mu H_2^\lambda(u) + \mu N(u)].$$

Note that the right hand side of the above equation is still equivariant under exp $JA\tau$. Now, we look for 2π-periodic solutions $u(\tau)$ of (3.2.7) via the Liapunov-Schmidt Reduction with symmetry.

Lemma 3.2.2.: The bifurcation function for (3.2.7) is just the right hand side of (3.2.7), i.e.,

(3.2.8) $V(a,\mu,\lambda) = (\mu-1)JAa + \mu JB(\lambda)a + \mu J\nabla N(a), \; a \in \mathbb{R}^{2n}$

and hence $V(a,\mu,\lambda)$ inherits the symmetry from (3.2.7), i.e.,

(3.2.9) $V(e^{JA\tau}a,\mu,\lambda) = e^{JA\tau}\, V(a,\mu,\lambda)$.

proof Consider the linearized equation of (3.2.7) at $u = 0$, $\mu = 1$, $\lambda = 0$.

$$\dot{u} = 0.$$

This equation has the 2π-periodic solutions u = constant. Let $\phi(t)$, and $\Psi(t)$ be the 2n x 2n matrix whose columns are linearly independent 2π-periodic solutions of $\dot{u} = 0$ and its adjoint equation respectively (the same as $\dot{u} = 0$ in this case). Then,

$$\phi(t) = \Psi(t) = I_{2n} = 2n \text{ x } 2n \text{ identity matrix.}$$

For any $f \in C^1_{2\pi}(\mathbb{R}, \mathbb{R}^{2n})$, define the projections P, Q onto the space of 2π-periodic solutions of $\dot{u} = 0$ and its adjoint equation by

$$Pf = Qf = \phi(t)\; b = I_{2n}\left(\int_0^{2\pi} \phi^*\phi\right)^{-1}\left(\int_0^{2\pi} \phi^* f\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt. \qquad \text{(see Hale [28])}$$

Then (3.2.7) is equivalent to

$$\begin{cases} (I-P)\dot{u} = (I-P)\,[(\mu-1)JAu + \mu JB(\lambda)u + \mu J\nabla N(u)] \\ P\dot{u} = P[(\mu-1)JAu + \mu JB(\lambda)u + \mu J\nabla N(u)] \end{cases}$$

or, equivalently,

$$(3.2.10) \quad \begin{cases} \text{(a)}\ u = a + K(I-P)\,[(\mu-1)JAu + \mu JB(\lambda)u + \mu J\nabla N(u)] \\ \text{(b)}\ 0 = P[(\mu-1)JAu + \mu JB(\lambda)u + \mu J\nabla N(u)], \end{cases}$$

where $K: (I-P)\, C^o_{2\pi} \to (I-P)C^o_{2\pi}$ such that Kg is the unique 2π-periodic solution of $\dot{u} = g(t)$ for $g \in (I-P)C^o_{2\pi}$ with $PKg = 0$ and $a \in \mathbb{R}^{2n}$ such that $a = Pu$.

Let $F(u,a,\mu,\lambda) = u - a - K(I-P)[(\mu-1)JAu + \mu JB(\lambda)u + \mu J\nabla N(u)]$.

Then, $F(0,0,1,0) = 0$

$$F_u(0,0,1,0) = I_{2n}.$$

Hence, by the Implicit function theorem, there exists a unique function $u^* = u^*(a,\mu,\lambda)$ for $|a| << 1$, $|\mu-1| << 1$, $|\lambda| << 1$ such that $F(u^*(a,\mu,\lambda), a,\mu,\lambda) = 0$ and $u^*(0,1,0) = 0$.

But, notice that $u^* = a$ satisfies (3.2.10)(a). By the uniqueness of the Implicit function theorem, it follows that (3.2.10)(a) has unique solution $u^* = a$. Substituting $u = a$ into (3.2.10)(b), we obtain the bifurcation equation.

$$0 = P[(\mu-1)JAa + \mu JB(\lambda)a + \mu J\nabla N(a)]$$

$$= (\mu-1)JAa + \mu JB(\lambda)a + \mu J\nabla N(a).$$

Therefore, the bifurcation function of (3.2.7) is

$$
\begin{aligned}
V(a,\mu,\lambda) &= (\mu-1)\ JA\ a + \mu JB(\lambda)a + \mu J\nabla N(a) \\
&= J\nabla[(\mu-1)H_2(a) + \mu H_2^{\lambda}(a) + \mu\ N(a)],\ a \in \mathbb{R}^{2n} \\
&= \text{Right hand side of equation (3.2.7).}
\end{aligned}
$$

That is, the solution set (a,μ,λ) of $V(a,\mu,\lambda) = 0$ is just the critical points of equation (3.2,7). Moreover, since the Right hand side of (3.2.7) is equivariant under $e^{JA\tau}$, clearly $V(a,\mu,\lambda)$ is also equivariant under $\exp JA\tau$. ///

Remark: Since $V(a,\mu,\lambda)$ can be expressed as

(3.2.11) $V(a,\mu,\lambda) = J\nabla S(a,\mu,\lambda)$

where $S(a,\mu,\lambda) = (\mu-1)H_2(a) + \mu H_2^{\lambda}(a) + \mu\ N(a)$, it follows that finding zeros (a,μ,λ) of $V(a,\mu,\lambda) = 0$ is equivalent to finding critical points of the real-valued function $S(a,\mu,\lambda)$ and each zero (a,μ,λ) of $V(a,\mu,\lambda) = 0$ corresponds locally in 1 - 1 fashion to each 2π-periodic solution $z(\tau) = (\exp JA\tau) \cdot a$ of (3.2.3) and so locally 1 - 1 corresponds to each $2\pi\mu$ - periodic solution of the original equation (3.2.1). From now on, we try to find the zero set of $V(a,\mu,\lambda)$, i.e., the critical points of the real-valued function $S(a,\mu,\lambda)$.

Note that $V(0,\mu,\lambda) = 0$ for all $\mu \approx 1$, $\lambda \approx 0$

$$
\begin{aligned}
D_aV(0,1,0) &= (\mu-1)JA + \mu JB(\lambda) + \mu JD^2N(a)\Big|_{a=0,\ \mu=1,\lambda=0} \\
&= 0 \qquad \text{where } N(a) = O(|a|^4)
\end{aligned}
$$

implies $(a,\mu,\lambda) = (0,1,0)$ is a singularity of $V(a,\mu,\lambda)$ and so is a possible bifurcation point. Furthermore,

$$D_{aa}V(0,1,0) = \mu J D^3 N(a)\Big|_{a=0} = 0,$$
$$D_{aaa}V(0,1,0) = \mu J D^4 N(a)\Big|_{a=0} \neq 0.$$

So, $V(a,\mu,\lambda) = O(|a|^3)$ as $a \to 0$, at $\mu = 1$ and $\lambda = 0$.

Now, returning to bifurcation equation (3.2.8),

$$V(a,\mu,\lambda) = (\mu-1)JAa + \mu JB(\lambda)a + \mu J\nabla N(a),$$

we first try to determine $\mu = \mu^*(a,\lambda)$ uniquely and continously so that $V(a,\mu^*(a,\lambda),\lambda) \overset{\text{def}}{=} \tilde{V}(a,\lambda)$ is orthogonal to JAa:

We put

$$\text{(3.2.12)} \qquad \begin{aligned} F(a,\mu,\lambda) &= \frac{1}{|JAa|^2} \langle JAa,\ V(a,\mu,\lambda) \rangle \quad \text{for } a \neq 0 \\ &= \mu-1 + \frac{\mu \langle JAa,\ JB(\lambda)a + J\nabla N(a) \rangle}{|JAa|^2}. \end{aligned}$$

Then we have

$$F(0,1,0) = 0 \text{ and } F_\mu(0,1,0) = 1.$$

Hence, by the Implicit function theorem,

there exists a unique C^1 function $\mu = \mu^*(a,\lambda)$ near $a = 0$, $\lambda = 0$ such that

$F(a,\ \mu^*(a,\lambda)\ ,\lambda) = 0$, $\mu(0,0) = 1$. i,e., (3.2.13) $\langle JAa, V(a,\mu^*(a,\lambda), \lambda) \rangle = 0$ for $0 < |a| << 1$, $|\lambda| << 1$.

In fact, from (3.2.12),

$$\mu\left(1 + \frac{\langle JAa, JB(\lambda)a + J\nabla N(a) \rangle}{|JAa|^2}\right) = 1.$$

Hence, $\mu^*(a,\lambda)$ is explicitly given by

$$\mu^*(a,\lambda) = \frac{|JAa|^2}{|JAa|^2 + \langle JAa, JB(\lambda)a + J\nabla N(a) \rangle}$$

$$= \frac{|a|^2}{|a|^2 + \langle Aa, B(\lambda)a + \nabla N(a) \rangle} \qquad (3.2.14)$$

$$= \frac{1}{1 + O(|\lambda| + |a|^2)} .$$

So, for $|a| << 1$ and $|\lambda| << 1$, $\mu^*(a,\lambda) \approx 1 - O(|\lambda| + |a|^2)$.

Notice that even though the formula (3.2.14) may be valid for all a,λ, we must restrict ourselves to a sufficiently small neighborhood of $(a,\lambda) = (0,0)$ to ensure that $|\mu^*(a,\lambda) - 1| << 1$.

<u>Lemma 3.2.3</u>: Let $\tilde{V}(a,\lambda) = V(a,\mu^*(a,\lambda), \lambda)$ for all $0 < |a| << 1$ and $|\lambda| << 1$, where $\mu^*(a,\lambda)$ is give by (3.2.14). Then, $\mu^*(a,\lambda)$ is invariant and $\tilde{V}(a,\lambda)$ is equivariant under the action of the 1-parameter group $\{\exp JAt: t \in \mathbb{R}\}$, that is,

$$\mu^*(\exp JAt\ a,\lambda) = \mu^*(a,\lambda)$$

$$\tilde{V}(\exp JAt\ a,\lambda) = \exp JAt\ \tilde{V}(a,\lambda).$$

<u>proof</u> Let $h(a,\lambda) = JB(\lambda)a + J\nabla N(a)$ in (3.2.14). Then from (3.2.14),

$$\mu^*((\exp JAt)\, a,\lambda) = \frac{|(\exp JAt)\, a|^2}{|\exp JAt\, a|^2 + \langle JA\, (\exp JAt)\, a,\ h(\exp JAt\, a,\lambda)\rangle}.$$

$|(\exp JAt)\, a|^2 = \langle (\exp JAt)\, a,\ \exp (JAt)\, a \rangle = \langle a,\ (\exp -JAt) \cdot (\exp JAt)\, a \rangle = |a|^2$. Since $h(a,\lambda)$ is equivariant under e^{JAt},

$$\begin{aligned} \langle JA \exp JAt\, a,\ h((\exp JAt)\, a,\lambda)\rangle &= \langle (\exp JAt)\, JAa,\ (\exp JAt)\, h(a,\lambda) \rangle \\ &= \langle JAa,\ h(a,\lambda) \rangle. \end{aligned}$$

Hence, $\mu^*(\exp JAt\, a,\lambda) = \mu^*(a,\lambda)$.

Also,

$$\begin{aligned} \tilde{V}(\exp JAt\, a,\lambda) &= V((\exp JAt)\, a,\ \mu^*((\exp JAt)\, a,\lambda),\ \lambda) \\ &= V((\exp JAt)\, a,\ \mu^*(a,\lambda),\ \lambda) \quad \text{(since } \mu^* \text{ is invariant)} \\ &= (\exp JAt)\, V(a,\ \mu^*(a,\lambda),\ \lambda) \quad \text{(since } V \text{ is equivariant)} \\ &= (\exp JAt)\, \tilde{V}(a,\lambda). \end{aligned}$$

///

Note that for $0 < |a| \ll 1$, $|\mu-1| \ll 1$,

if $V(a,\mu,\lambda) = 0$ then $F(a,\mu,\lambda) = 0$ and by the uniqueness of μ^* we must have

$$\mu = \mu^*(a,\lambda) \text{ with } \mu^*(0,0) = 1.$$

So, $V(a,\mu,\lambda) = 0$ iff $\tilde{V}(a,\lambda) = V(a,\mu^*(a,\lambda),\ \lambda) = 0$.

Hence,

(a,λ) is a zero of $\tilde{V}(a,\lambda) = 0$

iff $(a,\mu^*(a,\lambda),\ \lambda)$ is a zero of $V(a,\mu,\lambda) = 0$

iff $(a,\ \mu^*(a,\lambda),\ \lambda)$ is a critical point of equation (3.2.7)

iff $z(\tau) = \exp JA\tau\, a$ is a 2π-periodic solution of (3.2.3)

iff $z(t) = (\exp JAt/\mu^*(a,\lambda))\, a$ is a $2\pi\, \mu^*(a,\lambda)$ - periodic solution of the orginal equation (3.2.1).

§3. Reduction to a gradient system

Now, our problem to study the periodic solution of (3.2.1) near those of the linearized equation is reduced to finding the zeros of the bifurcation equation $\tilde{V}(a,\lambda) = 0$ which is the 2n x 2n finite system. Furthermore, since $V(a,\mu,\lambda) = J\nabla S(a,\mu,\lambda)$ by (3.2.11) where $S(a,\mu,\lambda) = (\mu-1)H_2(a) + \mu H_2^\lambda(a) + \mu N(a) = \mu H^\lambda(a) - H_2(a)$, we can easily express $\tilde{V}(a,\lambda)$ as a gradient-like system as above.

Lemma 3.3.1: Let $\tilde{S}(a,\lambda) = H_2(a) - \mu^*(a,\lambda)\ [H^\lambda(a) - c]$, for any constant c. Then $J\tilde{V}(a,\lambda) = \nabla\tilde{S}(a,\lambda)$ on the energy surface $H^\lambda(a) = c$.

proof $V(a,\mu,\lambda) = J\nabla[\mu H^\lambda(a) - H_2(a)]$.

$$JV(a,\mu,\lambda) = \nabla[H_2(a) - \mu H^\lambda(a)]$$
$$= \nabla H_2(a) - \mu\ \ \nabla[H^\lambda(a) - c] \text{ for any fixed constant c.}$$

If $\mu = \mu^*(a,\lambda)$, then

$$\nabla\tilde{S}(a,\lambda) = \nabla[H_2(a) - \mu^*(a,\lambda) \cdot (H^\lambda(a) - c)]$$
$$= \nabla H_2(a) - \mu^*(a,\lambda)\ \nabla(H^\lambda(a) - c) - \nabla\mu^*(a,\lambda) \cdot (H^\lambda(a) - c)$$
$$= JV(a,\mu^*(a,\lambda),\ \lambda) \text{ on } H^\lambda(a) = c$$
$$= J\tilde{V}(a,\lambda).$$

///

Remark: This Lemma 3.3.1 (and Lemma 3.3.2 in the following) are due to Chow and Mallet-Paret [14], but in our case these Lemmas are trivial consequences of the fact that our Hamiltonian function is in normal form with respect to H_2 and so the bifurcation function $V(a,\mu,\lambda)$ is just the right hand side of equation (3.2.7), which is again a

Hamiltonian vector field.

Now, by Lemma 3.3.1, the problem of finding zeros of $\tilde{V}(a,\lambda) = 0$ is again reduced to find the critical points of the potential function $\tilde{S}(a,\lambda) = H_2(a) - \mu^*(a,\lambda) \cdot [H^\lambda(a) - c]$ on the energy surface $H^\lambda(a) = c$. Thus, we must solve the two equations

(3.3.1)
$$\begin{cases} \nabla\tilde{S}(a,\lambda) = \nabla H_2(a) - \mu^*(a,\lambda)\nabla H^\lambda(a) = 0 \\ H^\lambda(a) = c \end{cases}$$

simultaneously for each given λ and c.

Let $\bar{a}(\lambda,c)$ be a critical point of $\tilde{S}(a,\lambda)$ on the energy surface $H^\lambda(a)=c$. Define G: $\mathbb{R}^{2n} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}$ by

$$G(a,\lambda,c) = \begin{bmatrix} \nabla H_2(a) - \mu^*(a,\lambda)\nabla H^\lambda(a) \\ H^\lambda(a) - c \end{bmatrix}.$$

Then, we know that $G(0,0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$, $D_aG(0,0,0) = 0$. Hence $(\lambda,c) = (0,0)$ is a possible bifurcation value for the zeros of the system (3.3.1).

Note that on each energy surface $H^\lambda(a) = c$, $\nabla S(a,\lambda) = \nabla H_2(a) - \mu^*(a,\lambda)\nabla H^\lambda(a)$ and this resembles the Lagrange multiplier method when we compute the critical points of $H_2(a)$ with the constraint $H^\lambda(a) = c$ in which case we solve the equation $\nabla H_2(a) - \eta \cdot \nabla H^\lambda(a) = 0$, e.g., for a in term of (η,λ) and then using the constraint $H^\lambda(a) = c$ we determine $\eta = \eta(\lambda,c)$ and so determine $a = a(\lambda,c)$. In the following Lemma, it turns out that the Lagrange multiplier η so obtained coincides with $\mu^*(a,\lambda)$.

<u>Lemma 3.3.2</u>: For each $|\lambda| << 1$ and $|c| << 1$, let $\bar{a} = \bar{a}(\lambda,c)$ be a nonzero critical point of the real-valued function

$$g(a,\lambda,\eta) = H_2(a) - \eta \cdot H^{\lambda}(a) \quad \text{with } H^{\lambda}(a) = c.$$

Then, $\eta = \mu^{*}(\bar{a},\lambda)$ in a sufficiently small neighborhood of (0,0).

<u>proof</u> Since $a = \bar{a}(\lambda,c)$ is a critical point of g with $H^{\lambda}(a) = c$ for each λ, c, we have

$$\nabla H_2(\bar{a}) - \eta \nabla H^{\lambda}(\bar{a}) = 0 \quad \text{where } \eta = \eta(\lambda,c).$$

Hence, $\nabla\tilde{S}(\bar{a},\lambda) = \nabla H_2(\bar{a}) - \mu^{*}(\bar{a},\lambda)\nabla H^{\lambda}(\bar{a}) = [\eta - \mu^{*}(\bar{a},\lambda)]\cdot \nabla H^{\lambda}(\bar{a})$.

Then, from (3.2.13) and Lemma 3.3.1, we have

$$0 = \langle JA\bar{a}, \tilde{V}(\bar{a},\lambda) \rangle$$

$$= \langle JA\bar{a}, J^{-1}\nabla\tilde{S}(\bar{a},\lambda) \rangle \text{ on } H^{\lambda}(\bar{a}) = c$$

$$= -[\eta - \mu^{*}(\bar{a},\lambda)] \langle JA\bar{a}, J\nabla H^{\lambda}(\bar{a}) \rangle \text{ (since } J^{-1} = -J).$$

Multiplying both sides by $\mu^{*}(\bar{a},\lambda)$, we can write

$$0 = -[\eta - \mu^{*}(\bar{a},\lambda)] \cdot \langle JA\bar{a}, \mu^{*}J\nabla H^{\lambda}(\bar{a}) - JA\bar{a} + JA\bar{a} \rangle$$

Since $\tilde{V}(a,\lambda) = \mu^{*}J\nabla H^{\lambda}(a) - JAa$ and $\langle JAa, \tilde{V}(a,\lambda) \rangle = 0$ for all $0 < |a| << 1$ and $|\lambda| << 1$, we have

$$0 = -\ [\eta - \mu^{*}(\bar{a},\lambda)] \cdot \langle\ JA\bar{a},\ \tilde{V}(a,\lambda) + JA\bar{a}\ \rangle$$
$$= -\ [\eta - \mu^{*}(\bar{a},\lambda)] \cdot\ |\bar{a}|^2$$

Hence, $\eta = \mu^{*}(\bar{a},\lambda)$ since $\bar{a} \neq 0$. ///

Therefore, finally our problem to study the periodic solutions of (3.2.1) has been reduced to finding the critical points of the real -valued function $g(a,\lambda,\eta) = H_2(a) - \eta \cdot H^{\lambda}(a)$ with $H^{\lambda}(a) = c$. So, if we solve the equation $\nabla(H_2(a) - \eta H^{\lambda}(a)) = 0$ with $H^{\lambda}(a) = c$ then η will be automatically determined as $\mu^{*}(a,\lambda)$.

Note that g is invariant and ∇g is equivariant under the action of the group $\{e^{JAt}:\ t \in \mathbb{R}\}$ and hence if $\bar{a} = \bar{a}(\lambda,c)$ is a solution of $\nabla g = 0$ with $H^{\lambda}(a) = c$, then $e^{JAt}\bar{a}$ are also critical points of g on the same energy surface for all time t. Now, we summarize all the above results in the following theorem, which will be a main theorem of this thesis.

<u>Theorem 3.3.3</u>: Consider a family of Hamiltonian functions

$$H^{\lambda}(z) = H_2(z) + H_2^{\lambda}(z) + N(z),\ z = (x,y) \in \mathbb{R}^{2n}$$

passing through 1: ... :1: -1: ... : -1, semisimple resonance at $\lambda = 0$, where $H_2(z)$ is given by

$$H_2(z) = \frac{1}{2}\sum_{j=1}^{\ell}(x_j^2 + y_j^2) - \frac{1}{2}\sum_{j=\ell+1}^{n}(x_j^2 + y_j^2) = \frac{1}{2}z^tAz$$

and $H_2^\lambda(z) = \frac{1}{2} z^t B(\lambda) z$ is a versal deformation of $H_2(z)$ and $N(z)$ is a higher order term. Suppose that $H^\lambda(z)$ is in H_2-normal form. Then, the periodic solution of the Hamiltonian system

$$\dot{z} = J\nabla H^\lambda(z)$$

on the energy surface $H^\lambda(z) = c$ are locally in a one-to-one correspondence to the critical points of the real-valued function

$$g(a,\lambda,\eta) = H_2(a) - \eta \cdot H^\lambda(a)$$

on $H^\lambda(a) = c$. More precisely, if $\bar{a} = \bar{a}(\lambda,c)$ is a critical point of g on $H^\lambda(a) = c$ for $|a|, |\lambda|, |c| << 1$, then the Hamiltonian system has a periodic solution

$$z(t) = e^{Jat/u^*(\bar{a},\lambda)} \cdot \bar{a}$$

with period $2\pi\mu^*$, where $\mu^*(a,\lambda)$ is given by

$$\mu^*(a,\lambda) = \frac{|a|^2}{|a|^2 + \langle Aa, B(\lambda)a + \nabla N(a) \rangle}$$

for $|a|, |\lambda| << 1$.

<u>proof</u>: Obvious from all the Lemmas in this chapter. ///

Hence, from now on, we concentrate only on the problem to find critical points of the real-valued function

$$g(a,\lambda,\eta) = H_2(a) - \eta\, H^{\lambda}(a)$$

on the energy surface $H^{\lambda}(a) = c$.

By using the invariance of g and the equivarience of ∇g under the action of $\{\exp JAt \mid t \in \mathbb{R}\}$, the problem to solve the equation $\nabla g = 0$ can be further reduced and we are going to work out this problem in the two degrees of freedom case explicity to show the bifurcation of the periodic orbits.

CHAPTER 4: TWO DEGREES OF FREEDOM 1: - 1 SEMISIMPLE RESONANCE PROBLEM

In this chapter, we apply the general theory of Chapter 2 and Chapter 3 to the Hamiltonian function of 2 degrees of freedom with the nonpositive definite quadratic form at 1: -1 semi-simple resonance, and study the bifurcations of periodic orbits as the parameter passes through the resonance.

§1. Normal form and Versal deformation

Consider the Hamiltonian H: $\mathbb{R}^4 \to \mathbb{R}$, C^∞, with the nonpositive definite quadratic form at 1: -1 semisimple resonance:

$$(4.1.1) \quad H_2(x,y) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2).$$

First, we find the normal form of H with respect to H_2 up to the 4th order.

In this case, the map $ad_{H_2}: C^\infty(\mathbb{R}^4, \mathbb{R}) \to C^\infty(\mathbb{R}^4, \mathbb{R})$ is given by

$$ad_{H_2} = \sum_{j=1}^{2} \left(\frac{\partial H_2}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial H_2}{\partial x_j} \frac{\partial}{\partial y_j} \right)$$

$$= \left(y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1}\right) - \left(y_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial y_2}\right).$$

In complex conjugate coordinates $(z,\bar{z}) = (z_1, z_2, \bar{z}_1, \bar{z}_2) \in \mathbb{C}^4$ with $z_j = x_j + iy_j$ $(j = 1,2)$, H_2 and ad_{H_2} can be rewritten as

$$H_2(x,y) = \tilde{H}_2(z,\bar{z}) = \frac{1}{2} z_1 \bar{z}_1 - \frac{1}{2} z_2 \bar{z}_2$$

$$ad_{\tilde{H}_2}(z,\bar{z}) = -i \left[\left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}\right) - \left(z_2 \frac{\partial}{\partial z_2} - \bar{z} \frac{\partial}{\partial \bar{z}_2}\right) \right].$$

The action of $ad_{\tilde{H}_2}$ on the basis monomial $z^k \bar{z}^\ell$ for the space $\tilde{\mathcal{P}}_n(z,\bar{z})$ of homogeous polynomial of degree n in $z,\bar{z}$ is

$$ad_{\tilde{H}_2}(z^k \bar{z}^\ell) = -i[(k_1 - \ell_1) - (k_2 - \ell_2)] (z^k \bar{z}^\ell).$$

Hence,

$$z^k \bar{z}^\ell \in \ker ad_{\tilde{H}_2} \text{ iff } k_1 - \ell_1 = k_2 - \ell_2 \text{ (resonance relation).}$$

It follows immediately that there are no third or higher odd-ordered terms in the normal form of H. The (Hilbert) generators for Ker $ad_{H_2}\big|_{\mathcal{P}_2}$ are given by

$$(4.1.2) \quad \rho_1 = |z_1|^2 = x_1^2 + y_1^2, \; \rho_2 = |z_2|^2 = x_2^2 + y_2^2,$$

$$\rho_3 = \mathrm{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2 \;, \; \rho_4 = \mathrm{Im}(z_1 z_2) = x_1 y_2 + x_2 y_1 .$$

Also, the generators for Ker $ad_{H_2}\big|_{\mathcal{P}_4}$ are given by

$$e_1 = z_1^2 \bar{z}_1^2 = (x_1^2 + y_1^2)^2$$

$$
\begin{aligned}
e_2 &= \mathrm{Re}\,(z_1^2\ z_2\ \bar{z}_1) = (x_1^2 + y_1^2)(x_1x_2 - y_1y_2) \\
e_3 &= \mathrm{Im}\,(z_1^2\ z_2\ \bar{z}_1) = (x_1^2 + y_1^2)(x_1y_2 + x_2y_1) \\
e_4 &= z_1z_2\bar{z}_1\bar{z}_2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) \\
e_5 &= \mathrm{Re}(z_1^2z_2^2) = (x_1^2 - y_1^2)(x_2^2 - y_2^2) - 4x_1x_2y_1y_2 \\
(4.1.3) \qquad e_6 &= \mathrm{Im}(z_1^2z_2^2) = 2x_1y_1(x_2^2 - y_2^2) + 2x_2y_2(x_1^2 - y_1^2) \\
e_7 &= \mathrm{Re}(z_1z_2^2\bar{z}_2) = (x_2^2 + y_2^2)(x_1x_2 - y_1y_2) \\
e_8 &= \mathrm{Im}(z_1z_1^2\bar{z}_2) = (x_2^2 + y_2^2)(x_1y_2 + x_2y_1) \\
e_9 &= z_2^2\bar{z}_2^2 = (x_2^2 + y_2^2)^2.
\end{aligned}
$$

Therefore, the normal form for $H(x,y)$ with respect to H_2 up to the fourth order is given by

$$(4.1.4) \quad H(x,y) = H_2(x,y) + H_4(x,y) + (\text{higher order terms})$$

where $H_4(x,y) = \sum_{j=1}^{9} a_j e_j$ and e_j's are given in (4.1.3). Moreover, from the general theory of Chapter 2, the versal deformation of $H(x,y)$ up to the second order i.e., in the space $\mathcal{P}_2(x,y)$ can be written as

$$(4.1.5) \quad H^{\lambda}(x,y) = H_2(x,y) + H_2^{\lambda}(x,y) + H_4(x,y) + O(|x,y|^5)$$

where

$$H_2^{\lambda}(x,y) = \frac{1}{2}\lambda_1(x_1^2 + y_1^2) - \frac{1}{2}\lambda_2(x_2^2 + y_2^2) + \lambda_3(x_1x_2 - y_1y_2) +$$
$$\lambda_4(x_1y_2 + x_2y_1).$$

Now, we consider the truncated Hamiltonain, denoted again by $H^{\lambda}(x,y)$, containing only a single fourth order term e_3.

$$(4.1.6) \quad H^{\lambda}(x,y) = H_2(x,y) + H_2^{\lambda}(x,y) + \bar{H}_4(x,y)$$

with

$$\bar{H}_4(x,y) = e_3 = (x_1^2 + y_1^2)(x_1y_2 + x_2y_1).$$

Remark: Here, we picked up a fourth order term e_3 randomly just for simplicity of calculation to show our method to get the bifurcation explicity in the presence of nonlinear terms. Even if we consider the full nine fourth order terms in $H_4(x,y)$, our methodology will be just the same except a slightly more involved computation.

Rewriting (4.1.6) in vector-matrix notation with $z = (x,y) \in \mathbb{R}^4$, we have

$$(4.1.7) \quad H^{\lambda}(z) = H_2(z) + H_2^{\lambda}(z) + \bar{H}_4(z) = \frac{1}{2} z^T A z + \frac{1}{2} z^T B(\lambda) z + \bar{H}_4(z),$$

where $A = \text{diag}(1, -1, 1, -1)$, and

$$B(\lambda) = \left[\begin{array}{cc|cc} \lambda_1 & \lambda_3 & 0 & \lambda_4 \\ \lambda_3 & -\lambda_2 & \lambda_4 & 0 \\ \hline 0 & \lambda_4 & \lambda_1 & -\lambda_3 \\ \lambda_4 & 0 & -\lambda_3 & -\lambda_2 \end{array}\right].$$

The corresponding Hamilton's equation is

$$(4.1.8) \quad \dot{z} = J\nabla H^{\lambda}(z) = JAz + JB(\lambda)z + J\nabla\bar{H}_4(z),$$

where

$$JA = \left[\begin{array}{cc|cc} & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & -1 \\ \hline -1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \end{array}\right], \qquad JB(\lambda) = \left[\begin{array}{cc|cc} 0 & \lambda_4 & \lambda_1 & -\lambda_3 \\ \lambda_4 & 0 & -\lambda_3 & -\lambda_2 \\ \hline -\lambda_1 & -\lambda_3 & 0 & -\lambda_4 \\ -\lambda_3 & \lambda_2 & -\lambda_4 & 0 \end{array}\right]$$

$$J\nabla\bar{H}_4(z) = \begin{bmatrix} 2y_1(x_1y_2 + x_2y_1) + x_2(x_1^2 + y_1^2) \\ x_1(x_1^2 + y_1^2) \\ -2x_1(x_1y_2 + x_2y_1) - y_2(x_1^2 + y_1^2) \\ -y_1(x_1^2 + y_1^2) \end{bmatrix}.$$

<u>Remark</u>: Since each term in the truncated Hamiltonian (4.1.7) is in the normal form with respect to H_2, $H^{\lambda}(z)$ is <u>invariant</u> under the action (rotation) of the one-parameter group of symplectic diffeomorphisms $\{\exp JAt: t \in \mathbb{R}\}$ generated by the flow of X_{H_2}. Hence, by Lemma 3.2.1 in Chapter 3, $X_{H^\lambda}(z) = J\nabla H^{\lambda}(z)$ is <u>equivariant</u> under the same action. Furthermore, since $\{H_2, H^{\lambda}\} = 0$, the system (4.1.8) is completely integrable with integrals $H^{\lambda}(z)$ and $H_2(z)$. Also note that the system (4.1.8) is a versal deformation of H in $\mathcal{P}_2(\mathbb{R}^4, \mathbb{R})$ with codimension 4 of the unperturbed system $\dot{z} = J\nabla H^{0}(z)$ preserving the Hamiltonian character. Since the number of parameters are too many to examine the qualitative behavior of (4.1.8), we are going to restrict ourselves to the codimension one bifurcations by choosing a suitable parameter and setting the other parameters to be zero.

§2. <u>Invariant manifolds of the linearized system</u>

The linearized Hamilton's equation of (4.1.8) at $z = 0$ and $\lambda = 0$

is

$$\dot{z} = JAz \tag{4.2.1}$$

with the solution $z(t,z_o) = (\exp JAt) \cdot z_o$ starting from the initial point $z_o \in \mathbb{R}^4$ at $t = 0$. Since the 4 x 4 matrix JA has the eigenvalues $\pm i$ each with multiplicty 2 and $\exp JAt = I(\cos t) + JA(\sin t)$, we have $|z(t)| = |\exp JAt \cdot z_o| = |z_o|$. Hence each solution curve $z(t,z_o) = (\exp JAt) \cdot z_o$ is a 2π-periodic circle lying on the 3-sphere S^3: $x_1^2 + y_1^2 + x_2^2 + y_2^2 = |z_o|^2$ in $\mathbb{R}^4$. Notice that the linear system (4.2.1) has the Hamiltonian

$$H(x.y) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2) \tag{4.2.2}$$

which is the sum of the energy functions of two harmonic oscillators running opposite in time both with frequency 1. Furthermore, the system (4.2.1) has another integral

$$L(x,y) = \frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{2}(x_2^2 + y_2^2) \tag{4.2.3}$$

as we can easily see from the fact that $\{H,L\} = 0$. The function $L(x,y)$ may be considered, up to canonical change of coordinates, as the angular momentum of these oscillators. Now, we may consider the so-called "energy - momentum mapping"

$$H \times L \overset{\text{def}}{=} (H,L) : \mathbb{R}^4 \to \mathbb{R}^2$$

defined by

$$(H \times L)(x_1,x_2,y_1,y_2) = (H(x,y),\ L(x,y)).$$

Then, each orbit, i.e., 2π-periodic circle, of the linear system (4.2.1) lies on the level set of the mapping H x L.

$$(4.2.4) \quad (H \times L)^{-1}(h,\ell) = \{(x,y) \in \mathbb{R}^4 \mid H(x,y) = h,\ L(x,y) = \ell\}$$

where $h \in \mathbb{R}$, $\ell \geq 0$.

If $(h,\ell) \in \mathbb{R}^2$ is a <u>regular value</u> of the mapping H x L, then the level set $(H \times L)^{-1}(h,\ell)$ defines a smooth 2 - dimensional invariant manifold of the system (4.2.1) in $\mathbb{R}^4$.

However, if $(h,\ell) \in \mathbb{R}^2$ is a <u>critical value</u> of the mapping H x L, that is, for some $z_o \in (H \times L)^{-1}(h,\ell)$, the derivative $D(H \times L)(z_o)$: $T_{z_o}\mathbb{R}^4 \rightarrow T_{(h,\ell)}\mathbb{R}^2$ is not surjective, then the level set $(H \times L)^{-1}(h,\ell)$ will be at most a 1 - dimensional critical manifold. To be more precise, let's find out the critical sets of the mapping H x L. Recall that a point $z = (x_1,x_2,y_1,y_2) \in \mathbb{R}^4$ is a critical point of H x L iff $D(H \times L)(z)$ is not surjective. Since $D(H \times L)(z) = (DH(z),\ DL(z))$, $z \in \mathbb{R}^4$ is a critical point of H x L if

(i) $DH(z) = 0$ or $DL(z) = 0$

or

(ii) $DL(z) + \lambda DH(z) = 0$ for some $\lambda \neq 0$, i.e., z is a critical point of $L|_{H^{-1}(h)}$ and λ is a Lagrange multiplier.

In the case (i), we have only the trivial critical point $z = 0$ with the energy $H = 0$ and the momentum $L = 0$. In the case (ii), for each $h \in \mathbb{R}$, the solutions of the system of equations for $\lambda \neq 0$,

$$\begin{cases} (1 + \lambda)x_1 = 0 \\ (1 - \lambda)x_2 = 0 \\ (1 + \lambda)y_1 = 0 \\ (1 - \lambda)y_2 = 0 \end{cases}$$

yield the critical circle

$$S_1 = \{(x_1, 0, y_1, 0) \in \mathbb{R}^4 \mid \tfrac{1}{2}(x_1^2 + y_1^2) = h\} \quad \text{for } h > 0$$

and

$$S_2 = \{(0, x_2, 0, y_2) \in \mathbb{R}^4 \mid \tfrac{1}{2}(x_2^2 + y_2^2) = -h\} \quad \text{for } h < 0$$

with the corresponding critical values $(H, L) = (h, h)$ for $h > 0$ and $(h, -h)$ for $h < 0$ respectively. For $h = 0$, (ii) yields only the trivial critical point $z = 0$. Therefore, for those critical values (h, ℓ) with $\ell = h$ for $h > 0$ and $\ell = -h$ for $h < 0$ of the mapping $H \times L$, the level set $(H \times L)^{-1}(h, \ell)$ is the 1 - dimensional circle lying in the (x_1, y_1) plane for $h > 0$ and in the (x_2, y_2) - plane for $h < 0$ respectively.

The foliation of the constant energy surface $H^{-1}(h)$ for each given $h \in \mathbb{R}$ with respect to the parameter values of ℓ can be easily examined by using polar coordinates.

Putting $x_1 = \gamma_1 \cos\theta_1, \quad y_1 = \gamma_1 \sin\theta_1$

$x_2 = \gamma_2 \cos\theta_2 \quad y_2 = \gamma_2 \sin\theta_2,$

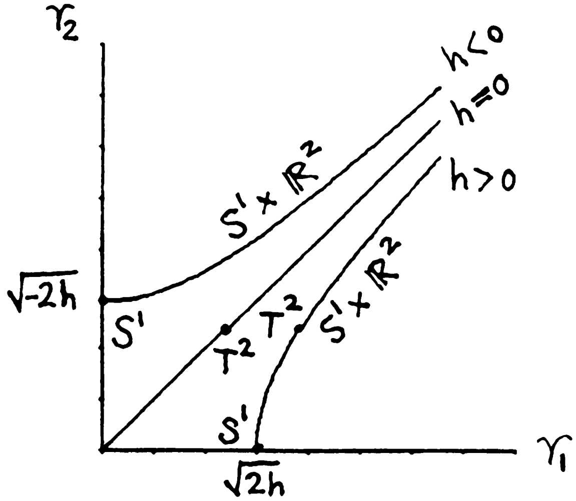
then the level set

$$H^{-1}(h) = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid \tfrac{1}{2}(x_1^2 + y_1^2) - \tfrac{1}{2}(x_2^2 + y_2^2) = h\}$$

can be expressed as

$$H^{-1}(h) = \{(\gamma_1,\gamma_2) \in \mathbb{R}^2 \mid \frac{1}{2}\gamma_1^2 - \frac{1}{2}\gamma_2^2 = h,\ \gamma_1,\gamma_2 \geq 0,\ h \in \mathbb{R}\},$$

which is a hyperbola for each $h \neq 0$ and a straight line for $h = 0$ in the (γ_1,γ_2)-plane as shown in < Figure 4.2.1 >.



< Figure 4.2.1 >

For $h \neq 0$, a constant energy surface $H^{-1}(h)$ is diffeomorphic to $S^1 \times \mathbb{R}^2$ and hence is not compact, while for $h = 0$, it is a cone $\{0\} \times \mathbb{R}^2$ over S^1 with vertex at the orgin. Also, the level set

$$L^{-1}(\ell) = \{(x_1,x_2,y_1,y_2) \in \mathbb{R}^4 \mid \frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{2}(x_2^2 + y_2^2) = \ell\}$$

can be rewritten as

$$L^{-1}(\ell) = \{(\gamma_1,\gamma_2) \in \mathbb{R}^2 \mid \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_2^2 = \ell,\quad \gamma_1,\gamma_2 \geq 0,\ \ell \geq 0\},$$

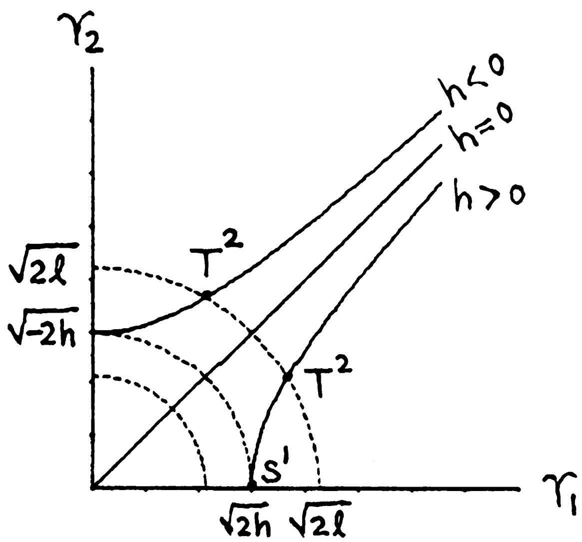
which is a quater-circle in the (γ_1,γ_2)-plane with radians $\sqrt{2\ell}$ as shown as dotted lines in < Figures 4.2.2 >.

Hence, the level set $(H \times L)^{-1}(h,\ell)$ may be expressed as the set of intersection points of the two curves

$$\gamma_1^2 - \gamma_2^2 = 2h \quad (h \in \mathbb{R})$$

$$\gamma_1^2 + \gamma_2^2 = 2\ell \quad (\ell \geq 0)$$

in the (γ_1,γ_2) -plane as is shown in < Figure 4.2.2 >



< Figure 4.2.2 >

Hence, for $h > 0$, if $\ell > h$ then $(H \times L)^{-1}(h,\ell) = S^1 \times S^1 = T^2$ and if $\ell = h$ (critical values of $H \times L$) then $(H \times L)^{-1}(h,\ell) = S^1 \times \{0\}$, which is a circle lying in the (x_1,y_1)-plane. While for $h < 0$, $(H \times L)^{-1}(h,\ell) = S^1 \times S^1 = T^2$ for $\ell > -h$ and $(H \times L)^{-1}(h,\ell) = \{0\} \times S^1$ for $\ell = -h$ (critical values of $H \times L$). If $0 \leq \ell < |h|$ then $(H \times L)^{-1}(h,\ell) = \phi$ for

$|h| \neq 0$.

For $h = 0$, $(H \times L)^{-1}(h,\ell) = \begin{cases} T^2 & \text{for } \ell > 0 \\ \{0\} & \text{for } \ell = 0 \end{cases}$

Therefore, we can conclude that every solution curve $z(t) = (\exp JAt) z_o$ of the linear system (4.2.1) is a (2π-periodic) circle lying on $S^1 \times \{0\}$, $\{0\} \times S^1$, or T^2 depending on the values of h and ℓ.

Remark: As van der Meer did in [35], we may also use the S^1-invariant variables defined by

$$\pi_1 = (x_1^2 + y_1^2) - (x_2^2 + y_2^2) = 2h \text{ (fixed)}$$

$$\pi_2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) = 2\ell \geq 0,$$

$$\pi_3 = 2(x_1x_2 + y_1y_2)$$

$$\pi_4 = 2(x_1y_2 - x_2y_1)$$

with relationship

$$\pi_1^2 + \pi_3^2 + \pi_4^2 = \pi_2^2$$

in order to describe the foliation of the constant energy surface $H^{-1}(h)$ for given $h \in \mathbb{R}$. Rewriting the identity as

$$\pi_2^2 - \pi_3^2 - \pi_4^2 = \pi_1^2 = (2h)^2 = \text{constant}, \ \pi_2 = 2\ell \geq 0,$$

then the mapping

$$F: \quad (x_1, x_2, y_1, y_2) \to (\pi_2, \pi_3, \pi_4) \text{ with } \pi_2 \geq 0$$

maps the constant energy surface $H^{-1}(h) = S^1 \times \mathbb{R}^2$ to a connected piece of two-sheeted hyperboloid in $\mathbb{R}^3$

$$\pi_2^2 - \pi_3^2 - \pi_4^2 = (2h)^2, \quad \pi_2 \geq 0$$

if $h \neq 0$ and to a half-cone if $h = 0$. The intersection of the hyperbold and the plane $\pi_2 = 2\ell \geq 0$ is a circle whose preimage under the mapping F is

$F^{-1}(2\ell, \pi_3, \pi_4) = (H \times L)^{-1}(h, \ell) =$ toroidal energy momentum surface.

§3. Eigenvalues of the perturbed linear system

From the above global analysis in Section 2 about the linear flow (4.2.1), we may expect that for $|z| \ll 1$ and $|\lambda| \ll 1$, the family of nonlinear Hamiltonian systems

$$(4.3.1) \quad \dot{z} = J\nabla H^\lambda(z) = JAz + JB(\lambda)z + J\nabla\bar{H}_4(z)$$

is close to the linear system (4.2.1) and hence may have small amplitude periodic solutions with period near 2π near the periodic solutions of the linear system (4.2.1). Now, the linearized equation

of (4.3.1) at $z = 0$ is

(4.3.2) $\dot{z} = J(A + B(\lambda))z \overset{\text{def}}{=} JA(\lambda)z,$

where

$$JA(\lambda) = \left[\begin{array}{cc|cc} 0 & \lambda_4 & 1+\lambda_1 & -\lambda_3 \\ \lambda_4 & 0 & -\lambda_3 & -1-\lambda_2 \\ \hline -1-\lambda_1 & -\lambda_3 & 0 & -\lambda_4 \\ -\lambda_3 & 1+\lambda_2 & -\lambda_4 & 0 \end{array}\right].$$

After a tedious calculation of the characteristic polynomial of $JA(\lambda)$, we find that

(4.3.3) $\det(\alpha I - JA(\lambda)) = \alpha^4 + \alpha^2[(1 + \lambda_1)^2 + (1 + \lambda_2)^2 - 2\lambda_3^2 - 2\lambda_4^2]$
$+ [\lambda_3^2 + \lambda_4^2 + (1 + \lambda_1)(1 + \lambda_2)]^2,$

and the eigenvalues are given by

(4.3.4) $\alpha^2 = -\frac{1}{2}[(1 + \lambda_1)^2 + (1 + \lambda_2)^2 - 2\lambda_3^2 - 2\lambda_4^2] \pm \frac{1}{2}(2 + \lambda_1 +$
$\lambda_2)\cdot\sqrt{(\lambda_1 - \lambda_2)^2 - 4(\lambda_3^2 + \lambda_4^2)}.$

From (4.3.4), we notice that

(i) when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, the eigenvalues of $JA(\lambda)$ are
$\alpha = i$(double), $\alpha = -i$(double) as they should be.

(ii) when $\lambda_3 = \lambda_4 = 0$, (4.3.3) becomes
$\alpha^4 + \alpha^2[(1 + \lambda_1)^2 + (1 + \lambda_2)^2] + (1 + \lambda_1)^2(1 + \lambda_2)^2 =$
$[\alpha^2 + (1 + \lambda_1)^2]\,[\alpha^2 + (1 + \lambda_2)^2].$

So, the eigenvalues of $JA(\lambda)$ are $\alpha = \pm\, i(1 + \lambda_1)$, $\alpha = \pm\, i(1 + \lambda_2)$.

(iii) when $\lambda_1 = \lambda_2 = 0$, (4.3.4) becomes

$$\alpha^2 = -\frac{1}{2}[2 - 2\lambda_3^2 - 2\lambda_4^2] \pm 2i\sqrt{\lambda_3^2 + \lambda_4^2}$$

$$= (-1 + \lambda_3^2 + \lambda_4^2) \pm 2i\sqrt{\lambda_3^2 + \lambda_4^2} = (\sqrt{\lambda_3^2 + \lambda_4^2} \pm i)^2.$$

So, the eigenvalues of $JA(\lambda)$ are $\alpha = \pm\,(\epsilon + i)$, $\alpha = \pm(\epsilon - i)$ where $\epsilon = \sqrt{\lambda_3^2 + \lambda_4^2}$.

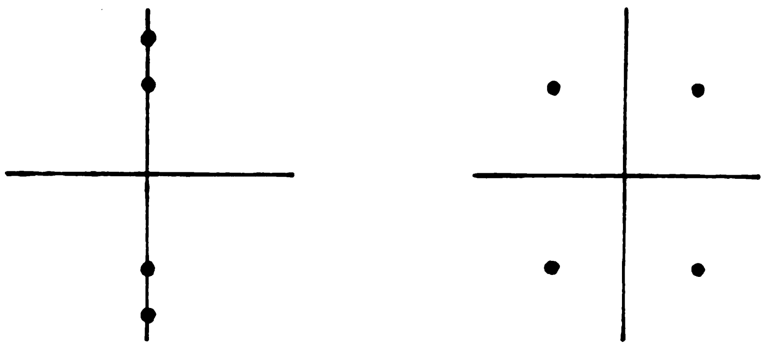
Since we are mainly interested in the cases when the eigenvalues vary along the imaginary axis, or across it, from now on, we restrict ourselves to the following two cases:

Case (a) $\lambda_2 = \lambda_3 = \lambda_4 = 0$ (or $\lambda_1 = \lambda_3 = \lambda_4 = 0$).

In this case, the eigenvalues of $JA(\lambda)$ are $\pm\, i$, $\pm i(1 + \lambda_1)$, i.e, the double eigenvalues $\pm\, i$ of $JA(0) = JA$ split <u>along</u> the imaginary axis.

Case (b) $\lambda_1 = \lambda_2 = \lambda_3 = 0$ (or $\lambda_1 = \lambda_2 = \lambda_4 = 0$).

In this case, the eigenvalues of $JA(\lambda)$ are $\pm(\lambda_4 + i)$, $\pm\,(\lambda_4 - i)$, i.e, the double eigenvalues $\pm\, i$ of $JA(0) = JA$ split <u>across</u> the imaginary axis.



< Figure 4.3.1 >

Note that in each case (a), (b), our Hamiltonian $H^{\lambda}(z)$ takes the form

$$H^{\lambda}(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2) + \frac{1}{2}\lambda_1(x_1^2 + y_1^2) + (x_1^2 + y_1^2)\cdot(x_1y_2 + x_2y_1)$$

$$H^{\lambda}(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2) + \lambda_4(x_1y_2 + x_2y_1) + (x_1^2 + y_1^2)(x_1y_2 + x_2y_1)$$

respectively.

Remark: By using the same method as in Section 2, it may be possible to study the bifurcations of invariant manifold $(H^{\lambda} \times H_2)^{-1}(c,m)$ as λ_1 (or λ_4) varies for various values of c and m. But, this does not give any detailed informations about the bifurcation of periodic orbits lying in those invariant manifolds. Hence, we will have to examine the local bifurcations of the periodic orbits themselves by other means. In the following, we will do this by using the method described already in Chapter 3.

In van der Meer's thesis [35], he examined the bifurcations of invariant manifold of the Hamiltonian system with the quadratic part $H_2 = (x_1y_2 + x_2y_1) + \frac{1}{2}(x_1^2 + x_2^2)$ which is 1: -1 <u>nonsemisimple</u> case. He considered the energy-momentum mapping H x S where S is the semisimple part of H_2 and obtained the standard form G x S and its unfolding G^ν x S by using singularity theory and finally examined the fibration $(G^\nu \times S)^{-1}(g,s)$ as ν varies for given g,s.

§4. <u>Local bifurcations of periodic orbits as the eigenvalues split along the imaginary axis</u>

Now, we follow the methods described in Chapter 3 with Hamiltonian

$$(4.4.1) \quad H^\lambda(z) = \frac{1}{2} z^T Az + \frac{1}{2} z^T B(\lambda)z + \overline{H}_4(z),$$

where A = diag (1, -1, 1, -1), $B(\lambda)$ = diag $(\lambda_1, 0, \lambda_1, 0)$, and

$$\overline{H}_4(z) = (x_1^2 + y_1^2)(x_1y_2 + x_2y_1).$$

The corresponding Hamiltonian system is

$$(4.4.2) \quad \dot{z} = J\nabla H^\lambda(z) = JAz + JB(\lambda)z + J\nabla\overline{H}_4(z),$$

where the linear part $A(\lambda) = A + B(\lambda)$ has eigenvalues, $\pm\, i$, and $(1 + \lambda_1)i$. After introducing the time scale $t = \mu\tau, |\mu - 1| << 1$, (4.4.2) becomes

$$(4.4.3) \quad \frac{dz}{d\tau} = \mu[JAz + JB(\lambda)z + J\nabla\overline{H}_4(z)]$$

and after putting this into rotation coordinates $z = e^{JA\tau}u$, $u \in \mathbb{R}^4$, equation (4.4.3) becomes

$$(4.4.4) \quad \frac{du}{d\tau} = (\mu - 1)JAu + \mu JB(\lambda)u + \mu J\nabla\overline{H}_4(z)$$

by Lemma 3.2.1. And the bifurcation function of (4.4.4) is given by

$$(4.4.5) \quad V(a,\mu,\lambda) = (\mu - 1)\ JAa + \mu JB(\lambda)a + \mu J\nabla\overline{H}_4(a),\ a \in \mathbb{R}^4$$

with the equivariant property

$$V(e^{JA\tau}a,\ \mu,\lambda) = e^{JA\tau}V(a,\mu,\lambda)$$

by Lemma 3.2.3. Also, we can choose $\mu = \mu^*(a,\lambda)$ uniquely and continuously so that

$$\langle\ JAa,\ V(a,\mu^*(a,\lambda),\lambda)\rangle = 0 \text{ for all } 0 < |a| << 1,\ |\lambda| << 1.$$

In fact, from (3.2.14), $\mu^*(a,\lambda)$ is given by

$$(4.4.6)\quad \mu^*(a,\lambda) = \frac{|a|^2}{|a|^2 + \langle Aa,\ B(\lambda)a + \nabla\overline{H}_4(\lambda)\rangle}\ .$$

With $B(\lambda) = \text{diag}(\lambda_1,\ 0,\ \lambda_1,\ 0)$, (4.4.6) becomes

$$(4.4.7)\quad \mu^*(a,\lambda) = \frac{|a|^2}{|a|^2 + \lambda_1(a_1^2 + a_3^2) + 2(a_1^2 + a_3^2)(a_1a_4 + a_2a_3)}$$

$$= \frac{1}{1 + \lambda_1\ p(a) + 2p(a)q(a)} = \frac{1}{1 + O(|\lambda_1|+|a|^2)} = 1 - O(|\lambda_1|+|a|^2)$$

as $|a| \to 0$ and $|\lambda_1| \to 0$,

where $p(a) = \dfrac{a_1^2 + a_3^2}{|a|^2}$, $0 \leq p(a) \leq 1$ for any $a \neq 0$,

$$q(a) = (a_1a_4 + a_2a_3) = O(|a|^2).$$

Further, letting $\tilde{V}(a,\lambda) = V(a,\mu^*(a,\lambda),\lambda)$, then by Lemma 3.2.3, $\tilde{V}(a,\lambda)$ is also equivariant under e^{JAt} and each zero (a,λ) of $\tilde{V}(a,\lambda)$ is locally in a one to one correspondence to the $2\pi\mu^*(a,\lambda)$ - periodic solution $z(t) = e^{JAt/u^*(a,\lambda)}a$ of our system (4.4.2). Also, by Lemma 3.3.1, each zero (a,λ) of $\tilde{V}(a,\lambda)$ is a critical point of the scalar-valued function

$$(4.4.8)\quad \tilde{S}(a,\lambda) = H_2(a) - \mu^*(a,\lambda)\cdot[H^\lambda(a) - c]$$

on the energy surface $H^\lambda(a) = c$. Moreover, by Lemma 3.3.2 in Chapter

3, we know that the solution of $\nabla\tilde{S}(a,\lambda) = 0$ with the constraint $H^{\lambda}(a) = c$ can be obtained by solving $\nabla g(a,\lambda,\eta) = 0$ with $H^{\lambda}(a) = c$, where

$$(4.4.9)\quad g(a,\lambda,\eta) = H_2(a) - \eta \cdot H^{\lambda}(a)$$

for $a = \bar{a}(\lambda,c)$ with $\eta = \mu^{*}(\bar{a},\lambda)$. Therefore, we concentrate on solving the equation $\nabla g(a,\lambda,\eta) = 0$ for $a = \bar{a}(\lambda,\eta)$ with $\eta = \mu^{*}(\bar{a},\lambda)$ given in (4.4.7). Then, by Theorem 3.3.3 this solution $\bar{a}(\lambda,\eta(\lambda,c))$ will be locally in 1 - 1 correspondence to the $2\pi\mu^{*}(\bar{a},\lambda)$-periodic solution

$$z(t) = e^{JAt/\mu^{*}(\bar{a},\lambda)}\bar{a}$$

of the orginal equation (4.4.2). Now,

$$\begin{aligned} g(a,\lambda,\eta) &= H_2(a) - \eta H^{\lambda}(a) \\ &= H_2(a) - \eta \cdot [H_2(a) + H_2^{\lambda}(a) + \bar{H}_4(a)] \\ &= (1 - \eta) \cdot \frac{1}{2} a^T A a - \eta \cdot \frac{1}{2} a^T B(\lambda) a - \eta \cdot \bar{H}_4(a) \\ &= (1 - \eta) \cdot \frac{1}{2}(a_1^2 + a_3^2 - a_2^2 - a_4^2) - \eta \cdot \frac{1}{2} \lambda_1 (a_1^2 + a_3^2) - \\ &\quad \eta(a_1^2 + a_3^2)\cdot(a_1 a_4 + a_2 a_3). \end{aligned}$$

Hence, the system of equations $\nabla g(a,\lambda,\eta) = 0$ becomes

$$
(4.4.10)\begin{cases}
(1) \quad (1-\eta)a_1 - \eta\lambda_1 a_1 - \eta[2a_1(a_1a_4 + a_2a_3) + a_4(a_1^2 + a_3^2)] = 0 \\
(2) \quad -(1-\eta)a_2 \qquad -\eta[a_3(a_1^2 + a_3^2)] = 0 \\
(3) \quad (1-\eta)a_3 - \eta\lambda_1 a_3 - \eta[2a_3(a_1a_4 + a_2a_3) + a_2(a_1^2 + a_3^2)] = 0 \\
(4) \quad -(1-\eta)a_4 \qquad - \eta[a_1(a_1^2 + a_3^2)] = 0 \\
\text{together with the energy condition } H^{\lambda}(a) = c, \\
(5) \quad \frac{1}{2}(a_1^2 + a_3^2) - \frac{1}{2}(a_2^2 + a_4^2) + \frac{1}{2}\lambda_1(a_1^2 + a_3^2) + (a_1^2 + a_3^2)\,(a_1a_4 + a_2a_3) = c.
\end{cases}
$$

Note that for each λ_1 and c,(4.4.10) is a system of 5 equations in 5 unknowns a_1, a_2, a_3, a_4, η and so we can solve (4.4.10) (1), (2), (3), (4), for $a = \bar{a}(\lambda,\eta)$ in terms of λ_1, η and make use of (4.4.10) (5) $H^{\lambda}(a) = c$ to determine $\eta = \eta(\lambda,c)$ and hence determine $a = \bar{a}(\lambda,\eta(\lambda,c))$. Furthermore $\eta(\lambda,c)$ will turn out to be $\eta(\lambda,c) = \mu^*(\bar{a},\lambda)$. Therefore, by using $a = \bar{a}(\lambda,\eta)$ and $\eta = \mu^*(\bar{a},\lambda)$ we can examine the number of solutions as the parameter λ_1 varies.

Also, note that system (4.4.10) is equivariant under the rotation exp JAt for all t and in particular invariant under the reflection $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$.

Clearly, $a = 0$ is a trivial solution of (4.4.10) for all λ_1 with $c = 0$.For $c \neq 0$, $a = 0$ is no longer a solution of (4.4.10).

Recall that $\eta = \mu^*(a,\lambda) \approx 1 - 0(|\lambda_1| + |a|^2)$ as $|a| \to 0$ and $|\lambda_1| \to 0$. Hence, for sufficiently small $|\lambda_1|$ and $|a|$, we always have $\eta > 0$ and $0 < \eta < 1$ for $\lambda_1 > 0$ and $q(a) > 0$ and $\eta > 1$ for $\lambda_1 < 0$ and $q(a) < 0$. Since $\mu^*(0,0) = 1$, η cannot be zero. If $\eta = 1$, then we have $\lambda_1(a_1^2 + a_3^2) + 2(a_1^2 + a_3^2)\cdot(a_1a_4 + a_2a_3) = 0$ and hence our system $\dot{z} = J\nabla H^{\lambda}(z)$ reduces to the linear system $\dot{z} = J\nabla H_2(z)$. Therefore we may assume $\eta \neq 1$.

Now, for $\eta \neq 0$ and $\eta \neq 1$, we can write (4.4.10) (2), (4) as

$$(4.4.11) \quad \begin{cases} a_2 = -\dfrac{\eta}{1-\eta}\, a_3(a_1^2 + a_3^2) \\ a_4 = -\dfrac{\eta}{1-\eta}\, a_1(a_1^2 + a_3^2). \end{cases}$$

Substituting (4.4.11) into (4.4.10) (1), (3), we have, by the reflection symmetry $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$,

$$(4.4.12) \quad \begin{cases} \text{(i)} \quad a_1[(1-\eta)\,(1-\eta - \eta\lambda_1) + 3\eta^2(a_1^2 + a_3^2)^2] = 0 \\ \text{(ii)} \quad a_3[(1-\eta)(1-\eta-\eta\lambda_1) + 3\eta^2(a_1^2 + a_3^2)^2] = 0. \end{cases}$$

Note that if we can solve the system (4.4.12) for a_1, a_3 in terms of λ_1, η, then by (4.4.11), a_2, a_4 are automatically determined and so we can determine the solution of the system (4.4.10). Hence, the 4 x 4 system (4.4.10) (1) - (4) has been reduced to solving the 2 x 2 system (4.4.12), which is entirely due to the equivariance of the orginal system (4.4.10).

Also notice that if $\lambda_1 = 0$ then (4.4.12) and hence (4.4.10) has only the trivial solution $a = 0$ with $c = 0$.

Now, we consider several cases:

case (i): $a_1 = 0$ and $a_3 = 0$.

This clearly satisfies (4.4.12) and from (4.4.11) we have $a_2 = a_4 = 0$ and from (4.4.10) (5), we have $c = 0$.

Hence, we get trivial solution $a = 0$ for all λ_1 with energy $c = 0$.

case (ii) $a_1 = 0$ and $a_3 \neq 0$

From (4.4.12)(ii), we have

$$(1-\eta)(1-\eta - \eta\lambda_1) + 3\eta^2 a_3^4 = 0.$$

or,

$$a_3^4 = -\frac{(1-\eta)(1-\eta - \eta\lambda_1)}{3\eta^2}.$$

From (4.4.11), we have

$$a_2 = \frac{\eta}{1-\eta} a_3^3, \quad a_4 = 0.$$

Hence, in this case, we have solutions of (4.4.10) of the form

$$(4.4.13) \quad \begin{cases} a_1 = a_4 = 0 \\ a_3^4 = -\dfrac{(1-\eta)(1-\eta - \eta\lambda_1)}{3\eta^2}, \quad a_2 = -\dfrac{\eta}{1-\eta} a_3^3. \end{cases}$$

case (iii): $a_3 = 0$ and $a_1 \neq 0$

By the reflection symmetry $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$, we have the solutions of (4.4.10) of the form

$$(4.4.14) \quad \begin{cases} a_3 = a_2 = 0 \\ a_1^4 = -\dfrac{(1-\eta)(1-\eta - \eta\lambda_1)}{3\eta^2}, \quad a_4 = -\dfrac{\eta}{1-\eta} a_1^3. \end{cases}$$

case (iv): $a_1 \neq 0$ and $a_3 \neq 0$.

From (4.4.12)(i)(ii), we have

$$(1-\eta)(1-\eta - \eta\lambda_1) + 3\eta^2(a_1^2 + a_3^2)^2 = 0$$

or,

$$(a_1^2 + a_3^2)^2 = -\frac{(1-\eta)(1-\eta - \eta\lambda_1)}{3\eta^2}.$$

Therefore, it follows from the above cases that the most general solution of (4.4.10) including cases (i)(ii)(iii)(iv) can be written as

$$(4.4.15)\quad \begin{cases} (a_1^2 + a_3^2)^2 = -\dfrac{(1-\eta)(1-\eta-\eta\lambda_1)}{3\eta^2} & \text{(a)} \\ a_2 = -\dfrac{\eta}{1-\eta}\, a_3(a_1^2 + a_3^2) & \text{(b)} \\ a_4 = -\dfrac{\eta}{1-\eta}\, a_1(a_1^2 + a_3^2) & \text{(c)} \end{cases}$$

where a_1, a_3 are allowed to be both zero and $\eta \neq 0, 1$ is given by (4.4.7):

$$\eta = \mu^*(a,\lambda) = \frac{1}{1+\lambda_1 p(a) + 2p(a)q(a)}$$

with $p(a) = \dfrac{a_1^2 + a_3^2}{|a|^2}$ and

$$q(a) = a_1 a_4 + a_2 a_3 = O(|a|^2)$$

and the energy corresponding to the solution (4.4.15) is given by

$$H^\lambda(a) = \frac{1}{2}(a_1^2 + a_3^2) - \frac{1}{2}(a_2^2 + a_4^2) + \frac{1}{2}\lambda_1(a_1^2 + a_3^2) + (a_1^2 + a_3^2)\cdot(a_1 a_4 + a_2 a_3) = c.$$

In order to put (4.4.15) into a simpler form, we set

$$a_1 = \gamma\cos\theta, \quad a_3 = \gamma\sin\theta$$

$$a_2 = \rho\cos\psi, \quad a_4 = \rho\sin\psi \qquad (\gamma, \rho \geq 0)$$

Then, (4.4.15)(a) becomes

$$\gamma^4 = -\frac{(1-\eta)(1-\eta-\eta\lambda_1)}{3\eta^2}, \text{ or}$$

$$(4.4.16)\quad \gamma = \left[-\frac{(1-\eta)(1-\eta-\eta\lambda_1)}{3\eta^2}\right]^{1/4} \text{ provided } [\ ] \geq 0$$

and from (4.4.15)(b),(c), we have

$$a_2^2 + a_4^2 = \left(\frac{\eta}{1-\eta}\right)^2 (a_1^2 + a_3^2)^3, \text{ or}$$

$$\rho^2 = \left(\frac{\eta}{1-\eta}\right)^2 \cdot \gamma^6, \text{ or}$$

$$(4.4.17) \quad \rho = \left|\frac{\eta}{1-\eta}\right| \cdot \gamma^3 = \left|\frac{\eta}{1-\eta}\right| \cdot \left[\frac{-(1-\eta)(1-\eta-\eta\lambda_1)}{3\eta^2}\right]^{3/4}$$

provided [ ] ≥ 0.

Hence, for those values of λ_1 and c satisfying [ ] ≥ 0, (4.4.16) and (4.4.17) show that the solution set of (4.4.10) forms a 2-dimensional torus $T^2 = S^1 \times S^1$ depending on λ_1 and c. Now, in order for γ to have real positive solutions, we need the condition

$$(4.4.18) \quad -\frac{(1-\eta)(1-\eta-\eta\lambda_1)}{3\eta^2} > 0$$

Now, since we can write

$$-\frac{(1-\eta)(1-\eta-\eta\lambda_1)}{3\eta^2} = -\frac{1}{3} \cdot \frac{1 - \left(\frac{\eta}{1-\eta}\right)\lambda_1}{\left(\frac{\eta}{1-\eta}\right)^2},$$

the condition (4.4.18) is equivalent to

$$\left(\frac{\eta}{1-\eta}\right)\lambda_1 > 1.$$

Hence, if $\lambda_1 > 0$, we need $\frac{\eta}{1-\eta} > \frac{1}{\lambda_1}$, i.e., $0 < \frac{1-\eta}{\eta} < \lambda_1$ and if $\lambda_1 < 0$, then we need $\frac{\eta}{1-\eta} < \frac{1}{\lambda_1}$, i.e. $0 > \frac{1-\eta}{\eta} > \lambda_1$. But, $\frac{1-\eta}{\eta} = \frac{1}{\eta} - 1 = \lambda_1$ p(a) +

$2p(a)q(a)$. Since $0 < p(a) < 1$ and $p(a) \to 0$ as $a_1^2 + a_3^2 \to 0$ and $q(a) = a_1a_4 + a_2a_3 = O(|a|^2)$, we know that when $\lambda_1 > 0$, the condition $0 < \lambda_1 p(a) + 2p(a)\, q(a) < \lambda_1$ is indeed satisfied for sufficiently small $|a|$. Similarly, when $\lambda_1 < 0$ and $|a|$ sufficiently small, the condition $0 > \lambda_1 p(a) + 2p(a) \cdot q(a) > \lambda_1$ is satisfied. Therefore, the solution (4.4.16) and (4.4.17) are valid for sufficiently small $|a|$. To obtain a direct relationship between γ and ρ, we eliminate η from (4.4.16) and (4.4.17). Rewrite (4.4.16) and (4.4.17) as

$$\begin{cases} \gamma^4 = -\dfrac{1-k\lambda_1}{3k^2}, & \text{where } k = k(\lambda_1, c) = \dfrac{\eta}{1-\eta} \\ \rho = |k|\gamma^3 . \end{cases}$$

Also, the energy condition (4.4.10)(5) can be rewritten in terms of polar coordinates as

$$\frac{1}{2}\gamma^2 - \frac{1}{2}\rho^2 + \frac{1}{2}\lambda_1\gamma^2 + \gamma^2(\gamma\rho \cos\theta \sin\psi + \gamma\rho \sin\theta \cos\psi) = c,$$

or

$$\gamma^2 - \rho^2 + \lambda_1\gamma^2 + 2\gamma^3\rho \sin(\theta + \psi) = 2c. \tag{4.4.18}$$

Now, recall that

$$\eta = \mu^*(\bar{a}, \lambda) = \frac{|\bar{a}|^2}{|\bar{a}|^2 + \lambda_1(\bar{a}_1^2 + \bar{a}_3^2) + 2(\bar{a}_1^2 + \bar{a}_3^2) \cdot (\bar{a}_1\bar{a}_4 + \bar{a}_2\bar{a}_3)}$$

and also recall that the solution $\bar{a}(\lambda_1, c)$ lies on the energy surface

$H^{\lambda}(\bar{a}) = c$ and on the momentum surface $H_2(\bar{a}) = m$. But, since

$$c-m = H^{\lambda}(\bar{a}) - H_2(\bar{a}) = \frac{1}{2}\lambda_1(\bar{a}_1^2+\bar{a}_3^2) + (\bar{a}_1^2+\bar{a}_3^2)\cdot(\bar{a}_1\bar{a}_4+\bar{a}_2\bar{a}_3),$$

we can write

$$\eta = \mu^*(\bar{a},\lambda_1) = \frac{|\bar{a}|^2}{|a|^2+2(c-m)}. \tag{4.4.19}$$

Thus, we know that if $c > m$ then $0 < \eta < 1$ and if $c < m$ then $\eta > 1$ in a sufficiently small neighborhood of the origin. If $c = m$ then $\eta = 1$ and so our system $\dot{z} = J\nabla H^{\lambda}(z)$ reduces to the linear system $\dot{z} = J\nabla H_2(z)$, which we have already considered in Section 2. Therefore, we can consider two cases:

Case (i): $c > m$

Then $0 < \eta < 1$, so $k = \frac{\eta}{1-\eta} > 0$. Hence, from (4.4.16) and (4.4.17), eliminating k, we have

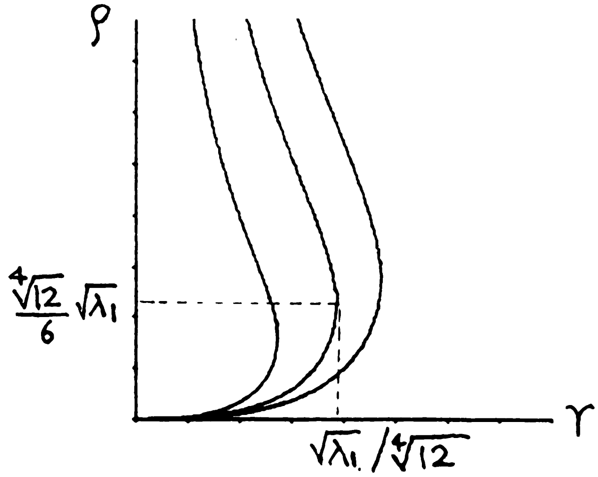
$$3\gamma\rho^2 - \lambda_1\rho + \gamma^3 = 0 \qquad (\gamma,\rho \geq 0). \tag{4.4.20}$$

Thus, when $c > m$, the critical points of g must lie on the curve (4.4.20) in the (γ,ρ) - plane. Notice that (4.4.20) is quadratic in ρ and so can be solved for ρ:

$$\rho = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - 12\gamma^4}}{6\gamma} \qquad \left(0 < \gamma \leq \sqrt[4]{\frac{\lambda_1^2}{12}},\ \rho > 0\right). \tag{4.4.21}$$

Since $\rho > 0$, we must have $\lambda_1 > 0$. If $\lambda_1 < 0$, then (4.4.20) has no

positive solution for ρ and hence the system (4.4.10) has no nontrivial solution, i.e., has only the trivial solution a = 0 with c = 0. Notice that if $\lambda_1 = 0$, then (4.4.20) has only the trivial solution a = 0 with c = 0. The graph of (4.4.21) with various values of $\lambda_1 > 0$ is shown in Figure 4.4.1 where $\frac{d\rho}{d\gamma}\Big|_{\gamma=0} = 0$.



< Figure 4.4.1 >

Now, the energy surface (4.4.18) can be rewritten as

$$(1 + \lambda_1)\gamma^2 - \rho^2 + 2\alpha\gamma^3\rho = 2c,$$

or

$$\rho^2 - 2\alpha\gamma^3\rho + 2c - (1+\lambda_1)\gamma^2 = 0 \quad (\lambda_1 > 0, \quad |\alpha| \le 1), \tag{4.4.22}$$

where $\alpha = \sin(\theta + \psi)$.

Also, (4.4.22) is quadratic in ρ and so can be solved for ρ:

$$\rho = \alpha\gamma^3 \pm \sqrt{\alpha^2\gamma^6 + (1 + \lambda_1)\gamma^2 - 2c} \quad (\lambda_1 > 0). \tag{4.4.23}$$

Now, we first consider the case c = 0. Then (4.4.23) becomes

$$\rho_o = \alpha\gamma^3 \pm \gamma\sqrt{1 + \lambda_1 + \alpha^2\gamma^4}\ .$$

Since $\rho > 0$, we must have

(4.4.24) $\rho_o = \alpha\gamma^3 + \gamma\sqrt{1 + \lambda_1 + \alpha^2\gamma^4}$ $(\lambda_1 > 0,\ |\alpha| \le 1)$.

Notice that

$$\left.\frac{d\rho_o}{d\gamma}\right|_{\gamma=0} = \sqrt{1+\lambda_1} \to 1 \text{ as } \lambda_1 \to 0+ \text{ and}$$

$$\rho_o \approx \gamma\sqrt{1+\lambda_1} \quad \text{for } \gamma \text{ sufficiently small.}$$

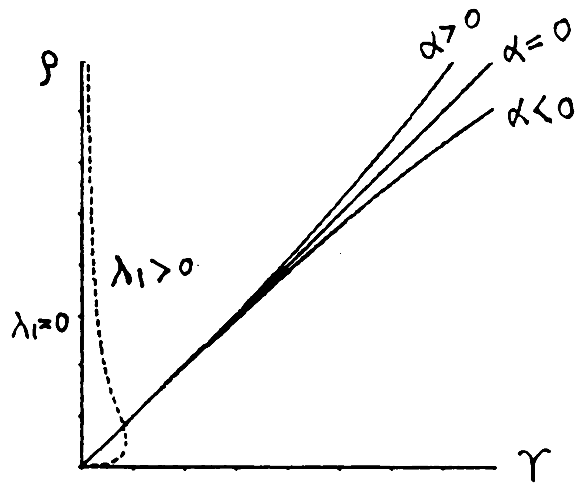
Also, when $\alpha > 0$, $\frac{d\rho_o}{d\gamma} > 1$ for all $\lambda_1 > 0$ and

$\rho_o \approx 2\alpha\gamma^3$ for $\gamma \gg 1$. When $\alpha < 0$, we have

$$\rho_o \approx \alpha\gamma^3 - \alpha\gamma^3 = 0 \quad \text{for } \gamma \gg 1.$$

If $\alpha = 0$, then (4.4.24) becomes $\rho_o = \gamma\sqrt{1+\lambda_1}$.

The graphs of (4.4.24) for $\alpha > 0$, $\alpha < 0$, and $\alpha = 0$ with $0 < \lambda_1 \ll 1$ are shown in Figure 4.4.2.



< Figure 4.4.2 >

Hence, from the graphs of (4.4.21) and (4.4.24) it is clear that for sufficiently small values $|a|$ and $\lambda_1 > 0$ they have a unique intersection point for any values of $|\alpha| \leq 1$, which indicates a torus $T^2 = S^1 \times S^1$ of critical points lying on the 3-dimensional energy surface $H^\lambda(a) = c = 0$ in the space $\mathbb{R}^4 = \{a_1, a_2, a_3, a_4)\}$.

Next, we considere the case $|c| \neq 0$ sufficiently small: If $c < 0$, then from (4.4.23), we have

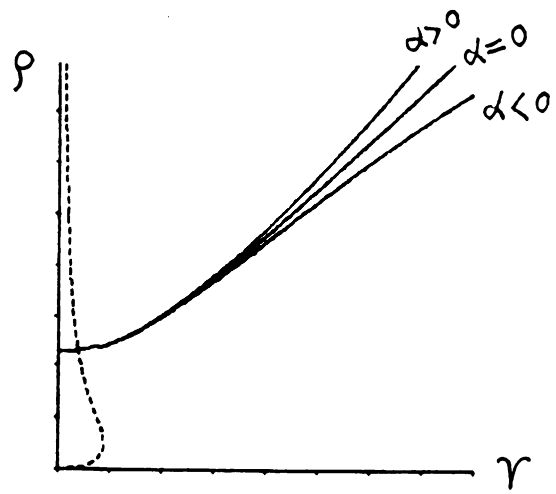
$$(4.4.25)\quad \rho_{c-} = \alpha\gamma^3 + \sqrt{\alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c} \qquad (c < 0,\ \lambda_1 > 0,\ |\alpha| \leq 1).$$

Note that $\rho_{c-}(0) = \sqrt{-2c} > 0$ and $\left.\frac{d\rho_{c-}}{d\gamma}\right|_{\gamma=0} = 0$. Also,

$\rho_{c-} > \rho_o$ and

$\rho_{c-} \approx \rho_o$ for $|c| << 1$.

The graphs of (4.4.25) for $\alpha > 0$, $\alpha = 0$, $\alpha < 0$ with $c < 0$, $\lambda_1 > 0$ are shown in Figure 4.4.3



< Figure 4.4.3 >

Hence, in the case of $c < 0$, as in the case of $c = 0$, clearly the graph of (4.4.25) intersects the graph of (4.4.21) exactly at one point for any values of α and hence we have a torus $T^2 = S^1 \times S^1$ of solution points in $\mathbb{R}^4$ for the system (4.4.10).

Now, we consider the case $c > 0$:

From (4.4.23), we have

$$(4.4.26)\quad \rho_{c+} = \alpha\gamma^3 \pm \sqrt{\alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c} \qquad (\lambda_1 > 0,\ c > 0,\ |\alpha| \leq 1).$$

Notice that the radicand $f(\gamma) = \alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c$ is an increasing function of γ for $\gamma \geq 0$ with $f(0) = -2c < 0$ and so $f(\gamma)$ has a unique positive zero $\bar{\gamma} = \bar{\gamma}(\lambda_1, c, \lambda)$ for any $\lambda_1 > 0$, $c > 0$, $|\alpha| \leq 1$ with the property that

$\bar{\gamma}(\lambda_1, c, \alpha) \to 0+$ as $c \to 0+$

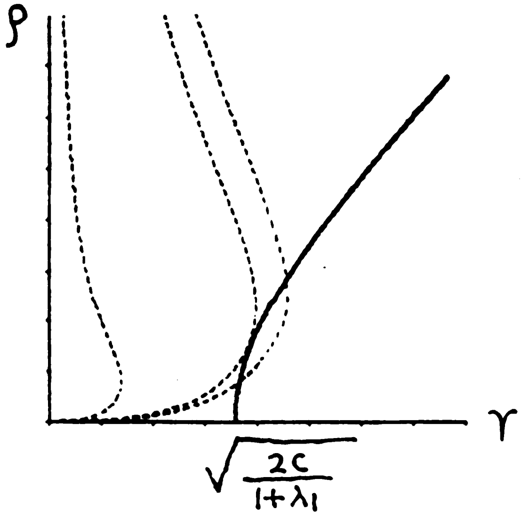
$\bar{\gamma}(\lambda_1, c, \alpha)$ decreases as λ_1 or $|\alpha|$ increases.

Also notice that even though $\bar{\gamma}(\lambda_1, c, \alpha)$ can be computed exactly by using the Cardan's formula, this expression is too complicated to be of any practical use for our purpose. Since ρ_{c+} must be $\rho_{c+} \geq 0$, we also note that if $\alpha \leq 0$, we must have

$$(4.4.27)\quad \rho_{c+} = \alpha\gamma^3 + \sqrt{\alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c} \qquad (\alpha \leq 0,\ \lambda_1 > 0, c > 0),$$

$$\approx \sqrt{(1+\lambda_1)\gamma^2 - 2c} \text{ for } \gamma \ll 1 \ \left(\gamma \geq \sqrt{\frac{2c}{1+\lambda_1}}\right).$$

Furthermore, in (4.4.27) with $\alpha \leq 0$, since $\rho_{c+} \geq 0$, the domain of ρ_{c+} must be $\gamma \geq \sqrt{\frac{2c}{1+\lambda_1}}$ and in this domain, ρ_{c+} is an increasing funtion of γ. The graph of (4.4.27) with $\alpha \leq 0$ for sufficiently small c, λ_1, γ is shown in Figure 4.4.4 together with the graph of (4.4.21) with the various values of λ_1.



< Figure 4.4.4 >

Hence, in this case $\alpha \lesssim 0$, it is clear from the graph that for each given $c > 0$ sufficiently small, there is a $\lambda_o = \lambda_o(c) > 0$ such that if $\lambda_1 < \lambda_o$, there is no intersection point and if $\lambda_1 = \lambda_o$, there is one intersection point and if $\lambda_1 > \lambda_o$, there are two intersection points. If $\alpha > 0$, we have two cases in (4.4.26).

(4.4.28) $\rho_{c+} = \alpha\gamma^3 \pm \sqrt{\alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c}$ $\quad (0 < \alpha \lesssim 1,\ \lambda_1 > 0,\ c > 0)$

In the + case, i.e.,

(4.4.28)(a) $\rho_{c+} = \alpha\gamma^3 + \sqrt{\alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c}$ $\quad (\alpha > 0,\ \lambda_1 > 0,\ c > 0)$

$$\approx \sqrt{(1+\lambda_1)\gamma^2 - 2c} \quad \text{for } \gamma << 1,$$

we know that ρ_{c+} is defined for $\gamma \geq \bar{\gamma}(\lambda_1, c, \alpha)$ and ρ_{c+} is an increasing function with range $\rho_{c+}(\gamma) \geq \alpha\bar{\gamma}^3$. In the - case in (4.4.28), i.e.,

(4.4.28)(b) $\rho_{c+} = \alpha\gamma^3 - \sqrt{\alpha^2\gamma^6 + (1+\lambda_1)\gamma^2 - 2c} \quad (\alpha > 0,\ \lambda_1 > 0,\ c > 0),$

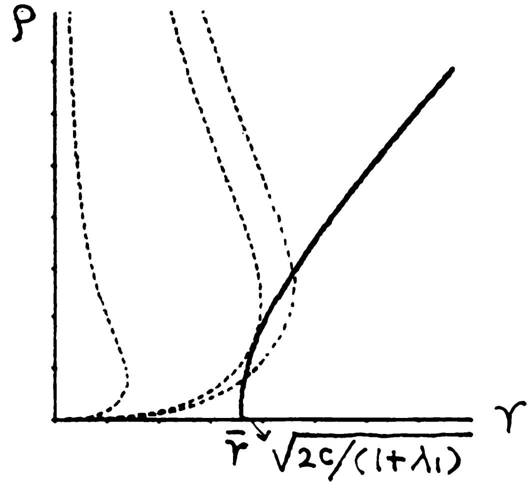
we notice that ρ_{c+} is a decreasing function in the domain

$$\bar{\gamma}(\lambda_1, c, \alpha) \leq \gamma \leq \sqrt{\frac{2c}{1+\lambda_1}} \qquad \text{with } \alpha\bar{\gamma}^3 \geq \rho_{c+} \geq 0.$$

But, since $\bar{\gamma}(\lambda_1, c, \alpha) \approx \sqrt{\frac{2c}{1+\lambda_1}}$ for $c > 0$ sufficiently small, the graph of (4.4.28)(b) exists in a very small interval and hence the overall graph of (4.4.28) shown in Figure 4.4.5 looks almost like that of (4.4.28)(a). Moreover, we can see that for γ sufficiently small

$$\alpha\gamma^3 < \frac{\lambda_1 - \sqrt{\lambda_1^2 - 12\gamma^4}}{6\gamma} \qquad \text{for all } 0 < \lambda_1 << 1$$

and so for sufficiently small c (i.e., sufficiently small $\bar{\gamma}$) the starting point $(\bar{\gamma}, \alpha\bar{\gamma}^3)$ of the graph of (4.4.28)(a) lies below the lower branch of the curve (4.4.21) as shown in Figure 4.4.5.



< Figure 4.4.5 >

Thus, even in the case $\alpha > 0$, we can still say that for each $c > 0$ sufficiently small there is a $\lambda_o = \lambda_o(c) > 0$ such that the graph of (4.4.21) intersects that of (4.4.28) at two points for $\lambda > \lambda_o$, at one point for $\lambda = \lambda_o$ and at no point for $\lambda < \lambda_o$.

Now, we consider the second case:

<u>(case ii): c < m</u>

Then, from (4.4.19) $\eta > 1$, so $k = \frac{\eta}{1-\eta} < 0$. Hence, from (4.4.16) and (4.4.17), eliminating k, we have

$$3\gamma\rho^2 + \lambda_1\rho + \gamma^3 = 0 \qquad (\gamma, \rho \geq 0), \tag{4.4.29}$$

or

$$\rho = \frac{-\lambda_1 \pm \sqrt{\lambda_1^2 - 12\gamma^4}}{6\gamma} \qquad (\lambda_1 < 0). \tag{4.4.30}$$

Note that since $\rho > 0$, we must have $\lambda_1 < 0$. If $\lambda_1 \geq 0$, then (4.4.29) and hence our system (4.4.10) has only the trivial solution a = 0 with c = 0. The graph of (4.4.30) with $\lambda_1 < 0$ is the same as that of (4.4.21) with $\lambda_1 > 0$. Now, the energy surface (4.4.23) becomes

(4.4.31) $\rho = \alpha\gamma^3 \pm \sqrt{\alpha^2\gamma^6 - 2c + (1+\lambda_1)\gamma^2} \qquad (\lambda_1 < 0,\ |\alpha| \leq 1).$

If c = 0, then (4.4.31) becomes

$$\rho = \alpha\gamma^3 \pm \sqrt{\alpha^2\gamma + (1+\lambda)\gamma^2} .$$

Since $\rho > 0$, we must have

(4.4.32) $\rho = \alpha\gamma^3 + \sqrt{\alpha^2\gamma^6 + (1+\lambda)\gamma^2} \quad (\lambda_1 < 0, |\lambda_1| << 1,\ |\alpha| << 1).$

Also, we notice that $\left.\frac{d\rho}{d\gamma}\right|_{r=0} = \sqrt{1 + \lambda_1} < 1$ and $\rho \approx \gamma\cdot\sqrt{1+\lambda_1}$ for $\gamma << 1$.

The graph of (4.4.32) is almost the same as that of Figure 4.4.2 as long as $|\lambda_1| << 1$ and $|\gamma| << 1$. Also, in the case of $c \neq 0$, the graphs of

(4.4.33) $\rho_{c-} = \alpha\gamma^3 + \sqrt{\alpha^2\gamma + (1+\lambda_1)\gamma^2 - 2c} \quad (-1 << c < 0,\ -1 << \lambda_1 < 0,$
$|\alpha| \leq 1)$

(4.4.34) $\rho_{c+} = \alpha\gamma^3 \pm \sqrt{\alpha^2\gamma + (1+\lambda_1)\gamma^2 - 2c} \quad (0 < c << 1,\ -1 << \lambda_1 < 0,$
$|\alpha| << 1)$

are almost the same as those of Figure 4.4.3 and Figure 4.4.5 for $|\lambda_1|$ << 1. Therefore, from the above analysis, we can state the following conclusion:

<u>Theorem 4.4.1</u> Consider the Hamiltonian system

$$\dot{z} = J\nabla H^{\lambda}(z) \tag{4.4.35}$$

with $H^{\lambda}(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2) + \lambda_1(x_1^2 + y_1^2) + (x_1^2 + y_1^2)(x_1y_2 + x_2y_1)$ in the normal form with respect to $H_2(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2)$. Let $H^{\lambda}(z) = c$ and $H_2(z) = m$. Then, in a sufficiently small neighborhood of the origin and for sufficiently small $|\lambda_1|$ and $|c|$, we have the following:

(i) when $c > m$ and $c \leq 0$, the system $\dot{z} = J\nabla H^{\lambda}(z)$ undergoes a supercritical bifurcation from the equilibrium solution $z = 0$ for $\lambda_1 \leq 0$ when $c = 0$ and from ϕ for $\lambda_1 < 0$ and S^1: $x_2^2 + y_2^2 = -2c$ for $\lambda_1 = 0$ when $c < 0$ to a continuous family of periodic solutions of the form

$$z(t) = e^{JAt/\mu^*(\bar{a},\lambda_1)}\bar{a}(\lambda_1, c) \tag{4.4.36}$$

for each small $\lambda_1 > 0$, lying on a torus with period $2\pi\mu^* < 2\pi$.

(ii) when $c > m$, and $c > 0$ sufficiently small, there is a $\lambda_o = \lambda_o(c) > 0$ sufficiently small such that if $\lambda < \lambda_o$, then the system (4.4.35) has no periodic solution and if $\lambda = \lambda_o$, then (4.4.35) has one continuous family of periodic solutions of the form (4.4.36) lying on the torus with period $2\pi\mu^* < 2\pi$ and if $\lambda_o < \lambda << 1$, then (4.4.35) has <u>two</u> disjoint continuous families of periodic solutions of the form (4.4.36), each lying on the corresponding torus with corresponding period $2\pi\mu^* < 2\pi$.

(iii) when $c < m$, we have the same kind of bifurcaiton as in (i) and (ii) except that in the case of (i), (4.4.35) undergoes a subcritical bifurcation from $z = 0$ for $\lambda_1 \geq 0$ to a torus for $\lambda_1 < 0$ and in the case of (ii), from no periodic solution for $\lambda_o < \lambda_1 < 0$ to a torus for $\lambda_1 = \lambda_o$ and to two disjoint tori for $-1 \ll \lambda_1 < \lambda_o < 0$ with corresponding period $2\pi\mu^* > 2\pi$.

§5. Local bifurcations of periodic orbits as the eigenvalues split across the imaginary axis.

Now, in this case our Hamiltonian $H^\lambda(z)$ takes the form

$$(4.5.1) \qquad H^\lambda(z) = \frac{1}{2} z^T A z + \frac{1}{2} z^T B(\lambda) z + \bar{H}_4(z),$$

where $A = \text{diag}\ (1, -1, 1, -1)$,

$$B(\lambda) = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & \lambda_4 \\ & & \lambda_4 & 0 \\ \hline 0 & \lambda_4 & \multicolumn{2}{c}{0} \\ \lambda_4 & 0 & & \end{array}\right],$$

$$\bar{H}_4(z) = (x_1^2 + y_1^2) \cdot (x_1 y_2 + x_2 y_1).$$

The corresponding Hamiltonian system becomes

$$(4.5.2) \qquad \dot{z} = J\nabla H^{\lambda}(z) = JAz + JB(\lambda)z + J\nabla\overline{H}_4(z).$$

Recall that $\mu^{*}(a,\lambda) = \dfrac{|a|^2}{|a|^2 + \langle Aa,\ B(\lambda)a + \nabla\overline{H}_4(a) \rangle}$.

Computing the right-hand side,

$Aa = (a_1,\ -a_2,\ a_3,\ -a_4)^T$,

$B(\lambda)a = \lambda_4(a_4, a_3, a_2, a_1)^T$,

$\langle Aa,\ B(\lambda)a \rangle = 0$ for all λ_4.

Hence, in this case $\mu^{*}(a,\lambda)$ becomes

$$(4.5.3) \qquad \mu^{*}(a,\lambda) = \frac{|a|^2}{|a|^2 + 2(a_1^2 + a_3^2)(a_1a_4 + a_2a_3)}$$

$$= \frac{1}{1 + 2p(a)\cdot q(a)} \quad ,$$

where $p(a) = \dfrac{a_1^2 + a_3^2}{|a|^2}$, $0 < p(a) < 1$

$q(a) = (a_1a_4 + a_2a_3)$.

Also, the funtion $g(a,\lambda,\eta)$ becomes

$$g(a,\lambda,\eta) = H_2(a) - \eta H^{\lambda}(a)$$

$$= H_2(a) - \eta[H_2(a) + H_2^{\lambda}(a) + \overline{H}_4(a)]$$

$$= (1-\eta)\cdot\frac{1}{2}\,a^T\,Aa - \eta\cdot\frac{1}{2}\,a^T\,B(\lambda)a - \eta\cdot\overline{H}_4(a)$$

$$= (1-\eta)\frac{1}{2}(a_1^2 - a_2^2 + a_3^2 - a_4^2) - \eta\cdot\lambda_4(a_1a_4 + a_2a_3)$$
$$-\eta(a_1^2 + a_3^2)\cdot(a_1a_4 + a_2a_3).$$

Hence, our gradiant system $\nabla g(a,\lambda,\eta) = 0$ becomes

$$(4.5.4)\quad \begin{cases} (1-\eta)a_1 - \eta\lambda_4a_4 - \eta\,[a_4(a_1^2 + a_3^2) + 2a_1(a_1a_4 + a_2a_3)] = 0 & (1) \\ -(1-\eta)a_2 - \eta\lambda_4a_3 - \eta\cdot a_3(a_1^2 + a_3^2) = 0 & (2) \\ (1-\eta)a_3 - \eta\lambda_4a_2 - \eta[a_2(a_1^2 + a_3^2) + 2a_3\,(a_1a_4 + a_2a_3)] = 0 & (3) \\ -(1-\eta)a_4 - \eta\lambda_4a_1 - \eta a_1(a_1^2 + a_3^2) = 0 & (4) \end{cases}$$

together with the energy

$$H^\lambda(a) = \frac{1}{2}(a_1^2 + a_3^2) - \frac{1}{2}\,(a_2^2 + a_4^2) + \lambda_4(a_1a_4 + a_2a_3) + (a_1^2 + a_3^2)(a_1a_4 + a_2a_3) = c. \quad (5)$$

Note that in this case $\eta = \mu^*(a,\lambda) = \frac{1}{1+2p(a)q(a)}$ does not depend on λ explicitly. Hence, if $q(a) = a_1a_4 + a_2a_3 > 0$, then $0 < \eta < 1$ and if $q(a) < 0$, then $\eta > 1$, and if $q(a) = 0$ then $\eta = 1$. Also system (4.6.4) has the trivial solution $a = 0$ for all λ_4 with energy $c = 0$ and still has the reflection symmetry $a_1 \leftrightarrow a_3$, $a_2 \leftrightarrow a_4$.

In the following, we consider several cases:

case (i): $\eta = 1$ (i.e., $(a_1^2+a_3^2)(a_1a_4 + a_2a_3) = 0$). Since $a_1^2 + a_3^2 = 0$ i.e., $a_1 = a_3 = 0$, is a special case of $a_1a_4 + a_2a_3 = 0$, we may only consider the general case $a_1a_4 + a_2a_3 = 0$. In this case, the system

(4.5.4) reduces to

$$(4.5.5)\quad \begin{cases} -\lambda_4 a_4 = a_4(a_1^2 + a_3^2) & (1) \\ -\lambda_4 a_3 = a_3(a_1^2 + a_3^2) & (2) \\ -\lambda_4 a_2 = a_2(a_1^2 + a_3^2) & (3) \\ -\lambda_4 a_1 = a_1(a_1^2 + a_3^2) & (4) \end{cases}$$

together with $H^{\lambda}(a) = \frac{1}{2}(a_1^2 + a_3^2) - \frac{1}{2}(a_2^2 + a_4^2) = c \quad (5)$

If $\lambda_4 = 0$, then (4.5.5) has the solution $a_1 = a_3 = 0$, a_2 and a_4 are arbitrary values with the energy $H^{\lambda} = -\frac{1}{2}(a_2^2 + a_4^2) = c$. Hence, when $q(a) = 0$ and $\lambda_4 = 0$, the solution set of (4.5.5) is

$$\begin{cases} S^1 = \{a \in \mathbb{R}^4 \mid a_1 = a_3 = 0, \ -\frac{1}{2}(a_2^2 + a_4^2) = c\} \text{ for each } c < 0 \\ a = 0 \text{ for } c = 0 \\ \phi \text{ for } c > 0 \ . \end{cases}$$

Therefore, when $q(a) = 0$ and $\lambda_4 = 0$, our Hamiltonian system (4.5.2) has the following 2π periodic solutions:

$$(4.5.6)\quad z(t) = \begin{cases} (\exp JAt)\, a \text{ with } a = (0, a_2, 0, a_4), -\frac{1}{2}(a_2^2 + a_4^2) = c \text{ foreach } c < 0 \\ 0 \text{ for } c = 0 \\ \phi \text{ for } c > 0 \end{cases}$$

Remark: In the case of $\lambda_4 = 0$ and $q(a) = 0$, the system (4.5.2) reduces to the linear system $\dot{z} = JAz$. The above result (4.5.6) agrees completely with that of global analysis given in Section 2. (See Figure 4.2.2). If $\lambda_4 = 0$ and $q(a) = 0$, we have $\ell = -h$ (in the notation of Section 2)(critical values of H x L) and so

$$(H \times L)^{-1}(h,-h) = \begin{cases} \{0\} \times S^1 & \text{for } h < 0 \\ \{0\} \text{ for } h = 0, & \phi \text{ for } h > 0 . \end{cases}$$

If $\lambda_4 \neq 0$ (and $q(a) = 0$), then from (4.5.5) (2),(4) we have

$$(4.5.7) \quad \begin{cases} a_3(\lambda_4 + a_1^2 + a_3^2) = 0 \\ a_1(\lambda_4 + a_1^2 + a_3^2) = 0 . \end{cases}$$

subcase 1: $a_1 = a_3 = 0$

Then from (4.6.5) (1),(3), we have only the trivial solution $a = 0$ for all λ_4 with energy $c = 0$.

subcase 2: $a_1 \neq 0$ or $a_3 \neq 0$. (and $\lambda_4 \neq 0$)

Then from (4.5.7), we have

$\lambda_4 + a_1^2 + a_3^2 = 0.$

If $\lambda_4 > 0$, we have no solutions a_1, a_3.

If $\lambda_4 < 0$, we have solution set for a_1, a_3 of the form

$$a_1^2 + a_3^2 = -\lambda_4$$

which forms a circle in the (a_1, a_3)-plane with radins $\sqrt{-\lambda_4}$.

Corresponding to these values of a_1, a_3, we can obtain a_2, a_4 from (4.5.5) (1) ,(3) which become

$-\lambda_4 a_4 = a_4 \cdot (-\lambda_4)$

$-\lambda_4 a_2 = a_2 \cdot (-\lambda_4)$

and hence a_2, a_4 may be arbitrary satisfying the energy condition

$$H^{\lambda}(a) = \frac{1}{2}(-\lambda_4) - \frac{1}{2}(a_2^2 + a_4^2) = c$$

or $\quad a_2^2 + a_4^2 = -2c - \lambda_4$

which form a circle on the (a_2,a_4)-plane if $c < -\frac{\lambda_4}{2}(\lambda_4 < 0)$, and $a_2 = a_4 = 0$ if $c = -\frac{\lambda_4}{2}$. For $\lambda_4 > 0$ or for $\lambda_4 < 0$ and $c > -\frac{\lambda_4}{2}$, we have no solutions for (4.5.5). Thus, we have the following conclusion.

<u>Theorem 4.5.1</u>: In the case of $q(a) = 0$ (i.e. linear system) the system (4.5.2) has the following 2π-periodic solutions as λ_4 varies for each energy level c.

(i) when $\lambda_4 = 0$

$$z(t) = \begin{cases} e^{JAt}a \text{ with initial points } a = (0,a_2,0,a_4) \text{ lying} \\ \quad \text{on the circle } -\frac{1}{2}(a_2^2 + a_4^2) = c \text{ for each } c < 0 \\ 0 \text{ for } c = 0 \\ \phi \text{ for } c > 0\ . \end{cases}$$

(ii) when $\lambda_4 < 0$

$$z(t) = \begin{cases} e^{JAt}a \text{ with initial points } a = (a_1,a_2,a_3,a_4) \text{ lying} \\ \quad \text{on torus } a_1^2 + a_3^2 = -\lambda_4 \text{ and } a_2^2 + a_4^2 = -2c - \lambda_4 \\ \text{if } -\lambda_4 > 2c \\ e^{JAt}a \text{ with } a_2 = a_4 = 0 \text{ and } a_1^2 + a_3^2 = -\lambda_4 \text{ if } -\lambda_4 = 2c \\ \\ \phi \quad \text{if } -\lambda_4 < 2c\ . \end{cases}$$

(iii) when $\lambda_4 > 0$, we have no periodic solutions except $z = 0$ with energy $c = 0$

In other words, in the case of linear system,

(a) For energy level $c < 0$, the system (4.5.2) undergoes a subcritical bifurcation from $\{0\} \times S^1$ to T^2 as λ_4 varies from 0 to $\lambda_4 < 0$.

(b) For $c = 0$, (4.5.2) undergoes a subcritical bifurcation from $\{0\}$ to T^2 as λ_4 varies from 0 to $\lambda_4 < 0$.

(c) For $c > 0$, (4.5.2) undergoes a saddle-node type subcritical bifurcation at $\lambda_4 = -2c$ from ϕ to T^2 via S^1 as λ_4 varies from 0 to $\lambda_4 < -2c$ via $\lambda_4 = -2c$. ///

Next, we consider the genuine nonlinear case:

case (ii): $\eta \neq 1$ (i.e., $q(a) = a_1a_4 + a_2a_3 \neq 0$)

In this case, we note that $0 < \eta < 1$ if $q(a) > 0$ and $\eta > 1$ for $q(a) < 0$. Also note that (4.5.4) still has trivial solution $a = 0$ for any λ_4 with energy $c = 0$. From the system (4.5.4) (2), (4), we have

$$(4.5.8)\quad \begin{cases} a_2 = -\dfrac{\eta}{1-\eta}\, a_3(\lambda_4 + a_1^2 + a_3^2) \\ a_4 = -\dfrac{\eta}{1-\eta}\, a_1(\lambda_4 + a_1^2 + a_3^2) \ . \end{cases}$$

Substituting (4.5.8) into (4.5.4) (1), (3) we have, by the refletion symmetry,

$$(4.5.9)\quad \begin{cases} a_1[(1-\eta)^2 + \eta^2(\lambda_4 + a_1^2 + a_3^2)\ (\ \lambda_4 + 3(a_1^2 + a_3^2))] = 0 \\ \\ a_3\cdot[(1-\eta)^2 + \eta^2(\lambda_4 + a_1^2 + a_3^2)(\lambda_4 + 3(a_1^2 + a_3^2))] = 0 \end{cases}$$

Thus, the 4×4 system (4.5.4) has been reduced to 2×2 system (4.5.9).

Now, notice that because of the assumption $(a_1^2 + a_3^2)(a_1a_4 + a_2a_3) \neq 0$ a_1 and a_3 cannot be both zero in (4.5.9). Hence, (4.5.9) reduces to the equivalent equation

$$(4.5.10)\quad (1-\eta)^2 + \eta^2(\lambda_4 + a_1^2 + a_3^2)(\lambda_4 + 3(a_1^2 + a_3^2)) = 0.$$

Also notice that if $\eta = 1$ then (4.5.10) includes the case $\lambda_4 + a_1^2 + a_3^2 = 0$, in which case our system (4.5.4) reduces to the linear case we have already considered. Let $X = a_1^2 + a_3^2 > 0$ and $Y = a_2^2 + a_4^2 > 0$, then (4.5.10) becomes

$$(4.5.11)\qquad 3X^2 + 4\lambda_4 \cdot X + \lambda_4^2 + \left[\frac{1-\eta}{\eta}\right]^2 = 0.$$

Also, from (4.5.8), we have

$$(4.5.12)\qquad Y = \left[\frac{\eta}{1-\eta}\right]^2 X \cdot (X + \lambda_4)^2.$$

Eliminating $\frac{1-\eta}{\eta}$ from (4.5.11) and (4.5.12), we have

$$(4.5.13)\qquad Y = \frac{-X(X+\lambda_4)^2}{(3X+\lambda_4)(X+\lambda_4)} = \frac{-X(X+\lambda_4)}{3X + \lambda_4}\ (X, Y > 0)$$

Here we assumed that $X + \lambda_4 \neq 0$ for if $X + \lambda_4 = 0$ then from (4.5.11) we have $\eta = 1$ and so our system reduces to the linear case. Notice that if $\lambda_4 = 0$, (4.5.11) and (4.5.12) has no solution except the trivial solution $a = 0$ with $c = 0$. Furthermore, since $X, Y \geq 0$, from (4.5.11) and (4.5.13), we must have

$$(4.5.14)\qquad \lambda_4 < 0 \text{ and } -\frac{\lambda_4}{3} < X \leq -\lambda_4$$

including the linear case $X = -\lambda_4$. If $\lambda_4 \geq 0$, then our system (4.5.4) has only the trivial solution $a = 0$ with $c = 0$. In terms of polar coordinates

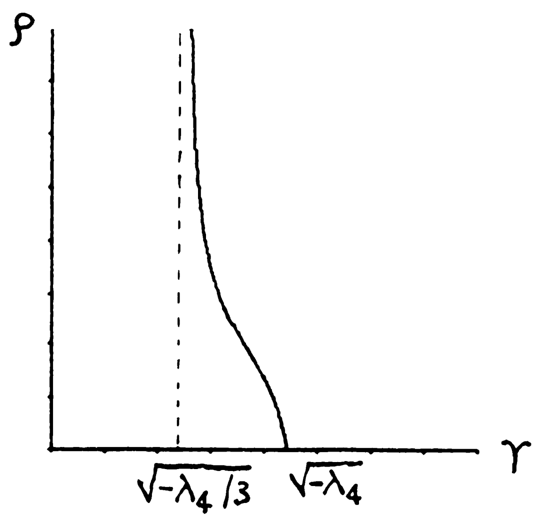
$$a_1 = \gamma \cos\theta,\ a_3 = \gamma \sin\theta,$$

$$a_2 = \rho \cos\psi,\ a_4 = \rho \sin\psi,$$

(4.5.13) becomes

$$(4.5.15) \qquad \rho = \gamma \cdot \sqrt{\frac{-(\gamma^2+\lambda_4}{3\gamma^2+\lambda_4}} \qquad (\lambda_4 < 0,\ \sqrt{\frac{-\lambda_4}{3}} < \gamma \leq \sqrt{-\lambda_4})$$

This is the equation of a curve in (γ, ρ) - plane which the solution points of (4.5.4) must satisfy. The graph of (4.5.15) is shown in Figure 4.5.1.



< Figure 4.5.1 >

Now, the energy surface (4.5.4)(5) can be rewritten as

$$\frac{1}{2}\gamma^2 - \frac{1}{2}\rho^2 + (\gamma^2 + \lambda_4)\ \gamma\rho\ \sin(\theta + \psi) = c,$$

or

(4.5.16) $\rho^2 - 2\alpha \cdot \gamma(\gamma^2 + \lambda_4)\rho + 2c - \gamma^2 = 0 \qquad (\lambda_4 < 0,\ |\alpha| \leq 1),$

where $\alpha = \sin(\theta + \psi)$.

Since (4.5.16) is quadratic in ρ, we can solve it for ρ:

(4.5.17) $\rho = \alpha\gamma(\gamma^2 + \lambda_4) \pm \sqrt{\alpha^2\gamma^2(\gamma^2+\lambda_4)^2 + \gamma^2 - 2c} \qquad (\lambda_4 < 0,\ |\alpha| \leq 1.)$

First, we consider the case $c = 0$:

Then (4.5.17) becomes

(4.5.18) $\rho_o = \alpha\gamma(\gamma^2 + \lambda_4) + \gamma \cdot \sqrt{\alpha^2(\gamma^2 + \lambda_4) + 1} \qquad (\lambda_4 < 0,\ |\alpha| \leq 1)$

since $\rho_o > 0$.

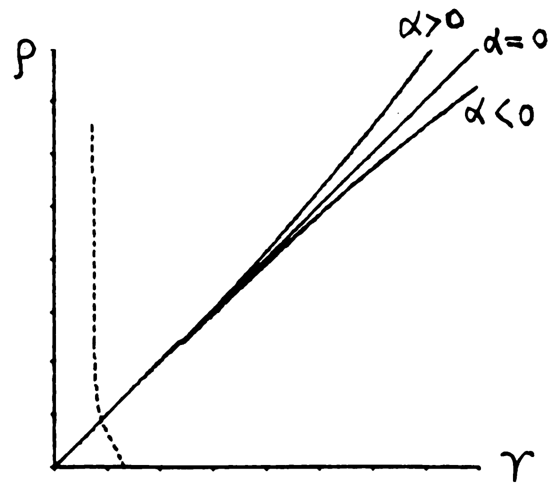
Notice that for $\lambda_4 < 0$ and $|\lambda_4| << 1$ and $|\gamma| << 1$, $\rho_o \approx \gamma$,

$$\left.\frac{d\rho_o}{d\gamma}\right|_{\gamma=0} = \alpha\lambda_4 + \sqrt{\alpha^2\lambda_4 + 1} \ (\lambda_4 < 0),$$

and so $\left.\frac{d\rho_o}{d\gamma}\right|_{\gamma=0} > 1$ for $\alpha < 0$, $= 1$ for $\alpha = 0$, 1 for $\alpha < 0$.

Note that if $\alpha = 0$ then $a_1a_4 + a_2a_3 = 0$ and hence our system reduces to the linear case.

The graph of (4.5.18) for $\alpha > 0$, $\alpha = 0$, $\alpha < 1$ with $-1 \ll \lambda_1 < 0$ is shown in Figure 4.5.2.



< Figure 4.5.2 >

Thus, it is clear from the graph of (4.5.15) and (4.5.18) that for $\lambda_4 < 0$ and $|\lambda_4| \ll 1$ and $c = 0$, they have a unique intersection point near $a = 0$, which corresponds to a torus $T^2 = S^1 \times S^1$ of critical points of g lying on the 3 - dimensional energy surface $H^{\lambda}(a) = c = 0$ in the space $\mathbb{R}^4$.

Next, we consider the case $|c| \neq 0$ sufficiently small: If $c < 0$, then from (4.5.17), we have

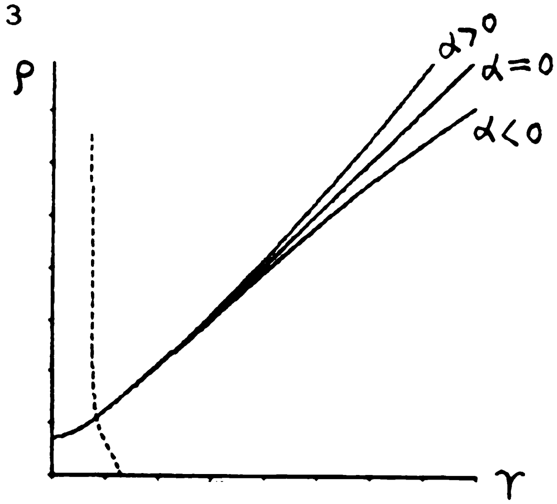
(4.5.19) $\rho_{c-} = \alpha\gamma(\gamma^2 + \lambda_4) + \sqrt{\alpha^2\gamma^2(\gamma^2 + \lambda_4)^2 + \gamma^2 - 2c}$ $\quad (c < 0,\ \lambda_4 < 0,\ |\alpha| \leq 1)$.

Notice that $\rho_{c-}(0) = \sqrt{-2c} > 0$ and $\left.\frac{d\rho_{c-}}{d\gamma}\right|_{\gamma = 0} = \alpha\lambda_4$ and so $\left.\frac{d\rho_{c-}}{d\gamma}\right|_{\gamma = 0} > 0$ for $\alpha < 0$, $= 0$ for $\alpha = 0$, < 0 for $\alpha > 0$. Also,

$\rho_{c-} > \rho_o$ and

$\rho_{c-} \approx \rho_o$ for $\gamma >> |c|$.

The graph of (4.5.19) for $\alpha > 0$, $\alpha = 0$, $\alpha < 0$ with $c < 0$, $\lambda_4 < 0$ is shown in Figure 4.5.3



< Figure 4.5.3 >

Hence, in the case of $c < 0$, as in the case of $c = 0$, clearly the graphs of (4.5.19) and (4.5.15) intersect exactly at one point, which corresponds to a torus $T^2 = S^1 \times S^1$ of critical points of g lying on the energy surface $H^{\lambda}(a) = c < 0$ in $\mathbb{R}^4$.

Now, we consider the case $c > 0$:

From (4.5.17), we have

(4.5.20) $\rho_{c+} = \alpha\gamma(\gamma^2 + \lambda_4) \pm \sqrt{\alpha^2\gamma^2(\gamma^2 + \lambda_4) + \gamma^2 - 2c}$ $(c > 0,\ \lambda_4 < 0,\ |\alpha| \leq 1)$.

Let $f(\gamma) = \alpha^2\gamma^2(\gamma^2 + \lambda_4)^2 + \gamma^2 - 2c$.

Then $f(0) = -2c < 0$ and

$$f'(\gamma) = 2\gamma[\alpha^2(\gamma^2 + \lambda_4)(3\gamma^2 + \lambda_4) + 1].$$

Hence $f'(\gamma) > 0$ for $\sqrt{\frac{-\lambda_4}{3}} \leq \gamma \leq \sqrt{-\lambda_4} \ll 1$, i.e.,

$$f'(\gamma) > 0 \text{ for all } \gamma > 0 \text{ if } \lambda_4 < 0, \ |\lambda_4| \ll 1.$$

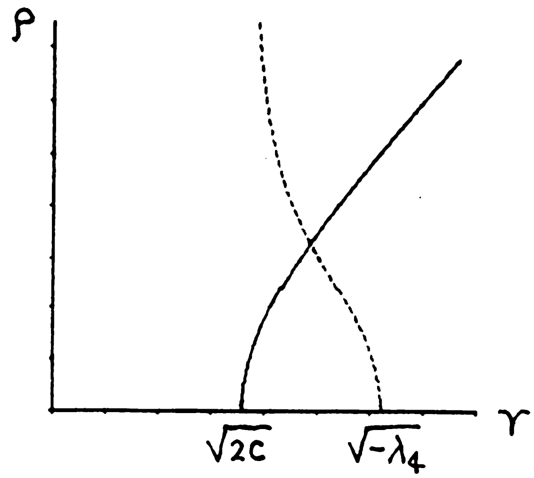
Thus, $f(\gamma)$ has a unique positive zero $\bar{\gamma}(\lambda_4, c, \alpha)$ with $\bar{\gamma}(\lambda_4, c, \alpha) \to 0+$ as $c \to 0+$. Moreover, since $f(\sqrt{-\lambda_4}) = -\lambda_4 - 2c$, if $-\lambda_4 = 2c \ll 1$ then $\bar{\gamma} = \sqrt{-\lambda_4}$ and so (4.5.15) and (4.5.21) have one intersection point $(\sqrt{-\lambda_4}, 0)$. If $-\lambda_4 > 2c$, then $\bar{\gamma} < \sqrt{-\lambda_4}$ so they have one intersection point $T^2 = S^1 \times S^1$. If $-\lambda_4 < 2c$, then $\bar{\gamma} > \sqrt{-\lambda_4}$ so they have no intersection point. Since $\sqrt{\frac{-\lambda_4}{3}} \leq \gamma \leq \sqrt{-\lambda_4}$ in (4.5.15), we may only consider ρ_{c+} for $\gamma \leq \sqrt{-\lambda_4}$. So, if we assume $\gamma \leq \sqrt{-\lambda_4}$ in (4.5.20), then for $\alpha \geq 0$, (4.5.20) must be

(4.5.21) $\rho_{c+} = \alpha\gamma(\gamma^2 + \lambda_4) + \sqrt{\alpha^2\gamma^2(\gamma^2 + \lambda_4)^2 + \gamma^2 - 2c}$ $(\alpha \geq 0,\ c > 0,\ \lambda_4 < 0)$.

Since $\rho_{c+}(\bar{\gamma}) \leq 0$ and $\rho_{c+}(\sqrt{2c}) = 0$, ρ_{c+} must have the domain $\gamma \geq \sqrt{2c}$. Also,

$$\rho_{c+} \approx \sqrt{\gamma^2 - 2c} \text{ for } r \ll 1 \text{ and } |\lambda_4| \ll 1.$$

The graph of (4.5.21) is shown in Figure 4.5.4.



< Figure 4.5.4 >

Thus, it is also clear from the graphs of (4.5.15) and (4.5.21) that for given $c > 0$ sufficiently small, they have one intersection point $T^2 = S^1 \times S^1$ if $-\lambda_4 > 2c$ and one intersection point S^1 on the γ-plane if $-\lambda_4 = 2c$ and no intersection point if $-\lambda_4 < 2c$. For $\alpha < 0$, we have from (4.5.20)

$$\text{(4.5.22)} \qquad \rho_{c+} = \alpha\gamma(\gamma^2 + \lambda_4) \pm \sqrt{\alpha^2\gamma^2(\gamma^2 + \lambda_4)^2 + \gamma^2 - 2c}$$

$(\alpha < 0,\ \lambda_4 < 0,\ c > 0)$.

In the + case, we have

$$\text{(4.5.22)(a)} \quad \rho^+_{c+} = \alpha\gamma(\gamma^2 + \lambda_4) + \sqrt{\alpha^2\gamma^2(\gamma^2 + \lambda_4)^2 + \gamma^2 - 2c} \qquad (\alpha < 0,\ \lambda_4 < 0,\ c > 0)$$

$$\approx \sqrt{\gamma^2 - 2c} \text{ for } \gamma \ll 1,\ |\lambda_4| \ll 1.$$

But, $\rho^+_{c+}(\bar{\gamma}) = \alpha\bar{\gamma}(\bar{\gamma}^2 + \lambda_4) \geq 0$ if $\bar{\gamma} \leq \sqrt{-\lambda_4}$, i.e., $-\lambda_4 \geq 2c$.

That is,

if $-\lambda_4 > 2c$, then $\bar{\gamma} < \sqrt{-\lambda_4}$ so $\rho^+_{c+}(\gamma) > 0$.

if $-\lambda_4 = 2c$, then $\bar{\gamma} = \sqrt{-\lambda_4} = \sqrt{2c}$ so $\rho^+_{c+}(\bar{\gamma}) = 0$.

if $-\lambda_4 < 2c$, then $\bar{\gamma} > \sqrt{-\lambda_4}$ so $\rho^+_{c+}(\bar{\gamma}) < 0$ and hence p^+_{c+} is defined for $\gamma \geq \sqrt{2c}$.

Also, in the - case, we have

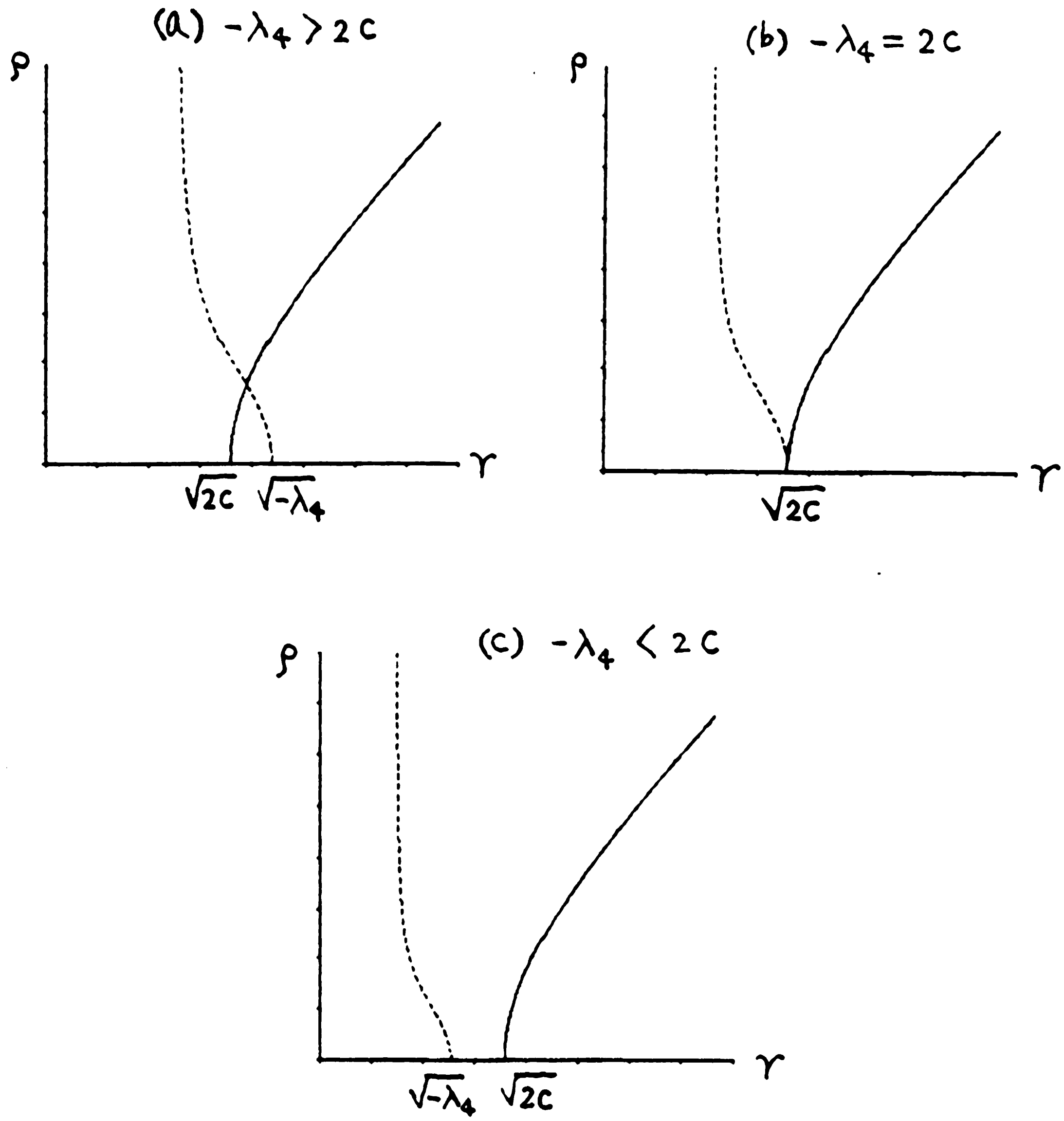
(4.5.22)(b) $\rho^-_{c+} = \alpha\gamma(\gamma^2 + \lambda_4) - \sqrt{\alpha^2\gamma^2(\gamma^2 + \lambda_4) + \gamma^2 - 2c}$ $(\alpha < 0,$ $\lambda_4 < 0,\ c > 0)$.

Similarly, we have

(i) if $-\lambda_4 > 2c$, then $\bar{\gamma} < \sqrt{-\lambda_4}$ so $p^-_{c+}(\gamma) = \rho^+_{c+}(\bar{\gamma}) > 0$ and $\rho^-_{c+}(\sqrt{2c}) = 0$. Hence in the case $-\lambda_4 < 2c$, ρ^-_{c+} is a decreasing function defined on the extremely small interval $[\gamma, \sqrt{2c}]$ with $\sqrt{2c} < \sqrt{-\lambda_4}$ and the combined graph of (4.5.22)(a) and (b) in this case looks like the one in Figure 4.5.5.(a).

(ii) if $-\lambda_4 = 2c$, then $\bar{\gamma} = \sqrt{-\lambda_4} = \sqrt{2c}$ so $\rho^-_{c+}(\bar{\gamma}) = 0$ and moreover for $\gamma > \bar{\gamma} = \sqrt{-\lambda_4} = \sqrt{2c}$, ρ^-_{c+} is not defined, in other words, the graph of ρ^-_{c+} is just one point $(\sqrt{2c}, 0)$ in this case. Hence, in the case $-\lambda_4 = 2c$, the combined graph of (4.5.22)(a) and (b) looks like the one in Figure 4.5.5(b).

(iii) if $-\lambda_4 < 2c$, then $\bar{\gamma} > \sqrt{-\lambda_4}$ so $\rho^-_{c+}(\bar{\gamma}) < 0$ and hence ρ^-_{c+} must be defined for $\gamma \geq \sqrt{2c}$ but for $\gamma > \sqrt{2c}$, $\rho^-_{c+} < 0$, that is, ρ^-_{c+} is defined just at one point $(\sqrt{2c}, 0)$. The combined graph of (4.5.22)(a) and (b) is shown in Figure 4.5.5(c).



< Figure 4.5.5 >

Therefore, when $c > 0$, even in the case of $\alpha < 0$, we can still say the same thing as in the case of $\alpha \geq 0$, that is, that if $-\lambda_4 > 2c$ then we have a torus $T^2 = S^1 \times S^1$ of critical points, if $-\lambda_4 = 2c$ then we have a circle S^1 of critical points on γ-plane, if $-\lambda_4 < 2c$ then we have no solutions for our system (4.5.4) on the energy surface $H^\lambda(a) = c > 0$. Thus, we can state the following general conclusions including the linear case.

Theorem 4.5.2: Consider the Hamiltonian system

$$(4.5.23) \qquad \dot{z} = J\nabla H^\lambda(z)$$

with $H^\lambda(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2) + \lambda_4(x_1y_2 + x_2y_1) + (x_1^2 + y_1^2)\cdot(x_1y_2 + x_2y_1)$ in the normal form with respect to $H_2(z) = \frac{1}{2}(x_1^2 + y_1^2) - \frac{1}{2}(x_2^2 + y_2^2)$. Let $H^\lambda(z) = c$. Then, in a sufficiently small neighborhood of the origin and for sufficiently small $|\lambda_4|$ and $|c|$, we have the following:

(i) when $c = 0$; the system (4.5.23) undergoes a subcritical bifurcation from $z = 0$ for $\lambda_4 \geq 0$ to a continuous family of periodic solution of the form

$$(4.5.24) \qquad z(t) = e^{JAt/\mu^*(\bar{a},\lambda_4)}\,\bar{a}$$

for $\lambda_4 < 0$ lying on a torus $T^2 = S^1 \times S^1$.

(ii) when $c < 0$; the system (4.5.23) undergoes a subcritical bifurcation from ϕ for $\lambda_4 > 0$ and from S^1 on (x_2,y_2) - plane for $\lambda_4 = 0$ to a continuous family of periodic solutions of the form (4.5.24) for $\lambda_4 < 0$, lying on a torus T^2.

(iii) when $c > 0$, then for $\lambda_4 > -2c$ the system (4.5.23) has no periodic solutions and for $\lambda_4 = -2c$ the system (4.5.23) has a periodic solution S^1: $x_1^2 + y_1^2 = 2c$ lying on the (x_1, y_1)-plane and for $\lambda_4 < -2c$ it has a continuous family of periodic solutions of the form (4.5.24) lying on a torus $T^2 = S^1 \times S^1$ in $\mathbb{R}^4$.

Furthermore, in each case (i), (ii), (iii), $\mu^* < 1, = 1, > 1$ depending on $x_1y_2 + x_2y_1 > 0, = 0, < 0$ respectively.

Remark: So far we have considered a truncated Hamiltonian containing only one fourth order term. However our methodology can still be extended to the case containing the whole nine fourth order term and can even be extended to a nearby nonintegrable system by combining Moser-Weinstein reduction. Furthermore, our method is so explicit that we can perform all the computations and graphics on the computer while the singularity theory method doesn't seem to work well in the semisimple 1: -1 resonance case.

Also, we may use the theory of equivariant vector field to express the Hamiltonian equation in terms of Hilbert generators and can study the bifurcation of equilibrium points of the new system expressed in terms of Hilbert generators.

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