ON THE TIME DOMAIN DESIGN OF LINEAR SAMPLED-DATA SYSTEMS

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THESIS

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ABSTRACT

ON THE TIME DOMAIN DESIGN OF LINEAR SAMPLED-DATA SYSTEMS

by Kensie Ross Johnson

The design problems associated with linear, sampled-data control systems have been extensively investigated in the past using, primarily, z-transform methods. Although the transform techniques have not been outmoded for linear control systems, it is felt that a time domain method, using discrete-state models, provides a common basis for the study of sampled-data control systems in general.

The discrete-state system model used in this thesis is easily obtained from the models of the subassemblies and appears as a set of first-order difference equations. The solution of this system of equations is accomplished by functions of matrices, and the response of systems to particular driving functions is discussed.

Procedures are developed for designing single input-single output, sampled-data systems with minimal, non-minimal and asymptotic time response. Considerable attention is devoted to the problem of plant saturation. An extended concept of plant controllability is also discussed in conjunction with asymptotic response.

A preliminary investigation is made of the problems associated with the design of state-controllable, multiport plants. Techniques for obtaining desired control functions are developed. It is felt that the area of multiport design is a most fruitful one for further research.

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I INTRODUCTION

In recent years, the field of automatic control systems has witnessed considerable interest in the representation of such control systems by mathematical models which are derived on the basis of the "state" of the system. In most general terms, the state of a dynamic system may be defined as:[1] "the minimum collection of numbers which must be specified at time $t = t_k$, in order to predict the entire system behavior at time $t \geq t_k$." In particular, the state of a system may appear as the solution to the set of first-order differential or difference equations which is characteristic of the system under consideration.

Prior to the investigations of Kalman[1, 2, 4], Bertram[2, 3, 4], Sarachik[3] and others[5], automatic control systems have been analyzed and designed using the Laplace or z-transform representation of the system equations. Although the transform techniques have not been outmoded for linear control systems, the system-state model provides a common basis for investigation of control systems in general; that is, continuous systems as well as discrete time systems, nonlinear as well as linear.

The shift in interest from Laplace-transform and z-transform representation of control systems to the system-state model is due in part to 1) the desire of a few investigators to arrive at a general time-domain formulation technique for the mathematical modeling of control systems; 2) recently translated technical journals and textbooks, espe-

cially from the Russian, by such authors as Lyapunov[6], Pontriagin[7] and Hahn[8] to name several. These contributions to the areas of optimal control processes and the stability of systems have featured systems modeled as a set of first-order differential or difference equations; that is, in state-model form. As a result, renewed interest has been fostered in the representation of systems by state-models in order to take advantage of the developments in these areas.

A sub-class of automatic control systems is that class of feedback control systems of the linear, time-stationary, digitally-controlled, sampled-data type. The analysis and design of these systems in the past has been carried out primarily by the use of a z-transform model and have been treated extensively in the literature[9]. The z-transform methods, however, do not extend to non-linear systems and also become quite unwieldy in certain linear applications. The use of a discrete-state model; that is, a system modeled as a set of first-order linear difference equations at least provides an alternate to the z-transform technique.

This thesis presents methods for the analysis and design of linear digitally controlled, sampled-data systems by means of the discrete-state model. Some attention is also devoted to the formulation of the system model from the discrete-state model of the sub-assemblies and methods of solution of the discrete-state equations.

II DISCRETE-STATE MODELS OF SAMPLED-DATA FEEDBACK CONTROL SYSTEMS

A. Mathematical Model For Systems With Negligible Computer Delay

Consider the system shown below which is made up of a digital controller D, a zero-order hold H and a continuous plant G which is to be controlled. The sampling operation is shown schematically by means of a switch. The digital controller is assumed to be one which receives a sequence of numbers (in time) at the input and operates arithmetically on the sequence to produce a number sequence at the output. The samplers in the system under consideration serve to indicate that the input and output of the controller are sequences of numbers. The sampling operation is assumed to be synchronous and any time delay in the computer for computation is neglected.

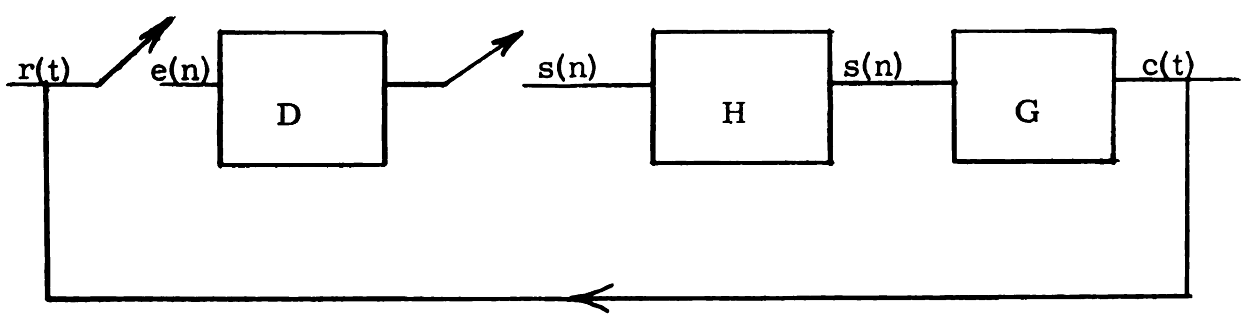

FIG. I

If the plant G can be modeled as a system of first-order linear differential equations; that is, as a continous state model of the form

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\, e(t)$$

then the linear discrete-state model for the plant is of the form[10]

$$\mathcal{X}_2(n) = \mathcal{A}_2\, \mathcal{X}_2(n-1) + \mathcal{B}_2 s(n-1)$$

where n is understood to signify nT, for n = 0, 1, 2, . . . , and $\mathcal{X}_2(n)$ represents the value of the plant state vector at t=nT. The scalar s(n-1) is the input to the plant from the zero-order hold and is of the form shown in Fig. II.

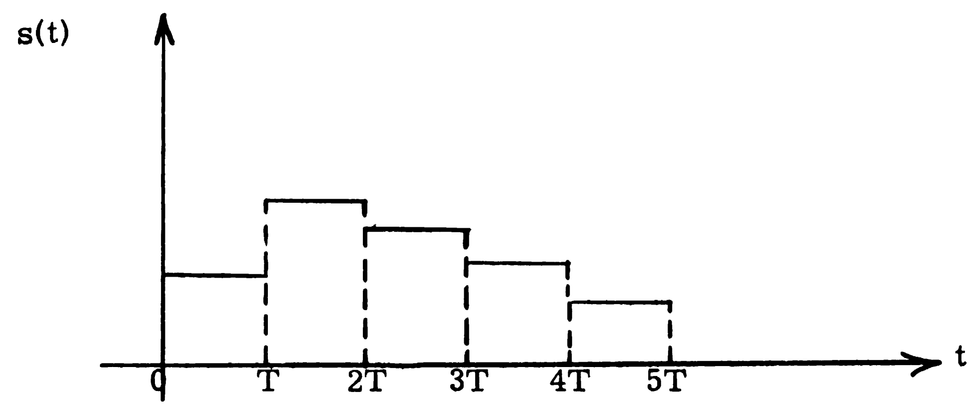

FIG. II

The discrete-state approximation to the continuous output c(t) may or may not be an element of the state vector. It is, however, assumed to be a linear combination of the state variables as indicated by the algebraic equation

$$c(n) = \mathcal{D}_2 \, \mathcal{X}_2(n) + f_2 s(n)$$

where $\mathcal{D}_2$ is a row vector and f_2 is a scalar. If c(n) is the first element of the state vector $\mathcal{X}_2(n)$, then $\mathcal{D}_2$ reduces to $[1, 0, \ldots, 0]$. In general, the discrete-state model of a linear, time-invariant plant is, therefore, taken as

$$\mathcal{X}_2(n) = \mathcal{A}_2 \, \mathcal{X}_2(n-1) + \mathcal{B}_2 s(n-1)$$

$$c(n) = \mathcal{D}_2 \, \mathcal{X}_2(n) + f_2 s(n) \qquad (1)$$

When $f_2 \neq 0$, the above model describes a plant having direct transmission.

If $f_2 = 0$, the plant is said to have no direct transmission.

The input-output characteristic of the digital controller D is considered to be linear[9] and given by the recursion formula,

$$c(nT) = a_0 r(nT) + a_1 r\left[(n-1)T\right] + \ldots + a_m r\left[(n-m)T\right]$$
$$-b_1 c\left[(n-1)T\right] - b_2 c\left[(n-2)T\right] - \ldots - b_k c\left[(n-k)T\right] \qquad (2)$$

where c(n) represents the output at a discrete time $t = nT$, r(n) represents the input at $t = nT$ and the a_i and b_i are constants. Note that in general the output at $t = nT$ is a linear combination of the k previous outputs and (m + 1) previous inputs. It can be shown[10] that (2) is easily transformed to the following state model for the controller.

$$\begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \\ \\ \vdots \\ x_k(n) \end{bmatrix} = \begin{bmatrix} -b_1 & -b_2 & -b_3 & \cdots & -b_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(n-1) \\ x_2(n-1) \\ \vdots \\ x_k(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} . r(n) \qquad (3a)$$

$$c(n) = \begin{bmatrix} a_0, & a_1, & a_2, \ldots, & a_m, & 0, \ldots, 0 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_m(n) \\ \vdots \\ x_k(n) \end{bmatrix} \quad \text{where } k > m \qquad (3b)$$

or

$$\mathcal{X}_1(n) = \mathcal{A}_1 \mathcal{X}_1(n-1) + \mathcal{B}_1 r(n) \qquad (4)$$

$$c(n) = \mathcal{D}_1 \mathcal{X}_1(n)$$

To establish a discrete-state model for the system shown in Fig. I, consider first the cascaded sub-assembly shown in Fig. III, (i.e., the system of Fig. I without feedback).

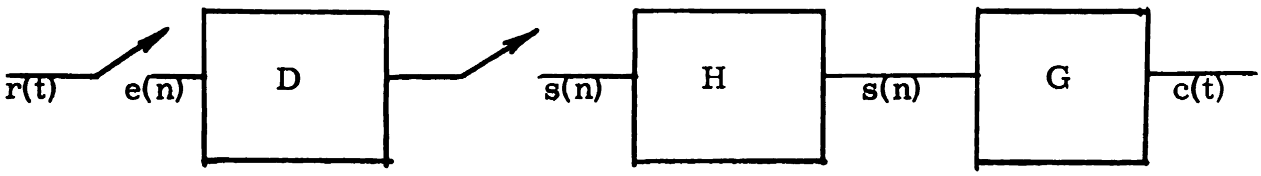

FIG. III

Rewriting (1) and (4), there results

$$\chi_2(n) = \mathcal{A}_2 \, \chi_2(n-1) + \mathcal{B}_2 \, s(n-1) \tag{5}$$

$$c(n) = \mathcal{D}_2 \, \chi_2(n) + f_2 \, s(n) \tag{6}$$

$$\chi_1(n) = \mathcal{A}_1 \, \chi_1(n-1) + \mathcal{B}_1 \, e(n) \tag{7}$$

$$s(n) = \mathcal{D}_1 \, \chi_1(n) \tag{8}$$

Substituting (8) into (5) and combining this result with (7), the cascade system model is established as

$$\begin{bmatrix} \chi_2(n) \\ \chi_1(n) \end{bmatrix} = \begin{bmatrix} a_2 & \mathcal{B}_2 \vartheta_1 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} \chi_2(n-1) \\ \chi_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_1 \end{bmatrix} e(n)$$

(9)

$$c(n) = \begin{bmatrix} \vartheta_2 & 0 \end{bmatrix} \begin{bmatrix} \chi_2(n) \\ \chi_1(n) \end{bmatrix} + f_2\, s(n)$$

The presence of the zero in the 2, 1 position of the coefficient matrix is characteristic of an open-loop system of this type. It will be shown later that the digital computer in the system of Fig. III cannot stabilize a plant for which the magnitude of the eigenvalues of the transition matrix are greater than unity.

When the feedback connection is made, there results

$$e(n) = r(n) - c(n) \tag{10}$$

When the plant output equation (6) is premultiplied by $-\mathcal{B}_1$, there results

$$-\mathcal{B}_1 c(n) = -\mathcal{B}_1 \vartheta_2 \chi_2(n) - \mathcal{B}_1 f_2 s(n)$$

or

$$-\mathcal{B}_1 c(n) = -\mathcal{B}_1 \vartheta_2 \chi_2(n) - \mathcal{B}_1 f_2 \vartheta_1 \chi_1(n)$$

(11)

Substituting (10) and (11) into the state equations yields

$$\begin{bmatrix} \mathcal{U} & 0 \\ \mathcal{B}_1 \mathcal{D}_2 & (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1) \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n) \\ \mathcal{X}_1(n) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ 0 & \mathcal{A}_1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n-1) \\ \mathcal{X}_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_1 \end{bmatrix} r(n)$$

Assuming that the inverse of the coefficient matrix exists*, then

$$\begin{bmatrix} \mathcal{X}_2(n) \\ \mathcal{X}_1(n) \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \left\{ \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ 0 & \mathcal{A}_1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n-1) \\ \mathcal{X}_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_1 \end{bmatrix} r(n) \right\} \tag{12}$$

where

$$\begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{U} & 0 \\ (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \mathcal{B}_1 \mathcal{D}_2 & (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \end{bmatrix}$$

*It can be shown that the inverse exists for all cases except $f_2 a_0 = -1$.

Equation (12) can also be written as

$$\begin{bmatrix} \mathcal{X}_2(n) \\ \mathcal{X}_1(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n-1) \\ \mathcal{X}_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \end{bmatrix} r(n) \tag{13}$$

where

$$\mathcal{R}_{11} = \mathcal{A}_2$$

$$\mathcal{R}_{12} = \mathcal{B}_2 \mathcal{D}_1$$

$$\mathcal{R}_{21} = (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \mathcal{B}_1 \mathcal{D}_2 \mathcal{A}_2$$

$$\mathcal{R}_{22} = (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \mathcal{B}_1 \mathcal{D}_2 \mathcal{B}_2 \mathcal{D}_1 + (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \mathcal{A}_1$$

Equation (13) along with

$$c(n) = \mathcal{D}_2 \mathcal{X}_2(n) + f_2 \; s(n)$$

represent the model for the entire feedback control system.

For notation purposes, let

$$\mathcal{X}(n) = \begin{bmatrix} \mathcal{X}_2(n) \\ \\ \mathcal{X}_1(n) \end{bmatrix}, \quad \Phi = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 \\ \\ (\mathcal{U} + \mathcal{B}_1 f_2 \mathcal{D}_1)^{-1} \end{bmatrix}$$

$$\text{and} \quad \mathcal{D} = \begin{bmatrix} \mathcal{D}_2 & 0 \end{bmatrix}$$

The discrete-state model of the system of Fig. I with negligible computer delay is then

$$\begin{aligned} \mathcal{X}(n) &= \Phi\, \mathcal{X}(n-1) + \beta\, r(n) \\ c(n) &= \mathcal{D}\, \mathcal{X}(n) + f_2\, s(n) \end{aligned} \qquad (14)$$

where the vector $\mathcal{X}(n)$ is referred to as the state vector of the system, the matrix Φ is the system state transition matrix and β is a vector associated with the sampled driving function $r(n)$. It is of interest to note that the term inside the braces of (12) is the same as (9) (the open-loop system), and the effect of the feedback loop shows explicitly as a premultiplication by a non-singular triangular matrix.

The special case of no direct transmission in the plant implies that $f_2 \equiv 0$ in (6). Equation (12) then reduces to

$$\begin{bmatrix} \mathcal{U} & 0 \\ \mathcal{B}_1 \mathcal{D}_2 & \mathcal{U} \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n) \\ \mathcal{X}_1(n) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ 0 & \mathcal{A}_1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n-1) \\ \mathcal{X}_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_1 \end{bmatrix} r(n)$$

or

$$\begin{bmatrix} \mathcal{X}_2(n) \\ \mathcal{X}_1(n) \end{bmatrix} = \begin{bmatrix} \mathcal{U} & 0 \\ -\mathcal{B}_1 \mathcal{D}_2 & \mathcal{U} \end{bmatrix} \left\{ \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ 0 & \mathcal{A}_1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n-1) \\ \mathcal{X}_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_1 \end{bmatrix} r(n) \right\} \tag{15}$$

$$= \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ -\mathcal{B}_1 \mathcal{D}_2 \mathcal{A}_2 & -\mathcal{B}_1 \mathcal{D}_2 \mathcal{B}_2 \mathcal{D}_1 + \mathcal{A}_1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_2(n-1) \\ \mathcal{X}_1(n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{B}_1 \end{bmatrix} r(n) \tag{16}$$

Equation (16) together with the relation

$$c(n) = \mathcal{D}_2 \; \mathcal{X}_2(n)$$

represent a discrete-state model of the entire feedback system providing the plant has no direct transmission. This model is characteristic of plants which do not respond instantaneously to changes in the input.

B. The Mathematical Model of a System With Computer Delay

The problem of computation delay in the digital computer can be thought of as an output delay; that is, the sampled input at $t = kT$ is not available for output until $t = (k + 1)\,T$. To incorporate a single unit of delay into the system model, note that it is only necessary to set $a_0 = 0$ in (2). Equation (2) then becomes

$$c(nT) = a_1 r\left[(n-1)T\right] + a_2 r\left[(n-2)T\right] + \ldots + a_m r\left[(n-m)T\right]$$
$$-b_1 c\left[(n-1)T\right] - b_2 c\left[(n-2)T\right] - \ldots - b_k c\left[(n-k)T\right]$$

It is easily seen that the output is not current. The input information is lagging one sample behind. The extension to higher order delays is trivial.

For the purposes of design, it is necessary to know the properties of the transition matrix of the system $\overline{\underline{\Phi}}$ as they relate to the properties of the transition matrices for the components. To establish these relations explicitly, let there be q variables in the plant state vector, and let the linear recursion formula for the digital computer as given by (2) contain (m+k) terms. For the sake of simplicity, consider the case of no direct transmission in the plant and no computer delay. The sub-matrices of the system transition matrix corresponding to the components (see Eq. (16)) are

$$\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(q+k)\times 1} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}}\right\} q \\ \left.\vphantom{\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix}}\right\} k \end{matrix}, \quad \mathcal{A}_2 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & & & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qq} \end{bmatrix}_{q\times q} \quad \mathcal{B}_2 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{q1} \end{bmatrix}_{q\times 1}$$

$$\mathcal{D}_2 = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1q} \end{bmatrix}_{1\times q}, \quad \mathcal{A}_1 = \begin{bmatrix} -b_1 & -b_2 & \cdots & -b_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{k\times k}$$

$$\mathcal{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{k\times 1} \qquad \mathcal{D}_1 = \begin{bmatrix} a_0, a_1 a_2 \cdots a_m, & 0, \ldots, 0 \end{bmatrix}_{1\times k}$$

The detailed form of Φ as obtained by performing the indicated matrix multiplications is:

$$
\Phi = \left[\begin{array}{cccc|ccccccc}
a_{11} & a_{12} & \cdots & a_{1q} & b_{11}a_0 & b_{11}a_1 & \cdots & b_{11}a_m & 0 & \cdots & 0 \\
\cdot & & & & b_{21}a_0 & b_{21}a_1 & \cdots & b_{21}a_m & 0 & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & & \vdots & & & \\
a_{q1} & \cdots & \cdots & a_{qq} & b_{q1}a_0 & b_{q1}a_1 & \cdots & b_{q1}a_m & 0 & \cdots & 0 \\
\hline
d'_{11} & d'_{12} & \cdots & d'_{1q} & p_{11} & p_{12} & \cdots & p_{1m} & -b_{m+1} & \cdots & -b_k \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots 1 & 0
\end{array}\right]_{(q+k)}
\tag{17}
$$

where $d'_{1i} = \sum_{j=1}^{q} c_{ij}\, a_{j1}$ and $p_{ij} = -b_j - a_{j+1} \sum_{i=1}^{m} c_{1i}\, b_{i1}$ for $j \leq m$

The following important points are to be noted:

1. The discrete-state transition matrix is related to the transition matrices of the components by a simple algorithm.

2. The column entries to the left of the partition lines are a function only of the entries in $\mathcal{A}_2$, $\mathcal{B}_1$, and $\mathcal{D}_2$. Since $\mathcal{B}_1$ contains only one non-zero entry, these columns are functions only of the plant parameters.

3. The column entries to the right of the partition lines are functions of the entries in both the plant and computer transition matrices.

4. The entries in the submatrix in the 1, 2 position are functions only of the controller a_i; the entries in the 2, 2 submatrix are linear combinations of the controller parameters a_i and b_i.

The transition matrices for systems with computer delay and direct transmission in the plant are formed in exactly the same manner as in the previous example. To avoid repetition, these forms are not included here.

III SOLUTION AND STABILITY OF DISCRETE-STATE MODELS

The primary performance characteristics of concern in the design of linear feedback control systems are:

1. System response time
2. System stability

To evaluate the system response the solution of the discrete-state model (14) must, of course, be obtained. There are several methods which can be used to solve (14). These are:

1. Direct computational method
2. Solution by functions of matrices
3. The augmented matrix method

These methods will be considered separately in the following discussion.

A. Direct Computation of the State Vector

Let (14) be written recursively as

$$
\begin{aligned}
\chi_{(1)} &= \Phi\, \chi_{(0)} + \beta_{r(1)} \\
\chi_{(2)} &= \Phi^2\, \chi_{(0)} + \Phi\, \beta_{r(1)} + \beta_{r(2)} \\
&\;\vdots \\
\chi_{(n)} &= \Phi^n\, \chi_{(0)} + \Phi^{n-1} \beta_{r(1)} + \dots + \Phi\, \beta_{r(n-1)} + \beta_{r(n)}
\end{aligned}
\tag{18}
$$

The response of the system is, of course, determined entirely by the behavior of the state vector. This follows directly from the definition of the system "state." The evaluation of $\chi(n)$ as given in (18) can be obtained by direct computation; that is, by calculating the indicated products of powers of the transition matrix. One measure of the behavior of the state vector as n increases without bound can be obtained by considering the norm of the state vector, $\|\chi\|$, where such a norm may be taken as any of the following:

1. $\|\chi\| = \max_i \; |x|_i$

2. $\|\chi\| = |x|_1 + |x|_2 + \dots + |x|_n$

3. $\|\chi\| = \left[|x|_1^2 + |x|_2^2 + \dots + |x|_n^2 \right]^{\frac{1}{2}}$

The third norm is the well-known Euclidian norm or length of a vector in n-space. It is stated here, without formal definition, that in order for the system to respond in a stable mode it is necessary that

the state vector $\chi(n)$ converge to some vector χ_0.

The following well-known theorem[11] establishes conditions for such a convergence:

For a sequence of vectors $\chi_{(1)}, \chi_{(2)}, \ldots, \chi(n), \ldots$ to converge to a vector χ_0 it is both necessary and sufficient that $\|\chi(k) - \chi_0\| \rightarrow 0$ as $k \rightarrow \infty$.

A test of the above norm, or equivalently, a test of $\|\chi(k)\|$ as $k \rightarrow \infty$, is sufficient to determine stability and rate of convergence. Although analysis by the direct method is possible, it involves excessive computation time for all but the most simple cases, particularly if the state vector does not converge rapidly. This difficulty is partially avoided by the use of the solution by functions of matrices. Since this approach is relatively new, considerable attention is devoted to it.

B. Solution of the Discrete-State Model by Functions of Matrices

Let (14) be written as

$$\chi(n+1) = \alpha^{n+1}\chi(0) + \alpha^{n}\beta\, r(0) + \alpha^{n-1}\beta\, r(1) + \ldots + \beta\, r(n)$$

When the eigenvalues of α are distinct, the above equation can be expanded by functions of matrices[12] to read

$$
\begin{aligned}
\mathcal{X}_{(n+1)} = {} & \left[\mathcal{Z}_{11}\lambda_1^{n+1} + \mathcal{Z}_{12}\lambda_2^{n+1} + \dots + \mathcal{Z}_{1k}\lambda_k^{n+1}\right] \cdot \mathcal{X}(0) \\
& + \left[\mathcal{Z}_{11}\lambda_1^{n} + \mathcal{Z}_{12}\lambda_2^{n} + \dots + \mathcal{Z}_{1k}\lambda_k^{n}\right] \beta\, r(0) \\
& + \left[\mathcal{Z}_{11}\lambda_1^{n-1} + \mathcal{Z}_{12}\lambda_2^{n-1} + \dots + \mathcal{Z}_{1k}\lambda_k^{n-1}\right] \beta\, r(1) \\
& + \quad \vdots \\
& + \left[\mathcal{Z}_{11}\lambda_1 + \mathcal{Z}_{12}\lambda_2 + \dots + \mathcal{Z}_{1k}\lambda_k\right] \beta\, r(n-1) \\
& + \beta\, r(n)
\end{aligned}
\tag{19}
$$

where the $\mathcal{Z}_{1i}$, $i = 1, 2, 3, \dots, k$, are the constituent matrices, and the λ_j, $j = 1, 2, 3, \dots, k$ are the k distinct roots of the polynomial equation $\det\left[\mathcal{A} - \lambda\mathcal{U}\right] = 0$. For certain classes of input signals (19) can be simplified somewhat as the following example indicates.

Example: Step-Function Input

Let the driving function r(n) be a unit step function, i.e.,

$$r(0) = r(1) = r(2) = \dots = r(n) = 1$$

For convenience let $\mathcal{A}^{n+1}\,\mathcal{X}(0) = \mathcal{Y}_0$. Equation (19) then becomes

$$\begin{aligned}\mathcal{X}_{(n+1)} = (\mathcal{Y}_0 + \beta) &+ \mathcal{V}_{11}(\lambda_1^n + \lambda_1^{n-1} + \ldots + \lambda_1^2 + \lambda_1) \\ &+ \mathcal{V}_{12}(\lambda_2^n + \lambda_2^{n-1} + \ldots + \lambda_2^2 + \lambda_2) \\ &+ \;\vdots \\ &+ \mathcal{V}_{1k}(\lambda_k^n + \lambda_k^{n-1} + \ldots + \lambda_k^2 + \lambda_k)\end{aligned} \tag{20}$$

where $\mathcal{V}_{1i} = \mathcal{Z}_{1i}\beta$, $i = 1, 2, \ldots, k$.

Letting $S_i = \frac{(\lambda_i^n - 1)}{\lambda_i - 1} = \lambda_i^n + \lambda_i^{n-1} + \ldots + \lambda_i, \quad \lambda_i \neq 1$

the general formula for the system response to a unit step forcing function is

$$\mathcal{X}_{(n+1)} = (\mathcal{Y}_0 + \beta) + \sum_{j=1}^{k} \mathcal{V}_{1j} S_j$$

If $\mathcal{A}$ has an eigenvalue of higher multiplicity than one, say m, then for a step input it can be shown by the procedures outlined above that

$$\mathcal{X}_{(n+1)} = (\mathcal{Y}_0 + \beta) + \sum_{j=1}^{m} \mathcal{V}_{1j} \frac{d^{(m)}}{d\lambda_j^{(m)}} S_1 + \sum_{i=1}^{k-m} \mathcal{V}_{1i} S_i$$

Example: Ramp-Function Input

If $\mathcal{A}$ has distinct eigenvalues but the input is a ramp function, that is, if

$$r(i) = Ki \quad \text{for } i = 1, 2, 3, \ldots, n \quad \text{and } r(0) = 0$$

then (19) reduces to

$$\begin{aligned}\mathcal{X}(n+1) = (\mathcal{Y}_o + \beta Kn) &+ \mathcal{V}_{11} K \left[(n-1)\lambda_1 + (n-2)\lambda_1^2 + \ldots + 2\lambda_1^{n-2} + \lambda_1^{n-1}\right] \\ &+ \mathcal{V}_{12} K \left[(n-1)\lambda_2 + (n-2)\lambda_2^2 + \ldots + 2\lambda_2^{n-2} + \lambda_2^{n-1}\right] \\ &+ \quad \vdots \\ &+ \mathcal{V}_{1k} K \left[(n-1)\lambda_k + (n-2)\lambda_k^2 + \ldots + 2\lambda_k^{n-2} + \lambda_k^{n-1}\right]\end{aligned}$$

where

$$S_i = \frac{\lambda_i^n - 1}{\lambda_i - 1}, \qquad \lambda_i \neq 1$$

or

$$\mathcal{X}(n+1) = (\mathcal{Y}_o + \beta Kn) + K \sum_{i=1}^{k} \mathcal{V}_{1i} \left[nS_i - \lambda_i \frac{d}{d\lambda_i} S_i\right]$$

Example: Constant Acceleration Input

Similarly, constant acceleration response, when $\mathcal{A}$ has distinct eigenvalues, is

$$\chi_{(n+1)} = (\mathcal{Y}_0 + \beta K n^2) + \sum_{i=1}^{k} \mathcal{V}_{1i} \left[n^2 S_i - 2n\lambda_i S_i' + \lambda_i \frac{d}{d\lambda_i}(\lambda_i S_i') \right]$$

This method of obtaining a closed form of the solution to (14) can be extended to include higher order inputs and repeated eigenvalues. Since the algebra is quite involved, the details are omitted here.

C. Solution by the Augmented Matrix Method

An alternate and sometimes more useful solution procedure results when the nonhomogeneous equations are first reduced to homogeneous form.

Consider

$$\chi_{(n+1)} = \mathcal{A}\,\chi_{(n)} + \mathcal{B}\, r(n) \text{ where } \chi_{(0)} = \chi_0 \quad (21)$$

Suppose (21) can be reduced to the homogeneous form

$$\begin{bmatrix} \mathcal{X}(n+1) \\ \mathcal{R}(n+1) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^* \\ 0 & \mathcal{X} \end{bmatrix} \begin{bmatrix} \mathcal{X}(n) \\ \mathcal{R}(n) \end{bmatrix}, \qquad \begin{bmatrix} \mathcal{X}(0) \\ \mathcal{R}(0) \end{bmatrix} = \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{R}_0 \end{bmatrix} \tag{22}$$

The obvious benefit derived from this transformation, if generally applicable, lies in the simplicity of the solution; namely, if (22) is written as

$$\mathcal{Y}(n+1) = \mathcal{H}\,\mathcal{Y}(n) \qquad \mathcal{Y}(0) = \mathcal{Y}_0$$

then the solution is

$$\mathcal{Y}(n+1) = \mathcal{H}^{n+1}\,\mathcal{Y}(0)$$

where

$$\mathcal{H} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^* \\ 0 & \mathcal{X} \end{bmatrix} \qquad \mathcal{Y}(n+1) = \begin{bmatrix} \mathcal{X}(n+1) \\ \mathcal{R}(n+1) \end{bmatrix} \quad \text{and } \mathcal{Y}(0) = \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{R}_0 \end{bmatrix}$$

This result is simple compared to the solution to the nonhomogeneous case which is

$$\mathcal{X}(n+1) = \mathcal{A}^{n+1}\,\mathcal{X}(0) + \sum_{j=1}^{n} \mathcal{A}^{n-j-1}\,\beta\; r(j-1) \tag{23}$$

It is immediately apparent that the benefit derived from reducing (23) to the form of (22) is the elimination of the summation term in (23). This benefit is realized, however, at the expense of increasing the order of the transition matrix. The question remaining is: under what conditions can (23) be reduced to the form of (22)? Several examples using input functions of particular interest will aid in determining these conditions.

Example 1:

If the driving function in (21) is taken as $r(n) = Kn^p$, where p is an integer and K a constant, then

$$r(n+1) = K(n+1)^p = K(n^p + pn^{p-1} + \dots + pn + 1)$$

This sequence represents the solution to the difference equations.

$$\begin{bmatrix} r(n+1) \\ r_p(n+1) \\ \cdot \\ \cdot \\ \cdot \\ r_2(n+1) \\ r_1(n+1) \end{bmatrix} = \begin{bmatrix} 1 & p & Kp & \dots & p & 1 \\ 0 & 1 & (p-1) & \dots & (p-1) & 1 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} r(n) \\ r_p(n) \\ \cdot \\ \cdot \\ \cdot \\ r_2(n) \\ r_1(n) \end{bmatrix}, \quad \begin{bmatrix} r(o) \\ r_p(o) \\ \cdot \\ \cdot \\ \cdot \\ r_2(o) \\ r_1(o) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ K \end{bmatrix}$$

and the nonhomogeneous form in (21) can be represented by the homogeneous form

$$\begin{bmatrix} \chi(n+1) \\ r_p(n+1) \\ \cdot \\ \cdot \\ \cdot \\ r_1(n+1) \end{bmatrix} = \begin{bmatrix} \alpha & \beta K & 0 & \ldots & 0 & 0 \\ 0 & 1 & p & \ldots & p & 1 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi(n) \\ r_p(n) \\ \cdot \\ \cdot \\ \cdot \\ r_1(n) \end{bmatrix}, \quad \begin{bmatrix} \chi(o) \\ r_p(o) \\ \cdot \\ \cdot \\ \cdot \\ r_1(o) \end{bmatrix} = \begin{bmatrix} \chi_o \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

Example 2:

If the driver in (21) is of the form

$$r(n) = \cos knT = \cos Kn$$

then

$$r(n) = \frac{e^{jKn} + e^{-jKn}}{2} = r_1(n) + \hat{r}_1(n)$$

and it follows that

$$r_1(n+1) = \frac{e^{jK(n+1)}}{2} = r_1(n)e^{jK} \text{ and } r_1(n+1) = r_1(n)e^{-jK}$$

The equivalent homogeneous state model is

$$\begin{bmatrix} \chi(n+1) \\ r_1(n+1) \\ \hat{r}_1(n+1) \end{bmatrix} = \begin{bmatrix} \alpha & \frac{1}{2}\beta & -\frac{1}{2}\beta \\ 0 & e^{jK} & 0 \\ 0 & 0 & e^{-jK} \end{bmatrix} \begin{bmatrix} \chi(n) \\ r_1(n) \\ \hat{r}_1(n) \end{bmatrix} \qquad \begin{bmatrix} \chi(o) \\ r_1(o) \\ \hat{r}_1(o) \end{bmatrix} = \begin{bmatrix} \chi_o \\ 1 \\ 1 \end{bmatrix}$$

Example 3:

If the driver is of the form

$$r(n) = ne^{Kn}$$

then define

$$r_1(n) = e^{Kn}$$

The resultant homogeneous form is

$$\begin{bmatrix} \chi(n+1) \\ r(n+1) \\ r_1(n+1) \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 \\ 0 & e^K & e^K \\ 0 & 0 & e^K \end{bmatrix} \begin{bmatrix} \chi(n) \\ r(n) \\ r_1(n) \end{bmatrix} \qquad \begin{bmatrix} \chi(o) \\ r(o) \\ r_1(o) \end{bmatrix} = \begin{bmatrix} \chi_o \\ 0 \\ 1 \end{bmatrix}$$

The above examples cover the driving functions frequently used as a basis of design. There remains, however, the general problem of what is to be done in the case of an input function which cannot be expressed analytically; that is, a driver whose values are known only as a function of n as shown in Fig. IV.

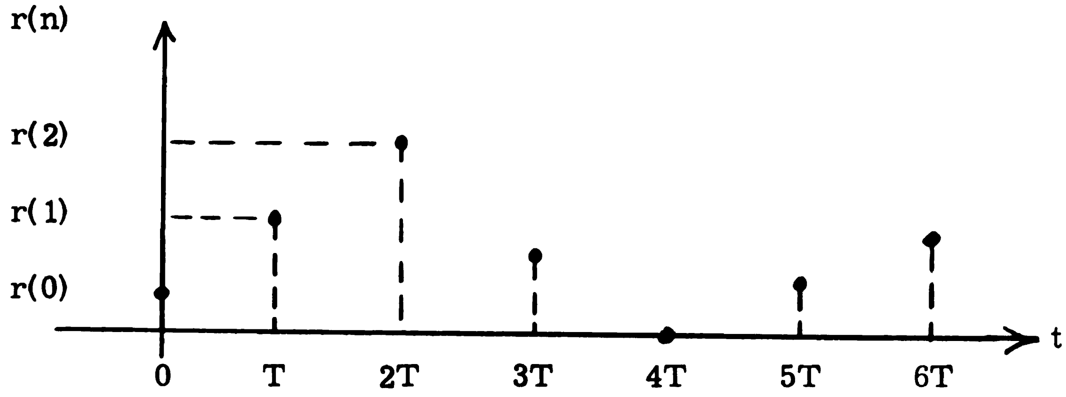

FIG. IV

In general if the (n+1) values of a function are known in an interval $0 \leq n \leq nT$, then a polynomial of at most degree n can be constructed to pass through the (n+1) known data points. For example, if the polynomial passing through three data points is

$$r(n) = K_1 n^2 + K_2 n + K_3$$

and

$$r_2(n) = n^2 \qquad r_1(n) = n \qquad r_0(n) = 1$$

then the nonhomogeneous equivalent of (21) is

$$\begin{bmatrix} \chi(n+1) \\ r_2(n+1) \\ r_1(n+1) \\ r_0(n+1) \end{bmatrix} = \begin{bmatrix} \alpha & \beta K_1 & \beta K_2 & \beta K_3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi(n) \\ r_2(n) \\ r_1(n) \\ r_0(n) \end{bmatrix} \qquad \begin{bmatrix} \chi(0) \\ r_2(0) \\ r_1(0) \\ r_0(0) \end{bmatrix} = \begin{bmatrix} \chi_0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In general, using the above technique, and any polynomial input to a system (providing the input is zero after n samples), the nonhomogeneous equation can be reduced to the augmented homogeneous form

$$\begin{bmatrix} \chi(n+1) \\ r_k(n+1) \\ r_{k-1}(n+1) \\ \cdot \\ \cdot \\ \cdot \\ r_1(n+1) \\ r_0(n+1) \end{bmatrix} = \begin{bmatrix} \alpha & \beta K_1 & \beta K_2 & \cdots & \beta K_{k-1} & \beta K_k \\ 0 & 1 & k & \cdots & k & 1 \\ 0 & 0 & 1 & \cdots & k-1 & 1 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi(n) \\ r_k(n) \\ r_{k-1}(n) \\ \cdot \\ \cdot \\ \cdot \\ r_1(n) \\ r_0(n) \end{bmatrix} \qquad \begin{bmatrix} \chi(0) \\ r_k(0) \\ r_{k-1}(0) \\ \cdot \\ \cdot \\ \cdot \\ r_1(0) \\ r_0(n) \end{bmatrix} = \begin{bmatrix} \chi_o \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

where the K_i are the coefficients of the polynomial driver

$$r(n) = K_1 n^k + K_2 n^{k-1} + \ldots + K_{k-1} n + K_k$$

and the non-zero entries in the i^{th} row are the coefficients of the polynomial $(n+1)^i$.

From the preceding examples, it is evident that if the driving functions can be written in the form

$$r(n) = \sum_{j=1}^{k} K_j \, r_j(n)$$

and the r_j can be regarded as the solution to a set of first order linear difference equations, then the nonhomogeneous system can be transformed to an equivalent homogeneous system. To show this, let

$$r(n) = K_1 r_1(n) + K_2 r_2(n) + \ldots + K_k r_k(n)$$

Since $r_j(n)$ represents the solution to the difference equations

$$\begin{bmatrix} r_k(n+1) \\ r_{k-1}(n+1) \\ \cdot \\ \cdot \\ \cdot \\ r_1(n+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} r_k(n) \\ r_{k-1}(n) \\ \cdot \\ \cdot \\ \cdot \\ r_1(n) \end{bmatrix}$$

or

$$\mathcal{R}(n+1) = \mathcal{T} \, \mathcal{R}(n),$$

it follows that the nonhomogeneous equation

$$\mathcal{X}(n+1) = \mathcal{A}\mathcal{X}(n) + \mathcal{B}\, r(n)$$

can be written as

$$\mathcal{X}(n+1) = \mathcal{A}\mathcal{X}(n) + \mathcal{B}\sum_{j=i}^{k} K_j\, r_j(n)$$

or

$$\mathcal{X}(n+1) = \mathcal{A}\mathcal{X}(n) + \left[\mathcal{B}K_1,\ \mathcal{B}K_2, \ldots\ \mathcal{B}K_k\right]\begin{bmatrix} r_1(n) \\ r_2(n) \\ \cdot \\ \cdot \\ \cdot \\ r_k(n) \end{bmatrix}$$

The homogeneous equivalent is

$$\left[\begin{array}{c} \mathcal{X}(n+1) \\ \hline r_1(n+1) \\ r_2(n+1) \\ \cdot \\ \cdot \\ \cdot \\ r_k(n+1) \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}K_1 \quad \mathcal{B}K_2 \quad \mathcal{B}K_3 \ \ldots \ \mathcal{B}K_k \\ \hline 0 & \\ 0 & \\ \cdot & \mathcal{T} \\ \cdot & \\ \cdot & \\ 0 & \end{array}\right] \left[\begin{array}{c} \mathcal{X}(n) \\ \hline r_1(n) \\ r_2(n) \\ \cdot \\ \cdot \\ \cdot \\ r_k(n) \end{array}\right]$$

and is the same form as (22). For the particular driving functions considered in the above examples, the matrix $\mathcal{T}$ is upper triangular.

Equation (22) can be reduced further, or "split," by a simple change of state variables. Let (22) be written as

$$\begin{bmatrix} \mathcal{Y}_{(n+1)} \\ \mathcal{R}_{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^* - \mathcal{W}\mathcal{T} \\ 0 & \mathcal{T} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{(n)} \\ \mathcal{R}_{(n)} \end{bmatrix} \qquad (24)$$

where

$$\mathcal{Y}_{(n+1)} = \mathcal{X}_{(n+1)} - \mathcal{W}\mathcal{R}_{(n+1)}$$

is a new set of state variables and $\mathcal{W}$ is a matrix undefined as yet.

Equation (24) can also be written as

$$\begin{bmatrix} \mathcal{Y}_{(n+1)} \\ \mathcal{R}_{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^* - \mathcal{W}\mathcal{T} + \mathcal{A}\mathcal{W} \\ 0 & \mathcal{T} \end{bmatrix} \begin{bmatrix} \mathcal{X}_{(n)} - \mathcal{W}\mathcal{R}_{(n)} \\ \mathcal{R}_{(n)} \end{bmatrix}$$

or

$$\begin{bmatrix} \mathcal{Y}_{(n+1)} \\ \mathcal{R}_{(n+1)} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^* - \mathcal{W}\mathcal{T} + \mathcal{A}\mathcal{W} \\ 0 & \mathcal{T} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_{(n)} \\ \mathcal{R}_{(n)} \end{bmatrix} \qquad (25)$$

The special form of (25) leads to the following:

Theorem I

If the nonhomogeneous set of difference equations

$$\mathcal{X}(n+1) = \mathcal{A}\mathcal{X}(n) + \mathcal{B}\, r(n)$$

can be written in the equivalent form

$$\begin{bmatrix} \mathcal{Y}(n+1) \\ \mathcal{R}(n+1) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^* - \mathcal{W}\mathcal{T} + \mathcal{A}\mathcal{W} \\ 0 & \mathcal{T} \end{bmatrix} \begin{bmatrix} \mathcal{Y}(n) \\ \mathcal{R}(n) \end{bmatrix}$$

then there exists a matrix $\mathcal{W}$ such that

$$\mathcal{B}^* - \mathcal{W}\mathcal{T} + \mathcal{A}\mathcal{W} = 0$$

where $\mathcal{W}$ is given by

$$\mathcal{W} = \sum_{j,\,k} \mathcal{Z}_{jk}\, \mathcal{B}^* (\mathcal{T} - \lambda_j \mathcal{U})^{-k}$$

and the $\mathcal{Z}_{jk}$ are the constituent matrices for $\mathcal{A}$ corresponding to the eigenvalue λ_j.

Proof: After Frame[13].

Note that such a selection of the matrix $\mathcal{W}$ leads to two separate homogeneous systems of equations, and that the original state variables $\mathcal{X}(n+1)$ are related to the new state variables $\mathcal{Y}(n+1)$ by the equation

$$\mathcal{X}(n+1) = \mathcal{Y}(n+1) + \mathcal{W}\mathcal{R}(n+1)$$

D. System Stability

The problem of system stability must be considered in the design of any linear feedback control system. Instability in a system represented by a discrete-state model is characterized by the fact that one or more of the state variables increases without bound when the system is displaced from an equilibrium state. The following stability theorem was proposed by Kalman with the statement that the proof can be carried out by reducing the transition matrix to its Jordan canonic form. An alternate proof is given here.

Theorem II

A stationary linear system is stable if and only if the n zeros of the polynomial equation det $\left[\Phi - \lambda \mathcal{U}\right] = 0$ satisfy $|\lambda_i| \leq 1$ for i = 1, 2, . . . , n. The $|\lambda_i|$ that satisfy $|\lambda_i| = 1$ must be of multiplicity one.

Proof: Let the system discrete-state model be written as

$$\mathcal{X}(n) = \Phi \mathcal{X}(n-1) + \mathcal{B}\, r(n) \qquad (26)$$

Since the system is assumed to be linear, the stability is independent of the driving functions; therefore, let r(n) = 0, then

$$\mathcal{X}(n) = \Phi\, \mathcal{X}(n-1) = \Phi^n \quad \mathcal{X}(0) \text{ for } \quad \mathcal{X}(0) \text{ arbitrary.}$$

The solution is of the form

$$\begin{aligned}\mathcal{X}(n) = \Big\{ & \mathfrak{z}_{11}\lambda_1^n + \mathfrak{z}_{12}\, n\, \lambda_1^{n-1} + \ldots + \mathfrak{z}_{1k}\, n(n-1)\ldots(n-k)\, \lambda_1^{n-k+1} \\ + & \mathfrak{z}_{21}\lambda_2^n + \mathfrak{z}_{22}\, n\, \lambda_2^{n-1} + \ldots + \mathfrak{z}_{2p}\, n(n-1)\ldots(n-p)\, \lambda_2^{n-p+1} \\ & \vdots \\ + & \mathfrak{z}_{m1}\lambda_k^n + \mathfrak{z}_{m2}\, n\, \lambda_k^{n-1} + \ldots + \mathfrak{z}_{mq}\, n(n-1)\ldots(n-q)\, \lambda_k^{n-q+1} \Big\} \mathcal{X}(0)\end{aligned}$$

or

$$\begin{aligned}\mathcal{X}(n) = & \mathcal{V}_{11}\lambda_1^n + \mathcal{V}_{12}\, n\, \lambda_1^{n-1} + \ldots + \mathcal{V}_{1k}\, n(n-1)\ldots(n-k)\, \lambda_1^{n-k+1} \\ + & \mathcal{V}_{21}\lambda_2^n + \mathcal{V}_{22}\, \lambda_2^{n-1} + \ldots + \mathcal{V}_{2p}\, n(n-1)\ldots(n-p)\, \lambda_2^{n-p+1} \\ & \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (26) \\ + & \mathcal{V}_{m1}\lambda_m^n + \mathcal{V}_{m2}\, n\, \lambda_m^{n-1} + \ldots + \mathcal{V}_{mq}\, n(n-1)\ldots(n-q)\, \lambda_m^{n-q+1}\end{aligned}$$

Since the vectors $\mathcal{V}_{ij}$ i = 1, 2, ... m, j = 1, 2, ... k, ... p, ... q. are constants (i.e. all entries are constants), consider the coefficients of the $\mathcal{V}_{ij}$.

In general, these coefficients are of the form

$$n(n-1)(n-2)\ \ldots\ (n-i)\ \lambda_k^{n-n_k+1}$$

where n_k is the multiplicity of the k^{th} eigenvalue λ_k. It is necessary, therefore, to prove the following:

Lemma

If $|z| < 1$, then $n(n-1)(n-2)\ldots(n-k)\, z^{n-(k+1)}$ tends to zero as $n \to \infty$, where k is a fixed positive number and $n > k+1$.

Proof:

Since

$$n(n-1)(n-2)\ldots(n-k)\; z^{\,n-(k+1)} < n^k \; z^{\,n-(k+1)}$$

and

$$\left|n^k \; z^{\,n-(k+1)}\right| = \left|n^k\right| \left|z^{\,n-(k+1)}\right|$$

it follows upon setting $|z| = \rho$ (the modulus of z) that

$$\left|n^k \; z^{\,n-(k+1)}\right| = n^k \; \rho^{\,n-(k+1)} = n^k \; \rho^n \; \rho^{-(k+1)}$$

Since $\rho^{-(k+1)}$ is a constant, consider only the behavior of

$$\lim_{n \to \infty} \left[n^k \; \rho^n \right] = \lim_{n \to \infty} \frac{n^k}{\rho^{-n}}$$

Applying L'Hopitals rule k-1 times, there results

$$\lim_{n\to\infty} \frac{k!}{(-1)^k(\ln \rho)^k \; \epsilon^{-n \ln}} = \lim_{n\to\infty} \frac{k! \, |z|^n}{(-1)^k(\ln \rho)^k} = 0$$

Applying the above Lemma to the terms in (26) it follows that the entries in the state vector tend toward zero if $|\lambda_i| < 1$ for all $i = 1, 2, \ldots, n$. If some $|\lambda_k| = 1$ and are of multiplicity one, then $\mathcal{V}_{km} = 0$ for $m > 1$. Consequently, (26) is stable for unit eigenvalues of multiplicity one.

To show that these conditions are also necessary, suppose any one of the λ_i, $i = 1, 2, \ldots, n$, exceed unity in absolute value. Obviously the entries in the vector $\mathcal{V}_{ij} \lambda_i^n$, associated with the eigenvalue $|\lambda_i| > 1$, increase without bound as $n \to \infty$. If any eigenvalue $|\lambda_j| = 1$, $j = 1, 2, \ldots, n$, and is of multiplicity $p > 1$, then (26) contains a vector sum

$$\mathcal{V}_{j1} \lambda_j^n + \mathcal{V}_{j2} n \lambda_j^{n-1} + \ldots + \mathcal{V}_{jp} n(n-1) \ldots (n-p) \lambda_j^{n-p+1}$$

which obviously increases without bound as $n \to \infty$.

It is of interest to note that the property of stability in a linear system represented by a discrete-state model implies that the entries in the transition matrix Φ^n tend toward zero uniformly in n.

Conceptually the stability and response problems are solved once the eigenvalues and the constituent matrices of the transition matrix are obtained. The problem of obtaining the constituent matrices and the eigenvalues will not be considered here. The reader is referred to the appropriate references in the bibliography[11, 12].

IV THE DESIGN OF DIGITALLY CONTROLLED SAMPLED-DATA SYSTEMS IN TERMS OF DISCRETE-STATE MODELS

The basic problem encountered in the design of systems such as that shown in Fig. I is the realization of a linear recursion formula for the digital controller in the form of equation (2). Whether or not this can be accomplished depends entirely upon the characteristics of the plant and the desired mode of response. Ragazzini and Franklin[9], and others[5, 14] have established design procedures for systems of this type using z-transform techniques. In these procedures, it is standard practice to take the initial conditions as zero and base the design on the response characteristics of standard input signals such as step-functions, ramp functions etc. In a design procedure based on state models, however, it is more convenient to consider the response of the system to an arbitrary set of initial conditions; i.e., an arbitrary state vector. In general, problems of inter-sampling ripple, and plant sensitivity are more tractable in the time domain than in the z-domain. For this and other reasons, a design based on state models is believed to give new insight to the design problem.

The control system design problem is primarily that of establishing a control sequence for the digital controller in the form (2) such that a system in some arbitrary initial state can be brought to equilibrium in

the shortest possible time without exceeding acceptable signal levels anywhere in the system.

Consider first the conditions under which an arbitrary system initial state vector can be reduced to the null vector. It is shown in the appendix that a "follow-up" system is reducible to this equivalent; i.e., the problem of reducing an initial state vector to a constant is equivalent to reducing an initial state vector to zero.

The design may be based on forcing an initial state vector to equilibrium along essentially three different modes:

1. Minimal-time response, in which the initial state vector is reduced to zero in the smallest number of sampling intervals N.
2. Non-minimal-time response, in which the initial state vector is reduced to zero in a finite number of sampling intervals $n > N$.
3. Asymptotic-time response, in which equilibrium is attained in the limit as $n \rightarrow \infty$.

A design procedure based on each of these three modes of response are considered in the order given.

A. The Design for Minimal-Time Response

The minimal-time response design has been considered by several investigators[1,4,5]. Two cases are of interest:

1. All of the system state variables are measurable.
2. Some of the state variables are not measurable.

Kalman and Bertram[4], and others[5], have given a solution to the problem for case 1. The more general problem stated as case 2 is considered here, i.e., given a plant in a feedback control configuration as in Fig. I, determine the conditions under which a digital controller recursion formula can be obtained which will force a plant in any arbitrary initial state to equilibrium in the minimum possible number of sampling intervals. It is assumed in the development which follows that none of the plant state variables is directly measurable.

Let the plant model be written as

$$\begin{aligned} \mathcal{X}_2(n) &= \mathcal{A}_2 \, \mathcal{X}_2(n-1) + \mathcal{B}_2 \, s(n) \\ c(n) &= \mathcal{D}_2 \, \mathcal{X}_2(n) \end{aligned} \qquad (27)$$

where s(n) is the control signal from the computer as derived from the sampled error, $\mathcal{X}_2(n)$ is the plant state vector, c(n) is the plant output, and $\mathcal{X}_2(0)$ is an arbitrary initial state vector.

Recursive application of equation (27) gives

$$\mathcal{X}_2(1) = \mathcal{A}_2 \mathcal{X}_2(0) + \mathcal{B}_2 s(1)$$

$$\mathcal{X}_2(2) = \mathcal{A}_2^2 \mathcal{X}_2(0) + \mathcal{A}_2 \mathcal{B}_2 s(1) + \mathcal{B}_2 s(2) \tag{28}$$

$$\mathcal{X}_2(3) = \mathcal{A}_2^3 \mathcal{X}_2(0) + \mathcal{A}_2^2 \mathcal{B}_2 s(1) + \mathcal{A}_2 \mathcal{B}_2 s(2) + \mathcal{B}_2 s(3)$$

$$\vdots$$

$$\mathcal{X}_2(k) = \mathcal{A}_2^k \mathcal{X}_2(0) + \mathcal{A}_2^{k-1} \mathcal{B}_2 s(1) + \ldots + \mathcal{B}_2 s(k)$$

Assuming that the control signals s(1), s(2), s(3), . . . s(k) are known for the present, premultiply both sides of the equations in (28) by $\mathcal{D}_2$ to obtain

$$c(1) = \mathcal{D}_2 \mathcal{A}_2 \mathcal{X}_2(0) + \mathcal{D}_2 \mathcal{B}_2 s(1)$$

$$c(2) = \mathcal{D}_2 \mathcal{A}_2^2 \mathcal{X}_2(0) + \mathcal{D}_2 \mathcal{A}_2 \mathcal{B}_2 s(1) + \mathcal{D}_2 \mathcal{B}_2 s(2)$$

$$\vdots \tag{29}$$

$$c(k) = \mathcal{D}_2 \mathcal{A}_2^k \mathcal{X}_2(0) + \mathcal{D}_2 \mathcal{A}_2^{k-1} \mathcal{B}_2 s(1) + \ldots + \mathcal{D}_2 \mathcal{B}_2 s(k)$$

The outputs c(i), i = 1, 2, . . . , k, and the inputs s(i), i = 1, 2, . . . , k are assumed to be measurable. The k equations in (29) are linear in the k unknowns $x_{01}, x_{02}, \ldots, x_{0k}$ and can be written as

$$\begin{bmatrix} c(1) \\ c(2) \\ c(3) \\ \cdot \\ \cdot \\ \cdot \\ c(k) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1k} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2k} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3k} \\ & & & & \\ & & & & \\ & & & & \\ f_{k1} & f_{k2} & f_{k3} & \cdots & f_{kk} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \\ \cdot \\ \cdot \\ \cdot \\ x_{0k} \end{bmatrix} + \begin{bmatrix} g_{11} & \cdots & g_{1k} \\ g_{21} & \cdots & g_{2k} \\ g_{31} & \cdots & g_{3k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_{k1} & \cdots & g_{kk} \end{bmatrix} \begin{bmatrix} s(1) \\ s(2) \\ s(3) \\ \cdot \\ \cdot \\ \cdot \\ s(k) \end{bmatrix} \tag{30}$$

A unique solution for the unknown initial state variables can be obtained if and only if the coefficient matrix $[f_{ij}]_{kk}$ in (30) is nonsingular. Any system which is described by a set of equations such as (30), in which the coefficient matrix $[f_{ij}]$ is nonsingular, is said to be observable[1].

Assuming the system to be observable, let (30) be written as

$$\mathcal{C}(k) = \mathcal{F}\,\mathcal{X}_2(0) + \mathcal{G}\,\mathcal{S}(k) \tag{31}$$

The solution for $\mathcal{X}_2(0)$ is of the form

$$\mathcal{X}_2(0) = \mathcal{F}^{-1}\,\mathcal{C}(k) - \mathcal{F}^{-1}\mathcal{G}\,\mathcal{S}(k) \tag{32}$$

and the system state vector $\mathcal{X}_2(k)$ is

$$\mathcal{X}_2(k) = \mathcal{A}_2^k\,\mathcal{X}_2(0) + \mathcal{A}_2^{k-1}\,\mathcal{B}_2\,s(1) + \ldots + \mathcal{A}_2\,\mathcal{B}_2\,s(n-1) + \mathcal{B}_2\,s(k)$$

Substituting (32) into the above equation yields

$$\mathcal{X}_2(k) = \mathcal{A}_2^k \mathcal{F}^{-1} \mathcal{C}(k) - \mathcal{A}_2^k \mathcal{F}^{-1} \mathcal{G} \mathcal{S}(k) + \left[\mathcal{A}_2^{k-1} \mathcal{B}_2, \ldots \mathcal{A}_2 \mathcal{B}_2, \mathcal{B}_2 \right] \mathcal{S}(k)$$

Letting

$$\mathcal{A}_2^k \mathcal{F}^{-1} \mathcal{C}(k) = \mathcal{P}$$

$$- \mathcal{A}_2^k \mathcal{F}^{-1} \mathcal{G} = \mathcal{Q}_1$$

$$\left[\mathcal{A}_2^{k-1} \mathcal{B}_2, \mathcal{A}_2^{k-2} \mathcal{B}_2, \ldots, \mathcal{A}_2 \mathcal{B}_2, \mathcal{B}_2 \right] = \mathcal{Q}_2$$

then

$$\mathcal{X}_2(k) = \mathcal{P} \mathcal{C}(k) + \left[\mathcal{Q}_1 + \mathcal{Q}_2 \right] \mathcal{S}(k)$$

$$= \mathcal{P} \mathcal{C}(k) + \mathcal{Q} \mathcal{S}(k)$$

$$= \mathcal{P} \begin{bmatrix} c(1) \\ c(2) \\ \cdot \\ \cdot \\ \cdot \\ c(k) \end{bmatrix} + \mathcal{Q} \begin{bmatrix} s(1) \\ s(2) \\ \cdot \\ \cdot \\ \cdot \\ s(k) \end{bmatrix} \qquad (33)$$

where $\mathcal{P}$ and $\mathcal{Q}$ are matrices of order k.

After k sampling periods, all components of the state vector at $t = nT$, $n \geq k$ are known, that is,

$$\mathcal{X}_2(n) = \mathcal{P} \begin{bmatrix} c(n-k) \\ c(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ c(n) \end{bmatrix} + Q \begin{bmatrix} s(n-k) \\ s(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ s(n) \end{bmatrix} \tag{34}$$

If the system is controllable, then from the discussion on page 101 of the appendix

$$s(n+1) = \begin{bmatrix} a_{11}, & a_{12}, \ldots, & a_{1k} \end{bmatrix} \begin{bmatrix} x_{n1} \\ x_{n2} \\ \cdot \\ \cdot \\ \cdot \\ x_{nk} \end{bmatrix} = \alpha \mathcal{X}_2(n) \tag{35}$$

where the a_{1j}, $j = 1, 2, \ldots k$, are real constants.

Substituting (34) into (35) gives

$$s(n+1) = \alpha \mathcal{P} \begin{bmatrix} c(n-k) \\ c(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ c(n) \end{bmatrix} + \alpha Q \begin{bmatrix} s(n-k) \\ s(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ s(n) \end{bmatrix}$$

or

$$s(n+1) = \mathcal{V} \begin{bmatrix} c(n-k) \\ c(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ c(n) \end{bmatrix} + \mathcal{W} \begin{bmatrix} s(n-k) \\ s(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ s(n) \end{bmatrix} \qquad (36)$$

where $\mathcal{V} = \alpha \mathcal{P}$, and $\mathcal{W} = \alpha Q$ are row vectors.

If the matrix multiplication indicated in (36) is carried out, there results

$$\begin{aligned} s(n+1) + w_1 s(n) + w_2 s(n-1) + \ldots + w_{k+1} s(n-k) \\ = v_1 c(n) + v_2 c(n-1) + \ldots + v_{k+1} c(n-k) \end{aligned} \qquad (37)$$

where the v_i, $i = 1, 2, \ldots, k+1$, and the w_j, $j = 1, 2, \ldots, k+1$, are real constants.

Since (37) is of the form (2), the derivation is complete. Equation (37) shows that, for any observable and controllable plant, the control sequence can be obtained as a linear combination of the measurable plant outputs c(i), i = (n-k), . . . , n. This control sequence will drive the system from any arbitrary initial state to equilibrium in a minimum number of sampling intervals. To show that this is actually a minimum, it is only necessary to note that, after the system state has been identified in k samples, it takes k intervals to reduce the system to equilibrium. It follows then, if none of the state variables is directly measurable, that a maximum of 2k sampling intervals are required to force the system to rest.

Suppose now that the matrix $\mathcal{F}$ in (30) is singular and of rank $r < k$. In this case, (30) can be rearranged, if necessary, and written as

$$\begin{bmatrix} \mathcal{C}_r \\ \hline \mathcal{C}_{k-r} \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \hline \mathcal{F}_{21} & \mathcal{F}_{22} \end{array}\right] \begin{bmatrix} \mathcal{X}_r(0) \\ \hline \mathcal{X}_{k-r}(0) \end{bmatrix} + \begin{bmatrix} g_{11} & \cdots & g_{1k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_{k1} & \cdots & g_{kk} \end{bmatrix} \begin{bmatrix} s(1) \\ s(2) \\ \cdot \\ \cdot \\ \cdot \\ s(n) \end{bmatrix}$$

where $\mathcal{F}_{11}$ is a nonsingular r x r submatrix. If the initial state variables $\mathcal{X}_{k-r}(0)$ can be measured directly, or approximated, then a solution for the remaining r variables in $\mathcal{X}_r(0)$ can be obtained. The system state can then be identified in r samples, where $r < k$. If the system is controllable, a recursion formula can again be obtained as above. Such a system may be described as being semi-observable.

If the variables $\mathcal{X}_{k-r}(0)$ cannot be measured directly or approximated, then the system is said to be unobservable and cannot be controlled. It follows then that a system must be controllable and at least semi-observable if a minimal-time response design is to be achieved.

Realization of a linear recursion formula such as (37) does not, however, guarantee that the controller will be a practical one. The derived control signals may overdrive the plant or they may be beyond the capabilities of the computer. If such is the case, one must turn to a different type of design such as that discussed in the next sections.

The following examples illustrate the design procedure in realizing a minimal-time response design.

Example 1:

The plant G shown in Fig. I is to be controlled by a digital controller D and a zero-order hold H. Design a controller which will force the system from an arbitrary initial state $X_{(0)}$ to equilibrium in the minimum number of sampling intervals if the plant transfer function is given as

$$G(s) = \frac{1}{s(s+1)}$$

The discrete-state model of the plant as obtained from the transfer function[10] is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T-1+e^{-T} \\ 1-e^{-T} \end{bmatrix} s(k)$$

If the sampling interval T = 1.0 second, the plant state model is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.632 \\ 0 & 0.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} s(k) \tag{38}$$

Obviously the plant is controllable since the transition matrix is non-singular.

The first variable $x_1(k+1)$ in the state vector is directly measurable since it represents the output. The variable $x_2(k+1)$ is not, however, directly measurable. The first sampling period gives

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 & 0.632 \\ 0 & 0.368 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} s(0)$$

and from a solution to the first equation

$$x_2(0) = 1.58 \left[x_1(1) - x_1(0) - 0.368\, s(0) \right]$$

Since the state of the system is known after the first sampling, $\mathcal{X}(k)$ is determined for $k \geq 1$. The requirement is that $\mathcal{X}(k+2) = 0$ for <u>all</u> $k \geq 1$. From (27)

$$\mathcal{X}(k+2) = 0 = \mathcal{A}^2 \mathcal{X}(k) + \mathcal{A}\mathcal{B}\, s(k) + \mathcal{B}\, s(k+1) \qquad k \geq 1$$

so

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.863 \\ 0 & 0.135 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.768 \\ 0.231 \end{bmatrix} s(k) + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} s(k+1)$$

Solving for s(k)

$$-s(k) = 1.58\, x_1\,(k) + 1.24\, x_2\,(k) \tag{39}$$

The variable $x_2(k)$ is not measurable and may be eliminated from (39) by using (38). Solving for $x_2(k)$ in (38) yields

$$x_2(k) = 0.583 \left[x_1(k) - x_1(k+1) \right] + 0.418\, s(k-1)$$

Substituting this into (39), there results

$$-s(k) - 0.519\, s(k-1) = 2.303\, x_1(k) - 0.732\, x_1(k-1) \tag{40}$$

Equation (40) is the desired recursion formula of the digital controller. This controller will force the plant G in an arbitrary initial state $\mathcal{X}_{(0)}$ to equilibrium in 3 sampling intervals.

It is of interest to consider the design based on the requirement that the controller will force the system to follow a unit step input with zero error after three sampling intervals.

Since $x_1(k)$ is the output variable and must follow the step input, the state vector $\mathcal{X}(k+2)$, for $k \geq 1$, must be

$$\mathcal{X}(k+2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathcal{A}^2 \mathcal{X}(k) + \mathcal{A}\mathcal{B}\, s(k) + \mathcal{B}\, s(k+1)$$

or from (38)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.863 \\ 0 & 0.136 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.768 \\ 0.231 \end{bmatrix} s(k) + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} s(k+1)$$

Solving the above, there results

$$-s(k) - 0.52\, s(k-1) = 2.303\, x_1(k) - 0.723\, x_1(k-1) - 1.58 \qquad (41)$$

From the feedback connection

$$x_1(k) = 1 - e(k) \qquad\qquad x_1(k-1) = 1 - e(k-1) \qquad (42)$$

Substituting (42) into (41) establishes the computer recursion formula

$$s(k) + 0.52\, s(k-1) = -\, 2.303\, e(k) + 0.723\, e(k-1)$$

Note that the results of a design based on the response to a step function and the state vector are identical for the above example.

The procedure given above for realizing a minimal-time response design with no constraints on the driving functions can be summarized as follows:

1. Test the system to be controlled in order to determine if it is both controllable and observable.

2. Having satisfied condition 1, allow enough sampling intervals to determine the initial state variables which are not directly measurable. This essentially allows the unknown forcing functions to be expressed as a linear combination of the measurable output or the sampled error.

3. From the relationships established in step 2, solve for the required linear recursion formula.

If step 1 in the above procedure shows that the plant is not controllable and/or not observable in any sense, then the minimal-time response design cannot be realized by this procedure. If for any reason the minimal design cannot be realized (e. g. the plant is not controllable), an asymptotic time response design may be used as discussed in Section C.

B. Design for Non-Minimal-Time Response

In non-minimal-time response systems, the state vector is forced to equilibrium in a finite number of sampling intervals n, where n is

greater than the minimum as determined by the procedures of the previous section. To establish under what conditions this type of design can be implemented, let the plant equations be written as

$$\mathcal{X}_2(n) = \mathcal{A}_2 \, \mathcal{X}_2(n-1) + \mathcal{B}_2 \, s(n-1) \tag{43}$$

or

$$\mathcal{X}_2(n) = \mathcal{A}_2^n \, \mathcal{X}_2(0) + \left[\mathcal{A}_2^{n-1} \mathcal{B}_2, \mathcal{A}_2^{n-2} \mathcal{B}_2, \ldots, \mathcal{B}_2\right] \begin{bmatrix} s(0) \\ s(1) \\ s(2) \\ \cdot \\ \cdot \\ \cdot \\ s(n-1) \end{bmatrix} \tag{44}$$

Assume that the system state vector has k elements, $k < n$, and that the system is both controllable and observable*. This amounts to taking more recursive steps in (43) than is necessary for minimal-time response. Partition (44) as

$$\mathcal{X}_2(n) = \mathcal{A}_2^n \, \mathcal{X}_2(0) + \left[\mathcal{H}_{11} \,\vdots\, \mathcal{H}_{12}\right] \begin{bmatrix} \mathcal{S}_a \\ --- \\ \mathcal{S}_m \end{bmatrix} \tag{45}$$

*It can be shown that the following development holds also for semi-observable systems.

where

$$\mathcal{H}_{11} = \left[\mathcal{A}_2^{n-1}\mathcal{B}_2,\ \mathcal{A}_2^{n-2}\mathcal{B}_2,\ \ldots,\ \mathcal{A}_2^{k}\mathcal{B}_2\right]$$

$$\mathcal{H}_{12} = \left[\mathcal{A}_2^{k-1}\mathcal{B}_2,\ \mathcal{A}_2^{k-2}\mathcal{B}_2,\ \ldots,\ \mathcal{B}_2\right]$$

and

$$\mathcal{S}_a = \begin{bmatrix} s(0) \\ s(1) \\ \cdot \\ \cdot \\ \cdot \\ s(n-k-1) \end{bmatrix} \quad \mathcal{S}_m = \begin{bmatrix} s(n-k) \\ s(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ s(n-1) \end{bmatrix}$$

If it is required that $\mathcal{X}_2(n) = 0$ (equilibrium), then from (45)

$$0 = \mathcal{A}_2^{n}\mathcal{X}_2(0) + \mathcal{H}_{11}\mathcal{S}_a + \mathcal{H}_{12}\mathcal{S}_m$$

Since the system is assumed to be controllable, $\mathcal{H}_{12}$ is nonsingular and it follows that

$$\mathcal{S}_m = -\mathcal{H}_{12}^{-1}\mathcal{A}_2^{n}\mathcal{X}_2(0) - \mathcal{H}_{12}^{-1}\mathcal{H}_{11}\mathcal{S}_a \qquad (46)$$

Equation (46) is a system of k linear equations in n unknowns, where $n > k$. Given any $\mathcal{X}_2(0)$, this system has a unique solution for $\mathcal{S}_m$ if the first (n-k) forcing functions $\mathcal{S}_a$ are specified. Obviously, there exists an infinity of non-trivial solutions for (46) since $\mathcal{S}_a$ may be specified arbitrarily. This additional freedom can be exploited in the design by selecting $\mathcal{S}_a$ and $\mathcal{S}_m$ in some optimal sense. One very practical optimizing scheme that can be used to specify $\mathcal{S}_a$ is to require that the squared norm of $\mathcal{S}(n-1)$ (i.e. $\mathcal{S}'\mathcal{S}$) shall be a minimum subject to the constraint that the plant output c(n) assumes the desired value $c^d(n)$. This can be accomplished by first premultiplying (45) by $\mathcal{D}_2$ to give

$$c(n) = \mathcal{D}_2 \mathcal{X}_2(n) = \mathcal{D}_2 \mathcal{A}_2^n \mathcal{X}_2(0) + \mathcal{D}_2 \mathcal{H}_{11} \mathcal{S}_a + \mathcal{D}_2 \mathcal{H}_{12} \mathcal{S}_m$$

or

$$c(n) = \mathcal{D}_2 \mathcal{A}_2^n \mathcal{X}_2(0) + \mathcal{D}_2 \mathcal{H} \mathcal{S}$$

Let the desired output at time $t = nT$ be specified as $c^d(n)$ and define the miss distance as

$$M = c^d(n) - \mathcal{D}_2 \mathcal{A}_2^n \mathcal{X}_2(0) - \mathcal{D}_2 \mathcal{H} \mathcal{S}$$

The variational method of LaGrange[15] can be applied to find the input sequence $s(i)$, $i = 0, 1, 2, \ldots, (n-1)$, such that the squared norm of the driving vector is minimized subject to the constraint that the miss distance is zero. This variational problem can be formulated as follows.

Let

$$e_n = c^d(n) - \mathcal{S}_2 \mathcal{A}_2^n \mathcal{X}_2(0)$$

and

$$u = \left[(\mathcal{S}'\mathcal{S}) + \lambda (e_n - \mathcal{S}_2 \mathcal{H} \mathcal{S}) \right]$$

where the λ is the LaGrange multiplier. The function u must now be minimized with respect to the elements of $\mathcal{S}$.

$$\min u = \min_{\mathcal{S}} \left[(\mathcal{S}'\mathcal{S}) + \lambda (e_n - \mathcal{S}_2 \mathcal{H} \mathcal{S}) \right]$$

Taking the partial derivatives with respect to the n variables in there results the n equations

$$\frac{\partial u}{\partial s_1} = 2 s_1 - \lambda p_{11}$$

$$\frac{\partial u}{\partial s_2} = 2 s_2 - \lambda p_{12}$$

$$\vdots$$

$$\frac{\partial u}{\partial s_n} = 2 s_n - \lambda p_{1n}$$

where the p_{1i}, i=1, 2, . . . , n, are constants.

The above n equations can be written as

$$\frac{\partial u}{\partial \mathcal{S}} = 2\,\mathcal{S} - \lambda\,\mathcal{P} \tag{47}$$

subject to the constraint

$$e_n - \mathcal{D}_2\,\mathcal{H}\,\mathcal{S} = 0 \tag{48}$$

If $\frac{\partial u}{\partial \mathcal{S}}$ is set equal to zero, then (47) gives

$$\mathcal{S} = \frac{\lambda}{2}\,\mathcal{P} \tag{49}$$

Substituting (49) into (48) gives

$$e_n - \mathcal{D}_2\,\mathcal{H}\,\frac{\lambda}{2}\,\mathcal{P} = 0$$

Assuming the scalar $\mathcal{D}_2\,\mathcal{H}\,\mathcal{P}$ is not identically zero

$$\lambda = 2\,e_n(\mathcal{D}_2\,\mathcal{H}\,\mathcal{P})^{-1} \tag{50}$$

Substituting (50) into (49), the vector of driving functions resulting from this particular optimizing process is

$$\mathcal{S} = e_n(\mathcal{D}_2\,\mathcal{H}\,\mathcal{P})^{-1}$$

<u>Example 1:</u>

Consider the plant represented by the discrete-state model

$$\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(n-1) \\ x_2(n-1) \end{bmatrix} + \begin{bmatrix} 0.693 \\ 0.5 \end{bmatrix} s(n-1)$$

Given that the initial state vector at $t = 0$ is

$$\chi(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

find the sequence of control functions that will take this second-order system to equilibrium in four sampling intervals in such a way that the square of the Euclidean norm of the input vector is minimized.

Let u be defined as

$$u = S'S + \lambda^T (e_N^d - H_N S)$$

where

$$e_N^d = \chi(N) - \alpha^N \chi(0) \quad \text{and} \quad S = \begin{bmatrix} s(0) \\ s(1) \\ s(2) \\ s(3) \end{bmatrix}$$

It is necessary to evaluate

$$\min_S u = \min_S \left\{ S'S + \lambda^T (e_N^d - H_N S) \right\} \tag{51}$$

subject to the constraint that

$$e_N^d - H_N S = 0 \tag{52}$$

Differentiating (51) with respect to S and setting equal to zero gives

$$0 = \frac{\partial}{\partial S} \left\{ S'S + \lambda^T (e_N^d - H_N S) \right\} \tag{53}$$

$$0 = 2S - H_N'$$

For N = 4, (52) and (53) give

$$\lambda = (H_4 H_4')^{-1} \; 2 e_4^d$$

Substituting λ into (53) gives

$$S = H_4' (H_4 H_4')^{-1} \; e_4^d \tag{54}$$

where for four sampling periods

$$H = \begin{bmatrix} 1.13 & 1.068 & 0.943 & 0.693 \\ 0.0625 & 0.125 & 0.25 & 0.5 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 1.13 & 0.0625 \\ 1.068 & 0.125 \\ 0.943 & 0.25 \\ 0.693 & 0.5 \end{bmatrix} \begin{bmatrix} 0.52 & -1.23 \\ -1.23 & 5.9 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 5.03 \\ 4.01 \\ 1.82 \\ -2.45 \end{bmatrix}$$

The above procedure assures that the norm of the driving vector will be a minimum for the sampling interval chosen and certainly helps to keep the input signals from overdriving the plant. It does not, however, assure that any single entry in $\mathcal{S}$ (n-1) will not overdrive the plant.

The individual entries in $\mathcal{S}$ (n-1) can be maintained below a safe, predetermined level <u>using the above technique</u> by using a multi-norm constraint on the driving vector. A discussion of this procedure is deferred until Chapter V.

If the input levels on the plant are firmly established, the alternate procedure given next can be used as an aid in selecting the driving functions such that none of the s(i), i=0, 1, . . . , n-1, will overdrive the plant. Let (46) be written as

$$\mathcal{S}_m = \mathcal{Y} + \mathcal{H}\mathcal{S}_a \tag{55}$$

where

$$\mathcal{Y} = -\mathcal{H}_{12}^{-1}\, \mathcal{A}_2^n\, \mathcal{X}_2(0)$$

and

$$\mathcal{H} = - \mathcal{H}_{12}^{-1} \mathcal{H}_{11}$$

The detailed form of (55) is

$$\begin{bmatrix} s(n-k) \\ s(n-k+1) \\ \cdot \\ \cdot \\ \cdot \\ s(n-1) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} & \cdots & h_1(n-k) \\ h_{21} & h_{22} & \cdots & h_2(n-k) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ h_{k1} & & \cdots & h_k(n-k) \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ \cdot \\ \cdot \\ \cdot \\ s(n-k-1) \end{bmatrix} \tag{56}$$

where the y_i, $i=1, 2, \ldots k$, and the k_{ij}, $i=1, 2, \ldots k$, $j=1, 2, \ldots, n-k)$, are real constants.

If the saturation requirements on the plant dictate that all $s(i)$, $i=0, 1, \ldots (n-1)$ shall not exceed $\pm k$, where k is a real constant, then the following set of inequalities must be satisfied.

$$\begin{array}{llll} |s(0)| \leq k, & |s(n-k)| & = & |y_1 + h_{11}\, s(0) + \ldots + h_1(n-k)\, s(n-k-1)| \leq k \\ |s(1)| \leq k, & |s(n-k+1)| & = & |y_2 + h_{21}\, s(0) + \ldots + h_2(n-k)\, s(n-k-1)| \leq k \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ |s(n-k-1)| \leq k, & |s(n-1)| & = & |y_k + h_{k1}\, s(0) + \ldots + h_k(n-k)\, s(n-k-1)| \leq k \end{array} \tag{57}$$

A sufficient condition for the existence of a solution to the inequalities in (57) can be obtained as follows.

Let $|y_i| \leq k$; this condition assures that the origin is included in the requirement space. If the equalities are taken in (57), the system of equations defines a set of hyperplanes bounding all possible solutions of (57). Consequently, when $|y_i| \leq k$, then there exists the particular solution

$$s(0) = s(1) = s(2) = \ldots = s(n-k-1) = s$$

and (57) becomes

$$\begin{array}{ccccc}
-k & \leq & y_1 + h_{11}\, s + h_{12}\, s + \ldots + h_1(n-k)\, s & \leq & k \\
-k & \leq & y_2 + h_{21}\, s + h_{22}\, s + \ldots + h_2(n-k)\, s & \leq & k \\
\vdots & & \vdots & & \vdots \\
-k & \leq & y_k + h_{k1}\, s + h_{k2}\, s + \ldots + h_k(n-k)\, s & \leq & k
\end{array}$$

or

$$\begin{array}{lcll}
-k - y_1 & \leq & s \sum_{i=1}^{n-k} h_{1i} & \leq k - y_1 \\
-k - y_2 & \leq & s \sum_{i=1}^{n-k} h_{2i} & \leq k - y_2 \\
\vdots & & \vdots & \vdots \\
-k - y_k & \leq & s \sum_{i-1}^{n-k} h_{ki} & \leq k - y_k
\end{array} \tag{58}$$

The set of normals to the hyperplanes bounding the solutions to (57) are

$$\frac{k - y_i}{\left[h_{i1}^2 + h_{i2}^2 + \dots + h_{i(n-k)}\right]^{\frac{1}{2}}} \qquad i = 1, 2, \dots, (n-k-1)$$

$$\frac{k + y_i}{\left[h_{i1}^2 + h_{i2}^2 + \dots + h_{i(n-k)}\right]^{\frac{1}{2}}} \qquad i = 1, 2, \dots, (n-k) \tag{59}$$

Let

$$r = \min_{i,j} \left[\frac{k - y_i}{\|h_{ij}\|} , \quad \frac{k + y_i}{\|h_{ij}\|} , \quad k \right] \qquad \begin{array}{l} i=0, 1, 2, \dots (n-k-1) \\ j=1, 2, \dots, (n-k) \end{array} \tag{60}$$

Equation (60) describes a hypersphere R, of radius $r > 0$, centered at the origin in the requirement space, which contains solutions to (57). It follows that there exists a set of s(i), i=0, 1, 2, . . . , n-1, which do not exceed the saturation level of the plant if only $|y_i| < k$.

Obviously, this procedure does not isolate all solutions to (57) but serves to determine if there exists any inside R. If the requirement is only to obtain a set of signals which do not overdrive the plant, any set of values inside of R will do.

It remains to inquire whether or not the requirement that $|y_i| < k$ is a realistic one. The vector $\mathcal{Y}$ is given by

$$\mathcal{Y} = - \mathcal{H}_{12}^{-1} \, \mathcal{A}_2^n \, \mathcal{X}_2(0)$$

The matrix $\mathcal{A}_2^n$ has entries which tend toward zero monotonically* as n increases (see Theorem II) while the entries in $-\mathcal{H}_{12}^{-1}$ and $\mathcal{X}_2(0)$ are constants; therefore, the entries in $\mathcal{Y}$ decrease if n is allowed to take on larger and larger values. The number of sampling intervals n can be selected arbitrarily by the designer which assures that for some n, the condition $|y_i| < k$ can be met.

If finite-response time is given in the specification of the design, then assuming the plant can be controlled, the response time can be made minimal or non-minimal as the designer so chooses. If minimal-response time is selected as a basis of design, there can be no adjustment of the input signals to conform to saturation requirements since the minimal design is unique. Only in non-minimal-response design is it possible to control the levels of the signals. The number of sampling intervals used to establish equilibrium is strictly a function of the particular control problem at hand. The following example illustrates how

*Providing $\mathcal{A}_2$ has no unit eigenvalues.

this procedure may be applied.

Example 2:

Consider the plant represented by the discrete-state model

$$\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.433 \\ 0 & 0.367 \end{bmatrix} \begin{bmatrix} x_1(n-1) \\ x_2(n-1) \end{bmatrix} + \begin{bmatrix} 0.567 \\ 0.433 \end{bmatrix} s(n-1)$$

Given that the initial state vector is

$$\chi(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

find the sequence of control functions that will take this system to equilibrium in such a manner that none of the control signals exceed $\pm$ 3.

Equation (55) for N = 4 is

$$\begin{bmatrix} s(2) \\ s(3) \end{bmatrix} = \begin{bmatrix} 1.9 \\ 0.7 \end{bmatrix} + \begin{bmatrix} -1.07 & -1.17 \\ 0.34 & 0.29 \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \end{bmatrix}$$

Since $|y_i| < 3$, s(0) and s(1) may be selected using the equation in (59).

Obviously, the minimum value of s will be generated by the form

$$\frac{3 - 1.9}{\sqrt{(1.07)^2 + (1.17)^2}} = 0.69$$

Letting s(0) = s(1) = s = 0.69, the solution for s(2) and s(3) is

$$\begin{bmatrix} s(2) \\ s(3) \end{bmatrix} = \begin{bmatrix} 1.9 \\ 0.7 \end{bmatrix} + \begin{bmatrix} -1.07 & -1.17 \\ 0.34 & 0.29 \end{bmatrix} \begin{bmatrix} 0.69 \\ 0.69 \end{bmatrix}$$

$$= \begin{bmatrix} 0.35 \\ 1.13 \end{bmatrix}$$

The above solution shows that none of the control functions exceed $\pm$ 3 as required.

C. <u>Asymptotic-Time Response</u>

For a number of reasons, a finite response may be impossible to obtain. Among these reasons are:

1. The plant may not be controllable.

2. The plant may not be observable in any sense.

In order to treat those systems which fall under the above classification, an extended concept of plant controllability is discussed which does not depend on the existence of the inverse of the matrix $\mathcal{H}$ as

outlined in the previous section. A design realized on the basis of the extended concept is not, however, a unique design.

The Stability Theorem II assures that the system to be controlled will respond to any bounded driving function in a stable mode as long as the eigenvalues of the transition matrix are less than unity in absolute value. Further, it is noted that the response is asymptotic if the eigenvalues are positive and oscillatory if the eigenvalues are negative real or complex. The mode of response in either case is asymptotic; that is, the equilibrium point in the state-space is attained in the limit as $t \rightarrow \infty$. To achieve a particular type of response, the eigenvalues of the system transition matrix must be specified. There appears to be no optimal way of selecting these eigenvalues other than by experience. Since this is the normal state of affairs in s-domain or z-domain design, nothing is gained or lost on this point by using state models as a basis of design.

Let it be assumed that the desired eigenvalues of the system transition matrix are given and that the discrete-state transition matrix of the plant is known. The problem is to select the digital controller constants so that the system transition matrix has the desired eigenvalues and constituent matrices. The general form of the system transition matrix Φ as developed in Chapter II is

$$\Phi = \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ -\mathcal{B}_1 \mathcal{D}_2 \mathcal{A}_2 & -\mathcal{B}_1 \mathcal{D}_2 \mathcal{B}_2 \mathcal{D}_1 + \mathcal{A}_1 \end{bmatrix}$$

for which the characteristic polynomial is

$$\det[\Phi - \lambda \mathcal{U}] = \det \begin{bmatrix} \mathcal{A}_2 - & \mathcal{B}_2 \mathcal{D}_1 \\ -\mathcal{B}_1 \mathcal{D}_2 \mathcal{A}_2 & -\mathcal{B}_1 \mathcal{D}_2 \mathcal{B}_2 \mathcal{D}_1 + \mathcal{A}_1 - \lambda \mathcal{U} \end{bmatrix} = 0 \tag{61}$$

Elementary row operations[11] on the determinant in (61) yields

$$\det[\Phi - \lambda \mathcal{U}] = \det \begin{bmatrix} \mathcal{A}_2 - \lambda \mathcal{U} & \mathcal{B}_2 \mathcal{D}_1 \\ -\lambda \mathcal{B}_1 \mathcal{D}_2 & \mathcal{A}_1 - \lambda \mathcal{U} \end{bmatrix} \tag{62}$$

which in detail is

$$\det \left[\begin{array}{cccccc|cccc} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1k} & & b_{11} & b_{11}a_1 & \cdots & b_{11}a_k \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk}-\lambda & & b_{k1}a_0 & b_{k1}a_1 & \cdots & b_{k1}a_k \\ \hline -\lambda d_{11} & -\lambda d_{12} & \cdots & \cdots & -\lambda d_{1k} & & -b_1-\lambda & -b_2 & \cdots & -b_k \\ 0 & 0 & 0 & \cdots & 0 & & 1 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & & 0 & \cdots & \cdots & 1-\lambda \end{array}\right] \tag{63}$$

To relate the eigenvalues of the system to the computer parameters, expand (62) by Laplace's expansion in terms of the submatrix $[\mathcal{A}_2 - \lambda \mathcal{U}]$.

For a sixth order system, the expansion is

$$D(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} \cdot (-1)^{S_{ij}} \begin{vmatrix} -b_1-\lambda & -b_2 & -b_3 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}-\lambda & a_{12} & b_{11} \\ a_{21} & a_{22}-\lambda & b_{21} \\ a_{31} & a_{32} & b_{31} \end{vmatrix} \cdot a_0(-1)^{S_{ij}} \begin{vmatrix} -\lambda a_3 & -b_2 & -b_3 \\ 0 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}-\lambda & a_{12} & b_{11} \\ a_{21} & a_{22}-\lambda & b_{21} \\ a_{31} & a_{32} & b_{31} \end{vmatrix} \cdot a_1(-1)^{S_{ij}} \begin{vmatrix} -\lambda a_3 & -b_1-\lambda & -b_3 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} a_{11}-\lambda & a_{13} & b_{11} \\ a_{21} & a_{23} & b_{21} \\ a_{31} & a_{33}-\lambda & b_{31} \end{vmatrix} \cdot a_2(-1)^{S_{ij}} \begin{vmatrix} -\lambda a_{12} & -b_1-\lambda & -b_2 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix} \tag{64}$$

Since the submatrix in the 1, 2 position in (62) is of rank one, any combination of one of the first three columns with any two of the last three columns yields a minor of zero value and $D(\lambda)$ reduces to

$$D(\lambda) = \left|\mathcal{a}_2 - \lambda\mathcal{U}\right|\left|\mathcal{a}_1 - \lambda\mathcal{U}\right| + \left[K_1\lambda^2 + K_2\lambda + K_3\right] c_{13}\left[a_0\lambda^3 + a_1\lambda^2 + a_2\lambda\right]$$
$$+ \left[K_4\lambda^2 + K_5\lambda + K_6\right] c_{12}\left[a_0\lambda^3 + a_1\lambda^2 + a_2\lambda\right]$$
$$+ \left[K_7\lambda^2 + K_8\lambda + K_9\right] c_{11}\left[a_0\lambda^3 + a_1\lambda^2 + a_2\lambda\right]$$

or

$$D(\lambda) = \left|\mathcal{a}_2 - \lambda\mathcal{U}\right|\left|\mathcal{a}_1 - \lambda\mathcal{U}\right| + \left[a_0\lambda^3 + a_1\lambda^2 + a_2\lambda\right]\left[K_1'\lambda^2 + K_2'\lambda + K_3'\right]$$

where the $K_1, K_2, \ldots, K_9, K_1', K_2', K_3$ are constants determined by the plant transition and output matrices.

Expanding the product of the first two terms,

$$D(\lambda) = \left[\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\right]\left[\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3\right]$$
$$+ \left[a_0\lambda^3 + a_1\lambda^2 + a_2\lambda\right]\left[K_1'\lambda^2 + K_2'\lambda + K_3'\right] \tag{65}$$

Equation (65) is the sum of two polynomials whose coefficients are functions of the computer parameters a_i and b_i. The first polynomial is of degree six and has coefficients which are linear combinations of the b_i. The second polynomial is of degree five and has coefficients which are linear combinations of the a_i. The sum of the two polynomials is monic and has coefficients which are linear combinations of the a_i and b_i.

Expanding the characteristic polynomial gives

$$D(\lambda) = \lambda^6 + (p_1+b_1+K_1'a_0)\,\lambda^5 + (p_2+p_1b_1 + b_2+K_1'a_1 + K_2'a_0)\,\lambda^4$$
$$+ (p_3+p_2b_1+p_1b_2+b_3+K_1'a_2 + K_2'a_1 + K_3'a_0)\,\lambda^3 \qquad (66)$$
$$+ (p_3b_1+p_2b_2+p_1b_3+K_2'a_2 + K_3'a_1)\,\lambda^2 + (p_3b_2+p_2b_3+K_3'a_2)\lambda$$
$$+ p_3b_3$$

Equation (66) is the desired form of the solution for the characteristic polynomial. Note that each coefficient can be controlled by the magnitude of the a_i and b_i in the computer transition matrix. This implies that, given sufficient freedom in the magnitude of the computer constants, it is possible to apply asymptotic time control to a plant which, for example, is not controllable in the sense of Kalman[1]. This is then an extended concept of controllability where the origin of the state-space, or the equilibrium point, is attained in the limit as $t \rightarrow \infty$. Note that nothing has been said about the stability of the uncontrolled plant. In fact, there is no reason why an unstable eigenvalue in the plant transition matrix cannot be digitally compensated. This property, however, is well known from z-transform control methods[9], but the attendant z-domain sensitivity problem associated with the cancellation of transfer function poles and zeros is avoided. The cancellation of transfer function poles and zeroes in frequency domain design procedures is usually considered as a fundamental problem[9]. The sensitivity problem, however, does not have a counterpart in discrete-state system design in terms of state models, i.e., concern over sensitivity in z-domain design has no foundation in fact*.

Suppose a set of eigenvalues have been specified for the sixth order system considered above. Let the desired characteristic polynomial for the system be written as

$$D(\lambda) = \lambda^6 + c_1 \lambda^5 + c_2 \lambda^4 + c_3 \lambda^3 + c_4 \lambda^2 + c_5 \lambda + c_6 \tag{67}$$

Equating (66) and (67), there results six independent equations in the six unknowns a_0, a_1, a_2, b_1, b_2, b_3, namely

$$\begin{bmatrix} 1 & 0 & 0 & K_1' & 0 & 0 \\ p_1 & 1 & 0 & K_2' & K_1' & 0 \\ p_2 & p_1 & 1 & K_3' & K_2' & K_1' \\ p_3 & p_2 & p_1 & 0 & K_3' & K_2' \\ 0 & p_3 & p_2 & 0 & 0 & K_3' \\ 0 & 0 & p_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} \tag{68}$$

*See reference 2 for a brief discussion on this point.

Equation (68) has a solution if the inverse of the coefficient matrix exists. Unfortunately, there is no convenient way to establish the correspondence between the entries in the plant transition matrix and the entries in (60); so very little can be said about the existence of the inverse.

This approach may be used in general for any order of system once the response has been specified. There is no way, however, to predict the magnitude of a_i and b_i resulting from a solution of (68). Consequently, there is no assurance that the resulting a_i and b_i will define a practical control scheme as discussed on page 48.

Omitting the tedious and somewhat lengthy matrix operations, the general form of the Laplace expansion of the system transition matrix for k elements in the plant state vector and k elements in the computer transition matrix is of the form

$$D(\lambda) = \left[\lambda^k + p_1\lambda^{k-1} + \ldots + p_k\right]\left[\lambda^k + b_1\lambda^{k-1} + \ldots + b_k\right]$$
$$+ \left[a_0\lambda^k + \ldots + a_{k-2}\lambda^2 + a_{k-1}\lambda\right]\left[K_1'\lambda^{k-1} + K_2'\lambda^{k-2} + \ldots + K_{k-1}'\lambda + K_k'\right] \quad (69)$$

The first term in the first product is the characteristic equation of the plant, the second term is the characteristic equation of the controller, and the second product will be called the control product*.

The following example illustrates how an asymptotic time response design is achieved.

Example 1:

Consider a plant which has the following discrete-state model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.693 \\ 0.5 \end{bmatrix} s(k)$$

*It will be noted that the stability of the open-loop system of Fig. III cannot be controlled by the digital computer. This is apparent when determinant $[\Phi - \lambda \mathcal{U}]$ is considered. For no feedback, the transition matrix has the form

$$\Phi = \begin{bmatrix} \mathcal{A}_2 & \mathcal{B}_2 \mathcal{D}_1 \\ 0 & \mathcal{A}_1 \end{bmatrix}$$

Then det $[\Phi - \lambda \mathcal{U}]$ = det $[\mathcal{A}_2 - \lambda \mathcal{U}]$ det $[\mathcal{A}_1 - \lambda \mathcal{U}]$

The characteristic equation can be written as

$$\det [\mathcal{A}_2 - \lambda \mathcal{U}] \det [\mathcal{A}_1 - \lambda \mathcal{U}] = [(\lambda - \lambda_1)(\lambda - \lambda_3)\ldots(\lambda - \lambda_k) \\ (\lambda - \lambda_2)(\lambda - \lambda_4)\ldots(\lambda - \lambda_m)]$$

which shows that if the plant has any eigenvalues in its transition matrix of modulus greater than unity it cannot be compensated by the controller and the system will respond in an unstable mode. This is not true, however, for the feedback system considered above.

It is desired to control this plant in a feedback arrangement as shown in Fig. I with the asymptotic response given by the system eigenvalues

$$\lambda_1 = 0.2, \quad \lambda_2 = 0.3, \quad \lambda_3 = 0.4, \quad \lambda_4 = 0.5$$

The transition matrix for the system is

$$\left[\begin{array}{cccc} 1-\lambda & 0.5 & 0.693a & 0.693a \\ 0 & 0.5-\lambda & 0.5a & 0.5a \\ \hdashline -\lambda & 0 & -b_1-\lambda & -b_2 \\ 0 & 0 & 1 & -\lambda \end{array}\right]$$

The Laplace expansion for the above determinant gives

$$\lambda^4 + (b_1 - 1.5 + 0.693a_0)\lambda^3 + (b_2 - 1.5b_1 + 0.5 - 0.097a_0 + 0.693a_1)\lambda^2$$
$$+ (0.5b_1 - 1.5b_2) - 0.097a_1)\lambda + 0.5b_2$$

From the specified eigenvalues, the desired characteristic polynomial is

$$\lambda^4 - 1.4\lambda^3 + 0.71\lambda^2 - 0.154\lambda + 0.012$$

Equating coefficients in the two polynomials there results

$$\begin{bmatrix} 0.5 & 0 & -0.097 \\ -1.5 & -0.097 & 0.693 \\ 1 & 0.693 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -0.118 \\ 0.186 \\ 0.1 \end{bmatrix}$$

For which the solution is

$$\begin{bmatrix} b_1 \\ a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -0.47 \\ 0.70 \\ 0.20 \end{bmatrix}$$

The recursion formula employing the above constants will force the system to respond asymptotically and, consequently, is an infinite-settling-time design. This design is analogous to the z-domain design using a so-called "staleness" factor[9].

V SYSTEMS WITH MULTIPLE INPUTS AND OUTPUTS

Consider a sampled-data control system having more than one input and output such as the system shown below.

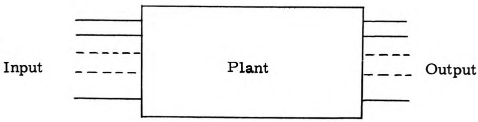

FIG. V

Let the state model of the plant be given as

$$\begin{aligned} \mathcal{X}(n) &= \mathcal{A}\mathcal{X}(n-1) + \mathcal{B}\mathcal{R}(n-1) \\ \mathcal{C}(n) &= \mathcal{D}\mathcal{X}(n) \end{aligned} \tag{70}$$

where $\mathcal{X}(n)$ is an n^{th} order state vector, $\mathcal{R}(n-1)$ is an m^{th} order vector of driving functions and $\mathcal{C}(n)$ is a k^{th} order vector of measurable outputs, $n \geq m, k$. The output vector expressed as an explicit function of the m inputs is

$$\mathcal{C}(n) = \mathcal{D}\mathcal{A}\mathcal{X}(n-1) + \mathcal{D}\mathcal{B}\mathcal{R}(n-1) \tag{71}$$

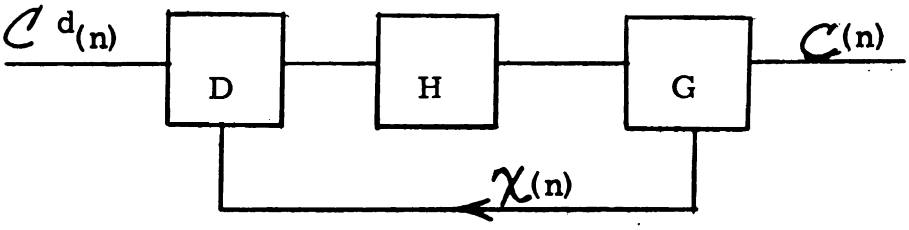

FIG. VI

It is required to design a controller in the multivariable feedback system of Fig. VI that will reduce a given initial state of the plant to zero in a finite time if possible.

Two cases are of interest. If all the state variables are measurable, the problem has already been considered by Bertram and Sarachik[3]. If all states are not measurable, they must either be approximated in some manner or sufficient control time must be provided to identify the unknown state variables. Suppose for example, that only the k outputs can be measured. From these outputs the n unknown initial state variables can be computed under certain conditions. To show this, let the first equation in (70) be written recursively as,

$$\mathcal{X}(1) = \mathcal{A}\mathcal{X}(0) + \mathcal{B}\mathcal{R}(0)$$

$$\mathcal{X}(2) = \mathcal{A}^2\mathcal{X}(0) + \mathcal{A}\mathcal{B}\mathcal{R}(0) + \mathcal{B}\mathcal{R}(1)$$

$$\vdots$$

$$\mathcal{X}(n) = \mathcal{A}^n\mathcal{X}(0) + \mathcal{A}^{n-1}\mathcal{B}\mathcal{R}(0) + \mathcal{A}^{n-2}\mathcal{B}\mathcal{R}(1) + \ldots + \mathcal{B}\mathcal{R}(n-1)$$

Premultiplying the above equations by $\mathcal{D}$ gives (ik) linear scalar equations in the n initial state variables, x_{10}, x_{20}, $x_{30}, \ldots, x_{n0}$.

$$\mathcal{C}(1) = \mathcal{D}\mathcal{A}\mathcal{X}(0) + \mathcal{D}\mathcal{B}\mathcal{R}(0)$$
$$\mathcal{C}(2) = \mathcal{D}\mathcal{A}^2\mathcal{X}(0) + \mathcal{D}\mathcal{A}\mathcal{B}\mathcal{R}(0) + \mathcal{D}\mathcal{B}\mathcal{R}(1)$$
$$\vdots$$
$$\mathcal{C}(i) = \mathcal{D}\mathcal{A}^i\mathcal{X}(0) + \mathcal{D}\mathcal{A}^{i-1}\mathcal{B}\mathcal{R}(0) + \ldots + \mathcal{D}\mathcal{B}\mathcal{R}(i-1)$$

Given $\mathcal{R}(0)$, $\mathcal{R}(1)$, ... $\mathcal{R}(i-1)$ and assuming the required inverse exists, the first n of these scalar equations can establish the initial state variables. At most i = $[n/k]$ * sampling intervals are required. As in the case of the single variable, this system is called observable.

Assuming then that the entries in the initial state vector can be established, the design is realized by writing (70) as

$$\mathcal{X}(k) = \mathcal{A}^k\mathcal{X}(0) + \left[\mathcal{A}^{k-1}\mathcal{B}, \mathcal{A}^{k-2}\mathcal{B}, \ldots, \mathcal{A}\mathcal{B}, \mathcal{B}\right]\begin{bmatrix}\mathcal{R}(0)\\ \mathcal{R}(1)\\ \mathcal{R}(2)\\ \vdots\\ \mathcal{R}(k-1)\end{bmatrix} \quad (72)$$

*read $[n/k]$ as, "the largest integer in n/k."

For convenience, let

$$\left[\mathcal{A}^{k-1}\,\mathcal{B}\,,\ \mathcal{A}^{k-2}\,\mathcal{B},\ldots,\ \mathcal{A}\mathcal{B},\mathcal{B}\right] = \underset{nx(km)}{\mathcal{H}_N}$$

In general, $\mathcal{H}_N$ is rectangular and as such does not have an inverse. Suppose N represents the minimum value of k for which the rank of $\mathcal{H}_N$ is maximum; i.e., there exists n independent vectors among the Nm columns of $\mathcal{H}_N$. Then by rearranging columns if necessary, $\mathcal{H}_N$ may be partitioned as

$$\mathcal{H}_N = \left[\mathcal{H}_{N1} \;\vdots\; \mathcal{H}_{N2}\right]$$

where $\mathcal{H}_{N2}$ is a non-singular nxn submatrix of $\mathcal{H}_N$. If this operation cannot be carried out, then the system is not controllable; that is, $\mathcal{H}_N$ must be of maximum rank for the system to be controllable.. When (72) is written as

$$\mathcal{X}_{(N)} = \mathcal{A}^N\,\mathcal{X}(0) + \underset{nx(Nm)}{\left[\mathcal{H}_{N1},\ \mathcal{H}_{N2}\right]}\ \underset{(Nmx1)}{\begin{bmatrix}\mathcal{R}_{N1}\\ \\ \mathcal{R}_{N2}\end{bmatrix}} \qquad (73)$$

the solution for $\mathcal{R}_{N2}$ when $\mathcal{X}_{(N)} = 0$ is

$$\mathcal{R}_{N2} = - \mathcal{H}_{N2}^{-1} \left[\mathcal{A}^N \mathcal{X}(0) + \mathcal{H}_{N1} \mathcal{R}_{N1} \right] \qquad (74)$$

where the set of inputs $\mathcal{R}_{N1}$ are arbitrary.

In the special case where n (the order of the state vector) is an integral multiple of m; i.e., n = qm, q a positive integer, the matrix $\mathcal{H}_N$ is square. If it is also nonsingular, (73) reduces to

$$\mathcal{R}_N = - \mathcal{H}_N^{-1} \mathcal{A}^N \mathcal{X}(0), \quad n = qm \qquad (75)$$

Therefore, when the number of inputs m is an integral multiple of the number of state variables, the set of forcing functions which will take any arbitrary state vector to zero is uniquely specified. This is also the minimal time solution if all the initial state variables are measurable. If the solution indicated in (74) exists, the system is said to be state controllable which, incidentally, implies output controllability; the converse, however, is not generally true.

Returning to the more general problem, let the initial state $\mathcal{X}(0)$ at time $t = t_0$ be given. Find the sequence of driving functions which will carry the system through some desired sequence of states $\mathcal{X}(1)$, $\mathcal{X}(2)$, $\mathcal{X}(3)$, . . ., $\mathcal{X}(n)$.

For $N \geq n$ the plant model can be written as

$$\mathcal{X}_{(N)} = \mathcal{A}^N \mathcal{X}(0) + \mathcal{H}_N \mathcal{R}_{(N)}, \tag{76}$$

where $\mathcal{H}_N$ is as defined after (72).

If $\mathcal{X}^d(N)$ is the desired state after N intervals, there results for (73)

$$\mathcal{X}^d(N) = \mathcal{A}^N \mathcal{X}(0) + \left[\mathcal{H}_{N1} \;\vdots\; \mathcal{H}_{N2}\right] \begin{bmatrix} \mathcal{R}_{N1} \\ \hdashline \mathcal{R}_{N2} \end{bmatrix} \tag{77}$$

If the system is state-controllable, then $\mathcal{H}_{N2}$ is nonsingular and (77) gives

$$\mathcal{R}_{N2} = \mathcal{H}_{N2}^{-1} \left[\mathcal{X}^d(N) - \mathcal{A}^N \mathcal{X}(0) - \mathcal{H}_{N1} \mathcal{R}_{N1}\right]$$

If $\mathcal{R}_{N1}$ is selected arbitrarily, then the driving functions $\mathcal{R}_{N2}$ are determined*. The question remaining is how to select the values of $\mathcal{R}_{N1}$.

*If $\mathcal{H}_N$ is a square matrix (the number of state variables being an integral multiple of the number of inputs) then, of course, $\mathcal{R}_N$ is unique providing the system is controllable.

Three mathematically tractable methods of selecting $\mathcal{R}_N$ which are of practical interest are:

1. selection of $\mathcal{R}_N$ such that the norm of the control vector is minimized subject to the constraint that the distance between the actual state and the desired state is zero. This may be accomplished using LaGrange multipliers to minimize the scalar function

$$\min_{\mathcal{R}_N} \left\{ \mathcal{R}_N^T \mathcal{U} \mathcal{R}_N + \lambda^T (\mathcal{X}^d(N) - \mathcal{A}^N \mathcal{X}(0) - \mathcal{H}_N \mathcal{R}_N) \right\} \tag{78}$$

2. selection of $\mathcal{R}_N$ such that the distance between the desired state and the actual state is minimized subject to the constraint that the norm of the control vector is equal to some predetermined constant. Again using LaGrange multipliers the problem reduces to minimizing the expression

$$\min_{\mathcal{R}_N} \left\{ (\mathcal{X}^d(N) - \mathcal{A}^N \mathcal{X}(0) - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{U} (\mathcal{X}^d(N) - \mathcal{A}^N \mathcal{X}(0) - \mathcal{H}_N \mathcal{R}_N) + \lambda \left[\mathcal{R}_N^T \mathcal{U} \mathcal{R}_N - K \right] \right\} \tag{79}$$

3. selection of $\mathcal{R}_N$ such that the distance between the desired state and the actual state is minimized subject to a multi-norm constraint on the control vector. Using LaGrange multipliers, the function to minimized is

$$\min_{\mathcal{R}_N} \left\{ (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{U} (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N) + \lambda_1 \mathcal{G}_1(\mathcal{R}_N) + \lambda_2 \mathcal{G}_2(\mathcal{R}_N) + \ldots + \lambda_k \mathcal{G}_k(\mathcal{R}_N) \right\} \quad (80)$$

where

$$\mathcal{G}_i(\mathcal{R}_N) = (\mathcal{R}_N^T \mathcal{Z}_i \mathcal{R}_N - k_i)$$

and $\mathcal{Z}_i$ is a diagonal matrix of order NmxNm and rank $r \leq Nm$, and k_i is the constant associated with the i^{th} constraint.

Design procedures based on each of the above constraints will be considered in the order given.

Letting $\mathcal{E}_N = \mathcal{X}^d(N) - \mathcal{Q}^N \mathcal{X}(0)$, (78) may be written as

$$\min_{\mathcal{R}_N} u = \min_{\mathcal{R}_N} \left\{ \mathcal{R}_N^T \mathcal{U} \mathcal{R}_N + \lambda^T (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N) \right\}$$

The partial derivative with respect to $\mathcal{R}_N$ is

$$\frac{\partial u}{\partial \mathcal{R}_N} = 2\,\mathcal{R}_N - \mathcal{H}_N^T \lambda$$

where $\mathcal{H}_N$ is defined after (72).

Setting $\partial u / \partial \mathcal{R}_N = 0$, it follows that a minimum occurs when

$$\mathcal{R}_N = \frac{1}{2}\,\mathcal{H}_N^T \lambda$$

and

$$\mathcal{E}_N - \mathcal{H}_N\,\mathcal{R}_N = 0$$

or

$$\mathcal{E}_N = \frac{1}{2}\,\mathcal{H}_N\,\mathcal{H}_N^T \lambda$$

If $(\mathcal{H}_N\,\mathcal{H}_N^T)^{-1}$ is nonsingular, then

$$\lambda = 2\,\mathcal{E}_N(\mathcal{H}_N\mathcal{H}_N^T)^{-1}$$

and the sequence of driving functions is

$$\begin{aligned}\mathcal{R}_N &= \mathcal{H}_N^T\,\mathcal{E}_N(\mathcal{H}_N\mathcal{H}_N^T)^{-1}\\ &= \mathcal{H}_N^T(\mathcal{X}^d(N) - \mathcal{A}^N\mathcal{X}(0))(\mathcal{H}_N\mathcal{H}_N^T)^{-1}\end{aligned}$$

and the state model of the computer has been determined. The norm of the plant driving functions is a minimum, subject to the constraint that the state vector achieves the predetermined desired value. The process depends on the existence of $(\mathcal{H}_N \mathcal{H}_N^T)^{-1}$. When $\mathcal{H}_N$ has real entries and is of maximum rank, the product $(\mathcal{H}_N \mathcal{H}_N^T)$ is positive definite and, therefore, nonsingular*. Note also that since the predetermined vector $\mathcal{X}^d(N)$ is achieved, the system is automatically stable. This procedure assures that the norm of the control vector is a minimum but does not assure that any individual entry in the control vector would not exceed some permissible safe upper limit.

In order to provide some control over the magnitude of the plant inputs, the second procedure outlined above may be used. The constraints on the individual plant inputs may be stated as

$$
\begin{aligned}
|r_{01}|^2 &\leq K_1 \\
|r_{02}|^2 &\leq K_2 \\
&\vdots \\
|r_{0Nm}|^2 &\leq K_{Nm}
\end{aligned}
$$

* $\mathcal{H}_N$ is of rank n, and order nxNm, so $\mathcal{H}_N \mathcal{H}_N^T$ is an nxn matrix of rank n, therefore positive definite.

where the Ki, i=0, 1, 2. . . , Nm, are the constants describing the safe upper limits for the respective inputs. The sum of these terms clearly represents the squared norm of $\mathcal{R}_N$. When such constraints are placed on the inputs, one can only minimize the distance between the desired state and the actual state. Again using LaGrange multipliers, the problem reduces to minimizing the expression in (79). If (79) is written as

$$\min_{\mathcal{R}_N} u = \min_{\mathcal{R}_N} \left\{ (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{U} (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N) + \lambda (\mathcal{R}_N^T \mathcal{R}_N) \right\}$$

Then upon differentiating with respect to $\mathcal{R}_N$, one has

$$\frac{\partial u}{\partial \mathcal{R}_N} = -\mathcal{H}_N^T (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N) + \lambda \mathcal{R}_N = 0 \tag{81}$$

subject to the requirement, that $\mathcal{R}_N^T \mathcal{R}_N = K.$ (82)

Solving (81) for $\mathcal{R}_N$ gives

$$\mathcal{R}_N = (\lambda \mathcal{U} + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N^T \mathcal{E}_N \tag{83}$$

Substituting this result into the constraint equation (82) gives a system of nonlinear algebraic equations in λ.

$$\mathcal{E}_N^T \mathcal{H}_N \left[(\lambda \mathcal{U} + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \right]^T \left[(\lambda \mathcal{U} + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \right] \mathcal{H}_N^T \mathcal{E}_N = K$$

If $\lambda = \lambda_0$ is the solution (if it exists), then from (83)

$$\mathcal{R}_N = \left[\lambda_0 \mathcal{U} + \mathcal{H}_N^T \mathcal{H}_N \right]^{-1} \mathcal{H}_N^T \mathcal{E}_N$$

and the control function has been determined which minimizes the miss distance subject to the one constraint in the driving function.

Additional and more restrictive constraints on the inputs can be introduced by considering a multi-norm constraint on the inputs as shown in the expression (80). Each of the constraints are of the form

$$\left| r_{0k} \right|^2 + \left| r_{0q} \right|^2 = K_p \qquad (84)$$

Here K_p is selected such that r_{0K} and r_{0q} do not exceed some pre-assigned safe upper limit. Letting $\mathcal{E}_N = \mathcal{X}^d(N) - \mathcal{Q}^N \mathcal{X}(0)$, (80) can be written as

$$\min_{\mathcal{R}_N} u = \min_{\mathcal{R}_N} \Big\{ (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{U} (\mathcal{E}_N - \mathcal{H}_N \mathcal{R}_N) + \lambda_1 \mathcal{R}_N^T \mathcal{Z}_1 \mathcal{R}_N + \lambda_2 \mathcal{R}_N^T \mathcal{Z}_2 \mathcal{R}_N + \dots + \lambda_k \mathcal{R}_N^T \mathcal{Z}_k \mathcal{R}_N \Big\}$$

Differentiating with respect to the entries of $\mathcal{R}_N$ gives

$$\frac{\partial u}{\partial r_1} = -2\,(e_N - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{H}_1 + 2\,\lambda_1 r_1$$

$$\frac{\partial u}{\partial r_2} = -2\,(e_N - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{H}_2 + 2\,\lambda_1 r_2$$

$$\vdots$$

$$\frac{\partial u}{\partial r_{n_m}} = -2\,(e_N - \mathcal{H}_N \mathcal{R}_N)^T \mathcal{H}_{nm} + 2\,\lambda_{\frac{nm}{2}}\, r_{nm}$$

or

$$\frac{\partial u}{\partial \mathcal{R}_N} = +2 \begin{bmatrix} \mathcal{H}_1^T(e_N - \mathcal{H}_N \mathcal{R}_N) \\ \mathcal{H}_2^T(e_N - \mathcal{H}_N \mathcal{R}_N) \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{H}_{nm}^T(e_N - \mathcal{H}_N \mathcal{R}_N) \end{bmatrix} + 2 \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_1 & & & & & \\ & & \lambda_2 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_{nm/2} & \\ & & & & & & \lambda_{nm/2} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_{nm} \end{bmatrix}$$

which for convenience is written as

$$\frac{\partial u}{\partial \mathcal{R}_N} = -2\,\mathcal{H}_N^T (e_N - \mathcal{H}_N \mathcal{R}_N) + 2\,\Lambda\,\mathcal{R}_N = 0 \qquad (85)$$

Equation (85) must now be solved subject to the constraints

$$\begin{bmatrix} \mathcal{R}_N^T & \mathcal{Z}_1 & \mathcal{R}_N \\ \mathcal{R}_N^T & \mathcal{Z}_2 & \mathcal{R}_N \\ & \vdots & \\ \mathcal{R}_N^T & \mathcal{Z}_{\frac{nm}{2}} & \mathcal{R}_N \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_{\frac{nm}{2}} \end{bmatrix} \tag{86}$$

Solving (85) for $\mathcal{R}_N$ and substituting (86) into the result gives

$$\left[(\Lambda + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N\, e_N\right]^T \mathcal{Z}_1 \left[(\Lambda + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N^T\, e_N\right] = K_1$$

$$\left[(\Lambda + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N\, e_N\right]^T \mathcal{Z}_2 \left[(\Lambda + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N^T\, e_N\right] = K_2$$

$$\vdots$$

$$\left[(\Lambda + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N\, e_N\right]^T \mathcal{Z}_{\frac{Nm}{2}} \left[(\Lambda + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N^T\, e_N\right] = K_{\frac{Nm}{2}}$$

The required minimization is realized as a solution to a system of non-linear equations. Let the solution (if it exists) be $\bar{\lambda}_1, \bar{\lambda}_2, \ldots \bar{\lambda}_{\frac{Nm}{2}}$.

The controller which minimizes the miss distance with pair-wise constraints on the drivers is then

$$\mathcal{R}_N = (\bar{\Lambda} + \mathcal{H}_N^T \mathcal{H}_N)^{-1} \mathcal{H}_N^T \quad e_N \tag{87}$$

The above discussion, which is certainly preliminary in nature, suggests several methods of approach which may be used to aid in the design of state control for a multiport plant. Unfortunately, the controller characteristic can no longer be modeled as a simple recursion formula. An effort to find a general mathematical model for the input-output characteristic of a controller having multiple inputs and outputs met with little success. Until such a model for the controller can be determined, the multiport system design problem cannot be handled with the same facility as the single input-single output system. The control functions may still be determined with the aid of the above formulas, but the method for determining them from the sampled state variables is unknown.

VI SUMMARY

The problems associated with the analysis and design of linear, sampled-data control systems have received much attention in the past decade. Until recently such analysis and design has been carried out using the z -transform as the mathematical tool. This thesis considers the design of linear, sampled-data control using the discrete-state model.

A state model of the system has been derived from the state models of the system components and a simple algorithm established relating the transition matrix of the system to the transition matrices of the components. This algorithm forms the basis for the design and can be considered as the counterpart of the z-domain algorithm

$$H(z) = \frac{D(z)\ G(z)}{1 + D(z)\ G(z)}$$

where D(z) and G(z) represent respectively the z-transform of the controller and plant characteristics and H(z) is the system transfer function.

Functions of matrices were used as a basis for studying the characteristics of the system model. It is believed that this is the first application of functions of a matrix in the design of sampled-data systems. Several general forms for the system-state response to particular drivers have been derived from both the homogeneous and nonhomogeneous forms of the system models and the fundamental stability theorem established in terms of functions of matrices. The major contribution of this thesis, however, is believed to be in the area of time domain design of sampled-data systems.

A design procedure for minimal-time response was developed for plants which are both observable and controllable in some sense. Such designs are shown to be unique and, consequently, provide no latitude for the control of the level of the plant input signal.

A design based on non-minimal response is shown to be more flexible. If the plant is observable and controllable, a design based on non-minimal response always exists. Several methods for determining the "best" control scheme, subject to certain constraints on the magnitude of the control sequence, are presented.

An extended concept of plant controllability has been developed in which the plant achieves equilibrium in the limit as $t \rightarrow \infty$. It was found that unstable eigenvalues in the plant transition matrix may be compensated without creating a problem in sensitivity, thereby showing that the so-called sensitivity problem in z-domain design has no foundation in fact.

A preliminary study of the design of systems with multiple inputs and outputs has been made, and the design procedure is shown to lead to nonlinear algebraic equations. It is felt that this area represents a fruitful area for further study.

APPENDIX A

Discussion of Controllability.

Let the plant state transition equation for k variables in the state vector be written as

$$\mathcal{X}_{(n+1)} = \mathcal{A}\mathcal{X}_{(n)} + \mathcal{B}\, r(n+1)$$

The state of the system at successive sampling instants is given by

$$\mathcal{X}_{(1)} = \mathcal{A}\mathcal{X}_{(0)} + \mathcal{B}\, r(1)$$
$$\mathcal{X}_{(2)} = \mathcal{A}^2\mathcal{X}_{(0)} + \mathcal{A}\mathcal{B}\, r(1) + \mathcal{B}\, r(2)$$
$$\vdots$$
$$\mathcal{X}_{(k)} = \mathcal{A}^k\mathcal{X}_{(0)} + \mathcal{A}^{k-1}\mathcal{B}\, r(1) + \ldots + \mathcal{A}\mathcal{B}\, r(k-1) + \mathcal{B}\, r(k)$$

Assuming $\mathcal{A}$ is non-singular, that $\mathcal{X}(0)$ is given and that it is required to establish r(1), r(2), ... r(k) such that $\mathcal{X}(k) = 0$, it follows from the last expression that

$$\mathcal{X}_{(0)} = -r(1)\,\mathcal{A}^{-1}\mathcal{B} - r(2)\,\mathcal{A}^{-2}\mathcal{B} - r(3)\,\mathcal{A}^{-3}\mathcal{B} - \ldots - r(k)\,\mathcal{A}^{-k}\mathcal{B}$$

or

$$\mathcal{X}_{(0)} = r(1)\,\mathcal{G}_1 + r(2)\,\mathcal{G}_2 + r(3)\,\mathcal{G}_3 + \ldots + r(k)\,\mathcal{G}_k$$

where

$$g_i = -\mathcal{A}^{-i}\mathcal{B}$$

The above matrix equation is a system of k scalar equations in the k unknowns r(1), r(2), ..., r(k). This system has a solution for the r_i for any $\mathcal{X}(0)$ if and only if the k vectors $g_1, g_2, g_3, \ldots g_k$ are linearly independent; i.e., if the vectors

$$\mathcal{B},\ \mathcal{A}\mathcal{B},\ \mathcal{A}^2\mathcal{B}, \ldots,\ \mathcal{A}^{k-1}\mathcal{B}$$

are linearly independent and span the k-dimensional vector space of $\mathcal{X}$. Such a system of vectors is said to define a system which is "controllable." Let the solution for the r(i) be written as

$$\begin{bmatrix} r(1) \\ r(2) \\ \cdot \\ \cdot \\ \cdot \\ r(k) \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1k} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \alpha_{k1} & \cdots & \alpha_{kk} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ \cdot \\ \cdot \\ \cdot \\ x_{0k} \end{bmatrix} \tag{A.1}$$

For any given $\mathcal{X}(0)$ the solution for the r_i is unique; therefore, a unique set of control functions exist that will take any arbitrary state $\mathcal{X}(0)$ to the origin in exactly k sampling intervals providing the system is controllable.

The requirement that $\mathcal{A}$ be non-singular is too restrictive for general controllability. Suppose that $\mathcal{A}$ is singular. Then as before, for k variables in the state vector,

$$\mathcal{X}(k) = \mathcal{A}^k \mathcal{X}(0) + \mathcal{A}^{k-1} \mathcal{B}\, r(1) + \ldots + \mathcal{A}\mathcal{B}\, r(k-1) + \mathcal{B}\, r(k)$$

Let

$$\mathcal{X}(n) = 0 \text{ for } n \geq k \text{ then}$$

$$-\mathcal{A}^k \mathcal{X}(0) = \mathcal{A}^{k-1}\mathcal{B}\, r(1) + \ldots + \mathcal{A}\mathcal{B}\, r(k-1) + \mathcal{B}\, r(k)$$

or

$$-\mathcal{A}^k \mathcal{X}(0) = \left[\mathcal{A}^{k-1}\mathcal{B},\ \mathcal{A}^{k-2}\mathcal{B}, \ldots, \mathcal{A}\mathcal{B}, \mathcal{B}\right] \begin{bmatrix} r(1) \\ r(2) \\ \cdot \\ \cdot \\ \cdot \\ r(k) \end{bmatrix}$$

Let $\left[\mathcal{A}^{k-1}\mathcal{B},\ \mathcal{A}^{k-2}\mathcal{B}, \ldots, \mathcal{A}\mathcal{B}, \mathcal{B}\right] = \Phi$

If the k vectors $\mathcal{A}^{k-1}\mathcal{B},\ \mathcal{A}^{k-2}\mathcal{B}, \ldots, \mathcal{A}\mathcal{B}, \mathcal{B}$ are linearly independent then Φ^{-1} exists and

$$\begin{bmatrix} r(1) \\ r(2) \\ \cdot \\ \cdot \\ \cdot \\ r(k) \end{bmatrix} = - \Phi^{-1} \mathcal{A}^k \begin{bmatrix} x_1(0) \\ x_2(0) \\ \cdot \\ \cdot \\ \cdot \\ x_k(0) \end{bmatrix}$$

The system represented by $\mathcal{A}$ is therefore controllable if $\mathcal{A}^{k-1}\mathcal{B}, \ldots, \mathcal{A}\mathcal{B}, \mathcal{B}$ are a linearly independent set of vectors.

A rather interesting situation arises if Φ is singular; that is, the vectors $\mathcal{A}^{k-1}\mathcal{B}, \ldots, \mathcal{A}\mathcal{B}, \mathcal{B}$ are linearly dependent. Assume Φ has rank $r \leq k$. Then it follows that

$$- \mathcal{A}^k \mathcal{X}(0) = \Phi \mathcal{R} \quad \text{where} \quad \mathcal{R} = \begin{bmatrix} r(1) \\ r(2) \\ \cdot \\ \cdot \\ \cdot \\ r(k) \end{bmatrix}$$

Since $-\mathcal{A}^k \mathcal{X}(0)$ is a known vector, let

$$- \mathcal{A}^k \mathcal{X}(0) = \mathcal{Y}(0)$$

and partition $\mathcal{Y}(0)$ and Φ to give

$$\begin{bmatrix} \mathcal{Y}_1(0) \\ \mathcal{Y}_2(0) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{bmatrix} \tag{A. 2}$$

Let $\mathcal{P}$ be an elementary transformation on the rows of Φ which reduces it to the form

$$\mathcal{P}\Phi = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ 0 & 0 \end{bmatrix}$$ where Γ_1 is a non-singular matrix of order r.

Setting $\mathcal{P}\mathcal{Y}(0) = \mathcal{Y}'(0) = -\mathcal{P}\,\mathcal{Q}^k\,\mathcal{X}(0)$, (A. 2) becomes

$$\begin{bmatrix} \mathcal{Y}_1'(0) \\ \mathcal{Y}_2'(0) \end{bmatrix} = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{bmatrix} \tag{A. 3}$$

The first equation in (A. 3) is a system of r scalar equations in the k unknowns $\mathcal{R}_1$, $\mathcal{R}_2$. This system has a unique solution for any arbitrarily specified set of k-r variables satisfying the second equation. There are two possibilities: either the initial conditions are such that $\mathcal{P}(-\mathcal{Q}^k)$ annihilates the last k-r initial state variables or it does not. If it does not, which probably is the case, then the system represented by Φ is not controllable in the sense of the above definition.

Assuming the system is controllable, then from (A. 1) the solution for the control functions is

$$r(1) = \alpha_{11} x_{01} + \alpha_{12} x_{02} + \dots + \alpha_{1n} x_{0n}$$
$$r(2) = \alpha_{21} x_{01} + \alpha_{22} x_{02} + \dots + \alpha_{2n} x_{0n}$$
$$\vdots$$
$$r(n) = \alpha_{n1} x_{01} + \alpha_{n2} x_{02} + \dots + \alpha_{nn} x_{0n}$$

This control sequence is unique for any x_{01}, x_{02}, ..., x_{0n}, and the state vector is reduced to zero in exactly n sampling periods. If the state $\chi(1)$ is considered as a new initial condition, then a new set of control functions $r'(1)$, $r'(2)$, $r'(3)$, ... $r'(n)$ can be selected to bring the state vector to zero in n sampling periods. Therefore,

$$r'(1) = \alpha_{11} x_{11} + \alpha_{12} x_{12} + \alpha_{13} x_{13} + \dots + \alpha_{1n} x_{1n}$$
$$r'(2) = \alpha_{21} x_{11} + \alpha_{22} x_{12} + \alpha_{23} x_{13} + \dots + \alpha_{2n} x_{1n}$$
$$\vdots$$
$$r'(n) = \alpha_{n1} x_{11} + \alpha_{n2} x_{12} + \alpha_{n3} x_{13} + \dots + \alpha_{nn} x_{1n}$$

If, however, $\chi(1)$ is known to be a system state between $\chi(0)$ and 0 then equilibrium can be reached in (n-1) sampling periods with a uniquely determined control sequence the first one being

$$r(2) = \alpha_{21} x_{01} + \alpha_{22} x_{02} + \dots + \alpha_{2n} x_{0n}$$

Since $\chi(1)$ _is_ an intermediate state and the control sequence _is_ unique, it follows that $r(1)' = r(2)$, $r(2)' = r(3) \ldots r(n-1)' = r(n)$, $r(n)' = r(n+1) = 0$. So $r(2) = \alpha_{11} x_{11} + \alpha_{12} x_{12} + \ldots + \alpha_{1n} x_{1n}$.

An extension of this same line of reasoning to $r(3)$, $r(4)$, ... $r(n)$, shows in general that $r(k) = \alpha_{11} x_{k-1,1} + \alpha_{12} x_{k-1,2} + \alpha_{13} x_{k-1,3} + \ldots + \alpha_{1n} x_{k-1,n}$. This means that if $r(1)$ is a linear combination of the state variables measured at time $t = 0$, then $r(k)$ is the same linear combination of the same state variables measured at time $t = (k-1)T$. This control can be derived by feedback which is linear and time-stationary.

Suppose $\mathcal{X}(k) \neq 0$ for any fixed k, but is a constant vector as is the case in a follow-up system. Then

$$\mathcal{X}(k) = \mathcal{A}^k \mathcal{X}(0) + \Phi \mathcal{R}'(k)$$

Let

$$\mathcal{X}(k) - \mathcal{A}^k \mathcal{X}(0) = \mathcal{Y}(k)$$

then

$$\mathcal{Y}(k) = \Phi \mathcal{R}'(k) \tag{A. 4}$$

Since (A. 4) is of the same form as (A. 3), this reduces to the same control problem as the one previously discussed. The only difference between the "follow-up" system and the above equilibrium problem is that the difference between the output and input is sampled instead of only the output.

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