APPROXIMATION OF
VECTOR-VALUED FUNCTIONS

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ABSTRACT

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by Lee Warren Johnson

Let X be a compact space and let C(X) be the set of all continuous functions on X to E^n. Let U be a closed convex subset of E^n, with the origin in U. For $f \in C(X)$, define M(f) by: $M(f) = \inf\{a \mid a \geq o,\ f(x) \in aU \text{ for all } x \in X.\}$ If no multiple of U contains the range of f, write $M(f) = \infty$.

Let LF be a linear subspace of C(X). If $p \in LF$, call p an approximation to f if $M(p-f) < \infty$. Call $p_o \in LF$ a best approximation if p_o is an approximation, and if $M(p_o - f) \leq M(p-f)$ for all p in LF.

Let $M(p-f) = a < \infty$. We say $x \in X$ is an extreme point of p-f if $[p-f](x)$ is in the boundary of aU. We examine some of the properties of the extreme points of best approximations. We define several auxiliary notions, such as a generalization of H-sets. Several characterizations of best approximations are given in terms of these auxiliary notions, and in terms of extreme points. Some applications and examples are given, including new proofs of several sell-known theorems, cast in the form of vector-valued approximations.

We show, for example, how the following theorem of

Moursund[1] can be treated as a problem in approximation of vector-valued functions.

<u>Theorem:</u> Let $f(x)$ be twice differentiable on $[a,b]$. Among all polynomials $a_n x^n + \cdots + a_1 x + a_0$, of degree n or less, let $p(x)$ be one that minimizes: $\max\{\|p-f\|, \|p'-f'\|\}$, where $\|g\| = \sup_{x\in[a,b]} |g(x)|$. If $q(x)$ is another such polynomial, then $q'=p'$.

We apply our results to this problem by setting:

a) $F(x)=(f(x),f'(x))$

b) $LF=\{P(x) \mid P(x)=(p(x),p'(x)),\ p(x) \text{ of degree n or less}\}$

c) $U=\{(x,y) \mid -1\le x\le 1,\ -1\le y\le 1\}$

d) $X=[a,b]$

Then: $M(P-F)=\max\{\|p-f\|, \|p'-f'\|\}$.

1. Moursund, D. G. "Chebyshev Approximation of a Function and its Derivatives" <u>Mathematics of Computation</u>, 18 (1964), 382-389.

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INTRODUCTION

The classical problem of uniform approximation is formulated as follows:

Let $f(x)$ be a continuous real-valued function on $[o,1]$. Among all polynomials $p(x)=a_n x^n+\cdots+a_1 x+a_0$, of degree n or less, find one that minimizes:

$$\max_{x\in[o,1]} |p(x)-f(x)| .$$

This problem, and many variants of it, have been studied extensively. In the problem above, it is well known that there is a solution, called a best uniform approximation, and that the solution is unique. Furthermore, if $p(x)$ is the best uniform approximation, then $p(x)$ can be characterized by the following theorem.

Theorem: $p(x)$ is a best uniform approximation to $f(x)$ if and only if there are $n+2$ points x_i in $[o,1]$, $x_i<x_{i+1}$, such that $p(x_i)-f(x_i)=(-1)^{i+1}d$, where $d=p(x_1)-f(x_1)$ and $|p(x)-f(x)|\leq|d|$ for all $x\in[o,1]$.

This problem can be extended by considering finite linear combinations $a_1g_1(x)+\cdots+a_ng_n(x)$ of continuous "base" functions, rather than polynomials. The problem can be further compounded by choosing the domain of $f(x)$ and $g_i(x)$ to be an arbitrary compact space, or perhaps a compact subset of E^n.

We have chosen to extend the problem in another direction, by allowing f(x) and $g_i(x)$ to take on values in E^n. In Chapter II we have given some examples to show that this is not a barren generalization.

In Chapter I, we treat the three fundamental questions of any theory of approximation. These are: existence of a best approximation; uniqueness of a best approximation; and the characterization of a best approximation. Of the three, the latter is perhaps the most important; if the characterization is strong enough, it enables one to construct algorithms to find the best approximation.

Notationally, we will write: $\left[x \middle| P(x) \right]$ for "the set of all x such that P(x) holds", (x_n) for a sequence, and S(y,R) for the closed n-ball of radius R centered at $y \in E^n$.

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CHAPTER I

APPROXIMATION OF VECTOR-VALUED FUNCTIONS

In this chapter we will consider the problem of approximating vector-valued functions. We will define what is to be meant by an approximation and derive some pertinent results. Having developed some of the necessary theory, we will turn to some applications and examples.

In what follows we will always have:

a) X a compact space

b) C(X) the set of all functions continuous on X to Euclidean n-space

c) LF a linear subspace of C(X).

Given f in C(X), we will be concerned mostly with questions of characterization and uniqueness of best approximations to f by functions in LF. It should be noted that some of the following theorems are valid in a more general setting where E^n is replaced by a real topological vector space, and LF by a convex subset of C(X). In the interest of cohesiveness, we have dispensed with some of this generality.

<u>Definition</u>: Let U be a closed convex set in E^n with a non-empty interior, and let the origin be in U. For $f \in C(X)$ we define M(f) by:

$$M(f)=\inf\left[a \,\middle|\, a \geq 0,\ f(x) \in aU \text{ for all } x \in X\right].$$

If it happens that no multiple of U contains the range of f, define $M(f)=\infty$.

We observe the following about M(f):

a) If $f(x)=o$ for all x in X, then $M(f)=o$

b) $M(tf)=tM(f)$ for t real, $t\geq o$

c) $M(f+g)\leq M(f)+M(g)$.

To prove the last observation we need the result: $(a+b)U=aU+bU$ when U convex and a,b are non-negative. To prove this, note first that we clearly have $(a+b)U\subseteq aU+bU$. If $a=b=o$, then $(a+b)U=aU+bU$. If one of a,b is non-zero, let au_1+bu_2 be any element in $aU+bU$. Choose u_3 by:

$u_3=\frac{a}{a+b}u_1+\frac{b}{a+b}u_2$; since U is convex, u_3 is in U. Then $(a+b)u_3\in(a+b)U$; and since $(a+b)u_3=au_1+bu_2$, we have shown $aU+bU\subseteq(a+b)U$.

Now, to prove c), let $M(f)=a$ and $M(g)=b$. We then have $f(X)\subset aU$ and $g(X)\subset bU$. Consequently, $(f+g)(X)\subseteq f(X)+g(X)\subseteq aU+bU$. As $(f+g)(X)\subseteq(a+b)U$, we see that $M(f+g)\leq a+b=M(f)+M(g)$.

If the convex set U is absorbent (that is, for each $y\in E^n$ some multiple of U contains y) and symmetric, then of course the function M defines a norm on C(X). However, we will not wish to restrict our considerations to just absorbent and symmetric sets U.

<u>Definition</u>: Let $f\in C(X)$, $p\in LF$. Then p is called an approximation to f (with respect to U) if $M(p-f)<\infty$.

As usual, $p_o \in LF$ is a best approximation if $M(p_o-f) \leq M(p-f)$ for all p in LF.

By way of example consider the following problem:

Let f(x) have a continuous derivative on [o,1]. Among all polynomials p(x) of degree n or less, find the ones that minimize: $\max\left[\|p-f\|, \|p'-f'\|\right]$, subject to $p(x) \geq f(x)$ for all $x \in [o,1]$. Here $\|p-f\|$ is the uniform norm, $\|p-f\| = \sup_{x\,[o,1]} |p(x)-f(x)|$.

We can place this problem in our context by defining $U = \left[(x,y) \mid o \leq x \leq 1, -1 \leq y \leq 1\right]$. Also, let $F(x) = (f(x), f'(x))$, $LF = \left[P(x) \mid P(x) = (p(x), p'(x)),\ p(x) \text{ is a polynomial of degree } n\right]$ Then if $X = [o,1]$, we have F maps X continuously into E^2, and LF is a linear subspace of C(X).

If $M(P-F) = a$, then for all $x \in [o,1]$, $(P-F)(x) \subseteq aU$. Hence, $(p(x)-f(x), p'(x)-f'(x)) \in aU$ for all $x \in [o,1]$, and no smaller a will suffice to contain the range of P-F.

But $(P-F)(x) \subseteq aU$ means that $a \geq p(x)-f(x) \geq o$, and that $a \geq p'(x)-f'(x) \geq -a$. Thus $M(p-f) = a$ gives us that $p(x) \geq f(x)$ for all $x \in [o,1]$, and that $a = \max\left[\|p-f\|, \|p'-f'\|\right]$. Therefore, finding $P \in LF$ that is a best approximation to F is equivalent to solving the problem as originally posed.

Many problems in approximation of real-valued functions, such as the one above, can be reformulated as problems in approximations of vector-valued functions. In Chapter 2 we will examine some more problems of this type, and we will attempt to apply results derived in this chapter to them.

A standard result in the theory of uniform approximation of real-valued functions is that the set of best approximations is convex when the set of approximating functions is convex. We can readily obtain a similar result.

Theorem 1: Let p and q be two best approximations to f. If $h=tp+(1-t)q$, $0<t<1$, then h is also a best approximation to f.

Proof: This follows immediately from the subadditivity of M. Let $M(p-f)=M(q-f)=a$. Then as p is a best approximation, $M(h-f)\geq a$.

But, $M(h-f)=M(tp+(1-t)q-f)\leq tM(p-f)+(1-t)M(q-f)=a$. Therefore, $M(h-f)=a$, and consequently, h is a best approxi- to f.

If p and f are real functions, and if p is an approximation to f, the extreme points of p-f are those points x such that $|p(x)-f(x)| =\|p-f\|$. Best uniform approximations can be characterized in terms of their extreme points. The extreme points also play an important role in deciding questions of uniqueness, and in the construction of algorithms to find best approximations. We will define an analog for vector-valued approximations.

In all that follows we will denote the topological boundary of U by B(U). We note the following which are obvious:

a) $B(tU)=tB(U)$ for t real, positive

b) $B(z+U)=z+B(U)$ for $z\in E^n$.

Definition: Let $M(p-f)=a<\infty$. Then $x_0\in X$ is an extreme point of p-f if $[p(x_0)-f(x_0)]\in B(aU)$. We denote the set of all extreme points of p-f by $E(p,f)$.

To avoid trivial cases, we will always assume that $B(U)$ is not empty; that is, that U is not all of E^n. We will also assume that $M(p-f)>o$, for the case $M(p-f)=o$ is ideal, and nothing of interest can be said.

Theorem 2: If p is an approximation to f, then $E(p,f)$ is not empty.

Proof: For $y\in E^n$ let $\|y\|$ be the usual Euclidean norm, $\|y\|=\sqrt{y_1^2+\cdots+y_n^2}$. Let $M(p-f)=a<\infty$. If $E(p,f)=\emptyset$, then $(p-f)(X)\subset aU-B(aU)=V$. Since X is compact, $(p-f)(X)$ is compact. As V is open and $(p-f)(X)$ is compact, the distance between $(p-f)(X)$ and the complement of V is positive; let $d>o$ be this distance.

Also, since $(p-f)(X)$ is compact, there is $b>o$ such that $\|y\|\leq b$ for all y in $(p-f)(X)$. Let m be any element of $(p-f)(X)$ and y any element of complement V. Then we have:

$\left\|(1+\frac{d}{2b})m-y\right\|\geq\left\|m-y\right\|-\frac{d}{2b}\left\|m\right\|\geq\left\|m-y\right\|-\frac{d}{2}\geq\frac{d}{2}$. Thus, the distance between $(1+\frac{d}{2b})\left[(p-f)(X)\right]$ and complement of V is positive. Therefore, $(1+\frac{d}{2b})\left[(p-f)(X)\right]\subseteq V\subseteq aU$. Hence, $(p-f)(X)\subset(\frac{2b}{2b+d})aU$, which means $M(p-f)<a$. This is contrary to our assumption, so it must be that $E(p,f)\neq\emptyset$.

We will now examine some of the properties of the extreme points of best approximations.

Theorem 3: Let p and q be two best approximations to f. Let $h=tp+(1-t)q$, where $o<t<1$. Then $E(h,f)\subseteq E(p,f)\cap E(q,f)$.

Proof: Let $M(p-f)=M(q-f)=a$. By Theorem 1, $M(h-f)=a$, since p and q are best approximations. Let $x\in E(h,f)$. Then $h(x)-f(x)$ is in $B(aU)$. Thus $tp(x)+(1-t)q(x)-f(x)=t[p(x)-f(x)]+(1-t)[q(x)-f(x)]$ is in $B(aU)$. It is well known [1], that if y,z are in a convex set W, and if y is in the interior of W, then the line segment [y,z] is contained in the interior of W with the possible exception of z.

We note that $h(x)-f(x)$ is in the interior of the line segment joining $p(x)-f(x)$ and $q(x)-f(x)$, both of which are in aU. In light of the above, we can have $h(x)-f(x)$ in $B(aU)$ only if both $p(x)-f(x)$ and $q(x)-f(x)$ are in $B(aU)$. Thus, we have $x\in E(p,f)$ and $x\in E(q,f)$.

Definition: We call U strictly convex if for y,z distinct, $y\in B(U)$, $z\in B(U)$ we have: $ty+(1-t)z$ is not in $B(U)$ when $o<t<1$.

Theorem 4: Let p and q be two best approximations to f. Let $h=tp+(1-t)q$, where $o<t<1$. If U is strictly convex, then $x\in E(h,f)$ if and only if $x\in E(p,f)\cap E(q,f)$ and $p(x)=q(x)$.

__Proof__: If $p(x)=q(x)$, and if $x\in E(p,f)\cap E(q,f)$, then $h(x)-f(x)=t[p(x)-f(x)]+(1-t)[q(x)-f(x)]=p(x)-f(x)$. Thus, since $p(x)-f(x)$ is in $B(aU)$, so is $h(x)-f(x)$. Hence, $x\in E(h,f)$.

Conversely, suppose $x\in E(h,f)$. From Theorem 3, we have that $x\in E(p,f)\cap E(q,f)$. If $p(x)$ and $q(x)$ are distinct, then so are $p(x)-f(x)$ and $q(x)-f(x)$. Both $p(x)-f(x)$ and $q(x)-f(x)$ are in $B(aU)$. Since U is strictly convex, the open line segment joining $p(x)-f(x)$ and $q(x)-f(x)$ would then lie in the interior of U. But $h(x)-f(x)$ lies on this open segment, and we are supposing that $h(x)-f(x)$ is in $B(aU)$. Hence, it must be that $p(x)=q(x)$.

To avoid introducing too many symbols, we will also use M in the following natural sense. For $y\in E^n$, let $M(y)=\inf\left[a \mid a\geq 0,\ y\in aU\right]$. $M(y)=\infty$, if $y\notin aU$ for all $a>0$. It will be clear in the context of the problem whether we are using M to measure $(p-f)\in C(X)$, or whether we are using M in the sense of a Minkowski functional defined on a portion of E^n.

In order to obtain our first characterization theorem, we will need to restrict the convex set U by requiring that the set $\left[y\in U \mid M(y)=1\right]$ be closed. In all that follows, we will assume that U meets this condition.

By imposing this restriction on U, we are excluding from consideration such convex sets as $V=\left[(x,y) \mid y\geq x^2\right]$. In V, the set of z such that $M(z)=1$ is precisely the graph of

$y=x^2$ with the origin removed. We do not, however, exclude such sets as $W=\left[(x,y)\mid o\leq x\leq 1,\ o\leq y\leq 1\right]$ which have "flat sides". In W, $\left[z\mid M(z)=1\right]$ consists of the two line segments, $t(o,1)+(1-t)(1,1)$ and $t(1,o)+(1-t)(1,1)$, where $o\leq t\leq 1$. Finally, if the origin is an interior point of U, then $\left[y\in U\mid M(y)=1\right]$ is easily seen to be closed. That this is true is apparent from the fact that the Minkowski functional is continuous on E^n (in fact, uniformly continuous) when the function is defined relative to a convex set containing the origin as an interior point.

<u>Lemma</u>: Let y, z be in E^n, and let $a=M(y)<M(z)=b<\infty$. Then, as t ranges in $[o,1]$, $M(ty+(1-t)z)$ ranges over all of $[a,b]$.

<u>Proof</u>: Again, we denote the closed line segment joining y and z by $[y,z]$. Choose $t\in(a,b)$, and let $G=[y,z]\cap tU$. G is not empty since $M(y)=a<t$, and thus $y\in tU$. Furthermore, since G is a closed convex set contained in $[y,z]$, G must have the form $[y,x]$.

Let $M(x)=c$. If $c<t$, we can choose $s>o$ such that $sM(z)+(1-s)M(x)=sb+(1-s)c<t$. If $x_1=sz+(1-s)x$, then $x_1\in(x,z]$. Also, $M(x_1)=M(sz+(1-s)x)\leq sM(z)+(1-s)M(x)<t$. But $M(x_1)<t$ would mean that $x_1\in tU$. This in turn gives $x\in[y,x_1)\subseteq tU$ which contradicts our assumption that $[y,z]\cap tU=[y,x]$. Consequently, it must be that $M(x)=c\geq t$. As $x\in tU$, we have $M(x)\leq t$. Thus, $M(x)=t$.

Theorem 5: Let $b>o$, $(y_n)\subseteq bU$. If $(y_n)\rightarrow y_o$, then $(M(y_n))\rightarrow M(y_o)$.

Proof: Since bU is closed, and since the sequence (y_n) is contained in bU, we have $y_o\in bU$. Suppose $M(y_o)=a$. Let $e>o$ be given, and suppose $a-e\geq o$. If y_n is such that $M(y_o)\geq M(y_n)+e$, we have $a-e\geq M(y_n)$. Hence, $y_n\in(a-e)U$. $(a-e)U$ is closed, and $y_o\notin(a-e)Y$; so we can have but finitely many y_n such that $M(y_o)\geq M(y_n)+e$. If $(a-e)<o$, then $M(y_n)+e>M(y_o)$ for all n. Thus, in any case, for all but finitely many y_n, we have $M(y_o)<M(y_n)+e$.

Suppose $M(y_o)+e\leq M(y_n)$. Then on the line segment $[y_o,y_n]$, we can choose z_n such that $M(z_n)=M(y_o)+e=a+e$. We know we can make this choice of z_n by the intermediate-value property of M, proven in the lemma.

If there were infinitely many y_n such that $M(y_o)+e\leq M(y_n)$, then the z_n chosen would form a sequence (z_n). This sequence (z_n) would of necessity converge to y_o. By our restriction on U, $\left[z|M(z)=a+e\right]$ is a closed set. We have chosen the z_n so that $M(z_n)=a+e$. If $(z_n)\rightarrow y_o$, it would mean that y_o is in the set $\left[z|M(z)=a+e\right]$. However, $M(y_o)=a$, and this then means there cannot be infinitely many y_n such that $M(y_o)+e\leq M(y_n)$. Hence, for all but finitely many y_n, we have $M(y_o)+e\geq M(y_n)$. Therefore, we have shown that $(M(y_n))\rightarrow M(y_o)$.

Corollary: Let $M(p-f)=a<\infty$. Then there is $x\in E(p,f)$ such that $M(p(x)-f(x))=a$.

Proof: $M(p-f)=a$ means for every n, $(p-f)(X)$ is not contained in $(a-\frac{1}{n})U$. So we can choose $x_n \in X$ such that $M(p(x_n)-f(x_n)) \geq a-\frac{1}{n}$. As X is compact, we may assume $(x_n) \to x_o$. $p-f$ continuous means $(p(x_n)-f(x_n)) \to p(x_o)-f(x_o)$. Then by Theorem 5 we have $M(p(x_o)-f(x_o))=a$.

If $p(x_o)-f(x_o)$ were not in $B(aU)$, we could extend the line segment joining o and $p(x_o)-f(x_o)$ slightly, and the extended segment would still be in aU. Formally, there would exist $t>o$ such that $(1+t)[p(x_o)-f(x_o)] \in aU$. Then we would have $p(x_o)-f(x_o)$ in $\frac{a}{1+t}U$; that is to say, $M(p(x_o)-f(x_o))<a$.

This establishes the corollary.

Theorem 6: p is a best approximation to f if and only if among all approximations to f, p is best on $E(p,f)$.

Proof: By p is best on $E(p,f)$ we mean, as usual, that given any approximation q, $M(q-f) \geq M(p-f)$ when the domain of $q-f$ and $p-f$ is restricted to $E(p,f) \subseteq X$. To denote this, we will write $M'(q-f)=d$, which will mean $M(q-f)=d$ when the domain of $q-f$ is restricted to $E(p,f)$. That is, $[q(x)-f(x)] \in dU$ for all $x \in E(p,f)$ and no smaller multiple of U will suffice.

Let $M(p-f)=a$. Suppose p is a best approximation on $E(p,f)$. Then $M'(p-f)=a$ (by the corollary above). If q is any other approximation, we have $M'(q-f) \geq a$. Clearly, $M(q-f) \geq M'(q-f)$. Since $M'(p-f)=M(p-f)$, we may assert $M(q-f) \geq M(p-f)$. Hence, p best on $E(p,f)$ means p is a best approximation.

Conversely, suppose p a best approximation, $M(p-f)=a$. If p is not best on $E(p,f)$, then there is an approximation q such that $M'(p-f)>M'(q-f)$. Let $M(q-f)=b$, and $M'(q-f)=a-c$, $c>o$.

Let $x\in E(p,f)$ and $t\in(o,1)$. Then, as $M(q(x)-f(x))\leq a-c$,

$$M(tq(x)+(1-t)p(x)-f(x))\leq tM(q(x)-f(x))+(1-t)M(p(x)-f(x))\leq t(a-c)+(1-t)a<a.$$

If $x\notin E(p,f)$, then $M(p(x)-f(x))<a$. Consequently, $M(tq(x)+(1-t)p(x)-f(x))\leq tb+(1-t)M(p(x)-f(x))$. As $M(p(x)-f(x))<a$, we can choose $t\in(o,1)$ so small that $tb+(1-t)M(p(x)-f(x))<a$. By the above, we see that for each $x\in X$, there is $t\in(o,1)$ such that $M(tq(x)+(1-t)p(x)-f(x))<a$.

Suppose we cannot choose a fixed t in $(o,1)$ that will work for all $x\in X$. Then for $t_n=\frac{1}{n}$, we have $x_n\in X$ such that $M(t_nq(x_n)+(1-t_n)p(x_n)-f(x_n))\geq a$. Since X is compact, we may assume $(x_n)\rightarrow x_o$, $x_o\in X$.

We show first that $x_o\notin E(p,f)$. If $x_o\in E(p,f)$, then $(M(q(x_n)-f(x_n)))\rightarrow M(q(x_o)-f(x_o))$. But, since $M(q(x_o)-f(x_o))\leq a-c$, we may choose N so that for $n\geq N$, $M(q(x_n)-f(x_n))\leq a-\frac{c}{2}$. This would give:

$$M(t_nq(x_n)+(1-t_n)p(x_n)-f(x_n))\leq t_n(a-\tfrac{c}{2})+(1-t_n)a=a-\tfrac{c}{2n}.$$ This

is contrary to our assumption that $M(t_nq(x_n)+(1-t_n)p(x_n)-f(x_n))\geq a$.

On the other hand, we see that:
$a \le M(t_n q(x_n)+(1-t_n)p(x_n)-f(x_n)) \le t_n b + (1-t_n)M(p(x_n)-f(x_n))$.
As $n \to \infty$, $(1-t_n)M(p(x_n)-f(x_n))$ goes to $M(p(x_o)-f(x_o))$.
This would mean $M(p(x_o)-f(x_o))=a$, or that $x_o \in E(p,f)$. But this cannot be, as shown above. So we are led finally to: there must be some $t \in (o,1)$ such that $M(tq(x)+(1-t)p(x)-f(x)) < a$ for all $x \in X$. This would say that $M(tq+(1-t)p-f) < a$ in contradiction to p being a best approximation. To avoid this contradiction, we must have that p is a best approximation on $E(p,f)$.

We turn our attention now to the convex set U. It is well known that for $U \subset E^n$, if $x \in B(U)$, there is a hyperplane H_x through x such that U lies on one side of H_x. Associated with this hyperplane H_x is a linear functional L_x such that either:

a) $L_x(z)=1$ for all $z \in H_x$, and if $u \in U$, then $L_x(u) \le 1$, or

b) $L_x(z)=o$ for all $z \in H_x$, and if $u \in U$, then $L_x(u) \le o$.

If the origin is an interior point of U, then b) never happens. Let Q be the set of all such "normalized" linear functionals L_x that are associated with supporting hyperplanes H_x of U.

We now define something which we will call an H-set, following the terminology of Collatz [2].

<u>Definition</u>: A set $[x_i] \subseteq X$, together with a correspond- set of functionals $[L_i] \subseteq Q$, is called an H-set for LF if there is no function q in LF such that $L_j(q(x_j)) > o$ for all $x_j \in [x_i]$.

As an example, let U be the closed unit disc in E^2. Let $LF = \left[(p(x), p'(x)) \middle| \; p(x) \text{ a polynomial, degree } p(x) \leq 1 \right]$, and $X=[o,1]$.

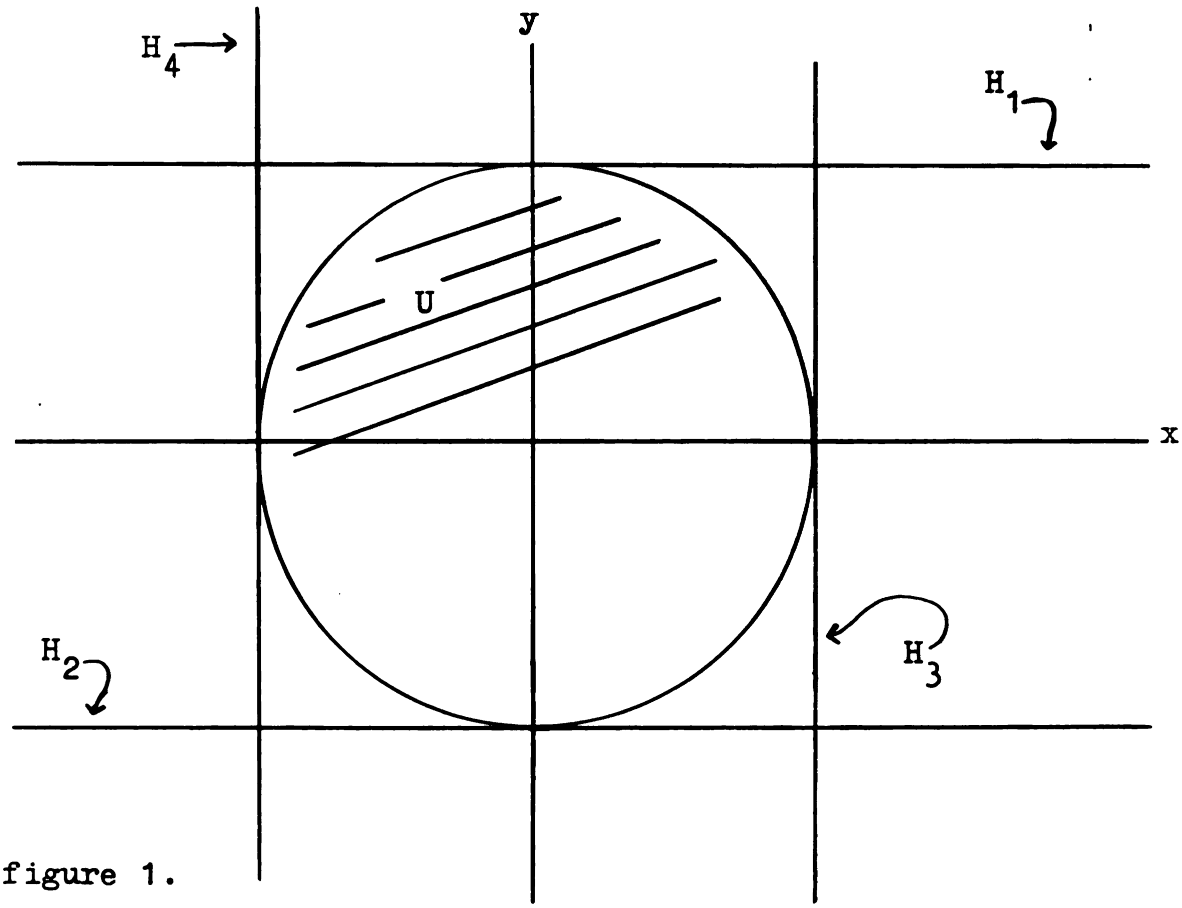

figure 1.

In the above figure:

H_1 a supporting hyperplane of U through (o,1)

H_2 a supporting hyperplane of U through (o,-1)

H_3 a supporting hyperplane of U through (1,o)

H_4 a supporting hyperplane of U through (-1,o)

The corresponding linear functionals in Q are: $L_1(x,y)=y$, $L_2(x,y)=-y$, $L_3(x,y)=x$, and $L_4(x,y)=-x$.

Since LF is the set of all functions $P(x):[o,1]\rightarrow E^2$ where $P(x)$ has the form $(ax+b,a)$, we may pick out H-sets with no difficulty.

For example, $\left[o,1\right]$ is an H-set if we correspond o with L_1, and 1 with L_2. If $Q\in LF$, then $Q(o)=(b,a)$ and $Q(1)=(a+b,a)$. $L_1(Q(o))=a$, $L_2(Q(1))=-a$, and hence we cannot have $L_1(Q(o))>o$ and $L_2(Q(1))>o$. A less trivial H-set is given by the correspondence: $o \leftrightarrow L_3$, $\frac{1}{2} \leftrightarrow L_1$, and $1 \leftrightarrow L_4$. If there were $P\in LF$, such that $L_3(P(o))>o$, $L_1(P(\frac{1}{2}))>o$, and $L_4(P(1))>o$, we would have for $P(x)=(p(x),p'(x))$ that $p(o)>o$, $p'(\frac{1}{2})>o$, and $p(1)<o$. But $p(x)$ is linear so there can be no such $P\in LF$. Finally, the correspondence $o \leftrightarrow L_3$, $\frac{1}{2} \leftrightarrow L_2$, and $1 \leftrightarrow L_4$ is not an H-set for LF, because the function $Q\in LF$, given by $Q(x)=(-x+\frac{1}{2},-1)$, does satisfy $L_3(Q(o))>o$, $L_2(Q(\frac{1}{2}))>o$ and $L_4(Q(1))>o$.

The following theorem is a natural extension of an important result in the theory of uniform approximation of real-valued functions. It gives some information about a lower bound for $M(p-f)$, where p is a best approximation to f.

<u>Theorem 7</u>: Let $\left[x_i\right]\subseteq X$, $\left[L_i\right]\subset Q$ be an H-set for LF, and let $p\in LF$ be a best approximation to f, $M(p-f)=a$. If $q\in LF$, and if $L_j(q(x_j)-f(x_j))\geq o$ for all $x_j\in\left[x_i\right]$, then for some $x_k\in\left[x_i\right]$, we have $L_k(q(x_k)-f(x_k))\leq a$.

Proof: We first note that if $y \in aU$, then $\frac{1}{a}y \in U$. Therefore, for all $L_i \in Q$ we have $L_i(\frac{1}{a}y) \leq 1$. Thus, $y \in aU$ means $L_i(y) \leq a$ for all $L_i \in Q$.

Having noted the above, we now suppose the conclusion of the theorem false. Then, for all $x_j \in \left[x_i\right]$, we have $L_j(q(x_j)-f(x_j)) > a$. $M(p-f)=a$ implies $L_j(p(x_j)-f(x_j)) \leq a$ by the observation above. This gives us $L_j(p(x_j)-f(x_j))-L_j(p(x_j)-f(x_j)) > o$. Using the linearity of L_j, we obtain $L_j(q(x_j)-p(x_j)) > o$ for all $x_j \in \left[x_i\right]$. But this contradicts our assumption that $\left[x_i\right], \left[L_i\right]$ is an H-set.

To obtain our next characterization theorem, we will ask that the convex set U have the following property: through each $y \in B(U)$ there is one and only one supporting hyperplane (essentially, that B(U) have no corners). If p(x) is an approximation to $f \in C(X)$, then there is a natural correspondence between $x_i \in E(p,f)$ and some $L_i \in Q$. If $M(p-f)=a$, then $x_i \in E(p,f)$ means that $\frac{1}{a}[p(x_i)-f(x_i)] \in B(U)$. Through $\frac{1}{a}[p(x_i)-f(x_i)]$, there is a unique supporting hyperplane, say H_i, with an associated linear functional $L_i \in Q$. The correspondence $x_i \leftrightarrow L_i$ is the one to which we refer in c) and d) of Theorem 8.

Theorem 8: Suppose U is compact, the origin is not in B(U), and that through each point of B(U) passes one and only one supporting hyperplane of U. Then, the following are equivalent:

a) p is a best approximation to f

b) p is a best approximation to f on E(p,f)

c) E(p,f) is an H-set

d) For each $q\in LF$, there is $x_j\in E(p,f)$ such that $L_j(q(x_j)-f(x_j))\geq M(p-f)$.

Proof: Before beginning the proof, we will make several observations and establish two rather obvious lemmas. First, we observe that for each $y\in E^n$, there is $a>o$ such that $by\in U$ for $a\geq b\geq o$. This is clear since the origin is in the interior of U. We should also note that that $[y\in U|\, M(y)=1]$ is precisely B(U) when the origin is in the interior of U. Consequently, $[y\in U|\, M(y)=1]$ is closed.

Lemma 1: $u\in U$ if and only if $L_i(u)\leq 1$ for all $L_i\in Q$.

Proof: By construction of Q, if $u\in U$, then $L_i(u)\leq 1$ for all $L_i\in Q$.

Suppose $L_i(y)\leq 1$ for all $L_i\in Q$. As remarked, there is $a>o$, such that $by\in U$ for all b such that $o\leq b\leq a$. Choose the largest such a, say a'. Then $a'y\in U$, but $(a'+t)y\notin U$ for all $t>o$. Clearly, there is such a largest value a', for U is compact, and hence, cannot contain a half-line emanating from the origin and going through y.

By our choice of a', we have that $a'y\in B(U)$. Let $L_i\in Q$ be the functional associated with the hyperplane through a'y. Then $L_i(a'y)=1$, or $a'L_i(y)=1$. But $L_i(y)\leq 1$, so that $a'\leq 1$. Since $by\in U$ for all b such that $o\leq b\leq a'$, and since $1\leq a'$, we have $y\in U$.

Every linear functional on E^n may be represented as an inner-product. Thus, if L is a functional, there is a fixed vector $k \in E^n$ such that $L(x)=(k,x)$. As usual, $(k,x)=k_1x_1+\cdots+k_nx_n$. We may define $\|L\|$ by $\|L\|=\sqrt{k_1^2+\cdots+k_n^2}$. Then with the Euclidean norm on E^n, we have that: $|L(x)| = |(k,x)| \leq \|L\|\|x\| = \|k\|\|x\|$.

<u>Lemma 2</u>: There is $T>o$ such that $\|L_i\| \leq T$ for all $L_i \in Q$.

<u>Proof</u>: This follows immediately, since the origin is an interior point of U. Choose $r>o$ such that an n-ball centered at the origin, with radius r, is contained in the interior of U. Denote this sphere by $S(o,r)$. Let $z \in B(U)$; and let $L \in Q$ be the functional associated with the unique supporting hyperplane through z. Then $L(z)=1$, and if $u \in U$, $L(u) \leq 1$. Suppose L is given by $L(y)=(k,y)$. The vector $\frac{r}{\|k\|}k$ is in $S(o,r) \subset U$; consequently, $L(\frac{r}{\|k\|}k) \leq 1$. Thus, $(k,\frac{r}{\|k\|}k)=r\|k\| \leq 1$; or $\|k\| \leq \frac{1}{r}$. Choosing $T=\frac{1}{r}$ will establish the lemma.

<u>Lemma 3</u>: Let $x \in B(U)$, H the unique supporting hyperplane of U through x, and let $L \in Q$ be the linear functional associated with H. Suppose $y \in E^n$, and $L(y)<o$. Then, there is $a>o$ such that the open segment $(x,x+ay)$ is contained in $U-B(U)$.

<u>Proof</u>: Let $L(y)=-b$, $b>o$; and, suppose the assertion is false. Then, for all n, $x+\frac{1}{n}y$ is not in $U-B(U)$. $L(x)=1$, so $L(x+\frac{1}{n}y)=L(x)+\frac{1}{n}L(y)=1-\frac{b}{n}$. As $x+\frac{1}{n}y$ is not in $U-B(U)$,

lemma 1 gives us that there is $L_n \in Q$, such that $L_n(x+\frac{1}{n}y) \geq 1$. Hence, we have $L_n(x+\frac{1}{n}y)-L(x+\frac{1}{n}y) \geq \frac{b}{n}$.

Since $x \in U$, we may assert $L_n(x)=1-d_n$, $d_n \geq o$. Using $L_n(x+\frac{1}{n}y)-L(x+\frac{1}{n}y) \geq \frac{b}{n}$, we obtain:

$L_n(x)+\frac{1}{n}L_n(y)-L(x)-\frac{1}{n}L(y)=(1-d_n)+\frac{1}{n}L_n(y)-(1-\frac{b}{n}) \geq \frac{b}{n}$. Thus, we have $\frac{1}{n}L_n(y) \geq d_n$. Suppose there is $d>o$ such that $d_n \geq d$, for all n sufficiently large. Then, we would have $L_n(y) \geq nd_n \geq nd$, for large n. But, by lemma 2, $|L_n(y)| \leq \|L_n\| \|y\| \leq K\|y\|$; so, we cannot have $L_n(y) \geq nd$ for large n. Thus, we may assume that $(d_n) \rightarrow o$. Because $L_n(x)=1-d_n$, $(L_n(x)) \rightarrow 1=L(x)$.

As before, for every m there is $k_m \in E^n$ such that $L_m(z)=(k_m,z)$, for all $z \in E^n$. By lemma 2, there is $K>o$ such that $\|k_m\| \leq K$. By passing to subsequences, if necessary, we may assume that the sequence (L_m) converges to a linear functional L'. As noted above, $L'(x)=1$. Furthermore, since $L_n \in Q$, $L_n(u) \leq 1$ for all $u \in U$. Thus, $H'=\left[z \,|\, L'(z)=1\right]$ is a supporting hyperplane of U, through $x \in B(U)$. By the supposed uniqueness of supporting hyperplanes, it must be that H'=H (H the support plane determined by L). Then, as $\left[z \,|\, L(x)=1\right]=\left[z \,|\, L'(z)=1\right]$, it must be that L=L'.

To gain our contradiction, we need only recall that $L_n(y) \geq nd_n \geq o$, and hence $L'(y) \geq o$. But, we had supposed that $L(y)=-b<o$. Hence, for some N, if $n \geq N$, $x+\frac{1}{n}y$ is in U-B(U). Therefore, there is $a>o$ such that $(x,x+ay) \subset U-B(U)$.

Now we are in a position to prove Theorem 8.

a)$\Rightarrow$b): we already have this from Theorem 6.

b)$\Rightarrow$c): let p be a best approximation to f on E(p,f), with M(p-f)=a. If E(p,f) were not an H-set, then there would be $q'\in LF$ such that $L_i(q'(x_i))>o$, for all $x_i\in E(p,f)$. Let $q=-q'$; then $L_i(q(x_i))<o$ for all $x_i\in E(p,f)$.

Since $p(x_i)-f(x_i)$ is in B(aU), we may apply Lemma 3. Thus, there is $a_i>o$ such that $p(x_i)+bq(x_i)-f(x_i)$ is in aU-B(aU), for all $b\in(o,a_i)$. Thus, we have $M(p(x_i)+bq(x_i)-f(x_i))<a$ when $b\in(o,a_i)$. Suppose there is no fixed a_j that will work for all $x_i\in E(p,f)$. That is, there is no $a_j>o$ such that for $b\in(o,a_j)$, all $x_i\in E(p,f)$, $p(x_i)+bq(x_i)-f(x_i)$ in aU-B(aU).

This being the case, for each n there is $x_n\in E(p,f)$, such that $p(x_n)+\frac{1}{n}q(x_n)-f(x_n)$ is not in aU-B(aU). It is evident that E(p,f) is closed, and hence compact, since $E(p,f)\subset X$ compact. We may assume $(x_n)\longrightarrow x_o$, $x_o\in E(p,f)$. Applying Lemma 3 to x_o, we have: there exists N such that for b in $(o,\frac{1}{N})$, $p(x_o)+bq(x_o)-f(x_o)$ is in aU-B(aU). In particular, $p(x_o)+\frac{1}{2N}q(x_o)-f(x_o)$ is in aU-B(aU). Hence, $M(p(x_o)+\frac{1}{2N}q(x_o)-f(x_o))<a$.

By the way the x_n were chosen, $p(x_n)+\frac{1}{n}q(x_n)-f(x_n)$ is not in aU-B(aU). Then, since aU-B(aU) is convex, $p(x_n)+bq(x_n)-f(x_n)$ cannot be in aU-B(aU) when $b\geq\frac{1}{n}$. For $n\geq 2N$, $\frac{1}{n}\leq\frac{1}{2N}$. Thus, $p(x_n)+\frac{1}{2N}q(x_n)-f(x_n)$ is not in aU-B(aU) when $n\geq 2N$. However, $p(x_o)+\frac{1}{2N}q(x_o)-f(x_o)$ is in aU-B(aU), and aU-B(aU) is an open set. Furthermore, since $(x_n) \rightarrow x_o$, we must have for infinitely many i, that $p(x_i)+\frac{1}{2N}q(x_i)-f(x_i)$ is in aU-B(aU). This is a contradiction; therefore, there is some $a_o>o$, such that $b \in (o,a_o)$ implies p(x)+bq(x)-f(x) is in aU-b(aU).

The image of E(p,f) under (p+bq-f) is compact, and is contained in aU-B(au). Since aU-B(aU) is open, we can show, as in Theorem 2, that M'(p+bq-f)<a. But this cannot be, since p is a best approximation on E(p,f). Thus, it must be that E(p,f) is an H-set.

c)⟹d): suppose there were $q \in LF$ such that $L_i(q(x_i)-f(x_i))<a$, for all $x_i \in E(p,f)$. Recall that if $x_i \in E(p,f)$, then $L_i(p(x_i)-f(x_i))=a$. This existence of $q \in LF$, as postulated above, would lead us to: $L_i(p(x_i)-f(x_i))-L_i(q(x_i)-f(x_i))>o$. This would mean $L_i(p(x_i)-q(x_i))>o$, for all $x_i \in E(p,f)$, contrary to c).

d)⟹a): this follows immediately. If $q \in LF$, and if $L_j(q(x_j)-f(x_j)) \geq a$, then $M(q(x_j)-f(x_j)) \geq a$. Since $M(q-f) \geq M(q(x_j)-f(x_j))$, $M(q-f) \geq a=M(p-f)$. Hence, p is a best approximation to f.

In the following group of theorems, we will consider some of the implications of non-uniqueness of best approximations. We can prove these theorems under weaker hypotheses than those that were needed for Theorem 8. That is, we need not require U to be compact, nor do we need unique supporting hyperplanes at each point of B(U). Specifically, we shall require that U be closed, convex, and have an interior point. Further, let $\left[y\in U \mid M(y)=1 \right]$ be a closed set.

Theorem 9: Let U be as above. If p and q are best approximations to f, then $E(p,f)\cap E(q,f)\neq \emptyset$.

Proof: Let h(x) be any convex combination of p(x) and q(x). By Theorem 1, h(x) is a best approximation to f(x). By Theorem 2, $E(h,f)\neq \emptyset$. From Theorem 3, $E(h,f)\subseteq E(p,f)\cap E(q,f)$, so the conclusion follows.

Theorem 10: If p and q are both best approximations to f, then p and q are best approximations on $E(p,f)\cap E(q,f)$.

Proof: Since $E(p,f)\cap E(q,f)\neq \emptyset$, the result is not meaningless. Let $M(p-f)=M(q-f)=a$. Suppose there is $h\in LF$, such that $M(h(x)-f(x))\leq b<a$ for all $x\in E(p,f)\cap E(q,f)$. Then, for $t\in(o,1)$, $M(th(x)+(1-t)q(x)-f(x))<a$, when $x\in E(p,f)\cap E(q,f)$. As in previous arguments, we suppose there is no $t\in(o,1)$ that will work for all of $E(p,f)$. This gives us a sequence $(x_n)\subseteq E(p,f)$, such that $M(\frac{1}{n}h(x_n)+(1-\frac{1}{n})q(x_n)-f(x_n))\geq a$. We can suppose that $(x_n)\rightarrow x_o$, $x_o\in E(p,f)$. Also, since $M(\frac{1}{n}h(x_n)+(1-\frac{1}{n})q(x_n)-f(x_n))$ goes to $M(q(x_o)-f(x_o))$ as $n\rightarrow \infty$, we see that $x_o\in E(q,f)$.

Since $M(h(x)-f(x))\leq b<a$ for $x\in E(p,f)\cap E(q,f)$, we have $M(h(x_o)-f(x_o))\leq b$. Then, for n large enough, $M(h(x_n)-f(x_n))<a$. Clearly, $M(q(x_n)-f(x_n))\leq a$ for all n. Using the subadditive property of M, we have for n large enough: $M(\frac{1}{n}h(x_n)+(1-\frac{1}{n})q(x_n)-f(x_n))<\frac{1}{n}a+(1-\frac{1}{n})a=a$. This is contrary to our assumption about the points x_n. Thus, there must be $t\in(o,1)$ such that $M(th(x)+(1-t)q(x)-f(x))<a$, for all $x\in E(p,f)$. But, by Theorem 6, p is a best approximation if

and only if p is a best approximation on E(p,f). Hence, we cannot have $t\in(o,1)$ such that $M(th(x)+(1-t)q(x)-f(x))<a$, for all $x\in E(p,f)$.

Thus, p must be a best approximation on $E(p,f)\cap E(q,f)$. The same argument will work for q; so the conclusion of Theorem 10 follows.

<u>Theorem 11</u>: Let p and q both be best approximations to f. Let $h=tp+(1-t)q$, $o<t<1$. Suppose $M(p-f)=M(q-f)=a$. If $x\in E(h,f)$, then $p(x)-f(x)$ and $q(x)-f(x)$ are on a common support plane of aU.

<u>Proof</u>: By Theorem 3, $E(h,f)\subseteq E(p,f)\cap E(q,f)$. Hence, if $x\in E(h,f)$, then $p(x)-f(x)$ and $q(x)-f(x)$ are both in B(aU). If $p(x)-f(x)=q(x)-f(x)$, we are finished.

Suppose then, that $p(x)-f(x)\neq q(x)-f(x)$. Let L be the line segment joining $p(x)-f(x)$ and $q(x)-f(x)$. Then $h(x)-f(x)$ is in the interior of L. Let H be a support plane of aU, through $h(x)-f(x)$. We would like to show that H contains L, and hence, contains $p(x)-f(x)$ and $q(x)-f(x)$.

There are only two possibilities: either $L\cap H=L$, or $L\cap H=h(x)-f(x)$. The latter cannot happen, for it would mean $p(x)-f(x)$ and $q(x)-f(x)$ are on different sides of H. As H is a support plane of aU, $p(x)-f(x)$ and $q(x)-f(x)$ must be on the same side of H. Hence, $L\cap H=L$, which proves Theorem 11.

<u>Corollary</u>: Suppose LF has the property: given any k-1 points $x_i\in X$ and k-1 points $f_i\in E^n$, there is $h\in LF$ such that $h(x_i)=f_i$, $i=1,\cdots k-1$.

Let p and q be best approximations to f, $M(p-f)=M(q-f)=a$. Then, there are at least k points x in $E(p,f)\cap E(q,f)$ such that $p(x)-f(x)$ and $q(x)-f(x)$ lie on the same support plane of aU.

<u>Proof</u>: Let h be any convex combination of p and q. Then h is a best approximation on $E(h,f)$. From the interpolatory property of LF, it follows that $E(h,f)$ has at least k points. The corollary follows then, from Theorem 11.

We now prove a theorem that will guarantee the existence of a best approximation in most of the problems we consider. The result will be easy to obtain, for we will ask that the set LF of approximating functions be a finite dimensional subspace of $C(X)$. Also let X be a metric space.

<u>Theorem 12</u>: Let LF be a finite dimensional subspace of $C(X)$, and let U be compact. For each $f\in C(X)$, if f has an approximation, then f has a best approximation.

<u>Proof</u>: For $y\in E^n$, let $\|y\|$ be the Euclidean norm. For $f\in C(X)$, let $\|f\|=\sup_{x\in X}\|f(x)\|$. Under this norm, $C(X)$ is a Banach space. Furthermore, $\|f\|\leq A$ if and only if $f(X)\subseteq S(o,A)$. Let $a=\inf_{p\in LF} M(p-f)$, $a<\infty$. For each n, we can choose $p_n\in LF$ such that $M(p_n-f)\leq a+\frac{1}{n}$. For all n, $(p_n-f)(X)\subseteq(a+1)U$. Thus, $p_n(X)\subseteq\left[(a+1)U+f(X)\right]$.

As U and f(x) are compact, we can choose $K>o$, such that $[(a+1)U+f(X)]\subseteq S(o,K)$. Hence, $\|p_n\|\leq K$ for all n.

Since LF is finite dimensional, a sphere of radius K in LF is compact. We may assume then, that $(p_n)\longrightarrow p_o$, $p_o\in LF$.

Under the norm of $C(X)$, $(p_n) \rightarrow p_o$ means $(p_n(x)) \rightarrow p_o(x)$ for each $x \in X$. This will allow us to show that p_o is an approximation, and the $M(p_o - f) = a$.

Let $x \in X$, since $[p_n(x) - f(x)] \in (a+1)U$; and since U is closed, it must be that $[p_o(x) - f(x)] \in (a+1)U$. By Theorem 5, we can assert $M(p_n(x) - f(x))$ goes to $M(p_o(x) - f(x))$ as $n \rightarrow \infty$. Since $M(p_n - f) \leq a - \frac{1}{n}$, $M(p_o - f) \leq a$. Hence, $M(p_o - f) \leq M(q - f)$ for all $q \in LF$.

Theorems 9, 10, 11, and the corollary to Theorem 11, find their greatest application in resolving questions of uniqueness of best approximations. Theorems 6 and 8 are theorems that characterize a best approximation, and are typically most useful when X is a finite point set. Theorem 7 finds its utility in measuring how close a given approximation is to a best approximation on an H-set.

CHAPTER II

APPLICATIONS

In this chapter, we will apply some of the results of Chapter I to several specific problems. As a first application, consider the problem below:

(A) Let $f(x)$ have a continuous k-th derivative on $[o,1]$. Among all polynomials $p(x)=a_n x^n+\cdots+a_1 x+a_o$, of degree n or less, find the one that minimizes:

$$\max_{x\in[o,1]} \sqrt{[p(x)-f(x)]^2+[p'(x)-f'(x)]^2+\cdots+[p^{(k)}(x)-f^{(k)}(x)]^2} .$$

This problem can be restated as:

(B) Let $F(x)=(f(x),f'(x),\cdots f^{(k)}(x))$, and let $X=[o,1]$. Let U be the closed unit sphere in E^{k+1}. Let

$$LF=\left[P(x)\middle|\ P(x)=(p(x),p'(x),\cdots,p^{(k)}(x)),\ p(x)\text{ a polynomial, degree } p(x)\leq n\right].$$

Find $P\in LF$ that is a best approximation to F.

Clearly, problem (A) and problem (B) are equivalent problems.

<u>Theorem</u>: Problem (B) has one and only one solution.

<u>Proof</u>: Since U is compact, and since LF is a finite dimensional subspace of $C(X)$, Theorem 12 will guarantee at least one solution.

Suppose there were two solutions, say P and Q, to (B). By Theorem 1, $H=\frac{1}{2}P+\frac{1}{2}Q$ is also a best approximation to F.

Since U is strictly convex, we may apply Theorem 4. That is, if $x' \in E(H,F)$, then $P(x')=Q(x')$.

We will suppose for the present that $n \geq k$. Using the generalized Vandermonde matrix, it is easy to show that we can find $S \in LF$ such that $S(x_i)=F(x_i)$, for any $\left[\frac{n+1}{k+1}\right]$ points x_i in [o,1]. Here, [r] means "the greatest integer in r."

Recalling Theorem 5, H is a best approximation to F if and only if H is a best approximation on E(H,F). In light of the above, E(H,F) must have at least $\left[\frac{n+1}{k+1}\right]+1$ points, for we can fit $\left[\frac{n+1}{k+1}\right]$ points exactly. Hence, $P(x)=Q(x)$ on $\left[\frac{n+1}{k+1}\right]+1$ points. Consider $P(x)-Q(x)=(p(x)-q(x),p'(x)-q'(x),\cdots,p^{(k)}(x)-q^{(k)}(x))$. If $P(x')=Q(x')$, then x' is a zero of multiplicity k+1 of p-q.

We note: $(k+1)\left[\frac{n+1}{k+1}\right] \geq n-k+1$. Counting multiplicity, p-q has $(k+1)\left(\left[\frac{n+1}{k+1}\right]+1\right)$ zeroes. But, $(k+1)\left(\left[\frac{n+1}{k+1}\right]+1\right) \geq (n-k+1)+(k+1)=n+2$. Hence, counting multiplicity, p-q has at least n+2 zeroes. As p,q have degree n, it must be that $p \equiv q$; and thus $P \equiv Q$.

If $n<k$, then $\left[\frac{n+1}{k+1}\right]=o$. However, by Theorem 2, $E(H,F) \neq \emptyset$, so let $x' \in E(H,F)$. Then, as above, $P(x')-Q(x')=o$. Since $k>n$, x' is a zero of p-q, of multiplicity at least n+1. So again, $P \equiv Q$. This establishes the uniqueness claimed in the theorem.

Using Theorem 8, we can characterize the best approximation in (B) as follows:

<u>Theorem</u>: P is a best approximation to F if and only if the system, $(P(x_i)-F(x_i),Q(x_i))>o$ $x_i \in E(P,F)$, is inconsistent for all $Q \in LF$.

<u>Proof</u>: For $y \in E^n$, let $\|y\|$ be the Euclidean norm. For $k \in E^n$, let $H=\left[z \,\middle|\, (k,z)=\|k\|^2\right]$. If $z \in S(o,\|k\|)$, then $|(k,z)| \leq \|k\|\,\|z\| \leq \|k\|^2$. Thus H is a hyperplane through k, such that $S(o,\|k\|)$ is on one side of H. The linear functional, associated with H, is given by $L(z)=(k,z)$.

In problem (B), U is the unit n-ball in E^n, $U=S(o,1)$. In particular, if $[P(x_i)-F(x_i)] \in B(aU)$, then $\|P(x_i)-F(x_i)\|=a$. In view of the above remarks, we may associate a linear functional L_i with x_i, where L_i is given by: $L_i(z)=\frac{1}{a}(P(x_i)-F(x_i),z)$. Then, $H_i=\left[z \,\middle|\, L_i(z)=1\right]$ is a hyperplane through $\frac{1}{a}[P(x_i)-F(x_i)]$, with U on one side of H_i. Here, of course, $\frac{1}{a}[P(x_i)-F(x_i)] \in B(U)$. Suppose now $M(P-F)=a$.

Since U obviously satisfies the conditions of Theorem 8, P is a best approximation to F if and only if E(P,F) is an H-set. We have shown above how to correspond a linear functional L_i to $x_i \in E(P,F)$. Given $Q \in LF$, $L_i(Q(x_i))= \frac{1}{a}(P(x_i)-F(x_i),Q(x_i))$. Hence, E(P,F) is an H-set if and only if the system $\frac{1}{a}(P(x_i)-F(x_i),Q(x_i))>o$, $x_i \in E(P,F)$, is inconsistent for all $Q \in LF$. Since $a>o$, the theorem is proved.

The theorem which follows is given by Meinardus and Schwedt [3]. It is the basic inclusion theorem in the theory of uniform approximation of complex functions. Although the theorem is not difficult to prove, the proof usually given sheds little light on the theorem. After some preliminary remarks, we will see that this result follows immediately from Theorem 7.

Let $C(X)$ be the set of continuous (complex) functions on X. For $f \in C(X)$, define $\|f\| = \sup_{x \in X} |f(x)|$. Let $LF \subset C(X)$, and suppose $a = \inf_{p \in LF} \|p-f\|$.

<u>Theorem</u>: Let $p_o \in LF$, and $e = p_o - f$. Let V be a fixed subset of X with the properties:

a) $|e(x)| > o$ for all $x \in X$

b) There is no $q \in LF$ such that $\mathrm{Re}(\bar{e}q) > o$ for all $x \in V$.

Then: $\inf_{x \in V} |e(x)| \leq a \leq \|e\|$.

<u>Proof</u>: It is clear that $a \leq \|e\|$. We now note the obvious: we may regard f as being given by $f(x) = f_1(x) + if_2(x) = (f_1(x), f_2(x))$, where f_1 and f_2 are real functions. Choose U as the unit disc in E^2, then $\|f\| = M(f) = M((f_1, f_2))$.

Let $e = p_o - f = w_1 + iw_2$, and for $q \in LF$, let $q = v_1 + iv_2$.

If $\mathrm{Re}[\overline{e(x)}q(x)] > o$, then $\mathrm{Re}[(w_1(x) - iw_2(x)) \cdot (v_1(x) + iv_2(x))] > o$. This is the same as $w_1(x)v_1(x) + w_2(x)v_2(x) > 0$, or $(w_1(x), w_2(x)) \cdot (v_1(x), v_2(x)) > o$. Thus, by requiring that there be no $q \in LF$ such that $\mathrm{Re}[\overline{e(x)}q(x)] > o$ for all $x \in V$, it must be that $(w_1(x), w_2(x)) \cdot (v_1(x), v_2(x)) > o$ cannot hold for all $x \in V$, $(v_1, v_2) \in LF$.

For $x_i \in V$, define L_i to be the linear functional given by the inner product: $L_i(z) = \frac{1}{|e(x_i)|}(e(x_i), z)$. Then, $\left[z \middle| L_i(z)=1\right]$ is a support plane of U, through $\frac{e(x_i)}{|e(x_i)|} \in B(U)$. Furthermore, by the remarks above, $L_i(q(x_i)) > o$ for all $x_i \in V$, does not hold for any $q \in LF$. Consequently, V is an H-set. As $L_i(e(x_i)) = |e(x_i)|$, and as $|e(x_i)| > o$ by a), we have from Theorem 7 that $\inf\left[L_i(e(x_i))\right] \leq a$. Thus, $\inf_{x_i \in V} |e(x_i)| \leq a$.

By letting C(X) be the set of real-valued continuous functions on X, and by setting $U=[-1,1]$, the results of Chapter I can be made to apply to the usual problems of uniform approximation. In this case, B(U) consists of the two real numbers 1 and -1. The linear functionals corresponding to B(U) are merely $L(x)=x$ and $L'(x)=-x$. Hence, if $V \subseteq X$ is an H-set, we can write V as $V=P \cup N$, where V has the property: there is no $q \in LF$ such that $q(x) > o$ for $x \in P$ and $q(x) < o$ for $x \in N$.

What follows is a new proof of a theorem proven by Collatz [4].

<u>Theorem</u>: Let f(x,y) be a real function defined on $W \subset E^2$. Let W be compact, and strictly convex. Further, suppose f(x,y) has continuous first partials in the interior of W. Then, f(x,y) has one and only one best uniform approximation of the form bx+cy+d.

<u>Proof</u>: That there is at least one best approximation is certain by Theorem 12. Let p(x,y)=bx+cy+d be one such

best approximation, and suppose $M(p-f)=a$.

We first obtain a lower bound on the number of points in $E(p,f)$. We will prove that $E(p,f)$ has at least three points. Theorem 8 is applicable to this problem, so we know that $E(p,f)$ must be an H-set. Suppose z_1 and z_2 are given, z_1 and z_2 in W. We will exhibit $q \in LF$ such that $q(z_1)>o$ and $q(z_2)<o$.

Let $\|y\|$ be the Euclidean norm, for $y \in E^2$. Consider first the case $\|z_1\| \neq \|z_2\|$. To begin; suppose that $\|z_1\|^2 > \|z_1\| \|z_2\|+e$, $e>o$. Let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$. Choose $q(x,y) \in LF$ by: $q(x,y)=x_1x+y_1y-\|z_1\|^2+e$. Then, $q(z_1)=\|z_1\|^2-\|z_1\|^2+e=e>o$, and $q(z_2)=(z_1,z_2)-\|z_1\|^2+e \leq |(z_1,z_2)| - \|z_1\|^2+e \leq \|z_1\| \| z_2\| - \|z_1\|^2+e<o$.

To handle the case $\|z_2\|^2 > \|z_1\| \| z_2\| +e$, reverse the roles of z_1 and z_2. This will give $q \in LF$ such that $q(z_2)>o$, and $q(z_1)<o$. Choosing $h=-q$ gives us $h(z_1)>o$, and $h(z_2)<o$.

Suppose $\|z_1\|=\|z_2\|$. If z_1 and z_2 are linearly independent, we can choose q LF by solving:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

. Then $q(x,y)=mx+ny$ has the desired property. Finally, if $\|z_1\|=\|z_2\|$, and z_1, z_2 are not linearly independent, then $z_1=-z_2$. Choosing $q \in LF$ by: $q(x,y)=x_1x+y_1y$, gives us $q(z_1)>o$ and $q(z_2)<o$. From these observations, if $E(p,f)$ is to be an H-set, it must be that $E(p,f)$ contains at least three points.

We now suppose there are two best approximations to f, say p and q. By Theorem 1, any convex combination h, of p and q is also a best approximation. From the above remarks, E(h,f) contains at least three points. Since U=[-1,1] is strictly convex, Theorem 4 tells us that $p(z_i)=q(z_i)$ for for all $z_i \in E(h,f)$. Thus, if $p(x,y)=bx+cy+r$, and if $q(x,y)=mx+ny+s$; then for $z_i \in E(h,f)$, $z_i=(x_i,y_i)$, we have $bx_i+cy_i+r=mx_i+ny_i+s$. Hence, $(b-m)x_i+(c-n)y_i+(r-s)=o$, for all $(x_i,y_i) \in E(h,f)$. This is sufficient to guarantee that all the $z_i \in E(h,f)$ are collinear in E^2. By hypothesis, W is strictly convex. Since the extreme points of h-f are collinear, and since there are at least three such extreme points, it must be that for some $z_k \in E(h,f)$, z_k is in the interior of W. As $z_k \in E(p,f) \cap E(q,f)$, again by Theorem 4, it must be that $|p(z_k)-f(z_k)|=|q(z_k)-f(z_k)|=a$.

Since p-f and q-f take their maximum at z_k, and since z_k is in the interior of W, it must be that all partials of p-f and q-f vanish at z_k. $p(x,y)-f(x,y)=bx+cy+r-f(x,y)$, and $q(x,y)-f(x,y)=mx+ny+s-f(x,y)$. Thus, we have: $m=f_x(z_k)=b$, and $n=f_y(z_k)=c$. As $p(z_k)=q(z_k)$, we must also have r=s. Thus, p(z)=q(z) for all z, proving uniqueness.

The above theorem is rather remarkable in that a similar theorem cannot be proved for quadratic polynomials. Rivlin and Shapiro showed [5], that given any plane region W, there is an C^∞ function on W which admits infinitely many best approximations. They also extended Collatz's result to C^1 functions defined on E^n.

The theorem below was proven by Moursund [6]. In the following, $\|f\|$ is the uniform norm.

Theorem: Let f(x) be twice differentiable on [o,1]. Among all polynomials $a_n x^n + \cdots + a_1 x + a_o$, of degree n or less, let p(x) be one that minimizes: $\max\left[\|p-f\|, \|p'-f'\|\right]$. If q(x) is another such polynomial, then $q'=p'$.

Proof: We reformulate the problem as follows: let $LF=\left[P(x) \mid P(x)=(p(x),p'(x)),\ \text{degree } p(x)\le n\right]$, and let $F(x)=(f(x),f'(x))$. Define $U=\left[(x,y) \mid -1\le x\le 1,\ -1\le y\le 1\right]$. If $P\in LF$ is such that $M(P-F)=a\le M(Q-F)$ for all $Q\in LF$, then clearly p(x) is a polynomial that minimizes: $\max\left[\|p-f\|, \|p'-f'\|\right]$.

We note that Theorem 12 guarantees at least one best approximation. As in previous problems, we first get a lower bound on the number of extreme points that a best approximation must have. For this, we again turn to the generalized Vandermonde matrix. Given any $\left[\frac{n+1}{2}\right]$ points x_i in [o,1], we can find p(x) of degree n or less, such that $p(x_i)=f(x_i)$ and $p'(x_i)=f'(x_i)$. Thus, if P(x) is a best approximation, we see by Theorem 6 that E(P,F) contains at least $\left[\frac{n+1}{2}\right]+1$ points.

We first prove the theorem in the special case where n is even and E(P,F) contains exactly $\left[\frac{n+1}{2}\right]+1$ points. The picture below will be helpful:

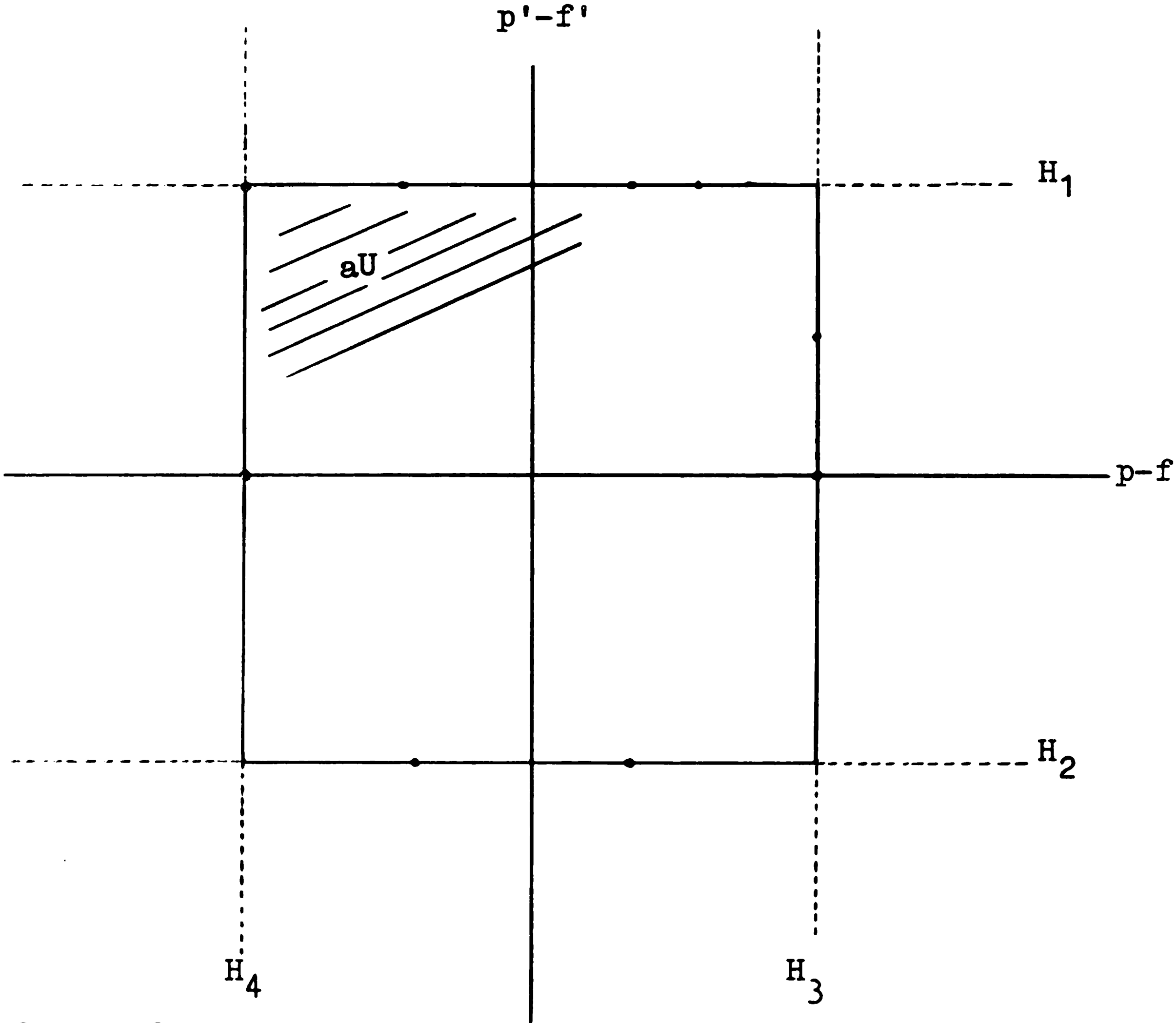

figure 2.

The heavy dots in B(aU) represent the images of extreme points under the mapping P-F. The lines H_1, H_2, H_3 and H_4 are support planes of aU.

If $P(x_i)-F(x_i)$ is in B(aU), it must be that either $|p(x_i)-f(x_i)|=a$, or $|p'(x_i)-f'(x_i)|=a$. If $x_i \neq o$, $x_i \neq 1$, then $|p(x_i)-f(x_i)|=a$ means $p'(x_i)-f(x_i)=o$, for $|p(x)-f(x)|\leq a$ when $x\in[o,1]$. Similarly, if $|p'(x_i)-f'(x_i)|=a$, and if $x_i\neq o$, $x_i\neq 1$, then $p''(x_i)-f''(x_i)=o$. From these observations, we have that only o and 1 can be mapped into the corners of aU, and that all other extreme points are either zeroes of $p'-f'$, or of $p''-f''$.

To consider the special case where n is even and $E(P,F)$ has exactly $\left[\frac{n+1}{2}\right]+1$ points, let $E(P,F)=[x_1,x_2,..,x_k]$ and $k=\left[\frac{n+1}{2}\right]+1$.

We observe first that $|p'(x_i)-f'(x_i)|=a$, for $i=1,2,\cdots,k$. If this were not the case, suppose, for instance, that $|p'(x_1)-f'(x_1)|<a$. Choose $q(x)=a_nx^n+\cdots+a_1x+a_o$ by the following:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ 0 & 1 & 2x_2 & \cdots & nx_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^n \\ 0 & 1 & 2x_k & \cdots & nx_k^{n-1} \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \quad \begin{bmatrix} f(x_1) \\ f(x_2) \\ f'(x_2) \\ \vdots \\ f(x_k) \\ f'(x_k) \end{bmatrix}$$

Then, $q(x_i)-f(x_i)=o$, $i=1,2,\cdots,k$. Also, $q'(x_i)-f'(x_i)=o$, for $i=2,3,\cdots,k$. Suppose $|q'(x_1)-f'(x_1)|=b$; if $|p'(x_1)-f'(x_1)|<a$, then we can choose $t>o$ such that $tb+(1-t)|p'(x_1)-f'(x_1)|<a$. Furthermore,

$t|q'(x_i)-f'(x_i)|+(1-t)|p'(x_i)-f'(x_i)|=o+(1-t)|p'(x_i)-f'(x_i)|<a$

for $i=2,3,...,k$. Similarly,

$t|q(x_i)-f(x_i)|+(1-t)|p(x_i)-f(x_i)|\leq(1-t)a<a.$

If we choose $Q=(tq+(1-t)p, tq'+(1-t)p')$, then Q would be a better approximation than P on E(P,F). Hence, in our special case, P a best approximation means that $|p'(x_i)-f'(x_i)|=a$ for $i=1,2,\cdots,k$.

If P is the only best approximation, the theorem is proven for our special case. Suppose then, that $Q=(q,q')$ is another best approximation. Let C be any convex combination of P and Q. Then C is also a best approximation. Since $E(C,F)\subseteq E(P,F)\cap E(Q,F)$, it must be that E(C,F) has no more than $\left[\frac{n+1}{2}\right]+1$ points. But if C is to be a best approximation, E(C,F) must have at least $\left[\frac{n+1}{2}\right]+1$ points. Hence E(C,F) has exactly $\left[\frac{n+1}{2}\right]+1$ points. Thus, we have, $E(C,F)=E(P,F)=[x_1,x_2,\cdots,x_k]$, $k=\left[\frac{n+1}{2}\right]+1$.

By our previous remarks, it must be that $|c'(x_i)-f'(x_i)|=a$, $i=1,2,\cdots,k$. Since $M(P-F)=M(Q-F)=a$, we know $|q'(x_i)-f'(x_i)|\leq a$. If $q'(x_i)-f'(x_i)\neq p'(x_i)-f'(x_i)$ for some i, we would have $|c'(x_i)-f'(x_i)|<a$. Hence, it must be that $q'(x_i)=p'(x_i)$ for $i=1,2,\cdots,k$. As we noted before, except for $x_i=o$ and $x_i=1$, we have $p''(x_i)-f''(x_i)=o$. Similarly, $q''(x_i)-f''(x_i)=o$, so $p''(x_i)=q''(x_i)$ when $x_i\neq o$, $x_i\neq 1$. Thus, $p'(x)-q'(x)$ has, counting multiplicity, at least $k+k-2$ zeroes. But $k-1=\left[\frac{n+1}{2}\right]=\frac{n}{2}$, so $2(k-1)=n$. As p' and q' are polynomials of degree less than n-1, we have $p'=q'$.

Having disposed with this troublesome special case, we can quickly prove the theorem by appealing to Theorem 11.

Again, if P is the only best approximation, we are finished. Suppose Q is another best approximation. By the corollary to Theorem 11, we have $\left[\frac{n+1}{2}\right]+1$ points x_i in $E(P,F)\cap E(Q,F)$ such that $P(x_i)-F(x_i)$ and $Q(x_i)-F(x_i)$ lie on the same support plane of aU.

From Figure 2, it is evident that this means, for each x_i, either $p(x_i)-f(x_i)=q(x_i)-f(x_i)$ or $p'(x_i)-f'(x_i)= q'(x_i)-f'(x_i)$.

Let $y_1,y_2,\cdots,y_s$ be those points x_i such that $p(x_i)-f(x_i)=q(x_i)-f(x_i)$, and also assume $y_i<y_{i+1}$. Let $z_1,z_2,\cdots,z_t$ be the remaining points x_i. Thus:

a) $p(y_i)=q(y_i)$; and $p'(y_i)=q'(y_i)$ when $y_i\neq o, y_i\neq 1$,

b) $p'(z_i)=q'(z_i)$; and $p''(z_i)=q''(z_i)$ when $z_i\neq o, z_i\neq 1$.

Furthermore, $s+t\geq\left[\frac{n+1}{2}\right]+1$.

Define j=2 if both o and 1 are among the y_i; define k=2 if both are among the z_i. Define j=1 (j=o), if one (neither) of o, 1 are among the y_i. Define k=1 (k=o), if one (neither) of o,1 are among the z_i. Then we have, $o\leq j+k\leq 2$. This definition will aid in counting the zeroes of $p'(x)-q'(x)$, for $p(x)-q(x)$ has s-j double zeroes, and $p'(x)-q'(x)$ has t-k double zeroes.

We can count the zeroes of $p'(x)-q'(x)$ as follows:

a) s-j, these are the double zeroes of $p(x)-q(x)$

b) t, these are the t zeroes z_i of $p'(x)-q'(x)$

c) t-k, these are the double zeroes of $p'(x)-q'(x)$

d) s-1

To justify d), we will show that there is a zero of

$p'(x)-q'(x)$, of odd multiplicity, in (y_i, y_{i+1}). For if not, $p'(x)-q'(x)$ does not change sign in (y_i, y_{i+1}); thus $p(x)-q(x)$ would be monotone in (y_i, y_{i+1}). But, as noted above, $p(y_i)-q(y_i)=o=p(y_{i+1})-q(y_{i+1})$. Thus, there must be a zero of odd multiplicity in (y_i, y_{i+1}). Since in b) and c) we have counted only t-k double zeroes of p'-q', we must add another zero for each of the s-1 intervals (y_i, y_{i+1}). In particular, if z_j is the only zero of p'-q' in (y_i, y_{i+1}), it must be that z_j is a zero of at least multiplicity three, of p'-q'.

Adding a), b), c) and d), we have that p'-q' has at least $2s+2t-j-k-1$ zeroes. As $j+k\leq 2$, p'-q' has at least $2s+2t-3=2(s-1)+2t-1$ zeroes. We recall that $s+t\geq\left[\frac{n+1}{2}\right]+1$, or $(s-1)+t\geq\left[\frac{n+1}{2}\right]$. If n is odd, $2(s-1)+2t\geq 2\left[\frac{n+1}{2}\right]=n+1$. Thus, for n odd, $2(s-1)+2t-1\geq n$; hence p'-q' has at least n zeroes. But p'-q' is a polynomial of degree n-1 or less, so it must be that p'=q'. We have disposed of the possibility: n even, $s+t=\left[\frac{n+1}{2}\right]+1$, in our special case. Thus, for n even, we need only consider $s+t\geq\left[\frac{n+1}{2}\right]+2$. Arguing as above, we get that p'-q' has at least n+1 zeroes; or that p'=q'. Hence, the theorem is proved.

The remainder of this chapter will be devoted to some general observations, and some more examples that may be treated as vector-valued approximation problems.

As we have already noted, the uniform approximation of real-valued functions may be cast in our terms by choosing U=[-1,1].

If we choose $U=[o,1]$, a best approximation is a best one-sided approximation. That is, if $M(p-f)\leq M(q-f)$ for all $q\in LF$, then $\|p-f\|\leq\|q-f\|$ for all $q\in LF$ such that $q(x)\geq f(x)$, for each $x\in X$.

If we wished to weight positive errors twice as heavily as negative errors, we could choose $U=[-2,1]$. In applying the results of Chapter I, the convex set $[-2,1]$ is no different from the convex set $[-1,1]$. Thus, if f is continuous and real on $[o,1]$, it is easy to derive some results that are similar to those for uniform approximation of continuous functions. In particular, if f is continuous, there is one and only one best polynomial approximation of degree n or less. This can be shown using Theorem 12 and the corollary to Theorem 11. An alternating sign characterization of the best approximation follows quickly from Theorem 8. Thus, if $M(p-f)=a$, p the best polynomial approximation of degree n or less, then there are at least $n+2$ points x_i in $E(p,f)$ such that:

a) $p(x_i)-f(x_i)=a \Rightarrow p(x_{i+1})-f(x_{i+1})=-2a$

b) $p(x_i)-f(x_i)=-2a \Rightarrow p(x_{i+1})-f(x_{i+1})=a$

In uniform approximation, positive weight functions $w(x)$ are sometimes introduced. Such a problem would take the form: find $p\in LF$ that minimizes $\max_{x\in X} w(x)|p(x)-f(x)|$.

We can handle such problems by letting:

$LF_1=\left[w(x)p(x)\middle| p(x)\in LF\right]$, and $f_1(x)=w(x)f(x)$. Then LF_1 is a linear subspace of $C(X)$, since LF is.

Some other problems that can be reformulated as problems in vector-valued approximation are given below. We let $f(x)$ be a continuous real function on $[o,1]$, LF a linear subspace of $C[o,1]$.

a) find $p \in LF$, such that among all functions in LF that are monotone increasing on $[o,1]$, p is a best uniform approximation.

b) let $x_1, x_2, \cdots, x_k$ in $[o,1]$, $b_1, b_2, \cdots, b_k$ be positive. Find the best uniform approximation to f among functions $q(x)$ in LF such that $|q(x_i)-f(x_i)| \leq b_i$, $i=1,2,\cdots,k$.

c) let $b>o$. Find the best uniform approximation to f among functions $q(x)$ in LF such that $|q''(x)| \leq b$, for all $x \in X$.

d) let g and f be continuous on $[o,1]$. Find $p \in LF$ that minimizes: $\max_{x \in [o,1]} \left[|p(x)-f(x)| + |p(x)-g(x)| \right]$.

The results of Chapter I can be brought to bear on such problems by suitably choosing the convex set U.

BIBLIOGRAPHY

1. Eggleston, H.G. Convexity. Cambridge: Cambridge University Press, 1958.

2. Garabedian, H.L. (ed.) Approximation of Functions. Amsterdam: Elsevier Publishing Co., 1965.

3. Meinardus, G., and D. Schwedt. "Nicht-lineare Approximation" Archive for Rational Mechanics and Analysis, 17 (1964), 297-326.

4. Collatz, L. "Approximation von Funktionen bei einer und bei mehreren unabhängigen Veränderlichen" Zeitschrift für Angewandte Mathematik und Mechanik, 36 (1956), 198-211.

5. Rivlin, T.J., and H.S. Shapiro. "Some Uniqueness Problems in Approximation Theory" Communications on Pure and Applied Mathematics, 13 (1960) 35-47.

6. Moursund, D.G. "Chebyshev Approximation of a Function and its Derivatives" Mathematics of Computation, 18 (1964), 382-389.