ON THE MICROECONOMIC THEORY OF OPTIMAL CAPITAL ACCUMULATION

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ABSTRACT

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By

Evan Jones

The theory of factor demand is in the process of generalization from a static to a dynamic framework. The fundamental contribution of the dynamic theory of factor demand is the interpretation of relative factor inflexibility in terms of relative costs of adjustment, and the incorporation of these dynamic costs in the determination of the profit optimum.

This dissertation makes three modest additions to the theory of optimal capital accumulation. Firstly, Marshall's concept of long run average costs, although not conducive to rigorous analysis, is firmly entrenched as a pedagogical device. An attempt is made to generalize the long run average cost curve to a dynamic framework. The rate of investment, through dynamic adjustment costs, becomes a determinant of long run average costs.

Secondly, selective problems are treated in the context of a dynamic neoclassical theory of the firm: the rationalization of a 'dynamic' representation of productive possibilities; the separation of adjustment costs internal and external to the firm; a comparison of dynamic and static capital equilibria; and the implications for returns to scale. Certain hypotheses produced from the

Marshallian generalization are examined within the specific context of this model.

Finally, the demand for investment goods has been interpreted as a two-part process - the determination both of static capital equilibria and the 'disequilibrium' acquisition of capital due to costs of adjustment. This two-part derivation of dynamic factor demand is theoretically inefficient. Statics is a special case of dynamics, and not vice versa. This same dichotomy is shown to have a macroeconomic parallel in the generally accepted interpretation of Keynes' concept of the marginal efficiency of capital.

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The two types of mind contrasted by Pascal are alike capable of subtlety and greatness, but the geometrician works in a closed universe, limited by his own axioms and definitions; the romanticist works in an open universe, limited by concrete imperfections - imperfections which have not all been charted, which may change, and which need not be the same for all men. Classicism is geometrical in its assumption that human shortcomings must be disregarded in order to be corrected, correctness being stated in the form of an exact rule. Romanticism is [intuitive] in the belief that exactitude is only a guide to thought, less important than fact, and never worthy of receiving human sacrifices. Classicism is therefore stability within known limits; romanticism is expansion within limits known and unknown.

Jacques Barzun - Classic Romantic and Modern

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CHAPTER 1

INTRODUCTION

1.1 Prologue

The scholar's task is not as simple as the extension of well-established truths, for few truths are so fortunate. Rather, he is faced with alternate representations of the same reality, each of which offers a partial understanding of the whole. In the confrontation that naturally follows, each alternative is found to have its own strengths and weaknesses. Theorizing is not a matter of separating Truth from Falsehood - of extending the boundaries of Truth, but the establishment of useful representations of reality. Some representations are useful for some purposes, and other representations for other purposes.

This dissertation aims, in passing, to question the myth that the neoclassical theory of the firm (as characterized by Samuelson's Foundations of Economic Analysis, for example) is a "well-established truth". As it happens, nothing of substance is destroyed, because it is not the practice of economics that is at fault, but its rationalization.

I refer to the status, undeserved in my opinion, that the deductive method enjoys in economics and, in particular, theory utilizing an axiom of behavioral optimization. The literature abounds with examples damming any theory which is "not based on

any explicit theory of optimization".[1] My argument is not with deduction, per se, or this work would be hypocritical. Rather it is with the presumption that deductions are from 'true' principles rather than from imaginary concepts. That certain concepts have proved useful in the understanding of certain phenomena does not make them universally applicable.

In the theory of the firm, the label "neoclassical" belies the deductive method. In a 1967 article,[2] D.W. Jorgenson implicitly identifies the term with its dependence upon a particular axiom of behavioral motivation, the familiar 'profit maximization' postulate. There is the presumption that this dependence automatically assures the imprimatur to the results of neoclassical reasoning.

But the axiom of profit maximization has no intrinsic value, just as the existence of God or of the atom is not postulated for its own sake. The purpose of positive economic analysis is not to prove that business maximizes (or intends to maximize) profits. Indeed, no observation can ever prove an axiom 'true'. Rather, the purpose is to offer an abstract representation of the regularities of economic behavior. The information content of the axiom is insignificant without an additional set of limiting assumptions which link 'profit maximization' to flesh-and-blood institutions - such as the definition of 'profit' in a dynamic, joint-stock enterprise, and a particular specification of technical possibilities.

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[1] A. Sandmo, 1971 (117), p. 1336.

[2] D.W. Jorgenson, 1967 (66).

Until recently, the representative neoclassical firm was also endowed with the following major qualities:

i. Its own actions have minimal effects on its environment, so that this interdependence could safely be ignored. The firm is 'perfectly competitive'.

ii. Its environment is known with certainty.

iii. It is always in 'equilibrium'.

An examination of the entire neoclassical model gives a hint of what should have long been obvious - the range of the model's applicability is by no means universal, but limited. Moreover, the consideration of an observable implication provides a measure of these limits. The neoclassical theory produced the interesting proposition that the firm is, of necessity, subject to decreasing returns to scale. Yet the existence of increasing returns to scale cannot be disputed. We have to look elsewhere for its explanation.

Clothed in this rather penetrable armor, neoclassicists are wont to criticize the empirical generalization. The Accelerator Principle is denigrated because it is not dependent upon any assumption of rational action, in spite of the fact[1] that output change, in econometric studies, in the single most significant explanatory variable of investment expenditures. The Keynesian theory of investment is labelled an "essentially ad hoc present value exercise",[2] in spite of (or perhaps because of) the

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[1] See, for example, E. Kuh, 1963 (87), Ch. 12.

[2] A. Sandmo, op. cit. (117), p. 1336.

businessman's constant use of present value techniques. And most recently, the use of particular distributed lag structures in econometric work has been criticized repeatedly as without theoretical foundation. The last assertion is correct, of course, and the reason is obvious. Economic theory had no distributed lags to offer, let alone particular forms. Until the Great Depression, there was also no unemployment in pure theory, which was no solace for those trying to understand this phenomenon. At least, no one had the courage to denigrate these ad hoc efforts. But out of the confrontation came a more broadly applicable theory.

The problem is that if Theory and Experience are to get together, a compromise must be made, since each speaks a different language. Each must forgo its purity in order to influence the other, and so improve itself. Two related examples of a failure to understand this polarity come to mind. They are the recent onslaught of profit maximization, and the heated debate over Marginalism in the 40's. That business does not attempt to maximize 'profits' or equate 'marginal revenues and costs' does not negate neoclassical theory, for this is the language of abstraction. On the other hand, profit maximization and the Marginal Proposition do not deny the efficacy of business practice. Without translation, the two sides remain irrelevant to each other. And it is the additional set of particularizing assumptions that provides the translation.

A fine example of the productive liason between theory and observation is Friedman's permanent income hypothesis. It is a well-conceived proposition that consumption is a simple function of permanent income. But the proposition is a useless (and irrefutable) abstraction until 'permanent income' is given an empirical definition.

In brief, the question one asks of a theory is not, "Is it true?", but "What is the range of its applicability?" Implicitly this latter question was asked when investment behavior became an important empirical concern. It was discovered that the neoclassical theory of the firm had absolutely nothing to say about investment. The reason is that the theory of the firm was set in a comparative-static framework and, if dated, within a single period framework. Yet investment is an intrinsically dynamic concept. 'Distributed lags' were born as a pragmatic attempt to extend the range of applicability of the neoclassical model. To criticize distributed lag techniques as without theoretical foundation is to misunderstand the nature of theoretical development. The fault lies not with distributed lags but with the limitations of the existing axiomatic theory of the firm. Thus the development of the microeconomic theory of investment over the last twenty years is significant from a methodological point of view, as a perfect example of the progress resulting from the deductive-inductive confrontation.

A secondary intention in this work is to highlight the devotion that microeconomists, both theorists and practitioners alike, have shown to static analysis. I would characterize static analysis as being primarily concerned with the description of stationary equilibrium states - the economic interrelations of the equilibrium itself, the qualitative influence of exogenous change, and the existence of stability. The notions of time and the passage of time, are outside its scope. But because of its tractability, the static method has permeated economic thought, and over-reached itself into the analysis of investment behavior. This distortion of an innately dynamic problem was forewarned by Marshall and Hicks, and is evidenced by Nerlove and Treadway.[1]

1.2 Content and Contribution.

Chapter 2 is an exposition of the confrontation of the static neoclassical theory of the firm with the fact of capital accumulation. It has been recognized that the classical theory of the firm makes no contribution to the theory of investment. Chapter 2, Section 1 is a painstaking explanation of why this was not immediately obvious. Marshall's celebrated concept of the 'long run' has the effect that the accumulation of capital, and its attendant problems, may be neglected. The study of the full adjustment of factors of production to economic conditions is sufficiently important of itself to warrant this initial neglect.

But somewhere along the line, theorists were not content merely to study the nature of stationary equilibrium. They had to ensure that their subject of inquiry was also in this happy state. The harmful assumption crept in the back door that economic adjustment to changing circumstances takes place completely without frictions. That this assumption could serve to elucidate the

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[1] A. Marshall, 1949 (99), Book V; J.R. Hicks, 1945 (56), Book III; 1965 (57), Part I; M. Nerlove, 1971 (104); A.B. Treadway, 1970 (132).

nature of adjustment is of course ludicrous. But the 'instantaneous' assumption had the effect of turning a theory of alternative equilibria into a theory of economic change. What were alternative states now become continuously substitutable stages of a dynamic process, depending on the movement of the external environment. This process was then given a time reference, locating this 'equilibrium' path in time.

The long run unit cost curve, in conjunction with a representation of demand, is one approach to the examination of static equilibrium. It is important both because of its simplicity, and because it still serves to introduce every student of economics to the theory of the firm. Chapter 2, Section 2 attempts to recognize explicitly the gradual adjustment of capital by introducing the time-dimension into Marshall's historical concept of long run costs.

The content of this section is the first contribution of this thesis. In a dynamic context, conclusions are drawn as to the nature of static and growth equilibrium, and the feasible range of returns to scale in each case. This analysis is to be taken in conjunction with the neoclassical theory of optimal capital accumulation, analyzed in Chapter 4, each of which has its own advantages. The Marshallian extension depends on an intuitive verbal analysis. The results are theorefore conjectural, but the extension is a fruitful source of examinable hypotheses. The neoclassical theory offers precise results within the limited framework of a specific model.

Specifically, the first step of the Marshallian extension associates each point on the long run cost curve with a (different) point in calendar time. The second step allows that the level of capital be changeable at differing rates, subject to the costs of adjustment. This implies the concept of a long run cost surface in the output-cost-time space, which includes the traditional long run curve. Thus the time taken for the realization of a new plant is assumed to influence long run costs. Given a dynamic profit criterion, static capital equilibrium is therefore dependent on dynamically determined costs. Dynamically determined costs are defined as those dependent on the rate of growth of the firm, while statically determined costs are dependent on the firm's size.

It is argued that this mixture of static and dynamic cost elements is not alien to the traditional notion of long run equilibrium, since the familiar 'managerial diseconomies' are given a dynamic interpretation. The factor management is typical of industrial factors in that it is not limited in absolute size, but in its rate of increase. Hence we distinguish reproducible and non-reproducible factors of production.

Traditionally, given factor prices, the slope of the long run unit cost curve is taken as indicating returns to scale. Since long run costs are also influenced by the rate of adjustment to equilibrium, the relationship between the slope of the long run unit cost curve and returns to scale is reexamined. And since these dynamic adjustment costs are a source of, at least, dynamic stability, they are used in an attempt to break down the classical incompatibility between the arch-rivals, perfect competition and constant or increasing returns to scale.

An attempt is also made to describe the adjustment to a constantly changing demand and the establishment of a growth equilibrium. But a verbal and graphical analysis is less than adequate to the task, since the capital decision itself influences future cost-output alternatives. A particular future demand growth is assumed, and an equilibrium path of long run costs results. From this we construct a long run equilibrium average cost curve.

Chapter 3 summarizes the historical transition from the (static) theory of the demand for capital to the theory of capital accumulation. Both the Accelerator Principle and the 'dynamized' neoclassical theory of Jorgenson represented the firm in continuous (long run) capital equilibrium, though the flesh-and-blood firm they sought to explain exhibited great resistance to change. Distributed lag manipulations with statistically desirable properties were then performed on these 'full equilibrium' theories to cope with the observed partial adjustment. The instantaneous adjustment assumption is used to put the firm into equilibrium commensurate with its surroundings, and the partial adjustment assumption is used to take the firm out of equilibrium. This loop was responsible for the suggestion of an artifical dichotomy - that there exists an optimal capital stock (determined by static factors) and an optimal rate of investment (determined purely by dynamic adjustment costs) to eliminate a stock discrepancy.

Obviously this procedure was not an efficient means of rationalizing the gradual adjustment of capital. It was then realized that the adjustment of capital is itself influenced by profitability considerations, and that capital adjustment should be theoretically determined as an integral part of the optimization process. This is the contribution of Eisner and Strotz.[1] Yet the adjustment of capital is still seen as a <u>disequilibrium</u> process, on the way to a stationary equilibrium capital stock.

Finally, the introduction of 'nonstationary' expectations or a constantly changing environment, by P.A. Tinsley[2] and others, does away with the preeminence of 'optimal capital stock'. Capital and net investment are merely different faces of the same coin. Maximizing net present value implies an optimal capital stock and an optimal rate of capital accumulation for all decision periods. Whereas the dynamic costs of capital were excluded from the comparative static theory of the firm, this model incorporates all costs of capital, and claims to approximate more closely the capital decisions of the living firm.

The neoclassical development is neatly summarized in the first-order equation for capital stock. Jorgenson's model reduces to an equation of the form

$$X_K = f(t) \qquad 1.1$$

where X_K is the (static) marginal productivity of capital and $f(t)$ is the time-variant cost of capital services. The Eisner-Strotz extension leads to a new form

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[1] R. Eisner and R. Strotz, 1963 (30).

[2] P.A. Tinsley, 1969 (125), 1970 (126).

$$\ddot{K} - r\dot{K} - gK = f.^{1} \qquad 1.2$$

The presence of $\ddot{K}$ and $\dot{K}$ follows from the introduction of adjustment costs into the optimization process. The equation's linearity follows from technological assumptions. The inhomogeneous constant, f,[2] represents a stationary environment. The coefficients $\underline{r}$ and $\underline{g}$ are constants. Putting $\ddot{K} = \dot{K} = 0$ in 1.2, the resulting equation is a linear representation of the traditional static capital equilibrium. Tinsley further allows for a non-stationary environment, and the capital equation becomes

$$\ddot{K} - r\dot{K} - gK = f(t) \quad . \qquad 1.3$$

A model of optimal capital accumulation is presented in detail in Chapter 4, leading to an equation of the form 1.3. The hypothesized firm is perfectly competitive in the output, labor and cost of funds markets, and subject to (external) adjustment costs in the capital goods market. The firm faces nonconstant, but certain, future prices. We thus neglect the profoundly important problem of uncertainty, but make the usual disclaimer that there are enough problems which precede uncertainty to warrant its neglect.

The second contribution of the thesis lies in the treatment of selective problems within the context of the dynamic neoclassical model. An explicit attempt is made to drain the mathematical model

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[1] After substitution of the first-order condition for labor.

[2] f is negative.

of its economic implications. First, a generalized 'dynamic' function for output in capital, labor and net investment, is postulated, consistent with Chapter 2 and the work of A.B. Treadway.[1] The terms in net investment house adjustment costs which are internalized in the firm, showing themselves not in higher prices (associated with external adjustment costs), but in lower output. This function is given a more specific formulation, whereby it is defended as a feasible dynamic representation of productive possibilities.

Second, internal and external adjustment costs are incorporated separately in the model, It is then possible to examine whether the separation influences the nature of the capital equilibrium. Generally, the influence is not symmetric, but our ultimate reliance upon the quadratic assumption for adjustment costs results in a symmetric influence.

Third, a comparison is made of the static and dynamic first-order equilibrium equations, under non-specific production conditions. The comparison obviates the special conditions under which equation 1.1 is the applicable representation of capital equilibrium.

Fourth, and most important, the (specific) model is examined for its implications for returns to scale. Returns to scale are defined according to the sign of the coefficient $\underline{g}$ in 1.3. The results are not as general as one would hope. It is found impossible to incorporate increasing returns to scale in this model and maintain constancy of $\underline{r}$ and $\underline{g}$. So it is difficult to

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[1] A.B. Treadway, 1969 (131).

examine whether the perfectly competitive firm may experience increasing returns, at least for some period in its expansion. But the case of constant returns to scale may be analyzed, giving the desired results. Perfect competition and constant returns are compatible in a period of expansion, the marriage being effected by dynamic adjustment costs. The classical incompatibility is confirmed for static equilibrium.

Chapter 5. Finally, an examination of the literature on the marginal efficiency of capital of Keynes indicates a parallel to the long-standing bias in the microeconomic theory of capital demand. The accepted interpretation of the MEC has been similarly distorted by the invalid dichotomy alluded to above. The recognition and examination of this macroeconomic parallel constitutes the third contribution of this thesis.

The demand for capital must be derived from static considerations, so the argument goes. Investment eliminates the capital stock discrepancy, and its level is determined independently. Keynes' MEC schedule was presumed to rationalize the disequilibrium rate of investment. In fact, the MEC schedule shows the demand for capital in each decision period. There is no capital-investment dichotomy. Note that Keynes utilizes the 'internal rate of return' concept while dynamic neoclassical theory utilizes maximum present value.[1] Were it not for this difference, Keynes' macroeconomic investment function would have a meaningful microeconomic foundation in the modern theory of optimal capital accumulation.

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[1] D.W. Jorgenson, 1967 (66).

CHAPTER 2

THE STATIC THEORY OF THE FIRM AND THE DEMAND FOR INVESTMENT

2.1 The Initial Controversy

I will begin by outlining the initial confrontation between the traditional neoclassical theory of the firm and those seeking a rationalization for investment behavior.

As a static theory of optimal behavior, the neoclassical theory is concerned with two things - the existence of a stable equilibrium position for the firm, and the qualitative effects on this equilibrium of parameter variations. Given output and factor input prices, if there exists a combination of output and factor inputs which maximizes its monetary return, the firm is assumed to choose that combination. Technological factors, by assumption, ensure that this optimum position is stable. When a price changes,[1] the theory indicates the qualitative influence on output and factor quantities.

What does this theory contribute to an understanding of the firm's accumulation of capital? Lerner, Haavelmo and Witte[2] have addressed themselves to the question of a meaningful investment

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1 Or a price-quantity curve, if the market is not perfectly competitive.

2 A.B. Lerner, 1944 (88), Ch. 25; T. Haavelmo, 1960 (43), Esp. Part iv; J.G. Witte, 1963 (139). See also G. Ackley, 1961 (1), p. 477f; D.W. Jorgenson, 1967 (66); A. Sandmo, 1971 (116).

demand function within this structure. According to accepted doctrine, the structure embodies a theory of optimum capital services.

At the outset we digress to make a classical assumption. The assumption is that there is a fixed relation between physical capital stock and the services this stock provides. This is a reasonable assumption for classical long run equilibrium, where there are no intertemporal profit tradeoffs, and optimum capacity utilization is a technical consideration. In the dynamic capital equilibrium of Chapter 2, Section 2, and Chapter 4, intertemporal profit tradeoffs influence the equilibrium itself. Here capacity utilization should legitimately be included in the 'dynamic' profit criterion. This is even more true of the model in Chapter 4 where perfectly divisible capital is assumed. Nevertheless, we shall avoid this complication in what follows. Henceforth, any reference to optimal capital stock hides a proportionate optimal level of capital services.

Lerner is generally credited with being the first to point out (albeit for the economy as a whole) that a theory of optimum capital services does not, of necessity, include a theory of investment. Capital is a stock, and investment is the time rate of change of this stock. Immediately, the mutual importance of both a variable and its time derivative indicates that a static theory will be lacking.[1] But we will be more explicit.

To analyze investment is to analyze capital in transition. Witte's statement is representative of the accepted position:

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[1] c/f P.A. Samuelson, 1948 (116), p. 314f.

> The theory of the firm, in its traditional comparative-statics form, yields only a capital-stock demand function and not a rate-of-investment demand function, because the firm's demand is for the services of capital goods, which services are proportional to a stock and not to the time rate of change of that stock. That is, the traditional theory of the firm permits the determination only of the optimum size of the capital stock and not the optimum rate of adjustment of that stock to a change in the external environment of the firm.[1]

Now, regarding static theory, the equilibrium itself gives little cause for misunderstanding, but not so the variation of parameters. There are two ways of analyzing parameter variation, or a change in the firm's environment. Firstly, for every set of values of the parameters there exists a set of optimum values for the decision variables. Parameter variation determines alternative states of equilibrium for the economic agent. Regardless of whether we are studying one economic agent or making comparisons, these alternative states are entirely independent. Secondly, let the agent face a change in a parameter. Its operations are now non-optimal, and it must make appropriate adjustments to reach a new state of equilibrium. These equilibrium states are linked in time.

Although the first treatment of parameter variation is the pure interpretation of the comparative static method, the second is the more usual interpretation. This is the first step of comparative statics onto dangerous ground. There is a difference between the two which is pertinent to the investment problem. The pure concept is of alternative states of equilibrium and is without

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[1] J.G. Witte, op. cit. (139), p. 205.

time reference. The second concept is formulated in the time dimension. One equilibrium follows the other. We shall see that the second concept is the source of some confusion regarding the existence of an investment demand function.

Consider the following specific problem. For the firm, the optimum level of capital stock, given parameter values, is indicated mathematically by the first-order conditions for a static equilibrium. This is represented by the equality of the value of the marginal productivity of each input with its price. Allow one of the parameters to vary, say a decrease in the rate of interest. Since the price of capital services is a positive function of the rate of interest, an increase in capital is indicated. Strictly, what is relevant is the qualtitative knowledge that the capital-interest rate partial derivative is negative, and in practice the quantitative extent of the influence. The time taken for the necessary acquisition of capital is outside the realm of the theoretical structure. By construction, the consideration of the passage of time is excluded.[1]

We may ask whether the famous Marshallian short run - long run dichotomy offers us any understanding of capital adjustment. In my opinion, the dichotomy exists precisely so that the act of adjustment itself may be neglected. The distinction recognizes that the various factors of production are capable of differing

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[1] However, as we shall see in Chapter 4, apart from the stationary state, the rate of acquisition of capital influences the nature of the equilibrium itself.

rates of adjustment.[1] So two[2] categories are created. But in both, the adjustment of all the decision variables may be ignored. In short run analyses, capital is arbitrarily assumed fixed. Only enough time need pass to allow the 'variable' factors to adjust. In long run analyses, capital itself must be allowed time to adjust.

The long run is defined as that period which enables the accumulation of capital. Presumably, the environment facing the firm has changed, and the existing stock is no longer optimal. During or after some (undefined) period, the firm makes up the deficiency in its stock, and is again at an optimum.

The examination of these optima is, of course, the realm of the 'statical method'. Utilizing this static framework, Marshall hoped to gain insight into the concept, normal value.[3] He observed that the economic world never stayed still long enough for 'normality' to be established. But for purposes of analysis, the two sides of the market need to be compatible. The root of the problem is that the acquisition of capital takes time - "the influence of changes in cost of production takes as a rule a longer time to work itself out than does the influence of changes in demand".[4]

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1 The case of variation is obviously an indication of that factor's cost of adjustment.

2 Or more precisely, four.

3 Normal value is "... the average value which economic forces would bring about if the general conditions of life were stationary for a run of time long enough to enable them all to work out their full effect". A. Marshall, op. cit. (99), p. 289.

4 ibid., p. 291.

Thus, if the firm is ever to achieve an optimum position (given uncertainty of the future), the fundamental requirement is that the level of demand, for which this capital is intended, must remain steady. Otherwise the firm will never own a capital stock perfect for its needs, and the ensuing supply price will differ from the product's normal value.

In short, the use of the static method requires that we impound demand into Caeteris Paribus. We throw out a huge chunk of time, as if it never existed. This undefined length of time is the long run.

There is an alternative means of coping with capital adjustment. It is to assume that adjustment to disequilibrium takes place without friction. Take, for example, Knight's requirement for his imaginary society:

> We must also assume complete absence of physical obstacle to the making, execution, and changing of plans at will; that is, there must be 'perfect mobility' in all economic adjustments, no cost involved in movements of changes ... The exchange of commodities must be virtually instantaneous and costless.[1]

Hence the adjustment of a capital stock disequilibrium is instantaneous. This supposedly harmless assumption, combined with the time-referenced interpretation of comparative statics, has the effect of turning alternative stationary states into dynamic equilibria.

For the Marshallian firm, equilibrium is a 'sometime thing'. Marshall is more concerned with the study of equilibrium as an

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[1] F. Knight, 1921 (82), p. 78

idea. In a temporal framework, the Knightian assumption implies that the firm is <u>always</u> in a state of long run equilibrium, one that is not necessarily stationary.[1]

It can be seen that both of the above assumptions, frictionless adjustment and the Marshallian long period, are subterfuges to overcome the timelessness of the static framework. They are really phrases signifying that the passage of time and the adjustment towards equilibrium is excluded from the analysis.

What does this mean for investment? Simply, there can be no concept of the rate of investment within the static theory of the firm. Compare Samuelson on this point: "It is a commonplace criticism of comparative statics that it does not do what it is not aimed to do, namely describe the transition paths between equilibria".[2] Haavelmo's much-quoted discovery from static theory is that the implicit rate of investment is (positively or negatively) <u>infinite</u>.[3] But this is to take for granted the Knightian assumption of instantaneous adjustment and to misunderstand its function. Haavelmo further states that there is no possibility for a finite, non-zero rate of net investment unless there is a continuous change in the parameters. For example, a positive rate could be explained by a negative rate of change of the price of capital goods or the rate of interest, a positive rate of change of the product price, or

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[1] There is the further implication of instantaneous adjustment, that whether one assumes future uncertainty or not is irrelevant for the conclusion of the static model.

[2] P.A. Samuelson, op. cit. (116), p. 263n.

[3] T. Haavelmo, 1960 (43), p. 173

continuing technological advances.[1] This is correct, but we are then outside the realm of static theory.

Marshall was content to remove the adjustment period, the long run from view. For him the rate of investment is certainly not infinite. Rather, it is simply neglected and therefore unknown. Marshall's albatross is not instantaneous adjustment, but an assumed constant demand. It is ironic that the analytical framework which is Marshall's legacy should be judged on its capacity to explain adjustment, since Marshall was constantly admonishing his reader that the statical method was a first approximation for the study of economic conditions, and was not to be applied unquestioningly to the practical world.

> But in the opinion of the present writer the problem of normal value belongs to economic Dynamics; partly because Statics is really but a branch of Dynamics, and partly because all suggestions as to economic rest, of which the hypothesis of a Stationary Sate is the chief, are merely provisional, used only to illustrate particular steps in the argument, and to be thrown aside when that is done.[2]

But to 'throw aside' statics is easier said than done. The static theory of the firm has been developed into a large and intricate network of precise results. By contrast, analytic dynamics is relatively undeveloped and potentially more complex.

Consequently, it seems natural that theorists would extend the static method beyond its capacity. The indicator of its misapplication is the time-dating of variables. Remember that, for Marshall, a market may never be in long run equilibrium, but

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1 ibid., p. 170f.

2 A. Marshall, op. cit (99), p. 304, n2. See also p. 315,5; p. 382.

he is content to study a tendency towards it. Date the series of equations pertaining to an optimum, and an equilibrium has been imposed on a particular time point whether it is warranted or not. Allow a multi-period model, and equilibrium has been imposed on every period.[1] Capital adjusts entirely within each period. And this model is labelled 'neoclassical dynamic equilibrium'. The label is clearly inappropriate.[2] Theoretically consistent models have been posited in which the period is defined "by the time required for the installation of new capital goods".[3] But it is erroneous to use this model empirically when the period naturally refers to calendar years or quarters.

It is understandable that this model was found inadequate for an applied analysis of capital accumulation, for it has circumvented Marshall's fundamental problem, that capital acquisition takes time. The development of distributed lag techniques by econometricians is a predictable reaction, because the lags compensate for the inadequacies of the theoretical model. Hence we arrive at the well-known empirical equation:

$$I_t = u(L)(K^*_t - K^*_{t-1}).^4$$

Investment in t is a weighted average of the change in (static) optimal capital stock. This is typified by Jorgenson's work,

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1 For example, P.A. Tinsley, 1970 (126), exp. p. 7.

2 Hicks anticipated this common error in (56), p. 116.

3 A. Sandmo, op. cit. (117), p. 1337.

4 L is the lag operator.

equation 3.19. K_t^* is derived from a multi-period static model while u(L) is derived empirically. Of this model, it has been said repeatedly[1] that the desired capital stock is determined rigorously while desired investment is not. This exemplifies that bias towards the deductive method claimed in the Prologue. The point is that the much-worshipped rigor has no value in the face of limited axioms. Methodologically, the static model and ad hoc distributed lags should be judged together, not separately. K_t^* is based upon the axiom of costless adjustment. The distributed lag u(L) exists <u>because</u> K_t^* is a poor approximation to the truth.

By this model, optimal capital stock, K_t^*, is acquired gradually. We have the strange construction whereby K_t^* is optimal for t but is never achieved in t. Moreover, if the demand schedule is nonconstant, by this process capital stock will <u>never</u> be optimal, except by chance. What kind of optimality is this, that can never be achieved? This paradox was recognized by Jorgenson in his own work, and an interesting rationalization is provided. The optimal capital stock is never achieved due to <u>unforeseen</u> delivery lags.[2] This is a nice try, but it will not do. The analytical framework is the problem.

The neoclassical theory of the firm has been variously interpreted as assuming instantaneous adjustment (Knight) or simply

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1 For example, see K. Wallis, 1969 (136).

2 D.W. Jorgenson and C.D. Siebert, 1968 (74), p. 1124-5. Jorgenson and Siebert are close to the truth. In practice, the continuous non-realization of optima can only be due to future uncertainty, but this is far more complex than delivery lags, which are probably predictable. On this point, see also M. Nerlove, 1971 (104).

ignoring the adjustment process (Marshall) in order to concentrate on the nature of static equilibrium. But the living firm is not concerned with stationary state factor prices and stationary state profit maximization. The living firm has until next year to accumulate capital to satisfy next year's expected demand. So the relevant cost of capital is the price the firm has to pay next year when the capital is needed, and not the price (presumably discounted) it can get away with if it waits till doomsday for purchase.

Some instances in the belief of these 'long run' prices can be found in Johnston.[1] One author 'corrects' the high wages paid for construction labor in the winter months because these are supposedly aberrations on a true wage. Yet summer wages aren't relevant to winter construction. In another study, Dean[2] corrects for the lower productivity of inferior factors at high short run outputs. Yet since the firm needs greater output in the short run, what is available in the long run, and at what price, is of no consequence.

We can now confront the problem of a non-achievable 'optimum'. The error is that a fundamental constraint is missing from the optimization process in time-referenced models. This constraint is the cost of acquisition of capital. If we are to have a feasible optimization process for each t, this additional constraint must be included. What results is a demand for capital

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[1] J. Johnston, 1960 (61), p. 27 and p. 138 respectively.

which can be realized in t. And the demand for realizable capital is nothing more than the demand for investment goods.

From an ideal standpoint, any attempt to rationalize separately the adjustment process characterized empirically by distributed lags is in error. To combine an optimal investment decision (derived dynamically) with an optimal capital decision (derived statically) is theoretically invalid.[1,2] Any problem in which both a variable (capital) and its time derivative (net investment) are intimately involved requires analysis by the variational calculus. This is the contribution of the 'adjustment cost' literature, Chapters 3 and 4.

2.2 A Reexamination and Extension of the Marshallian Long Run

I propose to reexamine the concept of the 'long run', with particular reference to the long run average cost curve of Marshall's representative firm. It is important, initially, that we clarify the accepted contribution of this simple concept to our understanding of the demand for capital. Our interest now lies explicitly in the investment process itself. But the notion of long run costs as a pedagogical device is likely to be with us for some time. It is therefore a meaningful excercise to attempt a more dynamic definition of long run costs - firstly in the hope of some understanding of capital accumulation without the loss of graphical simplicity, and secondly to suggest fruitful propositions

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[1] c/f P.A. Tinsley, op. cit. (126), p. 8, n5. "... that the firm is moving along an 'optimal' inter-optima path ... is a contradiction in terms".

[2] The same error has been made in macroeconomic theory. See Chapter 5.

for examination in the theory of capital accumulation advanced in Chapter 4.

2.2.1 A Time Variant Long Run

Text-book definitions notwithstanding[1], the meaning of the long run average cost curve, hereafter called LRAC, is not at all clear. Marshall's development of the long run is set out mainly in Chapters III, V and XII of Book V and in Appendix H of the Principles. The reader will have to tolerate extensive references to these sections.

Consider the short run and the long run average cost curves SRAC and LRAC. They are both planning curves showing minimum unit costs of producing alternative outputs. In the short run, it seems a reasonable approximation to assume that the passage of time necessary for switching from one output to another is sufficiently short as can be ignored. We make the 'instantaneous adjustment' assumption for the short run. It follows that the short run output levels are alternatives at the same point in time. Unfortunately, this assumption is more hazardous for the long run, for each point on the LRAC allows full adjustment of capital to the efficient production of that output. Though the SRAC and the LRAC are conceptually similar, the passage of time is more important to the LRAC, so the LRAC is more open to misinterpretation.

Thus in the short run, we freeze time and ignore it. But it is fundamental to long run costs that time must pass before

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1 c/f J.M. Henderson and R.E. Quandt, 1971 (52), p. 76. "The entrepreneur's long-run total cost function gives the minimum cost of producing each output level if he is free to vary the size of his plant".

the capital necessary to each 'alternative' output can be acquired.[1] How long this time must be is never made explicit. For Marshall, it must be of sufficient length so that

> "... all investments of capital and effort in providing the material plant and the organization of a business, and in acquiring trade knowledge and specialized ability, have time to be adjusted to the incomes which are expected to be earned by them ..."[2]

Viner, in his definitive article on cost curves, can manage only the more indefinite proposition: "... a period long enough to permit each producer to make such technologically possible changes in the size of his plant as he desires".[3]

Assume, for the moment, that our firm is in the planning stage. Consider the 'alternative outputs'. Is the long run for a $100,000 investment the same as the long run for a $1,000,000 investment? It seems plausible to assume that the larger are the dimensions of the considered investment, the larger is the time required for its fruition. I propose therefore that each point on the LRAC is associated with a point in calendar time.[4]

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[1] c/f A. Marshall, op. cit. (99), p. 285. "... both the material capital of machinery and other business plant, and the immaterial capital of business skill and ability and organization, are of slow growth and slow decay." Also c/f H.B. Malmgren, 1960 (95), p. 413. "... every point on the Marshallian long-run curve has <u>minimum</u> date attached to it." Emphasis mine.

[2] ibid., p. 313.

[3] J. Viner, 1952 (135), p. 205.

[4] Marshall attempts this in Appendix H. "We should have made a great advance if we could represent the normal demand price and supply price as functions both of the amount normally produced and of the time at which that amount became normal."

Suppose that the output x^* in Figure 2.1 is associated with time t_n. It takes t_n before x^* can be produced at minimum cost, and similarly for other levels of output.[1] Minimum cost is a fundamental property of unit cost curves, but one that has usually been associated with technical aspects of factor organization. Here, minimum cost is also made a function of the passage of time, introducing a dimension of factor organization usually neglected.

The two-dimensional LRAC hides a third dimension, that of time.[2] Our LRAC is strictly a curve in three dimensions, LRAC(t), and what we usually see is the projection of this curve onto the \$-x plane. Figure 2.2 shows LRAC(x,t) in three dimensions, and LRAC as the projection of LRAC(x,t) onto \$-x. LRAC(x,t) represents the minimum unit cost of producing the output 'x' as seen from (x_o, t_o). 'x_o' may be zero, as above. Purely as a pedagogical device, g(x,t) is seen as the projection of LRAC(x,t) onto the x-t plane.

Now assume that the firm is in existence and is located at 'x_o' on the LRAC, Figures 2.1 and 2.2. No radical changes are required. The three-dimensional representation still holds,

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[1] c/f H.B. Malmgren ., op. cit. (95). "A point at output x would have to be defined according to how long the firm was willing to wait", p. 413. This note contains the kernel of many ideas expressed here, but little further development.

[2] c/f R. Frisch, 1950 (35). "For simplicity in the graphical representation we may sometimes measure both short run output and long run output along the same axis, but it must always be remembered that these magnitudes are two distinct variables", p. 80. See also pp. 88, 92. The work of A. Alchian, 1959 (2), also implies a three-dimensional LRAC, but his use of rate of output, x, and volume of output, V, is confusing.

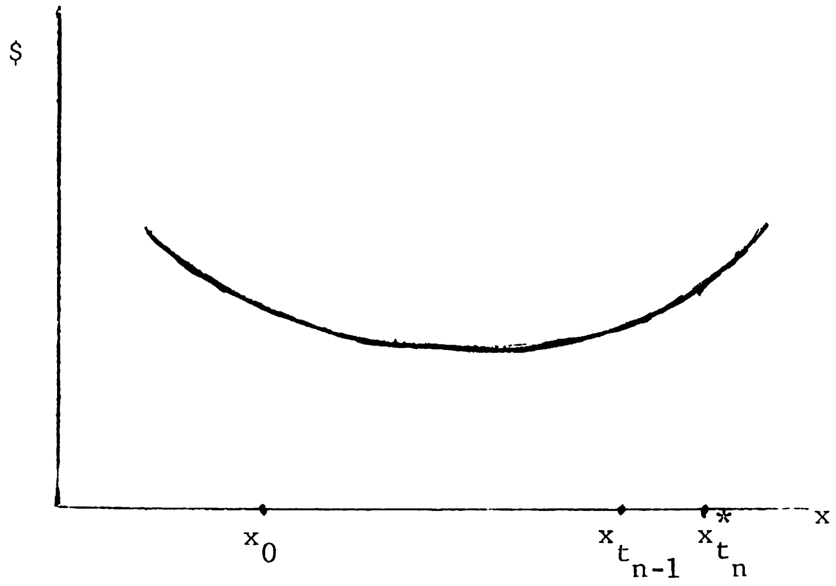

Figure 2.1 The Traditional Long Run Average Cost Curve

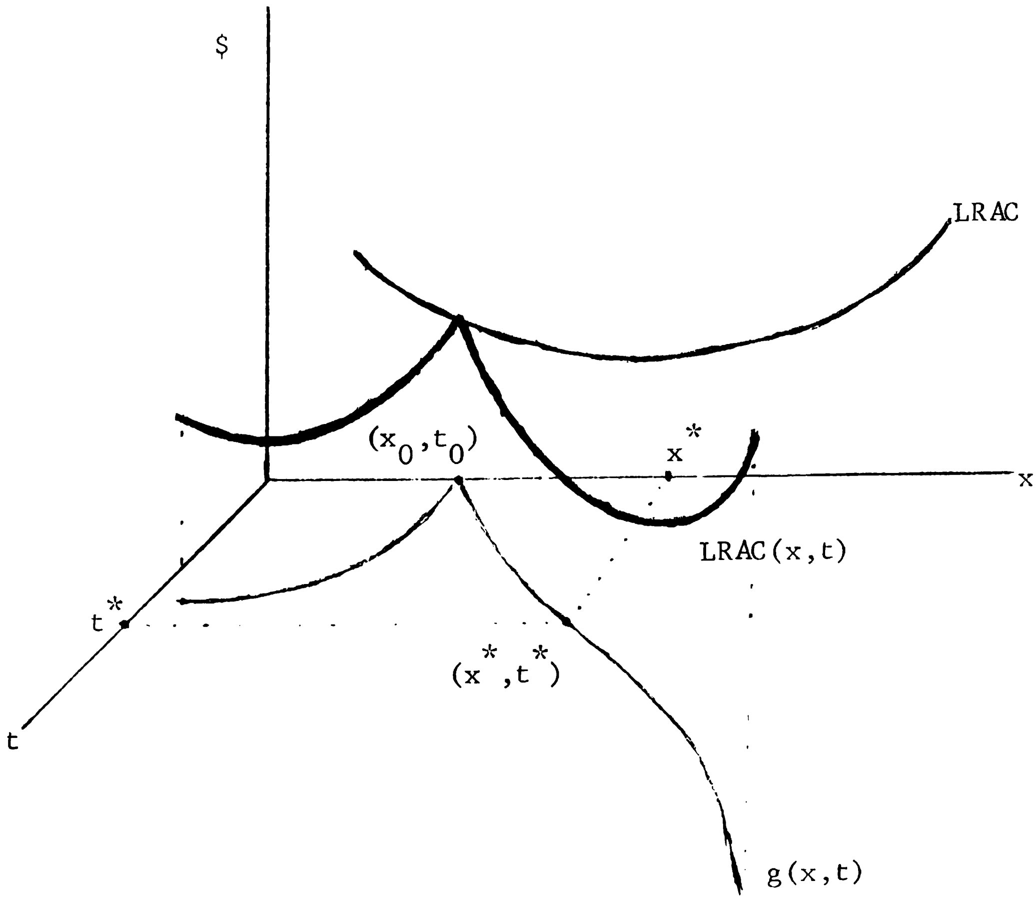

Figure 2.2 The Time-Dependent Long Run Average Cost Curve LRAC(**x**,t)

but now the function will be displaced in the time dimension. On the drawing borad, one is faced with a necessarily gradual acquisition of capital. With the firm in existence, one has also to cope with a capital stock presumably non-optimal for present and future demand. In this case, the time argument incorporates both the gradual acquisition of desired capital and the gradual disposal of undesired capital.

Briefly, let us make explicit the properties of the long run average cost curve, as understood above.

i. The LRAC is defined in three-space. In economic terms, no capital expansion takes place in the x-direction. The passage of time is integral to the capital decision. Further, both capital expansion and capital disposal take place in the t-direction.

ii. We are agreed that a minimum time must pass before a given plant can be efficiently installed and operated. We have also to assume that there is a definite time limit to the minimum-cost purchase and installation of plant, and to the development of administrative expertise associated with this plant. This lower and upper bound on t, respectively, defines a unique t^* for every x^*, such that (x^*,t^*) is a point in $g(x,t)$, Figure 2.2.

iii. There is a SRAC erected on every point in the LRAC, perpendicular to the t-axis. Short run decisions are 'timeless' decisions.[1]

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[1] The more sophisticated literature on adjustment costs, with good reason, treats labor also as a 'quasi-fixed' factor. Property iii maintains instantaneous adjustment for the factor labor, to facilitate comparison with the Marshallian framework. The simplifying assumption is carried over to the analysis of Chapter 4.

When the representative firm faces a stationary demand
nction, its confrontation with the LRAC provides the optimum
ationary output. If one is interested purely in this equilibrium,
e time taken for its achievement is irrelevant. It is precisely
is limited interest which allows for the neglect of the time
mension of the LRAC and its very considerable problems. The
aditional \$-x representation of the LRAC is a feasible representa-
on of long run costs only when one's concern is the special case
stationary equilibrium. But we are now interested more in the
namic representation of firm behavior.

What can we do with the traditional LRAC in a dynamic world?
re we are exposed to intertemporal tradeoffs. The relevant
iterion for the firm is not simple profit maximization but (we
all assume) the maximization of present value. It matters
ether this new level of demand is satisfied now or later. Sales
e being lost while the long run runs its course. Thus it may
that profitability considerations make the traditional LRAC
complete to the capital decision.

In general, we need a construction of long run costs which
s a particular time reference. The capital decision must be
de from alternatives that are each feasible to satisfy demand
<u>a given time</u>. Let us then construct a long run cost <u>surface</u>
$,t)$, where $A(x_i,t_j)$ is the minimum cost of producing x_i if
is allowed for factor adjustment. Let us further take cross
ctions of $A(x,t)$, perpendicular to the t-axis, and label them
C_i. $LRAC_i$ shows the minimum cost of producing alternative
puts at time t_i. As seen from t_o, t_i is the time allowed

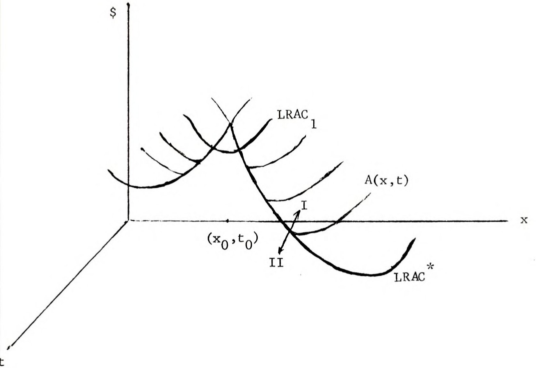

Figure 2.3a The Long Run Average Cost Surface A(x,t)

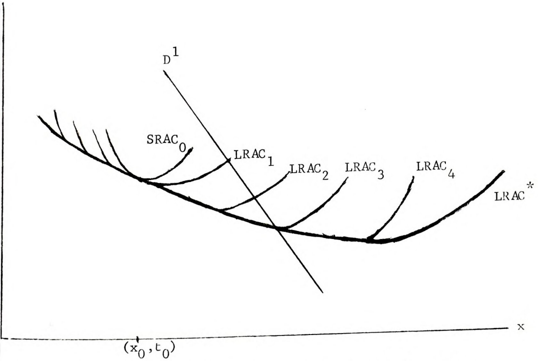

Figure 2.3b The Projection of A(x,t) in the \$-x Plane

r adjustment of productive capacity to that new optimum level quired by the change in demand. For example, $LRAC_1$ is shown the cost surface $A(x,t)$ in Figure 2.3a. $LRAC_1$ summarizes e cost-output possibilities if one period is allowed for adstment of capacity, and so on. $LRAC_i$ will be identical to e $LRAC^*$ for all points (x,t) such that $t < t_i$. This is own in Figure 2.3b, which shows the projection of the series $LRAC_i$ onto the \$-x plane. In the series of $LRAC_i$, we have eated a period of variable range, which we shall call the medium ng run.

Properties ii. and iii., 30 above, are replaced by[1]

ii'. $A(x^*,t) > A(x^*,t^*)$, $t < t^*$ 2.3

ere (x^*,t^*) is any point on the traditional LRAC, henceforth lled $LRAC^*$. Capital adjustment is possible at differing rates, : minimum cost of production is achieved when t^* is allowed : fruition. This is shown in region I in Figure 2.3a.

$$A(x^*,t) \to \infty \quad \text{as} \quad t \to 0 \; . \qquad 2.4$$

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here is no gain from assuming continuity of A, since the function h economically meaningful properties is not differentiable. Assume t A is continuous at (x_o,t_o). Then the directional derivative any direction s is approximated as a linear function of the :ial derivatives in the x and t directions. $\frac{dA}{dS} = \frac{\partial A}{\partial x} \cdot \frac{dx}{dS} + \frac{dt}{dS} \cdot$ $\frac{\partial A}{\partial t} \leq 0$, $\frac{\partial A}{\partial x}$ is infinite. But in the case of increasing rns, expansion results in $\frac{dA}{dS}$ lying outside the range given he coordinate partials. Therefore, A is non-differentiable (x_o,t_o).

stantaneous capital adjustment is made only at an infinite cost.
3 and 2.4 provide a definition of the 'quasi-fixed' factor - e whose increased utilization in t is possible but at an creasing marginal cost.

$$A(x^*,t) = A(x^*,t^*), \quad t > t^* . \qquad 2.5$$

is merely formalizes the assumption that secular influences cost, technical change, etc. are held constant. This is shown region II in Figure 2.3a.

iii'. There is a SRAC erected on every point in $A(x,t)$, rpendicular to the t-axis.

Figure 2.3b summarizes output-cost possibilities for the rm facing a once-for-all (or assumed once-for-all) change in nand. It is apparent that the $SRAC_o$ and the series of $LRAC_i$ rm a family of planning curves, each differentiated by the time lowed for the adjustment of capital. The limiting form of the nily is the traditional $LRAC^*$, where time is limited only by ular considerations. Thus the $LRAC^*$ is an envelope of elopes.

Suppose the new demand curve is represented by D^1, Figure b. Corresponding to the series of $LRAC_i$, there exists a series marginal cost curves $LRMC_i$. Maximizing present value results he equating of marginal revenue with a <u>particular</u> $LRMC_i$.[1] unit cost of this optimum output is then derived from the esponding $LRAC_i$.

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is optimum point can be shown graphically, but not derived nically, since the relevant $LRMC_i$ is itself derived from the nization procedure.

This means that the new equilibrium to be established will pend upon dynamic as well as static factors, since the former e necessarily present in the family of long run cost curves.

What factors are responsible for the slope of the cost rves? The $SRAC_o$ is as traditionally understood. One can, in e space of a very short time, increase output but only by the rther application of a <u>subset</u> of factors of production. Costs crease rapidly at the margin because of the limited substitut-ility of factors.

Next take the $LRAC^*$. The firm may be subject to internal onomies or diseconomies, due to the technical aspect of pro-ction, and external economies or diseconomies, due to the move-nt of factor prices,[1] both as size increases. The first category termines the returns to scale of the firm, and both categories fer to static phenomena. We distinguish static and dynamic fluences on costs as those resulting from the <u>size</u> of the firm l the <u>rate of growth</u> of the firm respectively. Traditionally, upward-sloping $LRAC^*$ is also assumed to result from managerial economies. This element I call a dynamic phenomenon. This is very familiar, except for the categorization of the managerial ment, which will be defended below.

The $LRAC_i$ involve an additional element, following the perty ii'. The more rapid acquisition and installation of tal requires the payment of higher capital costs. That is, stment costs are a positive function of the <u>rate</u> of adjustment

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refer here purely to (dis)economies external <u>to the firm</u>, but rnal to the industry.

capital. Typically, the adjustment cost literature assumes a nlinear function, so that adjustment costs increase dispropor-onately with the rate of adjustment. Higher capital costs have en excluded from the LRAC* by design. The LRAC$_i$ also depend on e above factors in the LRAC*, but in varying degrees of relative portance. The shorter the horizon, the more do the dynamic ements influence the shape of the LRAC$_i$.[1] At the margin, capital ods costs and managerial diseconomies are at their greatest for = 1 and diminish as i increases. The LRAC*, involving the ngest horizon under consideration, is subject most to static ctors, and least to dynamic factors.

Managerial diseconomies have been called upon to rationalize LRAC* increasing with output. The increase in unit costs result om the limited organizational abilities of a 'fixed' quantity of nagement. This characterization results from Marshall's dis-ction of time into arbitrary sections where some factors are mpletely variable and the rest completely fixed. Realistically, factor 'management' is not fixed during our long run period may be changed less rapidly than capital. It follows here that management diseconomies are the effective constraint on capital umulation, it is a constraint that is diminished through time management itself is given time to expand its capacities.

Though its implications have been neglected, this conclusion not new. G.B. Richardson writes in 1960:[2]

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his rationalizes, on a microeconomic level, the troublesome sage from Keynes' General Theory, explaining the slope of the MEC edule. See below, p. 117.

.B. Richardson, 1960 (109), p. 59

"... managerial diseconomies may indeed be more plausibly associated with the rate of growth of a company's operations, rather than, as is more usual, with their existing scale ... There is then a wide variety of factors which limit the supply potential of the individual firm, in the sense not of the size to which it might ultimately attain but of the maximum increase in capacity and output which it could effectively realize in a limited time".

owever, the earlier and probably more significant contribution s that of Penrose.[1]

"Thus the making of expansion plans, in which a firm has the requisite degree of confidence requires services which can only be produced within the firm. The production of these services requires time, and this limits the scope of a firm's expansion at any given time, but permits continued extension of these plans through time."

We may then conceive of the traditional LRAC* as the last an in a family of long run cost curves all of which will, at east turn up for dynamic reasons. The faster the firm tries to xpand output, the more must rising investment costs and organizaional problems figure in its total cost of expansion. Both these ynamic costs diminish as the planning horizon lengthens. This iterpretation rationalizes the existence of General Motors, and her super-giants. As Penrose notes, managerial problems proribe a limit to the size of the firm at any time, but these are ittled away with the passage of time.

The firm is so constrained because some factors causing creasing costs are themselves not limited in size, but in their te of growth. It is important to distinguish between reproducible d non-reproducible factors of production. If a factor is truly

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E. Penrose, 1955 (106), p. 535.

imited in size, the user will be subject to decreasing returns o scale. Land is the perfect classical example. Marshall used he classical framework for an analysis of <u>industry</u>. In his short un, an arbitrarily fixed capital stock plays the part of land[1] nd is responsible for diminishing returns. In the long run we re not so analytically fortunate.

Industrial factors of production are more typically re-roducible,[2] which are not limited in size, as is land, but rather n their rate of increase. Capital equipment, raw materials, labor nd managerial expertise all fall into this category.

We could conceivably draw up a cost curve for alternative utputs in which <u>all</u> reproducible factors have time to adjust to ach output. Call this curve $LRAC_n$, as the ultimate in the family f $LRAC_i$. The slope of the $LRAC_n$ is determined purely by static actors and indicates returns to scale. Presumably it must turn eventually from the limited supply of non-reproducible factors. uch a curve could have no operational significance since it sumes almost unlimited fore-knowledge. It nevertheless highlights e fact that the $LRAC^*$, like the SRAC, is defined for <u>arbitrarily</u> xed factors.

What have we established so far? Given a dynamic profit iterion, a once-over change in demand results in the establish-nt of a new equilibrium level of output, located in time, and self dependent upon dynamic adjustment costs which limits the

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And so earns a quasi-rent.

Excepting energy and environmental factors, which until recently ve not been binding on the firm's operations.

fitable accumulation of capital. This conclusion is given airing in Chapter 4, albeit in the context of a specific model capital accumulation, where some interesting results emerge.

What we cannot say from the above model is how fast this 'ly desired capital stock will be acquired. Suppose that the :imum level of output is being produced in three years. The :m will want to stagger its purchases of capital goods over : three years, but it is not possible to represent this accumula- ɔn diagrammatically.

2.2 Returns to Scale

It is possible to tackle a fundamental problem in the theory the firm, albeit rather heuristically. This is the problem of turns to scale, and the nature of long run unit costs. How is eturns to scale" defined? Mathematically, the definition is ıcise - returns to scale are increasing, constant or decreasing cording to whether the hessian of second partials of the pro- :tion function is negative, zero or positive. Given special ıditions,[1] returns to scale is given a familiar graphical re- :sentation in the sign of the slope of the LRAC*.

But an LRAC* increasing with output may be produced by ıagerial diseconomies, and this phenomenon does not indicate :reasing returns to scale. Only a subset of production factors perfectly variable. The factor management is the exception. may counter with the argument that returns to scale <u>should</u>

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onstant factor prices.

be defined over a subset of factors,[1] and that it is the ultimate fixity of some additional factor that provides decreasing returns to free variation of the chosen subset. This seems to be the correct understanding of scale returns, but I question the use of the factor management as that limiting factor. As has already been notes, actual firms have reached unprecedented size without running into some absolutely finite management barrier. It is management capacity limited <u>in time t</u> which interests us in practice, and which influences the slope of the LRAC*.

We are left with the conclusion that, even given factor prices, the slope of the LRAC* is not a pure representation of returns to scale. Dynamic adjustment costs also influence this slope.

What holds for the LRAC* holds for the LRAC$_i$ with greater certainty. Dynamic adjustment costs enter the LRAC$_i$, acting to offset increasing returns and reinforce decreasing returns to scale. We have already decided that the series of LRAC$_i$ eventually slope upwards of necessity from dynamic adjustment costs. The series of LRAC$_i$ shows, not decreasing returns to scale, but decreasing returns to growth.

Suppose a (medium) long run equilibrium has been established as on 34 above. That point on the particular LRAC$_i$ corresponding to the optimal plant chosen may be on either the downward-sloping or the upward-sloping segment of the LRAC$_i$, depending on the

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[1] This is the nature of the mathematical definition, though it follows indirectly from the inclusion of a limited number of measurable factors as arguments in the functional form.

elasticity of demand. If the equilibrium point is on the upward-sloping segment, nothing can be implied about returns to scale in equilibrium. Returns to scale may be either decreasing, constant or increasing. What is more, this may be true even if the firm is perfectly competitive.

2.2.3 Perfect Competition and Returns to Scale

Here we digress to confront a bugbear that has given theorists a difficult time. For the perfectly competitive firm facing constant returns to scale, the optimum size of the firm is indeterminate. With increasing returns to scale, there is no limit to the optimum size of the perfectly competitive firm, which is a contradiction in terms. Perfect competition is a powerful analytical tool, yet constant and increasing returns to scale are observed in practice. So a dilemma is posed.

It is informative to observe how the incompatibility between constant and increasing returns to scale and perfect competition was resolved historically. A good account of this is given in G.L.S. Shackle's The Years of High Theory,[1] Chapters 3 to 6. The essence of our problem is, what limits the size of the firm? What factors ensure that the profit-maximizing output is finite? For the perfectly competitive firm, demand is infinitely elastic, so the stabilizing element must come from the cost side. Factor prices are constant. Decreasing returns to scale does the trick, constant and increasing returns to scale do not.

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[1] G.L.S. Shackle, 1967 (120).

The early solution to the conflict lay in replacing perfect competition and its horizontal demand curve. The downward-sloping demand curve was introduced with increasing precision by authors from Cournot to Robinson and Chamberlin. Marshall saw that demand could be differentiated, and that the 'particular market' for a firm's product would render its demand inelastic.[1] In effect, limited demand prevents the firm from ever taking advantage of known economies, whose existence is admitted but irrelevant. Sraffa concurs.[2] The businessman is faced with decreasing costs, but demand is price-inelastic, and the greater economies are unattainable. Robertson and Shove, in a 1930 Symposium,[3] kept the horizontal demand curve by jumping verbal hops. Cost economies were supposedly attained only by new firms, existing firms being too conservative to take advantage of them. Each entering firm was larger and more efficient than the former. Thus a 'representative' firm (Marshall's concept abused) experiences decreasing costs, but each particular firm remains perfectly competitive. But this is not perfect competition as we usually understand it. Joan Robinson provided the last word on the subject.[4] The perfectly competitive firm can only be in full long run equilibrium when price equals minimum average cost, so the cost curve must turn upwards, ruling out continually decreasing costs. Long run equilibrium for the imperfectly competitive firm must be on a downward-

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[1] A. Marshall, op. cit. (99), p. 379.

[2] P. Sraffa, 1926 (122). See also G.L.S. Shackle, op. cit. (120), Ch.3.

[3] D.H. Robertson, et. al., 1930 (118).

[4] G.L.S. Shackle, op. cit. (120), Ch. 5.

sloping segment of the LRAC. Whether returns to scale are ultimately decreasing is not relevant to the imperfectly competitive firm.

After the war, the American domination of economic theory led to a new treatment of the incompatibility, one initiated by Robinson. In symbolic reasoning, perfect competition is a powerful analytical device which reduces the dimensions of uncertainty and enables tractable functional forms. This carries over to dynamic analysis. Consequently the recent trend has reversed the early direction, reinstated perfect competition, and thrown out increasing returns. Neither solution is wholly attractive. The first solution introduces a stable equilibrium from the demand side. The second solution requires that stability comes from an unnecessarily restrictive assumption on the cost side, ignoring evidence on increasing returns.

In a static framework, the dilemma remains. A neat compromise follows from the introduction of a dynamic framework, manifest in the series of $LRAC_i$.

We may assume perfect competition in the capital goods market if we like, since the sharply rising $LRAC_i$ depends also on internal problems of efficiency, which we have labelled 'managerial'. In the goods market, demand may be perfectly elastic, without denying the possible existence of the 'medium' long run equilibrium. We expect that the $LRAC_i$ to turn up from dynamic costs, so that long run costs offer a unique equilibrium for the perfectly competitive firm. But the equilibrium may not be characterized by the minimum point of the particular $LRAC_i$, since the firm may or may not be

earning normal profits in this dynamic equilibrium. With respect to normal profits, the medium long run is analogous to the short run.

The problem of normal profits[1] introduces a difficult question as to the meaning of a perfectly competitive firm in a dynamic world. We would like to maintain as assumptions the non-influence of prices, and freedom of entry. Now we have to encompass that huge area of adjustment to changing demand which Marshall and his descendants so conveniently ignored. Freedom of entry implies the absence of institutional barriers to entry. That it does not also imply zero adjustment costs for the potential entrant, is shown by the possibility of short run super-normal profits for the existing firm. The acknowledgement of adjustment costs means that any firm is free to enter but its <u>rate</u> of entry is limited. Thus the existing firm may earn supernormal profits in the medium long run as well as in the short run. The size of these profits depends entirely on how fast new firms can enter to rake off the surplus. If demand remains stable, the number of firms will eventually stablize, and our firm will be pushed to a profit level just permitting it to stay in business. This is the classical proposition. Normal profits only exist for the perfectly competitive firm in the stationary state. In a steady state, with demand growing at a constant rate, entering firms will always lag behind existing firms, and the latter will earn supernormal profits <u>permanently</u>.

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[1] That is, just enough for continued survival.

What is to prevent our firm from growing until it does reach a position of influence in its own market? The novel reply is, not decreasing returns to scale, but decreasing returns to forwth. 'Decreasing returns to growth' is the obverse of increasing costs to adjustment, by analogy with static terminology. Both dynamic adjustment costs and the rate of new entry[1] limit the firm's growth.

It appears from the nature of the medium long run equilibrium that the perfectly competitive firm may experience constant or increasing returns to scale, contrary to classical tenets. For a once-over change in demand, there are two new elements in the analysis. Firstly, there is a dynamic profit criterion, making rapid adjustment more profitable. Secondly, there are dynamic adjustment costs, making rapid adjustment less profitable. These, coupled with the indivisibility of capital, may confound the classical incompatibility.

As it happens, we are able to examine this assertion within the confines of the special model developed in Chapter 4. The novel possibility is not upheld for a once-over change in demand. This negative conclusion depends upon the particular assumptions used, which include quadratic adjustment costs[2] and perfectly divisible capital. For a constantly changing demand the situation is more promising, and the model allows the coexistence of perfect competition and constant returns to scale.

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[1] The rate of decrease of product price is a positive function of the rate of new entry.

[2] External adjustment costs are quadratic in gross investment, and internal adjustment costs are quadratic in net investment.

2.2.4 Long Run Growth

We have so far avoided any analysis of a constantly changing demand, which presents us with innumerable problems at this level. A fluctuating demand within the feasible region of short run output is simply handled, since adjustment is instantaneous. From the above construction of long run costs with an explicit time dimension, this possibility is excluded, by definition, for capital adjustment. Moreover, labor is rehired every period, so there are no initial stocks. Again capital is out of luck.

Herein lies the beauty of assuming otherwise - that all factors may be adjusted instantaneously. Then, Marshall's timeless representation of alternative stationary states is also a feasible representation of the expansion process. When the demand function changes, the firm moves immediately to a new optimum output, characterized by the equality of the marginal revenue with long run marginal cost $LRMC^*$. But whereas all points on the demand curve are atemporal alternatives, this is not so of points on the $LRMC^*$. The $LRMC^*$ shows the marginal cost of producing an extra unit of output if all factors are adjusted optimally. This is clearly not relevant to a growing firm with sunk capital. The true long run marginal cost lies anywhere in the range from $LRMC^*$ to infinity, depending on the rate of adjustment. So the $LRAC^*$ and $LRMC^*$ are not the appropriate cost curves for the living, growing firm. This is only to repeat the conclusion reached in Chapter 2, Section 1 that one cannot assume the firm to be in stationary equilibrium in every period with a fluctuating demand.

Refer again to Figure 2.3a. Instantaneous adjustment is equivalent to assuming that each point on the cost surface $A(x,t)$ can be reached from any other point. This is obviously fallacious in the case of an output reduction (not shown) since $A(x_i,t_j) \neq A(x_i,t_j + \tau)$. The time dimension merely highlights the fact that long run costs are non-reversible. Strictly, all points on $A(x,t)$ are alternatives and are with reference to (x_o,t_o).

If we want to deal with the growth process and the minimum cost of successive outputs, the construction of a cost tree is required - $A(x,t_j + \tau)/x_i,t_j$ where the decision to produce x_i in t_j changes all future cost-output-time combinations. An algebraic treatment would probably involve recourse to a recursive procedure, like dynamic programming, and will be avoided here. And a reasonable diagrammatic representation is impossible. Nevertheless a first approximation will be made, in the hope of gaining insight into the important problem of returns to scale.

Suppose that demand grows at a certain percentage per annum, and this is known with certainty. Suppose also that, from the dynamic profit criterion, new plants are constructed in periods t_a,t_b,t_c, etc. As a point of clarification, Figure 2.3b represents only the choice of plant 'a' for efficient production in t_a. We are now stringing a series of (medium) long run choices together, resulting in the 'equilibrium' long run cost curve $LRAC^e$. Unfortunately, the graphical derivation of the $LRAC^e$ is out of the question because of intertemporal profit tradeoffs. We could assume a "myopic" strategy so that the firm moves through a series of 'temporary' optima as described on p. 34. But even then the equilibrium point cannot be derived graphically.

A rigorous derivation of optimal firm expansion is provided in Chapter 4, although the emphasis is on capital accumulation. Here we shall be content to assume that a moving equilibrium exists. This is shown in Figure 2.4.

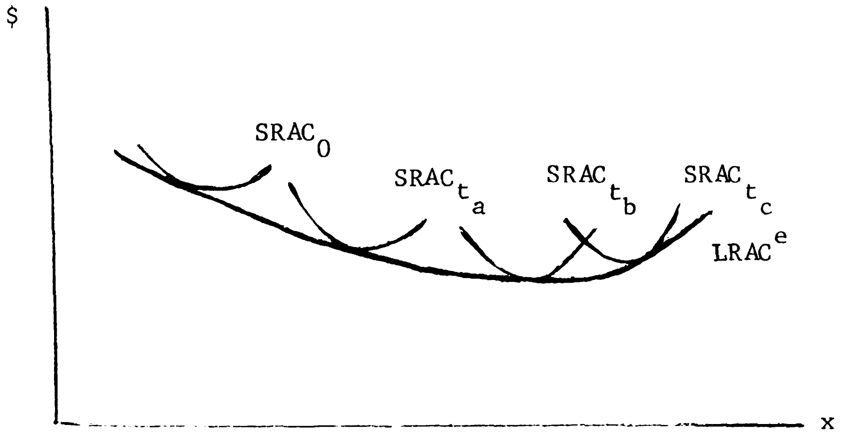

Figure 2.4 The Long Run Equilibrium Average Cost Curve

Though this diagram looks familiar, its interpretation is not. For the series of $SRAC_{t_i}$ represent unit costs for plants actually chosen. The envelope of these curves shows long run unit costs actually observable,[1] $LRAC^e$. The $LRAC^e$ is one possible branching process from the infinite number summarized in the cost tree $A(x, t_j + \tau)/x_i, t_j$. It is determined from the confrontation of cost alternatives with the known movement in demand.

There is evidence that Marshall tried to achieve something like this. The relevant note is sufficiently important to warrant its quoting almost in entirety. The problem at issue is increasing returns to scale, but the analysis is not dependent upon that particular assumption.

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[1] The observations might occur in the future.

> "We could get much nearer to nature if we allowed ourselves a more complex illustration. We might take a series of curves, of which the first allowed for the economies likely to be introduced as the result of each increase in the scale of production during one year, a second curve doing the same for two years, a third for three years, and so on. Cutting them out of cardboard, and standing them up side by side, we should obtain a surface, of which the three dimensions represented amount, price, and time respectively. If we had marked on each curve the point corresponding to that amount which, so far as can be foreseen, seems likely to be the normal amount for the year to which that curve related, then these points would form a curve on the surface, and that curve would be a fairly true long-period normal supply curve for a commodity obeying the law of increasing returns".[1]

Marshall has constructed a series of curves which appear to be identical to my $LRAC_i$ above, and confronted them with an estimated growth in demand. But it appears that Marshall intends his cost surface to represent planning alternatives which remain unchanged as expansion is actually carried out, ignoring the branching of cost-output alternatives. His "Long run normal supply curve" also purports to show long run supply price, which can only be true if the firm is perfectly competitive, and earning normal profits through time.

Again, the nature of returns to scale is not immediately evident from the $LRAC^e$, since the $LRAC^e$ also includes adjustment costs. Suppose that the adjustment cost schedule is time-invariant and that demand grows at a constant rate. This combination allows for a constant influence of adjustment costs in each capital equilibrium, and the slope of the resultant $LRAC^e$ would indicate returns to scale observed over time. There seems to be no reason why any shape of the $LRAC^e$ should be excluded a priori. The $LRAC^e$

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1 A. Marshall, op. cit. (99), p. 667, fn2.

might very well be horizontal, or monotonically decreasing, indicating permanent constant or increasing returns to scale respectively. The analysis of Chapter 4 shows that (even) the perfectly competitive firm may experience such permanent constant returns to scale in growth equilibrium.

Remember that the $LRAC^e$ shows successive equilibrium cost-output combinations, and is dependent upon a particular expected demand growth. A more rapidly increasing demand[1] will push the firm up the $LRAC_i$, involving the construction of higher cost plant. The resulting short run costs, $SRAC_{t_a}$, are greater than those from a slower demand growth, $SRAC_{t_b}$, implying a $LRAC^e$ greater in slope. In Figure 2.5, $SRAC_{t_a}$ shows the new short run cost alternatives following a rapid growth in demand. $SRAC_{t_b}$ shows the short run cost alternatives following a slower demand growth. $LRAC^e_a$ shows the minimum unit costs of successive equilibrium outputs given the demand growth for the first case, and similarly for the $LRAC^e_b$.

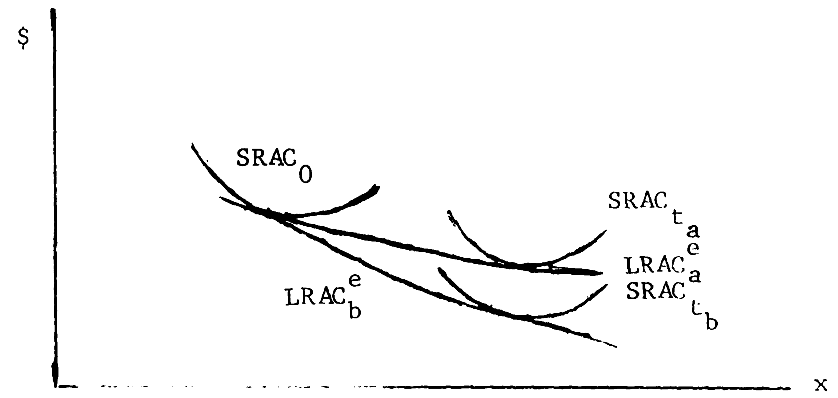

Figure 2.5 The Influence of Demand Growth on the Long Run Equilibrium Average Cost Curve

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1 The first discrete increment would be shown in Figure 2.3b.

It is meaningful to ask whether the firm's rate of growth would ever be such that adjustment costs completely offset increasing returns to scale, should they exist.[1] This is analogous to the situation in the short run, when the great restriction on time greatly inflates adjustment costs. This underlies the contention that increasing returns to scale can only be realized in time. Increasing returns to scale can only drag down unit costs by also overwhelming the additional costs of growth. Intuitively, it seems unlikely that the extra revenue gained from rapid expansion would be sufficient to compensate for the loss of scale economies, though this offsetting could hold for a short time with more continuous capital accumulation.[2] In any case, the relation between long run costs and returns to scale cannot be developed within the framework of the model in Chapter 4, and will have to remain conjectural.

Nevertheless a declining $LRAC^e$ indicates the firm experiences increasing returns to scale which are not offset by the higher costs of more rapid growth. This is a plausible state of affairs in practice, when rapidly growing firms, with known economies, can afford to pay inordinately high prices for skilled labor and additional management.

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1 In this case increasing returns to scale would be hidden by an upward-sloping $LRAC^e$.

2 "Thus a continuously progressive increase in demand may raise the supply price of a thing even for several years together; though a steady increase of demand for that thing, at a rate not too great fro supply to keep pace with it, would lower price." A. Marshall, op. cit. (99), p. 398, n2. More evidence of Marshall's groping towards the concept of a growth equilibrium supply price.

In a static framework, the stability of the firm is assured either by a declining demand curve on the demand side, or decreasing returns to scale on the cost side. But there is an important type of firm in practice which faces neither demand limitations, since the demand is predominantly self-induced,[1] nor decreasing returns to scale. Whey then are these firms not infinitely large? The rapidly increasing cost of expansion means that greater demand can only be catered to, and increasing returns only realized, with the passage of time. The size of the firm in time t is both determinate and finite.

[1] R. Marris holds that if the demand for any product is subject to rapid growth, it is more probably induced by the manufacturer. 1964 (97), Ch. 4.

CHAPTER 3

THE DEVELOPMENT OF A DYNAMIC THEORY OF INVESTMENT: A BRIEF SYNTHESIS OF THE LITERATURE

3.1 The Accelerator Principle

The Accelerator Principle provides the theory of investment with the major alternative starting-point to the neoclassical approach. The accelerator is also methodologically opposed to neoclassical theory, for it is a perfect example of inductive theorizing. It is based not upon an axiom of economic motivation but upon an empirical generalization. It is a reduced form equation which is itself postulated, rather than derived from higher order postulates. Aspects of a structural form consistent with the reduced form may be inferred, but the structural form is non-unique.

It has been presumed that a theory of investment derived from neoclassical propositions is intrinsically superior to the accelerator principle. This must be so, goes the argument, because the neoclassical theory is based on an axiom of economic motivation, in particular, the assumption of profit maximization.

But a usable deductive theory requires a set of specifying assumptions. In particular, the neoclassical structure has been partial to perfect competition and, until recently, the stationary state - two decidely unrealistic restrictions. Thus, it is by no means obvious that the neoclassical theory of investment should be a priori superior.

In one sense, neoclassical and accelerator versions complement each other. In the former, the level of demand enters through the back door. The demand influence on product price depends upon the unknown relation of the firm to its industry, and there is no attention to how this price might change over time. Emphasis is predominantly upon factor dependence on relative factor prices. With respect to capital, the neoclassical theory of the firm is fundamentally a theory of capital-deepening. Demand is precisely the emphasis of the accelerator theory of investment. To its own discredit, it omits the possibility of relative price changes. Though this may be a feasible representation when time is limited, or if the underlying technology is of the Leontief type (making short term relative factor prices irrelevant). Basically, the accelerator provides a theory of capital-widening. Each theory therefore is a partial explanation of investment demand, and their explanatory power depends upon the empirical importance of relative price changes and output. The work of Jorgenson and his co-authors has been an important first step in comparing the predictive power of the few major claimants to a useful theory of investment.[1]

Historically, the accelerator has its origins in the loose technical relation between an extra quantity of output (capital or

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[1] D.W. Jorgenson and C.D. Siebert, 1968 (73), 1968 (74; D.W. Jorgenson, J. Hunter and M.I. Nadiri, 1970 (75,76), at the level of the firm. For a summary of the entire literature see D.W. Jorgenson, 1971 (70).

consumption goods) and the amount of capital goods required to produce this output economically. Simply, an increase in output, 0, requires a proportionate increase in the output of capital goods, I. Though initially formulated loosely at the microeconomic level[1], it was intended as a partial explanation of the nature of industrial crises. It fits easily into the 'over-investment' theories of the business cycle.[2] In particular, it has received greatest exposure in mechanical business cycle theories in conjunction with the multiplier concept. But it has also been the recent subject of much testing at the firm and industry level.[3]

In its simplest form,

$$I_t = a_t \cdot \Delta O_t \qquad 3.1$$

where a_t[4] is the marginal capital-output ratio and ΔO_t is the change in output in t. This precise algebraic form has traditionally been taken as the Accelerator Principle. Thus, in distilling the essence of the notion of accelerated demand, a number of important qualifications have been avoided. In particular, 3.1 implies four (related) restrictive assumptions.

I. Net Induced Investment. In 3.1, output change is the sole determinant of investment. Replacement investment is given no special treatment, and there is no autonomous investment. In

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[1] G.H. Fisher has a useful survey on the early history of the accelerator, 1952 (33).

[2] See G. Haberler, 1958 (45), Ch. 3.

[3] Again, see D.W. Jorgenson, op. cit., (70).

[4] This parameter is henceforth underlined, viz a.

the words of Knox, the accelerator "... relates to investment in existing firms that produce existing types of goods for existing markets."[1]

II. Linearity. $\underline{a_t}$ is usually taken as a constant. If it is not assumed constant over time, its variation must be explained. That is, a non-constant $\underline{a}$ embodies all the non-accelerator determinants of desired investment. The explanatory power of 3.1 lies in the hypothesis that changes in $\underline{a}$ are small relative to changes in output.

Neoclassical theory posits the existence of an optimum combination of factor inputs given the technical production function and the factor prices. It follows that $\underline{a}$ is some function of the technical parameters and factor prices. Specifically, constant $\underline{a}$ requires constant parameters and prices. When applied at the macroeconomic level, the aggregate $\underline{a}$ is a weighted function of individual capital-output ratios. A constant aggregate $\underline{a}$ requires a constant composition of the individual ratios.

Further, it is held that for two successive increments in output of equal size to call forth the same investment, the production function must be subject to constant returns to scale. As an example, the following is taken from D. Hambert.[2]

Assume a Cobb-Douglas production function $O = AK^{\alpha}L^{\beta}$. From this and the static first-order conditions for a maximum profit, solve for K as a function of O to get:

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1 A.D. Knox, 1952 (84), p. 271.

2 D. Hamberg, 1971 (47), p. 23.

$$K = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta+\alpha}} \left(\frac{w}{r}\right)^{\frac{\beta}{\beta+\alpha}} A^{-\frac{1}{\beta+\alpha}} \right] \cdot O^{\frac{1}{\beta+\alpha}} \qquad 3.2$$

where w and r are the factor rentals. K is a linear function of O only if $\alpha + \beta = 1$, i.e. constant returns to scale. a equals the expression in the brackets.

Take $\alpha + \beta = 1$, and replace $A^{-1}\left(\frac{\alpha}{\beta}\right)^{\beta}$ by θ. Hamberg takes the total derivative with respect to time:

$$\frac{dK}{dt} = \theta\left(\frac{w}{r}\right)^{\beta} \frac{dO}{dt} + \theta \cdot O \cdot \beta\left(\frac{w}{r}\right)^{\beta-1} \frac{d\left(\frac{w}{r}\right)}{dt} \quad . \qquad 3.3a$$

Investment is attributed to two causes - that due to an expansion (capital-widening) effect, given by the first term on the right, and that due to a capital-deepening effect, given by the second term. If the factor-price ratio is constant, the second term is zero, and 3.3a reduces to 3.1. Given the assumed static framework, the accelerator 3.1 does imply a production function with constant returns to scale.[1]

We have been able to infer certain structural characteristics consistent with the simple accelerator, but these remain only possible characteristics. Indeed the inferred structure relies upon a familiar error. We cannot take the time derivative of an equation of stationary equilibrium, without also assuming instantaneous factor adjustment. One can legitimately take the total differential of 3.2, such that

---

[1] The requirement of a constant factor price ratio depends on the homogeneity of the assumed production function. More generally, the reduction would require constant factor prices.

$$dK = \theta \cdot (\frac{w}{r})^{\beta} dO + \theta \cdot O \cdot (\frac{w}{r})^{\beta-1} d(\frac{w}{r}) \qquad 3.3b$$

where the equilibrium increment in capital depends on the change in relative factor prices and output. The variables have no time-reference, and 3.3b is meaningless outside of stationary equilibrium. Nothing can be inferred about investment in t from 3.3b. The ideal procedure is to solve for optimum K_t from <u>dynamic</u> first-order conditions[1] and, by substitution, include output (determined simultaneously in t) as an argument in this equation. This procedure is exceedingly difficult to effect, and so will be ignored here.

It is, however, important to note that 3.1 does not represent a production function, and may not necessarily imply a fixed coefficient production function. A production function consistent with 3.1 will depend upon the entire inferred structural form.

III. <u>Optimum Capacity Utilization</u>. From 3.1, for every level of an increment of output there is an optimum level of investment. That is, I_t is desired investment. If 3.1 is taken as an empirical relation between output and investment, then actual investment is implicitly identified as optimal. Investment expenditure is always optimally adjusted. Theoretically, this may be deduced either from perfect fore-knowledge or from instantaneous adjustment to change, neither of which are attractive, as we have found. Immediate acquisition implies perfectly elastic supply from capital goods producers. Immediate disposal implies infinite depreciation of a segment of capital goods.

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[1] As in Chapter 4.

Of course, the firm is never in stationary equilibrium, so that either excess or deficient capacity may exist. In times of expansion, output change indicates desirable capital accumulation only when capacity is being well- or over-utilized. With economic decay, there are constraints on the rate of disinvestment, so that capital does not decrease as fast as output change would warrant. In the earlier literature, these qualifications were taken as implying an inoperative accelerator. At critical points the accelerator switches on and off, so that the mechanism is non-linear. The fault, however, lies in equating desired with actual investment. The parameter <u>a</u> cannot fill two roles as both the desired and the actual capital-output ratio. If we take <u>a</u> as defining the former, an additional explanation of actual investment is required.

This qualification was made originally by H.B. Chenery.[1] He observed that there is an optimum static relationship between capital and output, given technology and prices. One cannot difference this relation, as implied by the naive accelerator, 3.1, without making an unrealistic assumption about the change in capital stock. Chenery noted that a lag must exist for both the acquisition and disposal of capital, and the response will be asymmetric, since different causal factors are responsible for the lags. Moreover, Chenery showed that if a firm experiences increasing returns to scale,[2] the optimum position will be characterized by a certain

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[1] H.B. Chenery, 1952 (16).

[2] And if demand is increasing.

degree of <u>overcapacity</u>. This situation introduces another dimension into the capital-output relation, providing another reason for the modification of equation 3.1.

IV. <u>Perfect Knowledge</u>. In 3.1, the output change in the t-th period is known with certainty. If the course of future output were known, the necessary delays in acquiring and disposing of capital could be accounted for. With perfect knowledge and technical adjustment delays, the investment decision could be made ahead of time. Capital would always be optimally adjusted. Imperfect knowledge of the future is also responsible for the divergence of the existing stock from its optimal level.

Much intellectual exchange has been wasted on the simple accelerator principle. As a determinant of aggregate investment behavior, 3.1 has been criticized both analytically and empirically. To be empirically useful, the principle needs to account for some of the limitations noted above.[1] Gradually, authors have tackled the gradual adjustment of capital to change, with the introduction of distributed lags, rationalized simply as 'adaptive expectation' and 'partial adjustment'. Evans[2] has given an admirable summary of the accelerator and its refinements.

Consider lagged adjustment to change. Frictions prevent immediate adjustment to an optimal capital stock. The stock-adjustment model, independently devised by Chenery and Goodwin[3], is a fundamental contribution. Here the simple accelerator explicitly

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[1] This should be self-evident. But as late as 1957, for example, Hickman subjected the highly abstract formulation 3.1 to empirical testing. B.G. Hickman, 1957 (54).

[2] M.K. Evans, 1969 (32), Ch. 4.

[3] H.B. Chenery, op. cit. (16); R.M. Goodwin, 1951 (36).

determines the desired stock K^*, but allowance is made for the frictions inherent in adjusting to changed circumstances. Hence

$$I_t = \delta(K_t^* - K_{t-1}) \qquad 3.4$$

[1]

or

$$I_t = \delta(a \cdot O_t - K_{t-1}) \qquad 3.5$$

where investment is some fraction δ of the discrepancy between desired and actual stock. There is an assumption implicit in referring to the 'stock' version of the accelerator, $K = a \cdot O$, rather than the more correct flow version with I_t as the dependent variable. a is now the average capital-output ratio.

δ is the partial adjustment parameter. When $\delta = 1$, 3.5 reduces to 3.1, or the simple accelerator. In that case, all the required investment takes place during the t-th period. When $\delta \neq 1$, either surplus or deficient capacity has been inherited from past periods. Both this and the capital needs of the current period are now relevant to the investment decision. The introduction of the lag parameter in this way is arbitrary, but it is empirically superior to the naive version which does not allow for a world in disequilibrium.

But we must still allow for an uncertain future. The new version takes the estimation of future demand for granted. In 3.5, O_t must be known with certainty. (Alternatively, O_t can be interpreted as a proxy for future demand.) Eckaus[2] brought this to

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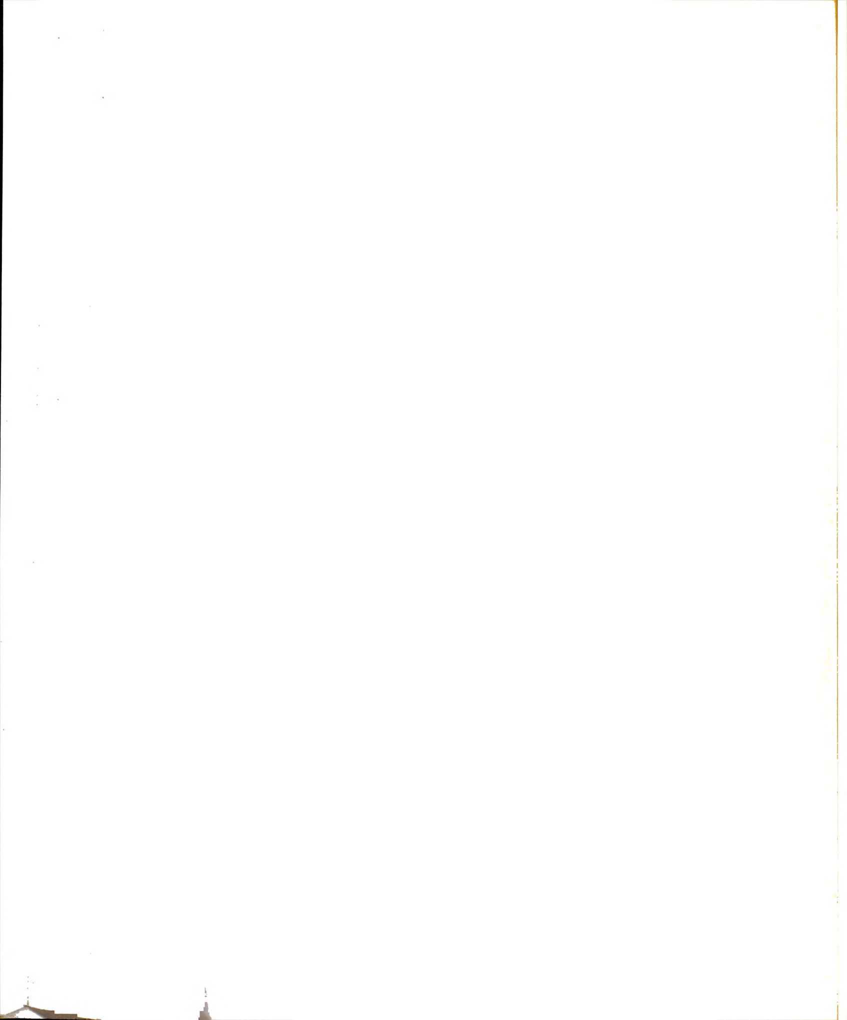
1 Where the variables are taken as measured at the end of the period.

2 R.S. Eckaus, 1952 (26).



attention. Koyck[1], however, has provided the classical introduction to this problem.

Suppose that demand is uncertain and is subject to cyclical fluctuations. The possibility arises that any increase in demand may be transitory rather than permanent. If investment is unwarranted, since the firm cannot easily divest itself of this capital, the cost of this undesired capacity must be borne for a considerable period. If the investment decision is delayed it can always be made the following 'year', and the associated losses are those of over-utilization of capacity and foregone sales. Moreover, the combination of increasing returns to scale and further increases in demand just around the corner may turn a hasty investment into an extremely unprofitable one.[2] It therefore "... seems plausible to state the hypothesis that there will be a lagged adjustment of capacity to cyclical rises in output as a consequence of the uncertainty of expectations".[3]

In corporating both the lag due to adjustment constraints and the lag due to the uncertainty of expectations, Koyck postulates a relation between capital stock and a weighted average of past period outputs. In particular,

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[1] L.M. Koyck, 1954 (86).

[2] For a comprehensive account of the effect of a time-variant demand on present-value calculations, see S.A. Marglin, 1963 (96).

[3] L.M. Koyck, op. cit. (86), p. 67.

$$K_t = \alpha(1-\lambda)[O_t + \lambda O_{t-1} + \lambda^2 O_{t-2} + \ldots] \quad (3.6)$$[1]

where the weights decline geometrically. This lag series is intuitively appealing, allocating less weight to outputs more removed in time. It is obvious that when $\lambda = 0$, 3.6 reduces to 3.1. The larger is λ, the more significant is past output in determining capital stock and the more divergent is this formulation from the simple accelerator. Through a well-known transformation, 3.6 reduces to

$$K_t = \alpha(1-\lambda)O_t + \lambda K_{t-1} \quad (3.7)$$

$$I_t = (1-\lambda)[\alpha O_t - K_{t-1}] \; . \quad (3.8)$$

This is algebraically equivalent to 3.5 when $\delta = 1-\lambda$. This equality does not, of course, provide any theoretical rationale for 3.5. There is even less information in the Koyck derivation. Koyck shows that a statistically simple equation in output and lagged capital stock may be derived from a general function in lagged output by choosing appropriate weights. The justification for 3.5 still lies in its empirical usefulness.

In principle it would be nice to distinguish between the technical and institutional lags (partial adjustment) and the lags due to uncertainty (adaptive expectations). Denote the former by δ and the latter by λ. Postulate that desired stock K^* is a weighted function of past levels of output with Koyck weights:

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[1] This is a simplified version of the Koyck equation. In particular, Koyck's equation is logarithmic, and includes a constant term and a time variable to allow for non-output determination of the capital stock. A more general version has the declining weights starting in a period later than the first.

$$K_t^* = a(1-\lambda)[O_{t-1} + \lambda O_{t-2} + \ldots]\ ^{1}. \qquad 3.9$$

Here the role of output is explicitly recognized as a proxy for future demand. Using a series of past outputs allows for the vagaries of demand. Add the partial adjustment equations:

$$I_t = \delta(K_t^* - K_{t-1}) \ . \qquad 3.5$$

Combining the two and applying the 'Koyck' transformation

$$I_t = \delta[a(1-\lambda)O_{t-1} - K_{t-1} + \lambda K_{t-2}] + \lambda I_{t-1} \ .$$

Replace I_{t-1} by $K_{t-1} - K_{t-2}$ [2]

$$I_t = \delta a(1-\lambda)O_{t-1} - (\delta-\lambda)K_{t-1} - \lambda(1-\delta)K_{t-2} \ . \qquad 3.10$$

Waud[3] has claimed that the statistical estimation of lagged adjustment from time series should involve both partial adjustment and adaptive expectations effects. In effect, he is arguing that to put to empirical test any theory of lagged adjustment which does not explicitly account for uncertainty is to invite specification error, and therefore inconsistent estimation of the parameters.

Equation 3.10 is therefore the general model for lagged adjustment, and the use of either equations 3.5 or 3.9 by themselves imply misspecification of the adjustment process. In particular,

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[1] We assume that O_t is unknown, and begin the series with O_{t-1}.

[2] Neglecting depreciation.

[3] R.N. Waud, 1966 (138).

the use of equation 3.5 (thereby neglecting uncertainty) imparts a substantial upward (asymptotic and small sample) bias to <u>a</u>, and a serious downward bias to δ[1]. In our discussion, <u>a</u> is the capital-output ratio and δ is the lag due to technical and institutional factors.

I would interpret the downward bias to δ as indicating that the estimated δ includes some of the neglected effect of uncertainty. This is quite reasonable. The true δ is apparently larger than the estimate from the commonly-used equation 3.5. Consequently the true lag due to 'partial adjustment' is shorter than that derived from equation 3.5.

It is interesting to note that in the usual specification of the equation with both lag effects, the two lag parameters, $(1-\lambda)$ and δ, enter symmetrically, and therefore cannot be identified.[2] In the formulation 3.10, the usual dependent variable is differenced (I_t instead of K_t), and the parameters enter the coefficient of K_{t-1} nonsymmetrically. This difference allows us to identify λ and δ.

3.2 Optimal Capital Accumulation without Adjustment Costs

The simple accelerator has been modified to account for the frictions associated with adjustment and the uncertainty of expectations but the influence of relative prices is avoided.[3] The prolific work of D.W. Jorgenson exists to satisfy this neglect. It has been

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1 R.N. Waud, op. cit. (138). Moreover, the bias from this source substantially outweighs that resulting from the presence of lagged dependent variables in the regressors.

2 c/f J. Johnston, 1972 (62), p. 303.

3 This influence should be observed both in the specification of optimal capital stock, and in the (constant) adjustment parameters, λ and δ.

Jorgenson's ambition to "derive a demand function for investment goods based on purely neoclassical considerations".[1] The emphasis is placed on relative prices and factor substitutability. In particular, investment demand is a significant function of the cost of capital - itself a function of the price of investment goods, the rate of change of this price, and the rate of interest. Jorgenson claims that previously Tinbergen, Roos and Klein have used neoclassical variables in the investment function but there was no systematic derivation of the way the variables entered the function. Further, their work was prior to the general use in investment analysis of distributed lags.

A model representative of Jorgenson's writings may be characterized as follows:[2]

i. The firm faces a certain future. Current and future prices, determined exogenously, are known with certainty.

ii. The set of technological possibilities facing the firm may be described by a production function, strictly convex, relating current output flows to current applications of labor and capital services, viz.

$$F(Q,L,K) = 0 \; . \qquad 3.11$$

iii. The quantity of capital services available is independent of the age distribution of the firm's capital stock. Further, it is assumed that replacement investment is proportional to the stock

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[1] D.W. Jorgenson, 1967 (66), p. 133.

[2] ibid.

of capital, the constant of proportionality given by γ. The net change in capital stock is given by

$$\dot{K}_t = I_t - \gamma \cdot K_t \quad . \qquad 3.12$$

iv. The firm choses that plant for production and purchase of capital services which maximizes its present value.

Denote net receipts at time t by

$$R_t = p_t \cdot Q_t - w_t \cdot L_t - q_t \cdot I_t \qquad 3.13$$

where Q, L, and I represent output, variable input (labor) and investment in durable goods, and p, w, q their prices. Present value W_t is defined as the integral of discounted net receipts

$$w_t = \int_0^\infty e^{-rt} R_t dt \qquad 3.14$$

where r is the constant time rate of discount, here the 'nominal' rate of interest. W is maximized subject to the constraints 3.11 and 3.12. From the Euler first-order conditions[1], the requirements familiar from static theory are derived. That is, the equality of the (real) marginal productivities of the factor services with the ratio of their cost to the product price. For capital,

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[1] The Euler conditions for a maximum are analogous to the first-order conditions in static theory. In this dynamic problem, a functional of a variable K, $\dot{K}$ and t is maximized. A work by D.A. Pierre, 1969 (105) is more comprehensible than most on the subject. The first book to deal at length with dynamic optimization for economists is G. Hadley and M.C. Kemp, 1971 (46). In any case, a brief and lucid exposition of the variational calculus is included in the Appendix to Chapter 4, courtesy of P.A. Tinsley.

$$\partial K/\partial O = \frac{c}{p} \quad .$$

There are two noteworthy aspects of these conditions. Firstly, the marginal productivity of capital equation uncovers the price of capital services c as a function of q, r and γ. Specifically,

$$c = q(r + \gamma) - \dot{q} \quad .$$

Secondly, the marginal conditions are usually associated with static theory and refer to a unique equilibrium point. In this dynamic model, the identical conditions hold at every point in time. Although Jorgenson utilizes the variational calculus, capital is implicitly assumed perfectly variable, and a static equilibrium is (illicitly) established for all t. This error is discussed at length in Chapter 2.2.

To summarize, Jorgenson's model consists of the two marginal productivity conditions and the two constraints, the production function and the net investment equation. From this are derived the optimum quantities of output, labor, capital and investment for each point in time. The demand for investment goods is an implicit function of prices, their rate of change and the rate of interest, as well as of output and the existing capital stock.

One of the advantages of this formulation is that it provides an opportunity to test the influence on investment of tax variables and inflation through their effect on relative price changes and especially on the price of capital services.[1]

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[1] In particular, see the work of R.E. Hall and D.W. Jorgenson, reported in D.W. Jorgenson, 1971 (70).

We have a theory dynamized to the extent that its variables are time-referenced. According to Jorgenson and Siebert[1], "the theory is simply the intertemporal analogue of the usual atemporal theory based on profit maximization". But the model is still a series of stationary optima strung together. Jorgenson rectifies this deficiency by treating the above purely as a model of 'optimal capital accumulation'. This set of dated equilibrium conditions gives the optimum growth path of the firm. But this is a spurious sort of optimum which would be attained if the market-place were without frictions, an optimum that is never achieved. We have the same dilemma that the accelerator theories confronted in equation 3.1. Jorgenson's solution is the same, to attach a statistically attractive distributed lag mechanism to the stationary optimal capital stock, albeit a more sophisticated one[2].

Specifically, Jorgenson assumes that the desired level of capital is not attained immediately because time is necessary for the initiation of projects, the appropriation of funds, the letting of contracts, etc. To the determination of the desired capital stock he adds the determination of actual investment expenditures.

Following a change in demand for capital, actual investments to satisfy this new demand are made over time according to a constant distribution pattern. Specifically, in every period the level of investment expenditures I_t^a is a weighted average of the level of projects initiated I^n in all previous periods:

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1 D.W. Jorgenson and C.D. Siebert, 1968 (74), p. 3.

2 See D.W. Jorgenson, 1966 (65).

$$I_t^a = \mu_1 I_t^n + \mu_1 I_{t-1}^n + \ldots = \mu(s) I_t^n \qquad 3.17$$

where u(s) is a power series in the lag operator S. The behavioral assumption implicit in this is that the increment in desired capital in time t is initiated in t as new projects:

$$I_t^n = K_t^* - K_{t-1}^* \,. \qquad 3.18$$

This is the pseudo-classical assumption. But because there is a lag in actualization, 3.18 must be substituted in 3.17. Hence:

$$I_t^a = \mu(s)[K_t^* - K_{t-1}^*] \,. \qquad 3.19$$

In the pseudo-classical case u(0) equals 1, with all succeeding terms equal to 0. There is no lag in adjustment. The Koyck lag is also a special case of this distributed lag, where u(S) is represented by $(1-\lambda)/(1-\lambda S)$. Here 3.19 reduces to 3.4 called the 'flexible' accelerator. In both the Koyck and the general function, actual stock always lags behind desired stock by the backlog of uncompleted projects. The firm is in a constant degree of disequilibrium[1] represented by the parameters of the lag distribution.

3.3 Optimal Capital Accumulation with Adjustment Costs

It is instructive to compare briefly the modified accelerator models 3.5 and 3.8 with the model of Jorgenson, 3.19. Both present an equation for desired capital stock, although in the first, output change is the sole determinant and in the second, relative

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[1] c/f J.R. Hicks, 1965 (57), p. 99.

prices predominate. Both 'tack on' to this equation a similar disequilibrium adjustment hypothesis, based on statistically desirable properties.

Eisner and Strotz, in their seminal essay,[1] take theoretical account of frictional adjustment to changed circumstances. This article marks a methodological turning point for the theory of the firm.[2] The characterization of factors of production as fixed or variable is seen not as based on some technically immutable condition, but on the cost associated with its rapid change. The level of the 'variable' factor is altered relatively cheaply, while the rapid adjustment of the 'fixed' factor, though possible, is prohibitive in cost. Transition from one state to another, is not without frictions, and is therefore not automatic but an object of decision based on economic considerations. In static theory, adjustment and disequilibrium are synonymous. In dynamic theory, the rate of adjustment is an integral part of the determination of the optimum position for the firm.

Again assume that future prices are known with certainty and, apart from the price of investment goods, are constant. For investment goods, they assume that a more rapid accumulation of capital involves a higher per-unit cost. Accordingly they postulate a 'cost-of-expansion' function such that "the adjustment path will be determined not by inflexible technological requirements but by

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[1] R. Eisner and R.H. Strotz, 1963 (30).

[2] Heralded by early contributions by Evans and Roos in the 20's and 30's.

the very principles of profit maximization which determines the equilibrium position itself."[1]

Specifically, let the cost of expansion be a function of the level of investment $\dot{K}$, and the decision-expenditure lag, t.

$$C(t) = C(\dot{K},t) \qquad C_{\dot{K}} > 0 \quad . \tag{3.20}$$

This function replaces the linear function $q \cdot I$ in equation 3.13. Eisner and Strotz (hereafter called E & S) further postulate a profit function P in the capital stock K, P(K),[2] where $P_k > 0$ and $P_{kk} < 0$.

Let the firm possess a (stationary) optimal capital stock and let a once-for-all change determine a new stationary equilibrium. What is the path of investment from one equilibrium to the other? In a special model, E & S posit a quadratic profit function P in K (a simplified convex production function) and a quadratic cost-of-expansion function C in $\dot{K}$[3], $C_{\dot{k}\dot{k}} > 0$. Although a distinction is made between internal and external adjustment costs, the C function incorporates both elements.

Under these special conditions, the Euler first-order conditions lead to an important second-order differential equation in K.

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[1] ibid. p. 481.

[2] P(K) is a 'reduced' production function from which the optimally-adjusted labor stock is eliminated. The second derivative P_{kk} should, more correctly, be read as the familiar Hessian in all factors.

[3] Here C is assumed independent of the decision-expenditure lag, t.

[4] The quadrature plays an essential part in making the problem analytically tractable, and producing a unique solution for capital.

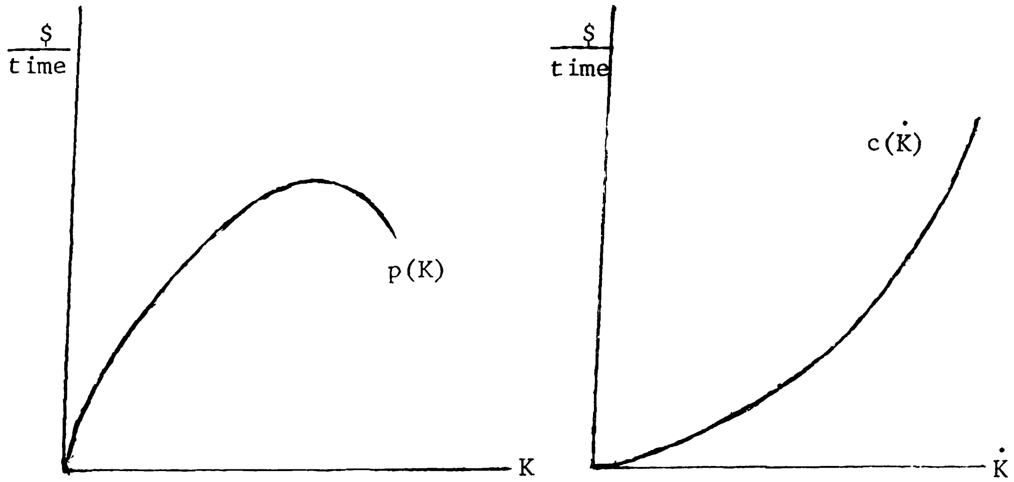

Figure 3.1

Eisner-Strotz Profit Function and Cost-of-Expansion Function.

$$\ddot{K} - r\dot{K} - \frac{p_{kk}}{c_{\dot{k}\dot{k}}} K + f = 0 \quad [1] \qquad 3.21$$

which solves to

$$K = (K_0 - K^*)e^{\lambda t} + K^* \quad [2] \qquad 3.22$$

where $K^* = (\frac{p_{kk}}{c_{kk}})^{-1} f$ and

$$\lambda = \frac{r}{2} - \sqrt{(\frac{r}{2})^2 - \frac{p_{kk}}{c_{\dot{k}\dot{k}}}} \qquad 3.23$$

is the negative (real) root of the homogeneous part of 3.21.

The expansion path of the firm is given by the optimum stationary stock minus a decaying exponential. Capital approaches its optimal level asymptotically. The adjustment parameter is e^{λ} where e^{λ} is the ratio of the (t+1) to the t-th period's investment.

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[1] Where f is a constant. f has no simple economic interpretation.

[2] For a detailed derivation of this solution see below, p. 98 .

When t is re-introduced as an argument in the C function, it is assumed to have the effect primarily of delaying the start of expansion. In this case the expansion path can be approximated by the above exponentially decaying function.[1]

If the adjustment period is split up discretely, presumably into short run decision periods, the weights applying to the level of investment in each sub-period constitute the declining geometric series of Koyck. Granted that the Koyck weights in the flexible accelerator 3.6 were proposed for their statistical convenience, it is commonly assumed that Eisner and Strotz have provided the missing theoretical rationale for this model. In fact, the two models differ.[2] The Koyck equation 3.6 has actual stock as a linear function of lagged outputs. The level of output has a geometrically declining influence upon the actual capital stock. If it is assumed that output bears a fixed relation to the desired capital stock, such that $K_t^* = a \cdot O_t$ (the simple accelerator applied to optimum capital), then 3.6 is equivalent to

$$I_t = (1-\lambda)[\Delta K_t^* + \lambda \Delta K_{t-1}^* + \lambda^2 \Delta K_{t-2}^* + \ldots] \qquad 3.24$$

which is the flexible accelerator case of Jorgenson's general equation 3.19.

By comparison, the investment equation of Eisner and Strotz is

$$I_t = (1-\xi)\xi^t \Delta K^*, \; \xi = e^{\lambda} \qquad 3.25$$

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[1] op. cit. p. 484.

[2] This has also been observed by P.A. Tinsley, 1969 (125), 1970 (126) who, by introducing a moving optimum, has correctly rationalized the Koyck lag.

where ΔK^* is a once-for-all change in optimum stock.

Clearly the two models are distinct. In Koyck's formulation capital decisions are made every period. Remember that because of the lag, there is always a gap between the optimum and the actual stock for any period. This is true unless output is constant, a possibility which is excluded by assumption in order to analyze uncertainty. In E & S, demand facing the firm is constant but it is never wholly met in disequilibrium because of expansion costs. These costs in a certain world determine the E & S lag.

Now Eisner and Strotz have taken the theory of investment one step further. By the explicit inclusion of nonlinear adjustment costs in the optimization process, they have rationalized the optimal adjustment of a firm to a once-for-all change in demand. This is, in fact, the problem we posed in Chapter 2.2 and attempted to gain insight from an extension of Marshallian unit cost curves.

But although capital and investment are merely different faces of the same variable, Eisner and Strotz create the impression that investment is a disequilibrium phenomenon in some way removed from the stationary capital stock. This is evidenced by their examination of the comparative dynamic effect of varying the interest rate. Since the adjustment process is derived from a maximum problem, the adjustment parameter λ is not a constant but a function of the rate of interest and second partial derivatives of the production function and the expansion cost function. In particular e^{λ} is a positive function of r, implying that a higher rate of interest will delay investment. This is quite true, and a worthy discovery. But E & S fail to realize that any parameter

variation which influences the rate of investment must also influence the ultimate capital stock. They are not determined independently.

The E & S model is still bound by its stationary state framework. We don't have an infinity to adjust capital before the parameters change again. With a regularly changing environment, the optimal capital stock itself is changing. We need a model of investment based on moving optima.

Nevertheless, the E & S contribution is an important vehicle of analysis, and it has been taken over and extended by others. R.E. Lucas[1] has generalized E & S to the case of multiple 'quasi-fixed' inputs. A non-diagonal adjustment matrix results, implying that the rate of adjustment of any factor depends on the divergence between the optimal and actual stock for all factors. Further, the need for analytical tractability manifest in linear differential equations limits our alternatives in the representation of production functions. E & S chose to assume a quadratic function. Lucas has taken the alternative route, that of a linear approximation to a general production function around the assumed stationary optimum.[2] He proves that the matrix generalization of the flexible accelerator is a valid approximation, near the optimal point, to the optimal paths of capital accumulation.

A.B. Treadway[3] has made the observation that the inclusion of adjustment costs in the optimization process may negate accepted

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[1] R.E. Lucas, 1967 (91).

[2] This is a procedure with historical precedence, attributed to Poincare in the analysis of physical dynamics. See M.J.P. Magill, 1970 (94).

[3] A.B. Treadway, 1970 (132).

comparative static theorems in the theory of the firm. In particular, the elasticity of demand for a factor is presumed to increase monotonically as a greater number of other factors are allowed to vary. In a dynamic world, some factors are varied more rapidly than others. Treadway shows that the elasticity of demand for the (relatively variable) factor labor may be greater in the short run than in the long run if labor contributes, not only to production in conjunction with capital, but also the process of capital adjustment.

All previous authors rationalize adjustment costs, both external and internal, by an imperfectly competitive capital goods market. Treadway also introduced the mathematical representation of adjustment costs internal to the firm,[1] adapting the production function itself to the incorporation of dynamic elements. Since this procedure is adopted in the model in Chapter 4, a more detailed exposition is delayed until then.

Given the assumption of constant future prices (the sole concern of this section), J.P. Gould[2] achieves some interesting results from specific assumptions. Gould specified a model involving quadratic adjustment costs, following E & S, but this time in gross investment, I. He cites evidence that the quadratic assumption is a reasonable approximation to a typical industrial adjustment cost structure. Gould further assumes the production function to be linear and homogeneous.

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1 A.B. Treadway, 1969 (131).

2 J.P. Gould, 1968 (38).

The model is structurally similar to the E & S model, 3.21. However, the marginal productivity of capital is a function only of the capital-labor ratio and, by substitution, of relative factor prices. Consequently, the coefficient of K[1] is independent of the production function and depends solely on nonlinear adjustment costs in replacement capital δK, as part of gross investment.

$$\ddot{K} - r\dot{K} - (r + \delta)\delta K + f = 0 \quad . \tag{3.26}$$

The resulting model is equivalent to 3.22 where the adjustment parameter λ equals minus the depreciation rate $-\delta$.

$$K_t = (K_0 - K^*)e^{-\delta t} + K^* \quad . \tag{3.27}$$

In essence, Gould's firm faces perfect competition in the goods market, and constant returns to scale - the dilemma confronted in Chapter 2. Stability comes from assuming external adjustment costs - call it imperfect competition in the capital goods market. The greater the investment, the greater the unit price. But since Gould uses I instead of $\dot{K}$ as the argument of the adjustment cost function, capital itself faces nonlinear adjustment. This results not only in a determinate rate of capital accumulation but also in a determinate optimal capital stock. This particular model will be discussed further in Chapter 4.

3.4 Optimal Capital Accumulation with Adjustment Costs and Nonconstant Prices

Eisner and Strotz and others have rationalized the gradual adjustment of capital to a once-for-all change in the environment.

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[1] The coefficient of K is important when considering returns to scale. See below, Chapter 4.3.

But the flesh-and-blood firm faces continual changes in its environment. It is therefore desirable to introduce nonconstant future prices into the framework of a model of capital accumulation. A possible solution lies in revising the assumed constant prices whenever prices actually change. The moving optimal stock would be a series of revised stationary targets. K^* is replaced by K^*_t. But this 'myopic' strategy uses only information from the current period. P.A. Tinsley shows that the explicit account of price expectations produces a superior approach to the myopic strategy.[1]

Tinsley is predominantly responsible for the respresentation of nonconstant prices in the adjustment cost literature.[2] Since the following model is based on Tinsley's work it would be repetitive to summarize it here.

It is sufficient to point out that the optimal adjustment of capital to a non-stationary target reduces to an equation[3] of the form:

$$\ddot{K} - r \cdot \dot{K} - CK + f(t) = 0 \qquad 3.28$$

where C is a function of the production and adjustment cost functions, and the inhomogeneous term in prices, f, is now a function of time. Compare equation 3.21.

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1 P.A. Tinsley, 1970 (126).

2 See also P.A. Tinsley, 1969 (125), 1971 (127).

3 Making commonly-used assumptions.

Obviously uncertainty is a paramount consideration limiting the analysis of expectations, but attempts at the stochastic representation of future prices are in their infancy.[1] This complex problem will be ignored here. In what follows, we assume a deterministic future - prices are nonconstant but known with certainty.

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1 See for example, P.A. Tinsley, 1969 (125), 1970 (126); R. Craine, 1971 (20); M. Nerlove, 1971 (103).

CHAPTER 4

A MODEL OF OPTIMAL CAPITAL ACCUMULATION

This chapter presents a mathematical formulation of the optimal growth of the firm. The firm chooses, simultaneously, optimal paths of output and factor acquisition. The framework follows naturally from the developments of optimal adjustment theory in Chapter 3. Certain simplifications are made. The future is deterministic, as previously stated. There is one perfectly variable factor, labor, and one quasi-fixed factor, capital. Though it is clear that all factors are variable to a greater or lesser degree, an analysis of multiple factor demand would unnecessarily complicate matters.

Though the details are selective, the framework of the model, including the specific quadratic production function 4.15, belongs to Tinsley. The generalized 'dynamic' production function 4.4 is similar to the work of Treadway.

4.1 The Model

1. The firm faces a demand function for a single product X such that:

$$P(t) = P_t \; . \qquad 4.1$$

Demand changes over time with population, tastes, etc., and the entry and exit of competitors. Price is given and known for each

decision period but is non-stationary.

2. The labor-supply function is given by

$$W(t) = W_t \ . \tag{4.2}$$

Again, the wage is given and known for each period, but is non-stationary. Labor is a perfectly variable factor. That is, labor adjusts optimally to any parameter change, instantaneously.

3. The capital goods supply function.

$$Q(t) = Q_{0 \cdot t} + Q_1 \cdot I(t), \quad Q > 0 \ . \tag{4.3}$$

$Q(t)$ is the price of investment goods in t. The greater the rate of investment in t, the greater the cost of investment goods in t. This results simply from a (linear) upward-sloping (short run) supply function in the capital goods industry. Equation 4.3 also implies a quadratic adjustment cost function, as postulated by Gould, for example.[1] At the same time, there are exogenous changes in this industry which, for simplicity, have been summarized in the intercept Q_0. Thus an increasing long run supply (shifting schedule) expresses itself in $\dot{Q}_0 < 0$, and vice versa.

Assumptions 1. and 3. together mean we have assumed perfect competition in the output market but not in the capital goods market. The only reason for this less than complete generalization is that the analysis of a monopolized goods market becomes hopelessly intractable.

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1 J.G. Gould, op. cit. (38), p. 48.

4. We postulate a dynamic production function

$$X = X(K,L,\dot{K}) \qquad 4.4$$

where X,K,L and $\dot{K}$ are all functions of time. Treadway's highly significant work[1] utilizes a function in these arguments. However, Treadway refuses to label 4.4 a 'production' function because of the added element, $\dot{K}$, although he earlier interprets 4.4 as a multi-product function in which the two products are X and $\dot{K}$.[2] We are forced to confront the meaning of the traditional implicit relation between inputs and outputs. If 4.4 represented a technical relationship, Treadway's aversion would be sound. But economists have never been consistent on the interpretation of the production function. At the process, and perhaps the plant level, the function may purport to be technical. But at the level of the firm, the function is not so stringently defined, since there is no ready way of measuring those organizational elements, without which the physical inputs would be valueless. The factor 'management' is hidden in the functional form $f(\cdot)$ itself. In fact, with entrepreneurial inputs included, the firm production function is, of necessity, a behavioral relation. This aspect is typically hidden, since the "efficiency" assumption of rational action plus perfect knowledge puts the firm always on its production frontier.

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[1] A.B. Treadway, 1969 (131), 1970 (132), 1971 (133).

[2] A.B. Treadway, op. cit. (133), p. 847; (131), respectively.

However, Marshak and Andrews,[1] for example, introduce a disturbance term allowing for entrepreneurial inefficiency. This exposes the behavioral nature of the production function. But Marshak and Andrews have in mind differential efficiency across firms. We are more interested in differential efficiency of the same firm across time.

Underlying the production function $X = f(K,L)$ is an implicit efficiency assumption. $f(\cdot)$ maps factor combinations into <u>maximum</u> output. This has some legitimacy in static theory. In a temporal context, this also requires an instantaneous learning process.[2] Here we can be more realistic and allow for managerial adaptation to change.

Retain $f(K_t, L_t)$ as the efficiency mapping, and introduce a learning (cost) function $L(\cdot)$ which detracts from output in t. We expect that the greater the rate of expansion the greater (absolutely) is $L(\cdot)$. We therefore assume that L is a positive function of $\dot{K}$. It is also possible that the <u>size</u> of the firm influences the learning adjustment cost, and probably favorably. Thus we assume

$$L = L(\dot{K},K), \quad L_{\dot{K}} > 0, \; L_{\dot{K}\dot{K}} > 0, \; L_{\dot{K}K} < 0 \qquad 4.4a$$

where K is a proxy for the firm's size, and $\dot{K}$ similarly proxies for the firm's rate of expansion.

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1 J.M. Marshak and W.H. Andrews, 1944 (98).

2 In textbook theory, movement across isoquants in the K-L plane is labelled an 'expansion path'. This is only strictly true with the above assumption.

The 'temporal' production function is then given by

$$X_t = f(K_t, L_t) - L(\dot{K}_t, K_t) \quad . \qquad 4.4b$$

This is a specific form of equation 4.4. Note that 4.4b exhibits non-separability[1] when K enters L. Treadway[2] devotes considerable space to the possibility of interaction between $\dot{K}$ and L. In particular he argues that labor may facilitate the expansion process, implying $X_{\dot{K}L} > 0$. This interaction does not seem to be of great importance[3] and will be ignored here.

Following the dichotomy of Chapter 2, it is apparent that external adjustment costs may be rationalized by the frictional adjustment of capital - assumption 3, while internal adjustment costs may be rationalized by the frictional adjustment of management - assumption 4. Of course, additional 'management' may be acquired externally, as is usually the case in rapidly expanding corporations. But the problem of management coordination remains, so that adjustment costs may not be completely externalized.[4]

Traditionally, we define returns to scale according to the sign of the hessian on f(K,L), the efficiency locus. Thus

$$H = f_{KK}f_{LL} - f_{KL}^2 < 0$$

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[1] Non-separability defines a non-zero cross product between a factor and any factor's rate of change.

[2] A.B. Treadway, 1970 (132).

[3] In any case, transitional labor may be included in the (external adjustment) cost of investment goods.

[4] This, in essence, is the contribution of E. Penrose, 1959 (107).

indicates increasing returns to scale, and so on. In the light of equation 4.4b we can gain some insight into the intuitive results of Chapter 2. The realization of returns to scale depends on the firm's growth rate, the costs of which are summarized in the learning function $L(\dot{K},K)$, and the capital goods supply function $Q(t)$.

5. Finally, we make the now customary assumption that capital stock depreciates exponentially.

$$I_t = \dot{K}_t + \delta K_t \qquad \delta \text{ constant} . \qquad 4.5$$

Gross investment equals net investment plus replacement.

This assumption has a more than minimal influence on the conclusions, but it simplifies the analysis considerably. The assumption is consistent with that made in Chapter 2, that the services of capital are proportional to capital stock. According to Jorgenson, the justification of 4.5 arises from a result of renewal theory. The result is that "replacement approaches an amount proportional to the accumulated stock of capital whatever the distribution of replacements for an individual piece of equipment, provided that the size of the capital stock is constant or that the stock is growing at a constant rate (in the probabilistic sense)".[1] Of course, neither in Jorgenson's model nor in the one following does capital stock satisfy these requirements. But the assumption is a better approximation the less does capital stock fluctuate with time. Nevertheless, Jorgenson devotes considerable

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[1] D.W. Jorgenson, 1967 (66), p. 139

effort to the defense of the exponential distribution of replacements.[1]

Taubman and Wilkinson[2] handle the more general case where the flow of services from a given stock of capital is not fixed, but is determined by profitability considerations. In this case, $\delta = \delta(H)$, where H is an index of capital utilization, and another dimension is added to the demand for capital stock.[3]

For the following model, we adhere to equation 4.5.

The firm maximizes net present value over an infinite horizon where the rate of discount is a constant r. Expectations, to repeat, are certain, but non-constant.

$$\mathcal{L} = \int_0^\infty E\ dt \quad \text{or}$$

$$\mathcal{L} = \int_0^\infty [e^{-rt}(P(t)\cdot X - W(t)\cdot L - Q(t)\cdot I) - \lambda_t(I - \dot{K} - \delta K)]dt. \qquad 4.6$$

Substituting 4.1, 4.2, 4.3, 4.4, Euler first-order conditions are derived for all t.[4]

$$\frac{\partial \mathcal{L}}{\partial L} = e^{-rt}[P \cdot X_L - W] = 0 \qquad 4.7$$

$$\frac{\partial \mathcal{L}}{\partial I} = -e^{-rt}[Q_0 + 2Q_1 I] - \lambda = 0 \qquad 4.8$$

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[1] D.W. Jorgenson, 1971 (70).

[2] P. Taubman and M. Wilkinson, 1970 (123).

[3] However, Taubman and Wilkinson limit their analysis to a "temporary and unexpected price inflation". Given factor inputs, output will increase (temporarily) if the money rate of interest fails to adjust fully to price expectations.

[4] Dropping the t subscripts.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - \dot{K} - \delta K = 0 \qquad 4.5$$

$$\frac{\partial \mathcal{L}}{\partial K} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{K}} = e^{-rt}P \cdot X_K + \delta \cdot \lambda - \frac{d}{dt}[e^{-rt}P \cdot X_{\dot{K}} + \lambda] = 0. \quad 4.9$$

Substitute 4.8 into 4.9, divide through by P, and differentiate.

$$X_K + (r - \frac{\dot{P}}{P})X_{\dot{K}} - X_{\dot{K}\dot{K}}\ddot{K} - X_{\dot{K}K}\dot{K}$$

$$= (q_0 + 2q_1 I)(r + \delta) - (\dot{q}_0 + 2q_1\dot{I}) \qquad 4.10$$

where lower-case q's are real prices.

The equations 4.4, 4.5, 4.7 and 4.10 determine simultaneously the optimal levels of X, I, L and K for each t. $\dot{K}$ is, of course, implicit in the solution for K.

In addition, we have the following requirements for maximization;

$$X_{\dot{K}\dot{K}} < 0 \qquad 4.11$$

$$\lim_{t \to \infty} e^{-rt}\mathcal{L}_{\dot{K}} = 0 . \qquad 4.12$$

These are the (second-order) Legendre condition and the (first-order) transversality condition respectively.[1]

4.2 The Euler Equation in Capital

Equation 4.10 is the Euler-Lagrange first-order condition for one quasi-fixed factor, capital, and deserves special attention. The left-hand side represents the marginal product of capital services at t. X_K is the familiar term. The additional

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[1] See Appendix A.

terms represent the influence on capital productivity of its rate of acquisition. $X_{\dot{K}}$ is the 'marginal product' of investment. Since $\dot{K}$ is a proxy for internal adjustment costs, we have assumed $X_{\dot{K}}$ to be negative. The whole term $(r - \frac{\dot{P}}{P})X_{\dot{K}}$, is the net (internal) marginal cost of investment. The second element of the coefficient of $X_{\dot{K}}$ is usually missing from analyses of optimal adjustment because of the traditional assumption of static expectations. If the expected future price of X is higher than the current price, $\dot{P}$ is positive. The second element, $- \frac{\dot{P}}{P} X_{\dot{K}}$, represents the 'capital gains' associated with making the investment now rather than in the immediate future, when the internal cost of investment will be greater (as a positive function of P). It is thus possible that $\dot{P}/P$ may be sufficiently large as to offset r for a limited time, so that the (real) internal marginal cost of investment is non-negative.[1]

$X_{\dot{K}\dot{K}}$ is the rate of change of the internal cost of investment as $\dot{K}$ increases. We expect this to be negative from Chapter 2. This is consistent with the mathematical requirement, expressed in the Legendre condition , 4.11. $X_{\dot{K}K}$ is the influence on $X_{\dot{K}}$ of the size of capital stock, and has been assumed positive. Of the 'dynamized' marginal product of capital, when $\ddot{K} < 0$, the three additional terms all act to offset X_K.

The right hand side of 4.10 gives the cost of capital services in t. External adjustment costs are incorporated in the cost of investment goods. The first term shows that the greater

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[1] The analysis of this important possibility is avoided here because of the complexity of the resulting differential equation, involving non-constant coefficients.

the rate of expansion in t, the greater is the resulting cost of capital services. This follows naturally from the assumption of quadratic adjustment costs, equation 4.3. The second term shows the influence of the immediate future price of investment goods on the current cost of capital services. If the expected future price is higher, there is a 'capital gain' associated with present purchases. This is reflected in the negative impact of the second term on capital costs.

Note that the elements $2q_1$ and $\dot{q}_0$ may have offsetting effects on costs. If $\dot{q}_0$ is positive, the higher cost of making a greater investment in t is compensated by the capital gains from a rising q_0. Thus, it may very well pay the firm to invest sooner rather than later, even in the face of rising adjustment costs,[1] if the exogenous movement of prices is compatible. On the other hand, the capital goods industry may be subject to increasing returns to scale, implying an expected decrease in q_0 over time. In this case, expected price changes compound adjustment diseconomies, providing another reason for a slower rate of investment.

Having given 4.10 some economic significance, let us rearrange the terms for mathematical convenience. From 4.5,

$$I = \delta K + \dot{K}$$

$$\dot{I} = \delta \dot{K} + \ddot{K} \; .$$

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[1] Since this has the same net effect as concave adjustment costs, adjustment would take the nature of a 'bang-bang' solution - or full adjustment to optimal capital stock. On this, see M. Rothschild, 1971 (113), pp. 605-622.

Substitute for I and $\dot{I}$ in 4.10, and arranging all terms in K and its derivatives on the left, we have

$$(-X_{\dot{K}\dot{K}} + 2q_1)\ddot{K} - (X_{\dot{K}K} + 2q_1 r)\dot{K} - 2q_1(r + \delta)\delta K \qquad 4.13$$
$$+ X_K + r'X_{\dot{K}} = q_0(r + \delta) - \dot{q}_0$$

where $r' = r - \dot{P}/P$.

The traditional first-order equation for capital is embedded in 4.11, viz.

$$X_K = q_0(r + \delta) \ . \qquad 4.14$$

Under what conditions is this reduction correct? In static equilibrium, $\ddot{K}$ and $\dot{K}$ are zero, and the first two terms drop out. The third term is the influence of replacement investment on capital costs. We have already decided that this is a scale factor, so that in the traditional 'long run' equilibrium, this factor would be subsumed in the element q_0. The term $r'\cdot X_{\dot{K}}$ is the real marginal internal cost of investment. q_0 is exogenous.

One thing is now apparent. In this model, external adjustment costs, represented here by q_1, do not enter into the static equilibrium itself, but determine only the adjustment path. The conclusion contradicts that of Chapter 2 where both managerial diseconomies <u>and</u> a rising supply price influence the static equilibrium. The conflict may lie in the differing set of assumptions. In Chapter 2, the firm is subject to the Marshallian assumption of <u>indivisible</u> capital. It is a prime consideration at what time the entire plant is assembled. The present model

allows continuous capital accumulation, and simultaneous output from infinitessimal capital increments. It appears that the extent of capital divisibility influences the nature of the static equilibrium itself in the form of a nonlinear supply price.

Returning to the present model, as long as $X_{\dot{K}}$ is not solely a linear function of $\dot{K}$, some elements of $r'.\ X_{\dot{K}}$ remain a determinant of the equilibrium. A specific example is given below, p. 98. Thus, in a dynamic world, static equilibrium is itself dependent upon (internal) adjustment costs.

Theoretically, we expect that the learning cost function tends to zero as conditions stabilize,[1] so that $X_{\dot{K}}$ also would disappear in the establishment of a stationary state. This may not be true of nonseparable elements (terms in $\dot{K}$ and K), but we shall take this up below.

In other words, the much-used equation 4.14 is strictly applicable only in the stationary state, in which all variables bear a constant relation to each other. In accordance with Chapter 2, Section 1, it is only in the stationary state that one can speak strictly of an optimal capital stock, derived from 4.14, and without reference ot its time acquisition path. It follows that it is fallacious to 'date' 4.14. The optimal capital stock for period t is determined from the general equation 4.13.

The arrangement of the capital equation as 4.13, and the consideration of 4.14 suggest another problem. Does it make any difference to optimal investment policies if adjustment costs

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[1] This tendency has not been built into the model.

are internal or external to the firm? To my knowledge, this issue has not been raised since both types are usually subsumed within the same symbolic notation. Here I have separated the two. External adjustment costs are a function of gross investment and are represented by terms in q_1. Those internal are a function of $\dot{K}$ and are represented by terms in $\dot{K}$.

The answer is not obvious since it depends upon the particular form of the production function. In general, there seems to be no reason why the influence should be symmetrical. On one point the answer is clear. $L(\dot{K},K)$ includes the non-separable term in $\dot{K}$ and K. There is no counterpart to this in external adjustment costs, at least in the extant literature. Nonseparable external adjustment terms are equally conceivable, however. These would result when the size of the firm influences the cost of capital goods.

Barring the nonseparable term $L_{\dot{K}K}$, in the special case of a quadratic production function, assumed below (or a linear approximation to any other), the influence is symmetric. The symmetric influence becomes obvious with the recognition that internal and external adjustment costs are mathematically equivalent in the special case. We therefore combine below, without loss, internal and external adjustment costs under the notation of the former.

Any more precise formulation of capital acquisition requires a specific assumption for the production function, and the difficulties of nonlinear differential equations forbids almost all but linear approximations to the true functional forms.

We therefore assume a quadratic production function, consistent with 4.4b.[1]

$$X = a'V + \tfrac{1}{2}V'AV \tag{4.15}$$

where $a' = (a_1\ a_2\ a_3),\ a_1, a_2 > 0;\ a_3 < 0$

$$V' = (K, L, \dot{K})$$

$$A = \{a_{ii}\}\ ,\ i = 1,2,3,\quad A \text{ symmetric;}$$

$$a_{ii} < 0;\ a_{23} = a_{32} = 0;\quad a_{31} > 0\ .$$

The quadratic assumption has been used most notably in Tinsley's work. In particular

$$X_L = a_2 + a_{21}K + a_{22}L$$

$$X_K = a_1 + a_{11}K + a_{12}L + a_{13}\dot{K}$$

$$X_{\dot{K}} = a_3 + a_{31}K + a_{33}\dot{K} = -L_{\dot{K}}$$

where $-L(\dot{K},K) = a_3K + a_{31}K\dot{K} + \tfrac{1}{2}a_{33}\dot{K}^2\ .$

We are now in a position to substitute the special production function 4.15 into the first-order equation 4.13.

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[1] At first sight, this functional form appears to be equivalent to the transcendental logarithmic production function. But X and V are logarithms of the output and factors respectively for the 'translog' function while, for the quadratic function, X and V are the quantities themselves. In the first case, there is a ready economic interpretation of the coefficients. The second form has been chosen for its analytical properties, and rationalized as the Taylor expansion of more general functional forms. Unfortunately, there is no means of replacing 4.15 with the translog function without introducing impossible nonlinearities into the capital equation 4.13.

Substitute for X_L into the equilibrium condition 4.7, and solve for L in terms of K and the parameters. Substitute for L in X_K. The resulting capital equation involves only terms in K and its time derivatives. From 4.13 and reversing the sign,

$$(a_{33} - 2q_1)\ddot{K} - (a_{13} + r'a_{33} - a_{31} - 2q_1 r)\dot{K}\ {}^{1}$$

$$- (H/a_{22} + r'a_{31} - 2q_1(r + \delta)\delta)K \qquad 4.16$$

$$= [(a_1 - \frac{a_{12}}{a_{22}}(a_2 - \frac{W}{P}) + r'a_3) - (q_0(r+\delta) - \dot{q}_0)]^2 = F(t)$$

where

$$H = a_{11}a_{22} - a_{12}^2 .$$

Since 4.16 is symmetric in internal and external adjustment costs, the terms in q_1 may be dropped without loss, to clarify the exposition. 4.16 reduces to

$$\ddot{K} - r'\dot{K} - gK = \frac{F(t)}{a_{33}} = -f(t) \qquad 4.17$$

where

$$g = (H + r'a_{31}a_{22})/a_{22}a_{33}$$

by now a familiar equation in the literature. We would, of course, prefer r' as a function of time, but non-constant coefficients result. $r' = r - \dot{P}/P$ is therefore assumed constant. Since r is also constant, $\dot{P}/P$ is constant and = p. Thus $r' = r - p$.

The general solution to 4.17 is

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[1] The cancellation of cross products in K and $\dot{K}$ is peculiar to the one-factor analysis. See A.B. Treadway, 1971 (133).

[2] F(t) is difficult to interpret heuristically. It is that part of marginal net revenue in t which is independent of K_t and $\dot{K}_t$.

$$K_t = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - d\int_0^t (e^{\lambda_2 (t-s)} - e^{\lambda_1 (t-s)}) f(s)^1 ds \qquad 4.18$$

where

$$d = 1/(\lambda_2 - \lambda_1)$$

and

$$\lambda_2, \lambda_1 = \frac{r'}{2} \pm \sqrt{(\frac{r'}{2})^2 + g} \qquad 4.19$$

and the two constants are determined from the initial condition K_0 and the end-point transversality condition 4.12.

Assume $r' > 0$. If $g > 0$, there is an elegant (saddle-point) solution to 4.17 with $\lambda_1 < 0 < r' < \lambda_2$. Solving for the constants, $C_1 = K_0 - C_2$. From 4.12 and 4.16

$$\lim_{t \to \infty} e^{-rt} (P_t \cdot X_{\dot{K}} - Q'_t \cdot \dot{K}) = 0 \ . \qquad 4.20a$$

The term in brackets

$$= P_t(-L_{\dot{K}}) - 2q_1 \cdot \dot{K} \ .$$

Symmetry allows the neglect of the second element. But since $\dot{P}/P=p$, $P_t = e^{pt}$. 4.20a reduces to

$$\lim_{t \to \infty} e^{-r't} \cdot (-L_{\dot{K}}) = 0 \qquad 4.20b$$

which we hope to solve for C_2.

Differentiate 4.18 with respect to time.

$$\dot{K}_t = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} - d\lambda_2 \int_0^t e^{\lambda_2 (t-s)} f(s) ds$$
$$+ d\lambda_1 \int_0^t e^{\lambda_1 (t-s)} f(s) ds \ .$$

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[1] For a comprehensible deduction of this solution see R. Bellman and R. Kalaba, 1965 (12), p. 30; and R. Bellman, 1968 (10), p. 70.

Substitute for K_t, 4.18, and $\dot{K}_t$ in

$$-L_{\dot{K}} = a_3 + a_{31}K + a_{33}\dot{K} \quad .$$

In the limit, both the constant term and terms in $e^{\lambda_1 t}$ [1] vanish from 4.20b. 4.20b reduces to

$$\lim_{t\to\infty} e^{(\lambda_2 - r')t}(a_{31} + \lambda_2 a_{33})(C_2 - d\int_0^t e^{-\lambda_2 s} f(s)ds) = 0 \ .$$

And since $\lambda_2 > r'$,

$$C_2 = d\int_0^\infty e^{-\lambda_2 s} f(s)ds = K_0^* \ . \qquad 4.21$$

Substitute for C_2 in 4.18.

$$K_t = (K_0 - K_0^*)e^{\lambda_1 t} + K_t^{*P} + K_t^{*f} \qquad 4.22$$

where
$$K_t^{*P} = d\int_0^t e^{\lambda_1(t-s)} f(s)ds$$

$$K_t^{*f} = d\int_t^\infty e^{\lambda_2(t-s)} f(s)ds \ .$$

It can be seen that the negative root discounts past value of $f(s)$ (seen from t) and the initial condition, while the positive root discounts expected future values of $f(s)$.

Differentiate 4.22 with respect to time for a continuous version of the stock adjustment mechanism

$$\begin{aligned}
\dot{K}_t &= \lambda_1 e^{\lambda_1 t}(K_0 - K_0^*) + \lambda_1 K_t^{*p} + \lambda_2 K_t^{*f} \\
&= \lambda_1[e^{\lambda_1 t}(K_0 - K_0^*) + K_t^{*p} + \frac{\lambda_2}{\lambda_1} K_t^{*f}] \\
&= \lambda_1[(K_t - K_t^{*f}) + \frac{\lambda_2}{\lambda_1} K_t^{*f}]
\end{aligned} \qquad 4.23$$

$$\Rightarrow \quad \dot{K}_t = |\lambda_1|[(1 + \frac{\lambda_2}{|\lambda_1|})K_t^{*f} - K_t] \quad \text{since } \lambda_1 < 0.$$

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[1] Since $\lambda_1 < 0$.

Finally this may be translated into discrete time giving the Koyck distributed lag

$$K_t = \lambda \sum_0^t (1 - \lambda)^i K^{**}_{t-i} + \lambda^t K_0 \qquad 4.24$$

where $\lambda = |\lambda_1|$ and $K^{**}_t = (1 + \frac{\lambda_2}{|\lambda_1|})K^{*f}_t$.

In the special case that $f(t) = f$, usually labelled 'static expectations', 4.18 reduces to

$$K_t = b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t} + \frac{f}{g} \quad . \qquad 4.25$$

The transversality condition constrains b_2 to zero, leaving

$$K_t = (K_0 - \bar{K})e^{\lambda_1 t} + \bar{K}, \quad \bar{K} = \frac{f}{g}.^1 \qquad 4.26$$

Differentiating 4.26

$$\dot{K}_t = |\lambda_1| (\bar{K} - K_t) \quad . \qquad 4.27$$

This, in essence, is the Eisner and Strotz model. As noted in Chapter 3, 4.27 rationalizes the Koyck geometric lag, but it is to a static optimum $\bar{K}$, precluding a "nontrivial distributed lag interpretation of the adjustment path".[2]

From Chapter 2 and p. 91 above, we were concerned with possible dynamic influences on a static equilibrium. Firstly, consider the temporal location of the equilibrium. 4.26 indicates that the optimal capital stock from a once-for-all change in the parameters is only achieved after an infinite passage of time.

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[1] Since $g = -\lambda_1 \cdot \lambda_2$.

[2] P.A. Tinsley, 1970 (126), p. 17.

This, of course, is practical nonsense, but it is a logical offshoot of the quadratic adjustment assumption. In the Marshallian model, our intuitive understanding is that full adjustment would take place in a limited time; but since, in 4.26, K_t approximates the optimal capital stock fairly rapidly, the quadratic assumption is quite reasonable on this point.

The optimum itself is a function of the two dynamic elements a_3 and a_{31}. $\partial\bar{K}/\partial a_3 < 0$. The greater is managerial resistance to change, the smaller is the optimal capital stock. This element we expect to tend to zero as conditions remain stable. We have the interesting proposition that the stationary state optimal stock is greater than that in a (temporarily achieved) static equilibrium as the firm moves onto its efficiency locus.

Further, $\partial\bar{K}/\partial a_{31} < 0$. a_{31} is the (assumed positive) contribution of scale to the costs of growth, or alternatively, the (positive) contribution of growth to the productivity of capital. The greater is a_{31} then, the less is the desired capital stock, caeteris paribus. Since this influence is intrinsic to the growth process itself, it seems unlikely that the output effect would disappear as conditions stabilize.

4.3 Returns to Scale

We return to a problem of special interest, that of returns to scale. Both the general 4.22 and the special 4.26 solutions depend on $g = (H + r'a_{31}a_{22})/a_{33}a_{22} > 0$. Since $a_{33}, a_{22} < 0$

$$\mathcal{H} = H + r'a_{31}a_{22} > 0 \qquad 4.28$$

or

$$(a_{11} + r'a_{31})a_{22} - a_{12}^{\ 2} > 0 \quad . \qquad 4.29$$

We have an accepted definition of returns to scale as depending on the sign of the hessian H in 4.28. But consider the entire left-hand side of 4,28, $\mathcal{V}$. a_{31} approximates the influence of the size of the firm on the marginal cost of investment. We have assumed this influence positive. There is a positive effect of scale on unit costs indirectly through a dynamic factor. Should this influence also be included in the concept of returns to scale? Clearly, this problem doesn't arise in static theory, but it is by no means a trivial question. Even in dynamic theory, the problem arises only through the existence of non-separability in the production function.

In a dynamic world, the decreasing marginal productivity of capital is partially offset by its contribution to growth. a_{31} is non-zero, and will be implicitly incorporated in a_{11} when it is not explicitly represented. If the influence were in the other direction - a_{31} negative - then scale would also contribute to the cost of growth. In reverse, this brings additional cost to size.

It seems that no clear case can be made for either definition of returns to scale - the sign of H or the sign of $\mathcal{V}$. In a stationary state, a_{31} (as part of the learning function L) eventually disappears, and the former definition prevails. The debate reduces to a matter of terminology. Since the sign of $\mathcal{V}$ heralds important structural differences in the dynamic capital equation 4.17, we shall opt for this modified definition. Any conclusions, of course, as to the relation of scale returns to

the stability of equilibrium are dependent upon this assumption.[1]

Under this second definition, $g > 0$ implies $\mathcal{K} > 0$, or decreasing returns to scale. The implication is that stability in this linear model requires not only $X_{\dot{K}\dot{K}} < 0$, but also, still, decreasing returns to scale, at every point on the firm's growth path. In spite of the dynamic framework, we can still not cope with the problem that plagued Marshall. This signifies a structural limitation in the model, since decreasing returns to scale is not a universal fact of economic life. We need to know why this limitation exists.

Suppose future prices are constant and a stationary optimum exists. Equations 4.19 and 4.26 are the relevant equations. With increasing returns to scale,[2] $g < 0$, λ_1 and λ_2 are both positive. λ_2 is eliminated by the end-point condition but a positive λ_1 remains. So the equilibrium is unstable. Increasing returns to scale denies convergence to the optimum $\bar{K}$. This is instability of the familiar static equilibrium, for which decreasing returns to scale is essential. If there is no growth, then the decreasing returns to growth cannot, of course, be a source of stability. Convex adjustment costs merely serve to make adjustment to equilibrium finite and unique.

It is established that perfect competition, a static optimum, and increasing returns to scale are incompatible. This is also observable by examination of 4.17. A static optimum requires $\ddot{K} = \dot{K} = 0$. Increasing returns to scale[3] requires $g < 0$.

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[1] See Appendix B.

[2,3] i.e. $\mathcal{K} < 0$.

This implies $\bar{K} = -\frac{f}{g}$, a negative optimal capital stock, which is meaningless.

Since something has to give, and since we are interested in increasing returns to scale, perhaps we should throw out the static optimum. It is intuitively reasonable that if the firm does face increasing returns to scale for some period, there will be no static optimum. Almost all the adjustment cost literature[1] has been written with the presupposition that a static optimum exists. Moreover, the exigencies of analysis require the exclusion of significant nonlinearities - relevant here are excluded non-linearities in the production function. The typical solution to the dilemma[2] is to take a linear approximation <u>about</u> the assumed static optimum. The resulting equation is only locally optimal. The above model avoids this local restriction by assuming a universally quadratic function, but this moves the unreality to higher-order assumption. The linearity remains. Since increasing returns to scale is non-viable at a static optimum, it remains non-viable in monotonic approaches to the static optimum.

In addition to the stringent requirements on the production function, we are also forced to assume perfect competition in the output market and the cost of funds market. A cursory examination of 4.6 is sufficient to indicate the impossible analytical task that would result from assuming imperfect competition and its attendant feedback. Our firm has unlimited borrowing capacity,

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[1] With the significant exception of Tinsley.

[2] As in Lucas and Treadway.

whereas in practice, capital funds are an important limitation to a firm's expansion. Similarly, our firm has unlimited selling possibilities in t. We could assume, to ensure stability, that the prices of output, P, and of funds, r, changes over time, without introducing the feedback mechanism inherent in imperfect competition. But this approach leads to non-constant coefficients in the capital equation, a problem we shall neglect for the moment. In general, the sources of the necessary stability for our hypothetical firm are quite limited.

Still considering the static model, there appears to be one way of incorporating increasing returns to scale. From 4.16, reintroduce <u>external</u> adjustment costs. In this case,

$$g' = [\aleph - 2q_1(r + \delta)\delta \cdot a_{22}]/(a_{33} - 2q_1)a_{22} \; . \qquad 4.30$$

Again the denominator is positive. The additional term in the numerator is positive, so it is feasible that $\aleph < 0$ while maintaining positive g'. This special result holds at a static optimum. It results from the use of <u>gross</u> investment as the argument in external adjustment costs, introducing an adverse nonlinearity in replacement capital δK, as well as in $\dot{K}$.[1] In this particular model, perfect competition and increasing returns to scale coexist. The firm is prevented from fully realizing scale returns by offsetting external diseconomies to size, endogenously induced. Nevertheless this model is theoretical

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[1] p. 78 above.

small fry, and we would like to produce increasing returns to scale under more general conditions.

Suppose that a stationary optimum does not exist - $\ddot{K}$ and $\dot{K}$ both not zero. For example, relative prices might be constantly changing. The general solution 4.22 applies.

In general, if returns are increasing, $g < 0$, and the roots satisfy $0 < \lambda_1 < \lambda_2 < r'$.[1] Here positive roots do not indicate instability since the dynamized 'forcing term' $f(t)$ has converted the capital equation into a growth path. However, with $\lambda_2 < r'$, the end-point condition 4.20b is always satisfied, and C_2 is arbitrary. A capital expansion path then exists for the firm facing increasing returns to scale, but it is non-unique. At no point in time will $\dot{K} = 0$, but the path is stable in the dynamic sense that $K_t < e^{r't}$ with r' constant.

The end-point condition is the problem. In economic terms, a unique solution requires that the marginal cost of investment, in the limit, increases at a rate faster than the discount rate r'. In this model, increasing returns to scale implies a growth rate of capital and net investment slower than decreasing returns to scale. Since the marginal cost of investment is a linear function of both capital and net investment, the rate of increase of this marginal cost is also reduced. The constant coefficients in 4.17 requires a production function fixed for the infinite horizon. The end-point restriction on the production function is therefore binding for the entire time path.

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[1] Assuming real roots.

This is confirmed by an examination of the full second-order condition for maximum present value, given in the Appendix, equation (a.4). The concavity condition applies to the summation of the hessian of E[1] over the horizon. If the elements of the hessian are constant, then the concavity applies for all t. If the elements are not constant, then this need not be so. The summation requires that concavity overcomes any temporal convexity, in the limit. Translated into our terminology, if the production function is not fixed for the entire horizon, condition (a.4) does not require $H = a_{11}a_{22} - a_{12}^2 > 0$ for all t.

Intuitively, there seems no reason why the firm should not face increasing returns to scale for a finite period of time, but the analysis of such a possibility involves even more complex differential equations than those treated above. A brief explanation is offered in Appendix C.

A.B. Treadway[2] offers the only treatment in the adjustment cost literature of increasing returns to scale. His treatment is predictable and offers no consolation. Treadway uses a linear approximation to the production function about an assumed static optimum. In the neighborhood of the static optimum, the above model is appropriate, and Treadway shows the well-known instability produced by increasing returns to scale around the static optimum. Elsewhere,[3] Treadway is forced to resort to phase plane analysis[4]

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[1] See equation 4.6.

[2] A.B. Treadway, 1969 (131).

[3] ibid., p. 231f.

[4] $\dot{K}$ is graphed against K. See G. Hadley and M.C. Kemp, 1971 (46).

for a graphical representation of a hybrid mixture of returns to scale possibilities. What can be imagined can also be drawn.

Fortunately, a compromise is possible. It is feasible to examine the case of constant returns to scale within the model, while producing a non-classical result, and some insight into the problems of Chapter 2.

Consider the possibility that $\aleph = 0$. Here $g = 0$, $\lambda_1 = 0$, $\lambda_2 = r'$. 4.18 reduces to

$$K_t = C_1 + C_2 e^{rt} - \frac{1}{r'} \int_0^t (e^{r'(t-s)} - 1) f(s) ds \; . \qquad 4.30$$

From 4.20b, C_2 is uniquely defined

$$C_2 = \frac{1}{r'} \int_0^\infty e^{-r's} f(s) ds = K_0^* \; . \qquad 4.31$$

Substituting for C_2

$$K_t = (K_0 - K_0^*) + \frac{1}{r'} \int_0^t f(s) ds + \frac{1}{r'} \int_t^\infty e^{r'(t-s)} f(s) ds \; . \qquad 4.32$$

Differentiating

$$\dot{K}_t = \int_t^\infty e^{r'(t-s)} f(s) ds \; . \qquad 4.33$$

Investment in t equals the present value of future $f(t)$,[1] and the past is ignored. In 4.32, optimal capital stock equals (minus) the initial discrepancy plus the present value of $f(t)$, capitalized at the (real) discount rate r'. Future $f(t)$ are discounted by r'; past values of $f(t)$ are non-discounted. By association, the constant d is shown to be the capitalization rate for $f(t)$ in the general solution 4.22.

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1 Remember that $f(t)$ does not represent net earnings in t.

There is a special case of 4.32, where $f(t)$ is constant.[1] Put $f(t) = f$ in 4.32, and integrate.

$$K_t = D_1 + D_2 e^{r't} + \frac{f}{r'} t \quad . \qquad 4.34$$

Alternatively, this solution may be found by putting $f(t) = f$, and $g = 0$ in 4.17, and then integrating.

Again the end-point condition 4.20b constrains $D_2 = 0$, so that

$$K_t = K_0 + \frac{f}{r'} t \; . \qquad 4.35$$

Differentiating,

$$\dot{K}_t = \frac{f}{r'} = \text{constant} \; . \qquad 4.36$$

4.32 and 4.35 are the capital equations for a perfectly competitive firm facing constant returns to scale, and do not imply the classical indeterminacy. The difference is best explained by reference to the differential equation 4.17.

In stationary equilibrium, $\ddot{K} = \dot{K} = 0$ and the <u>further</u> assumption $g = 0$ implies an indeterminate capital stock. This is obviously correct. But since the full equation 4.17 is a better approximation to the living firm's demand for capital, the classical indeterminacy need not concern us in practice. In general $\ddot{K}$ and $\dot{K} \neq 0$ and a vanishing g implies equation 4.32.

We have a perfectly competitive firm facing constant returns to scale. What results is a determinate rate of capital

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[1] c/f A.B. Treadway, 1969 (131).

accumulation 4.33 and a determinate level of optimal capital stock <u>for all</u> t, 4.32. This is the general solution. There is no <u>stationary</u> stock at any point in time, but this is compatible with the moving optima implied by the nonconstant 'environment' f(t). The determinacy results from the existence of (both internal and external) costs to adjustment. This makes good economic sense.

By contrast, the special case 4.35 and 4.36 does not make economic sense. The firm faces a constant environment, yet never settles down to a static equilibrium. Net investment is constant forever, and independent of the initial stock. The particular combination of assumptions result in a 'freak model', which has no empirical counterpart but which is theoretically interesting. Dynamic adjustment costs imply a unique adjustment path, but constant returns to scale implies an indeterminate optimal capital stock. Our deluded firm is adjusting optimally to nowhere in particular. The classical indeterminacy reigns again, this time in a dynamic version.

The conclusion is that perfect competition and constant returns to scale are compatible in <u>growth</u>, and if we could introduce a switching of technology to decreasing returns to scale near the static equilibrium, we would have a sensible model. Again, the intractability of nonlinear assumptions rules out this possibility.

Appendix A

A Brief Description of Conditions for Maximum Present Value

Assume $\underline{X(t)}$ are the factor trajectories that maximize the present value defined by the integral of forward (i.e., discounted) cash earnings.

$$V = \int_0^\infty E^*(\dot{Y},Y,t)\,dt \quad . \tag{a.1}$$

Let admissible curves in the ϵ-neighborhood of the optimal trajectories be represented by

$$X(t) + \epsilon\eta(t) , \tag{a.2}$$

where η is an arbitrary vector with the property that $\underline{V(\eta)}$ exists. Substitute a.2 for $\underline{Y}$ in the integral a.1 and approximate the ϵ-region of the maximum by a power series expansion about $\underline{\epsilon = 0}$.

$$V(\epsilon) = V(0) + V'(0)\epsilon + \tfrac{1}{2}V''(0)\epsilon^2 \quad . \tag{a.1'}$$

From the definition of $\underline{X(t)}$ it is known that $\underline{V(\epsilon)}$ attains its maximum at $\underline{V(0)}$ requiring the second term on the right hand side of a.1' (the "first variation") to vanish for all $\underline{\epsilon}$.

$$V'(0) = \int_0^\infty [\eta' E^*_x + \dot{\eta}' E^*_{\dot{x}}]dt = 0 \quad . \tag{a.3}$$

Integrating by parts gives

$$V'(0) = \int_0^\infty \eta'[E^*_x - \frac{d}{dt} E^*_{\dot{x}}]dt + \dot{\eta}' E^*_{\dot{x}}\Big|_0^\infty = 0 \quad . \tag{a.3'}$$

Since the factor trajectories must originate from the initial stocks $\underline{X(0)}$ we must require $\underline{\eta(0) = 0}$. The first order con-

condition a.3' can then be restated as the requirement that the factor trajectories defined by solutions of the Euler-Lagrange equation

$$E^*_x - \frac{d}{dt} E^*_{\dot{x}} = 0$$

must satisfy the end point conditions provided by the initial stocks $X(0)$ and the transversality condition.

$$\lim_{t \to \infty} E^*_{\dot{x}} = 0 \ .$$

The second order condition for maximizing present value requires the coefficient of ϵ^2 in a.1' (the "second variation") to be negative

$$V''(0) = \int_0^\infty [\eta' \ \dot{\eta}'] H^* \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} dt < 0, \tag{a.4}$$

where H is the Hessian of the discounted earnings function.

$$H^* = \begin{bmatrix} E^*_{xx} & E^*_{x\dot{x}} \\ E^*_{\dot{x}x} & E^*_{\dot{x}\dot{x}} \end{bmatrix} \ .$$

The condition that the gross receipts function is strictly concave not only satisfies a.4 but encompasses both the familiar second order condition for stationary optimization of variable factor models

$$z' E^*_{xx} \, z < 0, \text{ for } z \neq 0,$$

and the necessary Legendre condition

$$z' E^*_{\dot{x}\dot{x}} \, z < 0 \ .$$

Appendix B

On the Revised Concept of Returns to Scale

The empirical examination of a dynamized concept of returns to scale, from Section 4.3, would be something of a nightmare, for two reasons. Firstly, the specification of a particular dynamic production function with desirable statistical properties is in its infancy. Secondly, the inclusion of a priori information via the first-order equilibrium conditions is no longer a simple matter, since a dynamic equilibrium has replaced a static equilibrium. Note that the forced inclusion of static equilibrium conditions in the production function introduces an element of misinformation, since the living firm will not be in static equilibrium. If the inclusion of dynamic marginal productivity conditions could be effected, the resulting specification of the production function would be superior to the static specification. Of course, this gain in efficiency could be offset by the assumptions necessary to specify the complex dynamic equilibrium, especially those regarding the nature of expectations.

Nevertheless, one observation is worth making. Suppose we have a production function of the form

$$X = AK^{\alpha}L^{\beta} + L(\dot{K},K) \qquad \text{(b.1)}$$

where $$L = a_3K + a_{31}\dot{K}K + \tfrac{1}{2}a_{33}\dot{K}^2 .$$

Using the terminology of Chapter 4,

$$H = X_{KK}X_{LL} - X_{LK}^{2}$$
$$= \left(\frac{\alpha\beta X}{LK}\right)^2(1 - \alpha - \beta).$$

This is the traditional interpretation of returns to scale, and is indicated by the sign of $1 - \alpha - \beta$. In the dynamic formulation,

$$\mathcal{X} = H + r'a_{31} \cdot \frac{\beta(\beta-1)}{L^2} X \text{ [1]}$$

$$= \frac{\beta X}{L^2}\left[\frac{\alpha X}{K^2}(1 - \alpha - \beta) - (1 - \beta)r'a_{31}\right] .$$

Here $\alpha + \beta = 1$ implies $\mathcal{X} < 0$ which implies increasing returns to scale. This indicates the positive contribution of the new term, the cross product of $\dot{K}$ and K. The static elements, by themselves, indicate constant returns to scale, but the negative influence of capital on the internal costs of growth makes for overall increasing returns to scale. Constant returns to scale ($\mathcal{X} = 0$) indicates $\alpha + \beta = S < 1$. What does this imply for the size of $(\alpha + \beta)$? Since $\frac{\alpha X}{K^2}(1 - \alpha - \beta)$ is constant, S itself is positively dependent upon the level of output of the firm. So the larger is the firm's output, the more closely does $\alpha + \beta = 1$ represent constant returns to scale, even given the dynamic formulation of returns to scale. It follows that, if equation b.1 is a reasonable representation of the firm's productive process, the additional dynamic element in the concept of returns to scale is unlikely to be of much practical importance.

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[1] Equation 4.28.

Appendix C

On the Analysis of the Possibility of Increasing Returns

An examination of the basic differential equation in capital, 4.17, indicates that a changing technology implies a non-constant g. Similarly, more interesting movements in the interest rate and output price implies a nonconstant r'.

The obvious way of specifying a changing technology is to make g a function of K. For example, returns to scale might be increasing for small K and decreasing for large K, which implies g as a positive function of K. But this specification introduces dreaded nonlinear terms in K into the equation. Alternatively, we could introduce a time-dependent function in r', $r'(t)$. This would allow us to cope with various rates of change in demand and various rates of entry of competitive firms, both of which enter the analysis through the term $\dot{P}$. We could even allow $r'(t)$ to be negative (the rate of inflation greater than the nominal interest rate) for a finite period. This extension leads to the following general form:

$$\ddot{K} - r'(t)\dot{K} - g(t)K = f(t) \ . \qquad (c.1)$$

So an attempt to allow for the possibility of increasing returns to scale leads to a second-order differential equation with <u>non-constant coefficients</u>. With nonconstant coefficients, it can be shown that no solution exists of the form 4.18, where

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[1] g is a function of r'.

the homogeneous part is a linear combination of exponentials. This form requires constant roots of the homogeneous equation.

Instead, a 'power series' solution may be found to c.1, given certain conditions. Let v_t be such a solution. If $r'(t)$, $g(t)$ and $f(t)$ are power series convergent in some finite range $t < T$,[1] then

$$v_t = \sum_{i=0}^{\infty} a_i t^i \tag{c.2}$$

which converges in $t < T$.

If $r'(t)$, $g(t)$ and $f(t)$ are not simple polynomials, then we must be able to express these functions as Maclaurin expansions around $t = 0$. The solution c.2 then approximates the actual solution in the neighborhood of $t = 0$.

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[1] $t > 0$.

X

CHAPTER 5

THE KEYNESIAN THEORY OF INVESTMENT

We have found that the dogmatic acceptance of long run static equilibrium as a natural foundation for economic analysis has led to distortions in the treatment of microeconomic investment behavior. There is an interesting parallel at the macroeconomic level, and for this we turn to the theory of investment in the General Theory of Keynes. The 'marginal efficiency of capital' is purportedly one of those universally known economic concepts. But in spite of its omnipresence in the textbooks, upon close inspection it is rather elusive.

I submit that the confusion arises in trying to force Keynes' work into a perfectly static framework. Specifically, A.P. Lerner, in <u>The Economics of Control</u>,[1] has provided the definitive interpretation of the marginal efficiency of capital. We will examine this below. The general result has been a two-part model, familiar from the analysis of Chapter 2. We have a demand for capital, derived from the (static) marginal productivity of capital function MPK. But nothing is said about its rate of acquisition. This omission is supposedly rectified by Keynes. Keynes provides us with the optimal rate of (disequilibrium) investment, the macroeconomic counterpart of Eisner and Strotz.

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[1] A.P. Lerner, 1944 (88).

Recall Keynes' definitions:

> ... I define the marginal efficiency of capital as being equal to that rate of discount which would make the present value of the series of annuities given by the returns expected from the capital-asset during its life just equal to its supply price ...
>
> If there is any increased investment in any given type of capital during any period of time, the marginal efficiency of that type of capital will diminish as the investment in it is increased, partly because the prospective yield will as the supply of that type of capital is increased, and partly because, as a rule, pressure on the facilities for producing that type of capital will cause its supply price to increase; the second of these factors being usually the more important in the short run, but the longer the period in view the more does the first factor take its place.
>
> Thus for each type of capital we can build up a schedule, showing by how much investment in it will have to increase within the period, in order that its marginal efficiency should fall to any given figure. We can then aggregate these schedules for all the different types of capital ...[1]

Changing the convention of past chapters, I will denote the marginal efficiency of capital (or the internal rate of return) by r, and the schedule relating r to the level of investment by MEC. The rate of interest will be denoted by i.

Thus the intersection of the MEC with the current market rate of interest (or cost of funds) determines the value of investment <u>for the period</u>. We are outside the scope of static analysis. Firms making up the economy don't have the luxury of the long run, when the environment conveniently stands still, in which to make good their capital needs.

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[1] J.M. Keynes, 1936 (78), pp. 135,6.

Consider the determination of r. The comparison of the discounted expected yield of an asset and its cost determines r from the familiar formula:

$$q = R_1/(1 + r) + R_2/(1 + r)^2 + \ldots + R_n/(1 + r)^n \qquad 5.1$$

so the two major factors implicit in r are the supply price and the expected yield series. According to Keynes, if the investment horizon is long, a decreasing net yield from the exploitation of investment opportunities is the major cause of the declining slope of the MEC. If the horizon is short, a rising supply price of the capital is the dominant factor in the declining slope.

This separation bears a striking resemblance to the treatment of long run costs in Chapter 2. In particular, consider Figure 2.3. If the horizon is short, say one year, average costs are given by $LRAC_1$. The rise in costs is associated with too rapid a rate of investment. There we were concerned with a more general specification of 'adjustment' costs, whereas Keynes only details a rising supply price. But the principle is the same. If the horizon is long, the influence of supply price is minimal, and the firm moves (or plans to move) to a point in the $LRAC^*$, whose profitability is determined predominantly by the existing demand and technical possibilities - in short, by the 'prospective yield' to expansion. Thus, in terms of Chapter 2, there is no incompatibility in the two factors underlying Keynes' MEC.

Now rewrite 5.1 as:

$$\sum_{i}^{n} \frac{1}{(1+r)^{i}} R_i(I) - q(I) = 0 \ . \qquad 5.2$$

5.2 reminds one of the objective function in microeconomic models. The major difference, of course, is that the latter are based on maximization of present value, while 5.2 utilizes the 'internal rate of return' concept. Nevertheless, again the two factors underlying the slope of the MEC are recognizable in the optimal adjustment models of the Eisner and Strotz variety - a static production function coupled with external adjustment costs.[1,2]

But in the macroeconomic literature, a changing supply price is the traditional source of confusion. Investment decisions are supposed to be made in the 'long run' when the supply price is a constant. Ackley, following Lerner, claims that the combination is an "unfortunate confusion of factors", since the first relates to the stock of capital and the second to the rate of investment.[3] Clearly we need to examine the accepted interpretation of the marginal efficiency of capital in more detail. Lerner's <u>The Economics of Control</u> is the first step.

Lerner defines the marginal productivity of capital as "the <u>permanent</u> increase in output"[4] resulting from the use of an

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[1] However, any micro model consistent with Keynes would also involve non-stationary expectations.

[2] Sandmo states that the Keynesian theory omits adjustment costs. On this point he is quite wrong. Dynamic period analysis necessarily includes adjustment costs, whether implicitly (as in Fisher) or explicitly (as in Keynes). The separate notion of adjustment costs again derives from our firm belief that stationary equilibrium is the end-all of economic action.

[3] G. Ackley, 1961 (1), p. 485.

[4] A.P. Lerner, op. cit. (88), p. 330. (Emphasis mine). He has in mind a stationary state, with no depreciation.

extra unit of capital equipment. He uses the following significant simplification:

> It is measured as so many per cent per annum... because the increment of capital is measured by the number of units of consumption goods that are devoted to increasing it. The marginal product compared is in the same units as the factor, so the relationship between the increment of capital and its marginal product can be expresses as a ratio or a percentage.[1]

Lerner also creates another concept called the "marginal efficiency of investment" MEI. It is never clearly defined in his 1944 work but we attempt to explain it as follows.

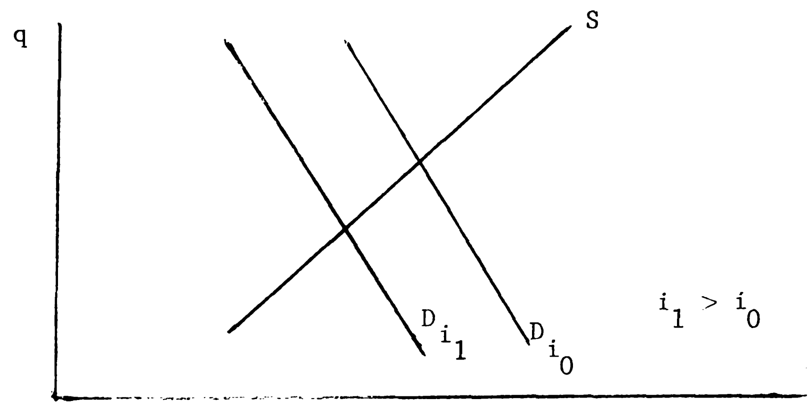

Figure 5.1 The Demand and Supply of Investment Goods

In Figure 5.1, S and D are the aggregate supply and demand for new capital goods, for a given period, as a function of their price q. The D curve is the partial derivative of the general demand function with respect to q. Its probable interpretation is that level of investment which, once acquired, equates MPK with the given market rate of interest i. This relationship is the MEI schedule and is held to be equivalent to

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1 ibid., p. 331.

Keynes' MEC,[1] only here the price of capital goods has been emphasized rather than the interest rate.

But how does the MPK fit into a growing economy? Lerner claims that the MPK equals the MEI when the rate of investment is zero. That is, the long run equilibrium, MPK equals MEI. This is shown in Lerner's diagram, Figure 5.2,[2] where equilibrium capital stock is bO and MPK equals MEI is OL. In other than long run equilibrium, MPK is greater than MEI.

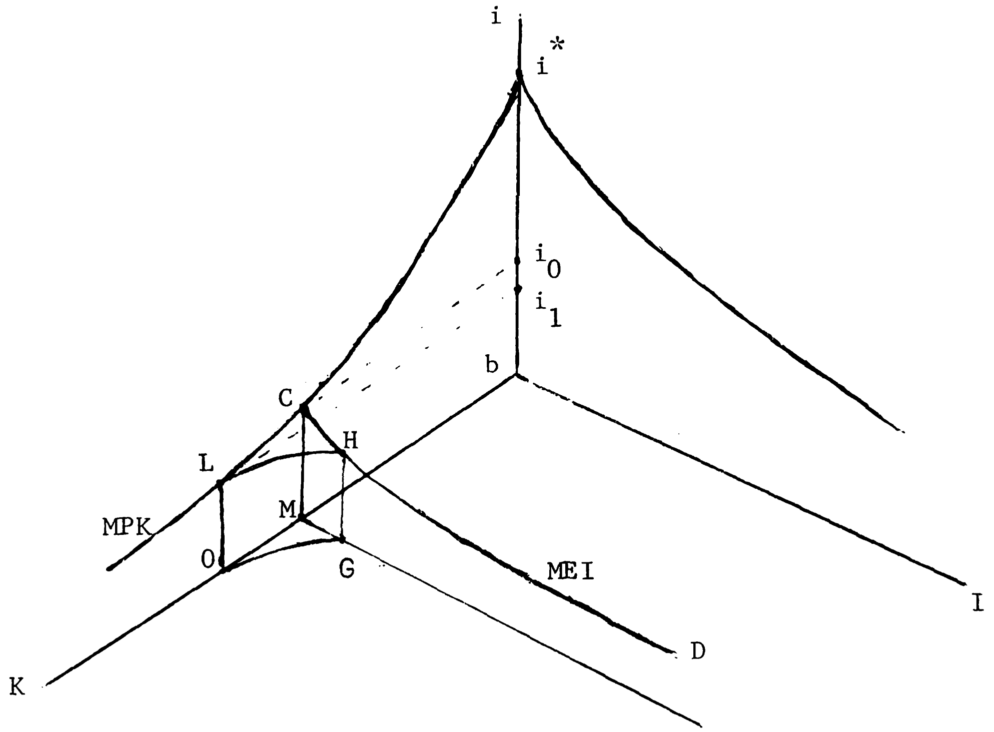

Figure 5.2 Lerner's Demand Functions for Capital Stock and Investment Goods

The intersection of the MEI with i gives the level of (or rate) of investment for the relevant time period. For example,

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[1]With one qualification, Keynes' MEC is derived from value terms.

[2] ibid., p. 336.

if the interest rate drops from i_0 to i_1 (equals HG), investment is represented by MG. In Lerner's interpretation, this quantity of investment is the first round in a series of investments. There is a series of MEI curves, those originating at points on the MPK curve between C and L, each of which is constructed after the investment from the preceding MEI has been made. The optimal capital stock is gradually made good by this series of investments. Total investment is given in the diagram by the area MGO. This area equals the length MO, or the increment to capital. (Here, as in the micro model of Eisner and Strotz, there is an implicit conception of an optimal capital stock, and an optimal rate of investment. A change in an exogenous variable, here i, results in distinct responses in both capital and investment.)

Since the identity of the MEI with the MEC concept is in question, the latter should also be interpreted on the Lerner diagram. Let the interest rate by i_1. Then total investment is represented by MG. And that is all. If no changes take place, there is no more investment. Optimal investment is given not by the area MGO but by the distance MG. Further, the distance MG equals MO. There is only one MEC schedule, and not a series, for every long run equilibrium.

For Keynes, investment is not a disequilibrium concept. The capital decision is made at every decision point. As in Chapter 2 potential investment is made up of possibilities with different horizons. The cost of capital goods thus reflects the length of the particular investment horizon.

But in questioning Lerner's analysis, we have to confront the issue of what is measured on the bK axis. How can it be the total capital stock? Is there any relevant rate of interest which would make a zero capital stock optimal? i^* would be so interpreted if bK is to measure the total capital stock. As an alternative, consider the MPK curve near the equilibrium point. Our interpretation leads to the conclusion that the MPK schedule below C and the MEI schedule CD are <u>identical</u>. Remember how Lerner's MPK is measured, p. 120. The percentage MPK is not as simple as it appears to be. A changing MPK hides either a changing physical yield, or a changing cost of capital (in terms of consumption goods), precisely the two factors embodied in the Keynesian MEC.

Lerner sees the MPK purely as a static concept. But because changing the level of capital necessarily involves pressure on prices (adjustment costs), a dynamic concept must be introduced to rationalize this - hence the MEI. But the percentage MPK already embodies these adjustment costs, and the MEI is superfluous. Indeed, Lerner recognizes this under conditions of general unemployment. On the macroeconomic level, this is the special case of friction - less adjustment which the usual static model is equipped to handle. Here there is no pressure on prices, relative prices are constant, and the MPK is purely static. The change in desired capital and investment are identical. Lerner cannot see that the MPK concept is equipped to handle the general case, because he will not disassociate it from a static optimum.

Certainly the use of the concept is misleading, and the confusion is cleared up in a later article in 1965.[1]

In this article, Lerner introduces another concept, the marginal productivity of investment, MPI. This is defined as "... the extra capital produced by diverting resources from making one unit of consumption goods to making capital goods".[2] Lerner has here segmented the two factors implicit in his 1944 'percentage' MPK. The MPK now has its more common meaning of physical output. In a complex many-good world, the MPI is more difficult to interpret. It is sufficient here to say that a decling MPI is associated with a rising supply price in the capital goods industries.

The product of MPK and MPI (in Lerner's consumption good-capital good model) gives the cost of the marginal consumption in terms of consumption goods, or the marginal capital productivity of consumption goods. This is the marginal efficiency of investment, MEI. Thus

$$MPK \cdot MEI = MEI \; . \qquad 5.3$$

Look again at equation 5.1. Allow revenue in each period to be equal, i.e. $R_i = R$, and let n tend to infinity. 5.1 reduces to

$$q = p.X/r \qquad 5.4$$

where $R = p.X$, and p is constant. In this terminology, MPK is X, MEI is r, and MPI is seen as the relative price of output and capital goods.

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1 A.P. Lerner, 1965 (90).

2 op. cit., p. 42.

The optimum growth, during the period, for the economy is given by the equality of its calculated MEI with the rate of interest. The essence of the dynamic approach is that changing the level of capital implies a change in both the components on the left hand side of 5.3. The accepted interpretation maintains a qualitative separation - the first factor determines the optimum stock, the second determines its optimum accumulation. From the accepted viewpoint, it is hard to see how costs influenced by the rate of investment can be an element in the determination of the optimum capital stock. As we saw in Chapter 4 (for the microeconomic case), it is true nonetheless.

For example, consider Ackley's treatment of the problem.[1] Ackley defines the demand for capital (incorrectly labelled MEC) in terms of that cost q which prevails at zero net investment. In Lerner's terminology, this curve is the MPK schedule, multiplied by the MPI prevailing in the previous (static) equilibrium. Ackley then superimposes upon this curve a declining MPI curve which determines the rate of investment.

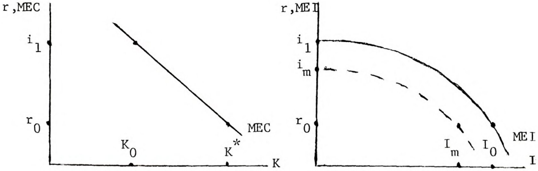

Figure 5.3 Ackley's Demand Functions for Capital Stock and Investment Goods.

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[1] G. Ackley, op. cit. (1), p. 477f.

The distinction is quite artifical. Of what possible relevance to the current investment rate is a price which prevailed in some past period? The rationalization for the distinction lies purely in the continued acceptance of the static equilibrium concept as the standard of analysis.

Witte, whose influential article appeared in 1963,[1] also insists on a capital stock-investment separation. Witte claims that no microeconomic foundation exists for Keynes' dynamic MEC and proceeds to supply one. His result relies upon a special model, with interesting properties, Figure 5.4. In particular, he assumes a perfect market in capital goods, and therefore that new capital goods are perfectly interchangeable with the existing

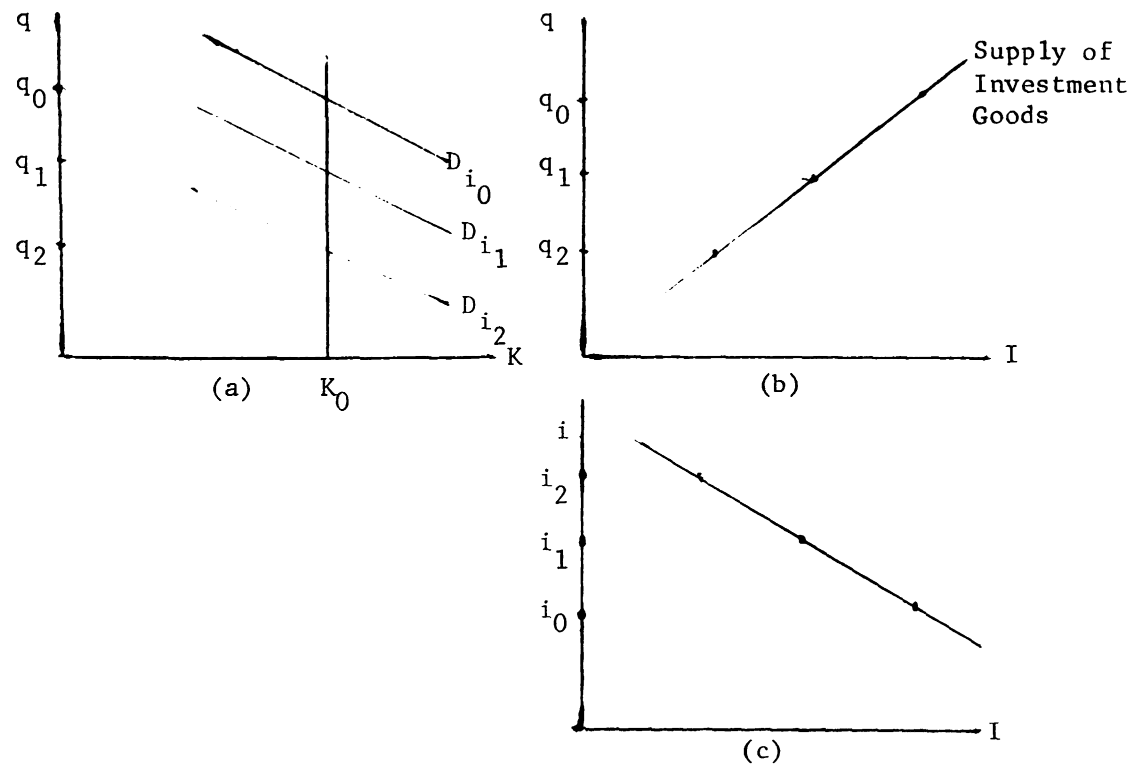

Figure 5.4 Witte's Demand Functions for Capital Stock and Investment Goods

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[1] J.G. Witte, op. cit. (139).

stock. This is Haavelmo's Law of Indifference. It follows that the supply of new capital goods is minimal compared to the stock, and its influence on the current price of capital goods negligible. The price of capital goods, and therefore the rate of return on capital, is purely demand-determined - Figure 5.4(a). The rate of accumulation is determined purely on the supply side by the profit-maximizing output response to the demand price - Figure 5.4(b). For Witte, this last factor rationalizes the slope of the MEC.

Fortunately, for we would like to do without it, the Indifference assumption is not essential to the analysis. If Indifference is present, the capital supply schedule, K_0, is certainly not price-inelastic, because reserve demand is price-elastic. Price is not purely demand determined. Moreover, there is a dimensional confusion. In Figure 5.4(a), we have a demand for capital in t. Figures (a) and (b) merge. With or without the presence of existing stock, there is a demand and supply of capital goods in t, which conjointly determine the price q. The series of demand curves reflects the greater investment made possible, of projects returning lower yields, when the interest rate is lower.

Witte's model appears to be no different to Keynes'. Witte's error lies in the assumption of a demand for capital without time reference. Like Lerner and Ackley, he wants to tackle a dynamic problem with a long run static equilibrium as his focal point.

As a digression, one other point of Witte's deserves mention. Witte claims that the Keynesian-type investment relation

is not a demand function at all, but a market equilibrium curve. In the capital goods market, this is true enough. Each point on the MEC schedule represents an equilibrium point in that market. But the MEC schedule _is_ a demand function for capital funds. In the usual macroeconomic setting, the comparison of the demand for and the supply of capital funds is relevant.

We interpret the Keynesian MEC as a dynamic replacement for the 'marginal productivity of capital'. The limitation of the latter concept arose indirectly through the problem of its dimensionality. The MPK involves the comparison of input with output, yet it is opposed to the rate of interest as the cost of capital, a pure number. Lerner has given an admirable discussion of the problem in a 1953 article.[1] Lerner's solution in 1944 was to measure the cost of capital in units of consumption goods, or output foregone. But his measuring rod is non-constant. The hidden ingredient is the relative price of capital goods and output, an essential element in the demand for capital. The natural tendency is to make this factor explicit in a term representing relative prices, reserving the MPK concept as a physical measure of output. This solution was adopted by Lerner in 1965.

But this presents another dilemma. The physical concept cannot cope with capital hetergeneity. This is a much more serious problem on the macroeconomic level. Obviously the next, and last, step is to measure the productivity of capital in value

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[1] A.P. Lerner, 1953 (89).

terms. Lerner spurns this alternative,[1] although his aversion rests upon a discussion of the marginal productivity of labor MPL, which differs in one important aspect from capital. In Lerner's example, the hiring of additional labor (due to an increased supply schedule) lowers the market wage which changes the value of the intra-marginal labor units. This decrement must be subtracted from the value of additional labor to give the corrected denominator of the 'value' MPL, thus distorting the measurement of labor's marginal productivity. Yet, one need value only the increment to capital stock. The current cost of capital is independent of the historical cost of existing capital.[2] Monetization does not distort the MPK concept by the revaluation of the intramarginal units. But it does cope with capital hetergeneity and allow explicitly for a change in factor and output prices.

By contrast, Keynes' MEC is derived from a comparison of net returns and costs to capital, both in value terms. As such it is a pure number, and comparable with the rate of interest. It overcomes the problems excluded from the traditional MPK concept, those of capital heterogeneity and a changing relative price of capital goods and output.

Moreover, the MEC introduces a form of symmetry between its two major components. A declining MEC results both from a

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[1] ibid., p. 1.

[2] It is true that the yield of existing capital will be implicit in the estimated yield of the new investment, but that is in the numerator of the calculations and presents no problem.

declining yield from the exploitation of investment opportunities, and from a rising supply price. The accepted Lerner-Ackley interpretation is that the first factor determines the demand for capital stock (but for what period nobody knows), and the second determines the optimum rate of accumulation of this previously determined stock. This is the same invalid dichotomy underlying much of microeconomic investment theory, embodied in the Flexible Accelerator. Here, as elsewhere, this separation results from the extension of the comparative static framework to dynamic problems.

To my knowledge, Conard[1] is the only author to recognize that the MEC of Keynes and the MEI of Lerner differ. Theoretically, the Keynesian formulation is certainly superior. Lerner interpreted Keynes as providing a theoretical rationalization for adjustment in (stationary) disequilibrium. In fact, Keynes thought as the businessman thought. The relevant consideration is the most profitable rate of investment <u>in the period t.</u>

Three factors limit the demand for investment goods - technical conditions, supply, and demand. These factors show themselves in:

i. expected increasing costs (decreasing physical MPK);

ii. increasing costs in the supplying industry (rising supply price) including the cost of funds;

iii. declining demand for the particular output (decreasing marginal rate of consumer substitution).

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[1] J.W. Conard, 1963 (18), p. 72f.

The relative influence of these factors on investment demand depends upon their relative movement through time, for each factor changes with time but at a different rate.

It was from this realization that Marshall created his periods of differing lengths, in order to separate the influence of each factor on 'normal value'. But Keynes has other interests and needs another framework. So a dynamic period analysis supplants comparative statics.

CHAPTER 6

CONCLUSION

The neoclassical theory of the firm has undergone a series of improvements, designed to incorporate, within the axiomatic optimization process itself, costs dependent on the nature of adjustment. With this we have confronted a heuristic model of capital expansion, based on the Marshallian long run unit cost curve. Both approaches postulate the existence of adjustment costs characterized, externally, by a rising supply price of capital goods; and, internally, by managerial diseconomies detracting from output. The juxtaposition has been fruitful, for it exposes the strengths and weaknesses of each approach.

The notion dies hard that all of economic analysis feeds on static equilibrium, even in the theory of capital, where inflexibility is a fact of economic life. The theory of investment attests to the gradual recognition that the nature of adjustment is an important subject in its own right, and that it is an integral consideration in the determination of capital stock. Both microeconomic and macroeconomic theories have suffered from the invalid assumption that capital and investment decisions result from separable static and dynamic optimization processes.

Both the neoclassical theory and the Marshallian extension imply that capital equilibrium, given a dynamic profit criterion,

is located in time, and itself dependent upon the costs of adjustment. But in the former, a rising supply price (external adjustment costs) is instrumental only in the determination of the adjustment path. We concluded that this was a consequence of the implicit assumption of perfect capital divisibility. This, usually neglected, assumption is not without its distortions on the derived paths of capital accumulation. The incorporation of a degree of capital indivisibility into the theory of the demand for capital is a source of potential improvement, though at first hand, it appears to be a difficult step.

A fundamental consideration in the 'construction' of a hypothetical firm is the stability of the firm's actions. The living firm is not large at one moment and small at the next, nor does it tend to infinite proportions. The hypothetical firm must be similarly constrained. For, not merely analytical simplicity but, the possibility of analysis, simplifying assumptions must be made which indirectly influence the nature of stability. Perfect competition is the main offender. It follows that the sources of stability for the hypothetical firm are limited to a subset of those facing the actual firm. In particular, the latter may enjoy increasing returns to scale, while the former may not.

It therefore came as an exciting possibility that dynamic adjustment costs might provide an alternative source of stability, letting returns to scale off the hook. But it is impossible to prove this contention in the simple graphical framework of Chapter 2. What is more important, for the case of increasing returns it was also impossible in the sophisticated framework of the neoclassical model.

But the reasons are clear. This limitation ensues firstly from the assumed linearities, essential to the analysis. Whereas the physical structure of a planet or a trajectory is fixed, the structure of an economy changes over time, and demands more complex analytical methods. The next step for the dynamic theory of capital accumulation lies in coping with a changing economic structure, and the attendant nonlinearities. My impression is that the source of this advance may lie in recursive decision procedures.

The second limitation arises from restricting the analysis to differential equations with constant coefficients. As stated in Appendix C, it is possible to solve an equation with nonconstant coefficients, given certain restrictions on the coefficients and the inhomogeneous element. This presents a definite possibility for future work on optimal capital accumulation, and it remains to be seen whether economically meaningful results can be gained in this direction.

Nevertheless, the present analysis provides a substantial advance on the subject of stability for the special case of constant returns to scale. Perfect competition and constant returns coexist for the firm in moving equilibrium where dynamic stability is provided by the limiting costs of growth.

It is apparent that strong assumptions have been necessary to ensure explicit analytic results in the dynamic neoclassical theory of the firm. For a linear differential equation, we had to assume perfect competition in the output and capital funds markets; and a quadratic production function, fixed over time. But perfect competition in the output market is assumed in static

theory, so no concession is made in this area. But the possible representations of technology are significantly reduced from those of static theory. So a trade-off has been effected in the movement from a static to a dynamic neoclassical theory of the firm. Whether this dynamic development is warranted depends ultimately on its empirical usefulness.

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A.E.R.	American Economic Review
E.J.	Economic Journal
E.R.	Economic Record
EKA	Econometrica
ECA	Economica, New Series
I.E.R.	International Economic Review
J.A.S.A.	Journal of the American Statistical Association
J.E.L.	Journal of Economic Literature
J.E.T.	Journal of Economic Theory
J.P.E.	Journal of Political Economy
Q.J.E.	Quarterly Journal of Economics
REStat	Review of Economics and Statistics
REStud	Review of Economic Studies
S.J.E.	Southern Journal of Economics
W.E.J.	Western Economic Journal

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