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ON THE MONOTONICITY AND CRITICAL POINTS OF THE PERIOD FUNCTION OF SOME SECOND ORDER EQUATIONS

By

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ABSTRACT

ON THE MONOTONICITY AND CRITICAL POINTS OF THE PERIOD FUNCTION OF SOME SECOND ORDER EQUATIONS

By

Duo Wang

The period function $p(c)$ of the equation $\ddot{x} + g(x) = 0$ is studied. We give sufficient conditions for the monotonicity of $p(c)$ in the cases where the closed orbits surround only one critical point and also more than one critical point. The boundedness of the number of critical points of the period function in an example where $g(x) = x^2(x - \alpha)(x - 1)$ $(0 \leq \alpha < 1)$ is a certain polynomial of degree 4 is established.

TO

MY PARENTS

AND

MY WIFE

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INTRODUCTION

Consider the scalar equation

$$\ddot{x} + g(x) = 0 , \qquad (\ddot{x} = \frac{d^2x}{dt^2}) \tag{0.1}$$

where $g(x)$ is smooth for all $x \in \mathbb{R}$. Let

$$G(x) = \int_0^x g(\xi)d\xi + C_o , \tag{0.2}$$

where C_0 is an arbitrary real number. If there exist $a < 0 < b$ such that $G(a) = G(b) = c$, $G(x) < c$ for all $a < x < b$ and $g(a) \cdot g(b) \neq 0$, then there exists a periodic orbit of (0.1) in the phase plane with energy c, intersecting the x-axis at $(a,0)$ and $(b,0)$. Let the least period of this periodic orbit be denoted by $p(c)$, which will be referred to as the period function in this note. It is well known that $p(c)$ is a smooth function of c (see [2]). In fact, if g is C^γ, $\gamma \geq 1$, then p is C^γ. Furthermore, $p(c)$ is given by the following formula

$$p(c) = \sqrt{2} \int_a^b \frac{dx}{\sqrt{c-G(x)}} . \tag{0.3}$$

Since the monotonicity of $p(c)$ plays a very important role in the study of subharmonic bifurcations from a planar Hamiltonian system (see [4]), there have been many authors who have studied the monotonicity of $p(c)$. See, for example, Loud [6], Obi [7], Opial [8], and Schaaf [9].

In this note, we will also discuss some of the properties of $p(c)$.

In Chapter I, we study the monotonicity of the period function of (0.1) for general g's. In §1.1, we derive some formulae for $p'(c)$ and $p''(c)$, and then give some useful theorems and corollaries for the monotonicity. In §1.2, we apply the results of §1.1 to some specific equations. We will prove the monotonicity of periodic function of equation

$$\ddot{x} + e^{x} - 1 = 0 , \tag{0.4}$$

that cannot be derived from previous results. This will complement the results of Wang [11] and will be useful for bifurcation problems [4].

Another important problem about the period function of (0.1) is: If $g(x)$ is a polynomial in x of degree n, is there a bound (depending only on n), denoted by $C(n)$, of the number of the critical points of period functions of (0.1)? This problem is raised by Chow and Sanders [5], and also by Smoller and Carr independently. It is related to the "weakened Hilbert's 16th problem" (Arnold [1], p. 303). In [5], Chow and Sanders proved that $C(2) = 0$ and $C(3) = 3$. For $n \geq 4$, this problem is still open. Even for $n = 4$ we do not know whether this bound exists.

If $g(x)$ is a quartic polynomial, then by scaling, all the cases under which the equation (0.1) has periodic solutions can be normalized to the following two cases:

(i) $g(x) = x(x - 1)(x^2 + \alpha x + \beta)$, $(\alpha^2 < 4\beta)$;

(ii) $g(x) = x(x - a)(x - b)(x - 1)$,

$$(0 \leq a \leq b \leq 1,\ a^2 + (b - 1)^2 \neq 0).$$

In Chapter II, we study the critical points of the period function of the equation

$$(0.5) \qquad x'' - x^2(x - \alpha)(x - 1) = 0 \ , \quad (0 \le \alpha < 1) \ ,$$

which is a special case of the case (ii) (b = 1). Our main result is that the number of critical points of the period functions $p_\alpha(c)$ of (0.5) for any $\alpha \in [0,1)$ is bounded (Theorem 2.2.24). The proof is mainly based on the analyticity of $p_\alpha(c)$ in α and c, which is proved by using the Picard-Fuchs equation.

In §2.1, we prove the analyticity of $p_\alpha(c)$ of (0.5) when (α,c) is in the domain $D = D_1 \cup D_2$ (see Figure 2.1.2), which is a bounded but not compact set in $\alpha - c$ plane.

In order to prove that there exists a bound for the number of critical points of $p_\alpha(c)$, we try to find a compact subdomain D_0 such that when $(\alpha,c) \in D - D_0$, $p_\alpha(c)$ has at most one critical point for any fixed α. So in §2.2, we first prove that $p_\alpha(c)$ is monotone for each $\alpha \in (0,1)$ and when c varies such that (α,c) is in D_2 (Lemma 2.2.1). Then we prove that for any $\alpha_0 \in (0,0.6)$ and $(\alpha_0,c_0) \in \partial D_1$ there is a neighborhood of (α_0,c_0) in which $p_\alpha(c)$ has no critical points (Lemma 2.2.6 and 2.2.10). The main difficulty in finding D_0 is to prove that there is a $\delta > 0$, such that for each $\alpha \in [0,\delta)$ and $\alpha \in (0.6 - \delta, 0.6)$, $p_\alpha(c)$ has exactly one critical point (Corollaries 2.2.19 and 2.2.23).

It does not seem easy to obtain the bound for the number of critical points of the period function $p_\alpha(c)$ of (0.5) for all $\alpha \in [0,1)$. The computer results suggest that $p_\alpha(c)$ may have 7 critical points for some $\alpha \in (0,0.6)$.

Chapter I

MONOTONICITY OF THE PERIOD FUNCTION OF $\ddot{x} + g(x) = 0$

§1.1. MAIN RESULTS

Let $g(x)$, $G(x)$ and $p(c)$ be as in the introduction.

Since we are interested mainly in either the monotonicity or the number of critical points of $p(c)$ (a critical point of $p(c)$ is the point c at which $p'(c) = 0$), we may assume that $g(x)$ has been scaled by $g(x) \to k\, g(\alpha x + \beta)$, where $k \cdot \alpha > 0$. Hence, we will assume $g(0) = 0$.

We consider now periodic orbits which contain only one critical point in their interior. In this case, we define $G(x)$ by (0.2) with $C_0 = 0$ and then consider the hypothesis:

(H1) There exist $-\infty \leq a^* < 0 < b^* \leq +\infty$, an integer $n \geq 0$ and a smooth function $h(x)$ such that

$$(1.1.1) \qquad \begin{aligned} h(x) &> 0\,, && a^* < x < b^*\,, \\ g(x) &= x^{2n+1}\, h(x)\,, && a^* < x < b^*\,, \end{aligned}$$

and

$$0 < G(a^*) = G(b^*) = c^* \leq +\infty\,.$$

Note that under the above hypothesis, the graph of $y = G(x)$ and the corresponding phase portrait of (0.1) are shown in Figure 1.1.1. Furthermore, $p(c)$ is defined for every $0 < c < c^*$.

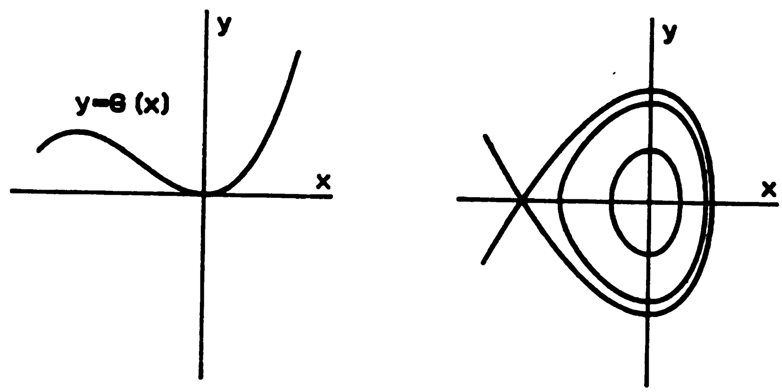

FIGURE 1.1.1

For simplicity, let

(1.1.2) $$\gamma(x,c) = 2(c - G(x)) .$$

Note that

(1.1.3) $$\frac{\partial \gamma}{\partial x} = -2g(x) ,$$

(1.1.4) $$p(c) = 2\int_a^b \frac{dx}{\sqrt{\gamma}} ,$$

where $a^* < a < 0 < b < b^*$, $\gamma(a,c) = \gamma(b,c) = 0$, $\gamma(x,c) > 0$ if $a < x < b$.

<u>**Theorem 1.1.1**</u>. Assume that (H1) holds. Then for any $0 < c < c^*$,

(1.1.5) $$cp'(c) = \int_a^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx\ , \qquad (p'(c) = \frac{dp}{dc})$$

where $a^* < a < 0 < b < b^*$, $G(a) = G(b) = c$, and

(1.1.6) $$R(x) = g^2(x) - 2G(x)g'(x)\ .$$

<u>**Proof**</u>: Let

$$I = \int_a^b \sqrt{\gamma}\, dx\ ,$$

and

(1.1.7) $$J = \int_a^b (\gamma - 2c)\sqrt{\gamma}\, dx\ .$$

Then

(1.1.8) $$I' = \int_a^b \frac{1}{\sqrt{\gamma}}\, dx\ ,$$

$$J' = \int_a^b \frac{\gamma - 2c}{\sqrt{\gamma}}\, dx = I - 2cI'\ .$$

Hence

(1.1.9) $$J'' = -I' - 2cI''\ .$$

On the other hand, integration by parts in (1.1.7) yields

$$J = \frac{2}{3}\int_a^b \frac{\gamma - 2c}{(\frac{\partial\gamma}{\partial x})}\, d\,\gamma^{3/2}$$

$$= -\frac{2}{3}\int_a^b \gamma^{3/2}\, d\left[\frac{\gamma - 2c}{(\frac{\partial\gamma}{\partial x})}\right]$$

$$= -\frac{2}{3}\int_a^b \frac{\gamma^{3/2}\,(g^2(x) - G(x)g'(x))}{g^2(x)}\, dx .$$

Differentiating the above equality with respect to c twice, we have

$$J'' = -2\int_a^b \frac{g^2(x) - G(x)g'(x)}{\sqrt{\gamma}\, g^2(x)}\, dx . \tag{1.1.10}$$

Then from (1.1.8), (1.1.9) and (1.1.10), we have

$$2cI'' = 2\int_a^b \frac{g^2(x) - G(x)g'(x)}{\sqrt{\gamma}\, g^2(x)}\, dx - I' \tag{1.1.11}$$

$$= \int_a^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx .$$

Note that $p(c) = 2I'$. Therefore (1.1.11) gives the desired result.

Remark 1.1.2. The hypothesis (H1) guarantees that all the integrations in the proof of Theorem 1.1.1 make sense.

Corollary 1.1.3. If (H1) holds and

$$x\, g''(x) < 0 \ (\text{or} > 0) , \quad x \neq 0 , \quad a^* < x < b^* ,$$

then

$$p'(c) > 0 \ (\text{or} < 0) , \quad 0 < c < c^* .$$

Proof: Since $R'(x) = -2G(x)g''(x)$, $R(0) = 0$, then $R(x) > 0$ (or < 0), $x \neq 0$, $a^* < x < b^*$.

Corollary 1.1.4. If (H1) holds and

$$\frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))} < 0 \quad (\text{or} > 0) \ , \quad a^* < x < 0 \ ,$$

where $R(x) = g^2(x) - 2G(x)g'(x)$ and $A(x)$ is defined by

$$G(A(x)) = G(x) \ , \quad a^* < x < 0 \ , \quad 0 < A(x) < b^* \ , \tag{1.1.12}$$

then

$$p'(c) > 0 \ (\text{or} < 0) \ , \quad 0 < c < c^* \ .$$

Proof: By the implicit function theorem, $A(x) \in C^1 \, (a^*,0)$ and

$$A'(x) = \frac{g(x)}{g(A(x))} \ , \quad a^* < x < 0 \ . \tag{1.1.13}$$

In the integral

$$\int_0^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)} \, dx \ ,$$

related to (1.1.5) we change variables by $x = A(y)$ to obtain

$$\int_0^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)} \, dx = \int_0^a \frac{R(A(x))}{\sqrt{\gamma}\, g^2(A(x))} \, A'(x) dx \ . \tag{1.1.14}$$

From Theorem 1.1.1 and (1.1.13), (1.1.14), we have

$$cp'(c) = \int_a^0 \frac{g(x)}{\sqrt{\gamma}} \left[\frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))}\right] dx \ .$$

Note that $g(x) < 0$, $a^* < x < 0$. The conclusion now follows.

<u>Corollary 1.1.5</u>. Suppose (H1) holds. If $g'(0) > 0$ and

$$(1.1.15) \qquad H(x) = g^2(x) + \frac{g''(0)}{3(g'(0))^2} g^3(x) - 2G(x)g'(x) > 0 \quad (\text{or } < 0) ,$$

$$x \neq 0 , \qquad a^* < x < b^* ,$$

then

$$p'(c) > 0 \ (\text{or } < 0) , \qquad 0 < c < c^* .$$

<u>Proof</u>: By L'Hopital's rule,

$$\lim_{x\to 0} \frac{R(x)}{g^3(x)} = -\frac{1}{3} \frac{g''(0)}{(g'(0))^2} .$$

Thus $H(x) > 0$ (or < 0) implies

$$\frac{R(x)}{g^3(x)} < -\frac{1}{3} \frac{g''(0)}{(g'(0))^2} < \frac{R(A(x))}{g^3(A(x))} , \quad a^* < x < 0 ,$$

$$(\text{or } \frac{R(x)}{g^3(x)} > -\frac{1}{3} \frac{g''(0)}{(g'(0))^2} > \frac{R(A(x))}{g^3(A(x))} , \quad a^* < x < 0) ,$$

since $A(x) > 0$, $x \in (a^*,0)$ and $x \cdot g(x) > 0$ for $x \neq 0$, $x \in (a^*,b^*)$.
By Corollary 1.1.4, we thus have

$$p'(c) > 0 \ (\text{or } < 0) , \qquad 0 < c < c^* .$$

<u>Corollary 1.1.6</u>. Suppose (H1) holds. If $g'(0) > 0$ and

$$(1.1.16) \qquad \nabla = 5(g''(0))^2 - 3g'(0)g'''(0) > 0 \quad (\text{or } < 0) ,$$

then there exists $\delta > 0$ such that

$$p'(c) > 0 \text{ (or } < 0) , \quad 0 < c < \delta .$$

<u>Proof</u>: By using Taylor's series we obtain,

$$H(x) = \frac{1}{12} x^4 \cdot \nabla + 0(|x|^5) , \quad \text{as} \quad |x| \to 0$$

The conclusion now follows from Corollary 1.1.5.

<u>Theorem 1.1.7</u>. Suppose (H1) holds. Then for any $0 < c < c^*$,

$$2c^2p''(c) = \int_a^b \frac{S(x)}{\sqrt{\gamma}\, g^4(x)} dx , \quad (p''(c) = \frac{d^2p}{dc^2})$$

where $a^* < a < 0 < b < b^*$, $G(a) = G(b) = c$, and

(1.1.17) $$S(x) = -g^4(x) - 4G(x)g^2(x)g'(x) - 4G^2(x)g(x)g''(x) + 12\, G^2(x)(g'(x))^2 .$$

<u>Proof</u>: Let

(1.1.18) $$K = \int_a^b \frac{R(x)\sqrt{\gamma}}{g^2(x)} dx ,$$

and

(1.1.19) $$L = \int_a^b \frac{2R(x)G(x)\sqrt{\gamma}}{g^2(x)} dx .$$

Differentiating (1.1.18) and (1.1.19) with respect to c, we obtain

(1.1.20)
$$K' = \int_a^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx \,,$$

$$L' = \int_a^b \frac{2R(x)\, G(x)}{\sqrt{\gamma}\, g^2(x)}\, dx \,.$$

By Theorem 1.1.1 it follows that $K' = cp'(c)$. Since $\gamma - 2c = -2G(x)$, $-L' = K - 2cK'$, we have

(1.1.21)
$$\begin{cases} p'(c) + cp''(c) = K'' \,, \\ K' + 2cK'' = L'' \,. \end{cases}$$

On the other hand, integration by parts in (1.1.8) yields

(1.1.22)
$$L = -\frac{2}{3}\int_a^b \frac{R(x)\, G(x)}{g^3(x)}\, d\,\gamma^{3/2}$$
$$= \frac{2}{3}\int_a^b \gamma^{3/2}\, d\left[\frac{R(x)\, G(x)}{g^3(x)}\right]$$
$$= \frac{2}{3}\int_a^b \frac{\gamma^{3/2}\, S_1(x)}{g^4(x)}\, dx \,,$$

where

(1.1.23)
$$S_1(x) = g^4(x) - 5G(x)g^2(x)g'(x) - 2G^2(x)g(x)g''(x) + 6G^2(x)(g'(x))^2 \,.$$

Differentiating (1.1.22) with respect to c twice, we then have

$$L'' = 2 \int_a^b \frac{S_1(x)}{\sqrt{\gamma}\, g^4(x)}\, dx\ . \tag{1.1.24}$$

From (1.1.20), (1.1.21) and (1.1.24) and by Theorem 1.1.1,

$$\begin{aligned} 2c^2 p''(c) &= L'' - 3cp'(c) \\ &= 2 \int_a^b \frac{S_1(x)}{\sqrt{\gamma}\, g^4(x)}\, dx - 3 \int_a^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx \\ &= \int_a^b \frac{2S_1(x) - 3R(x)g^2(x)}{\sqrt{\gamma}\, g^4(x)}\, dx\ . \end{aligned}$$

The desired result now follows from (1.1.6) and (1.1.23).

Remark 1.1.8. The hypothesis (H1) guarantees that all the integrations in the above proof make sense.

We now extend the previous results to periodic orbits whose interior may contain more than one critical points.

Note that we can also define $G(x)$ as follows:

$$G(x) = \int_0^x g(\xi)d\xi + c_o\ ,$$

where c_o can be any real number.

We need the following hypothesis:

(H2) There exist $-\infty \le a^* < \alpha \le 0 \le \beta < b^* \le +\infty$, integers $m \ge 0$, $n \ge 0$ and a smooth function $h(x)$ such that

$$h(x) \geq 0\ , \qquad a^* < x < b^*\ ,$$

$$x\, g(x) > 0\ , \qquad a^* < x < \alpha\ , \qquad \beta < x < b^*\ ,$$

$$0 < G(a^*) = G(b^*) = c^* \leq +\infty\ ,$$

and

(1.1.25) $$G(x) = (x - \alpha)^{2m+1}\,(x - \beta)^{2n+1}\,h(x), \quad a^* < x < b^*\ .$$

The graph of $y = G(x)$ and the corresponding phase portrait of (0.1) are shown in Figure 1.1.2.

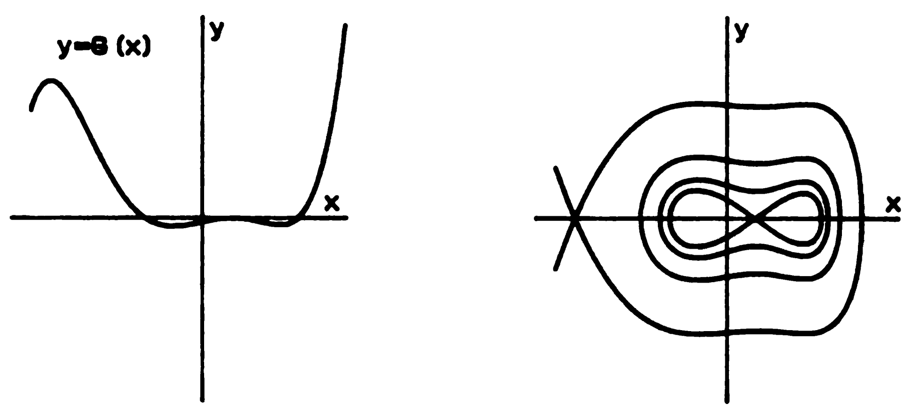

FIGURE 1.1.2

Theorem 1.1.9. Suppose (H2) holds. Then for $0 < c < c^*$,

$$\text{(1.1.26)} \qquad cp'(c) = \left(\int_a^\alpha + \int_\beta^b \right) \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx - 2c \int_\alpha^\beta \frac{dx}{\gamma^{3/2}} ,$$

where $a^* < a < \alpha$, $\beta < b < b^*$, $G(a) = G(b) = c$, $R(x)$, $\gamma(x,c)$ are the same as those in Theorem 1.1.1.

Proof: Define

$$J = \int_a^b (\gamma - 2c) \sqrt{\gamma}\, dx .$$

Note that $G(\alpha) = G(\beta) = 0$. Hence

$$J = \frac{2}{3} \left(\int_a^\alpha + \int_\beta^b \right) \frac{\gamma - 2c}{(\frac{\partial \gamma}{\partial x})}\, d\gamma^{3/2} + \int_0^\beta (\gamma - 2c) \sqrt{\gamma}\, dx$$

$$= -\frac{2}{3} \left(\int_a^\alpha + \int_\beta^b \right) \frac{\gamma^{3/2}(g^2(x) - G(x)g'(x))}{g^2(x)}\, dx$$

$$+ \int_0^\beta (\gamma - 2c) \sqrt{\gamma}\, dx .$$

The rest of the proof is similar to that of Theorem 1.1.1.

Remark 1.1.10. Hypothesis (H2) guarantees that all the integrations in the above proof make sense.

Corollary 1.1.11. If (H2) holds and

$$\text{(1.1.27)} \qquad \frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))} > 0 , \qquad a^* < x < \alpha ,$$

where $A(x)$ is defined by

$$G(A(x)) = G(x) , \quad a^* < x < \alpha , \quad \beta < A(x) < b^* ,$$

and $R(x) = g^2(x) - 2G(x)g'(x)$, then

$$p'(c) < 0 , \quad 0 < c < c^* .$$

<u>Proof</u>. It is similar to that of Corollary 1.1.4.

<u>Corollary 1.1.12</u>. Suppose (H2) holds. If

(i) $g(x)$ is odd,

(ii) $g(\alpha) = 0$,

(iii) $g''(x) \le 0 , \quad a^* < x < \alpha ,$

then

$$p'(c) < 0 , \quad 0 < c < c^* .$$

<u>Proof</u>: By the oddness of $g(x)$, (1.1.26) becomes

$$(1.1.28) \qquad cp'(c) = \int_a^\alpha \frac{2R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx - 2c \int_\alpha^{-\alpha} \frac{dx}{\gamma^{3/2}} .$$

Because $g''(x) \le 0$, so $R'(x) \ge 0$, $a^* < x < \alpha$. Therefore

$$R(x) \le R(\alpha) = g^2(\alpha) - 2G(\alpha)g'(\alpha) = 0, \quad a^* < x < \alpha .$$

From (1.1.28), the conclusion of the corollary is obvious.

<u>Theorem 1.1.13</u>. Suppose (H2) holds. Then for any $0 < c < c^*$,

$$2c^2 p''(c) = \left[\int_a^\alpha + \int_\beta^b \right] \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx + 12\, c^2 \int_\alpha^\beta \frac{dx}{\gamma^{5/2}} ,$$

where $s(x)$, $\gamma(x,c)$, a, b, c are the same as those in Theorem 1.1.7.

Corollary 1.1.14. Suppose (H2) holds and

$$\frac{S(x)}{g^5(x)} - \frac{S(A(x))}{g^5(A(x))} < 0 , \quad a^* < x < \alpha ,$$

where $A(x)$ is the same as that in Corollary 1.1.11, $S(x)$ is the same as that in Theorem 1.1.7, then

$$p''(c) > 0 , \quad 0 < c < c^* .$$

Remark 1.1.15. Theorems 1.1.1 and 1.1.7 are special cases of Theorems 1.1.9 and 1.1.13 respectively.

§1.2. APPLICATIONS

In this section, the results of §1.1 will be applied to several examples to show the monotonicity of the period function $p(c)$. The following theorem is useful in applications.

Theorem 1.2.1. Suppose (H1) holds. If $g'(0) > 0$, $g''(0) \geq 0$, then each of the following conditions implies $H(x) > 0$ for $x \neq 0$, $x \in (a_1,b_1)$ (see (1.1.15)).

(i) $g''(x) > 0$ and

$$\Delta(x) = x\,(g''(0)g'(x) - g'(0)g''(x)) \geq 0, \quad x \in (a_1,b_1) ,$$

where $a^* \leq a_1 \leq 0 \leq b_1 \leq b^*$.

(ii) $g''(x) > 0$, $g'''(x) \leq 0$, $x \in (a_1,b_1)$, where $a^* \leq a_1 \leq 0 \leq b_1 \leq b^*$.

(iii) $g''(x) < 0$, $g'(x) \geq 0$, $0 \leq a_1 < x < b_1 \leq b^*$, and $H(a_1) \geq 0$.

(iv) $g'(x) \leq 0$, $0 < a_1 < x < b_1 \leq b^*$.

(v) $g''(x) < 0$, $g'''(x) \geq 0$, $a^* \leq a_1 < x < b_1 < 0$ and $H(a_1) \geq 0$, $H(b_1) \geq 0$.

Example 1. Let

$$g(x) = e^x - 1 , \quad -\infty < x < +\infty .$$

Since $g'(x) = g''(x) = e^x > 0$, $-\infty < x < +\infty$, and $\Delta(x) = x(g''(0)g'(x) - g'(0)g''(x)) \equiv 0$. By Theorem 1.2.1 (i) and Corollary 1.1.5,

$$p'(c) > 0 , \quad 0 < c < +\infty .$$

From the results of Opial [8],

$$\lim_{c\to 0^+} p(c) = 2\pi , \quad \lim_{c\to +\infty} p(c) = +\infty .$$

Remark 1.2.2. The above result does not follow from the monotonicity results in [6], [7], [8] and [9].

Example 2. Let $g(x)$ be a quadratic polynomial. We may consider the normal form [5]:

$$g(x) = x(x + 1) , \quad -1 < x < +\infty .$$

Since $g''(x) = 2$, $g'''(x) \equiv 0$, by Theorem 1.2.1 (ii) and Corollary 1.1.5,

$$p'(c) > 0 , \quad 0 < c < c^* = \frac{1}{6} .$$

Since c^* corresponds to a homoclinic orbit, so $\lim_{c\to c^*-} p(c) = +\infty$. By the result of Opial [8], $\lim_{c\to 0+} p(c) = 2\pi$.

Example 3. Let $g(x)$ be a cubic polynomial. For periodic orbits with only one critical point in the interior, we may consider the following normal forms:

(3.a) $\qquad g(x) = -(x + a)x(x - 1), \quad 0 < a \leq 1, \quad -a < x < 1 .$

(3.b) $\qquad g(x) = x(x + a)(x + 1), \quad 0 < a \leq 1, \quad -a < x < +\infty .$

(3.c) $g(x) = x(x^2 + bx + 1), \quad 0 \le b < 2, \quad -\infty < x < +\infty .$

(3.d) $g(x) = x^3 , \quad -\infty < x < +\infty .$

For (3.a), $g(x) = -x^3 + (1 - a)x^2 + ax$. Then

$$g''(x) = -6x + 2(1 - a) ,$$
$$g'''(x) = -6 < 0 .$$

By Theorem 1.2.1 (ii), (iii), (iv) and Corollary 1.1.5,

$$p'(c) > 0 , \quad 0 < c < c^* = G(-a) .$$

For (3.b), $g(x) = x^3 + (1 + a)x^2 + ax$. Then

$$g''(x) = 6x + 2(1 + a) ,$$
$$g'''(x) = 6 > 0 .$$

Hence $g''(x) > 0$ if and only if $x > -\frac{1}{3}(1 + a)$.

$$\begin{aligned} \Delta(x) &= x^2[6(1 + a)x + 4(1 + a)^2 - 6a] \\ &\ge x^2[6(1 + a)(-\frac{1}{3}(1 + a)) + 4(1 + a)^2 - 6a] \\ &= x^2[2(1 + a)^2 - 6a] \\ &\ge 0 , \quad x > -\frac{1}{3}(1 + a) . \end{aligned}$$

By Theorem 1.2.1 (i), (v) and Corollary 1.1.5, we conclude that

$$p'(c) > 0 , \quad 0 < c < c^* = G(-a) .$$

For (3.c), if $b = 0$, then $g(x) = x^3 + x$, $g''(x) = 6x$. Thus by Corollary 1.1.3,

$$p'(c) < 0 , \quad 0 < c < +\infty .$$

If $b > \sqrt{9/10}$, then

$$\begin{aligned} \nabla &= 5(g''(0))^2 - 3g'(0)g'''(0) \\ &= 20b^2 - 18 > 0 . \end{aligned}$$

By Corollary 1.1.6, there exists $\delta > 0$ such that $p'(c) > 0$, $0 < c < \delta$. On the other hand, by a result of Opial [8], $p(c) \to 0$ as $c \to +\infty$. This implies that $p(c)$ is not monotone.

For (3.d), $g''(x) = 6x$. Then by Corollary 1.1.3,

$$p'(c) < 0 , \quad 0 < c < +\infty .$$

__Remark 1.2.3__. In [5], Chow and Sanders proved that there are at most three critical points of period function when $g(x)$ is a polynomial of degree three.

__Example 4__. Let $g(x) = -x^4 + x^3$, $-\infty < x < +1$. A direct calculation shows that

$$S(x) = \frac{4}{25} (156 x^{16} - 624 x^{15} + 896 x^{14} - 550 x^{13} + 125 x^{12}) .$$

Hence $S(x) > 0$ for $x < 0$. Furthermore,

$$\begin{aligned} S(x) = \frac{4}{25} x^{12}[&156(x - 0.65)^4 + 218.4(x - 0.65)^2(1-x) \\ &+ 4.879025(1 - x) + 2.439025(1 - x)x + 0.659025 x^2] \\ &> 0, \quad 0 < x < 1 . \end{aligned}$$

By Theorem 1.1.7,

$$p''(c) > 0 , \quad 0 < c < c^* = G(1) = \frac{1}{20} .$$

Since

$$\lim_{c\to 0^+} p(c) = \lim_{c\to c*^-} p(c) = +\infty ,$$

$p(c)$ has exactly one critical point.

Example 5. Let $g(x) = x(x^2 - 1)^2$ and

$$G(x) = \int_0^x g(\xi)d\xi - \frac{1}{6}$$

Then

$$G(x) = \frac{1}{6}(x + 1)^3 (x - 1)^3 .$$

Since $g(x)$ is odd, $g(-1) = G(-1) = 0$, and $g''(x) = 20 x^3 - 12 x < 0$, $x < -1$, by Corollary 1.1.12, the period function of the periodic orbits with three critical points in their interior is decreasing for $c \in (0,+\infty)$.

If we let

$$G(x) = \int_0^x g(\xi)d\xi ,$$

then by Theorem 1.2.1 (i), (iii), (iv), (v), we have $H(x) > 0$ for $x \neq 0$, $x \in (-1,1)$. Therefore

$$p'(c) > 0, \qquad 0 < c < c* = \frac{1}{6} .$$

We may thus conclude that there are no critical points of the period function of equation

$$\ddot{x} + x(x^2 - 1)^2 = 0 .$$

Chapter II

CRITICAL POINTS OF THE PERIOD FUNCTION OF $\ddot{x} - x^2(x - \alpha)(x - 1) = 0 \quad (0 \leq \alpha < 1)$

§2.1. ANALYTICITY OF THE PERIOD FUNCTION.

Let

$$g(x) = -x^2(x - \alpha)(x - 1),$$

$$G(x) = \int_0^x g(\xi)d\xi = -\frac{1}{5}x^5 + \frac{1}{4}(\alpha + 1)x^4 - \frac{1}{3}\alpha x^3.$$

The curves $y = g(x)$, $y = G(x)$ and the corresponding phase portraits are shown in Figure 2.1.1.

Let

$$(2.1.1) \qquad I_n = \int_a^b x^n y\, dx, \quad n = 0,1,2,3,$$

where $a < \alpha < b < 1$, $y = (2c - 2G(x))^{1/2}$, $y(a) = y(b) = 0$.

Lemma 2.1.1.

$$I_n' = \int_a^b \frac{x^n}{y}\, dx\ ,$$

where "'" denotes the differentiation with respect to c.

Proof. Since

$$y^2 = 2c - 2G(x)\ ,$$

then

$$yy' = 1\ .$$

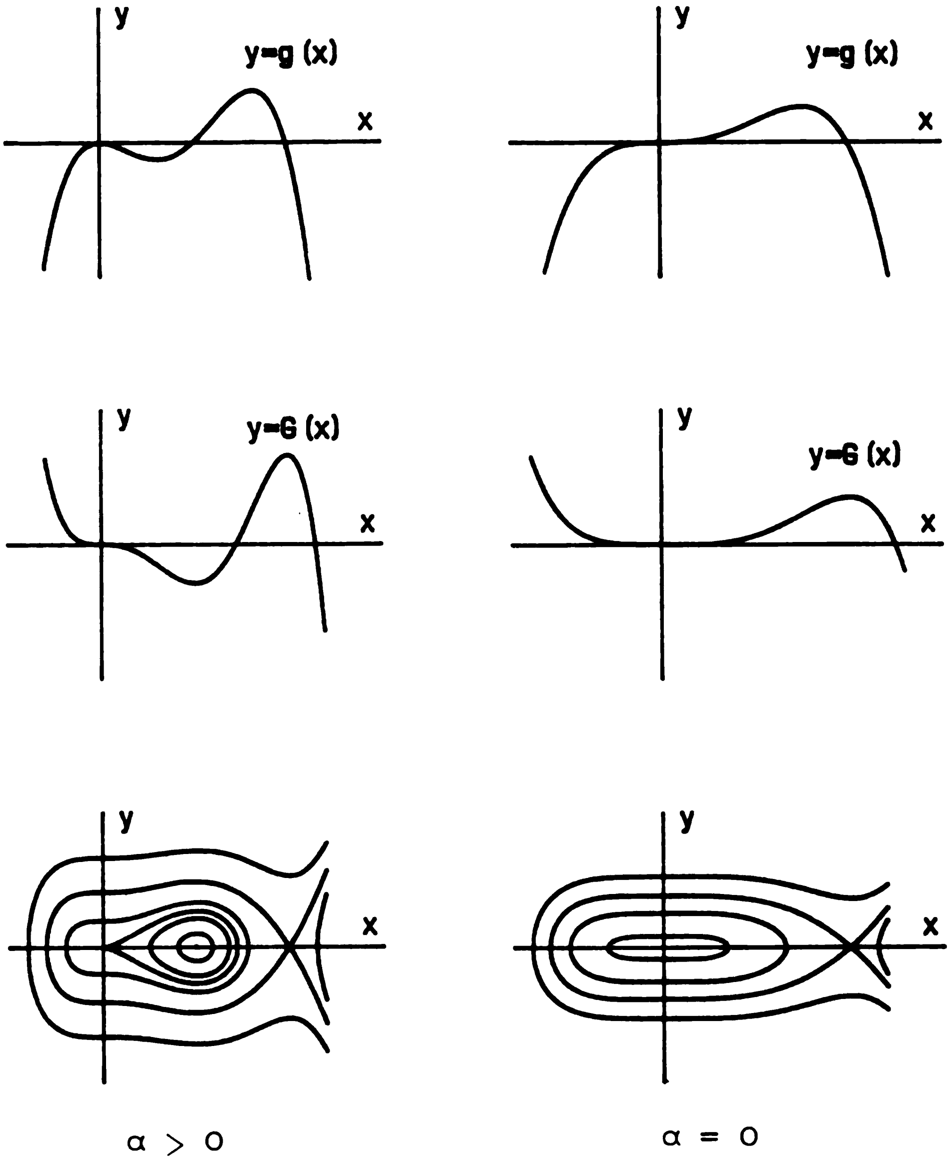

Figure 2.1.1

Lemma 2.1.2. Let $I = (I_0, I_1, I_2, I_3)^T$. Then

$$\Psi I' = \Phi I , \tag{2.1.2}$$

where

$$\Phi = \begin{bmatrix} 42 & 0 & 0 & 0 \\ -(3\alpha + 3) & 54 & 0 & 0 \\ -(3\alpha^2 - 2\alpha + 3) & -(6\alpha + 6) & 66 & 0 \\ -(3\alpha^3 - 2\alpha^2 - 2\alpha + 3) & -(6\alpha^2 - 4\alpha + 6) & -(9\alpha + 9) & 78 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} 60c & 0 & 3\alpha^2+3\alpha & -3\alpha^2+2\alpha-3 \\ 0 & 60c & 3\alpha^3-2\alpha^2+3\alpha & -3\alpha^3+2\alpha^2+2\alpha-3 \\ 0 & 0 & 60c+3\alpha^4-2\alpha^3-2\alpha^2+3\alpha & -3\alpha^4+2\alpha^3+2\alpha^2+2\alpha-3 \\ 0 & 0 & 3\alpha^5-2\alpha^4-2\alpha^3-2\alpha^2+3\alpha & 60c-3\alpha^5+2\alpha^4+2\alpha^3+2\alpha^2+2\alpha-3 \end{bmatrix}.$$

__Proof.__ Since

$$yy_x = -g(x) = x^4 - (\alpha + 1)x^3 + \alpha x^2 ,$$

$$x^4 = yy_x + (\alpha + 1)x^3 - \alpha x^2 .$$

It follows then

$$I_0 = \int_a^b y \, dx$$

$$= \int_a^b \frac{y^2}{y} \, dx$$

$$= \int_a^b \frac{2c + 2/5x^5 - 1/2(\alpha + 1)x^4 + 2/3\alpha x^3}{y} \, dx .$$

$$30 I_0 = 60 \, cI_0' + 12 \int_a^b \frac{x(yy_x + (\alpha + 1)x^3 - \alpha x^2)}{y} \, dx$$

$$- 15(\alpha + 1) \int_a^b \frac{x^4}{y} \, dx + 20 \, \alpha I_3'$$

$$= 60\ cI_0' + 12 \int_a^b xy_x dx - 12\ \alpha I_3'$$

$$- 3(\alpha + 1) \int_a^b \frac{yy_x + (\alpha + 1)x^3 - \alpha x^2}{y} dx + 20\ \alpha I_3'$$

$$= 60\ cI_0' - 12\ I_0 + 8\alpha\ I_3' - 3(\alpha + 1)^2 I_3' + 3\alpha(\alpha + 1)I_2' .$$

Hence

(2.1.3) $$42\ I_0 = 60\ cI_0' + (3\alpha^2 + 3\alpha)I_2' - (3\alpha^2 - 2\alpha + 3)I_3' .$$

Similarly, we have

(2.1.4) $$-(3\alpha + 3)I_0 + 54\ I_1 = $$
$$= 60\ cI_1' + (3\alpha^3 - 2\alpha^2 + 3\alpha)I_2' - (3\alpha^3 - 2\alpha^2 - 2\alpha + 3)I_3'$$

(2.1.5) $$-(3\alpha^2-2\alpha+3)I_0 - (6\alpha+6)I_1 + 66I_2 =$$
$$= (60c+3\alpha^4-2\alpha^3-2\alpha^2+3\alpha)I_2' - (3\alpha^4-2\alpha^3-2\alpha^2-2\alpha+3)I_3'$$

(2.1.6) $$-(3\alpha^3-2\alpha^2-2\alpha+3)I_0 - (6\alpha^2-4\alpha+6)I_1 - (9\alpha+9)I_2 + 78I_3 =$$
$$= (3\alpha^5-2\alpha^4-2\alpha^3-2\alpha^2+3\alpha)I_2' + (60c-3\alpha^5+2\alpha^4+2\alpha^3+2\alpha^2+2\alpha-3)I_3'.$$

Combining (2.1.3) ~ (2.1.6), we have the desired result.

The equation

(2.1.7) $$\ddot{x} - x^2(x - \alpha)(x - 1) = 0$$

has three critical points: $(0,0)$, $(\alpha,0)$ and $(1,0)$, which have energy constants $c_0 = G(0)$, $c_1 = c_1(\alpha) = G(\alpha) = 1/20\ \alpha^5 - 1/12\ \alpha^4$ and $c_2 = c_2(\alpha) = G(1) = 1/20 - 1/12\ \alpha$ respectively. Then $c_1(\alpha)$ and $c_2(\alpha)$ are both strictly decreasing, $c_1(\alpha) < c_2(\alpha)$, $0 \le \alpha < 1$. Furthermore, $c_1(0) = 0$, $c_2(0.6) = 0$.

Although $g(x)$, $G(x)$ and the period function of (2.1.7) depend on α, for simplicity, we usually suppress the subscript α in $g_\alpha(x)$, $G_\alpha(x)$ and $p_\alpha(x)$.

<u>Theorem 2.1.3</u>. $p(c)$ is analytic in α and c, provided (α,c) is in the domain $D = D_1 \cup D_2$, where

$$D_1 = \{(\alpha,c) \in R^2 \mid 0 \le \alpha < 0.6,\ 0 < c < c_2(\alpha)\}$$

$$D_2 = \{(\alpha,c) \in R^2 \mid 0 < \alpha < 1,\ c_1(\alpha) < c < \min(0,c_2(\alpha))\}$$

(see Figure 2.1.2).

<u>Proof</u>. A direct calculation shows that

$$\det \Phi = 60^4\, c^2(c - c_1(\alpha))(c - c_2(\alpha)) \ .$$

Therefore (2.1.2) is equivalent to

$$(2.1.8) \qquad I' = \frac{1}{\det \Phi} \Psi \Phi I, \quad \text{if} \quad c \neq 0, \quad c \neq c_1(\alpha), \quad c \neq c_2(\alpha),$$

where Ψ is the adjoint matrix of Φ. So the right hand side of (2.1.8) is analytic in I, α, c, when $(\alpha,c) \in D$, $-\infty < I_j < \infty$, $j = 0,1,2,3$. Therefore any solution of (2.1.8) $I = (I_0,I_1,I_2,I_3)^T$ is analytic in α, c, when $(\alpha,c) \in D$ (see, e.g., [10]). Since $P(c) = 2I_0'$, the conclusion now follows.

<u>Remark 2.1.4</u>. (2.1.8) is generally called the Picard-Fuchs equation to the algebraic curve $\frac{1}{2} y^2 + G(x) = C$.

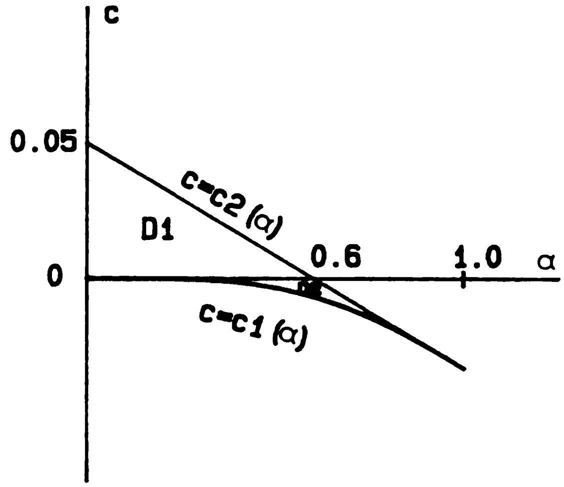

Figure 2.1.2

§2.2. BOUNDEDNESS OF NUMBER OF CRITICAL POINTS OF THE PERIOD FUNCTION.

Consider the period function $p(c)$ of the equation

$$\ddot{x} - x^2(x - \alpha)(x - 1) = 0, \quad 0 \leq \alpha < 1 . \tag{2.2.1}$$

We have

Lemma 2.2.1. For any $\alpha \in (0,1)$, $p(c)$ is strictly increasing when c varies from $c_1(\alpha)$ to $\min(0,c_2(\alpha))$.

Proof. By scaling $g(x)$, we have

$$\begin{aligned}\tilde{g}(x) &= -\frac{1}{(1-\alpha)^4}\, g(-(1 - \alpha)x + \alpha) \\ &= x(x + 1)(x - a)^2 ,\end{aligned}$$

where $a = \alpha/1-\alpha$. Let

$$\tilde{G}(x) = \int_0^x \tilde{g}(\xi)d\xi \ .$$

Thus it suffices to prove $\tilde{p}'(c) > 0$, $0 < c < c^* = \min(\tilde{G}(-1),\tilde{G}(a))$.

(1) Suppose $a \geq 2$. Then

$$\begin{aligned}\tilde{g}'(x) &= 4x^3 + 3(1 - 2a)x^2 + 2(a^2 - 2a)x + a^2 , \\ \tilde{g}''(x) &= 12x^2 + 6(1 - 2a)x + 2(a^2 - 2a) , \\ \tilde{g}'''(x) &= 24x + 6(1 - 2a).\end{aligned}$$

It is easy to check

(i) $\tilde{g}''(x) > 0$, $\tilde{g}'''(x) < 0$ for $-1 < x < x_1$, where

$$x_1 = \frac{1}{4}(1 - 2a) - \frac{1}{12}\sqrt{9(1-2a)^2 - 24(a^2-2a)} \geq 0 .$$

(ii) there exists $x_0 \in (x_1, a)$ such that

$$\tilde{g}''(x) < 0, \quad \tilde{g}'(x) > 0, \quad x \in (x_1, x_0) \text{ and}$$

$$\tilde{g}'(x) < 0, \qquad x \in (x_0, a) .$$

Then by Theorem 1.2.1 (ii), (iii), (iv) and Corollary 1.1.5,

$$\tilde{p}'(c) > 0, \qquad 0 < c < c^* .$$

(2) Suppose $0 < a < 2$. We consider

$$\tilde{g}_1(x) = -\tilde{g}(-x) = -x(x - 1)(x + a)^2 .$$

Then

$$\tilde{g}_1'(x) = -4x^3 + 3(1 - 2a)x^2 + (4a - 2a^2)x + a^2 ,$$

$$\tilde{g}_1''(x) = -12x^2 + 6(1 - 2a)x + (4a - 2a^2) ,$$

$$\tilde{g}_1'''(x) = -24x + 6(1 - 2a) .$$

It is easy to check

(i) $\tilde{g}_1''(x) > 0$, if $x_1 < x < x_2$, where

$$x_1 = \frac{1}{4}(1 - 2a) - \frac{1}{12}\sqrt{9(1-2a)^2 - 24(a^2-2a)} ,$$

$$x_2 = \frac{1}{4}(1 - 2a) + \frac{1}{12}\sqrt{9(1-2a)^2 - 24(a^2-2a)} ,$$

and $-a < x_1 < 0 < x_2 < 1$;

(ii) There exists $x_0 \in (x_2,1)$ such that

$$\tilde{g}_1''(x) < 0, \quad \tilde{g}_1'(x) > 0, \quad x \in (x_2,x_0) ,$$

$$\tilde{g}_1'(x) < 0, \quad x \in (x_0,1) ;$$

(iii) $\tilde{g}_1''(x) < 0, \quad \tilde{g}_1'''(x) > 0, \quad x \in (-a,x_1).$

(iv) $\tilde{g}_1'''(x) < 0, \quad \text{if} \quad x > \frac{1}{4}(1 - 2a)$.

We rewrite $\Delta(x) = x^2Q(x)$, where

$$Q(x) = 8(a^2 - 2a)x^2 - [6(a^2 - 2a)(1 - 2a) - 12a^2]x + 4(a^2 - 2a)^2 - 6a^2(1 - 2a) .$$

Note that $Q(x)$ is quadratic and $8(a^2 - 2a) < 0$. So $Q(x) > 0$ if and only if $x \in (\tilde{x}_1,\tilde{x}_2)$, where $\tilde{x}_1 < 0 < \tilde{x}_2$ are the two real roots of $Q(x)$. Therefore $\Delta(x) \geq 0$ if and only if $x \in [\tilde{x}_1,\tilde{x}_2]$. Thus to prove $\Delta(x) \geq 0$, $x_1 \leq x \leq x_2$, it suffices to show $\Delta(x_1) \geq 0$ and $\Delta(x_2) \geq 0$. By definition, $\Delta(x) = x(\tilde{g}_1''(0)\tilde{g}_1'(x) - \tilde{g}_1'(0)\tilde{g}_1''(x))$. Since x_1 and x_2 are the roots of $\tilde{g}_1''(x)$, $\Delta(x_i) = \tilde{g}_1''(0) \cdot x_i\, \tilde{g}_1'(x_i)$, $i = 1,2$, in this case. Note that $\tilde{g}_1''(0) > 0$. The sign of $\Delta(x_i)$ is the same as that of $x_i \cdot \tilde{g}_1'(x_i)$ $(i = 1,2)$. It is easy to see $\tilde{g}_1'(x_1) < 0$, $\tilde{g}_1''(x_2) > 0$ and $x_1 < 0 < x_2$. Then $\Delta(x) \geq 0$ $(x_1 < x < x_2)$ follows.

Then by Theorem 1.2.1 (i), (iii), (iv), (v) and Corollary 1.1.5, we conclude

$$\tilde{P}_1'(c) > 0, \qquad 0 < c < c^* .$$

<u>Remark 2.2.2</u>. We can prove in a similar manner that the period function p(c) of periodic orbits which contain only one critical point in their interiors of the equation

$$\ddot{x} + x(x-1)(x-a)(x-b) = 0, \qquad 0 \leq a \leq b \leq 1,$$
$$a^2 + (1-b)^2 \neq 0 \, .$$

is strictly increasing.

<u>Remark 2.2.3</u>. Lemma 2.2.1 implies that p(c) has no critical points when (α,c) is in D_2. Thus, in order to study the critical points of p(c), we can turn our attention on the case $(\alpha,c) \in D_1$, which corresponds to the periodic orbits of (2.2.1) that contain more than one critical points in their interior. We will henceforth always assume $(\alpha,c) \in D_1$.

<u>Lemma 2.2.4</u>. When $(\alpha,c) \in D_1$,

$$(2.2.2) \qquad cp'(c) = \left(\int_a^0 + \int_\beta^b \right) \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx - 2c \int_0^\beta \frac{dx}{\gamma^{3/2}} \, ,$$

$$(2.2.3) \qquad 2c^2 p''(c) = \left(\int_a^0 + \int_\beta^b \right) \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx + 12\, c^2 \int_0^\beta \frac{dx}{\gamma^{3/2}} \, ,$$

where $a < 0 < b < 1$ such that $G(a) = G(b) = c$; $R(x)$, $S(x)$, $\gamma = \gamma(x,c)$ are the same as those in Theorem 1.1.9 and 1.1.13; $\alpha \leq \beta < 1$ such that $G(\beta) = 0$.

<u>Proof</u>: (2.2.2) and (2.2.3) follow from Theorem 1.1.9 and 1.1.13 respectively.

<u>**Remark 2.2.5.**</u> By the implicit function theorem, the functions $\beta = \beta(\alpha)$, $a = a(\alpha,c)$ and $b = b(\alpha,c)$ implicitly defined by $G(\beta) = 0$ $(\alpha \le \beta < 1)$, $G(a) = G(b) = c$ $(a < 0 < b < 1)$ are continuous.

<u>**Lemma 2.2.6.**</u> For any $\alpha_0 \in (0, 0.6)$, there exists $\delta > 0$ such that

$$p'(c) < 0, \quad |\alpha - \alpha_0| < \delta, \quad 0 < c < \delta .$$

<u>**Proof:**</u> Let $A(x)$ be the function implicitly defined by

$$(2.2.4) \qquad G(A(x)) = G(x), \quad \text{for } x < 0, \quad A(x) > \beta.$$

Then $A(x) \in C^1(a^*,0)$, where $a^* < 0$ such that $G(a^*) = G(1)$, and

$$A'(x) = \frac{g(x)}{g(A(x))}, \qquad a^* < x < 0.$$

Since

$$\lim_{\substack{x\to 0^- \\ \alpha\to\alpha_0}} g(A(x)) = g(\beta(\alpha_0)) \neq 0 ,$$

$$\lim_{\substack{x\to 0^- \\ \alpha\to\alpha_0}} g(x) = 0 ,$$

we have

$$\lim_{\substack{x\to 0^- \\ \alpha\to\alpha_0}} \frac{R(A(x))}{g^2(A(x))} A'(x) = 0 .$$

On the other hand, it is easy to show that

(2.2.5) $$\frac{R(x)}{g^2(x)} < -\frac{1}{3}, \qquad x < 0 .$$

Then there exists $\delta_1 > 0$ such that

$$\frac{R(x)}{g^2(x)} - \frac{R(A(x))}{g^2(A(x))} A'(x) < -\frac{1}{3} - (-\frac{1}{4}) < 0,$$

$$\text{if } |\alpha - \alpha_0| < \delta_1, \qquad -\delta_1 < x < 0 .$$

Hence

$$\frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))} > 0, \quad |\alpha - \alpha_0| < \delta_1, \quad -\delta_1 < x < 0 .$$

Take

$$\delta = \min(\delta_1, \min_{|\alpha-\alpha_0|\le\delta_1} G(-\delta_1)) .$$

Now by Corollary 1.1.11, we have

$$P'(c) < 0, \qquad |\alpha - \alpha_0| < \delta, \qquad 0 < c < \delta .$$

<u>Lemma 2.2.7</u>. For any $\alpha_0 \in [0,0.6)$, there exist $\delta > 0$, $M > 0$ such that

$$\int_a^0 \frac{R(x)}{\sqrt{\gamma}\, g^2(x)} dx > -M, \qquad |\alpha - \alpha_0| < \delta, \qquad |c - c_2(\alpha_0)| < \delta .$$

<u>Proof</u>: Let $\xi = 1/2\, a(\alpha_0,c_2(\alpha_0))$. Then $\xi < 0$. By continuity of $a(\alpha,c)$, there exists $\delta_1 > 0$ such that

$$a(\alpha,c) < \xi, \qquad |\alpha - \alpha_0| < \delta_1, \qquad |c - c_2(\alpha_0)| < \delta_1 .$$

It follows that

$$c = G(a(\alpha,c)) > G(\xi), \quad |\alpha - \alpha_0| < \delta_1, \quad |c - c_2(\alpha_0)| < \delta_1 .$$

Therefore

$$\int_\xi^0 \frac{dx}{\sqrt{\gamma}} = \frac{1}{\sqrt{2}} \int_\xi^0 \frac{dx}{\sqrt{c-G(x)}} < \sqrt{2} \int_\xi^{A(\xi)} \frac{dx}{\sqrt{G(\xi)-G(x)}}$$

$$= P(G(\xi)) \to P_{\alpha_o}(G_{\alpha_o}(\xi)), \qquad \text{as} \quad \alpha \to \alpha_o ,$$

where $A(x)$ is defined by (2.2.4). On the other hand, by the mean value theorem,

$$\int_a^\xi \frac{dx}{\sqrt{\gamma}} = \frac{1}{\sqrt{2}} \int_a^\xi \frac{dx}{\sqrt{G(a)-G(x)}} = \frac{1}{\sqrt{2}} \int_a^\xi \frac{dx}{\sqrt{g(\eta)(a-x)}}$$

$$< \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{-g(\xi)}} (x - a)^{1/2} \Big|_a^\xi = \frac{\sqrt{2(\xi-a)}}{\sqrt{-g(\xi)}} , \quad a < \eta < \xi .$$

Note that $a(c,\alpha) > -1$ always. Then

$$\int_a^\xi \frac{dx}{\sqrt{\gamma}} < \frac{\sqrt{2(\xi+1)}}{\sqrt{-g(\xi)}} \to \frac{\sqrt{2(\xi+1)}}{\sqrt{-g_{\alpha_o}(\xi)}} , \quad \text{as} \quad \alpha \to \alpha_0 .$$

Therefore there exist $M > 0$, $\delta > 0$ $(\leq \delta_1)$ such that

$$\int_a^0 \frac{dx}{\sqrt{\gamma}} = \left(\int_a^\xi + \int_\xi^0\right) \frac{dx}{\sqrt{\gamma}} < M, \quad |\alpha - \alpha_o| < \delta,$$

$$|c - c_2(\alpha_0)| < \delta .$$

It is easy to show that

$$\frac{R(x)}{g^2(x)} > -1, \qquad x < 0$$

Hence the conclusion now follows.

Lemma 2.2.8. For any $\alpha_0 \in [0, 0.6)$ and $\gamma_0 \in (\beta(\alpha_0),1)$, there exist $\delta > 0$, $M > 0$ such that if $|\alpha - \alpha_0| < \delta$, $|c - c_2(\alpha_0)| < \delta$, then

(i) $\beta(\alpha) < \gamma_0$, $b(c,\alpha) > \gamma_0$,

(ii) $\displaystyle\int_\beta^{\gamma_0} \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx > -M$.

Proof: (i) is obvious. Now suppose (i) holds if $|\alpha - \alpha_0| < \delta_1$, $|c - c_2(\alpha_0)| < \delta_1$ for some $\delta_1 > 0$.

Claim: There exist $\delta_2 > 0$ $(\leq \delta_1)$, $M_1 > 0$ such that

(2.2.6) $$\frac{R(x)}{g^2(x)} > -M_1, \quad |\alpha - \alpha_0| < \delta_2, \quad \beta(\alpha) < x < \gamma_0 .$$

In fact, if $\alpha_0 \neq 0$, then $\beta(\alpha_0) > \alpha_0$, and (2.2.6) holds by continuity. When $\alpha_0 = 0$, we let

$$\left(\frac{R(x)}{g^2(x)} + 1\right) = \frac{x^4}{6g^2(x)} \cdot f(x) ,$$

where $f(x) = 4\alpha^2 + 3x^2 - 6\alpha x + 0((|x| + |\alpha|)^3)$ as $|x| \to 0$, $\alpha \to 0$. Since $4\alpha^2 - 6\alpha x + 3x^2$ is positive definite, there exists $\bar{\delta}_2 > 0$ $(\leq \delta_1)$ such that

$$f(x) > 0, \qquad 0 \leq \alpha < \bar{\delta}_2, \qquad 0 < x < \bar{\delta}_2 .$$

Therefore

$$(2.2.7) \qquad \frac{R(x)}{g^2(x)} > -1, \quad 0 \leq \alpha < \bar{\delta}_2, \quad 0 = \beta(0) < x < \bar{\delta}_2 .$$

Take $\delta_2 = 1/2\ \bar{\delta}_2$. Then

$$g(x) \neq 0, \quad 0 \leq \alpha \leq \delta_2, \quad \bar{\delta}_2 \leq x \leq \gamma_0.$$

Therefore there exists $M_1 > 1$ such that

$$(2.2.8) \qquad \frac{R(x)}{g^2(x)} > -M_1, \quad 0 \leq \alpha \leq \delta_2, \quad \bar{\delta}_2 \leq x \leq \gamma_0 .$$

Combining (2.27) and (2.28), we obtain (2.26).

Proof of (ii). Since

$$G(\gamma_0) \to G_{\alpha_0}(\gamma_0) < G_{\alpha_0}(1) = C_2(\alpha_0), \quad \text{as} \quad \alpha \to \alpha_0 ,$$

there exists $\delta_3 > 0$ ($\leq \delta_2$) such that

$$G(\gamma_0) < \frac{c_2(\alpha_0) + 2G_{\alpha_0}(\gamma_0)}{3}, \quad |\alpha - \alpha_0| < \delta_3 .$$

Take

$$\delta = \min\left\{\delta_3, \frac{c_2(\alpha_0) - G_{\alpha_0}(\gamma_0)}{3}\right\},$$

$$M = \frac{M_1}{\sqrt{2}} \cdot \frac{1}{\left[\frac{c_2(\alpha_0) - G_{\alpha_0}(\gamma_0)}{3}\right]^{1/2}}.$$

Then

$$\int_{\beta}^{\gamma_0} \frac{R(x)}{\sqrt{\gamma}\, g^2(x)} > -\frac{M_1}{\sqrt{2}} \int_{\beta}^{\gamma_0} \frac{dx}{\sqrt{\frac{2c_2(\alpha_0) + G_{\alpha_0}(\gamma_0)}{3} - G(\gamma_0)}}$$

$$> -\frac{M_1}{\sqrt{2}} \cdot \frac{(\gamma_0 - \beta)}{\sqrt{\frac{2c_2(\alpha_0)+G_{\alpha_0}(\gamma_0)}{3} - \frac{c_2(\alpha_0)+2G_{\alpha_0}(\gamma_0)}{3}}}$$

$$> -M, \quad |\alpha - \alpha_0| < \delta, \quad |c - c_2(\alpha_0)| < \delta .$$

__Lemma 2.2.9.__ Suppose $\alpha_0 \in [0,0.6)$. If there exist $\gamma_0 \in (\beta(\alpha_0),1)$ and $\delta_0 > 0$ such that

$$g'(x) < 0, \quad \gamma_0 < x \leq 1, \quad |\alpha - \alpha_0| < \delta_0,$$

then for any $M > 0$, there exists $\delta > 0$ such that

$$\int_{\gamma_0}^{b} \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx > M, \quad |\alpha - \alpha_0| < \delta, \quad |c - c_2(\alpha_0)| < \delta .$$

__Proof__: Since

$$\lim_{\alpha\to\alpha_0} \beta(\alpha) = \beta(\alpha_0) < \gamma_0 ,$$

$$\lim_{\substack{\alpha\to\alpha_0 \\ x\to 1^-}} \frac{R(x)}{g^2(x)} = +\infty ,$$

$$\lim_{\substack{\alpha\to\alpha_0 \\ x\to 1^-}} \frac{g(x)}{1-x} = 1 - \alpha_0 > 0 ,$$

and

$$\lim_{\substack{\alpha\to\alpha_0 \\ c\to c_2(\alpha_0)}} b(\alpha,c) = 1 ,$$

there exist $\delta > 0$, $\gamma_1 \in (\gamma_0, 1)$ such that

$$\beta(\alpha) < \gamma_0 ,$$

$$\frac{R(x)}{g^2(x)} > \sqrt{2}\, M \sqrt{1 - \alpha_0} ,$$

$$\left(\frac{g(\gamma_1)}{1-\gamma_1}\right)^{1/2} < \sqrt{2}\, \sqrt{1 - \alpha_0},$$

$$b = b(\alpha, c) > \frac{1 + \gamma_1}{2} , \quad \text{if} \quad |\alpha - \alpha_0| < \delta,$$

$$|c - c_2(\alpha_0)| < \delta, \quad \text{and} \quad \gamma_1 \leq x < 1 .$$

Note that $R(x) > 0$, $\gamma_0 \leq x \leq b$, $|\alpha - \alpha_0| < \delta_0$ under the condition of Lemma 2.2.9. Therefore

$$\int_{\gamma_0}^{b} \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx > \int_{\gamma_1}^{b} \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx$$

$$> M \sqrt{1 - \alpha_0} \int_{\gamma_1}^{b} \frac{dx}{\sqrt{G(b) - G(x)}}$$

$$= M \sqrt{1 - \alpha_0} \int_{\gamma_1}^{b} \frac{dx}{\sqrt{g(\xi)(b-x)}} \quad (x \leq \xi \leq b)$$

$$> 2M \sqrt{1 - \alpha_0} \sqrt{\frac{1-\gamma_1}{2g(\gamma_1)}}$$

$$> M, \quad \text{if} \quad |\alpha - \alpha_0| < \delta, \quad |c - c_2(\alpha_0)| < \delta .$$

<u>**Lemma 2.2.10**</u>. For any $\alpha_0 \in [0, 0.6)$, there exists $\delta > 0$ such that

$$p'(c) > 0, \quad |\alpha - \alpha_0| < \delta, \quad |c - c_2(\alpha_0)| < \delta .$$

Proof: Since $g'(x) \to g'_\alpha(1) = \alpha_0 - 1 < 0$, as $x \to 1^-$, $\alpha \to \alpha_0$, there exist $\delta_1 > 0$ and $\gamma_0 \in (\beta(\alpha_0),1)$ such that $g'(x) < 0$, $|\alpha - \alpha_0| < \delta_1$, $\gamma_0 < x < 1$.

It is not difficult to show that there exists $M_1 > 0$ such that

$$\int_0^\beta \frac{2c}{\gamma^{3/2}}\, dx < M_1, \quad \text{if} \quad c > \frac{1}{2} c_2(\alpha_0), \quad |\alpha - \alpha_0| < \delta_1 .$$

By Lemma 2.2.7 and 2.2.8, there exist $M_2 > 0$, $M_3 > 0$ and $\delta_2 > 0$ ($\leq \min(\delta_1, 1/2\, c_2(\alpha_0))$) such that

$$\int_a^0 \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx > -M_2$$

$$\int_\beta^{\gamma_0} \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx > -M_3 , \quad \text{if} \quad |\alpha - \alpha_0| < \delta_2 ,$$

$$|c - c_2(\alpha_0)| < \delta_2 .$$

By Lemma 2.2.9, there exists $\delta > 0$ ($\leq \delta_2$) such that

$$\int_{\gamma_0}^b \frac{R(x)}{\sqrt{\gamma}\, g^2(x)}\, dx > M_1 + M_2 + M_3 ,$$

$$|\alpha - \alpha_0| < \delta , \quad |c - c_2(\alpha_0)| < \delta .$$

It then follows from (2.2.2) that,

$$p'(c) > 0, \quad |\alpha - \alpha_0| < \delta, \quad |c - c_2(\alpha_0)| < \delta .$$

Lemma 2.2.11. There exists $\delta > 0$ such that

(i) $4/3\, \alpha \leq \beta(\alpha) \leq 1.33334\alpha$, if $0 \leq \alpha < \delta$,

(ii) $g'(x) > 0$, $g''(x) > 0$, $0 \leq \alpha < \delta$, $\alpha < x < 1/3$,

(iii) $R(x) = g^2(x) - 2G(x)g'(x) < 0, \quad 0 \le \alpha < \delta,$

$1.579\alpha \le x < 1/3.$

Proof: Since

$$G(k\alpha) = -\frac{1}{60}(k\alpha)^3 \cdot \alpha \cdot [3k\alpha(4k-5) + 5(4-3k)] ,$$

So (i) is obvious from the definition of $\beta(\alpha)$.

(ii) is trivial. Therefore, there exists $\delta_1 > 0$ such that

$$R'(x) = -2G(x)g''(x) < 0, \quad 0 \le \alpha < \delta_1, \quad \beta(\alpha) < x < \frac{1}{3} .$$

Note that

$$R(1.579\alpha) = 1.579^4\alpha^6[-0.000954 + 0(1)], \quad \text{as} \quad \alpha \to 0.$$

Then (iii) holds.

Lemma 2.2.12. If $k_1, k_2 > 0$ and

$$H_1(k_1,k_2) = (15k_1^{\,4} + 20k_1^{\,3}) - (15k_2^{\,4} - 20k_2^{\,3}) > 0 ,$$

then there exists $\delta > 0$ such that

$$G(-k_1\alpha) \ge G(k_2\alpha), \quad 0 \le \alpha < \delta .$$

Proof: It follows from the fact that

$$G(-k_1\alpha) - G(k_2\alpha)$$
$$= \frac{\alpha^4}{60}[H_1(k_1,k_2) + o(1)], \quad \text{as} \quad \alpha \to 0 .$$

<u>**Lemma 2.2.13**</u>. If $k_1, k_2 > 0$, and

$$H_2(k_1,k_2) = -1.40475(k_1^3 + k_1^2) + (k_2^3 - k_2^2)^3 > 0 ,$$

then there exists $\delta > 0$ such that

$$4g^2(\beta)g(-k_1\alpha) + g^3(k_2\alpha) \geq 0, \qquad 0 \leq \alpha < \delta .$$

<u>**Proof**</u>: By Lemma 2.2.11 there exists $\delta_1 > 0$ such that

$$\beta(\alpha) \leq 1.33334\alpha < \frac{1}{3} , \qquad 0 \leq \alpha < \delta_1 ,$$

$$g'(x) > 0 \qquad , \qquad 0 \leq \alpha < \delta_1, \qquad \beta(\alpha) \leq x < \frac{1}{3} .$$

Therefore, since $g(-k_1\alpha) < 0$, we have

$$\begin{aligned} &4g^2(\beta)g(-k_1\alpha) + g^3(k_2\alpha) \\ &\quad \geq 4g^2(1.33334\alpha)g(-k_1\alpha) + g^3(k_2\alpha) \\ &\quad \geq \alpha^9[H_2(k_1,k_2) + 0(1)], \qquad \text{as} \quad \alpha \to 0 . \end{aligned}$$

<u>**Lemma 2.2.14**</u>. If

(i) $k_{12} > k_{11} \geq 0, \qquad k_{21} > 1$,

(ii) $H_1(k_{11},k_{21}) > 0$,

(iii) $H_2(k_{12},k_{21}) > 0$,

where H_1, H_2 are as defined in the previous lemmas, then there exists $\delta > 0$ such that

$$4g^2(\beta)g(x) + g^3(A(x)) \geq 0, \qquad 0 \leq \alpha < \delta,$$
$$-k_{12}\alpha \leq x \leq -k_{11}\alpha ,$$

where $A(x)$ is the function implicitly defined by (2.2.4).

Proof: By Lemma 2.2.12 and (ii), there exists $\delta_1 > 0$ such that

$$G(-k_{11}\alpha) \geq G(k_{21}\alpha), \qquad 0 \leq \alpha < \delta_1.$$

Then

$$G(A(x)) = G(x) \geq G(-k_{11}\alpha) \geq G(k_{21}\alpha) \ ,$$

$0 \leq \alpha < \delta_1$, $-k_{12}\alpha \leq x \leq -k_{11}\alpha$. So

$$A(x) \geq k_{21}\alpha, \qquad 0 \leq \alpha < \delta_1, \qquad -k_{12}\alpha \leq x \leq -k_{11}\alpha \ .$$

Since

$$A(x) \to 0, \quad \text{as} \quad \alpha \to 0^+, \qquad x \to 0^-,$$

there exists $\delta_2 > 0$ $(\leq \delta_1)$ such that

$$A(x) < \frac{1}{3} \ , \qquad 0 \leq \alpha < \delta_2, \qquad -k_{12}\alpha \leq x \leq -k_{11}\alpha \ .$$

Since

$$g'(x) > 0, \qquad x < 0,$$

and by Lemma 2.2.11, there exists $\delta_3 > 0$ $(\leq \delta_2)$ such that

$$g'(x) > 0, \qquad 0 \leq \alpha < \delta_3, \qquad \alpha < x < \frac{1}{3} \ ,$$

we have

$$4g^2(\beta)g(x) + g^3(A(x)) \geq 4g^2(\beta)g(-k_{12}\alpha) + g^3(k_{21}\alpha)$$

$$0 \leq \alpha < \delta_3, \qquad -k_{12}\alpha \leq x \leq -k_{11}\alpha \ .$$

By Lemma 2.2.13 and (iii), the conclusion now follows.

Lemma 2.2.15. There exist $\delta > 0$, $x_0 > 0$ such that

$$\frac{R(A(x))}{g^2(A(x))} A'(x) \geq -\frac{1}{4}, \qquad 0 \leq \alpha < \delta, \qquad -x_0 < x < 0.$$

Proof: It suffices to show that there exist $\delta > 0$ and $x_0 > 0$ such that

$$f(x) = 4R(A(x))g(x) + g^3(A(x)) \geq 0, \quad 0 \leq \alpha < \delta, \quad -x_0 < x < 0.$$

From Lemma 2.2.11, there exists $\delta_1 > 0$ such that

$$\beta(\alpha) \leq 1.579\alpha < \frac{1}{3},$$

$$g''(x) > 0, \qquad 0 \leq \alpha < \delta_1, \qquad \alpha < x < \frac{1}{3},$$

and

$$R(A(x)) < 0, \qquad 0 \leq \alpha < \delta, \qquad 1.579\alpha \leq A(x) < \frac{1}{3}.$$

Therefore

$$f(x) \geq 0, \qquad 0 \leq \alpha < \delta_1, \qquad 1,579\alpha \leq A(x) < \frac{1}{3}. \tag{2.2.9}$$

On the other hand,

$$R'(x) = -2G(x)g''(x) < 0, \qquad 0 \leq \alpha < \delta_1, \qquad \beta(\alpha) < x < \frac{1}{3}.$$

Thus

$$R(x) \leq R(\beta(\alpha)) = g^2(\beta(\alpha)), \qquad 0 \leq \alpha < \delta_1, \qquad \beta(\alpha) < x < \frac{1}{3}.$$

Hence

$$f(x) \geq 4g^2(\beta(\alpha))g(x) + g^3(A(x)), \qquad 0 \leq \alpha < \delta_1, \qquad \beta(\alpha) < A(x) < \frac{1}{3}. \tag{2.2.10}$$

Let

$$0 = k_1(0) < k_1(1) = 0.33256 < k_1(2) = 0.38391$$
$$< k_1(3) = 0.41305 < k_1(4) = 0.43379 < k_1(5) = 0.45058$$
$$< k_1(6) = 0.46549 < k_1(7) = 0.47973 < k_1(8) = 0.49426$$
$$< k_1(9) = 0.51004 < k_1(10) = 0.52830 < k_1(11) = 0.55094$$
$$< k_1(12) = 0.58129 < k_1(13) = 0.62617 < k_1(14) = 0.70162$$
$$< k_1(15) = 0.77200 .$$

Correspondingly, let

$$1.333 = k_2(0) < k_2(1) = 1.35771 < k_2(2) = 1.37094$$
$$< k_2(3) = 1.38005 < k_2(4) = 1.38725 < k_2(5) = 1.39353$$
$$< k_2(6) = 1.39942 < k_2(7) = 1.40534 < k_2(8) = 1.41166$$
$$< k_2(9) = 1.41884 < k_2(10) = 1.42756 < k_2(11) = 1.43896$$
$$< k_2(12) = 1.45524 < k_2(13) = 1.48123 < k_2(14) = 1.52958$$
$$< k_2(15) = 1.57900 .$$

Computer results show that

$$H_1(k_1(I),k_2(I)) > 0, \qquad I = 0,1,2,\dots,14,15.$$
$$H_2(k_1(I+1),k_2(I)) > 0, \quad I = 0,1,2,\dots,14.$$

By Lemma 2.2.14, there exist $\delta(I) > 0$, $(I = 0,\dots,14)$, such that

$$4g^2(\beta(\alpha))g(x)^3 + g\,(A(x)) \geq 0, \text{ if } 0 \leq \alpha < \delta(I) \text{ and}$$
$$-k_1(I+1)\alpha \leq x \leq -k_1(I)\alpha, \qquad I = 0,1,\dots,14 .$$

Taking $\delta_2 = \min(\delta_1,\delta(1),\dots,\delta(14))$, we have

$$(2.2.11) \qquad 4g^2(\beta(\alpha))g(x) + g^3(A(x)) \geq 0, \text{ if } 0 \leq \alpha < \delta_2 ,$$
$$-0.772\alpha < x < 0 .$$

By Lemma 2.2.12, there exists $\delta_3 > 0$ ($\le \delta_2$) such that

$$G(A(-0.772\alpha)) = G(-0.772\alpha) \ge G(1.579\alpha), \quad 0 \le \alpha < \delta_3 ,$$

i.e.

$$A(-0.772\alpha) \ge 1.579\alpha, \quad 0 \le \alpha < \delta_3 ,$$

(2.2.11) then implies that

$$(2.2.12) \qquad 4g^2(\beta)g(x) + g^3(A(x)) \ge 0, \quad 0 \le \alpha < \delta_3 , \quad \beta(\alpha) \le A(x) < 1.579\alpha .$$

Take $\delta > 0$ ($\le \delta_3$) and $x_0 > 0$ such that

$$G(-x_0) \le G(\tfrac{1}{3}), \quad 0 \le \alpha < \delta .$$

The conclusion follows from (2.2.9), (2.2.10)) and (2.2.12).

Lemma 2.2.16. There exists $\delta > 0$ such that

$$p'(c) < 0, \quad 0 \le \alpha < \delta, \quad 0 < c < \delta .$$

Proof: From (2.2.5) and Lemma 2.2.15, there exist $\delta_1 > 0$ and $x_0 > 0$ such that

$$\frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))} > (-\frac{1}{3} + \frac{1}{4}) \cdot \frac{1}{g(x)} > 0 , \quad 0 \le \alpha \le \delta_1 , -x_0 < x < 0.$$

Take

$$\delta = \min(\delta_1, \min_{0 \le \alpha \le \delta_1} G(-x_0)) .$$

By Corollary 1.1.11, we have

$$p'(c) < 0, \quad 0 \leq \alpha < \delta, \quad 0 < c < \delta.$$

<u>Lemma 2.2.17</u>. There exists $\delta > 0$ such that

$$\frac{S(x)}{g^4(x)} > -4, \quad 0 \leq \alpha < \delta, \quad \frac{4}{3}\alpha < x < \delta ,$$

where $S(x)$ is as defined in Theorem 1.1.7.

<u>Proof</u>: It is equivalent to show that

$$\frac{1}{4} S(x) + g^4(x) > 0, \quad 0 \leq \alpha < \delta, \quad \frac{4}{3}\alpha < x < \delta .$$

A direct calculation shows that

$$\frac{1}{4} S(x) + g^4(x)$$
$$= \frac{1}{60^2} x^8[x^4(4300(\frac{\alpha}{x} - \frac{3}{4})^4 - 4900(\frac{\alpha}{x} - \frac{3}{4})^3 + 2212 \frac{1}{2}(\frac{\alpha}{x} - \frac{3}{4})^2$$
$$- 56 \frac{1}{4}(\frac{\alpha}{x} - \frac{3}{4}) + 10 \frac{35}{64}) + R(x,\alpha)] ,$$

where $R(x,\alpha)$ is a polynomial containing higher order terms only. The conclusion now follows.

<u>Lemma 2.2.18</u>. There exist $\delta > 0$, $\bar{c} > 0$ ($< 1/20$) such that

$$p'(c) < 0, \quad 0 \leq \alpha < \delta, \quad 0 < c \leq \bar{c}$$

$$p''(c) > 0, \quad 0 \leq \alpha < \delta, \quad \bar{c} < c < c_2(\alpha).$$

<u>Proof</u>.

(i) By Lemma 2.2.16, there exist $\delta_1 > 0$, $\bar{c} > 0$ ($< 1/20$) such that

$$p'(c) < 0, \qquad 0 \le \alpha < \delta_1, \qquad 0 < c \le \bar{c}.$$

(ii) $a(\bar{c},\alpha) \to a(\bar{c},0) < 0$, $c_2(\alpha) \to 1/20$, as $\alpha \to 0$. Therefore, there exist $a_1 < 0$, $\delta_2 > 0 (\le \delta_1)$ such that

$$a(\bar{c},\alpha) < a_1, \qquad c_2(\alpha) > \bar{c}, \qquad 0 \le \alpha \le \delta_2 .$$

It is not difficult to show that $S(x) > 0$, if $x < 0$. Then

$$\int_a^0 \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx > \int_{a_1}^{\frac{1}{2} a_1} \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx, \qquad 0 \le \alpha < \delta_2 ,$$

$$\bar{c} < c < c_2(\alpha) .$$

Since

$$\int_{a_1}^{\frac{1}{2} a_1} \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx$$

is continuous in α and c when $0 \le \alpha \le \delta_2$, $\bar{c} \le c \le c_2(\alpha)$, there exists $m > 0$, such that

$$\int_{a_1}^{\frac{1}{2} a_1} \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx \ge m > 0, \qquad 0 \le \alpha < \delta_2, \qquad \bar{c} < c < c_2(\alpha).$$

Therefore

$$\int_a^0 \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx > m > 0, \qquad 0 \le \alpha < \delta_2, \qquad \bar{c} < c < c_2(\alpha).$$

(iii) $b(\bar{c},\alpha) \to b(\bar{c},0) > 0$, as $\alpha \to 0$

$G(x) \to 0$, as $\alpha \to 0$, $x \to 0$.

Therefore there exist $\delta_3 > 0$ $(\le \delta_2)$, $b_1 > 0$ such that

$$b(\bar{c},\alpha) > b_1, \qquad G(x) < \frac{3}{4}\bar{c}, \qquad 0 \leq \alpha < \delta_3, \qquad 0 < x < \delta_3.$$

From Lemma 2.2.17, there exists $\delta_4 > 0$ $(\leq \delta_3)$ and

$$\gamma_0 > 0 \ (\leq \min(\delta_3, b_1, \tfrac{m}{8}\sqrt{\bar{c}}))$$

such that

$$\beta(\alpha) < \gamma_0 \qquad 0 \leq \alpha < \delta_4 ,$$

and

$$\frac{S(x)}{g^4(x)} > -4, \qquad 0 \leq \alpha < \delta_4, \qquad \beta(\alpha) < x < \gamma_0 .$$

Then

$$\int_\beta^{\gamma_0} \frac{S(x)}{\sqrt{\gamma}\ g^4(x)}\, dx > -4 \int_0^{\gamma_0} \frac{dx}{\sqrt{c-G(x)}}$$

$$> -4 \int_0^{\gamma_0} \frac{dx}{\sqrt{\bar{c} - \frac{3}{4}\bar{c}}} = -8 \frac{\gamma_0}{\sqrt{\bar{c}}} > -m$$

$$0 \leq \alpha < \delta_4, \qquad \bar{c} < c < c_2(\alpha) .$$

Therefore,

$$\left(\int_a^0 + \int_\beta^{\gamma_0} \right) \frac{S(x)}{\sqrt{\gamma}\ g^4(x)}\, dx > m - m = 0, \qquad 0 \leq \alpha < \delta_4,$$

$$\bar{c} < c < c_2(\alpha) .$$

(iv) $S(x)$ is continuous in α and x, and from Example 4 of §1.2, when $\alpha = 0$,

$$S(x) > 0, \qquad 0 < \gamma_0 \leq x \leq 1 .$$

Then there exists $\delta > 0$ $(\leq \delta_4)$ such that

$$S(x) > 0, \quad 0 \le \alpha < \delta, \quad \gamma_0 \le x \le 1 .$$

Therefore

$$2c^2 p''(c) = \left(\int_a^0 + \int_\beta^{\gamma_0} \right) \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx + \int_{\gamma_0}^b \frac{S(x)}{\sqrt{\gamma}\, g^4(x)}\, dx$$

$$+ 12c^2 \int_0^\beta \frac{dx}{\gamma^{5/2}}\, dx > 0, \quad 0 \le \alpha < \delta, \quad \bar{c} < c < c_2(\alpha).$$

__Corollary 2.2.19__. There exists $\delta > 0$ such that for any $\alpha \in [0,\delta)$, $p(c)$ has exactly one critical point.

__Proof__. The proof follows from Lemma 2.2.10 and Lemma 2.2.18.

__Lemma 2.2.20__. There exists $\delta > 0$ such that

(i) $g'(x) < 0, \quad g''(x) < 0, \quad g'''(x) < 0, \quad 1 - \delta < x < 1,$

(ii) $S(1 - 0.54\sqrt{0.6 - \alpha}) > 0,$

(iii) $G(-0.51(0.6 - \alpha)^{1/3}) \ge G(1 - 0.54\sqrt{0.6 - \alpha}),$

(iv) $1 - 0.54\sqrt{0.6 - \alpha} > \beta(\alpha) > 1 - 0.65\sqrt{0.6 - \alpha},$

provided $0.6 - \delta < \alpha < 0.6,$ where $S(x)$ is as defined in Theorem 1.1.3.

__Proof:__ (i) is trivial. (ii), (iii) and (iv) are proved by using the Taylor's series.

__Lemma 2.2.21__. There exists $\delta > 0$ such that

$$\frac{S(A(x))}{g^4(A(x))} A'(x) < \frac{2}{3}, \quad \text{if} \quad \beta(\alpha) < A(x) < 1,$$

$$0.6 - \delta < \alpha < 0.6,$$

where $A(x)$ is as defined by (2.2.4).

Proof: From Lemma 2.2.20(i) it is easy to show that there exists $\delta_1 > 0$ such that

$$S'(x) > 0, \quad \text{if } 0.6 - \delta_1 < \alpha < 0.6, \quad \beta(\alpha) < x < 1.$$

Therefore it suffices to show that there exists $\delta > 0$ such that

$$3g^4(\beta(\alpha))g(x) + 2g^5(A(x)) > 0, \quad 0.6 - \delta < \alpha < 0.6,$$
$$\beta(\alpha) < A(x) < 1 - 0.54\sqrt{0.6 - \alpha},$$

since $S(1 - 0.54\sqrt{0.6 - \alpha}) > 0$ when α is close to 0.6 by Lemma 2.2.20(ii).

Let

$$f(x,y) = 3g^4(\beta(\alpha))g(x) + 2g^5(y).$$

Then

$$\frac{\partial f}{\partial x}(x,y) = 3g^4(\beta(\alpha))g'(x) > 0, \quad \text{if } x < 0.$$

From Lemma 2.2.20(i), there exists $\delta_2 > 0$ $(\leq \delta_1)$ such that

$$\frac{\partial f}{\partial y}(x,y) = 10g^4(y)g'(y) < 0, \quad \beta(\alpha) < y < 1, \quad 0.6 - \delta_2 < \alpha < 0.6$$

Therefore, if $x_0 < x < 0$, $\beta(\alpha) < y < A_0 < 1$, then

$$f(x,y) > 3g^4(\beta(\alpha))g(x_0) + 2g^5(A_0), \qquad (2.2.13)$$
$$0.6 - \delta_2 < \alpha < 0.6.$$

Let $A_0(\alpha) = 1 - 0.54\sqrt{0.6 - \alpha}$. Take $\gamma(\alpha) < 0$ such that $G(\gamma(\alpha)) = G(A_0(\alpha))$. Then by Lemma 2.2.20(iii), there exists $\delta_3 > 0$

($\leq \delta_2$) such that

(2.2.14) $\quad \gamma(\alpha) \geq -0.51\ (0.6-\alpha)^{1/3}, \quad \text{if} \quad 0.6-\delta_3 < \alpha < 0.6\ .$

By (2.2.13), (2.2.14) and Lemma 2.2.20(i), (iv), there exists $\delta_4 > 0$ ($\leq \delta_3$) such that if $\beta(\alpha) < A(x) < 1 - 0.54\sqrt{0.6-\alpha}$, $0.6 - \delta_4 < \alpha < 0.6$, then

$$x > -0.51(0.6-\alpha)^{1/3}\ ,$$

and

$$\begin{aligned} 3g^4(\beta(\alpha))g(x) + 2g^5(A(x)) &= f(x,A(x)) \\ &> 3g^4(1-0.65\sqrt{0.6-\alpha})g(-0.51(0.6-\alpha)^{1/3}) \\ &\qquad + 2g^5(1-0.54\sqrt{0.6-\alpha}) \\ &= 2\cdot 0.216^5(0.6-\alpha)^{5/2} + o(0.6-\alpha)^{5/2}), \quad \text{as} \quad \alpha \to 0.6. \end{aligned}$$

Therefore there exists $\delta > 0$ ($\leq \delta_4$) such that

$$3g^4(\beta(\alpha))g(x) + 2g^5(A(x)) > 0, \quad \beta(\alpha) < A(x) < 1 - 0.54\sqrt{0.6-\alpha}$$
$$0.6 - \delta < \alpha < 0.6\ .$$

<u>Lemma 2.2.22</u>. There exists $\delta > 0$ such that

$$p''(c) > 0, \quad 0.6-\delta < \alpha < 0.6, \quad 0 < c < c_2(\alpha)\ .$$

<u>Proof</u>: Since

$$\frac{S(x)}{g^4(x)} \to \frac{7}{9} > \frac{2}{3}\ , \quad \text{as} \quad x \to 0^-, \quad \alpha \to 0.6^-\ ,$$

there exists $\delta_1 > 0$ such that

$$\frac{S(x)}{g^4(x)} > \frac{2}{3}, \qquad 0.6 - \delta_1 < \alpha < 0.6, \qquad -\delta_1 < x < 0 .$$

By Lemma 2.2.21, there exists $\delta_2 > 0$ ($\le \delta_1$) such that

$$\frac{S(A(x))}{g^4(A(x))} A'(x) < \frac{2}{3}, \qquad 0.6 - \delta_2 < \alpha < 0.6, \quad \beta(\alpha) < A(x) < 1.$$

Note $A(x) \to 1^-$ as $x \to 0^-$ and $\alpha \to 0.6^-$. There exists $\delta > 0$ ($\le \delta_2$) such that

$$\frac{S(x)}{g^4(x)} - \frac{S(A(x))}{g^4(A(x))} A'(x) > \frac{2}{3} - \frac{2}{3} = 0 ,$$

if $0.6 - \delta < \alpha < 0.6$, $-\delta < x < 0$. Then by Corollary 1.1.14 and note that $c_2(\alpha) \to 0$ as $\alpha \to 0.6$, the conclusion now follows.

Corollary 2.2.23. There exists $\delta > 0$ such that $p(c)$ has exactly one critical point if $0.6 - \delta < \alpha < 0.6$.

Proof. The proof follows from Lemmas 2.2.6, 2.2.10 and 2.2.22.

Theorem 2.2.24. There exists a uniform bound for the number of critical points of the period function $p(c)$ of equation (2.2.1) for $0 \le \alpha < 1$.

Proof: From Lemmas 2.2.1, 2.2.6, 2.2.10 and Corollaries 2.2.19 and 2.2.23, there exists $\delta > 0$ such that $p(c)$ has at most one critical point when $(\alpha,c) \in D_\delta \cup D_2$, where

$$D_\delta = \{(\alpha,c) \in D_1 \mid \mathrm{dist}((\alpha,c),\partial D_1) < \delta\} .$$

Theorem 2.1.3 shows that $p(c)$ is analytic in α and c when

$(\alpha,c) \in D_1 \setminus D_\delta$. We claim that there exists a uniform bound for the number of zeros of $p'(c)$ for any $\alpha \in [\delta, 0.6 - \delta]$ and c varying from δ to $c_2(\alpha) - \delta$.

Suppose not. Then there is a sequence $\{\alpha_n\} \subset [\delta, 0.6 - \delta]$ such that

$$\alpha_n \to \alpha_o, \quad N_n \to \infty, \quad \text{as } n \to +\infty, \tag{2.2.15}$$

where N_n denotes the number of zeros of $p'_{\alpha_n}(c)$ in $[\delta, c_2(\alpha_n) - \delta]$.

Since $p'_{\alpha_0}(c)$ is analytic in c, it has a finite number N (counting by multiplicity) of zeros in $[\delta, c_2(\alpha_0) - \delta]$. From the analyticity of $p'_\alpha(c)$ and the compactness of $[\delta, c_2(\alpha_0) - \delta]$, it is clear that there is a $\delta_1 > 0$ such that for any $\alpha \in (\alpha_0 - \delta_1, \alpha_0 + \delta_1)$, $p'_\alpha(c)$ has at most N distinct zeros when c varies from δ to $c_2(\alpha) - \delta$. This contradicts (2.2.15).

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