EMBEDDINGS IN SOME SINGULAR MANIFOLDS

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This is to certify that the

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ABSTRACT

EMBEDDINGS IN SOME SINGULAR MANIFOLDS

BY

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The singular 2-manifolds in which we will be embedding various 1-, and 2-dimensional spaces are 3-books, a space which is the one point compactification of an open 3-book, and finite 2-complexes. The question of the effects of the singularities on the classical topology of manifolds is investigated in a logical-historical manner for dimensions 2 and 3.

Beginning with A. Schoenflies it was found that Cantor sets were equivalently embedded in the plane, and hence they lie on an arc. It is shown in Chapter II that a Cantor set in a 3-book lies on an arc in the book.

Somewhat later R. L. Moore investigated uncountable mutually exclusive collections and found no such collections of triods in E^2. This type of question has been carried over to E^3 by G. S. Young and others. Chapter III deals with this question for an n-book where it is shown that a flat n-book does not contain an uncountable collection of mutually exclusive flat (n+1)-frames for $n \geq 2$, but it does contain an uncountable collection of mutually exclusive (flat) n-frames.

From the notion of connected im kleinen,[1] R. L. Moore started a study of uniform local connectedness (0-ULC) to characterize a curve in E^2. Ultimately a k-LC property was introduced. The 0-LC and 1-LC properties of the complement of a polygonal arc in a flat 3-book are studied in Chapter IV. The results that are obtained are

intermediate between those results already established in E^2 and E^3.

In Chapter V Eilenberg's thesis is used to study cut sets of the one point compactification of an open 3-book, $\hat{B}^3$. An example is given of a locally connected, G_δ set that cuts $\hat{B}^3$ between two points z' and z", but it contains no θ-curve that cuts $\hat{B}^3$, nor does it contain an irreducible cut of $\hat{B}^3$. Furthermore, with modifications this example becomes a locally connected irreducible cut of $\hat{B}^3$ between two points z' and z" that is not a θ-curve.

With Antoine the study of wild embeddings began, and one condition that has led in this field is cellularity. Homologically trivial 2-complexes and cellularity are studied in Chapter VI. It is shown that a homologically trivial finite 2-complex in a flat 3-book B^3 is cellular in E^3. An attempt is made to justify defining a cellular arc in B^3 as one that has a homologically trivial finite 2-complex neighborhood in the book contained in each neighborhood of the arc. An example is given of a wild arc that has such a neighborhood, and a proof that any wild arc that is bad at just one endpoint doesn't have such a neighborhood. It is also shown that any "cellular" arc in B^3 is cellular in E^3.

The work of C. Persinger and G. Atneosen allows a study of wild sets in 2-complexes. Chapter VII contains a characterization theorem of those finite 2-complexes in E^3 that contain a wild arc. Finally, the existence of knots in a 3-book naturally leads to questions about the Schoenflies Theorem in a 2-complex. A characterization theorem of those finite 2-complexes in which polygonal simple closed curves

bound a disk is proved in Chapter VIII.

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[1]For a definition of connected im kleinen see page 233 of R. L. Moore, "Concerning connectedness im kleinen and a related property," Fund. Math., Vol. 3 (1922) pp. 232-237.

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CHAPTER I

INTRODUCTION

The singular manifolds in which we will be embedding various 1-, and 2-dimensional spaces are 3-books, and more particularly flat 3-books, as well as a space which is the one point compactification of an open 3-book, and finite 2-complexes.

The embeddings will be primarily embeddings in a flat 3-book but since previous results are in a more general setting we need the following more general definition of an n-book. An <u>n-book</u> B^n is the union of n-closed disks in E^3 such that each pair of disks meets precisely on a single arc B on the boundary of each. The disks are called the <u>leaves</u> of B^n, and are denoted by D_i, $i = 1,2,\ldots,n$ and the arc is called its <u>back</u>.

This investigation of n-books was initiated by P. H. Doyle in [12] when he extended an earlier result [11], and showed that if each of the leaves of an n-book topologically embedded in E^3 is tame then the n-book is tame. C. A. Persinger continued the investigation of extrinsic properties of subsets of n-books in [28, 29, 30], and finally G. Atneosen in her thesis [2] investigated the embeddability of compacta in n-books from two different viewpoints: the intrinsic properties of n-books, and the extrinsic properties of n-books in E^3. In this paper in Chapters II, III, IV and VI we will investigate some embeddings in flat 3-books to answer questions of embeddability but more importantly to obtain some results in B^3 that are intermediate between the results already established in E^2 and E^3. In Chapter V we will investigate embeddings in the one point compactification of an open 3-book in an attempt to extend some of the results of Eilenberg's [16] to this space, and, finally in Chapters VII and VIII we will characterize those finite 2-complexes that contain wild arcs, and those finite 2-complexes for which any polygonal simple closed curve bounds a disk in the 2-complex.

The following comments on notation and definitions will be included for the reading of this paper. The notation will be given in a general setting although we will be using only the particular cases for n = 1,2, and 3 in this paper.

$$E^n = \{x | x = (x_1, x_2, \ldots, x_n) \text{ an n-tuple of real numbers}\}.$$

E^n is assumed to have the topology determined by the Euclidean metric d_n.

$$E^n_+ = \{x | x = (x_1, x_2, \ldots, x_n)\ x_n \geq 0\}.$$

$$S^{n-1} = \{x \in E^n | d_n(x,0) = 1 \text{ where } 0 = (0,0,\ldots,0)\}.$$

A homeomorphic image of S^1 is called a <u>simple closed curve</u>, and the homeomorphic image of $S^1 \cup [-1,1]$ is called a <u>θ-curve</u>.

$$\text{Closed n-cell} = \{x \in E^n | d_n(x,0) \leq 1\}.$$

$$\text{Open n-cell} = \{x \in E^n | d_n(x,0) < 1\}.$$

A <u>disk</u> is the homeomorphic image of a closed 2-cell, and an <u>arc</u> is the homeomorphic image of a closed 1-cell. If A and B are topological spaces, a homeomorphism of A into B is called an <u>embedding</u>.

By a <u>n-dimensional manifold</u> M is meant a separable metric space such that each point has a neighborhood whose closure is homeomorphic

to a closed n-cell. The <u>interior of M</u>, Int M, consists of those points which have neighborhoods homeomorphic to an open n-cell; the <u>boundary of M</u>, Bd M, is defined to be M - Int M. By the <u>interior of an n-book</u> is meant the set $\bigcup_{i=1}^{n}$ Int D_i ∪ Int B, which will be called an <u>open n-book</u>. The <u>interior of S^n</u>, Int S^n, will be the bounded complementary domain of S^n in E^{n+1} and the <u>exterior of S^n</u>, Ext S^n, will be the other component of $E^{n+1} - S^n$.

We also need some terminology from combinatorial topology. The definitions are essentially those of Zeeman [37].

By a <u>n-simplex</u> A, $0 \le n$, is meant the convex hull of n+1 linearly independent points (the vertices) $\{x_j | j = 0,1,\ldots,n\}$ in E^p, $n \le p$, where the <u>convex hull</u> of a set R is the intersection of all convex sets containing R. $\dot{A}$ will denote the boundary of A. By a <u>r-face</u> B of A, denoted by $B < A$, is meant the convex hull of r+1 distinct points of $\{x_j | j = 0,1,2,\ldots,n\}$. A <u>simplicial complex</u>, K of E^p, $p \ge 1$, is a finite collection of simplexes of E^p such that:

(1) if $A \in K$, then all the faces of A are in K,

(2) if $A, B \in K$, then $A \cap B$ is a common face of A and B.

The underlying point set of a simplicial complex K, denoted by $|K|$, is called a <u>Euclidean polyhedron</u> or <u>polyhedron</u>.

The phrase finite Euclidean polyhedron is used to emphasize the fact that we are only considering simplicial complexes consisting of finitely many simplexes. L is called a <u>subdivision</u> of K if $|L| = |K|$ and every simplex of L is contained in some simplex of K. L is a <u>subcomplex</u> of K if L is a simplicial complex and $L \subset K$. The subcomplex of K consisting of all q-simplexes of K, where $q \le m$, is called the <u>m-skeleton</u> of K, $K^{(m)}$. If P is a Euclidean polyhedron then a simplicial complex K such that $|K| = P$ is called a <u>triangulation</u> of P, and P is said to be the <u>carrier</u> of K. The <u>mesh</u> of a triangulation, K, is the supremum of the diameters of all the simplexes of K. The <u>dimension</u> of a simplicial complex K is the largest interger n such that K contains an n-simplex, and we call K an <u>n-complex</u>. The carrier of a 1-dimensional complex is called a <u>graph</u>.

If A and B are simplexes in E^p such that the union of their vertices forms a linearly independent set of points in E^p, then A and B are <u>joinable</u>, and the join of A and B, denoted by A*B, is defined to be the simplex spanned by the union of their vertices.

The <u>star</u> and <u>link</u> of a simplex A of a simplicial complex K are defined:

$$St(A,K) = \{B | A < B\}, \quad lk(A,K) = \{B | B*A \in K\}.$$

A subdivision L of K is said to be obtained from K by <u>starring A at a</u>, if $a \in \text{Int } A$ and $L = \{K - St(A,K)\} \cup \{a*\dot{A}*lk(A,K)\}$. A <u>first derived</u> subdivision of K is obtained by starring all the simplexes of K in some order such that if $\sigma_1 > \sigma_2$ then σ_1 precedes σ_2. An <u>r^{th} derived</u> subdivision of K is defined inductively as the first derived of an $(r-1)^{th}$ derived.

If $K \supset L$, we say there is an <u>elementary simplicial</u> collapse from K to L if K - L consists of a principal simplex A of K, i.e. A is not the proper face of any simplex in K, together with a free face. We say K <u>simplicially collapses</u> to L, written $K \overset{s}{\searrow} L$, if there is a sequence of elementary simplicial collapses from K to L. If L is a point we say that K is <u>collapsible</u>.

A topological polyhedron P in E^n is <u>tamely embedded</u> in E^n if there is a space homeomorphism that carries P onto a finite Euclidean polyhedron. Otherwise P is <u>wildly embedded</u>. A set X in E^n is to be <u>locally tame at a point p</u> of X if there is a neighborhood N of p and a homeomorphism h of $\bar{N}$ (the closure of N) onto a polyhedron in E^n such that $h(\bar{N} \cap X)$ is a finite Euclidean polyhedron. A set X is said to be <u>wild at a point p</u> if it is not locally tame at p. These definitions of tame and locally tame are due to Fox and Artin [18] and Bing [5], respectively. A set P in E^n is <u>locally polyhedral at a point x</u> of P if there is a neighborhood of x whose closure meets P in a finite Euclidean polyhedron.

The notions of wild and tame can also be applied to spaces that are not polyhedrons. By a <u>Cantor set</u> is meant any homeomorphic image of the classical Cantor ternary set, i.e., any compact, perfect, 0-dimensional, non-empty metric space is a Cantor set. A Cantor set $A \subset E^n$ is said to be tame if it lies on a tame arc in E^n; otherwise A is said to be wild.

Examples of wild arcs in E^3 were known as early as 1921 when Antoine [1] constructed a wild Cantor set in E^3, and thus an arc through this Cantor set is wild - the arc is called an Antoine's Necklace. In 1948 Fox and Artin [18] gave a number of examples of wild arcs and spheres in E^3 with one or two wild points.

In connection with n-books, C. A. Persinger proved in [30] the following two theorems.

<u>Theorem 1.1</u> No wild Cantor sets lie in a tame n-book in E^3.

<u>Theorem 1.2</u> There exists wild arcs and disks in tame n-books in E^3, $n > 2$.

A set C in E^n is said to be <u>cellular</u> if there exists a sequence of topological closed n-cells $\{C_i\}$ such that $C_{i+1} \subset \text{Int } C_i$ and $C = \bigcap_{i=1}^{\infty} C_i$. This notion was defined by M. Brown [8]. An arc A is <u>p-shrinkable</u> if A has an end point q and in each open set U containing q in E^n, there is a closed n-cell $V \subset U$ such that q lies in Int V with Bd V meeting A in exactly one point. This notion was defined by P. H. Doyle in [13] where he shows that all arcs that are bad at just one end point are cellular.

A k-cell in E^n, $k \leq n$, is said to be <u>flatly embedded</u> or <u>flat</u>, if there is a space homeomorphism of E^n onto itself mapping it onto a k-simplex. A <u>flat n-book</u> is a n-book such that each of its leaves is a Euclidean 2-simplex.

A topological space is <u>separated</u> if it is the union of two disjoint, non-empty open sets. Let S be a connected space then a closed subset C <u>separates</u> S if S - C is a separated set. A point p of S is called a <u>cut point</u> of S if S - p is separated. These definitions are in Hocking and Young [21].

Finally, in Chapter IV, we need the following definitions from homotopy theory which are in Dugundji [15]. If X and Y are two topological spaces and I is the unit interval, then two maps $f,g:X \to Y$ are called <u>homotopic</u> (written $f \simeq g$) if there exists a continuous function $H:X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$. An $f:X \to Y$ homotopic to a constant map is called <u>nullhomotopic</u>, written $f \simeq 0$. A space Y is <u>contractible</u> if the identity map on Y is nullhomotopic. The image of H is called a <u>homotopy path</u>.

One theorem that should be mentioned because we will continually refer to it is due to Schoenflies [32]. We will refer to it as the plane Schoenflies Theorem.

Theorem 1.3 If J is a simple closed curve in E^2, and h is a homeomorphism of J onto the unit circle S^1 in E^2, then h can be extended to a homeomorphism of E^2 onto itself.

CHAPTER II

CANTOR SETS IN B^3 LIE ON ARCS

One of the original proofs that was given to show that a Cantor set in the plane lies on an (tame) arc was by Antoine [1]. However, the proof of the generalization of that result to a flat 3-book follows the outline of a proof by A. S. Besicovitch [4].

We need the following two lemmas in order to prove the theorem. The first is a theorem[1] of Dugundji [15], and the second is a generalization of Lemma 1 of A. S. Besicovitch [4] to a 3-book, where the proof follows the proof of Lemma 1 found there.

<u>Lemma 2.1</u> If A is a compact subset, and B is a closed subset of a metric space X, and if $A \cap B \neq \emptyset$ then $d(A,B) = \varepsilon > 0$.

<u>Lemma 2.2</u> Given a perfect set F in a 3-book, all of whose components are points and a positive number d, there exists a finite set of disjoint closed disks and "closed" 3-books of diameter less than d containing F in their interior and each of them contain a perfect subset of F.

<u>Proof</u>: As in the proof of Lemma 1 of [4], if we let $\{F \cap B, \delta\}$ be the set of points at most δ distance from $F \cap B$ in the back of the 3-book, B, then there exists a $\delta' > 0$ such that all the components of

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[1]Theorem 4.4, Chapter XI, page 234.

$\{F \cap B, \delta'\}$ are of diameter less than d. Clearly these components are intervals of B.

Let I be one of the intervals of B and construct a closed "spherical" neighborhood U of the midpoint of I of diameter the length of I in the book. The Bd U might contain points of F off the back of the book but inside U there exists a "closed" 3-book neighborhood whose boundary misses F and intersects B in I, since the perfect set F is a Cantor set and Cantor sets in the plane are tame. Do this for each interval of $\{F \cap B, \delta'\}$.

Finally, by Lemma 1 of [4], in the interior of each leaf of the book the remainder of F can be covered by open disks whose closures are disjoint, of diameter less than d, and disjoint from the "closed" 3-books constructed above.

Clearly each of these closed disks and "closed" 3-books contain a perfect subset of F since their boundaries miss F.

Theorem 2.3 Let C be a Cantor set in the interior of a 3-book B^3 then there exists an arc in B^3 such that C is contained in the arc.

Proof: Let $\varepsilon_1 = d(C, \text{Bd } B^3)$ then $\varepsilon_1 > 0$ by Lemma 2.1.

By Lemma 2.2, there exists a finite collection of disjoint closed disks and "closed" 3-books of diameter less than $\min\{\varepsilon_1, 1\}$ containing C in their interior and each of them contains a Cantor set. Denote this collection by $\{D_{1i}\}$ $(1 \le i \le n_1)$. Now let $x_{1i}, x'_{1i} \in \text{Bd } D_{1i}$ $x_{1i} \neq x'_{1i}$ and connect x'_{1i} to x_{1i+1} $(1 \le i \le n_1-1)$ by an arc in Int $B^3 - \bigcup_{i=1}^{n_1} D_{1i}$. This can be done because the D_{1i}'s which are closed, are contained in the interior of B^3. For convenience let $a = x_1$ and $b = x'_{n_1}$.

Now let $\varepsilon_2 = \min\{d(C \cap D_{1i}, \text{Bd } D_{1i})\}$, then $\varepsilon_2 > 0$ since $d(C \cap D_{1i}, \text{Bd } D_{1i}) > 0$ by Lemma 2.1 for each i and there are only a finite number of i's.

Since each D_{1i} contains a Cantor set of points of C, by applying Lemma 2.2 to each D_{1i}, there exists a finite collection of disjoint closed disks and "closed" 3-books of diameter less than $\min\{\varepsilon_2, \frac{1}{2}\}$ containing C in their interior and each of them contains a Cantor set. Denote this collection by $\{D_{2j}\}$ $(1 \le j \le n_2)$, where $D_{2j} \subset D_{1i}$ $(n_i' \le j < n_{i+1}')$ with $n_1' = 1$ and $n_{n_1}' = n_2$. Now let $x_{2j}, x_{2j}' \in \text{Bd } D_{2j}$, $x_{2j} \neq x_{2j}'$ and connect x_{2j}' to x_{2j+1} if $n_i' \le j < n_{i+1}'-1$, x_{1i} to $x_{2n_i'}$ and $x_{2n_{i+1}'-1}'$ to x_{1i}' by arcs in $D_{1i} - \bigcup_{j=n_i'}^{n_{i+1}'-1} D_{2j}$. This can be done as above.

If we continue in this manner we want to show that the limit of this process, call it K, is an arc which clearly contains C.

By a lemma[2] of [21], K is a continuum containing a and b so by a theorem[3] of [21], we can show that K is an arc if we can show K has only two non-cut points, namely, a and b.

To this end let $x \in K - \{a,b\}$. If at some stage of the construction of K, x is a point on an arc that connects a D_{ni} to D_{ni+1} for some n then it is clear that x is a cut point of K. Hence we can suppose that $x \in D_{ni_n}$ for all n. Now let P_n by the union of all D_{ni} with $i < i_n$ and all arcs up to and including the arc connecting D_{ni_n-1} to D_{ni_n} except the point x_{ni_n}. Similarly let F_n be the union of all D_{ni} with $i > i_n$ and all arcs starting with the arc from D_{ni_n} to D_{ni_n+1}

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[2] Lemma 2-8, page 43.

[3] Theorem 2-27, page 54.

except the point x'_{ni_n}. Now let $P = \bigcup_{n=1}^{\infty} P_n \cap K$ and $F = \bigcup_{n=1}^{\infty} F_n \cap K$. Then P and F are disjoint relatively open non-empty subsets of K, and each point of K - x lies in one or the other. Therefore x is a cut point of K, and hence K is an arc from a to b.

CHAPTER III

UNCOUNTABLE MUTUALLY EXCLUSIVE COLLECTIONS IN B^n

In this chapter will will use a result of R. L. Moore [27] to point out some differences between a closed disk, the Euclidean plane, and flat n-books for different n's. In particular we will show that a flat n-book does not contain an uncountable collection of mutually exclusive flat (n+1)-frames for $n \geq 2$, but does contain an uncountable collection of n-frames.

An _n-frame_ is the homeomorphic image of n-line segments which are joined at a common end point. The homeomorphic image of each line segment is called a _branch_ of the n-frame, and the vertex of index n will be called the _branch point_. If in addition the branches of an n-frame are line segments we will say that the n-frame is _flat_. In the particular case when n = 3 we will call the 3-frame a _triod_, and if the branches of the triod are line segments we will call it a _flat triod_.

Because we will be referring to the result of R. L. Moore [27] we will state it now for easy reference.

Lemma 3.1 There does not exist an uncountable set of mutually exclusive triods in the Euclidean plane.

Using this lemma we can prove a result, which points out some differences between a closed disk and the Euclidean plane, and that will help shorten some of the proofs that follow.

Lemma 3.2 A closed disk does not contain an uncountable collection of mutually exclusive arcs that intersect the boundary of the disk in an interior point of the arc.

Proof: Suppose we have such an uncountable collection of arcs. Since there exists a homeomorphism of the closed disk onto the unit disk in E^2 we can construct an uncountable collection of mutually exclusive triods in the Euclidean plane in the following manner. Construct a line segment normal to the unit circle and in the complement of the unit disk in E^2 from each point of intersection of one of these arcs with the boundary of the disk. But this contradicts Lemma 3.1.

However, there exists an uncountable collection of mutually exclusive triods in a 3-book, as we can clearly see by the following construction.

Let B_o^3 be the open 3-book which is the union of the xy-plane and one-half of the yz-plane, and let B_1^3 be the 3-book contained in B_o^3 that has square leaves that intersect in the interval $[-1,1]$ on the y-axis. We can obtain uncountably many mutually exclusive triods in B_1^3 by joining three line segments perpendicular to the back of the book, one in each leaf of B_1^3, at each irrational number in $[-1,1]$. Since any 3-book is a homeomorphic image of B_1^3, it contains uncountably many mutually exclusive triods.

Furthermore this is essentially the only way we can get uncountably many mutually exclusive flat triods in a flat 3-book.

<u>Lemma 3.3</u> If there exists uncountably many mutually exclusive flat triods in a flat 3-book, then there exists an uncountable subcollection of these triods with the branch point on the back of the book and with one line segment in each leaf of the book. Moreover, all the branches of the triod have length at least ε, for some $\varepsilon > 0$.

<u>Proof</u>: The branch point has to lie on the back of the book for uncountably many of the triods for otherwise one of the leaves of the book would contain uncountably many mutually exclusive triods, which would contradict Lemma 3.1.

There exists an $\varepsilon > 0$ such that there are uncountably many of these triods with all their line segments of length greater than or equal to ε, for if not there then would only be a countable number of these triods with all their line segments of length at least $\frac{1}{n}$ for each n. But since the countable union of countable sets is countable we have a contradiction of the fact that there are uncountably many triods.

In this uncountable subcollection of the triods let us suppose that there are only a countable number of these triods with one line segment in each leaf of the book. Now since each line segment of these triods has length at least ε, there would exist at most a finite number of these triods with one or two line segments in the back of the book. Thus there would exist an uncountable subcollection of these triods with one leaf of the book containing at least two line segments of the triods. But this would contradict Lemma 3.2.

We can also use a similar argument to show that there exist uncountably many mutually exclusive θ-curves in a 3-book. To obtain uncountably many mutually exclusive θ-curves in B_1^3 just intersect the spheres centered at the origin with radius an irrational number between 0 and 1 with B_1^3. Again since each 3-book is a homeomorphic image of B_1^3, each contains an uncountable collection of mutually exclusive θ-curves.

In general we can embed uncountably many mutually exclusive polygonal knots and links in a flat book.

To construct this uncountable collection, we will first embed an arbitrary polygonal knot in a flat 3-book by the method of C. A. Persinger [30]. At one of the points where the knot crosses the back of the book we will alter the knot as in Figure 3.1, by replacing the line segment(s) of the knot by the three dotted line segments. In the third leaf construct a polygonal arc connecting any two of the three points on the back of the book, and meeting the new knot and the back of the book in only these two points.

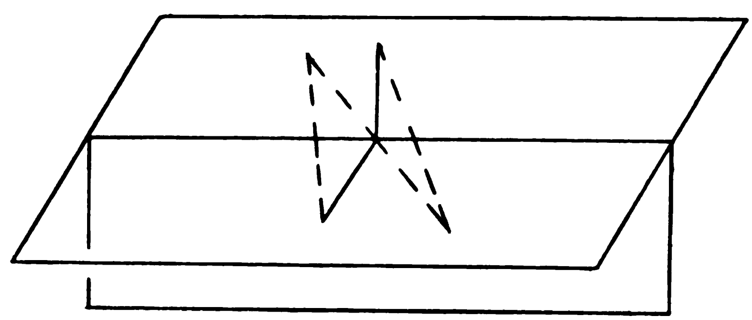

Figure 3.1

Now construct a "tubular" neighborhood of this polygonal knot and link in the book, and let d_o be the distance from the boundary of the neighborhood to the knot and link.

Let r_o be an irrational number between 0 and d_o, and construct a set of line segments in the book parallel to the line segments of the

knot and link and at a distance r_o from them. By shortening or lengthening these line segments, if necessary, we obtain two polygonal knots and links in the "tubular" neighborhood of the original knot and link. If we do this for each irrational number between 0 and d_o we have an uncountable collection of mutually exclusive polygonal knots and links in the book.

Remark: It is clear that the two points of index 3 can lie off the back of the book for only countable many of any uncountable collection of θ-curves in a 3-book.

We need the following definitions that are found in Whyburn's book Analytic Topology, to complete the proof of the next theorem.

Definition 3.1 A point P is said to be a condensation point of a set M provided every neighborhood of P contains uncountably many points of M.

Definition 3.2 A collection G of disjoint subsets of M will be called non-separated provided that no element of G separates in M two points belonging to any other single element of G.

Definition 3.3 A non-separating collection [G] of subsets of M will be said to be saturated provided that if $G \in [G]$ and P is any point of M - G, there exists at least one element G' of [G] which separates P and G in M.

Theorem 3.4 A flat 3-book doesn't contain an uncountable collection of disjoint 4-frames if the branches of the 4-frames are polygonal or at most one of the branches is an arbitrary arc.

Proof: Suppose we have an uncountable collection of mutually exclusive 4-frames in a flat 3-book. Let us consider the 4-frames consisting of the first line segment in each of the three polygonal branches of the 4-frame starting from the branch point and the arbitrary arc. Now by Lemma 3.3 there exists an uncountable subcollection of these 4-frames with the branch point on the back of the book and with one line segment in each leaf of the book. Moreover all the branches of the flat triod contained in the 4-frame have length at least ε, for some $\varepsilon > 0$. Furthermore, there exists an uncountable subcollection of these 4-frames with the distance from the branch point to the other end point of the arbitrary arc at least $\varepsilon' > 0$, where ε' exists for the same reason that ε exists in Lemma 3.3.

Since there exists an uncountable number of branch points in the back of the book, there exists a branch point, call it v_o, that is a point of condensation of the set of all branch points. Let U be the ε-neighborhood of v_o in the book. Now the collection of all flat triods that consist of the three line segments of the 4-frame which intersect U, is an uncountable non-separated collection, and each member separates U. By a theorem[1] of [35], this collection contains an uncountable saturated subcollection. But if we take an element of the saturated subcollection there must be an arc joining a point not on it to the branch point (since this element is a 4-frame and $\varepsilon' > 0$) which is a contradiction. Hence we have the desired conclusion.

Remark: The above proof cannot be generalized to an uncountable collection of 4-frames because a triod with the branch point on the

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[1] Theorem 2.2, Chapter 3, page 45.

back of a 3-book does not necessarily locally separate the 3-book. Thus the question, is there an uncountable number of 4-frames in a 3-book, is still an open question.

This theorem, and Lemmas 3.1 and 3.2 lead us to the following theorem which points out differences between n-books for different n.

Theorem 3.5 A flat n-book does not contain an uncountable collection of mutually exclusive flat (n+1)-frames for $n \geq 2$, but does contain an uncountable collection of n-frames.

Proof: If $n = 2$, we have the desired conclusion by Lemma 3.1.

If $n = 3$, the previous theorem gives us the desired conclusion.

Let $n > 3$, and suppose we have an uncountable collection of mutually exclusive flat (n+1)-frames. As before there exists an $\varepsilon > 0$ such that there is an uncountable subcollection of these flat (n+1)-frames with the length of each branch at least ε, and with the branch point on the back of the book. Now there are clearly only a finite number of these (n+1)-frames with one of the branches along the back of the book, so let us assume that all the branches have only one point in common with the back of the book. Then there must exist an uncountable subcollection with at least two branches in one leaf of B^n, which contradicts Lemma 3.2. Hence we have the desired conclusion.

We can clearly construct an uncountable collection of (flat) n-frames in B^n as we did in the case $n = 3$.

CHAPTER IV

THE 1-LC PROPERTY OF ARCS IN B^3

In this chapter we will investigate the effects that the singularities of a flat 3-book have on the 1-LC property of the complement of a polygonal arc in a flat 3-book. In this regard we will prove some intermediate results between the 1-LC property of the complement of a polygonal arc in E^2 and the complement of a polygonal arc in E^3.

The k-LC property was first defined by S. Eilenberg and R. L. Wilder [17], and is included here for completeness.

Definition 4.1 Let Y be a set in a metric space M. If $f:X \to Y$ is a map, let $B(f)$ be the infimum of the diameters of all homotopy paths in Y shrinking f to a constant. If f is not nullhomotopic $B(f) = \infty$. Y is said to be k-LC at a point x in M if given any sequence $\{f_n\}$ of maps of the k-sphere S^k into Y, $f_n(S^k) \to x$, then $B(f_n) \to 0$.

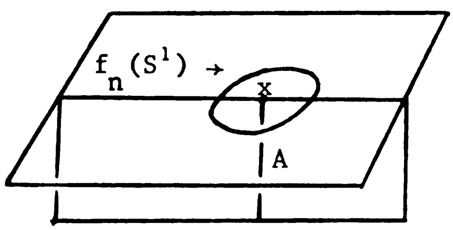
The following definition is a modification of a definition due to E. E. Moise [26], and will be used to facilitate the statement and proof of the following theorem.

Definition 4.2 Let B be the back of a flat 3-book B^3, and let A be an arc in Int B^3 such that $B \cap A$ is a point p. If for each sufficiently small open neighborhood U of p in B^3, U - B is the union of three disjoint open sets two of which intersects the component of $U \cap A$ that contains p, then A *pierces* B at p. If the component of $U \cap A$ that

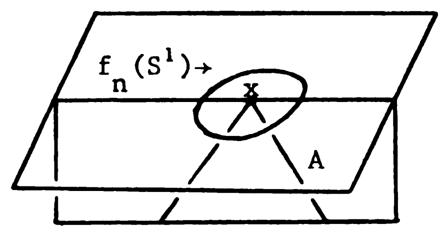
contains p intersects only one of the three disjoint open sets of $U - B$, then A will be said to be tangent to B at p.

Theorem 4.1 Let A be a polygonal arc in the interior of a flat 3-book B^3. $B^3 - A$ is not 1-LC at $x \in A$ if and only if there exists a subarc of A that is tangent to B at x.

Proof: If there exists a subarc of A that is tangent to B at x, then Figures 4.1 a and b indicate that if x is an end point or an interior point of A that there exists a sequence of functions $f_n: S^1 \to B^3 - A$ that are not nullhomotopic in $B^3 - A$. Hence $B^3 - A$ is not 1-LC at these points.



(a)



(b)

Figure 4.1

Conversely, if $x \in A$ is not a point where any subarc of A is tangent to B, then x belongs to one of the following three cases:

Case A: $x \in A$ and $x \notin B$. Since x is in one of the leaves of B^3, for $n \geq N$, for some N, $f_n(S^1)$ will be in the complement of A in one leaf of B^3, and thus $B^3 - A$ is clearly 1-LC at x.

Case B: x is a point of an interval of $A \cap B$. If the diameter of $f_n(S^1)$ is less than the length of the interval in $A \cap B$, then we can clearly see that $f_n: S^1 \to B^3 - A$ is nullhomotopic. See Figure 4.2 a. Hence $B^3 - A$ is clearly 1-LC at x.

Case C: There exists a subarc of A that pierces B at x. Then locally $B^3 - A$ looks like Figure 4.2 b, since A is polygonal.

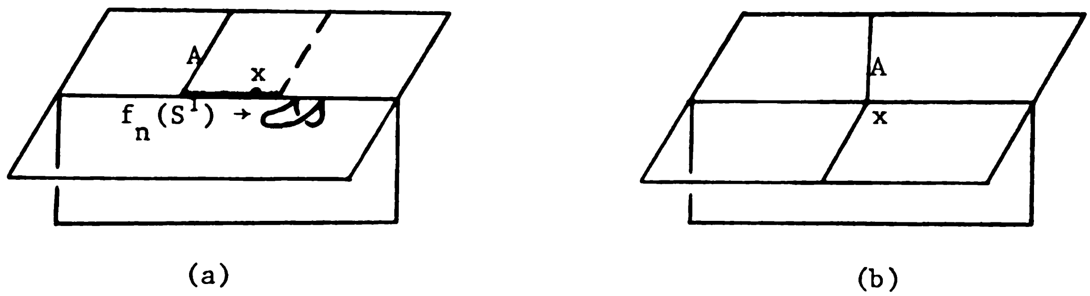


Figure 4.2

It is clear that by collapsing the two leaves that contain a portion of A first and then the other leaf that this neighborhood of x is contractible. Hence $B^3 - A$ is 1-LC at x.

Now continuing with the discussion of differences between the plane, a flat 3-book, and 3-space, the previous theorem provides a little additional information. It is clear that the complement of a polygonal arc in the plane is 1-LC at every point on the arc. In the interior of a flat 3-book, B^3, the complement of a polygonal arc in B^3 is not 1-LC only at points where some subarc of the arc is tangent to the back of the book, and is 1-LC at all other points of A. Finally, in 3-space the complement of a polygonal arc is not 1-LC at every interior point of the arc, and is 1-LC only at the end points of the arc.

Furthermore, if we allow the arc A to be wild and locally polygonal except at one end point then D. R. McMillian [25][1] has shown that $E^3 - A$ is not 1-LC at the bad point. A similar result can be proved for the complement of such an arc in a flat 3-book.

[1] Lemma 4, page 524.

Theorem 4.2 Let A be a wild arc in the interior of a flat 3-book B^3. If A is locally polygonal except at one end point of A, then $B^3 - A$ is not 1-LC at this end point.

Proof: Let P be the bad end point of A, then P is clearly in the back of the book, and let U_n be a neighborhood of P in B^3 of diameter $\frac{1}{n}$ starting with n sufficiently large so that $U_n \subset B^3$. In one leaf of B^3 we can obtain a sequence of disjoint arcs, $\{A_n\}$, that converge to P by picking A_n in U_n. This sequence exists for if there were only a finite number of these arcs near P in this leaf then the arc would be locally tame at P since locally it would lie in a plane.

Case A: There exists an infinite subset of these arcs, call them $\{A_{n'}\}$ where at one end point, $a_{n'}$, of these arcs in the back of the book a subarc of A is tangent to B at $a_{n'}$ or at $a_{n'}$ the arc A intersects the back of the book in an interval and then A goes back into the same leaf of B^3. Now since A is locally polygonal at $a_{n'}$ we can construct a simple closed curve about $a_{n'}$ in $(B^3 - A) \cap U_{n'}$ that isn't contractible in $B^3 - A$. See Figure 4.3. Clearly these simple closed curves converge to P, and hence the arc is not 1-LC at P.

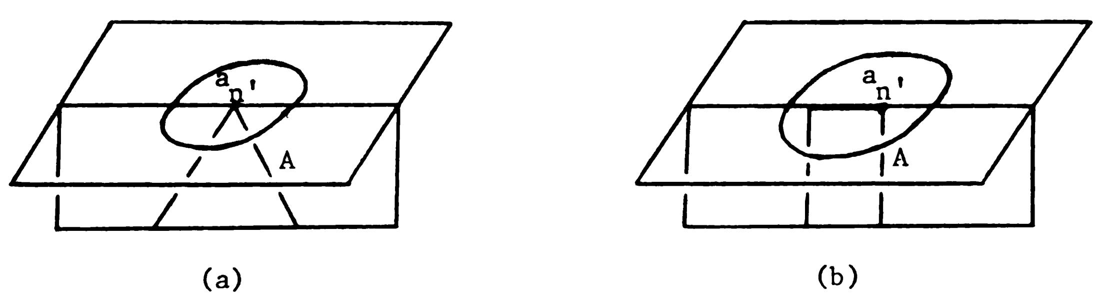


Figure 4.3

Case B: There doesn't exist an infinite subset of $\{A_n\}$ as in Case A. Then there exists an infinite subset of $\{A_n\}$ call them $\{A_{n''}\}$

where the arc pierces the back of the book at the end point(s) in the back of the book or intersects the back of the book in an interval and then goes into one of the other two leaves of B^3. Now as in Figure 4.4 we can construct a simple closed curve in $(B^3 - A) \cap U_{n''}$, since A is locally polygonal at the end points of $A_{n''}$, that isn't contractible in $B^3 - A$. These simple closed curves clearly converse to P, and hence $B^3 - A$ is not 1-LC at P.

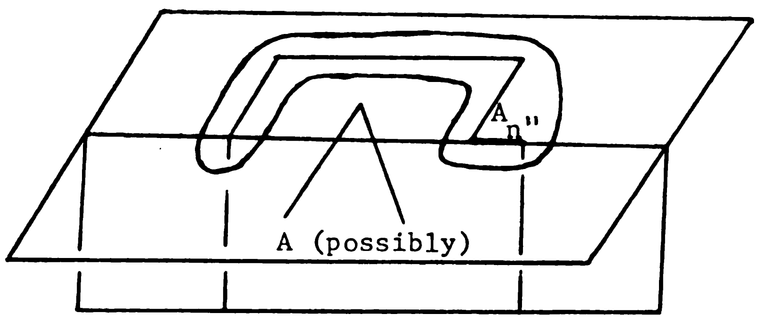


Figure 4.4

Note: P could lie on the interval in the back of the book between the end points of $A_{n''}$ but that won't make any difference as indicated in Figure 4.4 by staying close to $A_{n''}$ with the simple closed curve.

Remark: Tangent points and other non-piercing points, can always be removed without changing the E^3-embedding of an arc. Thus for an arc that is locally polygonal except at an end point P there is an embedding in which all points on the back except P are piercing points, and the arc has the same E^3-embedding.

This study could just as well have been made with the 0-LC property of the complement of the arc rather than the 1-LC property. The only reason for doing the latter is that it was what was initially looked at.

For the case of the complement of a polygonal arc A in the plane it is clear that it is 0-LC only at the end points, and $E^3 - A$ is clearly 0-LC at all points of A. The result for a flat 3-book would be the following.

Theorem 4.3 Let A be a polygonal arc in the interior of a flat 3-book B^3. $B^3 - A$ is 0-LC at $x \in A$ if and only if x satisfies one of the following two cases:

(1) x is an end point of A, or

(2) there exists a subarc of A that pierces B at x.

Proof: If x is an end point of A then clearly $B^3 - A$ is 0-LC at x, so let us suppose there exists a subarc of A that pierces B at x. By looking at Figure 4.5 it is clear that the arc connecting the two points of $f_n(S^0)$ is a homotopy path which can be made arbitrarily small for n sufficiently large.

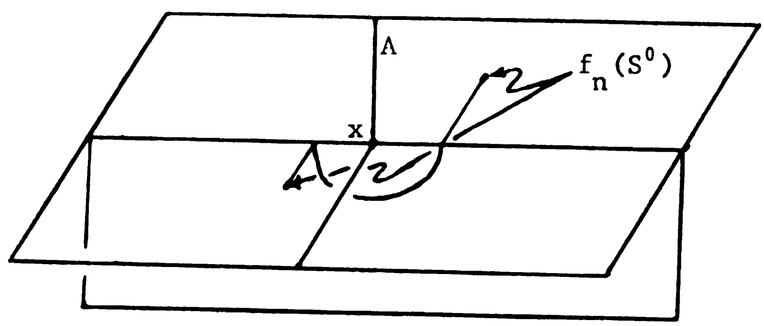


Figure 4.5

Conversely, there are three cases to consider:

Case A: $x \in \text{Int } A$ and $x \notin B$. Then locally the arc lies in a plane, and hence $B^3 - A$ is not 0-LC at x.

Case B: $x \in \text{Int } A$ and there exists a subarc of A that is tangent to B at x. By mapping S^0 as in Figure 4.6 a it is clear that the map is not nullhomotopic in this neighborhood, and there exists a sequence of these maps $\{f_n\}$ such that $f_n(S^0) \to x$. Hence $B^3 - A$ is not 0-LC at x.

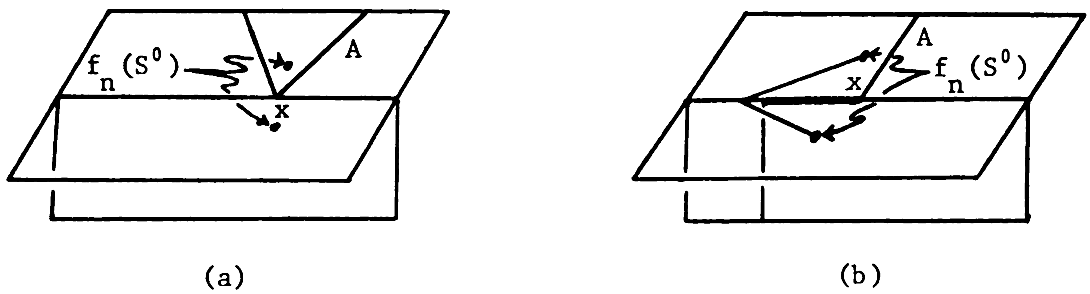


Figure 4.6

Case C: $x \in \text{Int } A$ and x is a point of an interval of $A \cap B$. If A is locally as in Figure 4.6 b by mapping S^0 as in the figure, it is clear that the map is nullhomotopic, but the length of the homotopy path is at least twice the distance from x to the farthest end point of this interval of $A \cap B$. Furthermore there exists a sequence of these maps $\{f_n\}$ such that $f_n(S^0) \to x$. Hence $B^3 - A$ is not 0-LC at x.

If A is locally as in Figure 4.2 a then there exists a sequence of maps of S^0 that are not nullhomotopic in this neighborhood as in Case B.

CHAPTER V

CUTS OF A 3-BOOK

During the translation of Eilenberg's thesis [16], it became apparent that some of the theorems on unicohérence could be extended to the one point compactification of an open 3-book $\hat{B}^3$, but that there are counterexamples for many of the generalizations of the theorems on cuts of S^2 to cuts of $\hat{B}^3$.

A connected space X is called <u>unicohérent</u>, when for any decomposition of X into two closed and connected sets X_1 and X_2, the set $X_1 \cap X_2$ is connected. A set Y contained in a space S <u>cuts</u> S between two points $s_1, s_2 \in S - Y$ if there doesn't exist a continuum $K \subset S - Y$ such that $s_1, s_2 \in K$.

<u>Note</u>: This definition of a cut due to Eilenberg is slightly less restrictive than the definition of a cut point and of cuttings found in Whyburn [35], and slightly more restrictive than the definition of cuttings found in Hocking & Young [21].

Let us define $\hat{B}^3$, and a restriction of a θ-curve in $\hat{B}^3$, which we will call a θ'-curve.

<u>Definition 5.1</u> Let C be a great circle on S^2, and let D_1 be the closed disk contained in the plane of C with boundary C. Then we will denote $S^2 \cup D_1$ by $\hat{B}^3$, and furthermore we will denote the closure of the two components of $\hat{B}^3 - D_1$ by D_2 and D_3.

Note: The D_i's were also used in the definition of a 3-book, but no confusion should arise if we use the same notation here.

Definition 5.2 $Z \subset \hat{B}^3$ is called a θ'-curve if Z is a θ-curve, $Z \cap D_i = Z_i$ is an arc, and $Z_i \cap C = \{a,b\}$ $a \neq b$ for each $i = 1,2,3$. We will denote the θ'-curve that is obtained by intersecting the plane that is perpendicular to the plane of C with $\hat{B}^3$ by θ'_o, and call it the standard θ'-curve.

Theorem 5.1 $\hat{B}^3$ is unicohérent.

Proof: By a theorem[1] of Eilenberg's [16] both S^2, and D_1 are unicohérent. Hence $\hat{B}^3 = S^2 \cup D_1$ is unicohérent, since $S^2 \cap D_1 = C$ is connected, and both S^2 and D_1 are closed and locally connected[2].

Remark: This is a generalization of Corollary I.6 of [16] to $\hat{B}^3$, which states in part that S^{n+1} is unicohérent for $n = 1,2,...$

The following theorem is a generalization of the plane Schoenflies Theorem to $\hat{B}^3$.

Theorem 5.2 If $Z \subset \hat{B}^3$ is a θ'-curve, then the homeomorphism $h:Z \to \theta'_o$ can be extended to a homeomorphism of $\hat{B}^3$ onto itself.

Proof: Clearly h exists, so suppose that a and b lie on a common diameter - if not we can easily find a homeomorphism $f \circlearrowleft \hat{B}^3$ such that $f(a)$ and $f(b)$ are on a common diameter. To that end let $f \circlearrowleft C$ such

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[1] Corollary I.6, page 73 of [16].

[2] Theorem I.4, page 72 of [16].

that f(a) and f(b) are on a common diameter, and extend f to D by mapping the radius $\overline{Oc}$ linearly onto $\overline{Of(c)}$ where O is the center of D and $c \in C$. Extend f to all of $\hat{B}^3$ by mapping each circle on S^2 that is obtained by intersecting S^2 with a plane parallel to D, as f maps C (map the "north" and "south" pole onto themselves).

Now, since Z is a θ'-curve, by extending $h|Z \cap D_1$ by the identity on C, we have a homeomorphism $h':(Z \cap D_1) \cup C \rightarrow (\theta'_o \cap D_1) \cup C$, where θ'_o is the standard θ'-curve that contains the points a and b. h' can be extended to $h'' \subsetneq D_1$ by a generalization of the plane Schoenflies Theorem for a θ-curve in the plane.[3]

Since we can do the same thing on D_2 and D_3 and since the three homeomorphisms agree on C we have an extension of h to $\hat{B}^3$.

Corollary 5.3 Each θ'-curve in $\hat{B}^3$ separates $\hat{B}^3$ into two open 3-books and it is their common boundary.

Remark: The following three examples show that the restrictions on a θ'-curve are necessary for the above theorem and corollary.

Example 5.4 $Z \cap C = \{a,b,x_o\}$, Figure 5.1 a. The complement of Z on the right is not an open 3-book. It is also clear that the homeomorphism of the theorem can't be extended to $\hat{B}^3$ since $h(x_o)$ would be an in interior point of D.

Example 5.5 $Z \cap C = \{a,b,x_o\}$, Figure 5.1 b. Z doesn't cut $\hat{B}^3$. It is also clear that the homeomorphism of the theorem can't be extended to $\hat{B}^3$ since $h(x_o)$ would be in $S^2 - C$.

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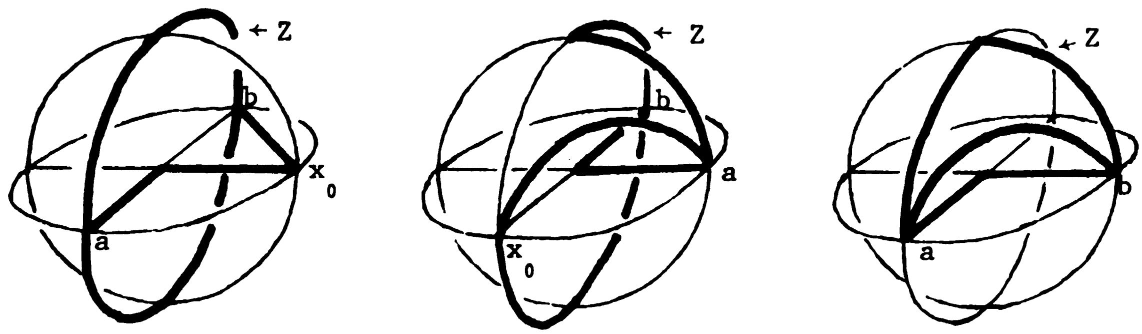
[3]The proof involves using Lemma 4.4, page 15 of [14].

<u>Example 5.6</u> $Z \cap D_i$ is a simple closed curve for $i = 2$ or 3 and $\{a,b\}$ for the other, Figure 5.1 c. The complement of Z has just two components but it is clear that neither is an open 3-book and that the homeomorphism of the theorem can't be extended to $\hat{B}^3$.



(a) (b) (c)

Figure 5.1

<u>Example 5.7</u> A locally connected, G_δ set that cuts $\hat{B}^3$ between two points z' and z'' on C. However, it contains no θ-curve that cuts $\hat{B}^3$ nor does it contain an irreducible cut of $\hat{B}^3$. This then is a counter-example of a generalization of Theorem II.17 of [16] for $\hat{B}^3$.

Let $x_i \to x_1$ be a sequence of points on C such that the lengths of the arcs $\widehat{x_i x_{i+1}}$ in C is $\frac{1}{2}$ the length of the arc $\widehat{x_{i-1} x_i}$ $(i > 1)$, and in D_2 join x_i to x_{i+1} by a "semicircle" of diameter the length of the chord $\overline{x_i x_{i+1}}$ $(i > 1)$. In D_2 also join the x_i's $(i > 1)$ to the "north pole" of D_2 by one quarter of great circles, and in D_1 join the center of D_1, call it 0, to the x_i's $(i > 1)$ by radii of D_1. Finally connect x_1 to x_2 by a great circle in D_3 and call the union of all these arcs X, Figure 5.2. Let z' and z'' be as in the figure.

Clearly X is locally connected since the only point to check is x_1 and any neighborhood of it in X contains a path connected neighborhood.

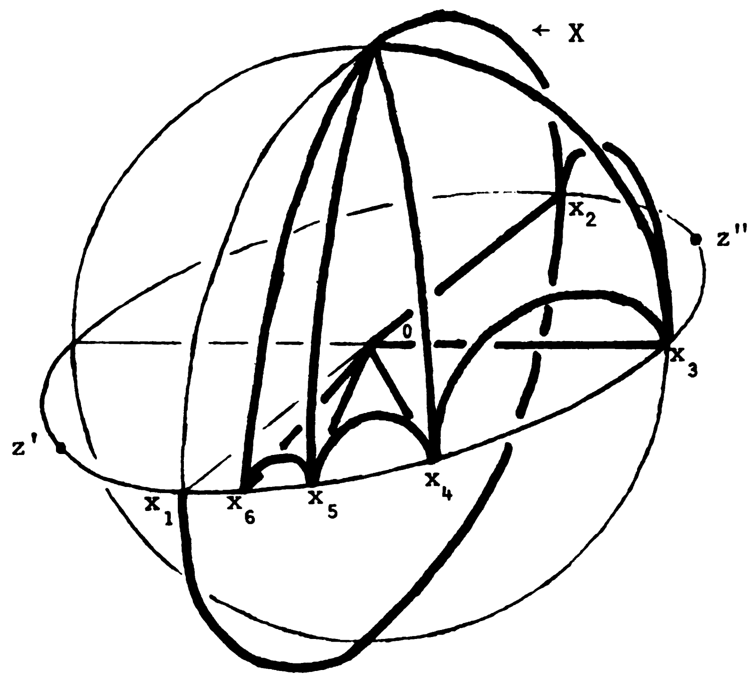


Figure 5.2

X is a G_δ set. If we stop the construction of X at X_j then that portion of X, call it X_j, is closed and by taking smaller and smaller "tubular" neighborhoods of X_j it is clear that X_j is a G_δ set. Thus $X = \bigcup_{j=1}^{\infty} X_j = \bigcup_{j=1}^{\infty}\{\bigcap_\alpha G_{\alpha j}\} = \bigcap_\alpha\{\bigcup_{j=1}^{\infty} G_{\alpha j}\}$ and hence X is a G_δ set since $\bigcup_{j=1}^{\infty} G_{\alpha j}$ is open for each α.

To show that X cuts $\hat{B}^3$ between z' and z" first let $r:\hat{B}^3 - X \to (S^2 - X) \cup x_1$ be the map defined by mapping the radii of D_1 onto their end points.

Suppose X doesn't cut $\hat{B}^3$ between z' and z"; then there exists a continuum $K \subset \hat{B}^3 - X$ such that z' and z" $\in$ K. Further if we suppose that $r(K) \subset S^2 - X$, then we get a continuum in $S^2 - X$ containing z' and z" so that $S^2 \cap X$ doesn't cut S^2 between z' and z". However, $S^2 \cap X$ contains a simple closed curve that cuts S^2 between z' and z". Hence $r(K) \supset \{x\}$

or equivalently $K \cap \overline{Ox_1} \neq \emptyset$.

Now suppose every neighborhood of some point $y \in K \cap \overline{Ox_1}$ contains points of K that are also in some sector of D determined by $0, x_j, x_{j+1}$ for some $j \geq 3$, then each sector of D determined by $0, x_i, x_{i+1}$ contains points of K for $i \geq N$.

Further, if $\widehat{x_k x_{k+1}} \cap K = \emptyset$ for some $k \geq N$, then it is clear that K is not connected since the sector $0, x_k, x_{k+1}$ and its complement in $\hat{B}^3$ intersected with K separates K into two open sets. Thus $\widehat{x_i x_{i+1}} \cap K \neq \emptyset$ for $i \geq N$. Let x_i' be a sequence of points such that $x_i' \in \widehat{x_i x_{i+1}}$, then $x_i' \to x_1$ so $x_1 \in K$ since K is closed. This is a contradiction.

Therefore for each $y \in K \cap \overline{Ox_1}$ there exists a neighborhood small enough so that it contains no points of K in these sectors of D_1, and a finite number of these cover $K \cap \overline{Ox_1}$ since this set is compact. We can move the points of $K \cap \overline{Ox_1}$ away from $\overline{Ox_1}$ by the following geometrical argument - we don't move 0 or x_1 since the distance from K to 0 and x_1 is positive.

First map the diameter through y onto the broken line through y' - see Figure 5.3 - and extend the map to the boundary by the identity map. Then extend the map by the plane Schoenflies Theorem for a θ-curve.

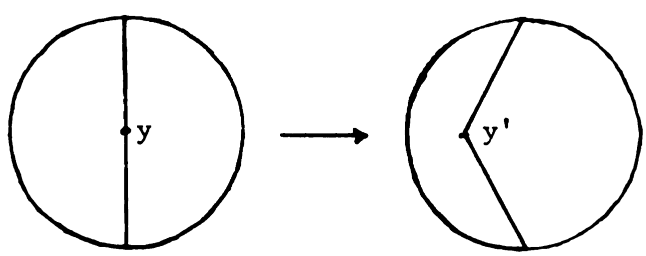


Figure 5.3

If we let K' be the continuum that is moved away from $\overline{Ox_1}$ by the above argument then $r(K') \subset S^2 - X$ and we obtain a contradiction of the fact that $S^2 \cap X$ cuts S^2 between z' and z". Hence X cuts $\hat{B}^3$ between z' and z".

It is clear no θ-curve in X cuts $\hat{B}^3$ by looking at Example 5.5.

To show that X contains no irreducible cut suppose that X' is an irreducible cut. Then X' contains a subsequence of $\{x_i\}$, call it $\{x_i'\}$ converging to $x_1 \in X'$; otherwise X' wouldn't cut $\hat{B}^3$. But now $(X' - \overline{0x_n'}) \cup \{0,x_n'\}$, i.e., just remove one of the radii of D_1 leaving the end points, cuts $\hat{B}^3$ between z' and z" if $n \geq 3$, which is a contradiction.

<u>Example 5.8</u> A locally connected irreducible cut of $\hat{B}^3$ between two points z' and z" on C that is not a θ-curve. This then is a counter-example of a generalization of Theorem II.19 of [16] for $\hat{B}^3$.

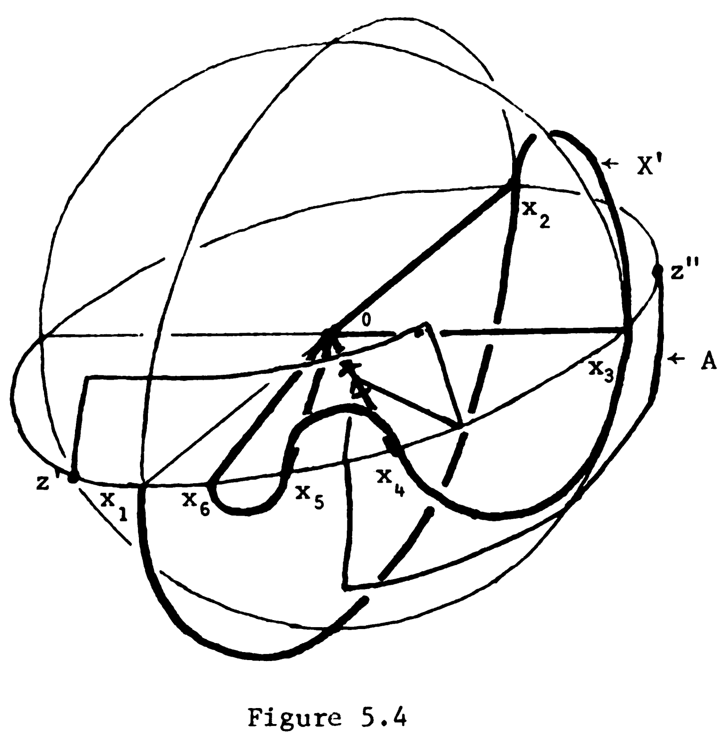


Figure 5.4

Let $x_i \to x_1$ be a sequence of points on C as in Example 5.7, and connect x_1 to x_2 by a "semicircle" of diameter the length of the

chord $\overline{x_1x_2}$ in D_3 as in Example 5.7. Then connect x_2 to x_3 by a "semicircle" in D_2, and continue alternating these "semicircles" between D_2 and D_3. Now connect the center of D_1 to each x_i $(i > 1)$ by a radius of D_1 and let X' be the resulting set, Figure 5.4. Let z' and z" be as in the figure.

We can show that X' cuts $\hat{B}^3$ between z' and z" by the same argument as in the previous example. Clearly X' is locally connected.

Now $X' \cap S^2$ is an irreducible cut of S^2 so if X' is not an irreducible cut of $\hat{B}^3$ we could take an open interval out of one of the radii of $X' \cap D_1$ and what would be left would still cut $\hat{B}^3$. But it is clear how one could construct an arc, like A in Figure 5.4, in $\hat{B}^3 - X''$ containing z' and z", which is a contradiction.

<u>Remark</u>: X' cuts $\hat{B}^3$ but it does not separate $\hat{B}^3$, for if $\hat{B}^3 - X'$ were the union of two disjoint closed sets then it is clear than one of the closed sets would contain most of the upper hemisphere of S^2, every other sector of D_1 determined by $0, x_{2i-1}, x_{2i}$ and the line segment from 0 to x_1 not including the end points. The other closed set would contain part of the lower hemisphere and every other sector of D_1 determined by $0, x_{2i}, x_{2i+1}$. But there exists a sequence of points in the sectors determined by $0, x_{2i}, x_{2i+1}$ converging to an interior point of $\overline{0x_1}$, which is a contradiction.

From the above constructions we can prove the following result which is the only theorem that we could prove for cuts of $\hat{B}^3$.

<u>Theorem 5.9</u> If $X \subset S^2 - \{z', z''\}$ is closed and cuts S^2 between z' and z" then there exists a component Y of X such that $Y \cup A$, where A is the set of line segments from 0 (the center of D) to $C \cap Y$, cuts $\hat{B}^3$ between

the two points z' and z" if z', z" $\in$ C.

Proof: Since X cuts S^2 between z' and z", then there exists a component Y of X that cuts S^2 between z' and z".[4]

Suppose Y $\cup$ A doesn't cut $\hat{B}^3$ between z' and z" then there exists a continuum K $\subset \hat{B}^3 - (Y \cup A)$ containing z' and z". Define $r'':\hat{B}^3 - (Y \cup A) \to S^2 - Y$ by mapping the radii of D - A onto their end points - r can be defined since C $\cap$ Y must contain at least one point in the two components of C - {z',z"} and thus A is nonvoid. Since r is continuous r(K) is a continuum in $S^2 - Y$ containing z' and z". This contradicts the fact that Y cuts S^2 between z' and z".

If we generalize the concept of local cuts to $\hat{B}^3$ we will also easily find counterexamples to the generalizations of the results of S. Eilenberg found on page 87 of [16].

A set $X \subset S^2$ is said to cut S^2 locally at a point $x \in S^2$, when X cuts each sufficiently small neighborhood U of x.

Example 5.10 A continuum $X \subset \hat{B}^3$ that doesn't locally cut $\hat{B}^3$ at a point $x_o \in X$ that cuts X. This is a counterexample of a generalization of Theorem II.22 of [16] for $\hat{B}^3$.

Let X be an arc in $\hat{B}^3$ that pierces C in the sense of Definition 3.2 at a point $x_o \in C$. X doesn't locally cut $\hat{B}^3$ at x_o, see Figure 4.5, but x_o clearly cuts X.

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[4] Theorem II.3, page 76, of [16].

CHAPTER VI

HOMOLOGICALLY TRIVIAL 2-COMPLEXES IN B^3

This chapter will deal with homologically trivial finite 2-complexes in a flat 3-book, and with homologically trivial finite 2-complex neighborhoods of arcs in a flat 3-book where the arcs are cellular when considered in E^3. The following remark indicates why we might want to look at homologically trivial finite 2-complexes in a 3-book, and their relation to a cellular arc.

Remark: If A is a cellular set in 3-space then by the definition of cellularity inside any neighborhood, U, of A there exists a topological 2-sphere, S^2, such that $A \subset \text{Int } S^2$. Hence by the Bing Approximation Theorem [6] there exists a polyhedral 2-sphere S_1^2 in U such that $A \subset \text{Int } S_1^2$.

To first of all prove that homologically trivial finite 2-complex in a flat 3-book are collapsible, we need the following lemma.

Lemma 6.1 Let K be a finite 2-complex in a flat 3-book B^3, then K has a free face.

Proof: Suppose that K has no free face, and consider a component of $K \cap D_1$, call it M, where D_1 is one of the leaves of B^3. Then the free faces of this simplicial complex are all on the back of the book.

To show that M is a 2-manifold with boundary, first let

$x \in M \cap \text{Int } D_i$. There are three cases to consider: if x is a point of the interior of a 2-simplex then it clearly has an open 2-cell neighborhood; if x is a point of a 1-simplex and not a vertex of the 2-complex, then x is contained in exactly two 2-simplexes because there are no free faces in the interior of D_i, and hence x has an open 2-cell neighborhood; finally, if x is a vertex of the 2-complex then x is contained in at least three 1-simplexes each of which is contained in exactly two 2-simplexes because again there are no free faces in $\text{Int } D_i$ and hence x has an open 2-cell neighborhood.

Now if we let $x \in M \cap B$, where B is the back of the book, there are two cases to consider: if x is not a vertex of the 2-complex then it clearly has a closed 2-cell neighborhood; if it is a vertex of K, by again using the fact that there are no free faces in $\text{Int } D_i$, x is contained in at least three 1-simplexes and two of these 1-simplexes are on the back of the book so x has a closed 2-cell neighborhood.

Thus M is a 2-manifold with nonvoid boundary. But the boundary of all compact 2-manifolds are disjoint simple closed curves[1]. Hence we have a contradiction since $\text{Bd } M \subset B$.

<u>Theorem 6.2</u> Any homologically trivial finite 2-complex K in a flat 3-book B^3 collapses to a point.[2]

<u>Proof:</u> By Lemma 6.1, K has a 2-simplex with a free face, and thus we can collapse this simplex to get a homologically trivial 2-complex

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[1] Chapter 1 of Massey [24].

[2] This theorem follows directly from an unpublished result of P. Dierker [10] : If K is a collapsible 2-complex and L is a homologically trivial subcomplex, then L is collapsible.

with one less 2-simplex. By a finite number of these elementary simplicial collapses we have a homologically trivial 1-complex, which is a connected tree. This tree can clearly be collapsed to a point.

Remark: To prove this theorem all we needed was the fact that K was simply connected.

By exactly the same arguments as in Lemma 6.1 and Theorem 6.2 we can show that a homological trivial finite 2-complex in the plane is collapsible. In fact, the complex could be thought of as lying in two of the leaves of a flat 3-book, and hence collapses to a point. However, the dunce cap provides a counterexample to the generalization of this result to E^3, because a triangulated dunce cap has no free faces but is homologically trivial.

Corollary 6.3 A dunce cap cannot be embedded in a flat 3-book.

Theorem 6.4 A homologically trivial finite 2-complex, K, contained in a flat 3-book B^3 is cellular in E^3.

Proof: First triangulate E^3 so that K or a subdivision of it is a subcomplex of the triangulation of E^3. To this end first of all triangulate E^2_+, and then take three copies of E^2_+ and join them along the x-axis so that they make an angle of 120^o with each other in E^3 - call the complex B_o. Since B^3 has planar leaves it can be embedded in B_o, thus $|K| \subset |B_o|$, and therefore there exists a subdivision of K and of B_o, call them K and B_o, such that K is a subcomplex of B_o. Now each pair of leaves D_i', D_j' of B_o separates E^3 into two copies of E^3_+. Let us consider the one that doesn't contain the other leaf of B_o. Since this domain is $E^1_+ \times (D_i' \cup D_j')$ it can be triangulated so that the triangulation is compatible with the triangulation of $D_i' \cup D_j'$. If we do this for each pair of leaves of B_o we have a triangulation of E^3 with K a subcomplex of it.

Let U be an open neighborhood of K in E^3, and take the r^{th} derived subdivision $(r \geq 2)$ so that the mesh of the triangulation of E^3 is less than $\frac{1}{2}d(\text{Bd } U, |K|)$.

The derived neighborhood N of K in E^3 is clearly contained in U, and since K is collapsible by Theorem 6.2, then N is a 3-cell[3].

An arc in a flat 3-book will be said to have a p-basis (pseudobasis) if for each open neighborhood U of the arc there exists a homologically trivial finite 2-complex neighborhood of the arc in the book contained in U.

Lemma 6.5 Any arc that is bad at just one end point doesn't have a p-basis even though it is cellular in E^3.

Proof: Persinger [30] has shown that we can embed the arc in a flat 3-book B^3, and since B^3 has planar leaves we can assume it is contained in B_o^3, where B_o^3 is the open 3-book which is the union of the xy-plane and one-half of the yz-plane.

Now we clearly can find a plane $y = y_o$ such that the plane doesn't intersect the arc, and since the arc doesn't separate B_o^3, we can extend the arc by a polygonal arc to get an arc A' that meets the plane $y = y_o$ in only one point, call it a, which is an end point of A'.

If we reflect A' in the plane $y = y_o$ to get A" and let $A = A' \cup A''$, Martin [23] has shown that A is not cellular. Therefore

[3]Theorem 2.11, page 57 of [22].

there exists an open neighborhood U of A in E^3 such that no 3-cell that contains A in its interior is contained in U.

If we now suppose the original arc has a p-basis, then there exists a homologically trivial finite 2-complex neighborhood of the arc, call it K_1, contained in B^3 such that $|K_1| \subset U$. If $A' \cap \mathrm{Bd}\, K_1$ is a vertex of K_1 move A' slightly so that the intersection is an interior point of a 1-simplex, again call this arc A'. We can construct a simplicial tubular neighborhood K_2 of $A' - (|K_1| \cup a)$ in B^3_o contained in U such that $K_1 \cap K_2$ is a 1-simplex and K_2 intersects the plane $y = y_o$ in a 1-simplex.

If we look at the Mayer-Vietoris sequence,

$$\cdots \to H_q(K_1 \cap K_2) \to H_q(K_1) \oplus H_q(K_2) \to H_q(K_1 \cup K_2) \to H_{q-1}(K_1 \cap K_2) \to \cdots$$

we can see that $H_q(K_1 \cup K_2) = 0$ for all $q > 0$ since $H_q(K_1 \cap K_2) = H_2(K_1) = H_q(K_2) = 0$ for all $q > 0$. Let $K_1^2 = K_1 \cup K_2$.

Similarly A" has a homologically trivial finite 2-complex neighborhood $|K_2^2| \subset U$ and K_2^2 intersects the plane $y = y_o$ is the same 1-simplex as K_1^2, and, therefore $K_1^2 \cup K_2^2$ is homologically trivial by the same argument as above. However $K_1^2 \cup K_2^2$ is then cellular by Theorem 6.3, and hence there exists a 3-cell inside U containing $|K_1^2 \cup K_2^2|$ in its interior, which contradicts the choice of U.

The fact that the original arc is cellular follows from the fact that every subarc of it is p-shrinkable and hence by a result of P. H. Doyle [13] the arc is cellular in E^3.

Remark: From this lemma it is clear that the p-shrinkable condition of [13] is not a sufficient condition for an arc in B^3 to have a p-basis. Hence a p-shrinkable argument of the type of [13] will not work in a flat 3-book.

At this point it seems we could define a cellular arc in a flat 3-book as one that has a p-basis. The polygonal arcs would have a p-basis by triangulating tubular neighborhoods of the arc in each leaf of the book and for points on the back of the book triangulate a small 3-book neighborhood of the point being sure that the triangulations are compatible. However, the following example of an embedding of a wild arc with a p-basis seems to indicate that how the arc is embedded is important. Since no general results could be obtained except in the converse sense, no definition of cellularity in a flat 3-book is put forward.

Example 6.6 An embedding of a wild arc which is locally polygonal except at one point in a flat 3-book B^3 such that the arc has a p-basis.

Let p be a point on the back of B^3 and let B_i^3 be a sequence of flat 3-book neighborhoods of p in B^3 such that $B_{i+1}^3 \subset \text{Int } B_i^3$ with $B_1^3 = B^3$. Let $E_i = \text{Int } B_i^3 - B_{i+1}^3$. Now embed a polygonal trefoil knot in $\overline{E}_i$ (the closure of E_i) for each i by the method of C. A. Persinger [30] so that the arc meets the Bd B_{i+1}^3 in exactly one point which is in only one of the leaves of B^3, and such that for $i > 1$ the arc intersects the Bd B_i^3 in the same point that the polygonal trefoil knot in $\overline{E}_{i-1}$ intersects Bd B_i^3. Thus by induction we have an embedding of a sequence of trefoil knots that converge to p and the resulting arc is locally polygonal except at p.

Now if U is any open neighborhood of the arc in B^3 there exists a B_j^3 for some j such that $B_j^3 \subset U$. Furthermore in $B^3 - B_j^3$ the arc is polygonal, and thus by a previous argument there exists a homologically trivial finite 2-complex neighborhood, K_1, of it in U, and since the

arc intersects B_j^3 in one point that is in only one leaf of B^3, K_1 can be made to intersect B_j^3 in a segment. To make $|K_1| \cup |B_j^3|$ a 2-complex just add the two vertices of the 1-simplex of K_1 in the 1-simplex of B_j^3 and extend the new triangulation of this 1-simplex to all of B_j^3. As in Lemma 6.5, $K_1 \cup B_j^3$ is homologically trivial.

Now since the back of the book has an orientation - its just a line segment - if we had embedded the arc so that all the points of the arc that are also on the back of the book were all on one side of p then we could extend the arc by a line segment along the back of the book, to get a wild arc that is locally polygonal except at p. If this line segment is not contained in B_j^3 we can use a similar argument to the one above to get the whole arc in a homologically trivial finite 2-complex inside U.

This example seems to depend quite strongly on the fact that the arc as constructed is L.P.U. at p. Thus in an attempt to obtain a general result that an arc in a flat 3-book that is locally polygonal except at one point has a p-basis, the restriction that it be L.P.U. at each point was added. Even with this additional restriction, the intersection of a polyhedral 2-sphere with a flat 3-book can be sufficiently bad to make it seem impossible to use this line of proof. However the following converse theorems can be proved.

<u>Theorem 6.7</u> Let A be an arc embedded in a flat 3-book B^3 such that it is locally polygonal except at one point p and at p there exists an element of the p-basis of p in each neighborhood of p whose boundary meets the arc in one point if p is an end point of A, and two points if p is an interior point of A. Then if we consider A as

being embedded in E^3, A is L.P.U. at p.

<u>Proof</u>: Suppose that p is an end point of A, let U be an open neighborhood of p in E^3, and let K be a homologically trivial finite 2-complex neighborhood of p in B^3 whose boundary meets A in one point, and $K \subset U \cap B^3$. Since A is locally polygonal at $A \cap \text{Bd } K = b$, we can retriangulate A so that b is a vertex of A.

We want to construct a derived neighborhood N of K in E^3 whose boundary meets A in just one point and $N \subset U$. Then Bd N will be a polyhedral 2-sphere that meets A in one point, since N is a 3-cell[4], and hence A is L.P.U. at p.

To this end let K_1 be the 1-simplex in A - K with vertex b, and then triangulate E^3 so that $K \cup K_1$ is a subcomplex of the triangulation as we did in Theorem 6.4. To make sure that the boundary of the derived neighborhood we construct misses the arc except on K_1 let $B' = A - (K \cup K_1)$, and take the r^{th} derived subdivision $(r \geq 2)$ of the triangulation of E^3 so that the mesh is less than or equal to $\frac{1}{2}d(K,B') > 0$. Also take the r^{th} derived subdivision of this triangulation of E^3 so that the mesh is less than or equal to $\frac{1}{2}d(\text{Bd } U, |K|)$.

Now it is clear by the definition of the r^{th} derived neighborhood N of K in E^3 that it would miss B' and hence its boundary would meet A in only one vertex of N which would be a point of K_1. Further $N \subset U$, and since $r \geq 2$ the r^{th} derived neighborhood of K in E^3 is a derived neighborhood of K in E^3.

It is clear that the same type of argument would work if p is an interior point of A, and if $A \cap \text{Bd } K$ is two points.

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[4]Theorem 2.11, page 57 of [22].

The following theorem is proved using Theorem 6.4.

Theorem 6.8 If an arc A in a flat 3-book has a p-basis, then A is cellular in E^3.

CHAPTER VII

A CHARACTERIZATION OF 2-COMPLEXES IN E^3 THAT CONTAIN WILD ARCS

In this chapter a characterization theorem is obtained for those finite 2-complexes in E^3 that contain wild arcs, where the restriction that is placed on the 2-complex is that it contains a subcomplex whose carrier is a 3-book. In order to facilitate this proof we need the following definitions of Whittlesey [34].

Let us assume initially that we are given a connected finite simplicial 2-complex, K', and in K' every edge is a face of some 2-simplex. Let P be an arbitrary point of K', and let St(P) be the star of the open simplex containing P.

Definition 7.1 If St(P) is homeomorphic to the Euclidean plane, then P is called regular. The points of K' which are not regular are called singular.

Definition 7.2 If St(P) is an open flat n-book ($n \neq 2$) then P is called line singular. If St(P) is topologically equivalent to the space obtained by identifying the origins of a certain m(>1) copies of the Euclidean plane, then P is called a conical point - St(P) is called a cone, and the components of its regular part are called the leaves of the cone. If St(P) is singular and neither conical nor line singular then P is called a node.

Definition 7.3 Let K be an arbitrary finite simplicial 2-complex.

If a point P of K is not in the closure of some 2-simplex, then we say P lies in the <u>1-dimensional part</u> of K.

<u>Remark</u>: If P is a node of K', then P is the vertex of at least one line singular 1-simplex which is either the back of a flat n-book ($n \geq 3$), or a free face of a 2-simplex. Furthermore, the components of the regular part of St(P) are either cone leaves or are topologically an open triangle with P as a vertex. If the component of the regular part of St(P) that is topologically an open triangle has two distinct singular edges with P as a vertex then that component is called a <u>fan</u>, and if there is only one singular edge with P as a vertex it is called a <u>cornet</u>. See Figure 7.1. In addition, from the figure it is clear that if the 2-complex contains no subcomplex whose carrier is a 3-book then the regular part of St(P), has no components that are cornets. Furthermore, the singular edges of a fan with P as a vertex must be free faces of some 2-simplexes.

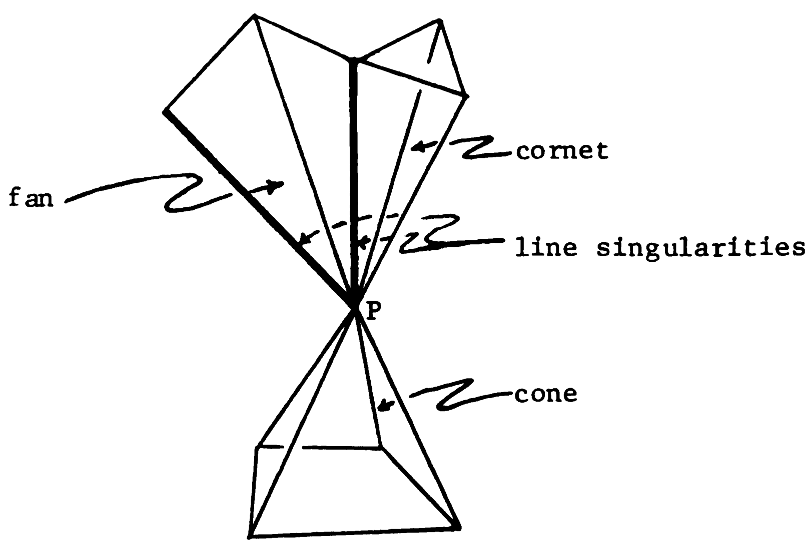

Figure 7.1

<u>Theorem 7.1</u> An arbitrary finite 2-complex K in E^3 contains a wild arc if and only if K has a subcomplex whose carrier is a 3-book.

<u>Proof</u>: If K has a subcomplex whose carrier is a 3-book, then C. A. Persinger [30] has shown that K contains a wild arc.

Conversely, suppose K has no subcomplex whose carrier is a 3-book, but K does contain a wild arc, A. By Bing's result that locally tame sets are tame [5], A is not tame at some point P.

To show that the arc is actually tame at P, first notice that P can't be an end point of A for otherwise near P the arc would lie in a plane and hence would be locally tame at P.

Now let us further suppose that every 1-simplex of K is the face of some 2-simplex, then by the previous definitions P clearly can't be a regular point or a line singular point of K, since then the arc would be locally planar at P. Thus by the previous remark P is either a conical point of K or P is a node of K where the components of the regular part of St(P) are either leaves of a cone or fans (the singular edges are free faces of some 2-simplex(es)).

Case A: P is a conical point of K. The St(P) must have at least two leaves, and furthermore the arc must intersect two of these leaves near P for otherwise the arc would be locally planar at P. Call these leaves L_1 and L_2. Since $K \subset E^3$, there clearly exists a homeomorphism of E^3 onto itself so that there exists a plane, H, through P that doesn't meet St(P) - P and L_1 is in one component of E^3- H and all the other leaves of St(P) - P are in the other component of E^3- H. Furthermore by a relative general position argument we can pick H so that it is not perpendicular to any 2-simplex of $L_1 \cup P$.

Let U" be a closed convex neighborhood of P in $L_1 \cup P$ contained in an ε-neighborhood of P, and sufficiently small so that Bd $U'' \cap A \neq \emptyset$. In fact, U" can be chosen so that Bd U" is n line segments, one on each 2-simplex of $L_1 \cup P$. Now let H' be a plane parallel to H and at a distance of ε from H in the same component of $E^3 - H$ as L_1. Consider the volume generated by lines perpendicular to H going through points of U" and bounded by H and H', call this volume U'.

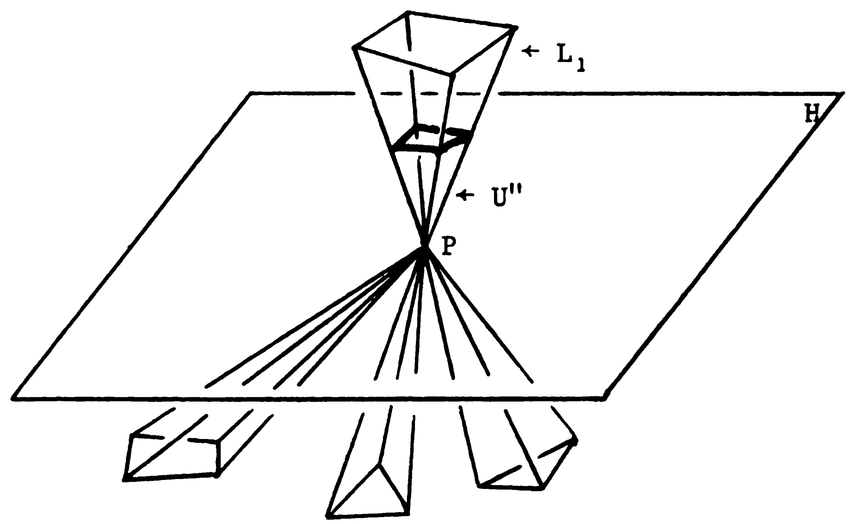

Figure 7.2

Let P' be the first point of A starting from P that is in Bd U" and let the subarc of A from P' to P by A'. Now in U"- A' connect P to another point P" $\in$ Bd U" by an arc A" (U"- A' is clearly path connected since A' is in fact tame). $A' \cup A''$ is homeomorphic to the line segments $\overline{PP'} \cup \overline{PP''}$ and this homeomorphism can be extended to the Bd U" by the identity, call it h". By applying the Schoenflies Theorem to the θ-curve, h" can be extended to U", call this homeomorphism h'.

To extend h' to the U', let U be the volume determined by U"

between H' and H'' where H'' is a plane parallel to H and at a distance ε from H with $H' \neq H''$. Further let $B \in H'' \cap \mathrm{Bd}\, U$, and $B' \in H' \cap \mathrm{Bd}\, U$. Now there exists a function $f: U' \to U$ such that f is a homeomorphism on $U' - P$ and $f(P) = \overline{PB}$, and h' can be extended to $h \not\subset U$ by first extending it to the Bd U by the identity and then by mapping $\overline{Bx}$ linearly onto $\overline{Bh'(x)}$ and $\overline{B'x}$ linearly onto $\overline{B'h'(x)}$ for all $x \in U''$. Extend by the identity to the remainder of U. Since $h'(P) = P$, $h|\overline{PB}$ is the identity on $\overline{PB}$, and therefore $f^{-1}hf$ is a homeomorphism on U' and an extension of h'. Furthermore, $h|\mathrm{Bd}\, U'$ is the identity, and thus none of the other 2-simplexes in the other component of $E^3 - H$ are moved, so if we do the same thing in $L_2 \cup P$, and extend by the identity to all of E^3 we have that A is in fact locally tame at P, which is a contradiction of our original assumption.

Case B: P is a node of K. Then the regular part of St(P) contains at least one fan, and another fan and/or a leaf of a cone. Again the arc must intersect two of the components of St(P) - P near P. Furthermore if the regular part of both of these components are fans then the arc is locally planar at P, and if both components are leaves of a cone we arrive at the same contradiction as in Case A. Thus the only case left to consider is if one of the components is a leaf of a cone, and the regular part of the other component is a fan. Call the fan plus the two singular 1-simplexes F_1.

As in Case A, there exists a homeomorphism of E^3 onto itself so that there exists a plane H such that $H \cap St(P) = P$ and F_1 is the only component of St(P) - P in one component of $E^3 - H$. Furthermore, the homeomorphism can be chosen so that F_1 is planar. Let V' be a closed neighborhood of P in F_1 contained in an ε-neighborhood of P, and such that Bd $V' \cap A \neq \emptyset$. Now if we pick two points one on each side of F' in this component of $E^3 - H$ and take the suspension of V' over these two points we obtain a closed neighborhood V in E^3. See Figure 7.3. As for the closed neighborhood U in Case A there exists a homeomorphism $h \circlearrowleft V$ such that h maps the component of $V' \cap A$ that contains P onto a line segment starting from P and $h|\text{Bd } V$ is the identity. For that portion of the arc in a leaf of a cone we can use Case A, and hence as in Case A we reach a contradiction of the assumption that A was not locally tame at P.

If K is an arbitrary 2-complex, we need only remark that the only additional case is if P is a node of K, where locally the arc lies in a leaf of a cone and a 1-simplex that is not a face of any 2-simplex, and then we reach a contradiction by Case A.

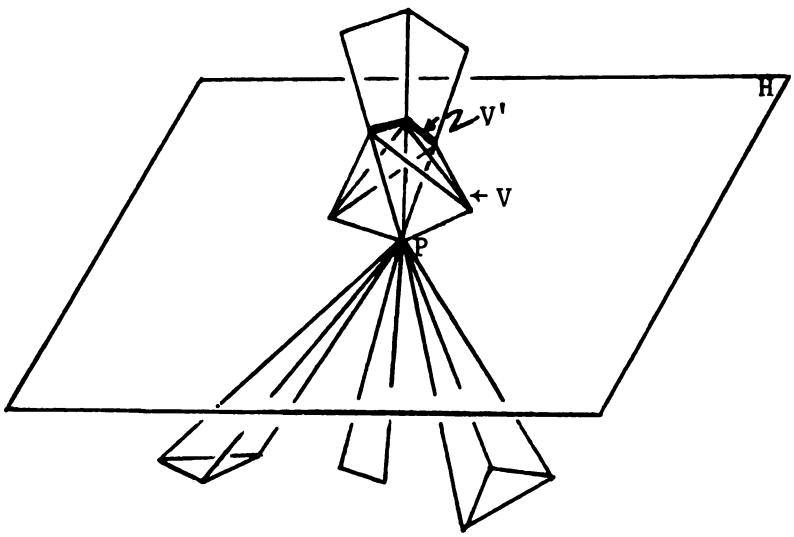

Figure 7.3

<u>Remark</u>: The complex K may or may not contain a node, as Figures 7.4 a and b illustrate. P is a node of a, but b has no nodes.

Figure 7.4

Corollary 7.2 An arbitrary finite 2-complex K in E^3 contains a wild arc if and only if for some vertex v of K the Lk(v) contains a subcomplex which is a triod.

Proof: If the Lk(v) contains a triod then K contains a subcomplex whose carrier is a 3-book, which can be seen by joining v to Lk(v). Hence K contains a wild arc by the previous theorem.

Conversely, if K contains a wild arc then it contains a subcomplex whose carrier is a 3-book. Hence if v is a vertex on the back of the book the Lk(v) contains a subcomplex which is a triod.

Corollary 7.3 Let S be the 1-dimensional part of an arbitrary finite 2-complex K in E^3. K contains a wild arc if and only if some component of $|K| - S$ contains a line singular point P where St(P) is an n-book with $n \geq 3$.

Proof: This is just a restatement of the previous theorem.

Corollary 7.4 A necessary and sufficient condition that a finite 2-complex in E^3 contains no wild arcs is that each homologically trivial subcomplex contains no wild arcs.

Proof: If the 2-complex contains no wild arcs then clearly each subcomplex contains none, and hence each homologically trivial subcomplex contains no wild arcs.

Conversely, suppose the 2-complex does contain a wild arc, then by the previous theorem the 2-complex contains a subcomplex whose carrier is a 3-book. This subcomplex is homologically trivial and contains a wild arc.

Corollary 7.5 A necessary and sufficient condition that a finite 2-complex in E^3 contains no wild arcs is that each collapsible subcomplex contains no wild arcs.

Proof: The proof is obtained from the proof of Corollary 7.4 by replacing "homological trivial" by "collapsible".

Having answered the questions of what types of finite 2-complexes contain wild arcs, let us turn to the question of which wild arcs can be embedded in a finite 2-complex that does contain a wild arc.

Theorem 7.6 Let K be a finite 2-complex in E^3 that contains a wild arc. An arbitrary arc A can be embedded in K if and only if it can be embedded in a flat 3-book.

Proof: Suppose A can be embedded in a flat 3-book. Since K contains a wild arc, then by the previous theorem K contains a subcomplex whose carrier is a 3-book and hence A can be embedded in K.

Conversely, suppose that A can be embedded in K. If the arbitrary arc A is tame, then A can be embedded in a 3-book by the method of C. A. Persinger [30]. Thus we can suppose that A is wild.

The points of A where A fails to be locally tame clearly lie in the singular set of K. In fact, by the previous theorem, it is clear these points all lie on the back of some 3-book which is contained in $|K|$.

Gail Atneosen [2] has shown that the set of wild points of A in each 3-book of $|K|$ is a compact, totally disconnected set, and hence the set of wild points of A is a compact, totally disconnected set since it is a finite union of such sets, call it C'.

First of all, let us connect up the various components of the line singular sets of K by line segments so that C' is contained in a finite connected graph, G -- G need not be contained in the 1-skeleton of K. Next add a Cantor set of points in G around each isolated point of C' to get a Cantor set, C, in G.

We want to show that C is locally tame and hence is tame by a result due to Bing [7]. If $c \in C$ is an interior point of a 1-simplex of G or a vertex of G of index at most 2 then G is clearly locally tame at C. Thus we need only check the vertices of G with index greater than 2. To show that the Cantor set is locally tame at such a vertex of G, construct a neighborhood, U, of the vertex whose boundary misses C (see Figure 7.5). Now the Cantor set inside this neighborhood is planar and hence is tame.

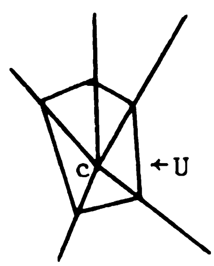

Figure 7.5

Thus there exists a tame arc that contains C, and therefore contains the set of wild points of A. By a theorem of Posey's [31] and the discussion that follows the statement of that theorem in C. A. Persinger's paper [30], we have that A can be embedded in a flat 3-book.

Remark: In the proof that an arbitrary arc A in K can be embedded in a flat 3-book we don't need the additional assumption that K contains a wild arc. However, the only interesting case is when A is a wild arc.

The proof of the theorem contains the following two corollaries.

Corollary 7.7 The set of wild points of an arc in a finite 2-complex in E^3 is a compact, totally disconnected set.

Corollary 7.8 Let T, and E be subsets of an arc A such that

(i) A is locally tame at each point of T.

(ii) $E = A - T$

(iii) each neighborhood of each point of E meets T in a non-empty set.

If E lies on a tame arc, then A can be embedded in a finite 2-complex in E^3 that contains a wild arc.

Remark: This is just a restatement of a result of Posey's [31] which was referred to in the proof of the theorem.

CHAPTER VIII

SIMPLE CLOSED CURVES THAT BOUND DISKS IN A 2-COMPLEX

Because of the existence of flat 3-books in the carrier of a 2-complex, and the existence of knots in a 3-book this naturally leads to the question of under what conditions on the 2-complexes does the plane Schoenflies Theorem hold. Clearly the theorem holds if the 2-complex is planar and has no "holes". The following theorem generalizes this to get a necessary and sufficient condition on a 2-complex so that each polygonal simple closed curve in it bounds a disk in the 2-complex. The outline of the proof is from Lemma 3.6 of G. Atneosen's Thesis [2].

Definition 8.1 The order of a 1-simplex, σ, in a 2-complex is the number of closed 2-simplexes that have σ as a face.

To this point in this paper it hasn't been necessary to specify which homology was being used nor what the coefficients were. However, in this theorem we will need simplicial homology with coefficients in Z_2, the group of integers modulo two. For an exposition on simplicial homology theory and related terminology see Chapter 6 of [21]. By a m-dimensional chain on a simplicial complex K with coefficients in Z_2 is meant a function c_m on the m-simplexes of K with values in Z_2. To facilitate the notation let c_m also denote the subcomplex of K which is all m-simplexes of K on which c_m has non-zero value. The carrier of this subcomplex will also be denoted by c_m rather than $|c_m|$. No confusion should arise since Z_2 coefficients are particularly well suited to geometric interpretations, and this notation will be used only in relation to chains.

Theorem 8.1 A necessary and sufficient condition that each polygonal simple closed curve in a finite collapsible 2-complex K bounds a disk in $|K|$ is that each 1-simplex in K is of order at most 2.

Proof: If there exists a 1-simplex in K of order 3 or more, then $|K|$ contains a 3-book. It is clear by looking at the following figure, which is Figure 3.1 a of [2], that there exists a polygonal simple closed curve in K that doesn't bound a disk.

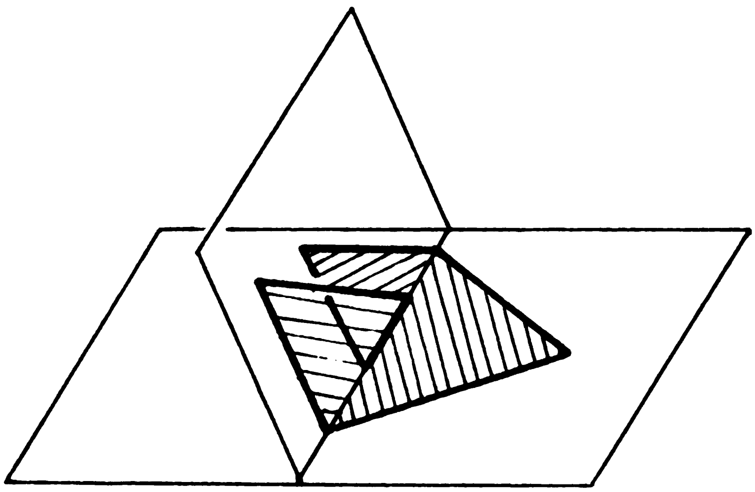

Figure 8.1

Conversely, suppose that each 1-simplex in K is of order at most 2 and let C be a polygonal simple closed curve in K. Now we can retriangulate K, which we will again call K, such that C is contained in the carrier of the 1-skeleton of K. Consider the simplicial homology of K with coefficients in Z_2, and let C also denote the 1-chain that has value 1 on all 1-simplexes of K that are contained in C and 0 on all other 1-simplexes of K. Since K is collapsible, $H_1(K,Z_2)$, the first simplicial homology group of B^3 with coefficients in Z_2, is trivial. Thus there exists a 2-chain M_c on K such that $\partial M_c = C$, since $\partial C = 0$. It will now be shown that M_c is a disk.

(i) M_c is compact because it is the point set union of a finite number of compact 2-simplexes. To show that M_c is connected, suppose that M_c can be expressed as the disjoint union of two closed sets, A_1 and A_2. Since C is connected it may be assumed that $C \subset A_1$. Let M_c' be the 2-chain on K that has non-zero value only on those 2-simplexes of K contained in A_2. Since $\partial M_c = C$, and since all the 1-simplexes of K are of order at most 2, each 1-simplex in A_2 is of order exactly 2. Thus $\partial M_c' = 0$, but $H_2(K,Z_2) = 0$, since K is collapsible, and hence there must be a 3-chain on K whose boundary is M_c'. Moreover, since the only 3-chain in K is the zero 3-chain, this implies that A_2 is empty and hence M_c is connected.

(ii) To show that M_c is a manifold, let us consider the following four cases for a point x in M_c.

Case A: x is contained in the interior of 2-simplex of M_c. Then x clearly has a neighborhood in M_c homeomorphic to an open disk.

Case B: x is contained in the interior of a 1-simplex in $M_c - C$. Since this 1-simplex does not lie in C, and since all 1-simplexes of K are of order at most 2, it is of order exactly 2. Hence x has a neighborhood in M_c homeomorphic to an open disk.

Case C: x is contained in the interior of a 1-simplex in C. Again since each simplex in K has order at most 2, and since 1-simplexes of C have odd order, the 1-simplex is of order exactly one. Thus x has a neighborhood in M_c that is homeomorphic to a closed disk.

Case D: x is a vertex of M_c. If in addition $x \in M_c - C$ then x can't be a node in M_c because each 1-simplex that has x as a vertex in M_c is of order exactly two, and hence x is conical point of M_c or a regular point of M_c. Similarly, if $x \in C$, then the regular part of $St(x)$ has exactly one fan - the two singular simplexes would be the two simplexes of C that have x as a vertex - and any other components of the regular part of $St(x)$ would be cones.

We wish to show that the x is regular if $x \in M_c - C$ and a line singular point with one fan in the regular part of $St(x)$ if $x \in C$ in order to complete the proof the M_c is a 2-manifold.

To this end let us consider $M_c - x$. There are two cases to consider.

Case 1: $M_c - x$ has at least two components. Then let C_1 be a component that doesn't contain C. Now since each 1-simplex in $C_1 \cup \{x\}$ is a face of exactly two 2-simplexes, we can't start in $C_1 \cup \{x\}$ when we collapse K, and in fact, we would never be able to collapse C_1 because it intersects the rest of K in just a vertex of K. This contradicts the fact that K is collapsible.

Case 2: $M_c - x$ is connected. Since there are at most a finite number of these points like x in M_c we have that M_c is topologically a compact, connected 2-manifold with boundary with a finite number of interior points identified, not necessarily to the same point, and a finite number of boundary and interior points identified, again not necessarily to the same point, and such that no two boundary points are identified.

Now if in fact there exists a conical point in $M_c - C$ or a node in C, then it is clear from the above discussion of what M_c would be, that there would exist a polygonal simple closed curve in M_c contained in the 1-skeleton of K through the conical point or node which will not bound a mod two 2-chain in M_c, but it must bound in K. However, every 1-simplex in M_c - C has all its 2-simplexes that have it as a face and thus the polygonal simple closed curve doesn't bound in K. Hence we have shown by contradicting the collapsibility of K that M_c can't have any nodes or conical points, or in other words that M_c is a 2-manifold.

Furthermore we can show by the argument above that M_c is homologically trivial for if a 1-cycle doesn't bound a mod two 2-chain in M_c it can't bound one in K. Thus M_c is a homologically trivial compact, connected 2-manifold with boundary C, and hence C bounds a disk in K.

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