# SEMIGROUPS HAVING QUASI-FROBENIUS ALGEBRAS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

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1965

This is to certify that the

thesis entitled

SEMIGROUPS HAVING QUASI-FROBENIUS ALGEBRAS

presented by

Ronald H. Wenger

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Major professor

Date May 13, 1965

O-169

# ABSTRACT

## SEMIGROUPS HAVING QUASI-FROBENIUS ALGEBRAS

by Ronald H. Wenger

Finite semigroups having semisimple algebras over fields have been characterized by Munn [7]. This characterization involves a restriction concerning the characteristic of the field and certain subgroups of the semigroup. The purpose here is to determine which finite semigroups have quasi-Frobenius algebras. The main result of section 2 is that this restriction on the field can be dropped and the semigroup algebra remains quasi-Frobenius.

<u>Theorem</u>: Let F be a field of characteristic c. If a semigroup S has a principal series in which each factor is c-nonsingular over F, then F(S) is quasi-Frobenius.

This theorem implies all finite inverse semigroups have quasi-Frobenius algebras over arbitrary fields.

The remaining sections are devoted to the investigation of semigroups S which have a principal series $S = S_o \supset S_1 \supset \cdots \supset S_{r+1}$, in which each principal factor $S_i/S_{i+1}$, is null, a group, or a group with zero. A semigroup S is defined to be of <u>type C</u> if S is a group or S satisfies the conditions:

1) S is commutative with zero,
2) S has a principal series $S = S_o \supset S_1 \supset \cdots \supset S_{r+1} = (0)$, with $S - S_1$ a group, and $S_1$ nilpotent,
3) $s \in sS \cap Ss$ for each $s \in S$.

The key result in this case is:

<u>Theorem</u>: Let S be a commutative semigroup and F a field. $F(S)$ is quasi-Frobenius if and only if S has an ideal series $S=I_0 \supset I_1 \supset \cdots \supset I_{q+1}$, with $I_i/I_{i+1}$ a semigroup of type C and with $F(I_i/I_{i+1})$ quasi-Frobenius for each $i = 0,1,\ldots,q$.

This theorem is used to reduce the commutative case to the simpler case of semigroups of type C. The group algebra $F(G)$ is a subalgebra of $F(S)$ for a type C semigroup. $F(G)$ is Frobenius and this is used to find conditions on $F(S)$. The primitive idempotents of $F(S)$ are found to be in $F(G)$. The result: If $e_1$ and $e_2$ are primitive idempotents of $F(G) \subseteq F(S)$, then $e_1F(G) \cong e_2F(G)$ as right $F(G)$-modules if and only if $e_1F(S) \cong e_2F(S)$ as right $F(S)$-modules, is proved. These results can be used to prove that if a semigroup S is of type C, then $F(S)$ is Frobenius if $F(S)$ is quasi-Frobenius.

The remaining sections are used to examine the question of when semigroups of type C have quasi-Frobenius algebras. This problem is reduced to that for semigroups of type C with one element in each J-class and these are said to be of type D. The most useful theorem then is:

If T is a semigroup of type D with a principal series $T = T_0 \supset T_1 \supset \cdots \supset T_{r+1} = (0)$, then $F(T)$ is quasi-Frobenius if and only if $r(F(T_1)) = F(T_r)$.

This result is used to extend the class of semigroups with quasi-Frobenius algebras.

In the last section, a construction of some semigroups of type C from semigroups of type D and abelian groups is given. This construction is used to form semigroups of type C with quasi-Frobenius algebras from semigroups of type D which are known to have quasi-Frobenius algebras.

# SEMIGROUPS HAVING QUASI-FROBENIUS ALGEBRAS

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1965

## ACKNOWLEDGEMENTS

The author wishes to thank Professor W. E. Deskins for his help with this thesis.

This work was supported, in part, by the National Science Foundation.

To

Sherry and my Parents

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## INTRODUCTION

Semigroups S with semisimple algebras F(S) over a field F have been characterized by Munn [7]. The goal here is to extend some of these results and to find others for semigroups for which F(S) may not be semisimple. As the theory of non-semisimple algebras in general is not very complete, a natural question is "For what semigroups S is F(S) quasi-Frobenius or Frobenius?" Quasi-Frobenius and Frobenius algebras have been studied rather extensively by Nakayama [8,9] and others so some reasonable characterizations for them are known. Another reason for asking this question is that some of the structure theory for semigroups is related to their subgroups and groups have Frobenius algebras over arbitrary fields. The following results are obtained:

In section 2, Munn's result for semisimple algebras is generalized. His result involves the restriction that the characteristic of the field does not divide the orders of certain groups associated with the semigroups. The main result of this section drops this condition and provides a large class of quasi-Frobenius semigroup algebras.

Theorem: Let F be a field of characteristic c. If a semigroup S has a principal series in which each factor is c-nonsingular over F, then F(S) is quasi-Frobenius.

This theorem is used to prove that finite inverse semigroups have quasi-Frobenius algebras over arbitrary fields. Such semigroups are being extensively studied by Munn and others. Then, in particular, a semigroup S with a principal series $S=S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ in which each $S_i/S_{i+1}$ is a group with zero, or a group, has a quasi-Frobenius algebra over an arbitrary field. The question then is raised "What happens if some of the $S_i/S_{i+1}$ are null semigroups?" The prototype for the semigroup for which each principal factor is null, a group with zero, or a group is the abelian semigroup.

Thus, in subsequent sections, semigroups for which each principal factor is either null, a group, or a group with zero are examined. The fundamental theorem for the whole investigation is:

Theorem: Let S be a commutative semigroup and F a field. F(S) is quasi-Frobenius if and only if S has an ideal series $S=I_0 \supset I_1 \supset \cdots \supset I_{q+1}$, with $I_i/I_{i+1}$ a semigroup of type C and with $F(I_i/I_{i+1})$ quasi-Frobenius for each $i=0,1,\ldots,q$.

Definition: A semigroup S is of type C' if S is a group or S satisfies the following conditions:

1) S has a zero,
2) S has a principal series $S=S_0 \supset S_1 \supset \cdots \supset S_{r+1}=(0)$ with $S-S_1$ a group G, and $S_1$ nilpotent,
3) $s \in Ss \cap sS$ for each $s \in S$.

S is said to be of type C if S is commutative and of type C'.

S is of type D if it is of type C and has precisely one element in each J-class.

This theorem reduces the problem to that for type-C semigroups. All of the remaining sections are devoted to studying semigroups of types C, C', and D, with the theorem above in mind. For if one can tell when semigroups of type C have quasi-Frobenius algebras then the theorem can be used to answer the question for all finite commutative semigroups.

Semigroups of type C with quasi-Frobenius algebras are not completely characterizedbut the following results are obtained.

Theorem: If S is of type C and F(S) is quasi-Frobenius then F(S) is Frobenius.

The problem for semigroups S of type C with group G can be related to that for type D semigroups when the characteristic of F does not divide the order of G. It is found that the set of J-classes for S forms a semigroup, say T, of type D. This leads to the investigation of semigroups of type D. The most useful theorem then is:

Theorem: Let T be a semigroup of type D with a principal series $T=T_0 \supset T_1 \supset \cdots \supset T_{r+1}=(0)$. Then F(T) is quasi-Frobenius if and only if $r(F(T_1))=F(T_r)$. (Here $r(F(T_1))=(a \in F(T): F(T_1)a=(0))$.

This theorem is used to prove various necessary and some sufficient conditions for a semigroup of type D to have a quasi-Frobenius algebra.

Section 4 shows how some semigroups of type C are constructed from semigroups of type D and abelian groups. This leads to other semigroups of type C with quasi-Frobenius algebras.

The only semigroups considered here are finite semigroups.

## 1. Preliminaries

Recent books will be cited as references whenever possible. All of the remarks about semigroups can be found in [2].

Notation: If B is a subset of A, then $A-B = (a \in A : a \notin B)$.

If F is a field, c(F) is the characteristic of F.

If X is a set, o(X) represents the cardinality of X.

$\emptyset$ will denote the empty set.

Semigroups: A semigroup is a nonempty set S with one binary, associative operation. If a semigroup S does not have an identity, one can be adjoined. Let $S \cup (1) = S^1$ with $1 \cdot 1 = 1$ and $1 \cdot a = a$ for each $a \in S$.

A subset I of S is a left ideal of S if $SI \subseteq I$, a right ideal of S if $IS \subseteq I$, and a two-sided ideal of S if it is both a left and a right ideal of S. By "ideal" is meant a two-sided ideal.

If S has a minimal ideal M, i.e., an ideal which does not properly contain an ideal of S, then M is unique and is called the kernel of S. If S has a zero, 0, then $M = (0)$.

A semigroup S with a zero is called nilpotent if $S^n = (0)$ for some positive integer n. S is called null if $S^2 = (0)$. A semigroup S with a zero is said to be 0-simple if $S^2 \neq (0)$ and S properly contains no nonzero ideal. S is called simple if S properly contains no two-sided ideals.

An element a of S is called regular if $a \in aSa$. S is said to be regular if each of its elements is regular. Two elements a and b of S are inverses of one another if $aba = a$ and $bab = b$.

S is <u>cyclic</u> if there is an $a \in S$ such that each element of S is a power of a.

A <u>congruence relation</u> R on a semigroup S is an equivalence relation on S with the additional property that if $a, b \in S$ and aRb then (ca)R(cb) and (ac)R(bc) for each c in S. If $[a] = (b \in S\colon bRa)$, the composition $[a][b] = [ab]$ gives $([a]\colon a \in S)$ the structure of a semigroup which will be denoted by S/R.

If I is an ideal of S, define the relation R on S as aRb if and only if a = b or both a and b are in I. R is a congruence and the semigroup S/R is written S/I, and is called the <u>Rees factor semigroup of S modulo I</u>. Because of its construction, S/I is often written as $(S-I) \cup I$ or $(S-I) \cup z$, where z is the zero of S/I and products are defined as: $ab = z$ if $ab \in I$, and ab is the product in S otherwise. This <u>identification</u> is sometimes convenient. The following theorem is often used.

<u>Theorem</u>: Let I be an ideal of S, and let $\pi$ be the natural homomorphism of S onto S/I. Then $\pi$ induces a one-to-one, inclusion-preserving mapping $A \to A\pi = A/I$, of the set of all ideals A of S containing I upon the set of all ideals of S/I, and $(S/I)/(A/I) \cong S/A$.

If a is in S, the set $(b \in S\colon S^1aS^1 = S^1bS^1) = J_a$ is called the <u>J-class</u> of S corresponding to a. (Note that if $a \in Sa \cap aS$ for each $a \in S$, then $S^1aS^1 = SaS$. If S is commutative, then $b \in J_a$ if and only if $bS^1 = aS^1$.) If $a \in S$, $S^1aS^1$ is called the <u>principal ideal generated by a</u>. $I(a) = S^1aS^1 - J_a$ is an ideal of S and $S^1aS^1/I(a)$ is called a <u>principal factor</u> of S.

For convenience, if S is a semigroup and $\emptyset$ is the empty set, let $S/\emptyset$ mean S.

An <u>ideal series</u> for a semigroup S is a chain

(1) $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1} \supseteq \emptyset$, with $S_i$ an ideal of S, $i = 0, 1, \cdots, r+1$.

An ideal series is called a <u>principal series</u> if there is no ideal of S strictly between $S_i$ and $S_{i+1}$, $i = 0, 1, \cdots, r+1$. If (1) is an ideal series for S, the semigroups $S_i/S_{i+1}$, $i = 0, 1, \cdots, r+1$, are called the <u>factors</u> of the series.

The following theorems are used.

<u>Theorem</u>: Let S be a semigroup admitting a principal series (1). Then the factors of (1) are isomorphic in some order to the principal factors of S. In particular, any two principal series of S have isomorphic factors. The last term in any principal series of S is the kernel.

The connection between the factors $S_i/S_{i+1}$ of a principal series for S and the principal factors can be written as $S_i/S_{i+1} \cong S^1aS^1/I(a)$ if $a \in S_i - S_{i+1}$, $S_i - S_{i+1} = J_a$. Thus, an ideal series (1) is a principal series if and only if $S_i - S_{i+1}$ is a J-class for each i.

<u>Theorem</u>: Each principal factor of any semigroup S is 0-simple, simple, or null. Only if S has a kernel is there a simple principal factor.

Note that all finite semigroups have principal series and that any ideal series can be refined to a principal series for such semigroups.

Let S be a finite semigroup and let F be a field. If X is a subset of S let $X' = X - (0)$. If S has no zero or $0 \notin X$, then $X' = X$.

In particular, let $S' = S-(0)$. Let $F(S) = (\sum_{s \in S'} a_s s : a_s \in F)$, and

define, for $\sum_{s \in S'} a_s s$ and $\sum_{s' \in S'} b_{s'} s'$ in $F(S)$,

$\sum_{s \in S'} a_s s = \sum_{s' \in S'} b_{s'} s'$ if and only if $a_s = b_s$ for each $s \in S'$.

Define also, $\sum_{s \in S'} a_s s + \sum_{s' \in S'} b_{s'} s' = \sum_{s \in S'} (a_s + b_s) s$, and

$$(\sum_{s \in S'} a_s s)(\sum_{s' \in S'} b_{s'} s') = \sum_{t \in S'} (\sum_{ss' = t} a_s b_{s'}) t.$$

With these definitions $F(S)$ is a finite dimensional, associative algebra with the elements of $S'$ as a basis. $S$ can be considered a subset of $F(S)$ if one identifies it with the set $(1 \cdot s : s \in S)$ (1 the identity of $F$), with the zero of $F$ identified with the zero of $S$ if $S$ has a zero. $F(S)$ is called the <u>semigroup</u> <u>algebra</u> (contracted) of $S$ over $F$.

<u>Theorem</u>: If $T$ is an ideal of $S$, then $F(T)$ is an ideal of $F(S)$ and $F(S)/F(T) \cong F(S/T)$ as algebras.

If $X$ is a subset of $S$, then $F(X)$ will denote the linear subspace $(\sum_{x \in X'} a_x x : a_x \in F)$ of $F(S)$.

<u>Algebras</u>: The following remarks are found in [3] unless otherwise indicated.

All algebras considered are finite dimensional, associative algebras over a field $F$.

If $X$ is a subset of an algebra $A$, then $r(X) = (a \in A : Xa = (0))$ is called the <u>right</u> <u>annihilator</u> of $X$ in $A$. In like manner, the set $l(X) = (a \in A : aX = (0))$ is called the <u>left</u> <u>annihilator</u> of $X$.

<u>Definition</u>: An algebra A with identity over a field F is <u>quasi-Frobenius</u> if a) $r(l(R)) = R$ and $l(r(L)) = L$ for each right ideal R and each left ideal L of A. A is <u>Frobenius</u> if, in addition, b) $(r(L):F) + (L:F) = (A:F)$ and $(l(R):F) + (R:F) = (A:F)$ for all left ideals L and right ideals R of A, where $(X:F)$ means the dimension of the linear space X over F.

Nakayama proved [8, Theorem 3, p. 619] that not all left and right ideals need be checked.

<u>Theorem</u>: Let N be the radical of A. If a) holds for every nilpotent simple left ideal L and every nilpotent simple right ideal R of A, as well as for the zero ideal and N itself, then A is quasi-Frobenius. If, moreover, b)holds for such ideals then A is Frobenius.

The following remarks can be made about quasi-Frobenius and Frobenius algebras, A. If $A = B + C$ is an algebra direct sum, then A is quasi-Frobenius (Frobenius) if and only if both B and C are quasi-Frobenius (Frobenius). Group algebras are Frobenius.

<u>Definition</u>: An idempotent f in an algebra A is called <u>primitive</u> if f cannot be written as the sum of two orthogonal idempotents. I.e., $f \neq f_1 + f_2$, $f_i^2 = f_i$, $i = 1,2$, $f_1 f_2 = f_2 f_1 = 0$.

If f is a primitive idempotent of A then Af is called a (left) <u>principal indecomposable module</u>. If N is the radical of A, then every irreducible left A-module is isomorphic to $Af/Nf$ for some primitive idempotent f of A. These modules are very important when discussing non-semisimple (or semisimple) algebras.

## 2. Munn algebras

The following theorem is proved in this section:

Theorem: Let F be a field of characteristic c. If a semigroup S has a principal series in which each factor is a c-nonsingular semigroup over F, then F(S) is quasi-Frobenius.

This result provides a large class of semigroups with quasi-Frobenius semigroup algebras. It is a partial generalization of a theorem by Munn [7, Theorem 6.4, pp. 10], in which semigroups with semisimple algebras are characterized. It will be used to prove that finite inverse semigroups have quasi-Frobenius algebras over arbitrary fields. Such semigroups have been studied extensively by Munn and others because they have properties similar to those of groups.

This theorem is proved by induction on the length of a principal series for S. This technique is used quite often and its success depends on the following lemma.

Lemma 2.1: Let B be an ideal in the algebra $\mathbf{A}$. If B and $\mathbf{A}/B$ are quasi-Frobenius (Frobenius), then $\mathbf{A}$ is quasi-Frobenius (Frobenius).

Proof: Let e be the identity of B. Then $\mathbf{A} = B+C$, an algebra direct sum, where $C = (c \in \mathbf{A}\colon ce = ec = 0)$ [1, Theorem 23, pp. 30]. But both B and $C \cong \mathbf{A}/B$ are quasi-Frobenius (Frobenius) by hypothesis so $\mathbf{A}$ is quasi-Frobenius (Frobenius) as their direct sum, by a remark in the preliminaries.

Many semigroup algebras can be related to total matric algebras over group algebras. The next result is useful in such cases. If $\mathbf{A}$ is an algebra, let $M_n(\mathbf{A})$ denote the algebra of nxn matrices over $\mathbf{A}$.

<u>Lemma 2.2</u>: $M_n(A)$ is quasi-Frobenius if and only if $A$ is quasi-Frobenius.

<u>Remark</u>: In this proof and elsewhere, if $X$ is a subset of an algebra, then $r(X)$ will mean the right annihilator of $X$ in the appropriate algebra. The same remark holds for left annihilators. For example, if $X$ is a subset of an algebra $A$ and $M_n(X)$ is the subset of $M_n(A)$ consisting of all matrices with entries from $X$, then $r(M_n(X))$ means the right annihilator of $M_n(X)$ in $M_n(A)$, while $M_n(r(X))$ means the set of all nxn matrices with entries from $r(X)$, the right annihilator of $X$ in $A$.

<u>Proof</u>: The fact that if $A$ is quasi-Frobenius then $M_n(A)$ is also, is the content of [9, lemma 2, p. 11].

Conversely, suppose $M_n(A)$ is quasi-Frobenius. If $L$ is a left ideal of $A$ such that $l(r(L)) \neq L$, consider $M_n(L)$. Clearly $r(M_n(L)) \supseteq M_n(r(L))$. Suppose $B \in r(M_n(L))$. Then $YB = 0$ for each $Y$ in $M_n(L)$. Let $(x)_{ki}$ denote the nxn matrix with $x$ as its $(k,i)$ entry and zeros elsewhere. If $b_{ij} \neq 0$ is the $(i,j)$ entry of $B$ then

$$(x)_{ki}B = \begin{bmatrix} & & 0 & \\ xb_{i1} & xb_{i2} & \cdots & xb_{in} \\ & & 0 & \end{bmatrix} = (0) \quad \text{for each } x \text{ in } L.$$

Then $xb_{ij} = 0$ for each $x$ in $L$ so $b_{ij}$ is in $r(L)$. Thus $B \in M_n(r(L))$ so that $r(M_n(L)) = M_n(r(L))$. Dually, $l(r(M_n(L))) = M_n(l(r(L)))$. Thus $l(r(L)) \neq L$ implies $l(r(M_n(L))) \neq M_n(L)$. This proves $l(r(L)) = L$ for each left ideal $L$ of $A$. Similarly, $r(l(R)) = R$ for each right ideal $R$ of $A$, so $A$ is quasi-Frobenius.

Lemma 2.2 has also been proved by the author for Frobenius algebras but the proof is a bit tedious and is unnecessary for the following development so it is not included here.

The semigroup algebras which are encountered are of a more general form than $M_n(A)$. Let $A$ be an algebra and let $M(A;m,n,;P)$ be the algebra obtained from the set of mxn matrices with the usual addition but with multiplication defined as $B \cdot C = BPC$, where $P$ is an nxm matrix over $A$.

Definition 2.1: $M(A;m,n;P)$ is called the Munn algebra over A with sandwich matrix P. [2, p. 162]

In general, if an algebra $A$ has an identity e, then an element a is said to be nonsingular if there is an element b in $A$ such that $ab = e$; then also $ba = e$ [1, p. 13]

Munn proved [7, Theorem 4.7, p. 7] that $M(A;m,n;P)$ is semisimple if and only if $P$ is nonsingular in $M_n(A)$ and $A$ is semisimple. In the process of proving this he actually proves [2, Lemma 5.18, p. 163]: $M(A;m,n;P)$ has an identity if and only if $A$ has an identity and $P$ is nonsingular. (Then $P^{-1}$ is the identity of $M(A;m,n;P)$.)

The result $M(A;n,n;P) \cong M(A;n,n;U_n) \cong M_n(A)$ if $P$ is nonsingular and $A$ has an identity e, is established and used in the proof of this theorem [7, 4.8, p.8], where $U_n$ is the nxn matrix with e's on the main diagonal and zeros elsewhere.

These relationships can be used to prove

Lemma 2.3: $M(A;m,n;P)$ is quasi-Frobenius if and only if i) $A$ is quasi-Frobenius and ii) $P$ is nonsingular in $M_n(A)$.

Proof: If $M(A;m,n;P)$ is quasi-Frobenius then it has an identity, namely $P^{-1}$. By the remarks above $M(A;n,n;P) \cong M_n(A)$. Then $M_n(A)$ is quasi-Frobenius so $A$ is quasi-Frobenius by lemma 2.2. Conversely, if $A$ is quasi-Frobenius, then $M_n(A)$ is quasi-Frobenius by lemma 2.2. If $P$ is nonsingular, then $M(A;n,n;P) \cong M_n(A)$ is quasi-Frobenius.

Conditions under which $P$ is nonsingular are given in [7] and in [2].

The semigroups which are of interest here are constructed as follows. Let $G$ be a group and $G^0 = G \cup (0)$. $G^0$ is a semigroup if $g \cdot 0 = 0 \cdot g = 0$, $0^2 = 0$, and other products are those in $G$. $G^0$ is called a group with zero. Let $m$ and $n$ be positive integers, and let $P$ be an $n \times m$ matrix with entries from $G^0$. Let $M^0(G;m,n;P)$ be the set of all $m \times n$ matrices with entries from $G^0$ with at most one nonzero entry. $(a;i,q)$ will represent the element of $M^0(G;m,n;P)$ with $a$ as the entry in the $(i,q)$ position and zeros elsewhere, $a \in G^0$. Identify the elements $(0;i,q)$ with $0$ for all $i$ and $q$. Let $P = (p_{qi})$. Then $M^0(G;m,n;P)$ is a semigroup if products are defined as $(a;i,q)(b;j,r) = (ap_{qj}b;i,r)$.

Definition 2.2: $M^0(G;m,n;P)$ is the Rees matrix semigroup over $G^0$ with sandwich matrix $P$. [2, p.87]

The same construction can be carried out with $G$ rather than $G^0$ and then the semigroup is denoted by $M(G;m,n;P)$.

The following things are known about such semigroups.

$M^0(G;m,n;P)$ is regular if and only if each row and each column of $P$ contains a nonzero entry [2, Lemma 3.1, p. 89].

A finite semigroup is $0$-simple (simple) if and only if it is isomorphic with a regular Rees matrix semigroup over a group with zero (without zero).

Thus, whenever a 0-simple semigroup arises, one may assume that it has a Rees matrix representation. This will be done.

It is clear by the construction of $M^0(G;m,n;P)$ that its contracted semigroup algebra over a field F is $M(F(G);m,n;P)$, a Munn algebra over a group algebra. Then as a corollary to lemma 2.3, one has:

<u>Corollary 1</u>: A 0-simple (simple) semigroup S has a quasi-Frobenius algebra over a field F if and only if $S \cong M^0(G;n,n;P)$ $(S \cong M(G;n,n;P))$ with P nonsingular in $M_n(F(G))$.

The corollary follows since group algebras are quasi-Frobenius.

<u>Definition 2.3</u>: A 0-simple semigroup S will be called <u>c-nonsingular</u> if $S \cong M^0(G;n,n;P)$, where P is nonsingular in $M_n(F(G))$ and $c(F) = c$. [7, p. 9]

A corresponding definition is given for simple semigroups.

Munn's main result [7, Theorem 6.4, p. 10] is:

<u>Theorem</u>: Let F(S) be the algebra of S over a field F with $c(F)=c$. Then F(S) is semisimple if and only if each of the principal factors of S is a c-nonsingular semigroup, and c, if nonzero, does not divide the orders of any of the corresponding groups.

This can be extended.

<u>Theorem 2.1</u>: Let $c(F) = c$. If a semigroup S has a principal series in which each factor is c-nonsingular over F then F(S) is quasi-Frobenius.

<u>Proof</u>: Let $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ be a principal series for S, with $S_{r+1} = (0)$ if S has a zero and $S_{r+1}$ empty otherwise. Assume $S_{r+1} = (0)$ in the proof for it is the same when $S_{r+1}$ is empty. If $r = 0$, then $S = S_0 \supset S_1 = (0)$ and $S = S_0$ is 0-simple and c-nonsingular

over $F$ so $F(S)$ is quasi-Frobenius by corollary 1. Suppose $r \geq 1$ and assume that the result is true for all semigroups of this type with a principal series of length less than $r+1$. Then $F(S_1)$ is quasi-Frobenius and $F(S)/F(S_1) \cong F(S/S_1)$ is quasi-Frobenius since $S/S_1$ is 0-simple and c-nonsingular. Then $F(S)$ is quasi-Frobenius by lemma 2.1.

The converse of this theorem — $F(S)$ quasi-Frobenius implies each principal factor of $S$ is c-nonsingular — is false as examples and further investigation will show. The following remark can be made, however.

<u>Corollary 1</u>: Let $S$ have a principal series $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1} = (0)$. If $F(S_i)$ has an identity for $i = 0,1,\ldots,r$, then $S_i/S_{i+1}$ is c-nonsingular for $i = 0,1,\ldots,r$, and $F(S)$ is quasi-Frobenius.

<u>Proof</u>: If $F(S_i)$ has an identity then $F(S_i)/F(S_{i+1}) \cong F(S_i/S_{i+1})$ has an identity. If $S_i/S_{i+1}$ is null then $S_i^2 \subseteq S_{i+1}$ so $F(S_i/S_{i+1})$ could not have an identity. Then $S_i/S_{i+1}$ is 0-simple (or simple). Suppose $S_i/S_{i+1} \cong M^0(G;m,n,:P)$. Then $F(S_i/S_{i+1}) \cong M(F(G);m,n;P)$ and this Munn algebra has an identity if and only if $m = n$ and $P$ is nonsingular. Thus $S_i/S_{i+1}$ is c-nonsingular for each $i$. Then $F(S)$ is quasi-Frobenius by theorem 2.1.

Another corollary to theorem 2.1 can be stated after these definitions.

<u>Definition 2.4</u>: A semigroup $S$ is a <u>Brandt</u> <u>semigroup</u> if $S \cong M^0(G;n,n;U_n)$. [2, Theorem 3.9, p. 102]

<u>Definition 2.5</u>: A semigroup $S$ is called an <u>inverse</u> <u>semigroup</u> if each element has a unique inverse. [2, p. 28]

In [7], Munn proves that each principal factor of a finite inverse semigroup is a Brandt semigroup.

Corollary 2: A finite inverse semigroup S has a quasi-Frobenius algebra over any field F.

Proof: Let $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ be a principal series for S. Then $S_i/S_{i+1}$ is a Brandt semigroup for $i = 0,1,\ldots,r$. Then $S_i/S_{i+1}$ is isomorphic to $M^0(G;n,n;U_n)$ by definition 2.4 ($M^0(G;n,n;U_n)$ depends on i.). But the identity matrix $U_n$ is nonsingular over F(G) for each field F so $S_i/S_{i+1}$ is a c-nonsingular semigroup for each $i = 0,1,\ldots,r$. Then by the theorem F(S) is quasi-Frobenius.

By the same argument, any semigroup for which each principal factor is a group with zero (or a group) has a quasi-Frobenius algebra over every field. Section 3 deals with a generalization of such a semigroup. There, all non-null principal factors are assumed to be groups with or without a zero.

## 3. A reduction

The main result of this section is:

Theorem: Let S be a commutative semigroup. If F(S) is quasi-Frobenius then S has a principal series with components $I_k/I_{k+1}$, $k = 0,1,\ldots,q$, such that $F(I_k/I_{k+1})$ is quasi-Frobenius for each $k = 0,1,\ldots,q$. Conversely, if S has such a principal series and $I_o = S$, then F(S) is quasi-Frobenius.

This theorem will reduce the study of commutative semigroups with quasi-Frobenius algebras to the study of semigroups of type C. This reduction is important because the structure of a semigroup S of type C is influenced by a certain subgroup G of S. Then the fact that F(G) is quasi-Frobenius over arbitrary fields can be used in studying F(S). The problem is more complicated than might be expected as a semigroup, unlike a group, can have a quasi-Frobenius algebra over a field of one characteristic but fail to have such an algebra over a field of a different characteristic. The other difficulty is that, as opposed to semisimple algebras, subalgebras and difference algebras of quasi-Frobenius algebras need not be quasi-Frobenius.

The results here are not generalizations of those in the preceding section. Here semigroups with null principal factors are permitted but the non-null factors are required to be groups (with or without zero). The prototype for this case is the commutative semigroup.

A few remarks can be made about an arbitrary finite semigroup S for which F(S) has an identity over some field F.

Lemma 3.1: If the semigroup algebra $F(S)$ has an identity $e$, then $s \in Ss \cap sS$ for each $s$ in $S$.

Proof: Consider $S$ as embedded in $F(S)$. Then in $F(S)$, $es = se = s$ for each $s$ in $S$. If $e = \sum_{t \in S'} a_t t$, $a_t \in F$, then $es = \sum_{t \in S'} a_t ts = s$ implies $\sum_{t \in S'} a_t ts - s = 0$ so, by the linear independence of the elements of $S'$ over $F$, $ts = s$ for some $t$ in $S$. In like manner, there is a $t'$ in $S$ such that $st' = s$, since $se = s$. Thus $s \in Ss \cap sS$ for each $s$ in $S$.

Lemma 3.2: If $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ is a principal series for $S$ and $F(S)$ has an identity $e$, then $S/S_1$ is a c-nonsingular semigroup, $c(F) = c$.

Proof: $F(S)$ has an identity implies $F(S/S_1) \cong F(S)/F(S_1)$ has an identity. A principal factor $S/S_1$ is null, 0-simple or simple (see the preliminaries). If $S/S_1$ is null $S^2 \subseteq S_1$. If $e = \sum_{t \in S'} a_t t$ , then

$e = \sum_{t \in S-S_1} a_t t + \sum_{t \in S_1'} a_t t$ and, for $s \in S-S_1$,

$s = es = \sum_{t \in S-S_1} a_t ts + \sum_{t \in S_1'} a_t ts$ is in $F(S_1)$ as $F(S_1)$ is an ideal of $F(S)$ and $S^2 \subseteq S_1$. This contradicts $s \in S-S_1$. Then $S/S_1$ is 0-simple (or simple). Then, as observed in the preceding section, $F(S/S_1)$ has an identity if and only if $S/S_1$ is a c-nonsingular semigroup.

In particular, these lemmas apply when $F(S)$ is quasi-Frobenius.

Lemma 3.3: If $S$ is a commutative 0-simple (simple) semigroup, then $S$ is an abelian group with zero (without zero).

<u>Proof</u>: If S is a 0-simple (simple) semigroup, then it has a Rees matrix representation, say $S \cong M^{0}(G;m,n;P)$. The elements of such a semigroup can be written as $(a;i,q)$ as in section 2. Then $(a;i,q)(b;j,r) = (ap_{qj}b;i,r)$ and $(b;j,r)(a;i,q) = (bp_{ri}a;j,q)$, where $p_{qj}$ is the $(q,j)$ entry of P and $p_{ri}$ is the $(r,i)$ entry. If S is commutative, then $m = n = 1$ and G must be abelian. Thus, $S \cong M^{0}(G;1,1;(p))$ ($p \neq 0$ since S is non-null). Then $M^{0}(G;1,1,(p)) \cong G^{0}$ under the mapping $(ap^{-1};1,1) \longrightarrow a$, a in $G^{0}$, since $(p^{-1};1,1)$ is the identity of $M^{0}(G;1,1;(p))$ and $(ap^{-1};1,1)(bp^{-1};1,1) = (ap^{-1}pbp^{-1};1,1) = ((ab)p^{-1};1,1)$. Thus S is a group with zero.

If S is commutative with a principal series $S = S_{0} \supset S_{1} \supset \cdots \supset S_{r+1}$ (Here $S_{r+1}$ will always be assumed to be zero if S has a zero and empty otherwise.) then $S_{i}/S_{i+1}$, $i = 0,1,\ldots,r$, is either null, an abelian group with zero, or an abelian group. This last case occurs only if $i = r$ and $S_{r+1}$ is empty. Then $S_{r}$ is the kernel of S (see the preliminaries). If S is commutative and F(S) is quasi-Frobenius then F(S) has an identity and lemma 3.2 implies $S/S_{1}$ is a group with zero (or without zero).

It is convenient to give certain classes of semigroups a special notation.

<u>Definition 3.1</u>: A semigroup S is of <u>type C'</u> if either S is a group or S satisfies the following conditions:

(1) S has a zero,

(2) S has a principal series $S = S_{0} \supset S_{1} \supset \cdots \supset S_{r+1} = (0)$, with $S - S_{1}$ a group and $S_{1}$ nilpotent,

(3) $s \in Ss \cap sS$ for each $s \in S$.

S is said to be of <u>type C</u> if S is a commutative semigroup of type C'.

S is said to be of <u>type D</u> if S is of type C and $S_i - S_{i+1}$ contains precisely one element, $i = 0,1,\ldots,r$. (I.e., S has precisely one element in each J-class.)

<u>Lemma 3.4</u>: If S is of type C' and R is a congruence on S, then S/R is of type C'. If S is of type C, then S/R is of type C.

<u>Proof</u>: Let $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ be a principal series for S. Then $S/R = S_0/R \supseteq S_1/R \supseteq \cdots \supseteq S_{r+1}/R$ is a principal series for S/R. $S/R - S_1/R = (S-S_1)/R$ is a group since the homomorphic image of a group is a group. If $S_1^n = (0)$, then $(S_1/R)^n = (S_1^n/R) = (0)$ so $S_1/R$ is nilpotent.

To see that S/R has property (3), let $s \in S$ and write sR for the image of s under R. $s \in Ss \cap sS$ since S is of type C'. Let $s = ts = st'$, t and t' in S. Then $sR = (tR)(sR) = (sR)(t'R) \in (S/R)(sR) \cap (sR)(S/R)$ for each sR in S/R. Thus S/R is of type C'.

A homomorphic image of a commutative semigroup is clearly commutative so if S is of type C, then S/R is of type C.

In what follows, let S be commutative unless otherwise specified. Let $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ be an arbitrary principal series for S with $S_{r+1} = (0)$ or $S_{r+1}$ empty. Q will denote the subset of the set $(0,1,\ldots,r)$ such that $S_i/S_{i+1}$ is non-null if and only if i is in Q. Suppose $Q = (i_0, i_1, \ldots, i_q)$, $i_k < i_n$ if $k < n$. Let $I_k = S_{i_k}$, $k = 0,1,\ldots,q$, and, for convenience, let $I_{q+1} = S_{r+1}$. Then $S \supseteq I_0 \supset I_1 \supset \cdots \supset I_q \supset I_{q+1}$ is an ideal series for S such that $I_k/I_{k+1}$ is a semigroup which satisfies all the conditions for a semigroup of type C except perhaps (3).

<u>Definition 3.2</u>: The factors $I_k/I_{k+1}$, $k = 0,1,\ldots,q$, are called the <u>components of S relative to the given principal series</u>.

These components depend upon the principal series chosen for S. Lemma 3.5 shows that the integer q is an invariant for S, however.

<u>Lemma 3.5</u>: The number of components in a principal series for S is the same for each principal series of S.

<u>Proof</u>: In the preliminaries a theorem was stated which said that given any two principal series for S, there is a one-to-one correspondence between their factors so that corresponding factors are isomorphic. Thus, any two principal series for S have the same number of non-null factors, say q+1. Then each principal series for S has q+1 components.

The following example illustrates the fact that the components for different principal series need not be isomorphic. Let S have the multiplication table:

| | e | a | b | 0 |
|---|---|---|---|---|
| e | e | a | b | 0 |
| a | a | a | 0 | 0 |
| b | b | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

S has two principal series:

(1) $S \supset (a,b,0) \supset (b,0) \supset (0)$, and

(2) $S \supset (a,b,0) \supset (a,0) \supset (0)$.

In (1), let $H_0 = S$, $H_1 = (a,b,0)$, $H_2 = (0)$. Then the components are $H_0/H_1 = (e,z)$ (using the identification described in the preliminaries), and $H_1/H_2 \cong H_1 = (a,b,0)$.

In (2) let $I_0 = S$, $I_1 = (a,0)$, $I_2 = (0)$. The components for this series are $I_0/I_1 = (e,b,z)$ and $I_1/I_2 = I_1 \cong (a,0)$.

(1) has no component isomorphic with $(e,b,z) = I_0/I_1$ of (2) so the components of (1) and (2) are not isomorphic. Note also that S satisfies property (3), $s \in sS$ for each $s \in S$, since S has an identity e. However, $H_1/H_2 \cong H_1 = (a,b,0)$ is a factor for (1) and $H_1$ does not satisfy (3) since $b \notin bH_1 = (0)$. Both components of (2) satisfy (3) as semigroups, and are therefore of type C. It is surprising that if a commutative semigroup S satisfies condition (3), then S has a principal series the components, $I_k/I_{k+1}$, of which also satisfy it. This series can be used to prove that if F(S) is quasi-Frobenius (Frobenius) then $F(I_k/I_{k+1})$ is quasi-Frobenius (Frobenius) for each k, and conversely. This reduces the question to investigating semigroups of type C, since the components $I_k/I_{k+1}$ for this series are of type C. In the next section it is found that semigroups of type C have quasi-Frobenius algebras only if they have Frobenius algebras.

Theorem 3.1: If S is a commutative semigroup having property (3), $s \in Ss$ for each s in S, then S has a principal series for which each component has this property.

Proof: The proof is by induction on q, where q+1 is the number of components of S. Note that property (3) is preserved by homomorphisms by lemma 3.4. Observe also that property (3) implies that in any principal series $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ for S, $S_0/S_1$ is non-null, for otherwise if $s \in S_0 - S_1$, then $s \notin sS \subseteq S_1$ since $S^2 \subseteq S_1$.

If q = 0, then $S = I_0$ is the only component of S so the result is true. Assume $q \geq 1$ and suppose the theorem is true for commutative semigroups with less than $q+1$ components. If $S=S_o\supset S_1\supset\cdots\supset S_{r+1}$ is a principal series for $S$ (as before $S_{r+1}=(0)$ or is empty), let $p$ be the maximal non-negative integer such that $S_p/S_{p+1}$ is non-null. Among all the principal series for $S$, choose one in which the corresponding $p$ is maximal. (I.e., the last non-null factor is "as close to the bottom" of the series as possible.) Fix such a series for $S$. Suppose $S$ has $q+1$ conponents, $q\geq 1$. $S/S_p$ has property (3) so it has a principal series $S/S_p=S_o'/S_p\supset S_1'/S_p\supset\cdots\supset S_p'/S_p=S_p/S_p$, with components $(I_k/S_p)/(I_{k+1}/S_p)$, $k=0,1,\ldots,(q-1)$, each having property (3) by the induction hypothesis applied to $S/S_p$. Then $(I_k/S_p)/(I_{k+1}/S_p)\cong I_k/I_{k+1}$, and a principal series for $S$ can be chosen with the $I_k/I_{k+1}$, $k=0,1,\ldots,(q-1)$, as its components, namely $S=S_o'\supset S_1'\supset\cdots\supset S_p'=S_p\supset S_{p+1}\supset\cdots\supset S_{r+1}$. This series still satisfies the maximality condition on $p$. Thus, all that needs to be done is to prove that the last component, $S_p/S_{r+1}\cong S_p$, has property (3). If $p=r$, this is true, for then $S_p=S_r$ is a group (or perhaps a group with zero) since $S_p/S_{p+1}$ is non-null. Thus, assume $p<r$. $S_p-S_{p+1}$ is a group, say with identity $e$, so $s=es\in Ss$ for each $s\in S_p-S_{p+1}$. If $S_p$ does not satisfy property (3), let $k$ be the minimum positive integer, $k>p$, such that there is an $a\in S_k-S_{k+1}$ with $a\notin aS_p$. $S_k/S_{k+1}$ is a null factor by the choice of $p$. Consider $S_p\supset\cdots\supset S_k\supset\cdots\supset S_{r+1}$. $S_k-S_{k+1}=J_a$ is the J-class containing $a$, by a remark in the preliminaries. That $S_p\dot{-}J_a=S_p-(S_k-S_{k+1})$ is an ideal of $S$ will be proved. Then a new principal series can be formed for $S$ which contradicts the maximality of p. In fact, $S_m - J_a = S_m - (S_k - S_{k+1})$, $p \leq m < k$, is an ideal of S. For let $b \in S_m - (S_k - S_{k+1})$ and suppose there is an s in S such that $bs \in J_a = (S_k - S_{k+1})$. If $b \in S_{k+1}$ then $bs \in S_{k+1} \subseteq S_m - (S_k - S_{k+1})$ as desired, so assume $b \in S_m - S_k$. $b \in S_m - S_k$ and $m < k$ imply $b \in bS_p$ by the minimality of k. Let $c \in S_p$ such that $b = bc$. $bs \in J_a = S_k - S_{k+1}$ means bs and a are in the same J-class. But S is commutative and has property (3) so bs and a in the same J-class means $(bs)S = aS$. Since $a \in aS$ there is an s' in S such that $(bs)s' = a$. But $b = bc$ and $bss' = a$ imply $a = (bs)s' = (bc)ss' = (bss')c = ac$ with $c \in S_p$, which contradicts the assumption that $a \notin aS_p$. Then $sb \in S_m - J_a = S_m - (S_k - S_{k+1})$ for each s in S and $S_m - (S_k - S_{k+1})$ is an ideal of S for $p \leq m < k$. Let $R_m = S_m - (S_k - S_{k+1})$, $p \leq m < k$. Form the ideal series $S = S'_0 \supset S'_1 \supset \cdots \supset S'_{p-1} \supset S'_p = S_p \supset R_p \supset \cdots \supset R_{k-1} \supset S_{r+1}$ for S. Then $S'_p - R_p = S_p - (S_p - (S_k - S_{k+1})) = S_k - S_{k+1}$ is a J-class of S since $S_k / S_{k+1}$ is a principal factor of S. Also $S'_p / R_p \cong S_k / S_{k+1}$ is null. $R_m - R_{m+1} = (S_m - (S_k - S_{k+1})) - (S_{m+1} - (S_k - S_{k+1})) = S_m - S_{m+1}$ is also a J-class of S for $m = p, \ldots, (k-1)$, so all factors of this series except perhaps $R_{k-1}/S_{r+1}$ are principal factors (see the preliminaries). Note that $R_p - R_{p+1} = S_p - S_{p+1}$ so $R_p / R_{p+1} \cong S_p / S_{p+1}$ is non-null and is the (p+1)st factor of the series. Then the maximality of p is contradicted by refining this series to a principal series for S. Thus $S_p$ must have had property (3). Then $S = S'_0 \supset S'_1 \supset \cdots \supset S'_p = S_p \supset S_{p+1} \supset \cdots \supset S_{r+1}$ is a principal series for S with components which have property (3).

<u>Corollary 1</u>: A commutative semigroup with property (3) has a principal series for which each component is a semigroup of type C.

Semigroups of type C' are examples of what Ljapin [6, p. 249] calls <u>semigroups with separating group part</u>. Such a semigroup is one which has an identity and in which the product of two elements which are not two-sidedly invertible is not two-sidedly invertible. (An element is two-sidedly invertible in S if it is a right and a left divisor of each element of S [6, p. 218].) That a semigroup of type C' has an identity can be proved as follows:

<u>Lemma 3.6</u>: Let S be a semigroup of type C' with corresponding group G. Then the identity of G is an identity for S.

<u>Proof</u>: This is clear if S is a group. The proof is by induction on r, where $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ is a principal series for S with $S-S_1=G$. The result is clear if $r = 0$. If $r = 1$, $S = S_0 \supset S_1 \supset S_2 = (0)$ and $S_1^2 \subseteq S_2 = (0)$ as $S_1$ is nilpotent. Then since S has property (3), if $a \in S_1-S_2$ there is a $g \in S_0-S_1$ such that $ga = a$. If e is the identity of G, $ea = e(ga) = (eg)a = ga = a$. In like manner $ae = a$. Assume $r > 1$, and suppose the lemma is true for all semigroups of type C' with principal series of length less than r+1. Homomorphic images of semigroups of type C' are of type C' by lemma 3.4 so, by the induction hypothesis, e is the identity for $S/S_r = (S-S_r) \cup z$ with the identification described in the preliminaries. I.e., if $s \in S-S_r$, then $es = se = s$. Let $a \in S_r$, $a \neq 0$. Then property (3) in S implies the existence of $s \in S$ such that $sa = a$. $s \in S-S_r$, for otherwise $a = sa \in S_{r+1} = (0)$. Then $ea = e(sa) = (es)a = sa = a$. In like manner $ae = a$. Thus e is the identity of S.

Theorem 3.1 and lemma 3.6 can be used to prove the reduction theorem stated at the beginning of the section.

Theorem 3.2: Let S be a commutative semigroup. If F(S) is quasi-Frobenius then S has a principal series with components $I_k/I_{k+1}$, $k=0,1,\ldots,q$, such that $F(I_k/I_{k+1})$ is quasi-Frobenius for each $k=0,1,\ldots,q$. Conversely, if S has such a principal series and $I_o = S$, then F(S) is quasi-Frobenius.

Proof: Suppose S has a principal series with components $I_k/I_{k+1}$, $k=0,1,\ldots,q$, and with $I_o=S$. Suppose also that $F(I_k/I_{k+1})$ is quasi-Frobenius for each k. Then F(S) is quasi-Frobenius by induction on q. If $q=0$, then $S=I_o \supset I_1$ and $I_o/I_1 \cong I_o = S$ is the only component so $F(S)=F(I_o)$ is a quasi-Frobenius algebra. Suppose the result is true for all such semigroups with less than q+1 components. Assume $q>0$. Then $F(S/I_q) \cong F(S)/F(I_q)$ is quasi-Frobenius since $S/I_q = I_o/I_q \supset \cdots \supset I_{q-1}/I_q \supset I_q/I_q$ is an ideal series for $S/I_q$ and $(I_k/I_q)/(I_{k+1}/I_q) \cong I_k/I_{k+1}$, $k=0,1,\ldots,(q-1)$, are components of $S/I_q$ corresponding to this series. $F(I_k/I_{k+1})$ is quasi-Frobenius, so by the induction hypotheses $F(S/I_q)$ is quasi-Frobenius. $F(I_q)$ is also quasi-Frobenius. Then lemma 2.1 together with $F(S/I_q) \cong F(S)/F(I_q)$ and $F(I_q)$ being quasi-Frobenius imply F(S) is quasi-Frobenius.

Conversely, suppose F(S) is quasi-Frobenius. Then S has property (3) by lemma 3.1 so it has a principal series with components $I_k/I_{k+1}$, $k=0,1,\ldots,q$, of type C by theorem 3.1. Also $I_o=S$ by lemma 3.2. Lemma 3.6 implies the existence of an identity for each component. The proof that $F(I_k/I_{k+1})$ is quasi-Frobenius is also by induction on q. If $q=0$, then $S = I_o \supset I_1$ implies $F(S) \cong F(I_o/I_1)$ is quasi-Frobenius, as $I_1 = (0)$ or $I_1$ is empty. Thus, assume $q \geq 1$ and suppose the result is true for all non-negative integers less than q. Let f be the identity of $F(I_q) \cong F(I_q/I_{q+1})$, and let e be the identity of $F(S)$. Then $F(S) \cong F(S)(e-f)+F(S)f=F(S)(e-f)+F(I_q)$ as an algebra direct sum, since $F(S)f=F(I_q)$ (because $F(I_q)$ is an ideal of $F(S)$ with identity f.). Then $F(I_q)$ and $F(S)(e-f)$ are direct summands of the quasi-Frobenius algebra $F(S)$ so they are quasi-Frobenius as observed in the preliminaries. But $F(S/I_q) \cong F(S)/F(I_q) \cong F(S)(e-f)$ quasi-Frobenius implies, by the induction hypothesis applied to $S/I_q$, that the components $(I_k/I_q)/(I_{k+1}/I_q)$ of $S/I_q$ have quasi-Frobenius algebras over F, $k=0,1,\ldots,(q-1)$. Thus $F(I_k/I_{k+1})$ is quasi-Frobenius for $k=0,1,\ldots,q$.

Precisely the same proof can be used for Frobenius algebras.

Theorem 3.2 is the central theorem of this development. All the results of sections 4 and 5 are devoted to finding conditions under which a semigroup of type C has a quasi-Frobenius algebra. These results should be viewed with this theorem in mind.

## 4. Semigroups of types C, C', and D and their algebras.

Theorem 3.2 reduces the problem of characterizing commutative semigroups with quasi-Frobenius algebras to that of describing semigroups of type C which have such algebras. Let S be a semigroup of type C with J-classes $J_i$, $i = 0,1,\ldots,(r+1)$. In this section it is found that the set $(J_i: i = 0,1,\ldots,(r+1))$ forms a semigroup T of type D, relative to the products of the complexes $J_i$ of S. Call T the J-class semigroup corresponding to S and let G be the group corresponding to S.

The main theorems of this section are:

Theorem: If S is of type C and F(S) is quasi-Frobenius, then F(S) is Frobenius.

Theorem: Let S be of type C with group G and J-class semigroup T. If $(c(F),o(G)) = 1$ and F(S) is quasi-Frobenius, then F(T) is quasi-Frobenius.

This theorem indicates that semigroups of type D should be studied. The following result is useful.

Theorem: Let T be a semigroup of type D with a principal series $T = T_0 \supset T_1 \supset \cdots \supset T_{r+1} = (0)$. Then F(T) is quasi-Frobenius if and only if $r(F(T_1)) = F(T_r)$.

The goals of the section then, are first, to reduce the "type C problem" to the investigation of semigroups of type D, and second, to find semigroups of type D which have quasi-Frobenius algebras. In section 5 the process is reversed. There a method of constructing some semigroups of type C from arbitrary abelian groups and arbitrary semigroups of type D is given.

If S is a semigroup of type C' with group G, then F(G) can be considered a subalgebra of F(S). This will be done. In what follows S is a semigroup of type C' (at least) with a principal series $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$, $S_{r+1} = (0)$ if S has a zero and $S_{r+1}$ is empty otherwise, $S - S_1 = G$, the group of S, and $S_1$ is nilpotent. Let $J_i$, $i = 0,1,\ldots,(r+1)$, be the J-classes of S, $J_i = S_i - S_{i+1}$, $i = 0,1,\ldots,r$, $J_{r+1} = (0)$ or is empty. The structure of S is greatly influenced by G.

<u>Lemma 4.1</u>: Let S be a semigroup of type C. If $a,b \in J_i$, $0 \leq i \leq r+1$, and $a = sb$, then s is in G, or $a = b = 0$. Then $S_i S_j \subseteq S_{j+1}$ if $r+1 > j \geq i \geq 1$.

<u>Proof</u>: Let $a,b \in J_i$. Then $Sa = Sb$ since S is of type C, so there are $s,s' \in S$ such that $a = sb$ and $s'a = b$. Then $a = sb = (ss')a$ so $a = (ss')^k a$ for each positive integer k. If either s or s' is in $S_1$, then $ss' \in S_1$ and $S_1$ is nilpotent so $a = b = 0$. Thus either $s,s' \in G$ or $a = b = 0$.

$S_i S_j \subseteq S_{j+1}$ is clear if $i = j \geq 1$, for then $S_i^2 \subseteq S_{i+1}$ since $S_1$ is nilpotent. If $r \geq j > i \geq 1$, let $a \in S_i$, $b \in S_j$. Then $ab \in S_j$ since $S_j$ is an ideal of S. If $b \notin S_j - S_{j+1}$, then $b \in S_{j+1}$ and $ab \in S_{j+1}$ as desired. Thus suppose $b \in S_j - S_{j+1} = J_j$. Then as $J_j$ is a J-class and S is of type C, $Sab = Sb$. This implies the existence of an $s \in S$ such that $s(ab) = b$. Then $(sa)b = b$ so $(sa)^k b = b$ for each positive integer k. But $S_1$ is nilpotent so if $b \neq 0$, then $sa \in S - S_1 = G$. But $a \in J_i = S_i - S_{i+1}$, $i > 0$, so a contradiction arises. Thus $ab \in S_{j+1}$.

The following observations will be needed.

Lemma 4.2: Let $S$ be a semigroup of type C. Let $g \in G$. Then

i) $gS = S$,

ii) $ga = gb$ if and only if $a = b$,

iii) $gJ_i = J_i$, for each $i = 0,1,\ldots,r+1$,

iv) if $a,b \in J_i$, then $ga = a$ implies $gb = b$,

v) $G$ is transitive as a permutation group on the set $J_i$, $i = 0,1,..,r+1$,

vi) if $G_i = (g \in G\colon ga = a,\ a \in J_i)$, then $o(G) = (o(G_i))(o(J_i))$, $i = 0,1,\ldots,r+1$.

Proof: i) will be a consequence of iii).

ii) $ga = gb$ implies $g^{-1}(ga) = g^{-1}(gb)$ so $a = b$, since the identity of $G$ is the identity of $S$ by lemma 3.6.

iii) If $a \in J_i$ and $ga \in S_{i+1}$, then $a \in g^{-1}S_{i+1} \subseteq S_{i+1}$, a contradiction. Thus $ga \in J_i$. Then $gJ_i \subseteq J_i$. Then, by ii), $gJ_i = J_i$ since $o(J_i)$ is finite.

iv) $J_i$ is a J-class of $S$. Thus if $a,b \in J_i$ then $sa = b$ for some $s \in S$ since $Sa = Sb$ and $b \in Sb$ by property (3). Then $gb = g(sa) = s(ga) = sa = b$.

v) $G$ is a permutation group on $J_i$ by ii) and iii). Let $a,b \in J_i$. Then $aS = bS$ since $J_i$ is a J-class of $S$, so there is an $s \in S$ such that $sa = b$. But $sa = b$ implies $s \in G$ by lemma 4.1. Thus $G$ is transitive on the set $J_i$.

vi) Since $G$ is transitive on $J_i$ and an element of $G$ either fixes each $a \in J_i$ or moves every such $a$, $(o(G_i))(o(J_i)) = o(G)$ by [10, Theorem 3.2, p. 5].

In what follows, the $G_i$ will have the same meaning as above, and S will be a semigroup of type C unless otherwise specified.

Lemma 4.3: i) If $a \in J_i$ and $aJ_j \cap J_k$ is nonempty, then $aJ_j = J_k$, $0 \le i,j \le r$. ii) If $a \in J_i$, $b \in J_j$ and $ab \in J_k$, then $G_i G_j \subseteq G_k$.

Proof: i) Let $b, b' \in J_j$ and $ab \in J_k$. Then there is a $g \in G$ such that $b' = gb$ by lemma 4.2 v). Then $ab' = a(gb) = g(ab) \in J_k$ by lemma 4.2 iii). Thus $aJ_j \subseteq J_k$. Let $c', c \in J_k$, $c = ab$. By lemma 4.2 v) there is an h in G such that $hc = c'$. Then $a(hb) = h(ab) = hc = c'$ and $hb \in J_j$ by lemma 4.2 iii), so $J_k \subseteq aJ_j$. Thus $aJ_j = J_k$.

ii) Let $g \in G_i$, $h \in G_j$ and $ab \in J_k$, say $ab = c$. Then $(gh)c = (gh)(ab) = (ga)(hb) = ab = c$. Thus $G_i G_j \subseteq G_k$.

Remark: As a result of lemma 4.3, the set $T = (J_i: i = 0,1,\ldots,r+1)$ is a semigroup relative to the multiplication of the complexes $J_i$ of S. T is a homomorphic image of S via the following congruence R: Define aRb if and only if $aS = bS$, $a,b \in S$. This is just the usual equivalence relation on S which yields the J-classes. But in the commutative case it is also a congruence. For $aS = bS$ certainly implies $c(aS) = c(bS)$ or $(ca)S = (cb)S$ and $(ac)S = (bc)S$ for each c in S.

This homomorphic image T of S was approached in an around-about way because the lemmas concerning the products of the J-classes $J_i$ will be used later. T is closely related to S in some ways.

Lemma 4.4: There is a one-to-one correspondence between the principal series for S and those for T. T is of type D.

Proof: Let $S \longrightarrow S/R = T$ be the natural homomorphism with $s \longrightarrow sR$. If $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ is a principal series for S with $J_i = S_i - S_{i+1}$, then $S/R = S_0/R \supset S_1/R \supset \cdots \supset S_{r+1}/R$ is a principal series for $S/R = T$, and conversely. $S/R = T$ is of type C as the homomorphic image of a semigroup of type C is of type C by lemma 3.4. But $S_i/R - S_{i+1}/R = (S_i - S_{i+1})/R = J_i/R$ are the J-classes of $S/R = T$, and $J_i/R$ consists of precisely one element since $a, b \in J_i$ if and only of $aRb$.

Before continuing with the connection between S and T and their algebras, the relationship between F(G) and F(S) is examined. As G is a subset of S, F(G) can be considered a subalgebra of F(S). Then $F(S) = F(G) + F(S)$, as algebras, (this is not a direct sum since F(G) is not an ideal of F(S) in general). $F(S_1)$ is a nilpotent ideal of F(S) so $F(S_1)$ is in the radical of F(S). Let N be the radical of F(G). If $(c(F), o(g)) = 1$, then $N = (0)$ since F(G) is semisimple [5, Theorem 16.4.1, p. 257] so the radical of F(S) is $F(S_1)$, and $F(S/S_1) \cong F(S)/F(S_1) \cong F(G)$.

<u>Lemma 4.5</u>: The radical of F(S) equals $F(S_1)$ + (radical (F(G))).

<u>Proof</u>: Let M denote the radical of F(S), N the radical of F(G). $F(S_1)$ is an ideal of F(S) and $F(S_1) \subseteq M$. Let $h: F(S) \longrightarrow F(S)/F(S_1)$ be the natural homomorphism. Then $(M)h$ is the radical of $(F(S))h = F(S)/F(S_1)$. But $(F(S))h = (F(G) + F(S_1))h = (F(G))h$, so $(M)h = (N)h$ and thus $M = N + F(S_1)$.

Idempotents play an important role in the decomposition of algebras so the following lemmas are of use.

<u>Lemma 4.6</u>: Let S be a semigroup of type C. If f is an idempotent of F(S) then $f \in F(G)$.

Proof: Suppose $f = a+b$, $a \in F(G)$, $b \in F(S_1)$. Then $f^2 = a^2+2ab+b^2 = a+b$ and $a^2 - a = b(e-2a-b)$. But $(a^2 - a) \in F(G)$ and $b(e-2a-b) \in F(S_1)$ so $a^2 = a$ and $b(e-2a-b) = 0$ by the linear independence of the semigroup elements over F. Then $(e-2a)b = b^2$. $b \in F(S_1)$ so b is a linear combination of elements from $S_1$. If $b \neq 0$, let i be the minimum positive integer such that the expression for b contains an element of $J_i$ with a nonzero coefficient. Let $b = b_1 + b_2$, $b_1 (\neq 0) \in F(J_i)$, $b_2 \in F(S_{i+1})$. Then $(e-2a)b_1 = 0$ since $(e-2a)b_1 \in F(J_i)$ and $(e-2a)b_2 \in F(S_{i+1})$. If the characteristic of F is 2 then $b_1 = 2ab_1 = 0$ and a contradiction arises unless $b = 0$ as desired. If the characteristic of F is not 2, then $2ab_1 = 2a(2ab_1) = 4a^2b_1 = 4ab_1$, so $b_1 = 2ab_1 = 0$, a contradiction. Thus $b = 0$ and $f = a \in F(G)$.

Then, in particular, the primitive idempotents of F(G) are primitive in F(S). This can also be proved for C'-semigroups.

Lemma 4.7: Let S be a C'-semigroup. Then each primitive idempotent in F(G) is primitive in F(S).

Proof: Let f be a primitive idempotent in F(G). Suppose $f = f_1 + f_2$, $f_1f_2 = f_2f_1 = 0$, $f_i^2 = f_i$, $i = 1,2$, in F(S). Let $f_i = a_i + b_i$, $a_i \in F(G)$, $b_i \in F(S_1)$, $i = 1,2$. Then $f_i^2 = f_i$ implies $a_i^2 + a_ib_i + b_ia_i + b_i^2 = a_i + b_i$. Then by linear independence, $a_i^2 = a_i$, $i = 1,2$, since all other terms are in $F(S_1)$. But $a_1a_2 = a_2a_1 = 0$ since $f_1f_2 = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 = 0$ and $a_1a_2$ is the only term in F(G). But $f = (a_1 + a_2) + (b_1 + b_2) \in F(G)$, so $b_1 + b_2 = 0$ and $f = a_1 + a_2$, which contradicts f being primitive in F(G). Thus f is also primitive in F(S).

Thus, a decomposition of e, the identity of $F(G) \subseteq F(S)$, into pairwise orthogonal, primitive idempotents of $F(G)$ gives rise to a decomposition of $F(S)$ into a sum of principal indecomposable modules when $S$ is of type C. Then every principal indecomposable module of $F(S)$ is isomorphic to one generated by a primitive idempotent in $F(G)$.

Before continuing with this, some remarks should be made about the decomposition of an arbitrary finite dimensional associative algebra $A$ with an identity $e$ [8, p. 612]. $e$ can be written as a sum of pairwise orthogonal primitive idempotents $e_{ij}$, and suppose these idempotents are enumerated as follows:

$e = e_{1,1}+ e_{1,2}+\dots+ e_{1,f(1)}+ e_{2,1}+\dots+ e_{2,f(2)}+\dots+ e_{n,1}+\dots+ e_{n,f(n)}$, where $Ae_{k,1}\cong Ae_{k,i}$ for $i = 1,\dots,f(k)$, and $Ae_{k,i}\not\cong Ae_{m,j}$ if $k \neq m$. Let $e_k= e_{k,1}$, $k = 1,\dots,n$, be "representatives" for the n pairwise non-isomorphic principal indecomposable modules (left) $Ae_{k,i}\cong Ae_k$, $i = 1,\dots,f(k)$; $f(k)$ is the number of $A$-isomorphic copies of $Ae_k$ which appear in the decomposition,

$A=Ae_{1,1}+\dots+Ae_{1,f(1)}+Ae_{2,1}+\dots+Ae_{2,f(2)}+\dots+Ae_{n,1}+\dots+Ae_{n,f(n)}$.

(Note: These remarks can be made for right $A$-modules also.)

Let $a\longrightarrow a'$ be the homomorphism of $A\longrightarrow A/\text{Radical } A$. Then one has the following characterization of quasi-Frobenius and Frobenius algebras $A$. [8, Lemma 2, p. 614]

<u>Theorem (Nakayama)</u>: Let $A$ have an identity. $A$ is quasi-Frobenius if and only if there is a permutation $p$ of the set $(1,\dots,n)$ such that for each $k$ in $(1,\dots,n)$:

i) $e_kA$ has a unique irreducible right $A$-submodule $r_k$ and $r_k\cong e'_{p(k)}A'$ as right $A$-modules,

ii) $Ae_{p(k)}$ has a unique irreducible left $A$-submodule $l_{p(k)}$ and $l_{p(k)} \cong A'e_k'$ as left $A$-modules.

Furthermore, $A$ is Frobenius if and only if there is a permutation p which also satisfies

iii) $f(k) = f(p(k))$ for each k in $(1,\ldots,n)$.

Returning now to semigroup algebras, let S be a semigroup of type C with group G. Let F(G) play the role of $A$ in the discussion above, and let $r_k$ be the unique irreducible right F(G)-submodule of $e_kF(G)$ with $r_k \cong e'_{p(k)}F(G)'$. F(G) is Frobenius so such modules and such a permutation exist and satisfy i), ii), and iii). Recall that the radical of F(S) is $F(S_1)$ + (radical F(G)). The next lemma indicates where the irreducible F(S)-submodules of $e_kF(S)$ can be found.

<u>Lemma 4.8</u>: Let $e_k$ be a primitive idempotent of $F(G) \subseteq F(S)$. If $s \in S$ such that $e_ksr_k \neq (0)$ and $e_ksr_kS_1 = (0)$, then $e_ksr_k$ is an irreducible right F(S)-submodule of $e_kF(S)$ and $e_ksr_k \cong e'_{p(k)}F(S)'$ as right F(S)-modules.

<u>Proof</u>: $(e_ksr_k)F(S) = e_ksr_k(F(G) + F(S_1)) = e_ksr_kF(G) = e_ksr_k$ since $e_ksr_kS_1 = (0)$ and $r_kF(G) = r_k$. Thus, $e_ksr_k$ is a right F(S)-submodule of $e_kF(S)$.

$e_ksr_k$ is irreducible. For let $x \in e_ksr_k$, say $x = e_ksy$, $y \in r_k$. Then $xF(S) = e_ksy(F(G) + F(S_1)) = e_ksyF(G) = e_ksr_k$, as $y \in r_k$ and $r_k$ is an irreducible F(G)-module so $yF(G) = r_k$.

Next, $e_ksr_k \cong e'_{p(k)}F(S)'$. Note first that $e'_{p(k)}F(S)' \cong e'_{p(k)}F(G)'$ as right F(S)-modules if one defines for $u \in F(G)$ and $x = x_1 + x_2$ in

$F(S) = F(G) + F(S_1)$, $x_1 \in F(G)$, $x_2 \in F(S_1)$, $(e_{p(k)}u)'x = (e_{p(k)}ux)' = (e_{p(k)}ux_1)'$, since $F(S_1) \subseteq$ Radical of $F(S)$ so $(e'_{p(k)}u')x_2 = (\text{rad.}F(S))' = 0'$.

Let $\pi$: $r_k \longrightarrow (e_{p(k)}F(G))'$ be the $F(G)$-isomorphism insured by $F(G)$ being quasi-Frobenius. $\pi$ will be used to construct an $F(S)$-isomorphism of $e_k s r_k$ onto $(e_{p(k)}F(G))' = (e_{p(k)}F(S))'$. Define $\sigma$: $e_k s r_k \longrightarrow (e_{p(k)}F(G))'$, as right $F(S)$-modules, with $(e_k su)\sigma = u\pi$, $u \in r_k$. $\sigma$ is well-defined and one-to-one as: If $e_k su = e_k su_1$, $u, u_1 \in r_k$, then $e_k s(u-u_1) = 0$. Then $0 = (e_k s(u-u_1))\sigma = (u-u_1)\pi = u\pi - u_1\pi$ so $u = u_1$ since $\pi$ is an $F(G)$-isomorphism.

$\sigma$ is an $F(S)$-homomorphism. It is clearly F-linear. If $x = x_1 + x_2$ is in $F(S) = F(G) + F(S_1)$, then

$$
\begin{aligned}
((e_k su)x)\sigma &= ((e_k su)x_1)\sigma && (\text{since } e_k s r_k S_1 = (0) \text{ and } u \in r_k) \\
&= (ux_1)\pi = (u\pi)x_1 && (\text{since } \pi \text{ is an } F(G)\text{-homomorphism}) \\
&= (u\pi)(x_1 + x_2) && (\text{by the definition of } (e_{p(k)}F(G))' \text{ as a right } F(S)\text{-module}) \\
&= (u\pi)x = ((e_k su)\sigma)x && (\text{by the definition of } \sigma).
\end{aligned}
$$

Thus $\sigma$ is an $F(S)$-isomorphism.

The same process can be carried out for the unique irreducible $F(G)$-submodule $l_{p(k)}$ of $F(G)e_{p(k)}$.

It is reasonable to ask whether $F(S)$ quasi-Frobenius implies $F(S)$ is Frobenius. The next lemma will be used to prove that this is the case for semigroups of type C. Since the decomposition of e into mutually orthogonal, primitive idempotents of $F(G) \subseteq F(S)$ yields a corresponding decomposition for $F(S)$ into principal indecomposable modules, the condition $f(k) = f(p(k))$ in $F(G)$ forces one to ask if it holds in $F(S)$. I.e., when are two principal indecomposable modules $e_1F(S)$ and $e_2F(S)$ isomorphic; $e_i$, $i = 1,2$, primitive idempotents of $F(G)$?

Lemma 4.9: If $e_1$ and $e_2$ are primitive idempotents of $F(G) \subseteq F(S)$, then $e_1F(G) \cong e_2F(G)$ as right $F(G)$-modules if and only if $e_1F(S) \cong e_2F(S)$ as right $F(S)$-modules.

Proof: In general, two principal indecomposable modules $e_1\mathbf{A}$ and $e_2\mathbf{A}$ are isomorphic if and only if $e_1/e_1(\text{rad.}\mathbf{A}) \cong e_2/e_2(\text{rad.}\mathbf{A})$ [3, Theorem 54.11, p. 372]. Also $e_i\mathbf{A}/e_i(\text{rad.}\mathbf{A}) \cong (e_i\mathbf{A})'$ as right $\mathbf{A}$-modules, under the isomorphism $e_ia + e_i(\text{rad.}\mathbf{A}) \longrightarrow e_ia + (\text{rad.}\mathbf{A})$, $a \in \mathbf{A}$, $i = 1,2$.

In particular, $e_iF(S)/e_i(\text{rad.}F(S)) \cong (e_iF(S))' = (e_iF(G))'$, where $(e_iF(G))'$ is considered a right $F(S)$-module as in the proof of lemma 4.8. Thus $e_1F(S) \cong e_2F(S)$ if and only if $(e_1F(G))' \cong (e_2F(G))'$ as right $F(S)$-modules by the remarks above. But if the operators are restricted from $F(S)$ to $F(G)$, then $(e_1F(G))' \cong (e_2F(G))'$ as $F(S)$-modules if and only if they are isomorphic as $F(G)$-modules. Then using the result stated above, $(e_1F(G))' \cong (e_2F(G))'$ as right $F(G)$-modules if and only if $e_1F(G)$ and $e_2F(G)$ are $F(G)$-isomorphic. Thus $e_1F(S) \cong e_2F(S)$ as $F(S)$-modules if and only if $e_1F(G) \cong e_2F(G)$ as $F(G)$-modules.

The same result holds for left modules since $F(S)$ is commutative.

Theorem 4.1: If $S$ is of type $C$ and $F(S)$ is quasi-Frobenius, then $F(S)$ is Frobenius.

<u>Proof</u>: Suppose $F(S)$ is quasi-Frobenius. Let $e = e_{1,1}+\dots+e_{1,f(1)}+e_{2,1}+\dots+e_{2,f(2)}+\dots+e_{n,1}+\dots+e_{n,f(n)}$, be a decomposition of the identity $e$ of $F(S)$ into a sum of pairwise orthogonal primitive idempotents $e_{k,i}$ of $F(G)\subseteq F(S)$ (this can be done as $e\in G\subset F(G)\subseteq F(S)$), with $e_{k,1}F(G)\cong e_{k,i}F(G)$ for $i = 1,\dots,f(1)$, $k = 1,\dots,n$, and $e_{k,i}F(G)\not\cong e_{m,j}F(G)$ if $k\neq m$, as before. The $e_{k,i}$ are primitive in $F(S)$ by lemma 4.7. $e_{k,i}F(S)\cong e_{m,j}F(S)$ if and only if $e_{k,i}F(G)\cong e_{m,j}F(G)$ by lemma 4.9, so $e_{k,i}F(S)\cong e_{m,j}F(S)$ if and only if $k = m$. Let $e_k = e_{k,1}$, $k = 1,\dots,n$. As $F(G)$ is Frobenius, the permutation $p$ of Nakayama's theorem can be chosen so that $f(k) = f(p(k))$ in $F(G)$. But this same permutation can be used for $F(S)$ by the preceding remarks, so $f(k) = f(p(k))$ in $F(S)$. Since $F(S)$ is quasi-Frobenius, conditions i) and ii) of Nakayama's theorem are also satisfied, and so by that theorem $F(S)$ is Frobenius.

Continuing now with semigroups of type C, let $e_k$ be a primitive idempotent of $F(G)\subseteq F(S)$. If $s\in S$, $e_ks\neq 0$, and $e_ksS_1 = (0)$, then $e_ksr_k$ is an irreducible right $F(S)$-submodule of $e_kF(S)$ if $r_k$ is the unique irreducible $F(G)$-submodule of $e_kF(G)$, and $e_ksr_k\cong(e_{p(k)}F(S))'$ as right $F(S)$-modules, by lemma 4.8. If $(c(F),o(G)) = 1$, then $r_k = e_kF(G)$ as then $F(G)$ is semisimple.

Let $F$ be fixed and let $S$ be of type C. Define $R_k = (s\in S\colon e_ks = 0)$.

<u>Lemma 4.10</u>: $R_k$ is an ideal of $S$ for each idempotent $e_k$ of $F(G)$. Let $s\in J_i$, let $g_1,\dots,g_m$, be coset representatives of $G_i$ in $G$, and let $e_k = \sum_{g\in G} a_g g$, $a_g\in F$. Then $s\in R_k$ if and only if $\sum_{g\in g_jG_i} a_g = 0$ for each $j = 1,\dots,m$.

Proof: $R_k$ is an ideal of S. For if $s \in R_k$ and $s' \in S$ then $e_k s = 0$ implies $e_k(s's) = e_k(ss') = (e_k s)s' = 0$.

Let $s \in J_i$. $se_k = \sum_{j=1}^{m} \sum_{g \in g_j G_i} a_g sg = \sum_{j=1}^{m} (\sum_{g \in g_j G_i} a_g) sg_j$ since $sg = sg'$ if and only if $g^{-1}g' \in G_i$, by lemma 4.2. Then $se_k = 0$ if and only if $\sum_{g \in g_j G_i} a_g = 0$ for each $j = 1,\ldots,m$, by the linear independence over F.

These ideals $R_k$ can be used to get a sufficient condition for a semigroup of type C to have a quasi-Frobenius algebra. Before proving this theorem note the following. The ideal series $S=S_0 \supset S_1 \supseteq R_k \supseteq (0)$ can be refined to a principal series for S, say $S=S_0 \supset S_1 \supset \cdots \supset S_{r+1} = (0)$, with $S_{p+1} = R_k$. Then if $s \in S-S_{p+1} = S-R_k$, $e_k s \neq 0$. If $a \in S_p - S_{p+1}$, then $e_k a \neq 0$ and $e_k a S_1 = (0)$ since $aS \subseteq R_k$, as in lemma 4.8. Keeping this notation one has:

Lemma 4.11: If F(S) is quasi-Frobenius, $(c(F),o(G)) = 1$, and $a \in S_p - S_{p+1} = S_p - R_k$, then $a \in sS$ for each $s \in S-R_k$.

Proof: If $s \in (S-R_k)$, then $e_k s \neq 0$ so $e_k sF(S) \neq (0)$. Then $e_k sF(S) \supseteq e_k aF(G)$ because $e_k aF(G)$ is the unique irreducible right F(S)-submodule of $e_k F(S)$ by lemma 4.8, and $e_k F(G) = r_k$ is the unique irreducible F(G)-submodule of $e_k F(G)$ since $(c(F),o(G)) = 1$. But then $e_k a \in e_k sF(S)$ and e, the identity of G, appears as a summand in $e_k$ so a appears as a summand in $e_k a$. Then there must be an element $s' \in S$ such that $a = ss'$, by the linear independence of the nonzero elements of S over F.

This is a necessary condition for F(S) to be quasi-Frobenius when $(c(F),o(G)) = 1$. A sufficient condition follows.

<u>Theorem 4.2</u>: If F is of type C and if $F(S/R_k)$ is quasi-Frobenius for each k in $(1,\ldots,n)$, then $F(S)$ is quasi-Frobenius.

<u>Proof</u>: One need only show that $e_kF(S)$ has a <u>unique</u> irreducible right $F(S)$-submodule for each k in $(1,\ldots,n)$ and then lemma 4.8 implies the rest. Suppose $e_kxF(S)$ is an irreducible $F(S)$-submodule of $e_kF(S)$. Then $e_kxS_1 = (0)$ so $e_kxF(S) = e_kxF(G)$. Then $e_kxF(G) \subseteq F(S-R_k)$ since $e_kR_k = (0)$. Let $h: S \longrightarrow S/R_k$ be the natural homomorphism of these semigroups and extend it linearly to a homomorphism $h: F(S) \longrightarrow F(S/R_k)$, i.e., $(\sum_{s \in S'} a_s s)h = \sum_{s \in S'} a_s(sh)$. Then $(e_kxF(G))h$ is an irreducible $F(S/R_k)$-submodule of $(e_kh)F(S/R_k)$. Thus, if $e_kxF(G)$ and $e_kyF(G)$ were two irreducible right $F(S)$-submodules of $e_kF(S)$, then i) $e_kxF(G)$ and $e_kyF(G)$ are contained in $F(S-R_k)$, and ii) $(e_kxF(G))h = (e_kyF(G))h$. But ii) implies $e_kxF(G) \subseteq e_kyF(G) + F(R_k)$ which is impossible by i) unless $e_kxF(G) = e_kyF(G)$. Thus $e_kxF(G)$ is unique for each k. Then by lemma 4.8 and Nakayama's theorem $F(S)$ is quasi-Frobenius.

More information about semigroups S of type C is needed. Some insight into the structure of S and its algebra $F(S)$ is provided by studying the relationship between S and T, its J-class semigroup. The next theorem will show that if $(c(F),o(G)) = 1$, then $F(T)$ is quasi-Frobenius if $F(S)$ is. The notation is the same as before; i.e., S is a semigroup of type C with group G and $G_i$, $i = 0,1,\ldots,r+1$, are the "fixing groups" for the J-classes $J_i$, $i = 0,1,\ldots,r+1$, of S. T is the corresponding J-class semigroup.

<u>Theorem 4.3</u>: If $(c(F),o(G)) = 1$, then $F(T) \cong F(S)E$, where $E = (o(G))^{-1} \sum_{g \in G} g$, $F(T)$ is isomorphic to a direct summand of $F(S)$, and $F(T)$ is quasi-Frobenius if $F(S)$ is quasi-Frobenius.

<u>Proof</u>: Let $(G:G_i) = o(J_i) = n_i$, $(G:G_i)$ the index of $G_i$ in G, and let $o(G_i) = m_i$, $i = 0,1,\ldots,r+1$. Then $m_i n_i = o(G) = n_o$. (This is justified by lemma 4.2 vi).) Let $E_i' = \sum_{s \in J_i} s$, $i = 0,1,\ldots,r+1$.

Then $E_i' = \sum_{s \in J_i} s = m_i^{-1} \sum_{g \in G} gs_i$, $s_i$ a fixed element of $J_i$, by lemma 4.2 iv).

The proof will consist of showing that the elements $E_i = n_i^{-1} E_i'$, $i = 0,1,\ldots,r+1$, form a semigroup which is isomorphic with T, are contained in $F(S)E = F(S)E_o$, and are linearly independent over F.

Consider $s_j E_i' = m_i^{-1} \sum_{g \in G} gs_j s_i$. If $s_j s_i \in J_k$, then

(1) $s_j E_i' = m_i^{-1} \sum_{g \in G} gs_k = m_k m_i^{-1} E_k'$ since $gs_k = hs_k$ if and only if $h^{-1} g \in G_k$. Then

(2) $E_j' E_i' = \sum_{s_j \in J_j} s_j E_i' = \sum_{s_j \in J_j} m_k m_i^{-1} E_k' = n_j m_k m_i^{-1} E_k'$ by using (1) and the distributive law. Then

$E_j E_i = (n_j n_i)^{-1} E_j' E_i'$, by the definition of $E_j$ and $E_i$,

$= m_k (n_i m_i)^{-1} E_k'$, by (2),

$= m_k n_o^{-1} E_k'$, since $n_i m_i = n_o$,

$= n_k^{-1} E_k' = E_k$, by the definition of $E_k$ and since $m_k n_o^{-1} = n_k^{-1}$.

Thus, the set $(E_i : i = 0,1,\ldots,r+1)$ is a semigroup. The elements $E_i$, $i = 0,1,\ldots,r$, are linearly independent over F since they are sums of elements from distinct J-classes of S and the nonzero elements of S are linearly independent over F. That $E_i \longrightarrow J_i$, $i = 0,1,\ldots,r+1$, is an isomorphism of the semigroup onto T is clear since, again, $E_i$ is the sum of elements from $J_i$ so the $E_i$ multiply as the $J_i$ do.

By (1), $s_iE'_o = m_im_o^{-1}E'_i = m_iE'_i$ since $m_o = 1$, so $s_iE_o = E_i$. Thus $E_i \in F(S)E_o = F(S)E$ for each $i = 0,1,\ldots,r+1$. These conditions imply $F(S)E \cong F(T)$. That $F(S) \cong F(S)E + F(S)(e - E)$ is a direct sum follows from $E(e - E) = 0$.

Then $F(T) \cong F(S)E$ is quasi-Frobenius if $F(S)$ is, since $F(S)E$ is a direct summand of the quasi-Frobenius algebra $F(S)$.

The converse of theorem 4.3 is false. Example (1) in the appendix is a counterexample.

Thus semigroups T which are of type D need to be studied. The remaining part of this section will be devoted to such semigroups. That each semigroup T of type D does occur as a J-class semigroup for some semigroup of type C is obvious since T itself is of type C. Later a method will be given by which other semigroups of type C can be constructed from T.

Let T be an arbitrary semigroup of type D. Let $T = T_o \supset T_1 \supset \cdots \supset T_{r+1}$ be a principal series for T, with $T_i - T_{i+1} = (t_i)$, $i = 0,1,\ldots,r$, $T_{r+1} = (0)$ if T has a zero and $T_{r+1}$ empty otherwise. $t_o$ is the identity of T and $(t_o) = T - T_1$ is the group corresponding to T. $T_1$ is nilpotent. If F is a field, $F(T_1)$ is the radical of $F(T)$ by lemma 4.5. $t_o$ is the only nonzero idempotent of $F(T)$ by lemma 4.6, so $t_oF(T) = F(T)$ is the only principal indecomposable $F(T)$-module. Then $F(T)/F(T_1) \cong F(t_o) \cong F$ is the only irreducible $F(T)$-module.

Lemma 4.12: If $F(T)$ is quasi-Frobenius then $t_r \in tT$ for each $t (\neq 0)$ in T. (I.e., each nonzero element of T divides $t_r$.) If $t \in T$ such that

$T_1 t = (0)$, then $t \in T_r$. In addition, if T is the J-class semigroup for a semigroup S of type C then S has a J-class $J_r$, each element of which is divisible by every nonzero element of S.

Proof: If F(T) is quasi-Frobenius, Nakayama's theorem implies F(T) has a unique minimal ideal or irreducible submodule, since F(T) is itself the only principal indecomposable module. $F(T_r) = F(t_r)$ is a minimal ideal of F(T) since $Tt_r = T_r = (t_r, 0)$. If $t(\neq 0)$ is in T, then F(tT) is an ideal of F(T) so it contains a minimal ideal of F(T). Thus $F(tT) \supseteq F(t_r)$ so $t_r \in tT$ for each $t(\neq 0)$ in T.

If $t \in T$ and $T_1 t = (0)$, then $Tt = (t,0)$ and F(t) is a minimal ideal of F(T). Then $t \in T_r$.

Let S be a semigroup of type C and let T be its corresponding J-class semigroup, G its group. If $(c(F),o(G)) = 1$ and F(T) is quasi-Frobenius, let $J_i$ be the J-class of S corresponding to $t_i$ in the homomorphism of S onto T, $i = 0,1,\ldots,r+1$. Let $a \in J_r$, $s \in J_i$, $i \leq r$. In T there exists a $t_k$ such that $t_k t_i = t_r$ by the first part of the lemma. Then $J_k J_i = J_r$. But lemma 4.3 i) implies $J_k s = J_r$ so there is an $s' \in J_k$ such that $s's = a$. This can be done for every such $a \in J_r$ and for each $s \in J_i$, $i = 0,1,\ldots,r$, so each element of $J_r$ is divisible by every element (nonzero) of S.

Lemma 4.12 does not provide a sufficient condition for F(T) to be quasi-Frobenius. Example (2) of the appendix is a counterexample. It is, in some sense, related to a sufficient condition however, in that F(T) is quasi-Frobenius if and only if $r(F(T_1)) = F(T_r)$. Before proving this, consider the following observation which is used several times in proofs.

<u>Lemma 4.13</u>: If $x \in F(T)$ and $t \in T_1$, $t \neq 0$, then $xt = 0$ implies $x \in F(T_1)$.

<u>Proof</u>: If $x = at_o + x'$, $a \in F$, $x' \in F(T_1)$, then $xt_i = at_i + x't_i = 0$ implies $a = 0$ by the linear independence of the nonzero elements of T over F. For by lemma 4.1, since $x' \in F(T_1)$, $x't \in F(T_{i+1})$ but $at_i$ is in $F(T_i - T_{i+1})$.

<u>Theorem 4.4</u>: F(T) is quasi-Frobenius if and only if $r(F(T_1)) = F(T_r)$.

<u>Proof</u>: F(T) is quasi-Frobenius if and only if $F(T_r)$ is the unique minimal ideal of F(T) by lemma 4.8 and Nakayama's theorem since F(T) is the only principal indecomposable module for F(T) and $F(T_r)$ is a minimal ideal of F(T). If $x \in F(T)$, then $xF(T)$ is a minimal ideal in F(T) if and only if $x \in r(F(T_1))$. For $xF(T_1) \subsetneq xF(T)$ by lemma 4.1, so $xF(T_1) = (0)$ if $xF(T)$ is minimal. Conversely, if $xF(T_1) = (0)$, then $xF(T) = xF(t_o) + xF(T_1) = xF(t_o) = F(x)$ is a minimal ideal in F(T). Then F(T) is quasi-Frobenius if and only if $xF(T) = F(T_r)$ for each $x \in r(F(T_1))$. Thus, F(T) is quasi-Frobenius if and only if $F(x) = F(T_r)$, i.e., $x \in F(T_r)$ for each $x \in r(F(T_1))$.

It is of interest to point out the broader context into which theorem 4.4 fits. In general, if **A** is an algebra and N is its radical, then r(N) is called the left socle of **A** and is the sum of all the irreducible left ideals of **A** [3, p. 394]. If f is an idempotent in **A**, then the dual module [3, p. 394] for the left **A**-module **A**f/Nf is isomorphic to the right **A**-module fr(N) [3, Lemma 58.4, p. 395]. **A** is quasi-Frobenius if and only if, for each primitive idempotent f, the dual module of $A f/Nf$ has composition length one; i.e., $fr(N)$ has a composition series of length one as a right $A$-module [3, Theorem 58.6 (ii), p. 396]. If $A = F(T)$, then $N = F(T_1)$ and $t_o$ is the only primitive idempotent of $F(T)$ so $F(T)t_o/Nt_o \cong F(t_o) \cong F$ is the only irreducible left $F(T)$-module for $F(T)$, as observed before. Its dual is $t_o r(N) = r(F(T_1))$. But $r(F(T_1)) \supseteq F(T_r)$ so if $r(F(T_1))$ is to have composition length one, $r(F(T_1)) = F(T_r)$ must hold.

It would be useful to have the following corollary for semigroups in general but it has not been proved nor has a counterexample been found.

Corollary 1: Suppose $T_1 t = 0$ implies $t \in T_r$. If $l(r(F(L))) = F(L)$ for each principal ideal $L$ of $T$ then $F(T)$ is quasi-Frobenius.

Proof: Suppose $x \in F(T)$ and $xF(T_1) = (0)$. Then by lemma 4.13, $x \in F(T_1)$ so $F(T)x \subseteq F(T_1)$. Let $x = \sum_{i=1}^{r} a_i t_i$, $a_i \in F$. If $x \notin F(T_r)$, let $j$ be the minimum positive integer such that $a_j \neq 0$. Then $j < r$. There exists a $k$, $j < k < r$, such that $a_k \neq 0$, for otherwise $x = a_j t_j + a_r t_r$ and $(0) = T_1 x = T_1(a_j t_j)$ since $T_1 t_r = (0)$, so $T_1 t_j = (0)$, contradicting the hypothesis. Consider $F(Tt_k)$. $r(F(Tt_k)) \subseteq F(T_1)$ by lemma 4.13, so $x \in l(r(F(Tt_k))) \supsetneq F(Tt_k)$, a contradiction to the hypothesis that the annihilator condition holds for principal ideals $L = Tt_k$ of $T$. Thus $x \in F(T_r)$ if $xF(T_1) = (0)$ and, by theorem 4.4, $F(T)$ is quasi-Frobenius.

Conditions on a semigroup $S$ which imply its algebra $F(S)$ is quasi-Frobenius are difficult to find because the annihilator relations in $F(S)$ can be very complicated even for "relatively nice" semigroups.

The next corollary is an example of such a condition for T.

Corollary 2: Suppose $T_1 t = (0)$ implies $t \in T_r$. Assume also that for each $t_i \in T_1 - T_r$, there exists a $t \in T_1$ such that $t_i t \neq 0$ and $t_i t \neq t_j t$ for each $j \neq i$. Then F(T) is quasi-Frobenius.

Proof: Suppose $x = \sum_{i=1}^{r} a_i t_i \in r(F(T_i))$, $a_i \in F$. (This summation need not start with $i = 0$ by lemma 4.13.) Then for $t \in T_1$,

$0 = xt = \sum_{i=1}^{r} a_i t_i t$ implies $a_i = 0$ whenever $t_i t \neq 0$ and $t_i t \neq t_j t$ for $j \neq i$, by the linear independence of the nonzero elements of T over F. If $i < r$, there is a $t \in T_1$ such that $t_i t \neq 0$ and $t_i t \neq t_j t$ for $j \neq i$, by hypothesis, so $a_i = 0$ for $i = 1,\ldots,r-1$, and $x \in F(T_r)$. Then F(T) is quasi-Frobenius by theorem 4.4.

Theorem 4.5: $F(T/T_i)$ is quasi-Frobenius for each $i = 1,\ldots,r+1$, if and only if $T_1$ is cyclic.

Proof: If $T_1$ is cyclic, then T satisfies the hypothesis of corollary 2. For if $T_1 = (t_1^i : i = 1,\ldots,r+1)$, $t_1^{r+1} = 0$, then $T_1 t_1^k = (t_1^{k+i} : i = 1,\ldots,r+1) = (0)$ implies $k+i \geq r+1$, for each $i = 1,..,r+1$. In particular, $k+1 \geq r+1$ so $k \geq r$ and $t_1^k \in T_r = (t_1^r, 0)$. Also $t_1^i t_1^k = t_1^j t_1^k \neq 0$ if and only if $i = j$. Thus F(T) is quasi-Frobenius.

Conversely, suppose $F(T/T_i) \cong F(T)/F(T_i)$ is quasi-Frobenius for each $i = 1,\ldots,r$. Then by lemma 4.12, for each $i = 1,\ldots,r-1$, $t_i \in Tt_j$ for each $j$, $0 \leq j \leq i$. $T_1$ cyclic means each element of $T_1$ is a power of $t_1$, say. If $r = 0$ or 1 then $T_1$ is clearly cyclic. Assume $r > 1$ and suppose the theorem is true for all integers less than r. Consider $T/T_r$.

$F((T/T_r)/(T_i/T_r)) \cong F(T/T_i)$ is quasi-Frobenius for each $i = 1,\ldots,r$. Then by the induction hypothesis, $T_1/T_r$ is cyclic. Since $F(T/T_{r+1}) \cong F(T)$ is quasi-Frobenius, lemma 4.12 implies $t_r \in tT_1$ for each $t \in T - T_r$. Let $t \in T_1 - T_r$, and suppose $t' \in T_1$ such that $t_r = tt'$. Both $t$ and $t'$ are powers of the generator of $T_1/T_r = (T_1 - T_r) \cup T_r$ so $t_r$ is also, and $T_1$ is cyclic.

In the next section, some semigroups of type C will be constructed from semigroups of type D and abelian groups. This construction will lead to other semigroups of type C with quasi-Frobenius algebras.

## 5. A construction for some semigroups of type C from semigroups of type D

Several relationships between a semigroup S of type C and its group G and J-class semigroup T have already been indicated. T was found to be of type D in lemma 4.4. In each case G and T resulted from the given semigroup S. In this section a method is given by which semigroups of type C can be constructed with an arbitrary abelian group G as their group and an arbitrary semigroup of type D as their J-class semigroup. This technique will be used to extend the class of semigroups of type C which have quasi-Frobenius algebras. The idea is motivated by lemma 4.3 and the remark which follows it. There it was found that if $J_i$, $J_j$, and $J_k$ are J-classes of a semigroup S of type C with corresponding fixing groups $G_i$, $G_j$, and $G_k$, and if $J_iJ_j = J_k$, then $G_iG_j \subseteq G_k$. The procedure can be carried out in a more general context.

Let $T = (t_i\colon i = 0,1,\ldots,r+1)$ be an arbitrary finite semigroup and let G be an arbitrary finite group. For each $t_i$ in T let $H_i$ be a normal subgroup of G, $i = 0,1,\ldots,r+1$.

Definition 5.1: The set $(H_i\colon i = 0,1,\ldots,r+1)$ of subgroups of G is said to be admissible relative to T if $H_iH_j \subseteq H_k$ whenever $t_it_j = t_k$ in T. In this case the notation $[H_i] = (H_i\colon i = 0,1,\ldots,r+1)$ is used.

It will be clear from the context what T is under consideration so the collection $[H_i]$ will often be called admissible.

Note that any group G contains admissible collections of subgroups relative to each semigroup T. Namely, $H_i = G$, $i = 0,1,\dots,r+1$, or $H_i = (e)$, $i = 0,1,\dots,r+1$ where e is the identity of G.

The main theorems of the section are:

<u>Theorem</u>: Let S be the semigroup constructed from a semigroup T and an admissible collection $[H_i]$ of subgroups of G. Then S is an inverse semigroup if and only if T is an inverse semigroup.

Returning to commutative semigroups one has:

<u>Theorem</u>: Let T be a semigroup of type D and let $[H_i]$ be a collection of subgroups of the abelian group G, which is admissible relative to T. Then there exists a semigroup S of type C with T as its J-class semigroup and with the $H_i$ as the fixing groups for its J-classes.

The following results indicate the influence that the characteristic of the field has on the algebra. They also provide examples for which the converse of theorem 4.3 is true.

<u>Theorem</u>: Let T be a semigroup of type D and let G be an abelian group. Let S be the semigroup obtained from T and the admissible collection $H_0 = (e)$, $H_i = G$, $i = 1,\dots,r+1$, $r \geq 0$. Then F(S) is quasi-Frobenius if and only if $(c(F),o(G)) = 1$ and F(T) is quasi-Frobenius.

<u>Theorem</u>: If T is of type D, let $T_1$ be the maximal nilpotent ideal of T. Suppose $T_1$ is cyclic and $[H_i]$ is an admissible collection of subgroups of the abelian group G. Let S be the semigroup obtained from T and $[H_i]$. If $(c(F),o(G)) = 1$ and F(T) is quasi-Frobenius, then F(S) is quasi-Frobenius.

The construction proceeds as follows. Let $T = (t_i: i = 0,1,\dots,r+1)$ be an arbitrary finite semigroup and let G be an arbitrary finite group.

Let $(G,T) = ((g,t): g \in G,\ t \in T)$. In the set $(G,T)$ define:

$(g,t) = (g',t')$ if and only if $g = g'$ and $t = t'$, and

$(g,t)(g',t') = (gg',tt')$, for $(g,t)$ and $(g',t')$ in $(G,T)$.

With these definitions $(G,T)$ is a semigroup. It is just the "direct product" of the semigroups G and T.

Next, let $[H_i]$ be an admissible collection of subgroups of G, relative to T. Define a relation R on the set $(G,T)$ as:

$$(g,t_i)R(h,t_j) \text{ if and only if } i = j \text{ and } g \equiv h(H_i), \text{ i.e., } g \in hH_i.$$

This is a congruence relation on the semigroup $(G,T)$. It is clearly reflexive and symmetric. It is also transitive, for if $(g,t_i)R(g',t_j)$ and $(g',t_j)R(g'',t_k)$, then $k = j = i$ so $H_k = H_j = H_i$ and $g \equiv g' \equiv g''(H_i)$. Thus $(g,t_i)R(g'',t_k)$.

R is a congruence. Suppose $(g,t_i)R(g',t_i)$ and $(h,t_j) \in (G,T)$. If $t_it_j = t_k$ in T, then $H_iH_j \subseteq H_k$ so $g \equiv g'(H_i)$ certainly implies $g \equiv g'(H_k)$ and $gh \equiv g'h(H_k)$. Thus $(gh,t_it_j)R(g'h,t_it_j)$. In like manner $(hg,t_jt_i)R(hg',t_jt_i)$. This proves that R is a congruence.

Then $(G,T)$ modulo R is a semigroup; call it S. For convenience let $[gt_i] = ((h,t_i) \in (G,T): (h,t_i)R(g,t_i)) = ((h,t_i) \in (G,T): h \in gH_i)$. Then S can be considered as the set $([gt_i]: g \in G,\ t_i \in T)$ with the identification $[gt_i] = [g't_j]$ if and only if $i = j$ and $g \in g'H_i$, and with products $[gt_i][g't_j] = [ht_k]$ if $t_it_j = t_k$ and $gg' \in hH_k$.

Note that the subset $([et_i]: t_i \in T)$ is isomorphic with T. This subset will be denoted by $[T]$.

A characterization of inverse semigroups is needed. Here S is arbitrary.

**Theorem:** S is an inverse semigroup if and only if S is regular and every pair of idempotent elements commute with one another. [2, Theorem 1.17, p. 28]

Theorem 5.1: Let S be the semigroup obtained from a semigroup T and a collection $[H_i]$ of subgroups of a group G, admissible relative to T, as above. Then S is an inverse semigroup if and only if T is an inverse semigroup.

Proof: First S is regular if and only if T is regular. In [T] let $[et_i] = [t_i]$. Suppose S is regular. Then $[t_i] = [t_i][gt_j][t_i]$ for some $[gt_j]$ in S. Then necessarily $t_i = t_i t_j t_i$ in T so T is regular.

Conversely, if T is regular, then for $t_i \in T$ there exists a $t_k$ in T such that $t_i = t_i t_k t_i$. Then $[gt_i] = [gt_i][g^{-1}t_k][gt_i]$ since $gg^{-1}g = g \equiv g(H_i)$.

Next consider the idempotents of S. $[gt_i][gt_i] = [gt_i]$ if and only if $t_i^2 = t_i$ in T and $g^2 \equiv g(H_i)$. Then $g \equiv e(H_i)$ so $g \in H_i$. Then $[gt_i] = [et_i] = [t_i]$. Thus all idempotents of S occur in [T]. Then clearly, the idempotents of S commute if and only if the idempotents of T commute. Thus, by the theorem stated above, S is an inverse semigroup if and only if T is an inverse semigroup.

Since finite inverse semigroups have quasi-Frobenius algebras over arbitrary fields by corollary 2 to theorem 2.1, this theorem shows how such semigroup algebras can be constructed from ones already known.

Nakayama has proved that if B and C are algebras over a field and $A = B \times C$ is their Kronecker product, then $A$ is quasi-Frobenius (Frobenius) if and only if both B and C are quasi-Frobenius (Frobenius)[9, Theorem 14, p. 14]. Thus, F(G,T) is quasi-Frobenius if and only if

$F(T)$ is quasi-Frobenius since $F(G,T) \cong F(G) \times F(T)$ and $F(G)$ is quasi-Frobenius. Note that $(G,T)$ itself can be described in terms of $T$ and the subgroups $H_i$ of $G$ with $H_i = (e)$, $i = 0,1,\ldots,r+1$.

Returning now to the commutative case it is clear how the following theorem should be proved.

Theorem 5.2: Let $T$ be a semigroup of type D and let $[H_i]$ be a collection of subgroups of the abelian group $G$, which is admissible relative to $T$. Then there is a semigroup $S$ of type C with $T$ as its J-class semigroup and with the $H_i$ as the fixing groups for its J-classes.

Proof: $(G,T)$ is a semigroup of type C, with group $G$ and fixing groups $H_i = (e)$, $i = 0,1,\ldots,r+1$. It is clearly commutative. If $T = T_0 \supset T_1 \supset \cdots \supset T_{r+1}$ is a principal series for $T$, $T_i - T_{i+1} = (t_i)$, then $K = (G,T)$ has the principal series $K = K_0 \supset K_1 \supset \cdots \supset K_{r+1}$, with $K_i - K_{i+1} = ((g,t_i)\colon g \in G)$. As $T_1$ is nilpotent $K_1$ is nilpotent, and $K - K_1 = ((g,t_0)\colon g \in G) \cong G$ since $t_0$ is the identity of $T$. $(e,t_0)$ is the identity of $(G,T)$ so $(G,T)$ has property (3).

Since $(G,T)$ is of type C, any homomorphic image of it is of type C by lemma 3.4. Let $[H_i]$ be an admissible collection of subgroups of $G$, relative to $T$ with $H_0 = (e)$ and let $R$ be the congruence on $(G,T)$ induced by $[H_i]$ as before. Let $(G,T)/R = S$. Then $S$ is of type C, has the $H_i$ as the fixing groups of its J-classes, and has $T$ as its J-class semigroup. The J-classes of $S$ are $J_i = ([gt_i]\colon g \in G)$ since $[t_i]S = [t_j]S$ implies $[t_i][T] = [t_j][T]$, where $[T] = ([et_i]\colon t_i \in T)$, which implies $[t_i] = [t_j]$ since $T$ is of type D. Also $[gt_i]S = [ht_i]S$ since $[gt_i][g^{-1}ht_0] = [ht_i]$ and $[ht_i][h^{-1}gt_0] = [gt_i]$. The group of $S$ is $G' = ([gt_0]\colon g \in G) \cong G$

since $H_o = (e)$. $[ht_o]$ is in the fixing group of $J_i = ([gt_i]: g \in G)$ if and only if $[ht_o][gt_i] = [gt_i]$. But $[ht_o][gt_i] = [gt_i]$ if and only if $hg \equiv g(H_i)$, i.e., if and only if $h \in H_i$. Thus the fixing group of $J_i$ is $H_i' = ([ht_o]: h \in H_i) \cong H_i$.

T is clearly the J-class semigroup for S since $J_i = ([gt_i]: g \in G)$, $i = 0,1,\ldots,r+1$, and the $J_i$ multiply as the elements of T do.

Thus, in what follows the notation $H_i = G_i$, $i = 0,1,\ldots,r+1$, can be used. For if $[H_i]$ is admissible relative to T and R is the congruence induced on (G,T) by $[H_i]$, then $(G,T)/R = S$ has fixing groups $G_i = H_i$, for its J-classes $J_i$. Here $J_i \longleftrightarrow t_i$, $i = 0,1,\ldots,r+1$, is the correspondence between the J-classes of S and the elements of T.

The following lemma is needed in the proof of theorem 5.3.

Lemma 5.1: Let $S = S_o \supset S_1 \supset \cdots \supset S_{r+1}$ be a principal series for a semigroup S of type C. If $s \in S_1$, $x_1 \in F(G)$ and $x_2 \in F(S_1)$ and if $(x_1 + x_2)s = 0$, then $x_1 s = 0$ and $x_2 s = 0$.

Proof: Let $s \in S_i - S_{i+1}$. Lemma 4.1 implies that $ss' \in S_{i+1}$ for each $s' \in S_1$. But $gs \in S_i - S_{i+1}$ for each $g \in G$ so $x_1 s \in F(J_i) = F(S_i - S_{i+1})$ and $x_2 s \in F(S_{i+1})$. Then, by the linear independence of the nonzero semigroup elements over F, $x_1 s = 0$. Then $x_2 s = 0$.

Theorem 5.3: Let T be a semigroup of type D and let G be an abelian group. Let S be the semigroup obtained from T and the admissible collection $G_o = (e)$, $G_i = G$, $i = 1,\ldots,r+1$, $r > 0$. Then F(S) is quasi-Frobenius if and only if $(c(F),o(G)) = 1$ and F(T) is quasi-Frobenius.

The following lemma will also be needed in the proof.

<u>Lemma 5.2</u>: Let $S$ be the semigroup in theorem 5.3. Let $M = (\sum_{g \in G} a_g g \colon \sum_{g \in G} a_g = 0) \subset F(G) \subseteq F(S)$ and let $E = \sum_{g \in G} g$. Then in $F(S)$, $l(M) = F(S)E$.

<u>Proof</u>: First note that the set of elements $(g-e)$, $g \in G-(e) = G'$, is a basis for $M$. For if $\sum_{g \in G} a_g g \in M$, $\sum_{g \in G} a_g = 0$. Then $\sum_{g \in G} a_g g = \sum_{g \in G'} a_g g + a_e e$. But $a_e = -\sum_{g \in G'} a_g$ since $\sum_{g \in G} a_g = 0$ so $\sum_{g \in G} a_g g = \sum_{g \in G'} a_g g - (\sum_{g \in G'} a_g)\, e = \sum_{g \in G'} a_g (g-e)$. The set $(g-e \colon g \in G')$ is clearly independent over $F$.

$l(M) \supseteq F(S)E$ for if $\sum_{g \in G} a_g g \in M$, then $E \sum_{g \in G} a_g g = \sum_{g \in G} a_g Eg = (\sum_{g \in G} a_g) E = 0$ as $\sum_{g \in G} a_g = 0$ and $Eg = E$ for each $g \in G$.

$l(M) \subseteq F(S)E$. Note that if $s \in S_1$, then $s = (o(G))^{-1} Es$ since $gs = s$ for each $s \in S_1$ (since $G_i = G$ for $i > 0$), so $F(S_1) \subseteq F(S)E$. Also $F(S_1) \subseteq l(M)$ for if $s \in S_1$ and $\sum_{g \in G} a_g g \in M$, then $\sum_{g \in G} a_g gs = (\sum_{g \in G} a_g) s = 0$ since $gs = s$ for $g \in G$ and $\sum_{g \in G} a_g = 0$.

Thus suppose $x = \sum_{h \in G} b_h h \in l(M)$, $x \in F(G)$. Then $x(g-e) = 0$ for each $g \in G$. Fix $g$. Then (1) $0 = x(g-e) = \sum_{h \in G} b_h h(e-g) = \sum_{h \in G} b_h h - \sum_{h \in G} b_h hg$.

Let $hg = k$ so $kg^{-1} = h$. Then as $k$ takes on all values in $G$, $h$ does, and conversely, so $\sum_{h \in G} b_h hg = \sum_{k \in G} b_{kg^{-1}} k$. Then substituting in (1),

$$0 = \sum_{h \in G} b_h h - \sum_{k \in G} b_{kg^{-1}} k = \sum_{h \in G} b_h h - \sum_{h \in G} b_{hg^{-1}} h = \sum_{h \in G} (b_h - b_{hg^{-1}}) h$$

for each $g \in G$. Then $b_h = b_{gh^{-1}}$ for each $h \in G$ by the linear independence of the elements of G over F. Then since g was arbitrary, $b_h = b_{hg^{-1}}$ for each $g \in G$ implies the existence of a $b \in F$ such that $b = b_h$ for each h. Thus $\sum_{h \in G} b_h h = b(\sum_{h \in G} h) = bE$ and $l(M) = F(S)E$. This completes the proof of the lemma.

<u>Proof</u>: (of theorem 5.3) The condition $r > 0$ is imposed so that S will not be a group or a group with zero, for such semigroups have quasi-Frobenius algebras over arbitrary fields.

Note that $F(S_1) \cong F(T_1)$ since $G_i = G$ for $i = 1,\dots,r+1$. Then in a principal series $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ for S, let $S - S_1 = G$, and $S_i - S_{i+1} = (s_i)$. (This is justifiable since $S_i - S_{i+1} = Gt_i = [t_i]$, so $S_i - S_{i+1}$ contains only one element. Let $s_i = [t_i]$.)

If F(S) is quasi-Frobenius, then $(c(F),o(G)) = 1$ will be proved. Then theorem 4.3 implies F(T) is quasi-Frobenius. Suppose c(F) divides o(G). Let $E = \sum_{g \in G} g \in F(G) \subseteq F(S)$. Then $Es = o(G)s = 0$ for $s \in S_1$ since $gs = s$ for each $g \in G$ and $s \in S_1$, and c(F) divides o(G). $E \in l(r(F(S_1)))$ will be proved. Then as $E \notin F(S_1)$, F(S) is not quasi-Frobenius. Let $x \in r(F(S_1))$, say $x = x_1 + x_2$, $x_1 \in F(G)$, $x_2 \in F(S_1)$. Then $sx = sx_1 + sx_2 = 0$ with $s \in S_1$, implies $sx_1 = 0$ and $sx_2 = 0$ by lemma 5.1. If $x_1 = \sum_{g \in G} a_g g$, $a_g \in F$, then $x_1 s = (\sum_{g \in G} a_g)s$ since $gs = s$ for each $s \in S_1$, so $x_1 s = (\sum_{g \in G} a_g)s = 0$ implies $\sum_{g \in G} a_g = 0$. Then $Ex_1 = \sum_{g \in G} a_g Eg = (\sum_{g \in G} a_g)E = 0$.

Thus, $x = x_1 + x_2 \in r(F(S_1))$ implies $Ex = Ex_1 + Ex_2 = 0$ so $E \in l(r(F(S_1)))$ and $E \notin F(S_1)$.

Conversely, suppose $(c(F),o(G)) = 1$ and $F(T)$ is quasi-Frobenius. $F(S) = F(G) + F(S_1)$ and $F(S_1) \cong F(T_1)$, in fact $T$ is isomorphic with a subsemigroup of $S$ as observed before. If $x = \sum_{g \in G} a_g g \in F(G) \subseteq F(S)$ and $y = \sum_{j=1}^{r} b_j s_j$, then $xy = (\sum_{g \in G} a_g)(\sum_{j=1}^{r} b_j s_j)$ since $gs = s$ for each $s \in S_1$. In what follows, if $B$ is a subset of $F(S_1)$, then let $r_1(B) = r(B) \cap F(S_1)$ and $l_1(B) = l(B) \cap F(S_1)$. By a theorem of Nakayama stated in the preliminaries, the annihilator conditions need only be established for ideals in the radical. Thus, let $L$ be an ideal of $F(S)$, $L \subseteq F(S_1) = \text{rad.}F(S)$, since $(c(F),o(G)) = 1$. $l(r(L)) = L$ will be verified. Let $M$ be as in lemma 5.2. Then $r(L) = M + r_1(L)$; for let $x = \sum_{i=1}^{r} c_i s_i \in F(S_1)$ and $y = \sum_{g \in G} a_g g + \sum_{j=1}^{r} a_j s_j$. Then $0 = xy = (\sum_{g \in G} a_g) \sum_{i=1}^{r} c_i s_i + (\sum_{i=1}^{r} c_i s_i)(\sum_{j=1}^{r} a_j s_j)$ implies either $x = 0$ or $\sum_{g \in G} a_g = 0$ by the same reasoning as in the proof of lemma 5.1, using lemma 4.1 and linear independence. Then also $(\sum_{i=1}^{r} c_i s_i)(\sum_{j=1}^{r} a_j s_j) = 0$ so $\sum_{j=1}^{r} a_j s_j \in r_1(x)$. Then $Ly = (0)$, $L \subseteq F(S_1)$, implies $y \in M + r_1(L)$. Thus $r(L) = M + r_1(L)$.

Then $l(r(L)) = l(M + r_1(L)) = l(M) \cap l(r_1(L))$. But $l(r_1(L)) = M + l_1(r_1(L))$ as before. $l_1(r_1(L)) = L$ since $F(S_1) \cong F(T_1)$, and $F(T)$ is quasi-Frobenius (for if $L'$ is an ideal of $F(T)$ contained in $F(T_1)$ then $r(L') \subseteq F(T_1)$

by lemma 4.13). Thus $l(r(L)) = l(M) \cap (M + L)$. $l(M) = F(S)E$ by lemma 5.2 so if $xE \in l(M) \cap (M + L)$, then $xE = mE + yE$ for some $m \in M$ and some $y \in L$ since $M \subset F(G)$ and $L \subseteq F(S_1)$. But $mE = 0$ so $xE = yE \in L$. I.e., $l(r(L)) \subseteq L$ so $l(r(L)) = L$ for each ideal in the radical $F(S_1)$ of $F(S)$. Thus $F(S)$ is quasi-Frobenius.

Now suppose $S$ is a semigroup obtained from a semigroup $T$ of type D and an admissible collection of subgroups $[G_i]$ of the abelian group $G$, with $G_0 = (e)$. The elements of $S$ can be written as $[gt_i]$, where $[gt_i] = [ht_j]$ if and only if $i = j$ and $g \equiv h(G_i)$. For convenience let $[gt_i] = gt_i$ and use the same identification; i.e., $gt_i = ht_j$ if and only if $i = j$ and $g \equiv h(G_i)$. The multiplication is $(gt_i)(g't_j) = ht_k$ if and only if $t_i t_j = t_k$ in $T$ and $gg' \equiv h(G_k)$. The J-classes are, as before, $J_i = Gt_i$, $i = 0,1,\ldots,r+1$, $J_0 = Gt_0 \cong G$. Let $gt_0 = g$ for each $g \in G$ so that $G$ can be considered a subsemigroup of $S$. If $T = T_0 \supset T_1 \supset \cdots \supset T_{r+1}$ is a principal series for $T$, then $S = S_0 \supset S_1 \supset \cdots \supset S_{r+1}$ with $S_i - S_{i+1} = Gt_i$, is a principal series for $S$.

Let $F$ be a field. The elements of $F(S)$ can be written in the form $\sum_{i=0}^{r} u_i t_i$, with $u_i \in F(G) \subseteq F(S)$, and multiplication is distributive, reducing products to those of the form $(gt_i)(g't_j)$, which have already been defined. This description may seem unduly complicated but it provides a convenient way of calculating in $F(S)$ and it will be used in the proof of theorem 5.4. $T = (et_i : t_i \in T) \subseteq S$, and let $et_i = t_i$ so $T$ can be considered a subsemigroup of $S$.

Theorem 5.4: If T is of type D, let $T_1$ be the maximal nilpotent ideal of T. Suppose $T_1$ is cyclic and $[G_i]$ is any admissible collection of subroups of the abelian group G. Let S be the semigroup obtained from T and $[G_i]$. If $(c(F), o(G)) = 1$, and F(T) is quasi-Frobenius, then F(S) is quasi-Frobenius.

Proof: Let $e_i$ be a primitive idempotent of F(G). One need only prove that $e_iF(S)$ has a unique irreducible F(S)-submodule for each i and then apply lemma 4.8 and Nakayama's theorem. Let $T_1$ be generated by t, $t^{r+1} = 0$, $t^r \neq 0$. Let $R_i = (s \in S: se_i = 0)$ as before. Suppose $S = S_o \supset S_1 \supset \cdots \supset S_{r+1}$, $S - S_1 = G$, $S_j - S_{j+1} = Gt^j$, is the unique principal series for S, and $S_{p+1} = R_i$ (the series is unique as $T_1$ is cyclic). Let $t^p \in S_p - R_i$, and let $t^o = e$, the identity of S, for convenience. If $R_i = S_1$, then $p = 0$ so $e_i t^p F(G) = e_i F(G)$, and $e_i F(G)$ is the unique irreducible submodule of $e_i F(S)$ since it is unique in $e_i F(G)$ and $e_i S_1 = (0)$. Suppose $p > 1$. $e_i t^p F(G)$ is an irreducible F(S)-submodule of $e_i F(S)$ as in lemma 4.8. Suppose $e_i x F(G)$ is another irreducible F(S)-submodule of $e_i F(S)$. $e_i x F(G) \supseteq e_i t^p F(G)$ will be proved. $e_i x S_1 = (0)$ since $e_i x F(S_1) \subsetneq e_i x F(G)$ by lemma 4.1, and $e_i x F(G)$ is irreducible. Let $x = \sum_{j=0}^{r} u_j t^j$, $u_j \in F(G)$.

Then $e_i x = \sum_{j=0}^{p} e_i u_j t^j$, since $e_i R_i = (0)$ implies $e_i t^j = 0$ for $j > p$.

Let q be the minimum non-negative integer such that $e_i u_q t^q \neq 0$.

Then $0 \leq q \leq p$ since otherwise $t^q \in R_i$ and $e_i x = \sum_{j=q}^{p} e_i u_j t^j$. $e_i F(G) = e_i u_q F(G)$ (since $(c(F), o(G)) = 1$ so $F(G)$ is semisimple and thus $e_i F(G)$ is irreducible) so there is a $v \in F(G)$ such that $e_i = e_i u_q v$. If $q = 0$, then $e_i x(vt^p) = e_i u_q vt^p = e_i t^p$, so $e_i xF(S) \supseteq e_i t^p F(G)$ as desired. If $q > 0$, then multiply $e_i x$ by $vt^{p-q}$ to get $(e_i x)vt^{p-q} = (\sum_{j=q}^{p} e_i u_j t^j)vt^{p-q} = e_i u_q vt^p$ (since $t^{j+p-q} \in R_i$ for $j > q$, and then $e_i t^{j+p-q} = 0$), which equals $e_i t^p$ since $e_i = e_i u_q v$. Thus, again $e_i xF(S) \supseteq e_i t^p F(G)$. Then $e_i t^p F(G)$ is the unique irreducible $F(S)$-submodule of $e_i F(S)$. This can be done for each primitive idempotent $e_i$ in $F(G)$. Then, by lemma 4.8 and Nakayama's Theorem, $F(S)$ is quasi-Frobenius.

Remark: The converse of theorem 5.4 is false. Example (3) in the appendix is a counterexample.

## 6. Appendix

**Example 1.** Let T be the semigroup of type D with the following multiplication table.

| | $t_o$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | 0 |
|---|---|---|---|---|---|---|
| $t_o$ | $t_o$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | 0 |
| $t_1$ | $t_1$ | $t_2$ | $t_4$ | 0 | 0 | 0 |
| $t_2$ | $t_2$ | $t_4$ | 0 | 0 | 0 | 0 |
| $t_3$ | $t_3$ | 0 | 0 | $t_4$ | 0 | 0 |
| $t_4$ | $t_4$ | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

F(T) is quasi-Frobenius. Let G be the group of order 2 with elements e, g, $g^2 = e$. Let $G_o = G_1 = G_2 = G_3 = (e)$, $G_4 = G$. Form S as in section 5. Then the multiplication table for S can be written as on the following page. Even if the characteristic of F is not 2, F(S) is not quasi-Frobenius. The primitive idempotents of F(S) are $e_o = 1/2(t_o + gt_o)$ and $e_1 = 1/2(t_o - gt_o)$. $e_1F(G)$ is an irreducible F(G)-module. Consider $e_1t_2F(S)$ and $e_1t_3F(S)$ as in lemma 4.8. $e_1t_2F(S) = F(t_2 - gt_2)$ and $e_1t_3F(S) = F(t_3 - gt_3)$ are both irreducible F(S)-submodules of $e_1F(S)$ since $e_1t_3S_1 = e_1t_2S_1 = (0)$, by lemma 4.8, so F(S) is not quasi-Frobenius even though F(T) is quasi-Frobenius.

The multiplication table for S is:

| | $t_0$ | $gt_0$ | $t_1$ | $gt_1$ | $t_2$ | $gt_2$ | $t_3$ | $gt_3$ | $t_4$ | 0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_0$ | $t_0$ | $gt_0$ | $t_1$ | $gt_1$ | $t_2$ | $gt_2$ | $t_3$ | $gt_3$ | $t_4$ | 0 |
| $gt_0$ | $gt_0$ | $t_0$ | $gt_1$ | $t_1$ | $gt_2$ | $t_2$ | $gt_3$ | $t_3$ | $t_4$ | 0 |
| $t_1$ | $t_1$ | $gt_1$ | $t_2$ | $gt_2$ | $t_4$ | $t_4$ | 0 | 0 | 0 | 0 |
| $gt_1$ | $gt_1$ | $t_1$ | $gt_2$ | $t_2$ | $t_4$ | $t_4$ | 0 | 0 | 0 | 0 |
| $t_2$ | $t_2$ | $gt_2$ | $t_4$ | $t_4$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $gt_2$ | $gt_2$ | $t_2$ | $t_4$ | $t_4$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $t_3$ | $t_3$ | $gt_3$ | 0 | 0 | 0 | 0 | $t_4$ | $t_4$ | 0 | 0 |
| $gt_3$ | $gt_3$ | $t_3$ | 0 | 0 | 0 | 0 | $t_4$ | $t_4$ | 0 | 0 |
| $t_4$ | $t_4$ | $t_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Example 2. This is an example of a semigroup T of type D with $t_r \in tT$, for each $t(\neq 0) \in T$. F(T) is not quasi-Frobenius, however.

| | $t_0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | 0 |
|---|---|---|---|---|---|---|
| $t_0$ | $t_0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | 0 |
| $t_1$ | $t_1$ | $t_3$ | $t_3$ | $t_4$ | 0 | 0 |
| $t_2$ | $t_2$ | $t_3$ | $t_3$ | $t_4$ | 0 | 0 |
| $t_3$ | $t_3$ | $t_4$ | $t_4$ | 0 | 0 | 0 |
| $t_4$ | $t_4$ | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

F(T) is not quasi-Frobenius since $t_1 - t_2 \in r(F(T_1))$ and $t_1 - t_2 \notin F(T_r) = F(t_4)$, by theorem 4.4.

Example 3. Theorem 5.4 says that if $(c(F),o(G)) = 1$ and $T_1$ is cyclic, then $F(S)$ is quasi-Frobenius, where S is obtained from T and subgroups $G_i$ of G. Without the condition $(c(F),o(G)) = 1$, this is false. Let G be of order 2, $G = (e,g\colon g^2 = e)$, let F be of characteristic 2, and let T have the multiplication table:

| | $t_o$ | $t_1$ | 0 |
|---|---|---|---|
| $t_o$ | $t_o$ | $t_1$ | 0 |
| $t_1$ | $t_1$ | 0 | 0 |
| 0 | 0 | 0 | 0 |

T is clearly of type D and $T_1 = (t_1,0)$ is cyclic. Form S with $G_o = (e)$, and $G_1 = G$. Then S has the multiplication table:

| | $t_o$ | $gt_o$ | $t_1$ | 0 |
|---|---|---|---|---|
| $t_o$ | $t_o$ | $gt_o$ | $t_1$ | 0 |
| $gt_o$ | $gt_o$ | $t_o$ | $t_1$ | 0 |
| $t_1$ | $t_1$ | $t_1$ | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

This semigroup S does not have a quasi-Frobenius algebra over a field of characteristic 2, by theorem 5.3. In fact, theorem 5.3 gives other examples of this situation if $T_1$ is cyclic.

Example 4. A reasonable conjecture might be that if T has ideals $I_1$ and $I_2$ such that $F(T/I_1)$ and $F(T/I_2)$ are quasi-Frobenius then $F(T/I_1 \cap I_2)$ is quasi-Frobenius. This conjecture is false. Let T be the semigroup of example 2. Let $I_1 = (t_1,t_3,t_4,0)$ and $I_2 = (t_2,t_3,t_4,0)$. Then $I_1 \cap I_2 = (t_3,t_4,0)$. $F(I_1)$ and $F(I_2)$ are quasi-Frobenius by corollary 2 to theorem 4.4. Then $T' = T/I_1 \cap I_2 = (t_o,t_1,t_2,z)$, with the identification described in the preliminaries. But $T_1' = T_1/I_1 \cap I_2 = (t_1,t_2,0)$ and $T_1'^2 = (z)$ so $t_1,t_2 \in r(F(T_1'))$ which implies $F(T')$ is not quasi-Frobenius, by theorem 4.4

T is also an example of a semigroup without a quasi-Frobenius algebra over F but with subsemigroups $R_1 = (t_o,t_1,t_3,t_4,0)$ and $R_2 = (t_o,t_2,t_3,t_4,0)$ with $R_1 \cap R_2 = (t_3,t_4,0)$ contained in $T_1$, such that $T = R_1R_2$ and $F(R_1)$ and $F(R_2)$ are quasi-Frobenius. Thus, decomposing a semigroup algebra into a product of quasi-Frobenius subalgebras seems to imply little about the algebra.

Conjectures:

1. Let F be a finite semigroup and let F be a field of characteristic c. If F(S) is quasi-Frobenius, then every principal factor of S is either c-nonsingular or null.

The converse certainly is false since every commutative semigroup is of this form.

2. Components can be defined for arbitrary semigroups just as they were defined for commutative semigroups in definition 3.2. The conjecture is that theorem 3.2 is true in this broader context.

The same proof given there shows that if S has a principal series for which the components have quasi-Frobenius algebras and $I_o = S$, then F(S) is quasi-Frobenius. The converse, however, has not been proved.

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