MONOTONE NORMS AND SEMINORMS ON AN ARCHIMEDEAN VECTOR LATTICE

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ABSTRACT

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By

Rosalind Reichard

Two distinct problems are considered in this paper. However, both are concerned with a discussion of monotone norms and monotone seminorms defined on an Archimedean vector lattice.

A norm (or seminorm) $\|\cdot\|$ on an Archimedean vector lattice E is said to be monotone whenever $|x| \le |y|$ in E implies that $\|x\| \le \|y\|$. An Archimedean vector lattice with a monotone norm defined on it $(E, \le, \|\cdot\|)$ is called a normed vector lattice. If $\|\cdot\|$ is complete $(E, \le, \|\cdot\|)$ is a Banach lattice. The Dedekind completion $\hat{E}$ of an Archimedean vector lattice E is a Dedekind complete vector lattice (i.e., the supremum of every subset of $\hat{E}$ which is bounded above in $\hat{E}$ must exist in $\hat{E}$) in which E is isomorphically (preserving algebraic and order relations) embedded and $\hat{x} = \sup(y\colon y \le \hat{x}, y \in E) = \inf(z\colon z \ge \hat{x}, z \in E)$ for any $\hat{x}$ in $\hat{E}$. Given a normed vector lattice $(E, \le, \|\cdot\|)$ an extension of $\|\cdot\|$ to $\hat{E}$ is a monotone norm $\|\cdot\|_1$ on $\hat{E}$ such that $\|x\| = \|x\|_1$ for all x in E.

Given any normed vector lattice $(E, \le, \|\cdot\|)$ all norm extensions of $\|\cdot\|$ to $\hat{E}$ are not necessarily equivalent.

The question that Chapter One answers is: What are some conditions under which all norm extensions of a Banach lattice are equivalent?

Section one of Chapter One indicates some restrictions under which norm extensions are unique. Whenever $\|\cdot\|$ is continuous (i.e., $0 \leq x_\alpha \downarrow 0$ implies that $\inf(\|x_\alpha\|: \alpha \in \mathcal{A}) = 0$) $\|\cdot\|$ extends uniquely to $\hat{E}$ (Theorem 1.2). However, examples of normed vector lattices are given where $\|\cdot\|$ is semi-continuous ($0 \leq x_\alpha \uparrow x$ implies that $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x\|$) or σ-continuous ($0 \leq x_n \downarrow 0$ implies that $\inf(\|x_n\|: n \in N) = 0$) and $\|\cdot\|$ does not extend uniquely to $\hat{E}$. One theorem in Section one is:

<u>Theorem 1.11</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that whenever $\{x_\alpha: \alpha \in \mathcal{A}\}$ is any subset of E^+ with $\sup(x_\alpha: \alpha \in \mathcal{A}) = x$ then $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x\|$. Also, if $\inf(w,\bar{e}) \in E$ whenever $w \in E$ and $\bar{e} = \sup(\hat{x} \in \hat{E}^+: \|\hat{x}\|^* \leq 1)$ (which exists in the universal completion of E) then $\|\cdot\|$ extends uniquely to $\hat{E}$.

The boundedness property is the topic for Section two. If a subset B of E is order bounded whenever $\{\lambda_n x_n\}$ order converges to zero for every sequence $\{x_n\} \subseteq B$ and every sequence $\{\lambda_n\}$ of positive real numbers decreasing to zero then E is said to have the <u>boundedness property</u>. The main result of this section is:

<u>Theorem 2.10</u>. Let $(E, \leq, \|\cdot\|)$ be a Banach lattice such that E has the boundedness property. Then all extensions of $\|\cdot\|$ to $\hat{E}$ are equivalent.

Normed vector lattices having the Egoroff property is the subject of Section three. An Archimedean vector lattice E has the <u>Egoroff property</u> whenever given any double sequence $\{x_{n,k}: n,k \in N\}$ in E such that $0 \le x_{n,k} \uparrow_k |x|$ for $n = 1,2,\ldots,$ there exists a sequence $\{u_m: m \in N\} \subseteq E^+$ such that $u_m \uparrow |x|$ and for every $n,m \in N$ there exists a positive integer $k(n,m)$ with $u_m \le x_{n,k}$. A discussion of the Egoroff property on normed vector lattices yields an example of a monotone norm $\|\cdot\|$ such that $\varphi(x) = \|x\| - \|x\|_L = \|x\| - \inf(\lim_{n\to\infty} \|x_n\|: 0 \le x_n \uparrow |x|)$ is a functional on E which is neither subadditive nor monotone. (This answers a question of W.A.J. Luxemburg.) We also obtain several theorems one of which is:

<u>Theorem 3.20</u>. Suppose $(E, \le, \|\cdot\|)$ is a normed vector lattice such that E is order separable and has the Egoroff property. If $\|\hat{x}\|_M = \inf(\|w\|: w \ge |\hat{x}|, w \in E)$ is equivalent to $\|\hat{x}\|_{ML} = \inf(\lim_{n\to\infty} \|\hat{x}_n\|_M : 0 \le \hat{x}_n \uparrow |\hat{x}| \text{ in } \hat{E})$ on $\hat{E}$, then all norm extensions of $\|\cdot\|$ to $\hat{E}$ are equivalent.

In Chapter Two we consider the problem of determining which universally complete vector lattices (i.e., Dedekind complete and $\sup(x_\alpha: \alpha \in \mathcal{A}) = x$ exists if $\inf(x_{\alpha_1}, x_{\alpha_2}) = 0$ whenever $\alpha_1 \ne \alpha_2$) have a countable collection of continuous, monotone seminorms defining a locally convex, Hausdorff topology. We find that the space of all real sequences and all finite dimensional spaces are the only such vector lattices.

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Rosalind Reichard

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INTRODUCTION

The theory of vector lattices, real vector spaces with compatible lattice structures, is one of the main cornerstones for functional analytic theory. Historically, the foundations for this theory were laid in the years from 1928 to 1936 by F. Riesz, L.V. Kantorovitch, and H. Freudenthal. For an excellent discussion of the development of the theory of vector lattices, one should read a paper by A.C. Zaanen [25].

An <u>Archimedean vector lattice</u> E is one in which $n|x| \le y$ for $n = 1,2,\ldots,$ with x and y elements in E implies that $x = 0$. If the supremum of a set B, written sup (B), always exists in E whenever B is a subset of E which is bounded above in E E is <u>Dedekind complete</u>. Every Archimedean vector lattice E has a <u>Dedekind completion</u> $\hat{E}$ ($\hat{E}$ is Dedekind complete, E is isomorphically embedded in $\hat{E}$ with preservation of the algebraic and order relations, and $\hat{x} = \sup(y\colon y \le \hat{x},\ y \in E) = \inf(z\colon z \ge \hat{x},\ z \in E)$ for every $\hat{x}$ in $\hat{E}$), and the Dedekind completion is unique up to isomorphism [13, Theorems 30.1, 2, 3].

In the study of the classical normed function spaces, such as the L_p spaces, the natural underlying structure is the theory of normed vector lattices. A <u>normed vector lattice</u> $(E, \le, \|\cdot\|)$ is a vector lattice with a norm $\|\cdot\|$ defined on it such that $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$. A norm defined on a vector lattice and satisfying this property will be called monotone. Whenever $\|\cdot\|$ is a complete norm, $(E, \leq, \|\cdot\|)$ is said to be a Banach lattice. A normed vector lattice is always Archimedean [23, p. 174]. And in the present paper, unless otherwise specified, we shall always be working with Archimedean vector lattices and all norms shall be monotone norms. Definitions for any of the terms used in this paper can be found in [23] or [16].

The question of extending a monotone norm from an Archimedean vector lattice E to its Dedekind completion $\hat{E}$ has been considered by B.Z. Vulich [23], V.A. Solovev [19, 20], and T. Nishiura [15]. An extension of a monotone norm $\|\cdot\|$ from E to $\hat{E}$ is defined to be a monotone norm $\|\cdot\|_1$ on $\hat{E}$ such that $\|x\| = \|x\|_1$ for all x in E. It is well known [23, p. 179] that every monotone norm $\|\cdot\|$ on a vector lattice E has an extension $\|\cdot\|_M$ to $\hat{E}$ defined by:

$$\|\hat{x}\|_M = \inf(\|y\|: y \geq |\hat{x}|, y \in E).$$

And if $\|\cdot\|_1$ is any other extension of $\|\cdot\|$ to $\hat{E}$, then $\|\hat{x}\|_1 \leq \|\hat{x}\|_M$ for all $\hat{x} \in \hat{E}$. We shall refer to $\|\cdot\|_M$ as the maximal extension.

As the following two examples will illustrate it is not true that norm extensions are unique nor is it true that all norm extensions are equivalent.

<u>Example 0.1</u>. Let $E = c$ (the real, convergent sequences) and $\|x\| = \sup(|x(n)| : n \in N) + \lim_{n\to\infty} |x(n)|$ where $x = (x(1), x(2), \ldots)$. Then $\hat{E} = \ell_\infty$ (the bounded, real sequences) and $\|\hat{x}\|_M = \sup(|\hat{x}(n)| : n \in N) + \overline{\lim_{n\to\infty}} |\hat{x}(n)|$. However, other extensions of $\|\cdot\|$ to $\hat{E}$ exist as $\|\hat{x}\|_1 = \sup(|\hat{x}(n)| : n \in N) + L(|\hat{x}|)$ is an extension where L is any Banach limit.

<u>Example 0.2</u>. Let $E = c$ and $\|x\| = \sum_{n=1}^{\infty} (|x(n)|/2^n) + \lim_{n\to\infty} |x(n)|$. Then $\hat{E} = \ell_\infty$ and $\|\hat{x}\|_M = \sum_{n=1}^{\infty} (|\hat{x}(n)|/2^n) + \overline{\lim_{n\to\infty}} |\hat{x}(n)|$. We shall show that $(E, \leq, \|\cdot\|)$ is not norm dense in $(\hat{E}, \leq, \|\cdot\|_M)$ but is norm dense in $(\hat{E}, \leq, \|\cdot\|_1)$ where $\|\hat{x}\|_1 = \sum_{n=1}^{\infty} (|\hat{x}(n)|/2^n) + L(|\hat{x}|)$ and L is any Banach limit. This not only indicates that $\|\cdot\|_M$ and $\|\cdot\|_1$ are not equivalent, but also answers a question of Nishiura [15, Problem 1].

In order to prove that $(E, \leq, \|\cdot\|)$ is norm dense in $(\hat{E}, \leq, \|\cdot\|_1)$ we note that for any $\hat{x} \in \hat{E} = \ell_\infty$ and any positive number $\epsilon > 0$, there exists a positive integer N such that $\sum_{n=N+1}^{\infty} 2M/2^n < \epsilon$ where M is a bound for the sequence $\hat{x}$. Now let $x = (x(1), x(2), \ldots) \in c$ be the element such that $x(n) = \hat{x}(n)$ for $n = 1, 2, \ldots, N$ and $x(n) = L(|\hat{x}|)$ for $n > N$. Then
$\|x - \hat{x}\|_1 \leq \sum_{n=N+1}^{\infty} 2M/2^n + 0 < \epsilon$.

However, we can show that $(E, \leq, \|\cdot\|)$ is not norm dense in $(\hat{E}, \leq, \|\cdot\|_M)$ by choosing $\hat{x} = (1,0,1,0,\ldots) \in \ell_\infty$ and noting that $L(\hat{x}) = 1/2$. If x is any element in c, it is easy to see that $\|x - \hat{x}\|_M \geq 1/2$.

As Example 0.2 illustrates, the norm extensions of a normed vector lattice need not be equivalent. However, the example given is not a Banach lattice. Therefore, we ask whether all extensions of a complete, monotone norm on a vector lattice E must be equivalent on $\hat{E}$. Chapter One of this paper is involved with an attempt at answering this question.

Nishiura [15, Theorem A] has shown that if $(E, \leq, \|\cdot\|)$ is a Banach lattice then $\|\cdot\|_M$ is complete on $\hat{E}$. Since all monotone, complete norms on a vector lattice are equivalent [14, Theorem 30.28] the above question asks whether every extension of a complete norm on E must be complete on $\hat{E}$.

In Section one we consider conditions under which norm extensions of a normed vector lattice are unique and apply these results to the question of equivalence of norm extensions for Banach lattices. The boundedness property and some of its consequences on normed vector lattices is the subject of Section two since this property plays an important role in the equivalence of norm extensions. In Section three we discuss the Egoroff property on normed vector lattices and we obtain a Riesz-Fischer Theorem as well as some results on norm extensions which aid us in answering the above question.

We shall need the following definitions in our discussion:

<u>Definitions 0.3</u>. A <u>subvector lattice</u> $M \subseteq E$ of a vector lattice E is a linear subspace of E such that $\sup(x,y)$ is in M whenever $x,y \in M$. A linear subspace M of E is an

ideal in E if y is an element of M whenever $|y| \leq x$ and $x \in M$. An ideal M in E is a band in E whenever $\sup(B) = x_o$ in E and $B \subseteq M$ implies that $x_o \in M$. A subvector lattice M of E is said to be order dense in E whenever for all $0 \leq x \in E$ there exists a directed system $0 \leq x_\alpha \uparrow$ (given any α_1, α_2, there exists an α_3 such that $x_{\alpha_3} \geq x_{\alpha_1}$ and $x_{\alpha_3} \geq x_{\alpha_2}$) of elements of M such that $\sup(x_\alpha : \alpha \in \mathcal{O}) = x$.

Definitions 0.4. A vector lattice E is universally complete if it is Dedekind complete and $\sup(x_\alpha : \alpha \in \mathcal{O}) = x_o$ exists in E whenever $\{x_\alpha : \alpha \in \mathcal{O}\} \subseteq E$ and $\inf(x_{\alpha_1}, x_{\alpha_2}) = 0$ for all $\alpha_1 \neq \alpha_2$. The vector lattice $E^{\#}$ is said to be a universal completion of E whenever $E^{\#}$ is universally complete and E can be isomorphically embedded as an order dense subvector lattice of $E^{\#}$.

A vector lattice E has a universal completion if and only if E is Archimedean and any two universal completions are isomorphic [13, Theorem 34.4].

In Theorem 39.1 of Modern Spectral Theory [13], H. Nakano characterizes a normed vector lattice $(E, \leq, \|\cdot\|)$ having the property that whenever $\{x_\alpha\}$ is a set of elements in E which is bounded above in E then $\sup(\|x_\alpha\| : \alpha \in \mathcal{O}) = \inf(\|y\| : y$ is an upper bound for $\{x_\alpha\})$. Such a normed vector lattice must be a norm dense subvector lattice of a space $C(K)$ of continuous, bounded functions, vanishing at infinity, and defined on a locally compact, Hausdorff space K. The first part of the proof of this theorem can be used to prove the following very useful result.

Theorem 0.5. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that whenever $\{x_\alpha\} \subseteq E^+$ (E^+ = the positive elements of E) and $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_o$ in E then $\sup(\|x_\alpha\| : \alpha \in \mathcal{A}) = \|x_o\|$. If $\{x_\alpha\} \subseteq E^+$ and $\|x_\alpha\| \leq M$ for all α, then $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_o$ exists in $E^\#$.

The proof of this theorem involves decomposing each x_α into the supremum of a disjoint collection of elements in $E^\#$ and in this way forcing an element of $E^\#$ to be above each of the x_α. We do not reproduce the proof here because it would involve introducing several terms that we shall not need elsewhere in this paper. However, for the convenience of the reader, the proof of Theorem 0.8, below, can be found in the Appendix.

The importance of Theorems 0.5 and 0.8 will become evident in Sections one and three when we apply these results to the problem of norm extensions. Before stating Theorem 0.8, we shall need the following definitions.

Definitions 0.6. A net $\{x_\alpha : \alpha \in \mathcal{A}\}$ in a vector lattice E is said to decrease (increase) to x_o in E (notation: $x_\alpha \downarrow x_o$, $x_\alpha \uparrow x_o$) if $x_o = \inf(x_\alpha : \alpha \in \mathcal{A})$ ($x_o = \sup(x_\alpha : \alpha \in \mathcal{A})$) and $x_\alpha \geq x_\beta$ whenever $\beta \geq \alpha$ ($x_\alpha \leq x_\beta$ whenever $\beta \geq \alpha$). A net $\{x_\alpha : \alpha \in \mathcal{A}\}$ is said to order converge to x_o in E (notation: $x_\alpha \xrightarrow{o} x_o$) if $\{x_\alpha\}$ is an order bounded subset of E and there exists a net $\{y_\alpha : \alpha \in \mathcal{A}\}$ in E that decreases to 0 and $|x_\alpha - x_0| \le y_\alpha$ for all $\alpha \in \mathcal{A}$.

Definitions 0.7. Let $(E, \le, \|\cdot\|)$ be a normed vector lattice. $\|\cdot\|$ is semi-continuous (σ-semi-continuous) whenever $0 \le x_\alpha \uparrow x_0$ in E ($0 \le x_n \uparrow x_0$ in E) implies that $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x_0\|$ $(\sup(\|x_n\|: n \in N) = \|x_0\|)$. A vector lattice E is said to be order separable whenever given any subset $A \subseteq E$ such that $\sup(A) = x_0$ exists in E, then there exists a countable subset $A' \subseteq A$ such that $\sup(A') = x_0$ in E.

Remark. Whenever $(E, \le, \|\cdot\|)$ is σ-semi-continuous and E is order separable it follows that $\|\cdot\|$ is semi-continuous. Examples of semi-continuous norms are frequently found; e.g., let $E = c$ and $\|x\| = \sup(x(n): n \in N)$.

The norm property stated in Theorem 0.5 is used very strongly at one point of the proof. However, we can obtain the following theorem using a slight variation.

Theorem 0.8. Let $(E, \le, \|\cdot\|)$ be a normed vector lattice such that $\|\cdot\|$ is semi-continuous (σ-semi-continuous). If $0 \le x_\alpha \uparrow$ ($0 \le x_n \uparrow$) in E and $\|x_\alpha\| \le M$ for all $\alpha \in \mathcal{A}$ ($\|x_n\| \le M$ for all $n \in N$) then $\sup(x_\alpha: \alpha \in \mathcal{A}) = x_0$ exists in $E^\#$ ($\sup(x_n: n \in N) = x_0$ exists in $E^\#$).

As noted above, applications of Theorems 0.5 and 0.8 can be found in Sections one and three. However, one immediate result is the following theorem which was proved by W.A.J. Luxemburg [7, Theorem 61.7] in a different manner.

<u>Definition 0.9</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice. $\|\cdot\|$ is said to be <u>locally norm complete</u> whenever $\{x_n\} \subseteq E$, $|x_n| \leq x_o \in E$ for $n = 1,2,\ldots$, and $\|x_n - x_m\| \to 0$ as $n,m \to \infty$ implies there exists an element x in E such that $\|x_n - x\| \to 0$ as $n \to \infty$.

<u>Theorem 0.10</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that E is Dedekind σ-complete and $\|\cdot\|$ is σ-semi-continuous. Then $(E, \leq, \|\cdot\|)$ must be locally norm complete.

<u>Proof</u>. Suppose $\{x_n\} \subseteq E$ is $\|\cdot\|$-Cauchy and $|x_n| \leq x_o \in E$ for all $n \in N$. We can assume without loss of generality that $\|x_n - x_{n+1}\| \leq 1/2^n$ for $n = 1,2,\ldots$. Consider the sequence $s_n = |x_2 - x_1| + |x_3 - x_2| + \cdots + |x_{n+1} - x_n|$ and note that by Theorem 0.8 $s_n \uparrow x^*$ in $E^{\#}$ since $\|s_n\| = \||x_2 - x_1| + \cdots + |x_{n+1} - x_n|\| \leq 1$ for all n and $\|\cdot\|$ is σ-semi-continuous. But then we must have $x_1 + (x_2 - x_1) + \cdots + (x_{n+1} - x_n) \xrightarrow{o} \bar{x} \in E^{\#}$. (Absolute convergence of a series implies convergence of the series in a Dedekind σ-complete vector lattice [23, p. 101]).

Therefore, we have $x_n \xrightarrow{o} \bar{x}$ in $E^{\#}$. However, since $|x_n| \leq x_o \in E \subseteq \hat{E}$ and $\hat{E}$ is an ideal in $E^{\#}$ we must have $\bar{x} \in \hat{E}$ and $x_n \xrightarrow{o} \bar{x}$ in $\hat{E}$ [23, p. 64]. Now let $y = \inf_n (\sup_{k \geq n} (x_k))$ which exists in E since E is Dedekind σ-complete and $\{x_k\}$ is bounded above in E. We note that by definition of y we must have $x_n \xrightarrow{o} y$ in E [16, p. 44]. However, since $x_n \xrightarrow{o} \bar{x}$ in $\hat{E}$ we must also have $\bar{x} = \inf_n (\sup_{k \geq n} (x_k))$

where the suprema and infima are taken in $\hat{E}$. But suprema and infima are preserved under the embedding $E \to \hat{E}$. Hence, we must have $\bar{x} = y$ and $x_n \xrightarrow{o} \bar{x}$ in E, $\bar{x} \in E$.

To complete the proof of the theorem we need only show that $x_n \to \bar{x}$ in norm. $\bar{x} = x_1 + (x_2 - x_1) + \cdots$ implies that $|\bar{x} - x_n| = |(x_1 + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) + \cdots) - (x_1 + (x_2 - x_1) + \cdots + (x_n - x_{n-1}))| \le \sum_{k=n}^{\infty} |x_{k+1} - x_k|$. It now follows that $\|\bar{x} - x_n\| \le \|\sum_{k=n}^{\infty} |x_{k+1} - x_k|\| \le \sum_{k=n}^{\infty} \|x_{k+1} - x_k\| \le 1/2^{n-1}$, with the second inequality holding because $\|\cdot\|$ is σ-semi-continuous. Therefore, $\|\bar{x} - x_n\| \to 0$ as $n \to \infty$.

The purpose of presenting the above theorem is to demonstrate some of the techniques used in proving theorems in the theory of vector lattices, and also to indicate the strength of Nakano's result.

As one reads Chapter One, he or she should keep in mind the main problem (i.e., when are norm extensions of a Banach lattice complete) as it is the motivation for the material presented here.

Chapter Two discusses a problem of a different nature and it is presented in this paper because it is an interesting result. In this last chapter we prove that every universally complete vector lattice with a countable collection of continuous, monotone seminorms defining a locally convex, Hausdorff topology must be the space of all real sequences or a finite dimensional space.

CHAPTER ONE

ARE THE NORM EXTENSIONS OF A BANACH LATTICE EQUIVALENT?

Section One. Uniqueness of Norm Extensions

Given a Banach lattice $(E, \leq, \|\cdot\|)$, we know that the maximal extension is always complete [15, Theorem A]. Therefore, whenever norm extensions are unique, the question of the completeness of all norm extensions can be easily answered. For this reason, it becomes important to determine conditions under which $\|\cdot\|_M$ is the only extension of a monotone norm $\|\cdot\|$ defined on E.

Definition 1.1. A norm $\|\cdot\|$ defined on a vector lattice E is said to be continuous (σ-continuous) whenever $0 \leq x_\alpha \downarrow 0$ ($0 \leq x_n \downarrow 0$) in E implies that $\inf(\|x_\alpha\|: \alpha \in \mathcal{A}) = 0$ ($\inf(\|x_n\|: n \in N) = 0$).

Remark. A σ-continuous norm is continuous whenever E is order separable. Also, a continuous norm is always semi-continuous. However, if $E = \ell_\infty$ and $\|x\| = \sup(|x(n)|: n \in N)$ then $\|\cdot\|$ is semi-continuous but not continuous.

We shall prove that whenever $(E, \leq, \|\cdot\|)$ is a normed vector lattice and $\|\cdot\|$ is continuous, $\|\cdot\|_M$ is the only extension of $\|\cdot\|$ to $\hat{E}$. This result was also proved by V.A. Solovev

[19, Theorem 4], but he obtained the result as a corollary to a combination of several theorems. We shall give a short direct proof.

Once we have the fact that continuous norms extend uniquely, it becomes a natural question to ask whether semi-continuous, σ-semi-continuous, or σ-continuous norms extend uniquely. The answer to all three questions is negative and several examples will be given to illustrate these points. At all times, we shall apply our results to the question concerning the equivalence of norm extensions.

<u>Theorem 1.2</u>. If $(E, \leq, \|\cdot\|)$ is a normed vector lattice and $\|\cdot\|$ is a continuous norm, then $\|\cdot\|$ extends uniquely to $\hat{E}$.

<u>Proof</u>. Let $\varphi(\hat{x}) = \sup(\|y\|: 0 \leq y \leq |\hat{x}|, y \in E)$ for all $\hat{x} \in \hat{E}$. If $\|\cdot\|_1$ is any extension of $\|\cdot\|$ to $\hat{E}$ we must have $\varphi(\hat{x}) \leq \|\hat{x}\|_1 \leq \|\hat{x}\|_M$ for all $\hat{x} \in \hat{E}$. Hence, to show that $\|\cdot\|$ extends uniquely to $\hat{E}$ we need only prove that $\varphi(\hat{x}) = \|\hat{x}\|_M$ for all $\hat{x} \in \hat{E}$.

Choose $\hat{x} \in \hat{E}$ and let $A = \{w - y: w \geq |\hat{x}|, 0 \leq y \leq |\hat{x}|, w,y \in E\}$. A is a directed set in E so we can let $A = \{u_\alpha: \alpha \in \mathcal{A}\}$ where $\mathcal{A}$ is a directed set. Then $0 \leq u_\alpha \downarrow 0$ since $\sup(y: 0 \leq y \leq |\hat{x}|, y \in E) = \inf(w: w \geq |\hat{x}|, w \in E) = |\hat{x}|$.

Using the hypothesis that $\|\cdot\|$ is continuous we must have $\inf(\|u_\alpha\|: \alpha \in \mathcal{A}) = 0$. Hence, given any $\epsilon > 0$ there exists an element $w \geq |\hat{x}|$ and an element $0 \leq y \leq |\hat{x}|$ with $w,y \in E$ such that $\|w - y\| < \epsilon$. However, $\|w\| - \|y\| \leq \|w - y\|$. Therefore, $\|w\| - \|y\| < \epsilon$ and it follows that $\varphi(\hat{x}) = \|\hat{x}\|_M$. This proves that $\|\cdot\|$ extends uniquely to $\hat{E}$.

<u>Corollary 1.3</u>. Let E be a vector lattice having any continuous, monotone, complete norm $\|\cdot\|_c$ defined on it. If $(E, \leq, \|\cdot\|)$ is a Banach lattice, every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

<u>Proof</u>. Since $\|\cdot\|_c$ and $\|\cdot\|$ are both complete they must be equivalent [14, Theorem 30.28]. Therefore, there exist real numbers $a,b > 0$ such that $\|x\| \leq a\|x\|_c \leq b\|x\|$ for all x in E. It easily follows that $\|\cdot\|$ is continuous because $\|\cdot\|_c$ is continuous. By the theorem, $\|\cdot\|$ extends uniquely to $\hat{E}$ and since $\|\cdot\|_M$ is complete, every extension of $\|\cdot\|$ to $\hat{E}$ is complete.

This corollary is also true if we assume that E is any Archimedean vector lattice such that $\hat{E}$ has a continuous, complete norm defined on it. For example, if E is any order dense subvector lattice of c_o (the real sequences which converge to zero) and $\hat{E} = c_o$ then if $\|\cdot\|$ is any complete norm on E every extension of $\|\cdot\|$ to $\hat{E} = c_o$ must be complete. This follows because $\|x\|_c = \sup(|x(n)|: n \in N)$ is a continuous, complete norm on c_o. Then since $\|\cdot\|_M$ is a complete norm on c_o, $\|\cdot\|_M$ must be equivalent to $\|\cdot\|_c$. This forces $\|\cdot\|_M$ to be continuous on $\hat{E} = c_o$. Then $\|\cdot\|$ must be continuous on E and the result follows.

Although we shall see that, in general, a semi-continuous, monotone norm on a vector lattice E need not extend uniquely to $\hat{E}$, for a very large class of semi-continuous norms we do get unique extensions.

Definition 1.4. If E is a vector lattice, $e \in E^+$ is said to be a strong unit for E if for every $x \in E$ there exists a real number $\lambda > 0$ such that $|x| \le \lambda e$. As in [23] we shall call a vector lattice with a strong unit a space of bounded elements.

If E is a space of bounded elements with strong unit e we can define a monotone seminorm on E by:

$$\|x\| = \inf(\lambda\colon \lambda e \ge |x|).$$

If E is Archimedean this seminorm is a norm and $\|\cdot\|$ is semi-continuous. To see this latter fact we choose a net $\{x_\alpha\colon \alpha \in \mathcal{A}\} \subseteq E$ such that $0 \le x_\alpha \uparrow x_o$ in E. By definition of $\|x_\alpha\|$ we have that $x_\alpha \le \|x_\alpha\|\cdot e$ for all $\alpha \in \mathcal{A}$. This implies that $\sup(\|x_\alpha\|\colon \alpha \in \mathcal{A})\cdot e = \sup(\|x_\alpha\|\cdot e\colon \alpha \in \mathcal{A}) \ge \sup(x_\alpha\colon \alpha \in \mathcal{A}) = x_o$. Then $\|\sup(\|x_\alpha\|\colon \alpha \in \mathcal{A})\cdot e\| = \sup(\|x_\alpha\|\colon \alpha \in \mathcal{A}) \ge \|x_o\|$. Since we always have that $\sup(\|x_\alpha\|\colon \alpha \in \mathcal{A}) \le \|x_o\|$ it follows that $\|\cdot\|$ is semi-continuous.

If E is an Archimedean space of bounded elements with strong unit e then $\hat{E}$ is a space of bounded elements with strong unit e and $\|\hat{x}\|_s = \inf(\lambda\colon \lambda e \ge |\hat{x}|)$ is a semi-continuous norm on $\hat{E}$. Using a characterization of $\hat{E}$ as a space of continuous, bounded functions defined on a compact set, V.A. Solovev [19, Theorem 5] proved that $\|\cdot\|$ extends uniquely to $\hat{E}$. We shall give a direct proof of the fact that semi-continuous norms of this type extend uniquely. However, first we need the following lemma and remark.

Lemma 1.5. Let $(E, \le, \|\cdot\|)$ be a normed vector lattice. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are extensions of $\|\cdot\|$ to $\hat{E}$ and both $\|\cdot\|_1$ and $\|\cdot\|_2$ are semi-continuous on $\hat{E}$ then $\|\hat{x}\|_1 = \|\hat{x}\|_2$ for all $\hat{x} \in \hat{E}$.

Proof. Choose any $\hat{x} \in \hat{E}$ and let $\{x_\alpha : \alpha \in \mathcal{A}\}$ be a net in E such that $0 \le x_\alpha \uparrow |\hat{x}|$ in $\hat{E}$. Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are semi-continuous we must have $\sup(\|x_\alpha\|_1 : \alpha \in \mathcal{A}) = \|\hat{x}\|_1$ and $\sup(\|x_\alpha\|_2 : \alpha \in \mathcal{A}) = \|\hat{x}\|_2$. But $\|\cdot\|_1$ and $\|\cdot\|_2$ agree on E. Therefore, $\|\hat{x}\|_2 = \sup(\|x_\alpha\|_2 : \alpha \in \mathcal{A}) = \sup(\|x_\alpha\|_1 : \alpha \in \mathcal{A}) = \|\hat{x}\|_1$.

Remark. It is known ([20, Lemma 1] and [6, Theorem 1.5]) that if $(E, \le, \|\cdot\|)$ is a normed vector lattice and $\|\cdot\|$ is semi-continuous then $\|\hat{x}\|^* = \sup(\|y\| : 0 \le y \le |\hat{x}|)$ is a semi-continuous norm extension of $\|\cdot\|$ to $\hat{E}$. As the example $E = c$ and $\|x\| = \sup(|x(n)| : n \in N) + \lim_{n\to\infty} |x(n)|$ demonstrates, this is not necessarily true when $\|\cdot\|$ is not semi-continuous. In this case $\|\hat{x}\|^* = \sup(|\hat{x}(n)|) + \underline{\lim}_{n\to\infty} |x(n)|$, which is not a norm because it is not sub-additive. Also whenever $\|\cdot\|$ is semi-continuous and we choose any $\hat{x} \in \hat{E}^+$ and any two nets $\{x_\alpha : \alpha \in \mathcal{A}\}$, $\{y_\beta : \beta \in \mathcal{B}\}$ in E such that $0 \le x_\alpha \uparrow \hat{x}$ and $0 \le y_\beta \uparrow \hat{x}$ then $\|\hat{x}\|^* = \sup(\|x_\alpha\| : \alpha \in \mathcal{A}) = \sup(\|y_\beta\| : \beta \in \mathcal{B})$. We can prove this latter statement by fixing some β_o and noting that $x_\alpha \wedge y_{\beta_o} \uparrow_\alpha \hat{x} \wedge y_{\beta_o} = y_{\beta_o}$. (The notation $x_\alpha \wedge y_{\beta_o}$ means $\inf(x_\alpha, y_{\beta_o})$). The semi-continuity of $\|\cdot\|$ implies that $\sup(\|x_\alpha \wedge y_{\beta_o}\| : \alpha \in \mathcal{A}) = \|y_{\beta_o}\|$. Then $x_\alpha \ge x_\alpha \wedge y_{\beta_o}$ for all α implies that $\sup(\|x_\alpha\| : \alpha \in \mathcal{A}) \ge \sup(\|x_\alpha \wedge y_{\beta_o}\| : \alpha \in \mathcal{A}) = \|y_{\beta_o}\|$ for all β_o. This shows that $\sup(\|x_\alpha\| : \alpha \in \mathcal{A}) \ge$

$\geq \sup(\|y_\beta\|: \beta \in \mathcal{B})$. A similar argument proves the other inequality.

Theorem 1.6. Let E be an Archimedean vector lattice with strong unit e. Define $\|x\| = \inf(\lambda: \lambda e \geq |x|)$. Then $\|\cdot\|$ extends uniquely to $\hat{E}$.

Proof. We note that $\|\cdot\|$ is semi-continuous on E so that $\|\cdot\|^*$ is a semi-continuous norm on $\hat{E}$. Hence, using Lemma 1.5 we need only show that $\|\cdot\|_M$ is semi-continuous on $\hat{E}$.

We shall prove that $\|\hat{x}\|_M = \inf(\lambda: \lambda e \geq |\hat{x}|)$ for all $\hat{x}$ in $\hat{E}$ because the right hand side is a semi-continuous norm on $\hat{E}$. Choose $\lambda > 0$ such that $\lambda e \geq |\hat{x}|$. Then by letting $w = \lambda e \in E$ and noting that $\|\lambda e\| = \lambda\|e\| = \lambda$ we have that $\{\lambda: \lambda e \geq |\hat{x}|\} \subseteq \{\|w\|: w \geq |\hat{x}|, w \in E\}$. Furthermore, if we choose $w \in E$ such that $w \geq |\hat{x}|$ it follows that $\|w\|\cdot e \geq |w| = w \geq |\hat{x}|$. And then $\{\|w\|: w \geq |\hat{x}|, w \in E\} \subseteq$ $\subseteq \{\lambda: \lambda e \geq |\hat{x}|\}$. Therefore, $\|\hat{x}\|_M = \inf(\lambda: \lambda\hat{e} \geq |\hat{x}|)$ and this completes the proof of the theorem.

Using Theorem 1.6 we know that whenever $(E, \leq, \|\cdot\|)$ is a Banach lattice of bounded elements every extension of $\|\cdot\|$ to $\hat{E}$ must be complete (because $\|\cdot\|_M$ is the only extension). However, it is also true if E is a vector lattice of bounded elements and $\|\cdot\|$ is any monotone norm which is complete on E, then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete. This result will follow from Corollary 1.12.

Having the results that a continuous norm on E must extend uniquely to $\hat{E}$ and a special class of semi-continuous norms extend uniquely to $\hat{E}$ we ask whether all semi-continuous norms extend uniquely to $\hat{E}$. We shall give an example of a normed vector lattice $(E, \leq, \|\cdot\|)$ such that $\|\cdot\|$ satisfies the property that if $\{x_\alpha : \alpha \in \mathcal{A}\}$ is any subset of E^+ (not necessarily directed upward) and $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_0$ then $\sup(\|x_\alpha\| : \alpha \in \mathcal{A}) = \|x_0\|$. But $\|\cdot\|$ does not extend uniquely to $\hat{E}$. This answers our question negatively since the stated property is stronger than semi-continuity. This example also shows that Theorem 3 in [20] is incorrect. Further, we shall prove that with an added condition on E, $\|\cdot\|$ does extend uniquely to $\hat{E}$.

<u>Example 1.7</u>. Let $E = c$ and $\hat{e} = (1,2,1,2,\ldots) \in \hat{E} = \ell_\infty$. $\hat{e}$ is a strong unit for ℓ_∞. Therefore, for any $\hat{x} \in \ell_\infty$ $\|\hat{x}\|_s = \inf(\lambda : \lambda\hat{e} \geq |\hat{x}|)$ defines a semi-continuous norm on ℓ_∞. It follows that $\|\cdot\| = \|\cdot\|_s|_E$ is a semi-continuous norm on E. We note that $\|\cdot\|$ also has the property that if $\{x_\alpha\} \subseteq E^+$ and $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_0$ then $\sup(\|x_\alpha\| : \alpha \in \mathcal{A}) = \|x_0\|$.

$$\|\hat{x}\|^* = \sup(\|y\| : 0 \leq y \leq |\hat{x}|,\ y \in E) = \|\hat{x}\|_s$$

since both $\|\cdot\|^*$ and $\|\cdot\|_s$ are semi-continuous norms on $\hat{E}$ which agree on E. (See Lemma 1.5). We shall show that $\|\hat{x}\|_M = \inf(\|w\| : w \geq |\hat{x}|,\ w \in E) \neq \|\hat{x}\|^*$ for some $\hat{x} \in \hat{E}$. Consider $\hat{e} = (1,2,1,2,\ldots)$. $\|\hat{e}\|^* = 1$. However, given any $w \in E = c$ such that $w \geq \hat{e}$ we must have $\lim_{n\to\infty} w(n) \geq 2$. It follows that $\|w\| \geq 2$ since $\|w\| = \inf(\lambda : \lambda\hat{e} \geq w)$. Therefore, $\|\hat{e}\|_M = 2 \neq 1 = \|\hat{e}\|^*$. This proves that $\|\cdot\|^*$ and $\|\cdot\|_M$ are two distinct extensions of $\|\cdot\|$ to $\hat{E}$.

Our previous example is one in which there exists a non semi-continuous extension, namely $\|\cdot\|_M$, to $\hat{E}$ of a semi-continuous norm $\|\cdot\|$ on E. However, $\|\cdot\|^*$ is an extension of $\|\cdot\|$ to $\hat{E}$. Of course, it is not always true that $\|\cdot\|^*$ is an extension of $\|\cdot\|$ to $\hat{E}$. As noted in a previous remark, if $E = c$ and $\|x\| = \sup(|x(n)|: n \in N) + \lim_{n\to\infty} |x(n)|$ then $\|\hat{x}\|^* = \sup(|x(n)|: n \in N) + \varliminf_{n\to\infty} |x(n)|$ is not a norm on $\hat{E} = \ell_\infty$. But in this example $\|\cdot\|$ is not σ-semi-continuous. Therefore, we ask whether there exists a normed vector lattice $(E, \leq, \|\cdot\|)$ such that $\|\cdot\|$ is σ-semi-continuous but $\|\cdot\|^*$ is not an extension of $\|\cdot\|$ to $\hat{E}$. Such an example would have to be non order separable.

The following example answers the above-mentioned question as well as shows that σ-semi-continuous norms need not extend uniquely.

Example 1.8. Let X be any uncountable set of points and let $E = \{f\colon f$ is a real-valued, bounded function on X and f is constant except possibly on a finite set of points in $X\}$. Then $\hat{E} = \{f\colon f$ is a real-valued, bounded function on $X\}$. For every $f \in E$ let $c(f)$ = that real number such that $\{x\colon f(x) = c(f)\}$ is uncountable. Also let $\|f\|_1 = \sup(|f(x)|: x \in X)$ and define $\|f\| = \|f\|_1 + 2|c(f)|$.

(i) $(E, \leq, \|\cdot\|)$ is a normed vector lattice and $\|\cdot\|$ is σ-semi-continuous on E. It is easy to see that $\|\cdot\|$ is a monotone norm on E. To show the σ-semi-continuity of $\|\cdot\|$ we choose any sequence $\{f_n\}$ in E such that $0 \leq f_n \uparrow f$ in E. If $c(f) = 0$ then $c(f_n) = 0$ for $n = 1,2,\ldots$. And

$\sup(\|f_n\|: n \in N) = \sup(\|f_n\|_1: n \in N) = \|f\|_1 = \|f\|$ since $\|\cdot\|_1$ is σ-semi-continuous. If $c(f) \neq 0$ and $\epsilon > 0$ is any real number, then there exists a positive integer N such that $c(f) - c(f_N) \leq \epsilon$. (It is crucial here that $\{f_n\}$ is a sequence and not an arbitrary net.) Hence, $2c(f_n) \geq 2c(f) - 2\epsilon$ for all $n \geq N$. Now, $\sup(\|f_n\|: n \in N) = \sup_n(\sup_x(f_n(x)) + 2c(f_n)) \geq \sup_n(\sup_x(f_n(x)) + 2c(f) - 2\epsilon) = \|f\|_1 + 2c(f) - 2\epsilon = \|f\| - 2\epsilon$. It follows that $\sup(\|f_n\|: n \in N) \geq \|f\|$. Since the other inequality always holds we must have $\sup(\|f_n\|: n \in N) = \|f\|$ and $\|\cdot\|$ is σ-semi-continuous.

(ii) $\|\hat{x}\|^* = \sup(\|y\|: 0 \leq y \leq |\hat{x}|)$ does not define a norm on $\hat{E}$. Let $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \phi$ and $\operatorname{card}(X_1) = \operatorname{card}(X_2) > \aleph_0$. Let $\chi_X, \chi_{X_1}, \chi_{X_2}$ denote the characteristic functions of X, X_1, and X_2, respectively. Then $\chi_X, \chi_{X_1}, \chi_{X_2}$ are elements of $\hat{E}$ and $\|\chi_X\|^* = 3$. However, $\|\chi_{X_1}\|^* = \|\chi_{X_2}\|^* = 1$ because both χ_{X_1} and χ_{X_2} are zero on an uncountable set. Hence, $\|\chi_{X_1} + \chi_{X_2}\|^* = 3 > 2 = \|\chi_{X_1}\|^* + \|\chi_{X_2}\|^*$ which shows that $\|\cdot\|^*$ is not a norm.

(iii) $\|\hat{x}\|_M = \inf(\|w\|: w \geq |\hat{x}|, w \in E)$ is not σ-semi-continuous. Let $\{x_n: n \in N\}$ be any countable set of points in X and define $f(x) = \begin{cases} 1 & \text{if } x = x_n \text{ for } n = 1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$

Then $f \in \hat{E}$. For $n = 1,2,\ldots,$ define

$$f_n(x) = \begin{cases} 1 & \text{if } x = x_k,\ k = 1,2,\ldots,n \\ 0 & \text{otherwise.} \end{cases}$$

Then $0 \leq f_n \uparrow f$ in $\hat{E}$ and $\sup(\|f_n\|_M: n \in N) = \sup(\|f_n\|: n \in N) = 1$. However, $\|f\|_M = 3$. Therefore, $\|f\|_M > \sup(\|f_n\|_M: n \in N)$. This proves that $\|\cdot\|_M$ is not σ-semi-continuous.

(iv) We now exhibit an extension of $\|\cdot\|$ distinct grom $\|\cdot\|_M$. Define $\|\hat{x}\|_{ML} = \inf(\lim_{n\to\infty} \|\hat{x}_n\|_M\colon 0 \le \hat{x}_n \uparrow |\hat{x}|$ in $\hat{E})$. Given any norm $\|\cdot\|$, $\|\cdot\|_L$ is always a seminorm. (See Section three for a thorough discussion of $\|\cdot\|_L$.) For our particular example we shall show that $\|\cdot\|_{ML}$ agrees with $\|\cdot\|$ on E. This fact forces $\|\cdot\|_{ML}$ to be a norm extension of $\|\cdot\|$ to $\hat{E}$. Obviously, $\|\cdot\|_{ML}$ cannot be equal to $\|\cdot\|_M$ on $\hat{E}$ because $\|\cdot\|_M$ is not σ-semi-continuous on $\hat{E}$. Hence, $\|\cdot\|_M$ and $\|\cdot\|_{ML}$ are two distinct extensions of $\|\cdot\|$ to $\hat{E}$.

Choose any f in E^+. If $c(f) = 0$ and $0 \le \hat{f}_n \uparrow f$ in $\hat{E}$ then we must have $\hat{f}_n \in E$ for $n = 1,2,\ldots$. And we know that $\sup(\|\hat{f}_n\|_M\colon n \in N) = \sup(\|\hat{f}_n\|\colon n \in N) = \|f\|$ since $\|\cdot\|$ is σ-semi-continuous on E. Suppose $c(f) \neq 0$. Then if $\epsilon > 0$ is any real number there exists a positive integer N such that $\{x\colon \hat{f}_N(x) \ge f(x) - \epsilon\}$ is an uncountable subset of X. Hence, $\hat{f}_N(x) \ge c(f) - \epsilon$ for uncountably many x in X. Then, $\|\hat{f}_N\|_M \ge \sup(\hat{f}_N(x)\colon x \in X) + 2c(f) - 2\epsilon$. We now have $\sup(\|\hat{f}_n\|_M\colon n \in N) \ge \sup_n(\sup_x(\hat{f}_n(x)) + 2c(f) - 2\epsilon) = \|f\|_1 +$ $+ 2c(f) - 2\epsilon = \|f\| - 2\epsilon$. It follows that for any sequence $\{\hat{f}_n\}$ in $\hat{E}$ with $0 \le \hat{f}_n \uparrow f$ we must have $\sup(\|\hat{f}_n\|_M\colon n \in N) =$ $= \|f\|$. Hence, $\|f\|_{ML} = \|f\|$ for all f in E.

We note that the norm defined in Example 1.8 is σ-continuous as well as σ-semi-continuous and $\|\cdot\|_M$ is not σ-continuous.

The following example is one in which $\|\cdot\|$ is σ-semi-continuous, $\|\cdot\|^*$ is not a norm, but $\|\cdot\|_M$ is σ-semi-continuous. We shall then indicate an interesting generalization of Examples 1.8 and 1.9.

Example 1.9. Let X be an uncountable set of points and $E = \{f: f$ is a real-valued, bounded function on X and f is constant except possibly on a countable set of points in $X\}$. Define $\|\cdot\|$ as in Example 1.8 and note that $\hat{E}$ is also as in Example 1.8. We merely remark that $\|\cdot\|$ is σ-semi-continuous and $\|\cdot\|^*$ is not a norm on $\hat{E}$ as the proofs are exactly like those in the previous example. However, $\|\cdot\|_M$ is σ-semi-continuous.

To show that $\|\cdot\|_M$ is σ-semi-continuous we first note that E is Dedekind σ-complete. Then we show that whenever $(E, \leq, \|\cdot\|)$ is a Dedekind σ-complete normed vector lattice and $\|\cdot\|$ is σ-semi-continuous, $\|\cdot\|_M$ must be σ-semi-continuous. The proof is due to V.A. Solovev [20, Theorem 1]. Suppose $0 \leq \hat{f}_n \uparrow \hat{f}$ in $\hat{E}$ and choose any real number $\epsilon > 0$. By definition of $\|\cdot\|_M$, there exists an element $x_n \in E$ such that $x_n \geq \hat{f}_n$ and $\|x_n\| \leq \|\hat{f}_n\|_M + \epsilon$ for $n = 1,2,\ldots$. We also choose an element x in E such that $x \geq \hat{f}$ and we can assume without loss of generality that $x_n \leq x$ for all $n \in N$. Define the element $y_n = \inf(x_k: k \geq n)$ in E, $n = 1,2,\ldots$. Since $y_n \leq x$ for all n there exists an element y in E such that $0 \leq y_n \uparrow y$. By the σ-semi-continuity of $\|\cdot\|$ we know that $\sup(\|y_n\|: n \in N) = \|y\|$. Also, $y \geq \hat{f}$ and $y_n \leq x_n$ for $n = 1,2,\ldots$

implies that $\|\hat{f}\|_M \le \|y\| = \sup(\|y_n\|: n \in N) \le \sup(\|\hat{f}_n\|_M: n \in N) + \epsilon$. Since $\epsilon > 0$ was arbitrary we have $\|\hat{f}\|_M \le \sup(\|\hat{f_n}\|_M: n \in N)$. It follows that $\|\cdot\|_M$ is σ-semi-continuous.

The previous two examples can be generalized in the following manner.

Example 1.10. Let $\aleph_1$ be any cardinal number such that $\aleph_1 > \aleph_0$. Choose a cardinal number $\aleph_2$ and a cardinal number $\aleph_3$ such that $\aleph_3 > \aleph_2 > \aleph_1$. Let X be a set such that $\operatorname{card}(X) = \aleph_3$ and let $E = \{f: f$ is a real-valued, bounded function on X and f is constant except possibly on a subset of X having cardinality less than or equal to $\aleph_1\}$. Then $\hat{E} = \{f: f$ is real-valued and bounded on $X\}$. For each f in E, let $c(f) =$ that real number such that $\{x: f(x) \ne c(f)\}$ has cardinality at most $\aleph_1$. For f in E let $\|f\|_1 =$ $= \sup(|f(x)|: x \in X)$ and define $\|f\| = \|f\|_1 + 2|c(f)|$. Again, it is easy to see that $(E, \le, \|\cdot\|)$ is a normed vector lattice.

(i) $\|\cdot\|$ is $\aleph_2$-semi-continuous on E. That is, if $\{\alpha: \alpha \in \mathcal{A}\}$ is any directed set such that its cardinality is less than or equal to $\aleph_2$ and $0 \le f_\alpha \uparrow f$ in E then $\sup(\|f_\alpha\|: \alpha \in \mathcal{A}) = \|f\|$. Suppose $0 \le f_\alpha \uparrow f$ in E as just described. If $c(f) = 0$ then $c(f_\alpha) = 0$ for all $\alpha \in \mathcal{A}$ and $\sup(\|f_\alpha\|: \alpha \in \mathcal{A}) = \|f\|$. If $c(f) \ne 0$ and $\epsilon > 0$ is any real number, there exists an index α_0 such that $c(f) - c(f_{\alpha_0}) < \epsilon$ (because $\operatorname{card}(\{\alpha: \alpha \in \mathcal{A}\}) \le \aleph_2 < \aleph_3$). Now, as in part (i) of Example 1.8, it follows that $\sup(\|f_\alpha\|: \alpha \in \mathcal{A}) =$

$\sup_\alpha(\sup_x(f_\alpha(x)) + 2c(f_\alpha)) \ge \sup_\alpha(\sup_x(f_\alpha(x)) + 2c(f) - 2\epsilon) =$

$= \|f\|_1 + 2c(f) - 2\epsilon = \|f\| - 2\epsilon$. Hence, $\sup(\|f_\alpha\|: \alpha \in \mathcal{A}) = \|f\|$.

(ii) $\|\cdot\|^*$ is not an extension of $\|\cdot\|$ to $\hat{E}$. Choose X_1, X_2 subsets of X such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and $\mathrm{card}(X_1) = \mathrm{card}(X_2) = \aleph_3$. The fact that $\|\cdot\|^*$ is not a norm follows just as in Example 1.8.

(iii) $\|\cdot\|_M$ is not $\aleph_2$-semi-continuous on $\hat{E}$. Let $\{x_\beta: \beta \in \mathcal{B}\}$ be a set of points in X having cardinality equal to $\aleph_2$ and define $f(x) = \begin{cases} 1 & \text{if } x = x_\beta \text{ for any } \beta \in \mathcal{B} \\ 0 & \text{otherwise.} \end{cases}$

Then $f \in \hat{E}$ and we shall define a net $\{f\lambda: \lambda \in \Lambda\}$ in E such that $\mathrm{card}(\{\lambda: \lambda \in \Lambda\}) = \aleph_2$ and $0 \leq f\lambda \uparrow f$ in $\hat{E}$. For each $\beta \in \mathcal{B}$ define $f_\beta(x) = \begin{cases} 1 & \text{if } x = x_\beta \\ 0 & \text{otherwise.} \end{cases}$

Then $\sup(f_\beta(x): \beta \in \mathcal{B}\} = f(x)$ for every $x \in X$. Consider the collection $\{f_\beta: \beta \in \mathcal{B}\}$ of elements in E and define $\{f\lambda: \lambda \in \Lambda\} = \{$collection of all finite suprema of the elements of $\{f_\beta\}\}$. Then $\{f\lambda: \lambda \in \Lambda\}$ is a directed net in E and $0 \leq f\lambda \uparrow f$ in $\hat{E}$. Also, $\mathrm{card}(\{\lambda: \lambda \in \Lambda\}) = \aleph_2$. It now follows that $\|\cdot\|_M$ is not $\aleph_2$-semi-continuous because $\sup(\|f\lambda\|_M: \lambda \in \Lambda) = \sup(\|f\lambda\|: \lambda \in \Lambda) = 1$ but $\|f\|_M = 3$.

If we let $E = \{f: f$ is a real-valued, bounded function on X and f is constant except possibly on a subset of X having cardinality at most $\aleph_2\}$ we would have an example of a normed vector lattice $(E, \leq, \|\cdot\|)$ such that $\|\cdot\|$ is $\aleph_2$-semi-continuous, $\|\cdot\|^*$ is not a norm, and $\|\cdot\|_M$ is $\aleph_2$-semi-continuous (noting that E is Dedekind $\aleph_2$-complete).

Although Examples 1.8 and 1.9 are special cases of Example 1.10 we present them all here so that we can see how the generalization in Example 1.10 can be motivated.

As we have seen by the previous examples, semi-continuity or σ-semi-continuity does not insure uniqueness of norm extensions. As Example 1.7 indicates, even the stronger condition that $\{x_\alpha\}$ is any collection of elements in E^+ (not necessarily directed upward) and $\sup(x_\alpha: \alpha \in \mathcal{A}) = x_o$ implies $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x_o\|$ does not force norm extensions to be unique. However, when we add the condition that $\bar{e} \wedge w \in E$ whenever $w \in E$ and $\bar{e} = \sup(\hat{x} \in \hat{E}^+: \|\hat{x}\|^* \leq 1)$ (which exists in $E^{\#}$ by applying Theorem 0.5) we shall have that $\|\cdot\|$ extends uniquely to $\hat{E}$. We note that this is precisely what fails to hold in Example 1.7. However, this condition is not a heavy restriction as several examples, including any space of bounded elements, will clearly indicate. After proving the following theorem, we shall apply it to the question of completeness of norm extensions.

<u>Theorem 1.11</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that $\{x_\alpha\} \subseteq E^+$ and $\sup(x_\alpha: \alpha \in \mathcal{A}) = x_o$ in E implies that $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x_o\|$. In addition, letting $\bar{e} = \sup(\hat{x} \in \hat{E}^+: \|\hat{x}\|^* \leq 1)$ (which exists in $E^{\#}$), we shall assume that $\bar{e} \wedge w \in E$ whenever $w \in E$. Then $\|\cdot\|$ extends uniquely to $\hat{E}$.

<u>Proof</u>. Since $\|\cdot\|$ is semi-continuous we can extend $\|\cdot\|$ to $\hat{E}$ by $\|\hat{x}\|^* = \sup(\|y\|: 0 \leq y \leq |\hat{x}|, y \in E)$. Also,

it is easy to see that for every $\{\hat{x}_\alpha : \alpha \in \mathcal{A}\} \subseteq \hat{E}^+$ such that $\sup(\hat{x}_\alpha : \alpha \in \mathcal{A}) = \hat{x}_o$ in $\hat{E}$ we must have $\sup(\|\hat{x}_\alpha\|^* : \alpha \in \mathcal{A}) = \|\hat{x}_o\|^*$. Hence, applying Theorem 0.5 we know that $\bar{e} = \sup(\hat{x} \in \hat{E}^+ : \|\hat{x}\|^* \le 1)$ exists in $E^\#$.

We extend $\|\cdot\|^*$ to $E^\#$ (allowing $\|\cdot\|^*$ to be infinite on some elements) as follows: for $\bar{x} \in E^\#$, define $\|\bar{x}\|^* = \sup(\|\hat{y}\|^* : 0 \le \hat{y} \le |\bar{x}|, \hat{y} \in \hat{E})$. Since $\|\cdot\|^*$ is semi-continuous on $\hat{E}$ it follows that if $\{\hat{y_\alpha} : \alpha \in \mathcal{A}\}$ is any net in $\hat{E}$ and $0 \le \hat{y_\alpha} \uparrow |\bar{x}|$ in $E^\#$ then $\|\bar{x}\|^* = \sup(\|\hat{y_\alpha}\|^* : \alpha \in \mathcal{A})$.

(i) $\|\bar{e}\|^* = 1$. For, let $A = \{\hat{x} \in \hat{E}^+ : \|\hat{x}\|^* \le 1\}$. A is a directed subset of $\hat{E}$ because given any $\hat{x}, \hat{y} \in A$ we must have $\|\hat{x} \vee \hat{y}\|^* = \|\hat{x}\|^* \vee \|\hat{y}\|^* \le 1$. Then $\hat{x} \vee \hat{y} \in A$ and $\hat{x} \vee \hat{y} \ge \hat{x}$, $\hat{x} \vee \hat{y} \ge \hat{y}$. Let $A = \{\hat{e}_\alpha : \alpha \in \mathcal{A}\} \subseteq \hat{E}$ where $\mathcal{A}$ is a directed set. Then $0 \le \hat{e}_\alpha \uparrow \bar{e}$ in $E^\#$ and since $\|\cdot\|^*$ is semi-continuous, $\sup(\|\hat{e_\alpha}\|^* : \alpha \in \mathcal{A}) = \|\bar{e}\|^*$. Now since $\|\hat{e}_\alpha\|^* \le 1$ for all α and some elements of A must have $\|\cdot\|^*$ equal to one we have that $\|\bar{e}\|^* = \sup(\|\hat{e}_\alpha\|^* : \alpha \in \mathcal{A}) = 1$.

(ii) For any $\bar{y} \in E^\#$, $|\bar{y}| \le \bar{e}$ if and only if $\|\bar{y}\|^* \le 1$. By the monotonicity of $\|\cdot\|^*$ on $E^\#$ we have that $|\bar{y}| \le \bar{e}$ implies that $\|\bar{y}\|^* \le 1$.

On the other hand, suppose $\|\bar{y}\|^* \le 1$ and let $0 \le \hat{x}_\alpha \uparrow |\bar{y}|$, $\{\hat{x_\alpha}\} \subseteq \hat{E}$. Then $\|\hat{x_\alpha}\|^* \le 1$ for all α so that $\hat{x}_\alpha \in A$ for all α. Therefore, $\bar{e} \ge \hat{x_\alpha}$ and we have that $|\bar{y}| = \sup(\hat{x_\alpha} : \alpha \in \mathcal{A}) \le \bar{e}$.

Now let $\bar{E} = \{\bar{x} \in E^\# : \|\bar{x}\|^* < +\infty\}$. Then $(\bar{E}, \le, \|\cdot\|^*)$ is a normed vector lattice with $\|\cdot\|^*$ semi-continuous, $\bar{E}$ is an ideal in $E^\#$ and $\bar{e} \in \bar{E}$. Also, we can show that $(\bar{E}, \le, \|\cdot\|^*)$ is a space of bounded elements with strong unit $\bar{e}$ and

$\|\bar{x}\|^* = \inf(\lambda\colon \lambda\bar{e} \geq |\bar{x}|)$. To prove this last statement we choose $\bar{x} \in \bar{E}$ and let $a = \|\bar{x}\|^*$. Then $\|1/a\, |\bar{x}|\|^* = 1$ and (ii) implies that $1/a\, |\bar{x}| \leq \bar{e}$. Therefore, $|\bar{x}| \leq a\bar{e}$. This shows that $\bar{e}$ is a strong unit for $\bar{E}$. Also, $\|\bar{x}\|^* = \inf(\lambda\colon \lambda\bar{e} \geq |\bar{x}|)$ since $|\bar{x}| \leq \|\bar{x}\|^* \cdot \bar{e}$, which follows easily from (ii).

In order to show that $\|\cdot\|$ extends uniquely to $\hat{E}$ we need only show that for every $\hat{x} \in \hat{E}$ $\|\hat{x}\|^* = \inf(\lambda\colon \lambda\bar{e} \geq |\hat{x}|) \geq$ $\geq \inf(\|w\|\colon w \geq |\hat{x}|,\ w \in E) = \|\hat{x}\|_M$. Choose any $\lambda > 0$ such that $\lambda\bar{e} \geq |\hat{x}|$. By hypothesis, $w \wedge \bar{e} \in E$ whenever $w \in E$. Hence, if $w \geq |\hat{x}|$ and $w \in E$ then $w' = w \wedge \lambda\bar{e} \in E$ and $w' \geq |\hat{x}|$. Since $\|w'\| = \|w'\|^* \leq \|\lambda\bar{e}\|^* = \lambda$ it follows that for all λ such that $\lambda\bar{e} \geq |\hat{x}|$ there exists an element $w' \geq |\hat{x}|$ with $w' \in E$ such that $\|w'\| \leq \lambda$. Hence, $\inf(\|w'\|\colon w' \geq |\hat{x}|,\ w' \in E) \leq$ $\leq \inf(\lambda\colon \lambda\bar{e} \geq |\hat{x}|)$. This completes the proof of the theorem.

__Corollary 1.12__. Let E be a vector lattice having any monotone, complete norm $\|\cdot\|$ defined on it such that $(E, \leq, \|\cdot\|)$ satisfies the hypotheses of the theorem. Then if $|||\cdot|||$ is any monotone, complete norm on E every extension of $|||\cdot|||$ to $\hat{E}$ must be complete.

__Proof__. Let $(E, \leq, \|\cdot\|)$ be any Banach lattice satisfying the hypotheses of the theorem and $|||\cdot|||$ any complete, monotone norm on E. Then $\|\cdot\|$ and $|||\cdot|||$ are equivalent. Hence, there exist constants $a,b > 0$ such that $\|x\| \leq a|||x||| \leq b\|x\|$ for all x in E. Let $|||\cdot|||_1$ be any extension of $|||\cdot|||$ to $\hat{E}$. Then $\|\hat{x}\|^* \leq a|||\hat{x}|||_1 \leq b\|\hat{x}\|_M$ for all $\hat{x} \in \hat{E}$. But $\|\hat{x}\|^* = \|\hat{x}\|_M$ so that $|||\cdot|||_1$ is equivalent to $\|\cdot\|_M$. This implies that $|||\cdot|||_1$ is complete on $\hat{E}$.

Example 1.13. Let $E = c$ and $\|x\| = \sup(|x(n)|: n \in N)$. Then $(E, \le, \|\cdot\|)$ satisfies the hypotheses of Theorem 1.11 and $\|\cdot\|$ is complete on E. Now defining $|||x||| = \sup(|x(n)|: n \in N) + \lim_{n\to\infty} |x(n)|$ we know that $(E, \le, |||\cdot|||)$ is a Banach lattice and several distinct extensions of $|||\cdot|||$ to $\hat{E}$ exist. (See Example 0,1.) However, according to Corollary 1.12, every extension of $|||\cdot|||$ to $\hat{E}$ must be complete.

Remark. Corollary 1.12 can be applied to any Archimedean vector lattice with strong unit e and any complete, monotone norm $|||\cdot|||$ defined on E. Let $\|x\| = \inf(\lambda: \lambda e \ge |x|)$. Then $|||\cdot|||$ and $\|\cdot\|$ are equivalent on E. To prove this latter fact we note that $\|\cdot\|$-convergence always coincides with (r)-convergence ($x_n \xrightarrow{r} x$ if there exists a z in E and a sequence of real numbers $\lambda_n \downarrow 0$ such that $|x - x_n| \le \lambda_n z$ for $n = 1,2,\ldots$) [23, Theorem VII.4.1]. And $x_n \to x$ in norm $|||\cdot|||$ and $|||\cdot|||$ complete implies every subsequence of $\{x_n\}$ has a subsequence which (r)-converges to x [22, Theorem VII.2.1].

Therefore, $(E, \le, \|\cdot\|)$ satisfies the hypotheses of Theorem 1.11. This forces every extension of $|||\cdot|||$ to $\hat{E}$ to be complete.

As illustrated in the foregoing paragraphs, if $(E, \le, \|\cdot\|)$ is a normed vector lattice various properties on $(E, \le, \|\cdot\|)$ force $\|\cdot\|$ to extend uniquely to $\hat{E}$. As indicated, these properties have applications to the problem of when every extension of a complete, monotone norm on E must be complete on $\hat{E}$. Therefore, we consider relationships among the following properties in order to clarify the preceding results:

(P1). Every set of elements $\{x_\alpha\} \subseteq E^+$ which is bounded above in E has an upper bound x_o in E such that $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x_o\|$.

(P2). Every set of elements $\{x_\alpha\} \subseteq E^+$ which is bounded above in E has the property that $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) =$ $= \inf(\|y\|: y$ is an upper bound for $\{x_\alpha\})$.

(P3). If $\{x_\alpha\}$ is any set of elements in E^+ such that $\sup(x_\alpha: \alpha \in \mathcal{A}) = x_o$ in E, then $\sup(\|x_\alpha\|: \alpha \in \mathcal{A}) = \|x_o\|$.

(P4). $\|\cdot\|$ is semi-continuous.

(P5). $\|\cdot\|$ is continuous.

Remarks.

(i). (P1) implies (P2). (P2) implies (P3). (P3) implies (P4). Also, (P5) implies (P4). However, $E = c$ and $\|x\| = \sup(|x(n)|: n \in N)$ demonstrates that (P4) does not imply (P5).

(ii). If E is Dedekind complete all of (P1), (P2), and (P3) are equivalent.

(iii). As the example $E = \ell_1$ (the real, summable sequences) and $\|x\| = \sum_{n=1}^{\infty} |x(n)|$ demonstrates, (P4) does not imply (P3) (and, therefore, does not imply (P2) or (P1)). This confirms the fact that (P3) is stronger than semi-continuity.

(iv). If $(E, \leq, \|\cdot\|)$ satisfies (P3), then $(\hat{E}, \leq, \|\cdot\|^*)$ also satisfies (P3). This fact was actually used in the beginning of the proof of Theorem 1.11. We shall now prove it for the convenience of the reader. Let $\{\hat{x}_\alpha: \alpha \in \mathcal{A}\}$ be a collection of elements in $\hat{E}^+$ such that $\sup(\hat{x}_\alpha : \alpha \in \mathcal{A}) = \hat{x}_o$ in $\hat{E}$. Since $\hat{E}$ is a Dedekind completion of E, for each $\alpha \in \mathcal{A}$ there exists a net $\{x_{\lambda,\alpha} : \lambda \in \Lambda\} \subseteq E$ such that $x_{\lambda,\alpha} \uparrow_\lambda \hat{x}_\alpha$. We know that $\|\cdot\|^*$ is semi-continuous on $\hat{E}$. Hence, $\sup(\|x_{\lambda,\alpha}\| : \lambda \in \Lambda) = \|\hat{x}_\alpha\|^*$ for all $\alpha \in \mathcal{A}$. Also, $\sup_\alpha(\sup_\lambda(x_{\lambda,\alpha})) = \hat{x}_o$ in $\hat{E}$. Now let $\{y_\beta : \beta \in \mathcal{B}\} =$ $=$ {the collection of all finite suprema of the elements $\{x_{\lambda,\alpha} : \alpha \in \mathcal{A}, \lambda \in \Lambda\}$}. Then $y_\beta \uparrow \hat{x}_o$ in $\hat{E}$ and $\sup(\|y_\beta\| : \beta \in \mathcal{B}) = \sup_\alpha(\sup_\lambda \|x_{\lambda,\alpha}\|)$ since $(E, \leq, \|\cdot\|)$ satisfies (P3). Also, $\|\cdot\|^*$ semi-continuous on $\hat{E}$ implies that $\sup(\|y_\beta\| : \beta \in \mathcal{B}) = \|\hat{x}_o\|^*$. Now, $\|\hat{x}_o\|^* = \sup(\|y_\beta\| : \beta \in \mathcal{B}) =$ $= \sup_\alpha(\sup_\lambda \|x_{\lambda,\alpha}\|) = \sup_\alpha(\|\hat{x}_\alpha\|^* : \alpha \in \mathcal{A})$. It follows that $(\hat{E}, \leq, \|\cdot\|^*)$ satisfies (P3).

Since (P3), (P2) and (P1) are all equivalent on a Dedekind complete vector lattice it follows that whenever $(E, \leq, \|\cdot\|)$ satisfies (P3), $(\hat{E}, \leq, \|\cdot\|^*)$ satisfies (P1), (P2) and (P3).

(v). If $(E, \leq, \|\cdot\|)$ satisfies (P2) then $\|\cdot\|$ extends uniquely to $\hat{E}$. However, as the example in (iii) indicates, $(E, \leq, \|\cdot\|)$ such that $\|\cdot\|$ extends uniquely to $\hat{E}$ does not necessarily imply (P2).

Most of the properties we have discussed up to this point have been restrictions on the norm rather than on the vector lattice. We shall now see that whenever E is "close to" $\hat{E}$ (in a sense that we shall define) and $(E, \leq, \|\cdot\|)$ is a normed vector lattice then $\|\cdot\|$ extends uniquely to $\hat{E}$.

<u>Definitions 1.14</u>. A band B in E is a <u>projection band</u> whenever given any $x \in E^+$ $\sup(y: y \in B$ and $y \leq x)$ exists in E. $\sup(y: y \in B$ and $y \leq x)$ is an element of B and is called the <u>projection of</u> x <u>onto</u> B. A vector lattice E is said to have the <u>projection property</u> whenever every band in E is a projection band.

Whenever a vector lattice E has the projection property, E is "close to" $\hat{E}$ in the sense that many properties which hold on E also hold on $\hat{E}$. (See [22] for a discussion of this.) We note that c does not have the projection property. The band B generated by the set $\{x_n = (1,0,1,0,\ldots,1,0,0,0,\ldots): n \in N\}$ and the element $x = (1,1,1,\ldots)$ in c demonstrate this fact since the projection of x onto B does not exist in c. $(\sup(y \in B: y \leq x) = (1,0,1,0,1,0,\ldots) \notin c.)$

<u>Theorem 1.15</u>. If $(E, \leq, \|\cdot\|)$ is a normed vector lattice and E has the projection property, $\|\cdot\|$ extends uniquely to $\hat{E}$.

<u>Proof</u>. A.I. Veksler has shown [22, Lemma 3] that whenever E has the projection property and $\hat{x} \in \hat{E}^+$ there exists a monotone sequence $\{x_n\} \subseteq E^+$ such that $0 \leq x_n \uparrow \hat{x}$ in $\hat{E}$ and x_n (r)-converges to $\hat{x}$ in $\hat{E}$. ($x_n \xrightarrow{(r)} \hat{x}$ if and only if there exists an element $\hat{z} \in \hat{E}^+$ and a sequence of positive real numbers $\lambda_n \downarrow 0$ such that $|x_n - \hat{x}| \leq \lambda_n \hat{z}$ for $n = 1,2,\ldots$.)

Let $\|\cdot\|_1$ be any extension of $\|\cdot\|$ to $\hat{E}$ and choose $\hat{x} \in \hat{E}$. Using Veksler's result we know there exists a monotone sequence $\{x_n\} \subseteq E^+$ such that $x_n \uparrow |\hat{x}|$ and $x_n \xrightarrow{(r)} |\hat{x}|$.

Hence, there exists an element $\hat{z} \in \hat{E}^+$ and a sequence $\lambda_n \downarrow 0$ such that $\big|\, |\hat{x}| - x_n \big| \le \lambda_n \hat{z}$ for $n = 1,2,\ldots$. Then $\| \, |\hat{x}| - x_n \|_1 \to 0$ as $n \to \infty$. Hence, $\|\hat{x}\|_1 - \|x_n\| \to 0$ as $n \to \infty$. But then $\sup(\|x_n\| : n \in N) = \|\hat{x}\|_1$ where $\|\cdot\|_1$ is any extension of $\|\cdot\|$ to $\hat{E}$. This implies that $\|\hat{x}\|_1 = \|\hat{x}\|_M$ for all $\hat{x} \in \hat{E}$ and $\|\cdot\|_M$ is the unique extension of $\|\cdot\|$ to $\hat{E}$.

In [21, Theorem 3] Veksler proved that if E is an Archimedean vector lattice of bounded elements not having the projection property there exists a positive, order bounded (order bounded = bounded on order intervals) linear functional φ on E such that φ has at least two distinct, positive, order bounded extensions φ_1 and φ_2 to $\hat{E}$. Then $\|x\| = \inf(\lambda\colon \lambda e \ge |x|) + \varphi(|x|)$ is a monotone norm on E and $\|\hat{x}\|_1 = \inf(\lambda\colon \lambda e \ge |\hat{x}|) + \varphi_1(|\hat{x}|)$ and $\|\hat{x}\|_2 = \inf(\lambda\colon \lambda e \ge |\hat{x}|) + \varphi_2(|\hat{x}|)$ are two distinct extensions of $\|\cdot\|$ to $\hat{E}$. However, as noted in the remark following Example 1.13, if $\|\cdot\|$ is complete on E, both $\|\cdot\|_1$ and $\|\cdot\|_2$ are complete on $\hat{E}$.

All of the previous results have aided us in determining when every norm extension of a Banach lattice is complete. In concluding this section we shall prove that under a specific restriction on E, if $(E, \le, \|\cdot\|)$ is a Banach lattice, every extension of $\|\cdot\|$ to $\hat{E}$ must be locally norm complete.

<u>Definitions 1.16</u>. The <u>principal ideal</u> in E generated by an element x in E is the ideal $\{y \in E\colon |y| \le \alpha\, |x|$ for some real $\alpha \ge 0\}$. The <u>principal band</u> in E generated by an element x in E is the smallest band in E containing x.

In [25, Theorem 6.3], A.C. Zaanen considers an Archimedean vector lattice E such that every principal band in E has a strong unit. If we restrict E to satisfy this condition we obtain the following theorem.

<u>Theorem 1.17</u>. Let $(E, \leq, \|\cdot\|)$ be a Banach lattice and suppose that every principal band in E has a strong unit. Then every extension of $\|\cdot\|$ to $\hat{E}$ must be locally norm complete.

<u>Proof</u>. Suppose $\|\cdot\|_1$ is an extension of $\|\cdot\|$ to $\hat{E}$, $\{\hat{x}_n\} \subseteq \hat{E}$ such that $\|\hat{x}_n - \hat{x}_m\|_1 \to 0$ as $n,m \to \infty$ and $|\hat{x}_n| \leq \hat{x}_o \in \hat{E}$ for all $n \in N$. Choose an element x in E such that $x \geq \hat{x}_o$. Let I_x be the ideal in E generated by x; i.e., $I_x = \{u \in E: |u| \leq \alpha x$ for some real number $\alpha \geq 0\}$. Let B_x be the order closure of I_x in E. Then B_x is the band in E generated by x. Since B_x is order closed in E, B_x is a norm closed subset of E [9, Theorem 35.5], and, therefore, $(B_x, \leq, \|\cdot\|)$ is a Banach lattice. Also, since B_x has a strong unit, every extension of $\|\cdot\|$ to $B_x^\wedge$ must be complete. (See the remark following Example 1.13.)

B_x an ideal in E implies that $B_x^\wedge \subseteq \hat{E}$ and $B_x^\wedge$ is an ideal in $\hat{E}$. Hence, $\hat{x}_o \in B_x^\wedge$ and $\hat{x}_n \in B_x^\wedge$ for $n = 1,2,\ldots$. Also, $\|\cdot\|_1|B_x^\wedge$ is an extension of $\|\cdot\|$ from B_x to $B_x^\wedge$. Therefore, $(B_x^\wedge, \leq, \|\cdot\|_1)$ must be a Banach lattice. This implies that there exists an element $\bar{x} \in B_x^\wedge$ such that $\|\bar{x} - \hat{x}_n\|_1 \to 0$ as $n \to \infty$. It follows that $(\hat{E}, \leq, \|\cdot\|_1)$ is a locally norm complete vector lattice.

<u>Remark</u>. If $\|\cdot\|$ is semi-continuous, $(\hat{E}, \leq, \|\cdot\|^*)$ is a Dedekind complete normed vector lattice with $\|\cdot\|^*$ semi-continuous. This implies that $\|\cdot\|^*$ is locally norm complete (Theorem 0.10). However, we do not necessarily have that every extension of $\|\cdot\|$ to $\hat{E}$ is locally norm complete. That is, $\|x\|_1 \leq \|x\|_2$ for all x and $\|\cdot\|_1$ locally norm complete does not necessarily imply that $\|\cdot\|_2$ is locally norm complete. As a counterexample to the latter statement let $E = \ell_\infty$, $\|x\|_1 = \sum_{n=1}^{\infty} (|x(n)|/2^n)$ and $\|x\|_2 = \sum_{n=1}^{\infty} (|x(n)|/2^n) + \overline{\lim_{n\to\infty}} |x(n)|$.

Section Two. The Boundedness Property.

In any normed space the norm boundedness of a set B can be characterized by the property that for any sequence $\{x_n\} \subseteq B$ and any sequence of positive real numbers $\lambda_n \downarrow 0$ we must have $\{\lambda_n x_n\}$ converging to zero in norm as $n \to \infty$. The question of whether an analogous property in an Archimedean vector lattice E can characterize the order boundedness of a subset B was first considered by L.V. Kantovovich [5]; i.e. given any Archimedean vector lattice E can we say that a subset B in E is order bounded if and only if given any sequence $\{x_n\} \subseteq B$ and any sequence of positive real numbers $\lambda_n \downarrow 0$ then $\{\lambda_n x_n\}$ order converges to zero as $n \to \infty$. Of course, if B is order bounded, $\{x_n\} \subseteq B$ and $\lambda_n \downarrow 0$ then $\{\lambda_n x_n\}$ order converges to zero because E is Archimedean. However, the subset $B = \{x_n = (1,1,1,\ldots,1,0,\ldots): n \in N\}$ is not order bounded in c_o (the real sequences converging to zero) and given any sequence of positive real numbers $\lambda_n \downarrow 0$, $\{\lambda_n x_n\} = \{(\lambda_n,\lambda_n,\ldots,\lambda_n,0,0,\ldots)\}$ does order converge to zero as $n \to \infty$. Therefore, in general, we cannot characterize the order boundedness of a set in this manner.

As in [16, p. 51] we shall say that E has the <u>boundedness property</u> if a subset B of E is order bounded whenever $\lambda_n x_n \xrightarrow{o} 0$ as $n \to \infty$ for every sequence $\{x_n\} \subseteq B$ and every sequence $\{\lambda_n\}$ of positive real numbers such that $\lambda_n \downarrow 0$. The importance of the boundedness property in the present paper centers around one of the main results of this section: If $(E, \leq, \|\cdot\|)$ is a Banach lattice and E has the boundedness property, then all extensions of $\|\cdot\|$ to $\hat{E}$ must be complete.

Definitions 2.1. A normed vector lattice $(E, \leq, \|\cdot\|)$ is said to be monotone complete (monotone bounded) whenever $\{x_\alpha : \alpha \in \mathcal{A}\}$ in E^+ and $0 \leq x_\alpha \uparrow$ in E with $\|x_\alpha\| \leq M$ for all α implies $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_o$ exists in E ($\{x_\alpha\}$ is a bounded set in E). Replacing arbitrary nets by sequences gives us the definitions for monotone σ-complete and monotone σ-bounded.

Remark. A normed vector lattice $(E, \leq, \|\cdot\|)$ is monotone bounded (monotone σ-bounded) if and only if $0 \leq x_\alpha \uparrow$ and $\{x_\alpha : \alpha \in \mathcal{A}\}$ an unbounded subset of E ($0 \leq x_n \uparrow$ and $\{x_n : n \in N\}$ an unbounded subset of E) implies that $\|x_\alpha\| \uparrow +\infty$ ($\|x_n\| \uparrow +\infty$). Also, if $\|\cdot\|$ is monotone complete (monotone σ-complete), E must be Dedekind complete (Dedekind σ-complete).

I. Amemiya [1] has proved that whenever $(E, \leq, \|\cdot\|)$ is a normed vector lattice and $\|\cdot\|$ is monotone σ-complete then $\|\cdot\|$ is norm complete. S. Yamamuro [24, Theorem 1.1] introduced a property similar to the boundedness property (i.e., property K: if $0 \leq x_n \uparrow$ and $\{x_n\}$ is unbounded in E, then there exists a sequence of positive real numbers $\lambda_n \downarrow 0$ such that $\{\lambda_n x_n\}$ is unbounded in E) and proved that whenever $(E, \leq, \|\cdot\|)$ is a Banach lattice and E has property K then $\|\cdot\|$ must be monotone σ-bounded. It is not difficult to show [19, Lemma 5] that if $(E, \leq, \|\cdot\|)$ is monotone σ-bounded and

E is order separable then every extension of $\|\cdot\|$ to $\hat{E}$ must be monotone σ-complete. Combining these three results we know that whenever $(E, \leq, \|\cdot\|)$ is a Banach lattice and E is order separable and has property K then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

We have indicated the importance of the boundedness property and property K to the question of equivalence of norm extensions of a Banach lattice. Before obtaining the results mentioned above we shall discuss the relationships among property K, the boundedness property and the following two properties on an Archimedean vector lattice E:

(A): Given any unbounded set $B \subseteq E$ B must have a countable unbounded subset.

(B): If $\{A_n\}$ is a countable collection of unbounded subsets in E, there exist finite subsets $A_n' \subseteq A_n$ $n = 1,2,\ldots,$ such that $\{\sup(A_n'): n = 1,2,\ldots\}$ is an unbounded set in E. If E satisfies this property we say that E is <u>finitely unbounded</u> [17].

<u>Remark</u>. The following are some well-known results about the above-mentioned properties:

(i). Any Archimedean vector lattice E with a countable subset $\{h_n: n \in N\} \subseteq E^+$ such that for any x in E there exists a positive integer $N(x)$ and a real number $\lambda(x) > 0$ such that $|x| \leq \lambda h_N$ must have the boundedness property [17, Proposition 1.8].

(ii). Any perfect sequence space must have the boundedness property [17, Proposition 1.7].

(iii). Any Archimedean vector lattice which is finitely unbounded has the boundedness property [12, Proposition 2.5].

(iv). Any Archimedean vector lattice with the boundedness property satisfies property (A) [23, Theorem VI.6.3].

We shall now prove that the boundedness property implies property K. However, property K does not imply the boundedness property.

<u>Theorem 2.2</u>. If E has the boundedness property then E must have property K.

<u>Proof</u>. Let $\{x_n\} \subseteq E^+$ such that $x_n \uparrow +\infty$ (i.e. $x_n \uparrow$ and $\{x_n\}$ is unbounded in E). We shall show that there exists a sequence of positive real numbers $\alpha_n \downarrow 0$ such that $\{\alpha_n x_n\}$ is unbounded in E. E having the boundedness property implies that there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and a sequence of positive numbers $\lambda_k \downarrow 0$ such that $\lambda_k x_{n_k} \not\xrightarrow{o} 0$. Let $\alpha_k = (\lambda_k)^{\frac{1}{2}}$. Then $\alpha_k \downarrow 0$ and $\{\alpha_k x_{n_k}\}$ is unbounded in E because if $\alpha_k x_{n_k} \leq z \in E$ for all $k \in N$ then $\lambda_k x_{n_k} \leq \alpha_k z$ which implies that $\lambda_k x_{n_k} \xrightarrow{o} 0$ as $k \to \infty$.

Now for $n = 1,2,\ldots$ define $\alpha_n = \alpha_k$ if $n_{k-1} < n \leq n_k$ $(n_o = 1)$. Then $\alpha_n \downarrow 0$ and $\{\alpha_n x_n\}$ is unbounded in E.

<u>Corollary 2.3</u>. If $(E, \leq, \|\cdot\|)$ is a Banach lattice and E has the boundedness property then $\|\cdot\|$ must be monotone σ-bounded.

<u>Proof</u>. This corollary follows directly from the theorem and Yamamuro's result [24, Theorem 1.1].

Example 2.4. Let $E = \{f: f$ is a real-valued, bounded function on $[0,1]$, $f(0) = 0$, and $f(0) \neq f(x)$ for at most countably many x in $[0,1]\}$. We shall show that (i) E does not satisfy the boundedness property, but (ii) E does have property K.

(i). Let $B = \{f_x \in E: f_x(t) = x$ if $t = x$ and $f_x(t) = 0$ if $t \neq x: x \in [0,1]\}$. B is an unbounded subset of E with no countable unbounded subset; i.e., E does not satisfy property (A). But the boundedness property implies property (A). (See Remark (iv)). Therefore, E does not have the boundedness property.

(ii). In order to show that E has property K we note that $\|f\| = \sup(|f(t)|: t \in [0,1])$ is a monotone σ-complete, monotone norm on E and we shall show that any normed vector lattice $(E, \leq, \|\cdot\|)$ which is monotone σ-bounded must have property K:

Let $0 \leq x_n \uparrow +\infty$ in E. Since $\|\cdot\|$ is monotone σ-bounded we must have $\|x_n\| \uparrow +\infty$. Let $\lambda_n = 1/(\|x_n\|)^{\frac{1}{2}}$ which is a sequence of positive real numbers decreasing to 0. Then $\|\lambda_n x_n\| = (\|x_n\|)^{\frac{1}{2}} \uparrow +\infty$. Hence, $\{\lambda_n x_n\}$ is unbounded in E. This proves that E has property K.

We also note that E is Dedekind complete and order separable, but $(E, \leq, \|\cdot\|)$ is not monotone bounded.

Since Example 2.4 does not satisfy property (A) we ask whether property K and property (A) together imply the boundedness property.

Theorem 2.5. An Archimedean vector lattice E has the boundedness property if and only if E has property K and property (A).

Proof. We already know that property K and property (A) are implied by the boundedness property. Therefore, we need only prove that property K and property (A) together imply the boundedness property.

Choose a set $B \subseteq E$ such that B is unbounded in E. E having property (A) implies there exists a countable unbounded subset $\{x_n\} \subseteq B^+$. Let $\{y_k\} = \{x_1 \vee x_2 \vee \cdots \vee x_k\}$ for $k = 1,2,\ldots$. Then $0 \le y_k \uparrow +\infty$ in E. Therefore, applying property K, there exists a sequence of positive real numbers $\lambda_k \downarrow 0$ such that $\{\lambda_k y_k\}$ is unbounded in E. Then $\{\lambda_k x_k\}$ is unbounded in E because if $\lambda_k x_k \le z \in E$ for $k = 1,2,\ldots$ then $\lambda_k y_k = \lambda_k x_1 \vee \lambda_k x_2 \vee \cdots \vee \lambda_k x_k \le$ $\le \lambda_1 x_1 \vee \lambda_2 x_2 \vee \cdots \vee \lambda_k x_k \le z$ for $k = 1,2,\ldots$. Now $\{\lambda_k x_k\}$ unbounded in E implies that $\lambda_k x_k \not\xrightarrow{o} 0$ in E. Therefore, E has the boundedness property.

Since we are interested in extending norms from E to $\hat{E}$ we shall ask questions concerning the extension of the above-mentioned properties from E to $\hat{E}$. In the following we shall prove that E has the boundedness property if and only if $\hat{E}$ has the boundedness property.

Lemma 2.6. If E is any Archimedean vector lattice having the boundedness property, then $\hat{E}$ must have property K.

Proof. Let $0 \leq \hat{x}_n \uparrow +\infty$ in $\hat{E}$ and for $n = 1,2,\ldots,$ choose a net $\{x_{\alpha,n}: \alpha \in \mathcal{A}\}$ in E such that $0 \leq x_{\alpha,n} \uparrow_\alpha \hat{x}_n$. Let $\{y_\beta: \beta \in \mathcal{B}\} = \{$collection of all finite suprema of the elements $\{x_{\alpha,n}: \alpha \in \mathcal{A}, n \in N\}\} \subseteq E$. Then $0 \leq y_\beta \uparrow +\infty$ in E. E having the boundedness property and $\{y_\beta: \beta \in \mathcal{B}\}$ being unbounded in E implies there exists a sequence $\{y_{\beta_k}\} \subseteq \{y_\beta\}$ which is unbounded in E. Then, there exists a sequence of positive numbers $\lambda_k \downarrow 0$ such that $\{\lambda_k y_{\beta_k}\}$ is unbounded in E.

For each k, choose $\hat{x_{n_k}} \geq y_{\beta_k}$. Then $\{\lambda_k \hat{x_{n_k}}\}$ is unbounded in $\hat{E}$. If we let $\lambda_n = \lambda_k$ for $n_{k-1} < n \leq n_k$ $(n_o = 1)$ we shall have $\lambda_n \downarrow 0$ and $\{\lambda_n \hat{x_n}\}$ is unbounded in $\hat{E}$. This completes the proof of the lemma.

Remark. If $\hat{E}$ has property K it easily follows that E has property K.

Lemma 2.7. An Archimedean vector lattice E has property (A) if and only if $\hat{E}$ has property (A).

Proof. Suppose $B \subseteq \hat{E}$ and B is unbounded in $\hat{E}$. Let $\bar{A} = \{\hat{x} \in \hat{E}: |\hat{x}| \leq \hat{y}$ for some $\hat{y} \in B\}$. $\bar{A}$ is unbounded in $\hat{E}$. Let $A = \bar{A} \cap E$. It then follows that A is unbounded in E because every $\hat{x} \in \bar{A}$ is such that $\hat{x} = \sup(x_\alpha: x_\alpha \leq \hat{x}, x_\alpha \in A)$ Assuming that E has property (A), there exists a sequence $\{x_n\} \subseteq A$ such that $\{x_n\}$ is unbounded in E. For $n = 1,2,\ldots,$ choose $\hat{y_n} \in B$ such that $\hat{y_n} \geq x_n$. Then $\{\hat{y_n}\}$ is a countable subset of B which is unbounded in $\hat{E}$. It follows that $\hat{E}$ has property (A) whenever E does.

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The fact that E has property (A) implies that $\hat{E}$ has property (A) is a direct result of the definition.

Theorem 2.8. An Archimedean vector lattice E has the boundedness property if and only if $\hat{E}$ has the boundedness property.

Proof. Recall that if E has the boundedness property then $\hat{E}$ has property K (Lemma 2.6). Also, property K and property (A) together imply the boundedness property (Theorem 2.5). Assuming that E has the boundedness property, it must have property (A) (Remark (iv)) and by the above lemma $\hat{E}$ has property (A). Therefore, $\hat{E}$ has property K and property (A) so it must have the boundedness property.

On the other hand, assume $\hat{E}$ has the boundedness property and let B be an unbounded subset of E. Let $\overline{B} = \{\hat{x} \in \hat{E}: |\hat{x}| \le |y|$ for some $y \in B\} \subseteq \hat{E}$. Then $\overline{B}$ is unbounded in $\hat{E}$. Therefore, there exists a sequence $\{\hat{x}_n\} \subseteq \overline{B}$ and a sequence of positive numbers $\lambda_n \downarrow 0$ such that $\lambda_n \hat{x}_n \overset{o}{\not\to} 0$. For $n = 1,2,\ldots$, choose $y_n \in B$ such that $|y_n| \ge |\hat{x}_n|$. Then $\lambda_n y_n \overset{o}{\not\to} 0$ because if $\lambda_n |y_n| \le z_n \in E$ and $z_n \downarrow 0$ then $\lambda_n |\hat{x}_n| \le \lambda_n |y_n| \le z_n$ which would imply that $\lambda_n \hat{x}_n \overset{o}{\to} 0$. Therefore, we have proved that E has the boundedness property.

Given an Archimedean vector lattice E which is finitely unbounded, it is easy to see that it also has property (A). However, S (the space of all real sequences) is not finitely unbounded [17, p. 324] but S does satisfy property (A). In fact, S has the boundedness property since S is a regular, extended space. (See [22, p. 165-172]). Hence, although every Archimedean vector lattice which is finitely unbounded must have the boundedness property (Remark (iii)), the implication does not reverse.

We now recall the example of a normed vector lattice $(E, \leq, \|\cdot\|)$ such that $\|\cdot\|$ was monotone σ-bounded but not monotone-bounded (Example 2.4). A natural question to ask at this point is which property (or properties) on E will force every monotone σ-bounded, monotone norm to be monotone bounded. We note that order separability of E will not do it since Example 2.4 is order separable.

<u>Theorem 2.9</u>. It E has property (A) and $(E, \leq, \|\cdot\|)$ is a monotone σ-bounded normed vector lattice, then $\|\cdot\|$ is monotone bounded.

<u>Proof</u>. Suppose $\{x_\alpha\}$ is a net in E^+ and $0 \leq x_\alpha \uparrow +\infty$ in E. E having property (A) implies there exists a sequence $\{x_{\alpha_n} : n \in N\} \subseteq \{x_\alpha : \alpha \in \mathcal{A}\}$ such that $\{x_{\alpha_n}\}$ is unbounded in E. If the set $\{x_{\alpha_n}\}$ is directed upward we must have $\|x_{\alpha_n}\| \uparrow +\infty$ because $\|\cdot\|$ is monotone σ-bounded and it would then follow that $\|x_\alpha\| \uparrow +\infty$.

If $\{x_{\alpha_n} : n \in N\}$ is not directed upward we define $\{y_n : n \in N\}$ as follows:

$$\begin{aligned} y_1 &= x_{\alpha_1} \\ y_2 &= x_{\alpha_1} \vee x_{\alpha_2} \\ &\vdots \\ y_n &= x_{\alpha_1} \vee \cdots \vee x_{\alpha_n} \\ &\vdots \end{aligned}$$

Then $0 \leq y_n \uparrow +\infty$ in E which implies that $\|y_n\| \uparrow +\infty$. For $n = 1,2,\ldots$, choose $z_{\alpha_n} \in \{x_\alpha\}$ such that $z_{\alpha_n} \geq y_n$. z_{α_n} exists because $\{x_\alpha\}$ is directed upward. Now $\{z_{\alpha_n}\} \subseteq \{x_\alpha\}$ and $\sup(\|z_{\alpha_n}\|: n \in N) = +\infty$. It follows that $\|x_\alpha\| \uparrow +\infty$ and $(E, \leq, \|\cdot\|)$ is monotone bounded.

V.A. Solovev proved [19, Theorem 5] that if $(E, \leq, \|\cdot\|)$ is a normed vector lattice and $\|\cdot\|$ is monotone bounded then every extension of $\|\cdot\|$ to $\hat{E}$ must be monotone complete. Combining this result with the other theorems quoted or proved in the preceding paragraphs we can state the main conclusions of this section.

<u>Theorem 2.10</u>. If E has the boundedness property and $(E, \leq, \|\cdot\|)$ is a Banach lattice then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

<u>Corollary 2.11</u>. If E is finitely unbounded and $(E, \leq, \|\cdot\|)$ is a Banach lattice then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

<u>Theorem 2.12</u>. If E is order separable and has property K and $(E, \leq, \|\cdot\|)$ is a Banach lattice, then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

<u>Corollary 2.13</u>. If E is order separable, $\hat{E}$ has property K and $(E, \leq, \|\cdot\|)$ is a Banach lattice then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

<u>Example 2.14</u>. Let $E = c$ and $\|x\| = \sup(|x(n)|) + \lim_{n\to\infty} |x(n)|$. E has the boundedness property because it has a strong unit. (See Remark (i)). Then since $\|\cdot\|$ is complete on c, by Theorem 2.10 we have that every extension of $\|\cdot\|$ to $\hat{E}$ must be complete. (We proved the same result in Example 1.13, but in a different manner.)

<u>Remark</u>. Let E be any Archimedean vector lattice with a strong unit e. E having the boundedness property implies that any complete, monotone norm on E has equivalent norm extensions. (Again, this was noted in the remark following Example 1.13.)

Also, $\|x\| = \inf(\lambda\colon \lambda e \geq |x|)$ is a monotone bounded norm on E. To show this we choose any net $\{x_\alpha\} \subseteq E$ such that $0 \leq x_\alpha \uparrow$ and $\|x_\alpha\| \leq M$ for all $\alpha \in \mathcal{A}$. Then since $|x_\alpha| \leq \|x_\alpha\| \cdot e$ for all α we have that $|x_\alpha| \leq M \cdot e$ which proves that $\|\cdot\|$ is monotone bounded. The fact that $\|\cdot\|$ is monotone bounded shows that whenever E is Dedekind σ-complete, $\|\cdot\|$ is complete.

One example of an Archimedean vector lattice not having property K (or the boundedness property) is c_0. As noted in Section one, however, if E is any Archimedean vector lattice such that $\hat{E} = c_0$ then every extension of a complete norm on E must be a complete norm on $\hat{E}$.

Other examples of vector lattices not having the boundedness property are $c \oplus c_0$, $\ell_\infty \oplus c_0$ or $\lambda \oplus c_0$ where λ is any sequence space.

Section Three. The Egoroff Property

In [9], W.A.J. Luxemburg and A.C. Zaanen discuss some results concerning Banach function spaces. Given a set X, a σ-complete Boolean algebra $\mathcal{B}$ of subsets of X which is a subalgebra of the power set of X, and a countably additive, non-negative measure μ on $\mathcal{B}$, let $M(X,\mathcal{B},\mu)$ denote the space of all extended, real-valued functions on X, measurable with respect to $\mathcal{B}$ with identification of almost everywhere equal functions. Let $\|\cdot\|$ be an extended, monotone seminorm defined on M and let $E = \{x \in M: \|x\| < +\infty\}$. If M is complete with respect to $\|\cdot\|$, then E is said to be a Banach function space. Luxemburg and Zaanen show that whenever μ is σ-finite, $\|x\|_L \equiv \inf\{\lim_{n\to\infty} \|x_n\|: 0 \le x_n \uparrow |x|\}$ is a σ-semi-continuous, extended, monotone seminorm on M, $\|x\|_L \le \|x\|$ for all $x \in M$, and $\|\cdot\|_L$ is the maximal seminorm on M having these properties. The condition on M that appears to be essential to their proofs is the Egoroff property. Hence, we shall show that if $(E, \le, \|\cdot\|)$ is any normed vector lattice and E has the Egoroff property then $\|\cdot\|_L$ as defined above satisfies the same properties and $\|\cdot\|_L$ is a norm. We shall study $\|\cdot\|_L$ in detail because the functional $\varphi(x) = \|x\| - \|x\|_L$ will aid us in determining when all extensions of $\|\cdot\|$ to $\hat{E}$ are equivalent.

Combining Theorem 0.8 in the introduction with some results we have about $\|\cdot\|_L$, we shall prove that completeness of $(E, \le, \|\cdot\|)$ is equivalent to a Riesz-Fischer property. Although this Riesz-Fischer Theorem does not bear directly on the problem of equivalence of norm extensions of a Banach lattice, we present it here as an interesting application of Theorem 0.8.

The main results of this section are: (1). If $(E, \leq, \|\cdot\|)$ is a Banach lattice, E is order separable and has the Egoroff property and $\varphi(\hat{x}) = \|\hat{x}\|_M - \|\hat{x}\|_{ML}$ is a continuous functional at $x = 0$ with respect to $\|\cdot\|_{ML}$ on $\hat{E}$, then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.
(2). If $(E, \leq, \|\cdot\|)$ is a Banach lattice which has the Egoroff property and $\varphi(x) = \|x\| - \|x\|_L$ is a continuous functional on E at $x = 0$ with respect to $\|\cdot\|_L$, then every extension of $\|\cdot\|$ to $\hat{E}$ is complete, provided that either $\|\cdot\|_L$ is continuous on E or $(E, \leq, \|\cdot\|_L)$ satisfies the hypotheses of Theorem 1.11.

<u>Definition 3.1</u>. A vector lattice E has the <u>Egoroff property</u> if for each $x \in E$ and for any double sequence $\{x_{n,k}: n,k \in N\}$ of elements of E^+ such that for $n = 1,2,\ldots,$ $0 \leq x_{n,k} \uparrow_k |x|$ in E, there exists a sequence $\{x_m\} \subseteq E$ such that $0 \leq x_m \uparrow |x|$ and for every $m,n \in N$ there exists a positive integer $k(n,m)$ such that $x_m \leq x_{n,k(n,m)}$.

For a discussion of the Egoroff property and its relationship with the classical Egoroff theorem, we refer the reader to [4].

<u>Theorem 3.2</u>. If $(E, \leq, \|\cdot\|)$ is a normed vector lattice and E has the Egoroff property, then $\|x\|_L = \inf\{\lim_{n\to\infty} \|x_n\|: 0 \leq x_n \uparrow |x|\}$ is a σ-semi-continuous norm on E, $\|x\|_L \leq \|x\|$

for all $x \in E$ and $\|\cdot\|_L$ is the largest such norm on E.

Proof. One can easily show that $\|\cdot\|_L$ is a seminorm as the triangle inequality can be proved by the usual method.

(i) We shall show that $\|\cdot\|_L$ is a norm on E. Choose $x \in E$ and for $n = 1,2,\ldots$, by definition of $\|x\|_L$ we can find a sequence $\{v_{n,k}: k \in N\}$ such that $0 \le v_{n,k} \uparrow_k |x|$ and $\|v_{n,k}\| \le \|x\|_L + 1/n$ for all $k \in N$. E having the Egoroff property implies there exists a sequence $\{x_m\} \subseteq E^+$ such that $x_m \uparrow |x|$ and for every $m,n \in N$ there exists an integer $k(n,m)$ such that $x_m \le v_{n,k(n,m)}$. Hence, $\|x_m\| \le \|v_{n,k}\| \le \|x\|_L + 1/n$ for all n,m. Therefore, $\|x_m\| \le \|x\|_L$ for all m and it follows that $\lim\limits_{m\to\infty} \|x_m\| \le \|x\|_L$.

We can now prove that $\|\cdot\|_L$ is a norm on E because if $\|x\|_L = 0$ then there exists a sequence $0 \le x_m \uparrow |x|$ with $\lim\limits_{m\to\infty} \|x_m\| = 0$. But since $\|\cdot\|$ is a norm we must have $x_m = 0$ for all $m \in N$ and then $x = 0$.

(ii) $\|\cdot\|_L$ is σ-semi-continuous. Let $0 \le x_n \uparrow x$ in E. We shall show that $\|x\|_L \le \lim\limits_{n\to\infty} \|x_n\|_L$ and this will imply that $\|x\|_L = \lim\limits_{n\to\infty} \|x_n\|_L$. As in (i), for $n = 1,2,\ldots$, choose a sequence $\{v_{n,k}: k \in N\}$ such that $0 \le v_{n,k} \uparrow_k x_n$ and $\|v_{n,k}\| \le \|x_n\|_L + 1/n$ for all $k \in N$. Let $u_{n,k} = v_{n,k} + (x - x_n)$ for all $k,n \in N$. Then $0 \le u_{n,k} \uparrow_k x$ for all n. E having the Egoroff property implies there exists a sequence $\{z_m\} \subseteq E^+$ such that $0 \le z_m \uparrow x$ and for all $n,m \in N$ there exists a positive integer $k(n,m)$ such that $z_m \le u_{n,k}$. Let $y_m = (z_m + x_m - x)^+$. Then $0 \le y_m \uparrow x$ and for each $n,m \in N$ there exists a $k(n,m)$ such that $y_m = (z_m + x_m - x)^+ \le$

$\leq u_{n,k} + x_m - x = v_{n,k}$. Now, $\|y_m\| \leq \|v_{n,k}\| \leq \|x_n\|_L + 1/n$ for all $n,m \in N$. Therefore, $\lim_{m\to\infty} \|y_m\| \leq \lim_{n\to\infty} \|x_n\|_L$. It follows that $\|x\|_L \leq \lim_{n\to\infty} \|x_n\|_L$ which proves that $\|\cdot\|_L$ is σ-semi-continuous.

The fact that $\|x\|_L \leq \|x\|$ for all $x \in E$ and $\|\cdot\|_L$ is the maximal norm on E satisfying the above properties follows easily, completing the proof of the theorem.

J.A.R. Holbrook has shown that if $\|\cdot\|$ is a monotone seminorm on an Archimedean vector lattice E and $\|\cdot\|_K$ is the maximal σ-semi-continuous seminorm below $\|\cdot\|$, then $\|\cdot\|_L = \|\cdot\|_K$ on E if and only if E has the Egoroff property [4, Theorem 5.2]. The above theorem shows that $\|\cdot\|_L$ is a norm if and only if $\|\cdot\|$ is a norm.

We shall now apply Theorem 0.8.

__Theorem 3.3__. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that E has the Egoroff property. If $0 \leq x_n \uparrow$ in E and $\|x_n\| \leq M < +\infty$ for all $n \in N$, then $\sup(x_n : n \in N) = x_0$ exists in $E^{\#}$.

__Proof__. Using Theorem 3.2 we know there exists a σ-semi-continuous norm $\|\cdot\|_L$ on E such that $\|x\|_L \leq \|x\|$ for all $x \in E$. Therefore, $0 \leq \|x_n\|_L \leq M < +\infty$ for all $n \in N$. Applying Theorem 0.8 to $(E, \leq, \|\cdot\|_L)$ it follows that $\sup(x_n : n \in N) = x_0$ in $E^{\#}$.

We are now motivated to make the following definition.

<u>Definition 3.4</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that E has the Egoroff property. $(E, \leq, \|\cdot\|)$ is said to have the <u>Riesz-Fischer property</u> whenever $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ and $\{x_n\} \subseteq E^+$ implies that $\sum_{n=1}^{\infty} x_n$ is an element in E. (Note that $\sum_{n=1}^{\infty} x_n$ exists in $E^{\#}$ whenever $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ by applying Theorem 3.3.)

In [9, Theorem 4.8] Luxemburg and Zaanen have shown that a normed function space $E \subseteq M(X,\mathcal{B},\mu)$ is complete if and only if $0 \leq x_n \in E$ for $n = 1,2,\ldots,$ and $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ implies that $\|\sum_{n=1}^{\infty} x_n\| < +\infty$. Of course, since E is a normed function space, $\sum_{n=1}^{\infty} x_n$ always exists in M. This is not necessarily true for a general normed vector lattice $(E, \leq, \|\cdot\|)$. However, whenever E has the Egoroff property and $\{x_n\} \subseteq E^+$ with $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ we know that $\sum_{n=1}^{\infty} x_n$ exists in $E^{\#}$. (Apply Theorem 3.3.) The proof of the following theorem is completely analogous to that of Luxemburg and Zaanen.

<u>Theorem 3.5</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that E has the Egoroff property. $(E, \leq, \|\cdot\|)$ is complete if and only if $(E, \leq, \|\cdot\|)$ has the Riesz-Fischer property.

<u>Proof</u>. (i) If $(E, \leq, \|\cdot\|)$ is complete and $\{x_n\} \subseteq E^+$ such that $\sum_{n=1}^{\infty} \|x_n\| < +\infty$, let $S_n = x_1 + x_2 + \cdots + x_n$, $n = 1,2,\ldots$. Then $\|S_n - S_m\| \to 0$ as $n,m \to \infty$. Hence, there exists an element $S_o \in E$ such that $\|S_n - S_o\| \to 0$ as $n \to \infty$. But $0 \leq S_n \uparrow$ and $\|S_n - S_o\| \to 0$ implies that $S_o = \sup(S_n: n \in N)$; i.e., $S_o = \sum_{n=1}^{\infty} x_n \in E$.

(ii). We now assume that $(E, \leq, \|\cdot\|)$ has the Riesz-Fischer property. We shall first show that whenever $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ then $\|\sum_{n=1}^{\infty} x_n\| \leq \sum_{n=1}^{\infty} \|x_n\|$. Assume that for some positive number $\epsilon > 0$ we have $\|\sum_{n=1}^{\infty} x_n\| > \epsilon + \sum_{n=1}^{\infty} \|x_n\|$. For $k = 1,2,\ldots,$ obtain a sequence $\{b_{n,k}: n \in N\}$ in E such that $\|\Sigma_n b_{n,k}\| > k + \Sigma_n\|b_{n,k}\|$. Then for every positive integer p $\|\sum_{n\geq p} b_{n,k}\| > k + \sum_{n\geq p} \|b_{n,k}\|$ and for some $p = p_k$ $\sum_{n\geq p} \|b_{n,k}\| < 1/k^2$. Then by reindexing we obtain for every $k = 1,2,\ldots,$ a sequence $\{z_{n,k}: n \in N\}$ in E such that $\Sigma_n\|z_{n,k}\| < 1/k^2$ and $\|\Sigma_n z_{n,k}\| > k$. Let $\{u_n: n \in N\}$ be the double sequence $\{z_{n,k}\}$ arranged in single order. Then $\Sigma_n\|u_n\| < \Sigma_k 1/k^2 < +\infty$ and $\|\Sigma_n u_n\| > \|\Sigma_n z_{n,k}\| > k$ for all $k \in N$. But $\Sigma_n\|u_n\| < +\infty$ and $\|\Sigma_n u_n\| = +\infty$ implies that $\Sigma_n u_n$ is not in E which contradicts the Riesz-Fischer property.

Using the above result we can prove that $(E, \leq, \|\cdot\|)$ is complete. Let $\{x_n\}$ be $\|\cdot\|$-Cauchy in E and assume without loss of generality that $\|x_{n+1} - x_n\| < 1/2^n$ for $n = 1,2,\ldots$. Then $x = \sum_{n=1}^{\infty} |x_{n+1} - x_n|$ is an element of E. Let $y_n = x_n - x_1 + \sum_{1}^{n-1} |x_{k+1} - x_k|$ for $n = 1,2,\ldots,$ and note that $y_1 = 0$ and $y_{n+1} - y_n = x_{n+1} - x_n + |x_{n+1} - x_n| \geq 0$. It follows that $0 \leq y_n \uparrow$. Since $y_{n+1} - y_n \leq 2|x_{n+1} - x_n|$ we have $\sum_{1}^{\infty} \|y_{n+1} - y_n\| < +\infty$. Then $\sup(y_n: n \in N) = y$ is an element of E. Let $\bar{x} = y - x + x_1$. Then $\bar{x} - x_n = y - x + x_1 - x_n =$
$= y - x + \sum_{1}^{n-1} |x_{k+1} - x_k| - y_n = y - y_n - \{x - \sum_{1}^{n-1} |x_{k+1} - x_k|\} =$
$= \sum_{n}^{\infty} (y_{k+1} - y_k) - \sum_{n}^{\infty} |x_{k+1} - x_k|$. Hence, $|\bar{x} - x_n| \leq \sum_{n}^{\infty} (y_{k+1} - y_k) -$
$- \sum_{n}^{\infty} |x_{k+1} - x_k| \leq 3 \sum_{n}^{\infty} |x_{k+1} - x_k|$. Now, $\|\bar{x} - x_n\| \leq$

$\leq 3\|\sum_{n}^{\infty} |x_{k+1} - x_k|\| \leq 3 \sum_{n}^{\infty} \|x_{k+1} - x_k\|$ the last inequality holding because of the result proved in the preceding paragraph. Hence, $\|\bar{x} - x_n\| \to 0$ as $n \to \infty$ which proves that $(E, \leq, \|\cdot\|)$ is complete in norm.

As noted in Section two, the monotone completeness of a normed vector lattice is a relatively strong property. For this reason, we ask whether a normed vector lattice $(E, \leq, \|\cdot\|)$ can be isomorphically and isometrically embedded in a monotone complete normed vector lattice. Although this question does not directly involve the problem of equivalence of norm extensions, the proof of the theorem which answers this question is a nice application of some of the results of this section as well as Theorem 0.8.

Theorem 3.6. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that E has the Egoroff property and E is order separable. If $0 \leq x_\alpha \uparrow$ in E and $\|x_\alpha\| \leq M$ for all $\alpha \in \mathcal{A}$, then $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_o$ exists in $E^\#$. Furthermore, if $\|\cdot\|$ is σ-semi-continuous $(E, \leq, \|\cdot\|)$ can be isomorphically and isometrically embedded in a monotone complete normed vector lattice.

Proof. As in Theorem 3.2, $\|\cdot\|_L$ is a σ-semi-continuous norm on E such that $\|x\|_L \leq \|x\|$ for all x in E. However, since E is order separable it follows that $\|\cdot\|_L$ is semi-continuous. We can now apply Theorem 0.8 to $\|\cdot\|_L$ and we have that whenever $0 \leq x_\alpha \uparrow$ and $\|x_\alpha\| \leq M$ for all $\alpha \in \mathcal{A}$ then $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_o$ in $E^\#$.

If $\|\cdot\|$ is σ-semi-continuous, $\|\cdot\| = \|\cdot\|_L$ and we can extend $\|\cdot\|$ to $E^{\#+}$ (allowing $\|\cdot\|$ to be infinite) by defining $\|\bar{x}\| = \sup(\|y\|: 0 \le y \le \bar{x}, y \in E)$. Let $\bar{E} = \{x \in E^{\#}: \|\bar{x}\| < +\infty\}$. One can easily show that $(\bar{E}, \le, \|\cdot\|)$ is a monotone complete normed vector lattice and $(E, \le, \|\cdot\|)$ is isomorphically and isometrically embedded in $(\bar{E}, \le, \|\cdot\|)$.

If $(E, \le, \|\cdot\|)$ is a normed vector lattice and $\hat{E}$ has the Egoroff property, $\|\cdot\|_{ML}$ can be defined on $\hat{E}$ where $\|\cdot\|_M$ is the maximal extension of $\|\cdot\|$ to $\hat{E}$ and $\|\cdot\|_{ML}$ is the maximal σ-semi-continuous norm below $\|\cdot\|_M$ on $\hat{E}$. A natural question to ask is whether $\|\cdot\|_{ML}$ is equal to $\|\cdot\|_L$ when $\|\cdot\|_{ML}$ is restricted to E. If E is order separable then E has the Egoroff property if and only if $\hat{E}$ has the Egoroff property [18, Chapter 1, Theorem 1.3]. Therefore, we consider the above-mentioned question for $(E, \le, \|\cdot\|)$ where E is order separable and has the Egoroff property. The impact of the following two theorems on the problem of equivalence of norm extensions of a Banach lattice will become evident in the corollary following them.

Theorem 3.7. Let $(E, \le, \|\cdot\|)$ be a normed vector lattice such that E is order separable and has the Egoroff property. Then $\|x\|_L = \inf(\lim_{n\to\infty} \|x_n\|: 0 \le x_n \uparrow x, \{x_n\} \subseteq E)$ is equal to $\|x\|_{KL} = \inf(\lim_{n\to\infty} \|\hat{z}_n\|_K: 0 \le \hat{z}_n \uparrow x, \{\hat{z}_n\} \subseteq \hat{E})$ for all x in E^+ where $\|\cdot\|_K$ is any extension of $\|\cdot\|$ to $\hat{E}$.

Proof. By the definition of $\|\cdot\|_L$ and $\|\cdot\|_{KL}$ we can see that $\|x\|_{KL} \le \|x\|_L$ for all x in E. To show that

$\|x\|_{KL} \geq \|x\|_L$ we prove that given any sequence $\{\hat{z}_n\} \subseteq \hat{E}$ such that $0 \leq \hat{z}_n \uparrow x$ in $\hat{E}^+$ there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \leq \hat{z}_n$ for $n = 1,2,\ldots,$ and $0 \leq x_n \uparrow x$.

Let $A_n = \{y \in E: 0 \leq y \leq \hat{z}_n\}$, $n = 1,2,\ldots,$ and let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is a directed set in E and $\sup(A) = x$. Since E is order separable, there exists a sequence $\{y_k\} \subseteq A$ such that $y_k \uparrow x$. ($\{y_k\}$ can be chosen directed upward beacuse A is a directed set.) By definition of A there exists a subsequence $\{\hat{z}_{n_k}\} \subseteq \{\hat{z}_n\}$ such that $\hat{z}_{n_k} \geq y_k$ for $k = 1,2,\ldots$. Choose $x_n = y_k$ for n such that $n_k \leq n < n_{k+1}$, $k = 1,2,\ldots$. For $1 \leq n < n_1$ let $x_n = x_o$ where $x_o \in E$ and $0 \leq x_o \leq \hat{z}_1$. Then for $n = 1,2,\ldots$ we must have $0 \leq x_n \leq \hat{z}_n$. And since $y_k \uparrow x$ it follows that $x_n \uparrow x$. This completes the proof of the theorem.

<u>Theorem 3.8</u>. Let $(E, \leq, \|\cdot\|)$ be order separable and have the Egoroff property. Then $\|\hat{x}\|_L^* = \|\hat{x}\|_{KL}$ for all $\hat{x}$ in $\hat{E}$ where $\|\cdot\|_K$ is any extension of $\|\cdot\|$ to $\hat{E}$.

<u>Proof</u>. Using the fact that E is order separable it follows that $\|\cdot\|_L$ is semi-continuous on E and then $\|\cdot\|_L^*$ is a semi-continuous extension of $\|\cdot\|_L$ to $\hat{E}$. Also, since $\hat{E}$ is order separable whenever E is [19, Lemma 3] $\|\cdot\|_{KL}$ is also a semi-continuous norm on $\hat{E}$. Now, $\|\cdot\|_L^*$ and $\|\cdot\|_{KL}$ are semi-continuous norms on $\hat{E}$ and by Theorem 3.7 $\|\cdot\|_L^*$ and $\|\cdot\|_{KL}$ agree on E. By Lemma 1.5, we have that $\|\cdot\|_L^*$ and $\|\cdot\|_{KL}$ agree on $\hat{E}$.

<u>Corollary 3.9</u>. Let $(E, \leq, \|\cdot\|)$ be a Banach lattice such that E is order separable and has the Egoroff property. If $\|\cdot\|_K$ is equivalent to $\|\cdot\|_{KL}$ on $\hat{E}$ where $\|\cdot\|_K$ is any extension of $\|\cdot\|$ to $\hat{E}$, every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

Before examining conditions under which $\|\cdot\|_K$ is equivalent to $\|\cdot\|_{KL}$ we present the following examples. The first example uses corollary 3.9 and the others illustrate what happens when $\hat{E}$ does not have the Egoroff property.

<u>Example 3.10</u>. Let $E = c$ and $\|x\| = \sup(|x(n)|: n \in N) + \lim_{n\to\infty} |x(n)|$. Then if $\|\cdot\|_K$ is any extension of $\|\cdot\|$ to $\hat{E} = \ell_\infty$ we must have $\|\hat{x}\|_{KL} = \sup(|x(n)|: n \in N)$ since $\|\hat{x}\|_L^* = \sup(|x(n)|: n \in N)$. Furthermore, since $\|\cdot\|_{ML}$ is equivalent to $\|\cdot\|_M$ $(\|\hat{x}\|_M = \sup(|\hat{x}(n)|: n \in N) + \overline{\lim_{n\to\infty}} |x(n)|)$ on $\hat{E}$ we must have that every extension of $\|\cdot\|$ to $\hat{E}$ is complete. Note that we have now proved this fact in three different ways. (Compare with Examples 1.13 and 2.14.)

<u>Example 3.11</u>. Theorems 3.7 and 3.8 were proved for E order separable and having the Egoroff property. We shall now give an example of a normed vector lattice $(E, \leq, \|\cdot\|)$ such that E has the Egoroff property but $\hat{E}$ does not.

Let X = the set of all sequences $x = (x(1), x(2), \ldots)$ with positive integer coordinates and let $E = \{f: f$ is a real-valued, bounded function on X such that f is constant except possibly on a finite number of points in $X\}$. J.J. Masterson and G. Crofts have shown [12, Example 1] that E is diagonalizable for any uncountable set X. (A vector lattice E is <u>**diagonalizable**</u> if whenever $\{x_{n,k}: n,k \in N\}$ is any double sequence in E^+, $x_{n,k} \xrightarrow[k]{o} x_n$ for each n and $x_n \xrightarrow{o} x$ in E, then there exists a strictly increasing sequence $\{k_n: n = 1,2,\ldots\}$ of positive integers such that $x_{n,k_n} \xrightarrow{o} x$ as $n \to \infty$.) It is easy to see that whenever E is diagonalizable it must have the Egoroff property [10, p. 185]. Therefore, the vector lattice E which we have defined above has the Egoroff property.

We shall now prove that $\hat{E} = \{f: f$ is a real-valued, bounded function on $X\}$ does not have the Egoroff property. The proof is due to J.A.R. Holbrook [4, Example 4.2]. Define subsets $A_{n,k}$ of X by letting $A_{n,k} = \{x \in X: x(n) \le k\}$ and let $f_{n,k} = \chi_{A_{n,k}}$, for $n,k = 1,2,\ldots$. Then $0 \le f_{n,k} \uparrow_k f$ in $\hat{E}$ for $n = 1,2,\ldots$, where $f = \chi_X$. However, for no choice of $k(n)$ do we have $\sup(f_{n,k(n)}: n \in N) = f$. To prove this latter statement we note that if we choose any sequence $\{k(n): n \in N\}$ of positive integers and let $\bar{x} = (x(1),x(2),\ldots)$ be the element in X such that $\bar{x}(n) = k(n) + 1$ for $n = 1,2,\ldots$, then $\bar{x}$ does not belong to any of the sets $A_{n,k(n)}$. It now follows that $\hat{E}$ does not have the Egoroff property for if we assume there exists a sequence $\{u_m\} \subseteq \hat{E}$ such that for all $n,m \in N$ there exists a positive integer $k(n,m)$ with $u_m \le f_{n,k}$ then there exists a $k(m)$ for each m such that $u_m \le f_{m,k(m)}$. Hence, $\sup(u_m: m \in N) \le \sup(f_{m,k(m)}: m \in N) \ne f$.

The element $f = \chi_X$ not having the Egoroff property implies there exists a monotone seminorm $\|\cdot\|$ on $\hat{E}$ such that

$\|f\|_L \neq \|f\|_{LL}$ [4, Theorem 5.2]; i.e. $\|\cdot\|$ is not σ-semi-continuous on $\hat{E}$. Then $\|g\|' = \sup(|g(t)|: t \in X) + \|g\|$ is a monotone norm on $\hat{E}$ and $\|f\|_L' = \|\chi_X\|_L' = 1 + \|f\|_L \neq 1 + \|f\|_{LL} = \|f\|_{LL}'$; i.e., $\|\cdot\|'$ is not σ-semi-continuous on $\hat{E}$. However, $\|\cdot\|'|E = \|\cdot\|_1$ is a monotone norm on E such that $\|\cdot\|_{1L}$ is σ-semi-continuous on E because E has the Egoroff property.

Although $\|\cdot\|_L$ need not be σ-semi-continuous whenever E does not have the Egoroff property, we can still consider $\|\cdot\|_L$ and $\|\cdot\|_{ML}$ and ask some questions about these seminorms.

<u>Example 3.12</u>. We shall give an example of a normed vector lattice $(E, \leq, \|\cdot\|)$ where $\|\cdot\|_{ML}$ is not equal to $\|\cdot\|_{LM}$ on $\hat{E}$.

Let X be an uncountable set of points and let $E = \{f: f$ is a real-valued, bounded function on X and f is constant except possibly on a finite set of points in $X\}$. Then $\hat{E} = \{f: f$ is real-valued and bounded on $X\}$. Assuming the continuum hypothesis, W.A.J. Luxemburg and A.C. Zaanen have shown that $\hat{E}$ does not have the Egoroff property [10, Theorem 43.3].

Let $\hat{g} \in \hat{E}$ be the element $\hat{g} = \chi_{X_1} + 2\chi_{X_2}$ where $X = X_1 \cup X_2$, $X_1 \cap X_2 = \phi$ and $\operatorname{card}(X_1) = \operatorname{card}(X_2) > \aleph_o$. Then $\hat{g}$ is a strong unit for $\hat{E}$. Hence, we can introduce a semi-continuous norm on E by defining $\|f\| = \inf(\lambda: \lambda\hat{g} \geq |f|)$ for every f in E.

(i). $\|\cdot\|^*$ is a semi-continuous extension of $\|\cdot\|$ to $\hat{E}$ and $\|\hat{h}\|^* = \inf(\lambda: \lambda\hat{g} \geq |\hat{h}|)$ for all $\hat{h}$ in $\hat{E}$.

(ii). $\|\cdot\|^* \neq \|\cdot\|_M$ on $\hat{E}$ because $\|\hat{g}\|^* = 1$, but if w is any element of E such that $w \geq \hat{g}$ then $w(x) \geq 2$ except possibly for a finite number of points in X. This implies that $\|\hat{g}\|_M = \inf(\|w\|: w \geq \hat{g}, w \in E) = 2$.

(iii). $\|\cdot\|_{ML} \neq \|\cdot\|^*$ on $\hat{E}$. We know that $\|\hat{g}\|^* = 1$. However, we shall show that $\|\hat{g}\|_{ML} = 2$. Let $\{f_n^\wedge\} \subseteq \hat{E}$ be a sequence of elements such that $0 \leq \hat{f}_n \uparrow \hat{g}$. Then for some positive integer N, $f_N^\wedge$ must be greater than one on an uncountable set in X (i.e., $f_N^\wedge \notin E$). Also, given any $\epsilon > 0$ there exists a positive integer $K > 0$ such that $\{x: |f_K^\wedge(x) - \hat{g}(x)| < \epsilon\}$ is an uncountable set in X. This implies that $\|f_K^\wedge\|_M > 2 - \epsilon$. Hence, $\sup_n(\|f_n^\wedge\|_M) = 2$ and we have that $\|\hat{g}\|_{ML} = 2$.

(iv). $\|\cdot\|_{ML} \neq \|\cdot\|_{LM}$ on $\hat{E}$. Since $\|\cdot\| = \|\cdot\|_L$ on E we must show that $\|\cdot\|_{ML} \neq \|\cdot\|_M$ on $\hat{E}$; i.e., $\|\cdot\|_M$ is not σ-semi-continuous on $\hat{E}$. Define $\hat{h} \in \hat{E}$ by choosing $\{x_k: k \in N\}$ a countable subset of X_2 and letting

$$\hat{h}(x) = \begin{cases} 1 & \text{for } x = x_k,\ k = 1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\hat{h}\|_M = \|\hat{h}\|_{LM} = 1$. However, taking

$$f_n(x) = \begin{cases} 1 & \text{for } x = x_k,\ k = 1,2,\ldots,n \\ 0 & \text{otherwise,} \end{cases}$$

we have $0 \leq f_n \uparrow \hat{h}$ in $\hat{E}$, $\{f_n\} \subseteq E$ and $\|f_n\|_M = \|f_n\| = 1/2$. Hence, $\|\hat{h}\|_{ML} = \inf(\lim_{n\to\infty} \|u_n^\wedge\|: 0 \leq u_n^\wedge \uparrow \hat{h}$ in $\hat{E}) \leq 1/2$. This proves that $\|\hat{h}\|_{ML} \neq \|\hat{h}\|_{LM}$.

Note that the previous example is similar to Example 1.7 in Section one in the sense that it is an example of a normed vector lattice with a semi-continuous norm which does not extend uniquely to $\hat{E}$. Although our present example is not order separable, Example 1.7 provides us with a normed vector lattice which is order separable and has the Egoroff property but $\|\cdot\|_{LM} \neq \|\cdot\|_{ML}$ on $\hat{E}$. In Example 1.7, $\|\cdot\|_{ML} = \|\cdot\|_L^*$ on $\hat{E}$. However, in our present example $\|\cdot\|_{ML} \neq \|\cdot\|_L^*$ on $\hat{E}$.

In Example 3.12 if we had chosen $E = \{f\colon f$ is real-valued and bounded on X and f is constant except possibly on a countable set of points in $X\}$ with the norm defined in the same way, $(E, \leq, \|\cdot\|)$ would be a normed vector lattice with $\|\cdot\|$ semi-continuous, $\|\cdot\|^*$ a semi-continuous norm extension, $\|\cdot\|^* \neq \|\cdot\|_M$, but $\|\cdot\|_M$ would be σ-semi-continuous on $\hat{E}$. (See Example 1.9.) Hence, we would have $\|\hat{x}\|_{ML} =$
$= \|\hat{x}\|_M = \|\hat{x}\|_{LM}$ on $\hat{E}$.

For the case when E is order separable and has the Egoroff property we know that $\|\cdot\|_{ML} = \|\cdot\|_L^*$ on $\hat{E}$. However, as the previous example illustrates, this is not necessarily true in other cases. We now ask whether $\|\cdot\|_{ML}$ is an extension of $\|\cdot\|_L$ (i.e., does $\|\cdot\|_{ML} = \|\cdot\|_L$ on E) even though $\|\cdot\|_{ML}$ may not be equal to $\|\cdot\|_L^*$ on $\hat{E}$.

<u>Example 3.12</u> (continued). We recall that $\|\cdot\|_{ML} \neq \|\cdot\|_L^*$ on $\hat{E}$. However, we shall show that $\|\cdot\|_{ML} = \|\cdot\|_L$ on E.

By definition of $\|\cdot\|_L$ and $\|\cdot\|_{ML}$ we always have $\|f\|_{ML} \leq \|f\|_L$ for every $f \in E$. Let $\{f_n^\wedge\colon n \in N\}$ be a sequence in $\hat{E}$ such that $0 \leq f_n^\wedge \uparrow f$ in $\hat{E}$. For $n = 1,2,\ldots$, let $A_n = \{y \in E\colon 0 \leq y \leq f_n^\wedge\}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. Then A is a directed set in E and $\sup(A) = f$. Since A is directed we can let $A = \{y\lambda: \lambda \in \Lambda\} \subseteq E$ where $0 \leq y\lambda \uparrow f$. Since $\|\cdot\|$ is semi-continuous on E we have $\sup(\|y\lambda\|: \lambda \in \Lambda) = \|f\|$. But, by definition of A, $\sup(\|y\lambda\|: \lambda \in \Lambda) \leq \sup(\|\hat{f_n}\|_M: n \in N)$. Hence, $\|f\| \leq \sup(\|\hat{f_n}\|_M: n \in N)$ which implies that $\|f\| = = \|f\|_L = \|f\|_{ML}$.

Obviously, if $E = \{f: f$ is real-valued and bounded on X and f is constant except possibly on a countable set of points in $X\}$ then $\|\cdot\|_{ML}$ is an extension of $\|\cdot\|_L$ since $\|\cdot\|_{ML} = \|\cdot\|_M$ and $\|\cdot\|_L = \|\cdot\|$.

<u>Example 3.13</u>. We now consider Examples 1.8 and 1.9 in Section one. In both of those examples we proved that $\|\cdot\|_L^* = \|\cdot\|^*$ was not a norm on $\hat{E}$. Hence, $\|\cdot\|_L^* \neq \|\cdot\|_{ML}$ on $\hat{E}$. However, we also showed in both cases that $\|\cdot\|_{ML}$ agreed with $\|\cdot\| = \|\cdot\|_L$ on E.

As noted in Corollary 3.9, whenever $(E, \leq, \|\cdot\|)$ is order separable and has the Egoroff property and $\|\cdot\|$ is complete, if we can show that $\|\cdot\|_{ML}$ is equivalent to $\|\cdot\|_M$ then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete. We note that $\|\cdot\|$ is not always equivalent to $\|\cdot\|_L$. (Let $E = \ell_\infty$ and let $\|x\| = \sum_{n=1}^{\infty} (|x(n)|/2^n) + \overline{\lim}_{n\to\infty} |x(n)|$. Then $\|x\|_L = \sum_{n=1}^{\infty} (|x(n)|/2^n)$ and $\|\cdot\|_L$ is locally norm complete while $\|\cdot\|$ is not. Therefore, $\|\cdot\|_L$ cannot be equivalent to $\|\cdot\|$.) However, in all of the examples that we presently have at hand, whenever $\|\cdot\|$ is complete, $\|\cdot\|_L$ is equivalent to $\|\cdot\|$. Proving that this is true in general depends on properties of the functional $\varphi(x) = \|x\| - \|x\|_L$. W.A.J. Luxemburg and

A.C. Zaanen considered this functional (defined on normed function spaces) in [9, Note IV, p. 262]. They noted that in all of their examples φ is a monotone seminorm. However, they could not prove that φ is even monotone, nor did they have a counterexample. We can use Example 1.7 from Section one and Theorem 3.8 of the present section to answer Luxemburg and Zaanen's question.

<u>Example 3.14</u>. Let $E = \ell_\infty$ and for each x in E let $\|x\| = \inf(\|w\|_0 \colon w \geq |x|,\ w \in c)$ where $\|w\|_0 = \inf(\lambda \colon \lambda\hat{e} \geq |w|)$ and $\hat{e} = (1,2,1,2,\ldots) \in \ell_\infty$. Note that $(\ell_\infty, \leq, \|\cdot\|)$ is a Banach function space. Recalling the form in which we stated Example 1.7 and applying Theorem 3.8 to this example it follows that $\|x\|_L = \inf(\lambda \colon \lambda\hat{e} \geq |x|)$.

We can now easily show that $\varphi(x) = \|x\| - \|x\|_L$ is not monotone on E. Let $y = (2,2,\ldots) \in c \leq \ell_\infty$. Then $y \geq \hat{e} = (1,2,1,2,\ldots)$ and $\varphi(y) = \|y\| - \|y\|_L = 0$ since $\|\cdot\|$ is equal to $\|\cdot\|_L$ on c. However, $\varphi(\hat{e}) = \|\hat{e}\| - \|\hat{e}\|_L = 2-1 = 1$. (Recall that $\|\hat{e}\| = 2$ and $\|\hat{e}\|_L = \|\hat{e}\|^* = 1$ as noted in Example 1.7.) Therefore, $\varphi(\hat{e}) > \varphi(y)$. Also, φ is not subadditive on E. Let $x = (1,1,1,\ldots)$ and $y = (0,1,0,1,\ldots)$. Then $x + y = \hat{e}$ and $\varphi(x + y) = 1$. But $\varphi(x) = 0$ while $\varphi(y) = \|y\|_M - \|y\|^* = 1 - \frac{1}{2} = \frac{1}{2}$.

We are motivated by Example 3.14 to prove the following theorem.

Theorem 3.15. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that E has the Egoroff property. Then if $\varphi(x) = \|x\| - \|x\|_L$ is monotone on E, $D = \{x \in E: \|x\| = \|x\|_L\}$ is a solid subset of E. (D is said to be a solid subset of E whenever $0 \leq y \leq |x|$ and $x \in D$ implies that $y \in D$).

Proof. Suppose D is not solid in E and choose x in D and $y \in E$ such that $0 \leq y \leq |x|$ but $y \notin D$. Then $\varphi(x) = \varphi(|x|) = 0$ but $\varphi(y) = \|y\| - \|y\|_L > 0$ which shows that φ is not monotone.

In [4, Section 6, Comment b], J.A.R. Holbrook asks if E having the almost Egoroff property (almost Egoroff = Egoroff whenever E is Archimedean) implies that $\|\cdot\|$ is equal to $\|\cdot\|_L$ on some super order dense ideal in E. (A subset $A \subseteq E$ is said to be super order dense in E whenever given any element x in E there exists a sequence $\{x_n\} \subseteq A$ such that $\sup(x_n: n \in A) = x$). Example 3.14 gives us an Egoroff vector lattice E and a monotone norm $\|\cdot\|$ such that $D = \{x: \|x\| = \|x\|_L\}$ is not a super order dense ideal in E because D is not solid in E. (An ideal is always a solid subset.) However, c_o is a super order dense ideal in E and $c_o \subseteq D$.

Holbrook notes [4, p. 77] that if $\|\cdot\| = \|\cdot\|_L$ on a super order dense ideal in E then $\|\cdot\|_L = \|\cdot\|_{LL}$; i.e., $\|\cdot\|_L$ must be σ-semi-continuous. This, of course, implies that E has the Egoroff property (provided that E is Archimedean) [4, Theorem 5.2]. Using Example 3.14 as motivation we can prove the following generalization of this latter statement.

Theorem 3.16. Let E be an order separable, Archimedean vector lattice. If $\|\cdot\|_L = \|\cdot\|$ on E where $\|\cdot\|$ is a monotone seminorm on $\hat{E}$, then $\|\cdot\|_{LL} = \|\cdot\|_L$ on $\hat{E}$.

Proof. $\|\hat{x}\|_{LL} = \inf(\lim_{n\to\infty} \|\hat{u}_n\|_L : 0 \le \hat{u}_n \uparrow |\hat{x}|) \le \|\hat{x}\|_L = \inf(\lim_{n\to\infty} \|\hat{u}_n\| : 0 \le \hat{u}_n \uparrow |\hat{x}|)$. Let $\{\hat{u}_n\}$ be any sequence in $\hat{E}$ such that $0 \le \hat{u}_n \uparrow |\hat{x}|$. As in the proof of Theorem 3.7 we can find a sequence $\{z_n\} \subseteq E$ such that $0 \le z_n \le \hat{u}_n$ for $n = 1,2,\ldots$, and $z_n \uparrow |\hat{x}|$. Now, $\lim_{n\to\infty} \|z_n\| = \lim_{n\to\infty} \|z_n\|_L \le$ $\le \lim_{n\to\infty} \|\hat{u}_n\|_L$. Therefore, $\|\hat{x}\|_{LL} \ge \|\hat{x}\|_L$ which completes the proof of the theorem.

Corollary 3.17. Let E be an order separable, Archimedean vector lattice and $\hat{E}$ its Dedekind completion. Then if for every monotone seminorm $\|\cdot\|$ on $\hat{E}$, either $\|\cdot\| = \|\cdot\|_L$ on a super order dense ideal in $\hat{E}$ or $\|\cdot\| = \|\cdot\|_L$ on E, then $\hat{E}$ must have the Egoroff property.

Although $\varphi(x) = \|\cdot\| - \|\cdot\|_L$ need not be monotone or subadditive, if we place a different kind of restriction on φ we can prove that $\|\cdot\|$ is equivalent to $\|\cdot\|_L$

Theorem 3.18. Let $(E, \le, \|\cdot\|)$ be a normed vector lattice having the Egoroff property. Then $\|\cdot\|$ is equivalent to $\|\cdot\|_L$ if and only if $\varphi(x) = \|x\| - \|x\|_L$ is continuous at $x = 0$ with respect to $\|\cdot\|_L$.

Proof. Suppose first that $\|\cdot\|$ is equivalent to $\|\cdot\|_L$ and let $\{x_n\}$ be a sequence in E such that $\|x_n\|_L \to 0$ as $n \to \infty$. Then $\|x_n\| \to 0$ as $n \to \infty$ so that

$\varphi(x_n) = \|x_n\| - \|x_n\|_L \to 0$ as $n \to \infty$.

On the other hand, assuming that φ is continuous at $x = 0$ with respect to $\|\cdot\|_L$ and choosing a sequence $\{x_n\}$ in E such that $\|x_n\|_L \to 0$ as $n \to \infty$, we must have $\varphi(x_n) \to 0$ as $n \to \infty$. Hence, $\|x_n\| \to 0$ as $n \to \infty$. Since $\|x\| \geq \|x\|_L$ for all $x \in E$ we have that $\|\cdot\|$ and $\|\cdot\|_L$ are equivalent.

Example 3.19. Let $E = \ell_\infty$ and $\|x\| = \sum_{n=1}^{\infty} (|x(n)|/2^n) + \overline{\lim_{n\to\infty}} |x(n)|$. $\varphi(x) = \overline{\lim_{n\to\infty}} |x(n)|$ is not continuous at $x = 0$ with respect to $\|\cdot\|_L$. To show this let $x_k = (0,0,\ldots,0,\underset{k}{1},1,\ldots)$. Then $\|x_k\|_L \to 0$ as $k \to \infty$ but $\varphi(x_k) = 1$ for all $k \in N$. However, $\|\cdot\|$ is not a complete norm.

In all of the examples which we presently have at hand, whenever $\|\cdot\|$ is complete φ is continuous at $x = 0$ with respect to $\|\cdot\|_L$. Note that even in Example 3.14, although φ is not monotone or subadditive φ is continuous with respect to $\|\cdot\|_L$ at $x = 0$. In this example $\|\cdot\|_L$ is a complete norm on ℓ_∞. (See the remark following Example 2.14 where we show that if E is Dedekind σ-complete and $\|x\|_L = \inf(\lambda: \lambda\hat{e} \geq |x|)$ and $\hat{e}$ is a strong unit for E then $\|\cdot\|_L$ is complete.) Also, $\|x\|_L \leq \|x\|$ for all x in E forces $\|\cdot\|$ to be complete [14, Theorem 30.28] as well. Hence, by our previous theorem φ is continuous at $x = 0$ with respect to $\|\cdot\|_L$.

We can now state the main results of this section.

Theorem 3.20. Let $(E, \leq, \|\cdot\|)$ be a Banach lattice such that E is order separable and has the Egoroff property.

If $\varphi(\hat{x}) = \|\hat{x}\|_M - \|\hat{x}\|_{ML}$ is a continuous functional at $x = 0$ with respect to $\|\cdot\|_{ML}$ on $\hat{E}$ then every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

<u>Theorem 3.21</u>. Let $(E, \leq, \|\cdot\|)$ be a Banach lattice such that E has the Egoroff property. Also, suppose $\varphi(x) = \|x\| - \|x\|_L$ is a continuous functional at $x = 0$ with respect to $\|\cdot\|_L$. Then if either $\|\cdot\|_L$ is continuous or $(E, \leq, \|\cdot\|_L)$ satisfies the hypotheses of Theorem 1.11, every extension of $\|\cdot\|$ to $\hat{E}$ must be complete.

In order to apply the results of this section to specific examples it would be nice to have several Archimedean vector lattices having the Egoroff property. We note that if λ is any Archimedean vector lattice which is also a sequence space (i.e., $\lambda \subseteq S$ and $\lambda \supseteq \varphi$ where S is the space of all real sequences and φ is the space of all eventually zero sequences) then λ must have the Egoroff property. In order to see this we first note that $\lambda^{\#} = S$ since S is universally complete and λ is order dense in S. (The fact that λ is order dense in S follows because $\lambda \supseteq \varphi$.) Obviously, S has the Egoroff property because it is the space of all measurable functions on the countable discrete space $\aleph_0$. Also, whenever $E^{\#}$ has the Egoroff property and is order separable it follows that E must be order separable and have the Egoroff property. To prove this latter statement recall that whenever $\hat{E}$ has the Egoroff property and is order separable E also has these two properties [18, Chapter 1, Theorem 1.3]. Therefore, we need only show that $\hat{E}$ has the Egoroff property and is order separable. Obviously, if $E^{\#}$ is order separable then $\hat{E}$ must be order separable. Choose a double sequence $\{x_{n,k}: n,k \in N\}$ in $\hat{E}^{+}$ such that $0 \le x_{n,k} \uparrow_k x$ in $\hat{E}$ for $n = 1,2,\ldots$. Since $\hat{E}$ is an ideal in $E^{\#}$ we have $0 \le x_{n,k} \uparrow_k x$ in $E^{\#}$. Hence, there exists a sequence $\{\bar{x}_m: m \in N\} \subseteq E^{\#}$ such that $0 \le \bar{x}_m \uparrow x$ in $E^{\#}$ and for each $n,m \in N$ there exists a positive integer $k(n,m)$ such that $\bar{x}_m \le x_{n,k}$. Using a construction similar to the one used in the proof of Theorem 3.7 we can find a sequence $\{y_m\} \subseteq \hat{E}^{+}$ such that $y_m \uparrow x$ in $\hat{E}$ and $y_m \le \bar{x}_m$ for all $m \in N$. Therefore, given any $n,m \in N$ there exists a $k(n,m)$ such that $y_m \le x_{n,k}$. It now follows that λ has the Egoroff property.

In concluding this section we shall prove the following theorem which relates some of the properties of Section two with the present section.

<u>Theorem 3.22</u>. Let $(E, \le, \|\cdot\|)$ be any Dedekind σ-complete Banach lattice which has the Egoroff property. If E also has property K, $\varphi(x) = \|x\| - \|x\|_L$ is continuous at $x = 0$ with respect to $\|\cdot\|_L$.

<u>Proof</u>. By Yamamuro's result [24, Theorem 1.1], $(E, \le, \|\cdot\|)$ must be monotone σ-complete. Amemiya [1, Theorem 1] has shown that whenever $\|\cdot\|$ is monotone σ-complete, there exists a real number $k > 0$ such that for every x in E and for every sequence $0 \le x_n \uparrow |x|$ we must have

$\|x\| \le k \sup(\|x_n\| : n \in N)$. Therefore, it follows that $\|x\| \le k\|x\|_L$ for all x in E. Hence, $\|\cdot\|$ and $\|\cdot\|_L$ are equivalent. The conclusion follows by applying Theorem 3.18.

<u>Corollary 3.23</u>. Let $(E, \le, \|\cdot\|)$ be any order separable Banach lattice which has the Egoroff property. If E also has property K, $\varphi(x) = \|x\| - \|x\|_L$ is continuous at $x = 0$ with respect to $\|\cdot\|_L$.

<u>Proof</u>. Note that $(\hat{E}, \le, \|\cdot\|_M)$ satisfies the hypotheses of the theorem. Therefore, $\|\cdot\|_M$ is equivalent to $\|\cdot\|_{ML}$. Also, since $\|\cdot\|_{ML}$ is an extension of $\|\cdot\|_L$ (i.e., $\|\cdot\|_{ML} = \|\cdot\|_L^*$) it follows that $\|\cdot\|_L$ is equivalent to $\|\cdot\|$ on E.

CHAPTER TWO

A CHARACTERIZATION OF THE SPACE OF ALL REAL SEQUENCES

While the subject of this chapter is not directly related to the problems discussed in the preceeding chapter, it is interesting in its own right and does involve the properties of monotone seminorms on an Archimedean vector lattice. We shall characterize a universally complete vector lattice E having a countable collection of continuous, monotone seminorms defining a locally convex, Hausdorff topology as either a finite dimensional space or the space S of all real sequences.

Every vector lattice E always has a collection $\{x_\alpha : \alpha \in \mathcal{A}\}$ of mutually disjoint (i.e., $x_{\alpha_1} \wedge x_{\alpha_2} = 0$ for $\alpha_1 \neq \alpha_2$) elements in E such that $x_\alpha \wedge y = 0$ for all $\alpha \in \mathcal{A}$ implies that $y = 0$. ($\{x_\alpha\}$ is said to be <u>complete system</u> in E.) [13, Theorem 4.6]. If E is also universally complete and $\{x_\alpha\}$ has at least countably many elements then E cannot have a monotone norm defined on it. For, if $\|\cdot\|$ is a norm on E and $\{x_{\alpha_n}\}$ is a countable subset of $\{x_\alpha\}$ then $\|x_{\alpha_n}\| > 0$ for all $n \in N$. Letting $y_n = (n/\|x_{\alpha_n}\|)\cdot x_{\alpha_n}$ we must have $\sup(y_n : n \in N) = y$ in E because E is universally complete. But $\|y_n\| = n$ for $n = 1,2,\ldots$, implies that $\|y\| = +\infty$, which is impossible if $y \in E$. Hence, no such monotone norm exists. Of course, if E is any Archimedean vector lattice such that any system of mutually disjoint non-zero elements is finite, then E is of finite dimension [25, Theorem 4.3]. Hence, a universally complete vector lattice which is not a finite dimensional space cannot have a monotone norm defined on it.

Also, many universally complete spaces do not even have monotone seminorms defined on them. (All of our norms and seminorms are assumed to be real-valued, i.e., they do not take on $+\infty$ at any points.) For example, the space of all equivalence classes of Lebesgue measurable functions defined on $[0,1]$ has no non-trivial monotone seminorm defined on it [2, Corollary 1, p. 119]. However, $p_n(x) = x(n)$ for $n = 1,2,\ldots,$ and $x = (x(1),x(2),\ldots)$ defines a continuous, monotone seminorm on S and $\{p_n : n \in N\}$ also defines a locally convex, Hausdorff topology on S. Therefore, we ask whether S and the finite dimensional spaces are the only universally complete vector lattices having such a topology.

We shall need the following preliminary results.

<u>Lemma 1.1</u>. Let E be an Archimedean vector lattice with a countable collection of monotone seminorms $\{\|\cdot\|_n : n \in N\}$ defining a locally convex, Hausdorff topology on E. If $\{f_n : n \in N\} \subseteq E$, $f_n \uparrow$ and there exists an element g in E such that $\|f_n - g\|_k \to 0$ $(k = 1,2,\ldots)$ as $n \to \infty$, then $\sup(f_n : n \in N) = g$.

<u>Proof</u>. For $n > m$ we have $|f_m - \inf(g,f_m)| = |\inf(f_n,f_m) - \inf(g,f_m)| \le |f_n - g|$. Therefore,

$\|f_m - \inf(g,f_m)\|_k = 0$ for $k = 1,2,\ldots$. But the collection $\{\|\cdot\|_k : k \in N\}$ defining a Hausdorff topology on E implies that $f_m = \inf(g,f_m) \le g$ for all $m \in N$. Hence, g is an upper bound for $\{f_n\}$.

Let h be another upper bound for $\{f_n\}$. Then $|f_n - \inf(g,h)| = |\inf(f_n,h) - \inf(g,h)| \le |f_n - g|$ and we must have $\|f_n - \inf(g,h)\|_k \to 0$ $(k = 1,2,\ldots)$ as $n \to \infty$. But since we also know that $\|f_n - g\|_k \to 0$ $(k = 1,2,\ldots)$ as $n \to \infty$, it follows that $g = \inf(g,h) \le h$. Hence, $g = \sup(f_n : n \in N)$.

Theorem 1.2. Let E be an Archimedean vector lattice with $\{\|\cdot\|_k : k \in N\}$ a countable collection of monotone, continuous seminorms defining a locally convex, Hausdorff topology on E. Then if $\{x_\alpha\}$ is any net in E such that $0 \le x_\alpha \uparrow x$, there exists a sequence $\{x_{\alpha_n}\} \subseteq \{x_\alpha\}$ such that $0 \le x_{\alpha_n} \uparrow x$ in E (i.e., E is order separable).

Proof. If $\{x_\alpha\}$ is any net in E such that $0 \le x_\alpha \uparrow x$ in E, then $\|x - x_\alpha\|_k \downarrow 0$ for $k = 1,2,\ldots$. For $n = 1,2,\ldots$ choose $x_{\alpha_n} \in \{x_\alpha\}$ such that $0 \le x_{\alpha_n} \uparrow_n$ and $\|x - x_{\alpha_n}\|_k < 1/n$ for $k = 1,2,\ldots,n$. Then for any $k \in N$ and any $\epsilon > 0$ we can choose $N > 0$ such that $1/N < \epsilon$ and $N \ge k$. Then $\|x - x_{\alpha_n}\|_k < 1/N < \epsilon$ for all $n \ge N$. Hence, $\|x - x_{\alpha_n}\|_k \downarrow 0$ for $k = 1,2,\ldots$. Then, by the above lemma, $0 \le x_{\alpha_n} \uparrow x$.

We shall need the following definition concerning linear functionals on an Archimedean vector lattice E.

__Definition 1.3__. A linear functional f defined on E is said to be __order continuous__ if $\inf(f(x_\alpha)\colon \alpha \in \mathcal{A}) = 0$ whenever $\{x_\alpha\}$ is a net in E such that $x_\alpha \overset{o}{\rightarrow} 0$.

Letting $\tilde{\Omega}(E)$ = the set of all order continuous linear functionals on E, it is known that $\tilde{\Omega}(E)$ is a Dedekind complete vector lattice [23, p. 247]. We shall denote the topological dual space of a locally convex space $(E, \leq, \{\|\cdot\|_\alpha\colon \alpha \in \mathcal{A}\})$ by E'.

__Theorem 1.4__. Let E be an Archimedean vector lattice with a collection $\{\|\cdot\|_\alpha\colon \alpha \in \mathcal{A}\}$ of monotone seminorms defining a locally convex, Hausdorff topology on E. Then $\|\cdot\|_\alpha$ is continuous for all $\alpha \in \mathcal{A}$ if and only if $E' \subset \tilde{\Omega}(E)$.

__Proof__. Assume every $\|\cdot\|_\alpha$ is continuous and let $f \in E'$, $f \geq 0$. Suppose $0 \leq x_\beta \overset{o}{\rightarrow} 0$. Then $\|x_\beta\|_\alpha \underset{\beta}{\rightarrow} 0$ for each $\alpha \in \mathcal{A}$. Therefore $f(x_\beta) \rightarrow 0$ and we have $f \in \tilde{\Omega}(E)$.

On the other hand, if $\tilde{\Omega}(E) \supset E'$ and we choose any net $\{x_\beta\colon \beta \in \mathcal{B}\}$ in E such that $0 \leq x_\beta \overset{o}{\rightarrow} 0$ and any $f \in E' \subset \tilde{\Omega}(E)$ we must have $f(x_\beta) \rightarrow 0$. Applying proposition 3.4 in [16, p. 91] we have that $\|x_\beta\|_\alpha \underset{\beta}{\rightarrow} 0$ for all $\alpha \in \mathcal{A}$. Therefore, $\|\cdot\|_\alpha$ is continuous for each $\alpha \in \mathcal{A}$.

John J. Masterson [11, Corollary 2] has proved that an Archimedean vector lattice E is isomorphic (algebraically and lattice) to a space $M(X,\mathcal{B},\mu)$ of equivalence classes of μ-measurable, almost everywhere finite-valued functions on the completely additive, σ-finite measure space $(X,\mathcal{B},\mu)$ if and only if E is order separable and universally complete and $\Gamma(E)$ is separating on E, where $\Gamma(E)$ is an extended dual space for E. For our purposes, all we need know about $\Gamma(E)$ is that $\tilde{\Omega}(E)$ separating on E implies that $\Gamma(E)$ is separating on E. (For a discussion of $\Gamma(E)$, see [8].) Therefore, if E is a universally complete vector lattice with a countable collection of continuous, monotone seminorms defining a locally convex, Hausdorff topology on E we know that $E \approx M(X,\mathcal{B},\mu)$ where $M(X,\mathcal{B},\mu)$ is described as above. (This result follows from Theorems 1.2 and 1.4 and Masterson's result.)

In order to show that under our assumptions on E, $M(X,\mathcal{B},\mu)$ is either a finite-dimensional space or the space of all real sequences we need some results of C. Goffman [2]. We shall refer the reader to Goffman's paper for the definitions of carriers of a lattice and the space of Caratheodory functions generated by a lattice. We note that if $(X,\mathcal{B},\mu)$ is any localizable measure space (i.e., the lattice of equivalence classes of μ-measurable subsets is complete) having the finite subset property (i.e., any set of μ-positive measure has a subset of μ-finite measure) and $\mathcal{B}^*$ is the Boolean algebra of equivalence classes of measurable subsets of X then $\mathcal{B}^*$ is the lattice of carriers of $M(X,\mathcal{B},\mu)$ and $M(X,\mathcal{B},\mu)$ is the space of Caratheodory functions generated by $\mathcal{B}^*$.

<u>Goffman's Criterion 2</u>. Suppose $x^* \in \mathcal{B}^*$ may be split as follows:

$$x^* = x_1^* \cup x_2^*, \text{ where } x_1^* \cap x_2^* = 0^*$$

$$x_i^* = x_{i1}^* \cup x_{i2}^*, \text{ where } x_{i1}^* \cap x_{i2}^* = 0^* \qquad (i = 1,2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{i_1 i_2 \ldots i_{n-1}}^* = x_{i_1 i_2 \ldots i_{n-1} 1}^* \cup x_{i_1 i_2 \ldots i_{n-1} 2}^*$$

$$\text{where } x_{i_1 i_2 \ldots i_{n-1} 1}^* \cap x_{i_1 i_2 \ldots i_{n-1} 2}^* = 0^* \quad (i_1, i_2, \ldots, i_{n-1} = 1,2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and so that for every subsequence $i_1, i_2, \ldots, i_n, \ldots = 1,2$ we have that $x_{i_1}^* \cap x_{i_1 i_2}^* \cap \ldots \cap x_{i_1 i_2 \ldots i_n}^* \cap \ldots = 0^*$. Then x^* is said to satisfy criterion 2.

<u>Goffman's Theorem</u>. If $x^* \in \mathcal{B}^*$ is the carrier of $x \in M(X,\mathcal{B},\mu)$ and x^* satisfies criterion 2, then there exists no monotone seminorm p on $M(X,\mathcal{B},\mu)$ such that $p(x) > 0$.

We shall show that if x^* does not contain an atom (an <u>atom</u> e^* in a Boolean algebra is a non-zero element such that whenever $e^* \supseteq y^* \supseteq 0^*$ then $e^* = y^*$ or $y^* = 0^*$) and $(X,\mathcal{B},\mu)$ is σ-finite then we can split x^* as in criterion 2.

<u>Lemma 1.5</u>. If $(X,\mathcal{B},\mu)$ is a σ-finite measure space, $x^* \in \mathcal{B}^*$ such that x^* contains no atom, and $0 \le \alpha$ is any extended real number with $0 \le \alpha \le \mu(x^*)$ then there exists an element $x_1^* \in \mathcal{B}^*$ such that $0 \subseteq x_1^* \subseteq x^*$ and $\mu(x_1^*) = \alpha$.

<u>Proof</u>. See [3], p. 174, exercise 2.

<u>Theorem 1.6</u>. If $(X,\mathcal{B},\mu)$ is a σ-finite measure space and x^* is an element of $\mathcal{B}^*$ such that $x^* \neq 0^*$ then x^* does not contain an atom if and only if x^* satisfies criterion 2.

Proof. The fact that x^* satisfying criterion 2 implies that x^* does not contain an atom follows directly from some of Goffman's results. (See criterion 1 and some of the following discussion.)

Therefore, we need only show that if x^* does not contain an atom then x^* satisfies criterion 2. We consider two cases.

Case 1: $\mu(x^*) < +\infty$. Choose $x_1^*, x_2^* \in \mathcal{B}^*$ such that $x^* = x_1^* \cup x_2^*$, $x_1^* \cap x_2^* = 0^*$ and $\mu(x_1^*) = \mu(x_2^*) = 1/2\mu(x^*)$. (We can do this using the previous lemma.) Continue splitting x^* so that for $n = 1,2,\ldots$, we choose $x^*_{i_1 i_2 \ldots i_{n-1} 1}$, $x^*_{i_1 i_2 \ldots i_{n-1} 2}$ with $x^*_{i_1 i_2 \ldots i_{n-1}} = x^*_{i_1 \ldots i_{n-1} 1} \cup x^*_{i_1 \ldots i_{n-1} 2}$, $x^*_{i_1 i_2 \ldots i_{n-1} 1} \cap x^*_{i_1 \ldots i_{n-1} 2} = 0^*$ and $\mu(x^*_{i_1 \ldots i_{n-1} 1}) =$ $= \mu(x^*_{i_1 \ldots i_{n-1} 2}) = 1/2\, \mu(x^*_{i_1 \ldots i_{n-1}})$ $(i_1, \ldots, i_{n-1} = 1,2)$.

Now for every sequence $i_1, i_2, \ldots, i_n, \ldots$ $(i_n = 1$ or 2, $n = 1,2,\ldots)$ we get a chain $x^*_{i_1 i_2} \supseteq x^*_{i_1 i_2} \supseteq \cdots \supseteq x^*_{i_1 i_2 \ldots i_n} \supseteq \cdots$. Therefore, $\mu(\bigcap_{n=1}^{\infty} x^*_{i_1 \ldots i_n}) = \lim_{n\to\infty} \mu(x^*_{i_1 \ldots i_n}) = \lim_{n\to\infty} 1/2^n\, \mu(x^*) = 0$. Hence, for any sequence $i_1, \ldots, i_n, \ldots$ $(i_n = 1,2,\ n = 1,2,\ldots)$ we get that $\bigcap_{n=1}^{\infty} x^*_{i_1 i_2 \ldots i_n} = 0^*$ and this proves that x^* satisfies criterion 2.

Case 2: $\mu(x^*) = +\infty$. Let $X^* = \bigcup_{n=1}^{\infty} X_n^*$ where $X_i^* \cap X_j^* = 0^*$ whenever $j \neq i$ and suppose $\mu(X_n^*) < +\infty$ for $n = 1,2,\ldots$. Let $\{n_k\colon k = 1,2,\ldots\}$ be the subsequence of positive integers such that $x^* \cap X^*_{n_k} \neq 0^*$ for $k = 1,2,\ldots$, and $x^* \cap X_m^* = 0^*$ if $m \neq n_k$ for each k. We split x^* as follows: Let

$x^* = x_1^* \cup x_2^*$ where $x_1^* = x^* \cap \bigcup_{k\neq 1} X_{n_k}^*$, $x_2^* = x_1^* \cap X_{n_1}^*$. Then $x_1^* \cap x_2^* = 0^*$ and $\mu(x_2^*) < +\infty$. We continue by splitting x_2^* as in case 1 and we split x_1^* into $x_1^* = x_{11}^* \cup x_{12}^*$ where $x_{11}^* = x^* \cap \bigcup_{k\neq 1,2} X_{n_k}^*$ and $x_{12}^* = x_1^* \cap X_{n_2}^*$. As $m = 1,2,\ldots$, we have at the m^{th} step: $x^*_{\underbrace{111\ldots1}_{(m-1)\text{ times}}} = x^*_{\underbrace{11\ldots1}_{m\text{ times}}} \cup x_{11\ldots12}$ where

$x_{11\ldots1}^* = x^* \cap \bigcup_{k\neq 1,2,\ldots,m} X_{n_k}^*$ and $x_{11\ldots12}^* = x_{11\ldots1}^* \cap X_{n_m}^*$ and $x^*_{i_1 i_2\ldots i_{m-1}}$ is split as in case 1 for any $i_1, i_2, \ldots, i_{m-1} = 1,2$ which are not all ones.

Now we consider any sequence $i_1, i_2, \ldots, i_n, \ldots$ ($i_n = 1$ or 2, $n = 1,2,\ldots$). We note that $x^*_{i_1} \cdot x^*_{i_1 i_2} \supset$ $\supset x^*_{i_1 i_2 i_3} \supset \cdots \supset x^*_{i_1 i_2 \ldots i_n} \supset \cdots$ forms a strictly decreasing chain. If for any n we have that $\mu(x^*_{i_1 i_2 \ldots i_n}) < +\infty$ then as in case 1 $\bigcap_{n=1}^{\infty} x^*_{i_1 i_2 \ldots i_n} = 0^*$. Hence, we need only consider the case $i_1, i_2, \ldots, i_n, \ldots = 1$. $x_1^* \supset x_{11}^* \supset x_{111}^* \supset \cdots \supset x_{111\ldots1}^* \supset \cdots$. Then $x_1^* \cap x_{11}^* \cap \cdots \cap x_{11\ldots1}^* \cap \cdots = \bigcap_{m=1} (x^* \cap (\bigcup_{k>m} X_{n_k}^*)) =$

$= x^* \cap [\bigcap_{m=1}^{\infty} (\bigcup_{k>m} X_{n_k}^*)] = 0^*$.

We are ready to prove the following theorem.

<u>Theorem 1.7</u>. If $(X,\mathcal{B},\mu)$ is a σ-finite measure space and $x \in M(X,\mathcal{B},\mu)$ then there exists a monotone seminorm p on $M(X,\mathcal{B},\mu)$ with $p(x) > 0$ if and only if x^* contains an atom.

<u>Proof</u>. Again, the proof that x^* contains an atom implies that there exists a monotone seminorm p on $M(X,\mathcal{B},\mu)$ such that $p(x) > 0$ follows directly from criterion 1 in Goffman's paper.

To prove the other direction we note that if x^* does not contain an atom then x^* must satisfy criterion 2 by the previous theorem. But then Goffman's Theorem shows that there exists no monotone seminorm p on M such that $p(x) > 0$.

Before proving the characterization theorem we give the following definitions.

Definitions 1.8. Let E be an Archimedean vector lattice. An element x in E is said to be discrete whenever there do not exist mutually disjoint elements $y > 0$ and $z > 0$ such that $y,z \le |x|$. Let x be any element in E^+ and B_x the band in E generated by x. Then if y is any element of E^+ the projection of y onto B_x is defined to be $\sup(y \wedge (nx): n \in N)$ whenever this supremum exists. (Notation: $[x]y = \sup(y \wedge (nx): n \in N)$.)

A discrete element x always satisfies the property that whenever $0 < z < |x|$ then $z = \lambda x$ for some real number $\lambda > 0$ [23, p. 73]. Also, note that whenever E is Dedekind σ-complete all projections onto principal bands must exist.

Theorem 1.8. E is a universally complete vector lattice having a countable collection $\{\|\cdot\|_n: n \in N\}$ of monotone, continuous seminorms defining a locally convex, Hausdorff topology on E if and only if E is a finite dimensional space or E is the space of all real sequences.

Proof. We have already shown that E must be order separable and $E \simeq M(X,\mathcal{B},\mu)$ where $(X,\mathcal{B},\mu)$ is σ-finite.

Since $\{\|\cdot\|_n : n \in N\}$ defines a Hausdorff topology on E, given any $0 \neq x \in E$ there exists a positive integer $n(x)$ such that $\|x\|_n > 0$. Hence, by the previous theorem, every $0 \neq x \in E$ is such that x^* (the carrier of x) contains an atom. $(X,\mathcal{B},\mu)$ σ-finite implies that $\mathcal{B}^*$ can have at most countably many atoms. Let $\{e_n^*\}$ be the collection (possibly finite) of distinct, and, therefore, disjoint atoms in $\mathcal{B}^*$. For each n let $e_n = \chi_{e_n^*}$ (the characteristic function of e_n^*) and note that $e_n \in M(X,\mathcal{B},\mu) = E$. Then $\{e_n\}$ forms a complete system of pairwise disjoint, discrete elements in E. Obviously, $e_n \wedge e_m = 0$ for $n \neq m$ because $e_n^* \cap e_m^* = 0^*$. To show that e_n is discrete for each n, choose any elements $x,y \in E$ such that $0 \leq x,y \leq e_n$ and show that either one of x and y must be zero or $x \wedge y$ is not zero. Since e_n^* is an atom and $0^* \subseteq x^*, y^* \subseteq e_n^*$ either one of x^* and y^* must be zero or both x^* and y^* are equal to e_n^*. If one of x^* or y^* is zero we are done, so we assume $x^* = y^* = e^*$. But then $x \wedge y$ cannot be zero. Also, the collection $\{e_n\}$ is complete in E because if $f \wedge e_n = 0$ for all n where $f \in E = M(X,\mathcal{B},\mu)$ then $f^* \cap e_n^* = 0^*$ for all n. But this is possible only if $f^* = 0^*$ because if $f^* \neq 0^*$ then f^* must contain an atom; i.e., one of the e_n^*. Hence, $f = 0$.

Now, E is a universally complete vector lattice and $\{e_n\}$ is a complete, countable (possibly finite) system of pairwise disjoint, discrete elements. We define a mapping $F: E \to R^{\aleph}$, a 1-1, onto, lattice and algebraic isomorphism where $\aleph = \aleph_o$ or $\aleph = N_o$ (some finite integer) as follows: For every x in E note that $[e_n]x = \alpha_n e_n$ for each n since e_n is a discrete element which forces B_{e_n} to be a one dimensional universally complete vector lattice [23, Theorem IV.12.1]. Define $F(x) = f_x \in R^{\aleph}$ where $f_x(n) = \alpha_n$. One can easily see that this is a lattice and algebraic isomorphism. Also, the mapping is onto since E is universally complete.

Hence, we have proved that every universally complete vector lattice satisfying the hypotheses of the theorem must be S or a finite dimensional space. The other direction of the theorem follows easily because $p_n(x) = x(n)$ for $x = (x(1),x(2),\ldots)$ in S (or $x = (x(1),x(2),\ldots,x(N))$ for the case of N-dimensional space) defines a continuous, monotone seminorm for $n = 1,2,\ldots$ $(n = 1,2,\ldots,N)$ and $\{p_n\}$ defines a locally convex, Hausdorff topology on E.

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APPENDIX

APPENDIX

Before proving Theorem 0.8 of the introduction we shall introduce some terms and state some results. The following discussion comes directly from [13, Sections 7 and 8].

Let E be a Dedekind σ-complete vector lattice. For any two elements a and p in E^+ define <u>$[p]a$</u> $= \sup(a \wedge (np): n \in N)$. For arbitrary elements a and p in E define $[p]a = [|p|]a^+ - [|p|]a^-$. $[p]$ is an operator on E and is called the <u>projector</u> of p. Following are some results concerning the projector operator. Any of the proofs can be found in [13].

(1). $[p](\alpha x + \beta y) = \alpha[p]x + \beta[p]y$ for all x,y,p in E and α,β real numbers.

(2). $|[p]x| \leq |x|$ for all x and p in E.

(3). $[p]x = 0$ if and only if $p \wedge x = 0$.

(4). $[p] = 0$ if and only if $p = 0$.

<u>Definition</u>. An element $e \neq 0$ of a vector lattice E is said to be a <u>complete element</u> in E if for any x in E $e \wedge x = 0$ implies $x = 0$.

A universally complete vector lattice always has a complete element [13, Theorem 32.4].

(5). $[p] = 1$ if and only if p is a complete element of E.

(6). $[p] = [\alpha p]$ for all real $\alpha \neq 0$.

(7). Given any a in E, $[a^+]a = a^+$.

(8). $[p][q] = [[p]q] = [|p| \wedge |q|]$.

For any two projectors $[p]$ and $[q]$ we shall write $[p] \geq [q]$ if and only if $[p]x \geq [q]x$ for every x in E^+.

(9). $[p] \geq [q]$ is equivalent to each of $[p][q] = [q]$ and $[p]q = q$.

(10). $|p| \geq |q|$ implies $[p] \geq [q]$.

Since $[p] \geq [q]$ as defined above defines an order relation on the set of projectors in a vector lattice we can define such terms as upper bound and least upper bound of a set of projectors just as in a vector lattice.

(11). If $|p| = \sup(|p\lambda| : \lambda \in \Lambda)$ then

$[p]a = \sup([p\lambda] : \lambda \in \Lambda)a$.

<u>Theorem 0.8</u>. Let $(E, \leq, \|\cdot\|)$ be a normed vector lattice such that $\|\cdot\|$ is semi-continuous. If $\{x_\alpha : \alpha \in \mathcal{A}\}$ is any increasing net in E^+ such that $\|x_\alpha\| \leq M$ for all $\alpha \in \mathcal{A}$, then $\sup x_\alpha = x_o$ exists in $E^{\#}$.

<u>Proof</u>. Let $\bar{e}$ be a complete element in $E^{\#}$. For $n = 0,1,2,\ldots$, define $\overline{p_n} = \sup([(x_\alpha - n\bar{e})^+]\bar{e} : \alpha \in \mathcal{A})$ in $E^{\#}$. Then $0 \leq \overline{p_n} \downarrow$. And by (10) it follows that $0 \leq [\overline{p_n}] \downarrow$. We shall show that $0 \leq [\overline{p_n}] \downarrow 0$.

Suppose there exists an element a in E such that $[\bar{p}]\bar{e} \geq a \geq 0$ where $[\bar{p}] = \inf([\overline{p_n}]: n \in N)$. We shall show that $a = 0$. Since $[(x_\alpha - n\bar{e})^+](x_\alpha - n\bar{e}) \geq 0$ for all α in $\mathcal{A}$ and all n (see (7)) we must have $[(x_\alpha - n\bar{e})^+]x_\alpha \geq$ $\geq [(x_\alpha - n\bar{e})^+](n\bar{e})$. But, using (2), we then have $x_\alpha \geq$ $\geq [(x_\alpha - n\bar{e})^+](n\bar{e})$. Since (2) also implies that $\bar{e} \geq [\bar{p}]\bar{e}$ we also have $n\bar{e} \geq n[\bar{p}]\bar{e} \geq na$. Combining these last two statements we obtain $x_\alpha \geq [(x_\alpha - n\bar{e})^+](na)$ and it now follows that $\sup(x_\alpha \wedge (na): \alpha \in \mathcal{A}) \geq \sup([(x_\alpha - n\bar{e})^+](na): \alpha \in \mathcal{A})$ for $n = 0,1,2,\ldots$.

On the other hand, using (11), (8), and (5), we have $\sup([(x_\alpha - n\bar{e})^+](na): \alpha \in \mathcal{A}) = [\overline{p_n}](na)$. Also, $[\overline{p_n}](na) \geq$ $\geq [\bar{p}](na)$. Applying (9) to the inequality $[\bar{p}] \geq [na]$ which holds because $[\bar{p}] \geq [[\bar{p}]\bar{e}] \geq [a]$ we obtain $[\bar{p}](na) = na$. Combining these inequalities we have $\sup([(x_\alpha - n\bar{e})^+](na):$ $\alpha \in \mathcal{A}) \geq [\bar{p}](na) = (na) \geq \sup(x_\alpha \wedge (na): \alpha \in \mathcal{A})$. It now follows that $(na) = \sup(x_\alpha \wedge (na): \alpha \in \mathcal{A})$ for $n = 0,1,2,\ldots$. Applying the semi-continuity of $\|\cdot\|$ to the net $x_\alpha \wedge (na) \uparrow_\alpha (na)$ we have $\|na\| = \sup(\|x_\alpha \wedge (na)\|: \alpha \in \mathcal{A}) \leq M$. But $\|na\| \leq M$ for all n implies $a = 0$. Hence, $[\bar{p}]\bar{e} = 0$ because $\bar{e} = \sup(a\lambda: \lambda \in \Lambda)$ for some set $\{a\lambda\} \subseteq E$. We now apply (4), (3) and the fact that $\bar{e}$ is a complete element to obtain $[\bar{p}] = 0$.

Since $E^{\#}$ is universally complete the supremum of any collection of mutually disjoint elements in $E^{\#}$ must exist in $E^{\#}$. Hence, $\bar{e}_o = \sum_{n=1}^{\infty} n([\overline{p_{n-1}}] - [\overline{p_n}])\bar{e} = \sup(n([\bar{p}_{n-1}] - [\bar{p}_n])\bar{e})$ exists in $E^{\#}$. Also, since $[\overline{p_n}] \downarrow 0$ and $[\bar{p}_o] \geq [x_\alpha][\bar{e}] = [x_\alpha]$

for all $\alpha \in \mathcal{A}$, we must have $x_\alpha = \sum_{n=1}^{\infty}(\overline{[p_{n-1}]} - \overline{[p_n]})x_\alpha$. Note that $([\bar{p}_{n-1}] - [\bar{p}_n])(x_\alpha - n\bar{e}) = [\bar{p}_{n-1}](1 - [\bar{p}_n])(x_\alpha - n\bar{e})$ since $[\bar{p}_{n-1}][\bar{p}_n] = [\bar{p}_n]$. (See (9)). And, $[\bar{p}_{n-1}](1 - [\bar{p}_n])(x_\alpha - n\bar{e}) =$
$= [\bar{p}_{n-1}](1 - [\bar{p}_n] + [\bar{p}_n][(x_\alpha - n\bar{e})^+] - [(x_\alpha - n\bar{e})^+])(x_\alpha - n\bar{e}) =$
$= [\bar{p}_{n-1}](1 - [\bar{p}_n])(1 - [(x_\alpha - n\bar{e})^+])(x_\alpha - n\bar{e})$ because $[\bar{p}_n] \geq [(x_\alpha - n\bar{e})^+]$. This last expression for $([\bar{p}_n] - [\bar{p}_{n-1}])(x_\alpha - n\bar{e})$ is less than or equal to zero. Hence, for all α in $\mathcal{A}$ and for $n = 0,1,2,\ldots$, $([\bar{p}_{n-1}] - [\bar{p}_n])x_\alpha \leq$
$\leq n([\bar{p}_{n-1}] - [\bar{p}_n])\bar{e}$. It follows that $x_\alpha \leq \bar{e}_o$ for all $\alpha \in \mathcal{A}$. Hence, $\sup(x_\alpha : \alpha \in \mathcal{A}) = x_o$ in $E^{\#}$.