BOUNDARY SOLUTION OF THE WAVE EQUATION

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

Richard J. Reid

1959

This is to certify that the

thesis entitled

BOUNDARY SOLUTION OF THE WAVE EQUATION

presented by

RICHARD J. REID

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in E. E.

L. W. Von Tersch
Major professor

Date August 21, 1959

BOUNDARY SOLUTION OF THE WAVE EQUATION

By

RICHARD J. REID

Dick Weed

A THESIS

Submitted to the School for Advanced Graduate Studies of Michigan State University of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1959

ABSTRACT

The solution of the wave equation subject to boundary conditions existing on irregularly shaped surfaces is obtained by assuming a solution function that is a truncated, n-dimensional Taylor's series about some point on the boundary or within the region where a solution is required.

The assumed series is of degree m_i (i = 1,2,...n) in each coordinate (i). There are then

$$\prod_{i=1}^{n} (m_i + 1)$$

coefficients of the series which are chosen to minimize the deviation of the solution function from the prescribed boundary values and also to minimize any residual remaining upon substitution of the assumed function into the wave equation.

The method of solution has the properties that:

(a) the error of the solution can be estimated;

(b) boundaries can be of any shape;

(c) calculations for the eigenvalues and eigenfunctions of the solution can be made exclusively at coordinates of the boundaries;

(d) the solution can be made to correspond to a least squares, Chebychev, or other type fit over the region of interest.

ACKNOWLEDGMENTS

The author wishes to acknowledge with gratitude the help and guidance received from his major professor Dr. L. W. Von Tersch. Grateful acknowledgment is also due Dr. G. P. Weeg for his helpful suggestions and assistance.

TABLE OF CONTENTS

		Page
I.	INTRODUCTION	1
II.	APPROXIMATING SOLUTION FUNCTION	4
III.	BOUNDARY SOLUTION FOR EIGENVALUES AND EIGENFUNCTIONS	10
IV.	EXTENDING THE NUMBER OF VARIABLES	15
V.	SIMULTANEOUS WAVE EQUATIONS	16
VI.	MODIFYING THE SOLUTION FUNCTION	18
VII.	EXAMPLE OF EIGENVALUE AND EIGENFUNCTION DETERMINATION	20
VIII.	DISCUSSION	24
IX.	REFERENCES CITED	25

I. INTRODUCTION

The solution of the electromagnetic wave equation subject to boundary conditions existing on surfaces deviating more than a slight amount from surfaces on which x, r, θ, et cetera, are constant,[1] requires the use of numerical methods to obtain approximate results.

The usual numerical method divides the space over which the equation and boundary conditions are to be satisfied into a mesh and produces at the nodes of the mesh a difference equation which approximates the wave equation. This difference equation is obtained by assuming a polynomial expansion of the solution function in each coordinate direction about each particular node. The degree of the polynomial is commonly the same as the order of the wave equation, and the wave equation is used to determine the relation among the coefficients of the polynomial. This method is described by Booth (3).

A more versatile approach to the solution of the wave equation and other partial differential equations has been contributed by

---

[1]Methods for small boundary perturbations are given by Slater (1) and by Schelkunoff (2).

Milne et al. (4), and assumes an n-dimensional polynomial (where n is the number of independent variables) about each node point, and allows the degree of the polynomial to be as large as practical from the considerations of required accuracy of solution and of the feasibility of calculating the coefficients for the polynomial from the given partial differential equation.

All these approaches require the simultaneous solution for the function value at each node subject to the function value satisfying all polynomials that have coefficients determined from the position of this node. An iterative method is usually employed for the solution of these equations. These methods thus require:

(a) special calculation of the polynomial coefficients and the function values for given boundary points which do not lie at the nodes of the mesh, or an approximation of the desired boundary that does pass through the nodes;

(b) a new determination of the polynomial coefficients and/or smaller mesh size and then iterating to a solution in order to realize more accuracy in the solution.

These will then yield an approximate solution at the nodes of the mesh in terms of a table of values corresponding to the field's component values.

The method to be investigated here is that of assuming a functional form for approximating the solution, since the functional form will allow:

(a) calculations giving the error in the solution;[2]

(b) boundaries to be of any shape since there is no mesh and node system to which they must correspond;

(c) calculations for the eigenvalues and eigenfunctions of the solution to be made only at coordinates of the boundaries.

---

[2]The methods previously discussed also permit error analysis.

II. APPROXIMATING SOLUTION FUNCTION

The wave equation that will be directly considered here is that resulting from Maxwell's equations for sinusoidal time variations describing one of the coordinate components of the electric or magnetic field in a charge-free, linear, isotropic medium where the parameters of the medium are constants; i.e.,

$$\nabla^2 E = -w^2 ueE \tag{1}$$

Equation (1) must be satisfied for some solution function, E, which normally must also satisfy boundary conditions, as in two dimensions

$$\frac{\partial E(x,y)}{\partial n} = 0 \tag{2}$$

along some curve C_1, where n is normal to C_1, and

$$E(x,y) = 0 \tag{3}$$

along some curve C_2.

Since Equation (1) has constant coefficients, if the boundaries C_1 and C_2 can be formed from a finite number of rectifiable arcs, then, as Bernstein (5) has shown, there exists a solution, E, to Equations (1), (2), and (3) that is of a class A^∞ (continuous in all derivatives) in the neighborhood of all points on boundaries C_1 and

C_2. This then allows a Taylor's series expansion for E about some point on the boundary, and it will be assumed that a finite number of terms of the series will give the solution function approximately for some maximum irregularity of the boundaries.

The truncated, two-dimensional Taylor's series, f(x,y) is given by the matrix product

$$f(x,y) = [Y]'[A][X] \tag{4}$$

where

$$[Y]' = [1,y,\ldots,y^{n-1}] \; ; \; y = y' - y_o$$

and

$$[X]' = [1,x,\ldots,x^{m-1}] \; ; \; x = x' - x_o$$

and [A] is an nxm coefficient matrix.

For the definition

$$f_{yy} = [Y]'_{yy}[A][X]$$

where

$$[Y]'_{yy} = [0,0,2,6y,\ldots,(n-1)(n-2)y^{n-3}]$$

and similarly for f_{xx}, Equation (1) becomes upon substitution of f(x,y), the E approximating function,

$$f_{xx} + f_{yy} + w^2uef = R_1(x,y)$$

in S, the region of solution, where $R_1(x,y)$ is the wave equation residual or error function. Also in Equations (2) and (3)

$$\frac{\partial f}{\partial n} = \frac{\partial f}{\partial n} \frac{1}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} - \frac{\partial f}{\partial y} \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = R_2(x,y)$$

along C_1, and

$$f = R_3(x,y)$$

along C_2.

A total residual function, R_T, can be defined as

$$R_T = \iint_S W_1(x,y) R_1^2(x,y)da + \int_{C_1} W_2(x,y) R_2^2(x,y)dl + \int_{C_3} W_3(x,y) R_3^2(x,y)dl \qquad (5)$$

where $W_1(x,y)$, $W_2(x,y)$, and $W_3(x,y)$ are nonnegative weighting functions introduced to allow variance in the degree of satisfaction of Equations (1), (2), and (3). For example, if $W_1(x,y)$ is chosen large with respect to $W_2(x,y)$ and $W_3(x,y)$, then the assumed function will more closely satisfy the wave equation than it will the boundary conditions. This allowance of weighting functions is particularly useful when examining small changes in a section of the boundary, as that boundary section can be relatively heavily weighted, thus elucidating its effect upon the solution.

The squares of the partial residual functions have been used in Equation (5) in defining the total residual function. This will lead to a solution function fitting the wave equation and boundary conditions on a least squares basis. Similarly, for other types of fit,

the magnitudes of the partial residual function could be used in Equation (5) or, for a Chebychev type fit, the single maximum magnitude of $R_1(x,y)$, $R_2(x,y)$, or $R_3(x,y)$ could be chosen as the total residual function. Of course, this maximum will shift among R_1, R_2, and R_3, and range over the region of solution as the elements of the matrix A are changed.

For any of these choices, the total residual function, R_T, will be considered as a function of the coefficients in the matrix A; i.e., $R_T(a_{11}, a_{12}, \ldots a_{1m}, \ldots, a_{nm})$.

When the least squares basis is used, as in Equation (5), the total residual function will be quadratic in the elements of the matrix A. In order to obtain the absolute minimum value for the total residual function, the requirements for a relative minimum will first be imposed; i.e.,

$$\sum_{j=1}^{m} \quad \sum_{i=1}^{n} \left| \frac{\partial R_T}{\partial a_{ij}} \right| = 0 \tag{6}$$

and the $(m.n) \times (m.n)$ matrix

$$\begin{bmatrix} \dfrac{\partial^2 R_T}{\partial a_{11}^2}, \cdots & \dfrac{\partial^2 R_T}{\partial a_{11} \partial a_{nm}} \\ \vdots & \\ \dfrac{\partial^2 R_T}{\partial a_{nm} \partial a_{11}}, \cdots & \dfrac{\partial^2 R_T}{\partial a_{nm}^2} \end{bmatrix} \tag{7}$$

must be positive definite.

Since R_T is of quadratic form in the a_{ij}'s, the elements of Equation (6), $\frac{\partial R_T}{\partial a_{ij}}$, will be linear and Equation (6) will be homogeneous. In addition, since R_T is quadratic and always positive, it is guaranteed of having a single minimum.

Let the solution of Equation (6) be a set of a_{ij}'s, $\bar{a}_{ij}$. The calculations involved in determining the $\bar{a}_{ij}$'s will be a check that

$$R_T(\bar{a}_{11}, \ldots, \bar{a}_{ij} \pm e, \ldots \bar{a}_{nm}) > R_T(\bar{a}_{11}, \ldots, \bar{a}_{ij}, \ldots, \bar{a}_{nm})$$

for all i and j and for some small e. The finite nature of e for this check can be allowed since $\frac{\partial R_T}{\partial a_{ij}}$ is linear and consequently can have no more than one root in the region $\bar{a}_{ij} \pm h$, $h \leq e$.

The function R_T can possibly be reduced to zero when the assumed solution form is the same as the exact solution form. Also, it is possible to obtain a zero for R_T by the numerical method of evaluating Equation (5). This misleading result can occur if the integrals are evaluated in a particular dimension at fewer points than the degree of the polynomial plus one, since the polynomial would then be nonunique. Careful attention to the evaluation of the integrals in Equation (5) will avoid this difficulty.

If, in a particular problem, it is required to determine the eigenvalues of Equation (1), then w^2ue must be included as one of

the independent variables of the function R_T. The total residual function will no longer be of quadratic form, however, and likewise, the $\frac{\partial R_T}{\partial a_{ij}}$ and $\frac{\partial R_T}{\partial (w^2 ue)}$ will not result in linear equations. This must be so, of course, since for the exact solution there will be an infinity of values for $w^2 ue$. The resulting difficulty of determining the $\bar{a}_{ij}$'s with $w^2 ue$ as a variable can be overcome by a knowledge of a good approximation to $\overline{w^2 ue}$. This approximation is originally introduced as a constant and the best values for the a_{ij}'s determined for the resulting linear case. Then the $w^2 ue$ is treated as a variable to obtain $\overline{w^2 ue}$.

III. BOUNDARY SOLUTION FOR EIGENVALUES AND EIGENFUNCTIONS

Consider an exact solution for Equations (1), (2), and (3), G(x,y) so that

$$\nabla^2 G(x,y) + w^2 ue\ G(x,y) = 0 \tag{8}$$

for all x and y in the region of solution S and on the boundary C. Let there be another function H(x,y) so that

$$G(x,y)\Big|_C = H(x,y)\Big|_C , \tag{9}$$

$$\frac{\partial G(x,y)}{\partial x}\Big|_C = \frac{\partial H(x,y)}{\partial x}\Big|_C , \tag{10}$$

$$\frac{\partial G(x,y)}{\partial y}\Big|_C = \frac{\partial H(x,y)}{\partial y}\Big|_C , \tag{11}$$

$$\frac{\partial^2 G(x,y)}{\partial x^2}\Big|_C = \frac{\partial^2 H(x,y)}{\partial x^2}\Big|_C , \tag{12}$$

and

$$\frac{\partial^2 G(x,y)}{\partial y^2}\Big|_C = \frac{\partial^2 H(x,y)}{\partial y^2}\Big|_C . \tag{13}$$

Higher order derivatives may also be equal when evaluated along C, but in general, could be quite different. If all orders of derivatives were equal along C, or along any other curve, or at

any point then the functions G(x,y) and H(x,y) would be identical in the neighborhood of this equivalence. This requirement is not to be imposed upon H(x,y) at this point. It can be seen from Equations (10) and (11) that H(x,y) will satisfy the normal derivative boundary condition. Of course, this first order derivative equivalence need only be satisfied upon that part of the curve $y = f(x)$ where the normal derivative boundary condition is to apply.

If the function H(x,y) is substituted into the wave equation, the resulting nonhomogeneous partial differential is

$$\nabla^2 H(x,y) + w^2 ue\ H(x,y) = J(x,y). \qquad (14)$$

In order to guarantee that the eigenvalues determined from Equations (8) and (14) are the same, it will be sufficient to force the function J(x,y) to negligible values throughout the region of solution. It is here assumed that the function H(x,y) is also forced to fit the proper boundary conditions so that J(x,y) is already negligible on the boundary.

The fact that J(x,y) must be reduced within the region of solution in order to determine the eigenvalues accurately is illustrated in the following example, although, as will be shown, it is possible to obtain useful approximations to the eigenvalues when the values for J(x,y) within the region of solution are not negligible. Consider the one-dimensional wave equation

$$\frac{\partial^2 G(x)}{\partial x^2} + w^2 ue\ G(x) = 0$$

subject to the boundary conditions

$$G(x) = 0 \text{ at } x = \pm 1.$$

The first imagined solution might be a parabola, say

$$H(x) = 1 - x^2$$

which satisfies the boundary conditions but does not satisfy the wave equation at the boundary. Therefore, first choose the second derivative of H(x) to be zero at the boundary, then construct H(x) itself to be zero at the boundary; i.e.,

$$\frac{\partial^2 H(x)}{\partial x^2} = 12x^2 - 12$$

$$\frac{\partial H(x)}{\partial x} = 4x^3 - 12x + c_1$$

$$H(x) = x^4 - 6x^2 + c_1 x + c_2.$$

Now by choosing c_1 equal to zero and c_2 equal to plus five, the function H(x) also satisfies the boundary conditions now as

$$H(x) = x^4 - 6x^2 + 5.$$

Substituting these results into the wave equation results in

$$12x^2 - 12 + w^2 ue(x^4 - 6x^2 + 5) = J(x).$$

Now, while J(x) has roots at x equal to plus or minus one, it can not be made zero over a finite interval for a constant value of

w^2ue. For the determination of possible roots of J(x)

$$w^2ue = \frac{12 - 12x^2}{x^4 - 6x^2 + 5} = \frac{12}{5 - x^2}$$

which has a maximum value of plus three at the boundary of the interval of solution and a minimum of 2.4 at x equal to zero. The average value over the interval is determined as

$$(w^2ue)_{avg} = \frac{1}{2}\int_{-1}^{1} \frac{12}{5 - x^2}\, dx = 2.59$$

The lowest ordered, exact solution of the one-dimensional wave equation for these boundaries is

$$G(x) = \cos \frac{\pi x}{2},$$

and the principal eigenvalue determined for this is

$$w^2ue = \left(\frac{\pi}{2}\right)^2 = 2.467$$

The deviations of the approximate eigenvalues from this correct value result because the function J(x) has not been sufficiently reduced within the interval of solution. While the previous results may in some cases be adequate or useful approximations, additional requirements which also can be imposed at the boundaries alone will yield the eigenvalues and the eigenfunctions to any desired accuracy.

It is noted that the nonhomogeneous part of Equation (14), J(x,y) corresponds to the partial residual function $R_1(x,y)$ in

Equation (5). However, the preceding development suggests that the integral of $R_1^2(x,y)$ over the region of solution be replaced in Equation (5) so that the total residual function is now determined as

$$R_T = \sum_{j=0}^{m} \sum_{i=0}^{n} W_{ij} \left[\frac{\partial^{i+j} R_1(x,y)}{\partial x^i \partial y^j} \right]^2_{(x_0,y_0)} + \int_{C_1} W_2(x,y) R_2^2(x,y) dl$$

$$+ \int_{C_2} W_3(x,y) R_3^2(x,y) dl$$

where the summation terms are the Taylor's series coefficients for the expansion of the function $R_1(x,y)$ about the point (x_0,y_0). Thus, if these coefficients are reduced sufficiently by the reduction of R_T to a minimum, the assumed solution function is automatically fitted to the differential equation in the region about (x_0,y_0).

IV. EXTENDING THE NUMBER OF VARIABLES

The case of Taylor's series in two dimension, Equation (4), can be extended to any number of dimensions by repeatedly applying the extension rule that

$$g(x,y,z) = [Z]'[B]$$

where

$$[Z]' = [1,z,\ldots,z^{p-1}] \; ; \; z = z' - z_o$$

and [B] is a column matrix of p elements such that each element b_i is given as

$$b_i = [Y]'[A_i][X]$$

$$i = 0,1,2,\ldots,p-1$$

and there will thus be p separate A_i matrices for the three-dimensional case.

For the case of n-dimensions such that an m_j degree polynomial is assumed in the j'th coordinate, there will then be a total of

$$\prod_{j=1}^{n} (m_j + 1)$$

elements belonging to A matrices.

Equation (6) would be applied to all the elements of the A's.

V. SIMULTANEOUS WAVE EQUATIONS

Although the wave function will often satisfy the condition that it is expandable in a Taylor's series about some point in the region of solution, when a coefficient of the wave equation (u or e) has discretely different values in the region of solution, this will no longer be possible.

Consider the case of Equation (1) where e must have two distinct values within a region and the boundary between the e values is given by some curve C_3. The solution in either of the regions (as determined by C_3 and the original boundaries) is continuous in all derivatives, and it is desired that the solution in each of these regions give the same values along the curve C_3. Since from Equation (1) if E(x,y) is to be continuous while e has a change in value, then the second derivatives of E(x,y) must be discontinuous which the assumed solution type will not provide.

For problems such as these it will be necessary to assume multiple solutions, one for each region where derivatives of all orders may exist. Thus for two such regions, a and b, separated

by a curve C_3 two series will be used, $f_a(x,y)$ and $f_b(x,y)$, and the residuals within each of these regions calculated. The total residual function for the entire problem is then calculated as

$$\begin{aligned} R_T = &\iint_a W_{1a}(x,y)R_{1a}^2(x,y)da + \iint_b W_{1b}(x,y)R_{1b}^2(x,y)da \\ &+ \int_{C_{1a}} W_{2a}(x,y)R_{2a}^2(x,y)dl + \int_{C_{1b}} W_{2b}(x,y)R_{2b}^2(x,y)dl \\ &+ \int_{C_{2a}} W_{3a}(x,y)R_{3a}^2(x,y)dl + \int_{C_{2b}} W_{3b}(x,y)R_{3b}^2(x,y)dl \\ &+ \int_{C_3} W_4(x,y)[f_a(x,y) - f_b(x,y)]^2 dl \end{aligned} \qquad (15)$$

where C_{1a} is the part of the boundary of region a to which the boundary condition of Equation (2) applies, similarly for C_{1b}; C_{2a} is the part of the boundary of region a to which the boundary condition of Equation (3) applies, similarly for C_{2b}; and $W_4(x,y)$ is a weighting function applied to the criterion of the degree satisfaction that both functions $f_a(x,y)$ and $f_b(x,y)$ have the same values along the curve C_3.

The coefficients of the functions $f_a(x,y)$ and $f_b(x,y)$ can now be calculated by applying Equation (6) to the residual function found from Equation (15).

VI. MODIFYING THE SOLUTION FUNCTION

Since it is often necessary to obtain solutions for the wave equation where it is known that the magnitude of the solution function varies over a considerable range in the region of interest, it is possible to modify the assumed solution function to correspond in form to the suspected general shape of the actual solution. The following is a list of these modifications and the type of problems to which they apply.

Partial radial symmetry

When there is some apparent degree of radial symmetry in cases of two or more dimensions, a more general assumed solution is possible that corresponds to the Method of Frobenius in one-dimensional problems with nonconstant coefficients. The analogous functions in two dimensions is given as

$$f(r,\theta) = [r^{p_0},\ \theta^{p_1},\ \theta^2 r^{p_2},\ldots,\ \theta^{n-1} r^{p_{n-1}}] \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^{m-1} \end{bmatrix}$$

where the p_i's will be additional variables in the total residual function and will be calculated in the same way as the a_{ij}'s;

i.e., by evaluating the partial derivatives as in Equations (6) and (7).

Of course, the ∇^2 operator must be in the same coordinate system as the assumed, independent variables.

Zero at infinity

The assumed function can be modified to yield a zero at infinity by multiplying the assumed Taylor's series by a weight function, so that the assumed function is

$$g(x,y) = e^{-(x^2 + y^2)} f(x,y)$$

where f(x,y) is the normal Taylor's series as given in Equation (4). The polynomials corresponding to g(x,y) in one dimension are the Hermite polynomials.

First approximation to the solution function

When an apparently good approximation to the solution function is known in terms of explicit functions, say h(x,y), then the assumed solution

$$g(x,y) = h(x,y) + f(x,y),$$

where f(x,y) is the Taylor's series, may result in a low degree for f(x,y) for good results in g(x,y). Thus g(x,y) may be a relatively simple explicit function easily used in following analytical investigations.

VII. EXAMPLE OF EIGENVALUE AND EIGENFUNCTION DETERMINATION

A complete, digital-computer program to obtain the approximate solution for the wave equation in two dimensions has been prepared for the Computer Laboratory (6). The program requires that the matrix A be square and have an order equal to or less than ten, so that polynomials through the ninth degree are possible. Mixed boundary conditions of $\frac{\partial E}{\partial n} = 0$ and $E = o$ are possible. The program user must supply the following information:

1. The order of the assumed matrix A up to and including ten.

2. The rectangular coordinates of

   (a) points in the region at which it is desired to have the assumed function satisfy the wave equation;

   (b) points on the boundary where the condition $E(x,y) = 0$ is to apply;

   (c) pairs of points which define a normal to the boundary where the condition $\frac{\partial E(x,y)}{\partial n}$ is to apply.

3. An initial approximation for the entries in the matrix A and for w^2ue.

The program will then determine the total residual function as defined in Equation (5) with the weight functions all taken as unity.

This function is minimized as a function of the elements of the matrix A and of w^2ue. The term a_{11} is not treated as a variable, however, and maintains the initial value it is given at the outset. This is done to prevent the matrix A from degenerating to the zero matrix during the minimization procedure. The center of the coordinate system therefore must be chosen to coincide with a point within or on the boundary where it is known that the solution function will have a nonzero value.

The solution for the following problem has been calculated using this program.

Given that the single boundary condition is E(x,y) = 0 along the circular boundary R = 0.5 meters. Find the smallest eigenvalue corresponding to this boundary condition and the corresponding eigenfunction.

For this problem the origin for the solution function was chosen to be at the center of the circular boundary, since the function will not be zero there.

The computer program was arranged to hold the initial approximation of the eigenvalue constant until the eigenfunction had been fitted relatively closely to the proper solution as indicated by a sufficiently small total residual function.

The initial conditions chosen were:

$$w_o^2 ue = 20$$

and

$$A_o = \begin{bmatrix} +0100 & +0000 & -0495 & +0000 & +0410 & +0000 & -0132 & +0000 & +0023 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ -0495 & +0000 & +2400 & +0000 & -2000 & +0000 & +0640 & +0000 & -0110 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ +0410 & +0000 & -2000 & +0000 & +1680 & +0000 & -0530 & +0000 & +0090 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ -0132 & +0000 & +0640 & +0000 & -0530 & +0000 & +0169 & +0000 & -0030 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ +0023 & +0000 & -0110 & +0000 & +0090 & +0000 & -0030 & +0000 & +0005 \end{bmatrix}$$

where the initial A_o was chosen to be the truncated series solution for the square boundary within which the circular boundary in this problem may be considered inscribed.

With the above as initial conditions, the eigenvalue and eigenfunction were calculated to be

$$w^2 ue = 23.183$$

and

$$A = \begin{bmatrix} +0100 & +0000 & -0595 & +0000 & +0709 & +0000 & +0368 & +0000 & -0027 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ -0595 & +0000 & +2700 & +0000 & -1900 & +0000 & -1910 & +0000 & +2640 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ +0709 & +0000 & -1900 & +0000 & +1880 & +0000 & -0181 & +0000 & +0742 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ +0368 & +0000 & -1910 & +0000 & -0181 & +0000 & +0170 & +0000 & -0038 \\ +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 & +0000 \\ -0027 & +0000 & +2640 & +0000 & +0742 & +0000 & -0038 & +0000 & +0000 \end{bmatrix}$$

The total residual function was reduced so that the a_{ij}'s above are correct to about the third decimal place. At this time the residual function numerically determined to approximate Equation (5) by replacing the integral by unweighted sums was

$$R_T = 0.10004.$$

The correct eigenvalue obtained from the analytic solution of the wave equation in cylindrical coordinates is

$$w^2_{ue} = 23.150$$

which is in close agreement with the value determined for this problem.

VIII. DISCUSSION

The methods developed here have been applied exclusively for solutions of the wave equation. These methods can also be used on any other partial differential equation where a solution is known to exist and adequate approximating functions are known. The same applies to ordinary differential equations where the development in Part III obviates the need for integrating throughout the intervals of solution. This should be most valuable in ordinary differential equations with two-point boundary conditions.

For the present, application of these methods is somewhat limited by computational speeds available. For the example worked herein, a total calculation time of about 20 hours was necessary in a machine with an add time of 100 microseconds. Thus, it seems for the wave equation, at least, only rough approximations could be obtained if three independent variables were included. The major factor in time-consumption is the inadequacy of the numerical methods used in finding minima of functions of several variables. Two numerical minimization methods were used on the example herein with no significant differences in the calculation times or the results. The methods were that of "steepest descent" and minimization with respect to one variable at a time.

IX. REFERENCES CITED

1. Slater, J. C. Microwave Electronics, pp. 80–83. D. Van Nostrand Co., 1950.

2. Schelkunoff, S. A. Applied Mathematics for Engineers and Scientists, p. 298. D. Van Nostrand Co., 1948.

3. Booth, A. D. Numerical Methods, pp. 110–37. Butterworth Scientific Publications, 1955.

4. Milne, W. E., et al. Wright Air Development Center Technical Report 57-556. Mathematics for Digital Computers, Volume I, Multivariate Interpolation, February, 1958.

5. Bernstein, D. L. Existence Theorems in Partial Differential Equations. Annals of Mathematics Studies No. 23. Princeton University Press, 1950.

6. Computer Laboratory. Library Program G1-M, Wave Equation Solution. Michigan State University Computer Laboratory, 1959.