AUTOMORPHISMS OF FINITE p-GROUPS

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ABSTRACT

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By

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In the second chapter certain subgroups of the automorphism group of a finite p-group are investigated with a view to determining upper bounds for the derived length and exponent of the subgroups.

In the third chapter, the full automorphism group of a finite p-group of maximal class is investigated and bounds for the derived length and order of the group are obtained.

Many of the subgroups of the automorphism group that have been studied are associated in some way with subgroups of G. In chapter II of this thesis, we first investigate those subgroups of the automorphism group of G inducing the identity on certain abelian subgroups of G. The final result is the following

Theorem If A is maximal subject to being abelian and of exponent p^t, then $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 \frac{s}{t} + c$ and $\exp\mathrm{Aut}(G|A) \leq p^{(c+1)s-t}$ where $\exp G = p^s$ and $\mathrm{cl}G = c$.

The proof of this result can be divided into 5 distinct steps; each succeeding step is dependent for its proof on the preceding one.

1st Step We obtain information about $\mathrm{Aut}(G|\Omega_t)$ for regular p-groups, p odd and for 2-groups of class not greater than 2.

2nd Step We investigate $\mathrm{Aut}(G|A)$ where A is maximal subject to being normal, abelian and of exponent p^t. Again we assume G is regular if p is odd and of class at most 2 if G is a 2-group.

3rd Step $\mathrm{Aut}(G|\Omega_t)$ is investigated, no restriction on the group G.

4th Step We investigate $\mathrm{Aut}(G|A)$, A still maximal subject to being normal, abelian and of exponent p^t.

5th Step We no longer require A to be normal and obtain the quoted theorem.

Finally, as a generalization of the preceding, we obtain information about those automorphism subgroups inducing power automorphisms on certain abelian subgroups of G.

In chapter III we prove the following

Theorem If G is a p-group of maximal class, $|G| = p^m$, $m > 3$, then

(i) $\lambda[\mathrm{Aut}(G)] \leq \log_2 8(m-1)$

(ii) $\mathrm{Aut}(G) = P.C$ where P is the p-sylow subgroup of $\mathrm{Aut}(G)$. Furthermore

$$p^m \leq |P| \leq p^{2m-3} \quad \text{and} \quad |C| \,|\, (p-1)^2.$$

Intermediate steps in proving this theorem are as follows:

1st Step $\mathrm{Aut}(G;\Gamma_2) \leq \mathrm{Aut}(G > \Gamma_2 > \ldots > \Gamma_{m-1} > 1)$ where Γ_i is the i^{th} term of the lower central series of G, G any nilpotent group.

2nd Step If E is an elementary abelian group of order p^2, generated by P and Q, then the group A of automorphisms given by

$$P \to P^i \qquad i = 1, \ldots, (p-1)$$

$$Q \to QP^{\ell} \qquad \ell = 0, \ldots, (p-1)$$

is of order $p(p-1)$ and of derived length 2.

3rd Step The quoted theorem follows easily from steps 1 and 2.

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To Sandy, Helen, Rhean and Jonathan

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INTRODUCTION

The automorphisms of a group may be studied from two distinct points of view. On the one hand we may postulate the existence of an automorphism or a group of automorphisms having certain properties and we may ask if we can determine to some extent the structure of a group G possessing such automorphisms. On the other hand, we may impose restrictions on the group G and we may then investigate the structure of $Aut(G)$ or of some of its subgroups. Knowledge of the structure of $Aut(G)$ may then be used to obtain further information of the structure of G.

It is from this latter point of view that we shall investigate certain subgroups of the automorphism group of a finite p-group in Chapter II. In Chapter III, information about the full automorphism group of a p-group of maximal class will be obtained.

Most of the subgroups of $Aut(G)$ that have been studied are those related in some way to a subgroup or chain of subgroups in G. Thus, for example, P. Hall has proved that the group of automorphisms stabilizing a chain of subgroups in G, ending in the identity, is nilpotent. Another example is a result of N. Blackburn which states that the automorphisms of a finite p-group, fixing elementwise a subgroup which is maximal subject to being abelian and of exponent p^t, is a p-group.

In this investigation, we shall be primarily concerned with those subgroups of $Aut(G)$ inducing the identity, or more generally,

inducing a power automorphism on certain subgroups of G. We shall obtain an upper bound for the derived length and for the exponent of the automorphism groups under investigation. In some cases, we shall even be able to obtain bounds for the order of the automorphism group.

CHAPTER I

Preliminary Results

In this chapter we introduce some of the concepts to be used in subsequent chapters. We shall also prove a number of theorems which form part of the folk-lore of the subject of automorphisms of groups.

Section 1

Definition I.1.1 Let G be a group and let x_i be elements of G. We write the expression $x_1^{-1}x_2^{-1}x_1x_2$ as $[x_1,x_2]$ and call it a simple commutator of weight 2. Define a simple commutator of weight n inductively by

$$[x_1,\ldots,x_{n-1},x_n] = [[x_1,\ldots,x_{n-1}],x_n].$$

Let $\Gamma_k(G) = \langle [x_1,\ldots,x_k] \mid x_i \in G \rangle$, $k \geq 2$. The series $G \geq \Gamma_2(G) \geq \ldots \geq \Gamma_k(G) \geq \ldots$ is called the lower central series of G. Clearly each Γ_i is fully invariant and it may be proved that $[\Gamma_i(G),G] = \langle [x\ y] \mid x \in \Gamma_i(G) \text{ and } y \in G \rangle = \Gamma_{i+1}(G)$ for $i = 1,\ldots$ provided $\Gamma_1(G)$ is interpreted as G.

Definition I.1.2 If $\Gamma_{c+1}(G) = 1$ and $\Gamma_c(G) \neq 1$, G is said to be nilpotent of class c.

Theorem I.1.1 (P. Hall)[12] If G is a nilpotent group, $[\Gamma_i,\Gamma_j] \leq \Gamma_{i+j}$ for $i,j = 2,3,\ldots$.

The following commutator relations will be used:

Lemma I.1.2 If $a,b,c \in G$, G any group, then

(i) $[a,bc] = [a,c][a,b][a,b,c]$

(ii) $[ab,c] = [a,c][a,c,b][b,c]$.

Proofs of these relations will be found in M. Hall [11].

The following theorem lists the elementary results we shall need about p-groups.

Theorem I.1.3 Let G be a finite p-group. Then

(i) G is nilpotent

(ii) If $H \lneqq G$, $N_G(H) \gneqq H$.

(iii) $\Phi(G)$, the Frattini subgroup of G is the intersection of all maximal subgroup of G. Φ is characterized by the property that it is the smallest subgroup H for which G/H is elementary abelian.

As a general reference for the theory of finite p-groups we give P. Hall's paper [12].

Section 2 In this section we list and prove some of the simpler theorems on automorphisms.

Notation (i) $\mathrm{Aut}(G)$ denotes the full group of automorphisms of G.

(ii) $\exp G$ is the least common multiple of the orders of the elements of G.

Definition I.1.3 Let $H \leqq G$. Then if $\alpha \in \mathrm{Aut}(G)$ and $x^{-1}x^{\alpha} \in H$ for all $x \in G$, α is said to be an H-automorphism. The set of all such automorphisms is denoted by $\mathrm{Aut}(G;H)$. This set is actually a group, and when $H \lhd G$, it is just the group of automorphisms of G inducing the identity automorphism on G/H. We sometimes say that α stabilizes G/H.

(b) Let $G = G_0 > G_1 > \ldots > G_m$ be a chain of subgroups. Then the set of automorphisms α having the property that for all

$x \in G_i$, $(i = 0,\ldots,m-1)$, $x^{-1}x^{\alpha} \in G_{i+1}$, will be denoted by $\mathrm{Aut}(G > G_1 > \ldots > G_m)$. This set again forms a group and we shall call this group the group of automorphisms stabilizing the chain $G > G_1 > \ldots > G_m$.

(c) Let $H \leqq G$. Then $\mathrm{Aut}(G|H)$ will denote the group of automorphisms fixing H elementwise. We note that if $G_m = 1$ in (b) above, $\mathrm{Aut}(G > G_1 > \ldots > G_{m-1} > 1) \leqq \mathrm{Aut}(G|G_{m-1})$.

Lemma I.2.1 Let G be a finite p-group and let $G = G_0 > G_1 > \ldots > G_{m-1} > 1$ be a chain of subgroups. Then $\mathrm{Aut}(G > G_1 > \ldots > G_{m-1} > 1)$ is a p-group.

Proof By induction on the length of the chain. If $m = 2$, and $\alpha \in \mathrm{Aut}(G > G_1 > 1)$, then $x^{\alpha} = xg_1, g_1 \in G_1$. Therefore $x^{\alpha^n} = xg_1^n$. Hence, if $n = \exp G$, $x^{\alpha^n} = x$ for all $x \in G$. We note that we have incidentally shown that $\exp\mathrm{Aut}(G > G_1 > 1) \leqq \exp G_1$. Assume inductively that the result is true if the length of the chain is less than m and let $\alpha \in \mathrm{Aut}(G > G_1 > \ldots > G_{m-1} > 1)$. Thus the restriction of α to G_1 is in $\mathrm{Aut}(G_1 > \ldots > 1)$ and by induction α^{p^k} fixes G_1 elementwise for some k. Therefore $\alpha^{p^k} \in \mathrm{Aut}(G > G_1 > 1)$ and by what we have shown above, α is of p-power order. By induction, we are done.

Lemma I.2.2 Let G be a p-group, $H \lhd G$ and suppose $C_G(H) \leqq H$. Then $\mathrm{Aut}(G|H) = \mathrm{Aut}(G > H > 1)$ and is an abelian p-group of exponent not greater than $\exp H$.

Proof Let $g \in G$ and $h \in H$. Then since $H \lhd G$, $[g,h^{-1}] \in H$. Let $\alpha \in \mathrm{Aut}(G|H)$. Then $[g,h^{-1}]^{\alpha} = [g,h^{-1}] = [g^{\alpha},h^{-1}]$. Therefore $g^{-1}h\,g\,h^{-1} = g^{-\alpha}h\,g^{\alpha}h^{-1}$. Hence $g^{\alpha}g^{-1}h = hg^{\alpha}g^{-1}$ for all $h \in H$. Thus $g^{\alpha}g^{-1} \in C_G(H) \leqq H$ for all $g \in G$ and the result follows.

By the previous lemma $\mathrm{Aut}(G|H)$ is a p-group. That it is abelian is proved as follows:

Let $\alpha, \beta \in \mathrm{Aut}(G|H)$.

Then $x^{\alpha\beta} = (xh_\alpha)^\beta = (xh_\beta)h_\alpha = xh_\alpha h_\beta = x^{\beta\alpha}$ since, from what we proved above, h_α and $h_\beta \in Z(H)$.

We shall have occasion to use a generalization of the above.

Theorem I.2.3 Let $G > G > \ldots > G = 1$ be a series of normal subgroups. Then $\mathrm{cl}\mathrm{Aut}(G > G > \ldots > 1) \leq m-1$. A proof of this will be found in [13].

CHAPTER II

Section 1 In this section we prove some theorems about certain subgroups of the automorphism group of two specific classes of p-groups. In the case p is odd, the results will be proved for regular p-groups and in the case p even for groups of nilpotent class at most 2.

When p is odd, the regular groups form a more comprehensive class of groups than that formed by the groups of nilpotent class at most 2. When $p = 2$, however, the latter class is bigger since regular 2-groups are abelian [14]. Later on, when we extend these results to arbitrary p-groups, we shall need the theorems to hold for groups of nilpotent class at most 2. Thus in order to obtain the maximum degree of generality, we are led to consider the 2 cases separately.

Definition II.1.1 A p-group is said to be regular if, given any positive integer n and any pair of elements a and b of G, it is always possible to find elements $c_1,\ldots,c_r$ all belonging to $\langle a,b\rangle'$ and satisfying the equation $(ab)^{p^n} = a^{p^n} b^{p^n} c_1^{p^n} \ldots c_r^{p^n}$.

P. Hall [12] proves the following

Theorem II.1.1 If a and b are any two elements of the regular p-group G and if $a^{p^n} = b^{p^n}$, then $(ab^{-1})^{p^n} = 1$, and conversely.

In the same paper, P. Hall proves that the order of the product of two elements of a regular p-group is no greater than the maximum of the orders of the factors. This means that $\Omega_t = \langle \Omega_t \rangle$ in a regular p-group.

It is on theorem 1.1 that the basic lemma for the case p odd will depend.

For the case $p = 2$, we shall depend on the following

Lemma II.1.2 [11] If G is a group of class not greater than 2, then for all $a,b \in G$ and for all integers n,

$$(ab)^n = a^n b^n [b,a]^{\binom{n}{2}} = a^n b^n [b^{\binom{n}{2}},a] = a^n b^n [b,a^{\binom{n}{2}}].$$

The following lemma is the cornerstone of many of the proofs:

Lemma II.1.3 (a) Let G be a regular p-group (p odd) and let $\exp G = p^s$. Then $\mathrm{Aut}(G|\Omega_t)$ stabilizes G/Ω_{s-t}

(b) Let G be a 2-group generated by $x_1,\ldots,x_n$ where $|x_i| \leq 2^s$ for all i. Let G be of class at most 2. Then

$$\mathrm{Aut}(G|\Omega_t) \leq \mathrm{Aut}(G;\langle \Omega_{s-t+1} \rangle)$$

Proof (a) Let $\alpha \in \mathrm{Aut}(G|\Omega_t)$ and let $x \in G$. Then $|x^{p^{s-t}}| \leq p^t$. Therefore $x^{p^{s-t}} \in \Omega_t$ and thus $(x^{p^{s-t}})^\alpha = x^{p^{s-t}}$. Therefore $(x^\alpha)^{p^{s-t}} = x^{p^{s-t}}$. By theorem II.1.1, $(x^\alpha x^{-1})^{p^{s-t}} = 1$ and hence $x^\alpha x^{-1} \in \Omega_{s-t}$.

(b) Let $\alpha \in \mathrm{Aut}(G|\Omega_t)$ and let $x_i \in \{x_1,\ldots,x_n\}$. Then $|x_i^{2^{s-t+1}}| \leq 2^{t-1}$ and $x_i^{2^{s-t+1}} \in \Omega_t$ as above. Therefore

$$(x_i^{-1}x_i^\alpha)^{2^{s-t+1}} = x_i^{-2^{s-t+1}} x_i^{2^{s-t+1}} [x_i^\alpha, x_i^{-1}]^{\binom{2^{s-t+1}}{2}}$$

$$= [x_i^{2^{s-t}}, x_i^{-1}]^{(2^{s-t+1}-1)} = 1.$$

Therefore $x_i^{-1}x_i^{\alpha} \in \Omega_{s-t+1}$.

Now let $x \in G$. Then $x = \prod_{i=1}^{r} y_i$ where $y_i \in \{x_1,\ldots,x_n\}$. When $r = 1$, $x^{-1}x^{\alpha} \in \Omega_{s-t+1} \leq \langle\Omega_{s-t+1}\rangle$ as we have just seen.

Assume inductively that when x is the product of at most r factors, $x^{-1}x^{\alpha} \in \langle\Omega_{s-t+1}\rangle$. Suppose now $x = \prod_{i=1}^{r+1} y_i$

Then $x^{-1}x^{\alpha} = x_{r+1}^{-1}y_k^{-1}\ldots y_1^{-1}y_1^{\alpha}y_2^{\alpha}\ldots y_r^{\alpha}y_{r+1}^{\alpha}$

$$= y_{r+1}^{-1}y^{-1}y^{\alpha}y_{r+1}^{\alpha} \quad \text{where} \quad y = \prod_{i=1}^{r} y_i$$

By induction, $y^{-1}y^{\alpha} \in \langle\Omega_{s-t+1}\rangle$. Therefore

$$x^{-1}x^{\alpha} = y_{r+1}^{-1}\,y_{r+1}^{\alpha}\,(y_{r+1}^{-\alpha}\,y^{-1}y^{\alpha}y_{r+1}^{\alpha}) \quad \text{which is in} \quad \langle\Omega_{s-t+1}\rangle.$$

Remark (b) cannot be improved to give the same result as in (a) as can be seen by considering the quaternion group. In this case Ω_1 is the center of the group and $\exp G = 2^2$. But $\mathrm{Aut}(G|\Omega_1) = \mathrm{Aut}(G) \simeq S_4$, the symmetric group on four letters. However, $\mathrm{Aut}(G|\Omega_1)$ cannot stabilize $G/\Omega_{2-1} = G/\Omega_1$, for if it did, $\mathrm{Aut}(G)$ would stabilize $G > \Omega_1 > 1$ and hence would be abelian by lemma 2.2 in chapter I.

Theorem II.1.4 (a) Let G be a regular p-group (p odd) and let $\exp G = p^s$. Then

$$\mathrm{Aut}(G|\Omega_t) \leq \mathrm{Aut}(G > \Omega_{s-t} > \ldots > \Omega_{s-kt} > 1) \quad \text{where} \quad k = \left\{\frac{s-t}{t}\right\}$$

and $\{x\}$ denotes the first integer greater than or equal to x. Hence, by the theorem of P. Hall I.2.3, $\mathrm{cl}[\mathrm{Aut}(G|\Omega_t)] \leq \left\{\frac{s-t}{t}\right\}$. Furthermore $\mathrm{Aut}(G|\Omega_t)$ is a p-group.

(b) Let G be a 2-group of class at most 2. Let $G = \langle x_1,\ldots,x_n\rangle$ where $|x_i| \leq 2^s$. Then if $t > 1$,

$$\mathrm{Aut}(G|\Omega_t) \leq \mathrm{Aut}(G > \langle\Omega_{s-t+1}\rangle > \ldots > \langle\Omega_{s-k(t-1)}\rangle > 1)$$

where $k = \{\frac{s-t}{t-1}\}$. Again, as in (a)

$$cl[Aut(G|\Omega_t)] \leq \{\frac{s-t}{t-1}\} \quad \text{and} \quad Aut(G|\Omega_t) \quad \text{is a 2-group.}$$

Proof (a) We have shown in lemma II.1.3(a) that if $\alpha \in Aut(G|\Omega_t)$, α stabilizes G/Ω_{s-t}. By restricting α to Ω_{s-rt} where r is any positive integer, we see that α must stabilize $\frac{\Omega_{s-rt}}{\Omega_{s-(r+1)t}}$.

Now let k be the first integer for which $\Omega_{s-kt} \leq \Omega_t$. Then α stabilizes $G > \Omega_{s-t} > \ldots > \Omega_{s-kt} > 1$. But k is the first integer for which $s-kt \leq t$ or, solving the inequality for k, the first integer for which $k \geq \frac{s-t}{t}$. Therefore $k = \{\frac{s-t}{t}\}$

(b) Since $\langle\Omega_{s-r(t-1)}\rangle$ is generated by those elements of G of order not greater than $2^{s-r(t-1)}$, lemma II.1.3(b) is applicable and shows that $Aut(G|\Omega_t)$ stabilizes

$\langle\Omega_{s-r(t-1)}\rangle/\langle\Omega_{s-(r+1)(t-1)}\rangle$

Therefore $Aut(G|\Omega_t) \leq Aut(G > \langle\Omega_{s-(t-1)}\rangle > \ldots > \langle\Omega_{s-k(t-1)}\rangle > 1)$ where this time $k = \{\frac{s-t}{t-1}\}$.

Remark Again the quaternion group shows that a similar theorem for $t = 1$ cannot be proved.

Definition II.1.2 Let G be a group and let $G^{(0)} = G$. Define inductively $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$. If $G^{(n)} = 1$ and $G^{(n-1)} \neq 1$, G is said to be solvable of derived length n. Symbolically, $\lambda(G) = n$. $G^{(i)}$ is called the i^{th} derived group of G.

P. Hall [12] shows that if the class of G is less than k, the derived length of G is no greater than $\log_2 k$. Hence we have the following

Corollary II.1.5 (a) If G is a regular p-group (p odd) $\exp G = p^s$, then $\lambda[\mathrm{Aut}(G|\Omega_t)] \leq \log_2 s/t$.

(b) If G is a 2-group of class at most 2 and G is generated by elements of order at most 2^s and $t \geq 2$, then $\lambda[\mathrm{Aut}(G|\Omega_t)] \leq \log_2 s-1/t-1$.

Proof (a) $\mathrm{cl}\,\mathrm{Aut}(G|\Omega_t) < \frac{s-t}{t} + 1$. Therefore $\lambda[\mathrm{Aut}(G|\Omega_t)] \leq \log_2 s/t$

(b) $\mathrm{cl}\,\mathrm{Aut}(G|\Omega_t) < \frac{s-t}{t-1} + 1$. Therefore $\lambda[\mathrm{Aut}(G|\Omega_t)] \leq \log_2 s-1/t-1$.

Corollary II.1.5 with a slightly coarser bound can be proved by using lemma II.1.3 and we shall give the proof in the following. The method of prood has the merit of making the structure of $\mathrm{Aut}(G|\Omega_t)$ more transparent.

Theorem II.1.6 (a) Let G be a regular p-group, (p odd), $\exp G = p^s$. Then $\mathrm{Aut}(G|\Omega_t) \geq \mathrm{Aut}(G|\Omega_{2t}) \geq \ldots \geq \mathrm{Aut}(G|\Omega_{2^k t}) \geq 1$ is a normal series with abelian factor groups. k is the first integer for which $2^k t \geq \frac{s}{2}$. Thus $\lambda[\mathrm{Aut}(G|\Omega_t)] \leq \{\log_2 s/t\}$

(b) Let G be a 2-group of class not greater than two. Then if $t > 1$,

$$\mathrm{Aut}(G|\langle\Omega_t\rangle) \geq \mathrm{Aut}(G|\langle\Omega_{2t-1}\rangle) \geq \ldots \geq \mathrm{Aut}(G|\langle\Omega_{2^k t - 2^k + 1}\rangle) \geq 1$$

is a normal series with abelian factor groups. k is the first integer for which $2^k t - 2^k + 1 \geq \frac{s+1}{2}$. Thus $\lambda[\mathrm{Aut}(G|\langle\Omega_t\rangle)] \leq \{\log_2 s-1/t-1\}$.

Proof (a) Consider μ: $\mathrm{Aut}(G|\Omega_{2^r t}) \to \mathrm{Aut}(\Omega_{2^{r+1} t})$ where μ is the restriction mapping. Then $\ker \mu = \mathrm{Aut}(G|\Omega_{2^{r+1} t})$ and $\mathrm{Im}\,\mu \leq \mathrm{Aut}(\Omega_{2^{r+1} t}|\Omega_{2^r t})$. But $\mathrm{Aut}(\Omega_{2^{r+1} t}|\Omega_{2^r t})$ stabilizes

$\Omega_{2^{r+1}t} > \Omega_{2^r t} > 1$.

Therefore $\mathrm{Aut}(G|\Omega_{2^r t})/\mathrm{Aut}(G|\Omega_{2^{r+1}t})$ is abelian and (a) is proved.

(b) is proved similarly, except that the restriction map must be taken from $\mathrm{Aut}(G|\langle\Omega_{2^r t-2^r+1}\rangle)$ to $\mathrm{Aut}(\langle\Omega_{2^{r+1}t-2^{r+1}+1}\rangle)$

Theorem II.1.7 (a) Let G be a regular p-group (p odd), $\exp G = p^s$. Then $\exp\mathrm{Aut}(G|\Omega_t) \leq p^{s-t}$.

(b) Let G be a 2-group of class at most 2, $G = \langle x_1,\ldots,x_n\rangle$ where $|x_i| \leq 2^s$. Then $\exp\mathrm{Aut}(G|\Omega_t) \leq 2^{3(s-t)}$ for $t > 1$.

Proof (a) We shall show that if $\alpha \in \mathrm{Aut}(G|\Omega_t)$, $\alpha^{p^k} \in \mathrm{Aut}(G|\Omega_{t+k})$ for all $k \geq 1$.

Let μ: $\mathrm{Aut}(G|\Omega_{t+k-1}) \to \mathrm{Aut}(\Omega_{t+k})$ be the restriction mapping. Then $\ker\mu \cong \mathrm{Aut}(G|\Omega_{t+k})$ and $\mathrm{Im}\,\mu \leq \mathrm{Aut}(\Omega_{t+k}|\Omega_{t+k-1})$. But by lemma II.1.3(a), $\mathrm{Aut}(\Omega_{t+k}|\Omega_{t+k-1})$ stabilizes $\Omega_{t+k}/\Omega_{t+k-(t+k-1)} = \Omega_{t+k}/\Omega_1$. But $\Omega_1 \leq \Omega_{t+k-1}$ for $k \geq 1$. Therefore $\mathrm{Aut}(\Omega_{t+k}|\Omega_{t+k-1})$ stabilizes $\Omega_{t+k} > \Omega_1 > 1$ and $\mathrm{Aut}(G|\Omega_{t+k-1})/\mathrm{Aut}(G|\Omega_{t+k})$ is elementary abelian. Hence if $\alpha \in \mathrm{Aut}(G|\Omega_t)$, $\alpha^{p^k} \in \mathrm{Aut}(G|\Omega_{t+k})$ and $\exp\mathrm{Aut}(G|\Omega_t) \leq p^{s-t}$

(b) (i) We show first that $\exp\langle\Omega_t\rangle \leq 2^{t+1}$. Let $x \in \langle\Omega_t\rangle$. Then $x = \prod_{i=1}^{r} x_i$, $|x_i| \leq 2^t$. If $r = 1$, $|x| \leq 2^t$. Assume inductively that if x is the product of at most r x_i's, $|x| \leq 2^{t+1}$. Let $x = \prod_{i=1}^{r+1} x_i$

Then
$$\begin{aligned} x^{2^{t+1}} &= (x_1,\ldots,x_r,x_{r+1})^{2^{t+1}} \\ &= (x_1,\ldots,x_r)^{2^{t+1}}\, x_{r+1}^{2^{t+1}}\, [x_{r+1}, x_1,\ldots,x_r]^{\binom{2^{t+1}}{2}} \\ &= [x_{r+1}^{2^t}, x_1,\ldots,x_r]^{(2^{t+1}-1)} \quad \text{by induction} \\ &= 1 \end{aligned}$$

(ii) Let μ: $\mathrm{Aut}(G|\Omega_t) \to \mathrm{Aut}(\langle\Omega_{t+1}\rangle)$ where μ is the restriction. $\ker\mu = \mathrm{Aut}(G|\langle\Omega_{t+1}\rangle)$ and $\mathrm{Im}\,\mu \leqq \mathrm{Aut}(\langle\Omega_{t+1}\rangle|\langle\Omega_t\rangle)$. By lemma II.1.3(b) $\mathrm{Aut}(\langle\Omega_{t+1}\rangle|\langle\Omega_t\rangle)$ stabilizes $\langle\Omega_{t+1}\rangle/\langle\Omega_2\rangle$. Since $t \geqq 2$, $\langle\Omega_2\rangle \leqq \langle\Omega_t\rangle$. Hence $\mathrm{Aut}(\langle\Omega_{t+1}\rangle|\langle\Omega_t\rangle)$ stabilizes $\langle\Omega_{t+1}\rangle > \langle\Omega_2\rangle > 1$. But $\exp\langle\Omega_2\rangle \leqq 2^3$ by (i) and hence $\exp\left[\frac{\mathrm{Aut}(G|\langle\Omega_t\rangle)}{\mathrm{Aut}(G|\langle\Omega_{t+1}\rangle)}\right] \leqq 2^3$. A simple induction then shows that $\exp\mathrm{Aut}(G|\langle\Omega_t\rangle) \leqq 2^{3(s-t)}$.

We next ask: Under what circumstances does $\mathrm{Aut}(G|\Omega_t)$ strictly contain $\mathrm{Aut}(G|\Omega_{t+1})$? We can only answer this when t is "big enough" for purely non-abelian groups.

Definition II.1.3 A non-abelian p-group G is said to be purely non-abelian if it has no abelian direct factor.

Theorem II.1.8 (a) Let G be a regular purely non-abelian p-group (p odd), $\exp G = p^s$. Let $\exp G' = p^n$ and $\exp Z(G) = p^z$ where $n < s$. Then $\mathrm{Aut}(G|\Omega_{t+1}) \nleqq \mathrm{Aut}(G|\Omega_t)$ if $t \geqq \max(n, s-z)$ and $t < s$.

(b) Let G be a purely non-abelian 2-group, $\mathrm{cl}G \leq 2$ and $G = \langle x_1, \ldots, x_m\rangle$, $|x_i| \leqq 2^s$. Let $\exp G' \leqq 2^n$ and let $\exp Z(G) = p^z$. Then $\mathrm{Aut}(G|\langle\Omega_{t+1}\rangle) \nleqq \mathrm{Aut}(G|\langle\Omega_t\rangle)$ if $t \geqq \max(n, s-z)$ and $\langle\Omega_t\rangle \neq G$.

Proof (a) $\mathrm{Exp}G/\Omega_t = p^{s-t} \leqq p^z$. There is therefore a homomorphism μ: $G/\Omega_t \to Z(G)$ which is a monomorphism when restricted to a cyclic factor of G/Ω_t. There is therefore an element $x\Omega_t \in G/\Omega_t$, $|x\Omega_t| = p$, such that $\mu(x\Omega_t) \neq 1$.

Let β: $G \to G/\Omega_t$ be the natural homomorphism and let $\rho = \mu\beta$. Then ρ is a homomorphism from G to $Z(G)$ which annihilates Ω_t but does not annihilate Ω_{t+1}.

Consider the mapping α: $G \to G$ given by α: $x \to x\rho(x)$. This is clearly always a homomorphism. Moreover, it has been shown [1] that when G is purely non-abelian, α is an automorphism. Clearly $\alpha \in \mathrm{Aut}(G|\Omega_t)$ but $\alpha \notin \mathrm{Aut}(G|\Omega_{t+1})$.

(b) The proof is similar to that given in (a) except that in this case $\exp\langle\Omega_t\rangle \leq 2^{t+1}$ and the element x whose order $\mathrm{mod}\langle\Omega_t\rangle$ is 2 and which is not in the kernel of μ: $G/\langle\Omega_t\rangle \to Z(G)$, has order not greater than 2^{t+2}. (We cannot say it has order $\leq 2^{t+1}$.)

Corollary II.1.9 (a) If G is a purely non-abelian regular p-group (p odd) such that $\exp G' < \exp G$ and $\Omega_t \neq G$, then $\mathrm{Aut}(G|\Omega_t) \neq 1$.

(b) If G is a purely non-abelian 2-group of class 2 such that $\exp G' < \exp G$ and $\langle\Omega_t\rangle \neq G$, then $\mathrm{Aut}(G|\langle\Omega_t\rangle) \neq 1$.

Proof Apply preceding theorem.

Corollary II.1.10 Let G be a regular purely non-abelian p-group (p odd) with $\exp Z(G) \geq p^{s-1}$ and $\exp G' = p$. Then $\mathrm{Aut}(G|\Omega_t) \ngeq \mathrm{Aut}(G|\Omega_{t+1})$ for all $s > t \geq 1$.

Proof $\max(n, s-z) = 1$. Hence result.

Finally we obtain a rough lower bound for $|\mathrm{Aut}(G|\langle\Omega_t\rangle)|$ when t is big enough. We first prove the following

Lemma II.1.11 Let A be an abelian p-group, $A = C(p^{\alpha_1}) \otimes \ldots \otimes C(p^{\alpha_i})$ and let $B = C(p^k)$ where $k \geq \alpha_i$ for all i. Let $\alpha = \Sigma\alpha_i$. Then there are at least p^{α} distinct homomorphisms of A into B [8].

Proof Clearly there are p^{α_t} homomorphisms of $C(p^{\alpha_i})$ into B. There are therefore at least $\prod_i p^{\alpha_i} = p^{\alpha}$ distinct homomorphisms of A into B.

Theorem II.1.12 (a) Let G be a regular purely non-abelian p-group (p odd), $\exp G = p^s$, $\exp G' = p^n$ and $\exp Z(G) = p^z$. Then if $t \geq \max(n, s-z)$, $|\mathrm{Aut}(G|\Omega_t)| \geq |G: \Omega_t|$

(b) Let G be a purely non-abelian 2-group of class 2, $G = \langle x_1, \ldots, x_m \rangle$, $|x_i| \leq 2^s$, $\exp G' \leq 2^n$. Then if $t \geq \max(n, s-z)$, $|\mathrm{Aut}(G|\langle \Omega_t \rangle)| \geq |G: \langle \Omega_t \rangle|$

Proof (a) $\mathrm{Exp} G/\Omega_t \leq \exp Z(G)$. Thus by the previous lemma there are at least $|G: \Omega_t|$ different homomorphisms of G/Ω_t into $Z(G)$, say $\mu_1, \ldots, \mu_k$ where $k = |G: \Omega_t|$. Then $\alpha_i: x \to x\mu_i\beta(x)$ (where β is the natural homomorphism from G to G/Ω_t) are distinct automorphisms in $\mathrm{Aut}(G|\Omega_t)$.

(b) is proved similarly.

We shall now obtain bounds for the derived length of $\mathrm{Aut}(G|A)$ where A is maximal subject to being normal, abelian and of exponent p^t.

Theorem II.1.13 (a) Let G be a regular p-group, (p odd), $\exp G = p^s$. Let A be maximal subject to being normal abelian and of exponent p^t. Then

$$\lambda[\mathrm{Aut}(G|A)] \leq \log_2 2s/t \quad \text{and}$$

$$\exp \mathrm{Aut}(G|A) \leq p^s$$

(b) Let G be a 2-group of class at most 2. Let $G = \langle x_1, \ldots, x_n \rangle$ where $|x_i| \leq 2^s$ for all i. Then if A is maximal subject to being normal abelian and of exponent p^t, $t > 1$,

$\lambda[\mathrm{Aut}(G|A)] \leqq \log_2 2(s-1/t-1)$ and

$\exp\mathrm{Aut}(G|A) \leqq 2^{4s-3t+1}$

Proof (a) Let μ: $\mathrm{Aut}(G|A) \to \mathrm{Aut}(\Omega_t)$ be the restriction mapping. Then $\ker\mu = \mathrm{Aut}(G|\Omega_t)$, $\mathrm{Im}\mu \leqq \mathrm{Aut}(\Omega_t|A)$. But by Alperin [2], $C_{\Omega_t}(A) \leqq A$, and hence by lemma I.2.2, $\mathrm{Aut}(\Omega_t|A)$ is abelian. By Corollary II.1.5(a), $\lambda[\mathrm{Aut}(G|\Omega_t)] \leqq \log_2 s/t$. Therefore $\lambda[\mathrm{Aut}(G|A)] \leqq \log_2 s/t + 1$.

Furthermore, by theorem II.1.7(a), $\exp\mathrm{Aut}(G|\Omega_t) \leqq p^{s-t}$ and by lemma I.2.2, $\exp\mathrm{Aut}(\Omega_t|A) \leqq p^t$. Therefore $\exp\mathrm{Aut}(G|A) \leqq p^s$.

(b) Let μ: $\mathrm{Aut}(G|A) \to \mathrm{Aut}(C_G(A)|A)$ be the restriction map. Since $C_G(A) \geqq A$, $C_G(C_G(A)) \leqq C_G(A)$. Therefore $\ker\mu = \mathrm{Aut}(G|C_G(A))$ is abelian by lemma I.2.2 and $\mathrm{Im}\mu \leqq \mathrm{Aut}(C_G(A)|A)$. But by a result of Alperin [2], $C_{\Omega_t}(A) \leqq A$. Therefore $\Omega_t[C_G(A)] = A$.

Therefore $\lambda[\mathrm{Im}\mu] \leqq \log_2(s-1/t-1)$ and hence $\lambda[\mathrm{Aut}(G|A)] \leqq \log_2(s-1/t-1) + 1$. Furthermore, since $\mathrm{Im}\mu \leqq \mathrm{Aut}(C_G(A)|A)$, $\exp\mathrm{Im}\mu \leqq 2^{3(s-t)}$ by theorem II.1.7(b) and $\exp\ker\mu \leqq 2^{s+1}$. Therefore $\exp\mathrm{Aut}(G|A) \leqq 2^{4s-3t+1}$.

In a regular p-group (p odd) there is an interesting connection between the exponent of A, (where A is maximal subject to being normal, abelian of exponent p) the exponent of G and the class of $C_G(A)$ as can be seen in the following: $A \leqq Z(C_G(A))$. Thus $C_G(A)/Z(C_G(A)) \cong \mathrm{Aut}(C_G(A)|A)$. But as we saw earlier, $\mathrm{cl}\,\mathrm{Aut}(C_G(A)|A) < s/t$, since $A = \Omega_t(C_G(A))$.

Therefore $\mathrm{cl}\,C_G(A) < s+t/t$. Thus for example, if $t \geqq s/2$, $C_G(A)$ is of class at most 2.

For the next few theorems we shall restrict our considerations to regular p-groups, p odd or even. Of course, when p is even we

are merely dealing with abelian groups.

Similar results cannot be proved by the same methods for 2-groups of class 2 because in general in such groups $\exp(\langle\Omega_t\rangle) \leqq p^{t+1}$ whereas we need $\exp(\langle\Omega_t\rangle) \leqq p^t$ as will be seen.

Notation $\mho_t$ will denote the set of elements which are p^t-th powers of elements of G. P. Hall [12] has shown that in a regular p-group this set actually forms a group.

Theorem II.1.14 If G is a regular p-group, p-odd or even, then $\mathrm{Aut}(G;\Omega_t) = \mathrm{Aut}(G|\mho_t)$.

Proof Let $\alpha \in \mathrm{Aut}(G|\mho_t)$. Then $x^{p^t} = (x^{p^t})^\alpha = (x^\alpha)^{p^t}$ for all $x \in G$. Therefore, by theorem II.1.1, $(x^{-1}x^\alpha)^{p^t} = 1$. Hence $\alpha \in \mathrm{Aut}(G;\Omega_t)$. Conversely, suppose $\alpha \in \mathrm{Aut}(G;\Omega_t)$. By the converse of the same theorem, since $(x^{-1}x^\alpha)^{p^t} = 1$, $x^{p^t} = (x^\alpha)^{p^t} = (x^{p^t})^\alpha$ and hence $\alpha \in \mathrm{Aut}(G|\mho_t)$. Therefore $\mathrm{Aut}(G;\Omega_t) = \mathrm{Aut}(G|\mho_t)$.

Corollary II.1.15 Let G be an abelian p-group, p odd or even, which has no direct factor which is cyclic of prime order. Then $\mathrm{Aut}(G;\Omega_1) = \mathrm{Aut}(G|\mho_1) = \mathrm{Aut}(G > \Omega_1 > 1)$ i.e. $\mathrm{Aut}(G;\Omega_1)$ is elementary abelian.

Proof Since G does not have a cyclic group of prime order as direct factor, $G = C(p^{\alpha_1}) \otimes \ldots \otimes C(p^{\alpha_n})$ where $\alpha_i > 1$. Therefore $\mho_1 = G^p \approx C(p^{\alpha_1 - 1}) \otimes \ldots \otimes C(p^{\alpha_n - 1})$. Hence $\mho_1 \geq \Omega_1$. By the preceding theorem, $\mathrm{Aut}(G;\Omega_1) = \mathrm{Aut}(G|\mho_1) \leqq \mathrm{Aut}(G|\Omega_1)$. Therefore $\mathrm{Aut}(G;\Omega_1) = \mathrm{Aut}(G > \Omega_1 > 1)$ and the theorem is proved.

Corollary II.1.16 Let G be a p-group with $\Omega_1 \leqq Z(G)$ and such that Z(G) has no direct factor which is a cyclic group of order p. Then $\lambda[\mathrm{Aut}(G;\Omega_1)] \leqq 2$ and $\exp\mathrm{Aut}(G;\Omega_1) \leqq p^2$.

Proof Let $\mu: \mathrm{Aut}(G;\Omega_1) \to \mathrm{Aut}(Z)$ where μ is the restriction map. Then $\mathrm{Im}\mu \leqq \mathrm{Aut}(Z;\Omega_1(Z))$ and by corollary II.1.16, the latter group is elementary abelian. Furthermore,

$\ker\mu = \mathrm{Aut}(G;\Omega_1) \cap \mathrm{Aut}(G|Z) \leqq \mathrm{Aut}(G > \Omega_1 > 1)$ since $\Omega_1 \leqq Z$.

Therefore $\lambda[\mathrm{Aut}(G;\Omega_1)] \leqq 2$ and

$$\exp\mathrm{Aut}(G;\Omega_1) \leqq p^2.$$

Section 2 In this section we extend many of the results of section 1 to arbitrary groups. In general the bounds obtained in this section will be coarser.

The transition from the special classes of groups we have been considering so far is effected by the use of the Feit-Thompson [7] "critical subgroup" of a p-group. This subgroup, which we shall henceforth denote by K, has the following properties:

(i) K is characteristic of class at most 2.

(ii) $C_G(K) \leqq K$ and $K/Z(K)$ is elementary abelian.

(iii) $[G,K] \leqq Z(K)$.

Theorem II.2.1 (a) Let G be a p-group (p odd), $\exp G = p^s$. Then $\lambda[\mathrm{Aut}(G|\Omega_t)] \leqq \log_2 2s/t$ and $\exp\mathrm{Aut}(G|\Omega_t) \leqq p^{2s-t}$.

(b) Let G be a 2-group, $\exp G = 2^s$. Then $\lambda[\mathrm{Aut}(G|\Omega_t)] \leqq \log_2 2(s-1/t-1)$ and $\exp\mathrm{Aut}(G|\Omega_t) \leqq 2^{4s-3t}$ where $t > 1$.

Proof (a) Let K be the Feit-Thompson critical subgroup of G and consider $\mu: \mathrm{Aut}(G|\Omega_t) \to \mathrm{Aut}(K)$ where μ is the restriction map. $\ker\mu = \mathrm{Aut}(G|\langle\Omega_t\rangle\cdot K) = \mathrm{Aut}(G > \langle\Omega_t\rangle\cdot K > 1)$ by lemma I.2.2 since $C_G(\langle\Omega_t\rangle\cdot K) \leqq K$. Therefore $\ker\mu$ is abelian. $\mathrm{Im}\mu \leqq \mathrm{Aut}(K|\Omega_t(K))$ and since K is of class 2 it is regular. Hence corollary II.1.5(a)

is applicable and $\lambda[\mathrm{Im}\mu] \leq \log_2 s/t$. Therefore $\lambda[\mathrm{Aut}(G|\Omega_t)] \leq \log_2 2s/t$.

Furthermore, $\mathrm{Aut}(G|\Omega_t)/\mathrm{Aut}(G > K.\langle\Omega_t> > 1) \cong \mathrm{Aut}(K|\Omega_t(K))$. But $\exp\mathrm{Aut}(K|\Omega_t(K)) \leq p^{s-t}$ by theorem II.1.7(a). By lemma I.2.2 $\exp\mathrm{Aut}(G > K.\langle\Omega_t> > 1) \leq p^s$. Therefore $\exp\mathrm{Aut}(G|\Omega_t) \leq p^{2s-t}$

(b) We have again that

$\mathrm{Aut}(G|\langle\Omega_t>)/\mathrm{Aut}(G > \langle\Omega_t> \cdot K > 1) \cong \mathrm{Aut}(K|\Omega_t(K))$. By theorem II.1.7(b), $\exp\mathrm{Aut}(K|\Omega_t(K)) \leq 2^{3(s-t)}$. Therefore $\exp\mathrm{Aut}(G|\langle\Omega_t>) \leq 2^{4s-3t}$.

Corollary II.2.2 (a) If G is a p-group (p odd) and if $\Omega_t \leq Z(G)$, then $\lambda(G) \leq \log_2 4s/t$

(b) If G is a 2-group and if $\Omega_t \leq Z(G)$, $t > 1$, then $\lambda(G) \leq \log_2 4(s-1)/(t-1)$.

Proof (a) and (b) Since $\Omega_t \leq Z(G)$, $G/Z \subset \mathrm{Aut}(G|\Omega_t)$. We note that if A is maximal subject to being normal, abelian of exponent p^t in G, then by Alperin [2] $\Omega_t[C_G(A)] = A \leq Z(C_G(A))$. We may then, using the corollary above, set a bound on the derived length of $C_G(A)$. In fact $\lambda(C_G(A)) \leq \log_2 4s/t$ in the case p is odd and $\lambda(C_G(A)) \leq \log 4(s-1/t-1)$ if $p = 2$, $t > 1$.

Corollary II.2.3 (a) If G is a p-group (p odd), $\exp G = p^s$ and if A is maximal subject to being normal, abelian of exponent p^t, then $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 4s/t$ and $\exp\mathrm{Aut}(G|A) \leq p^{3s-t}$

(b) If G is a 2-group, $\exp G = 2^s$ and A is as above, then $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 4(s-1/t-1)$ $t > 1$ and $\exp\mathrm{Aut}(G|A) \leq 2^{5s-3t}$ $t > 1$.

Proof (a) Let μ: $\mathrm{Aut}(G|A) \to \mathrm{Aut}(C_G(A))$ where μ is the restriction. $\ker\mu$ is abelian since $C_G(C_G(A)) \leq C_G(A)$ $\mathrm{Im}\mu \leq \mathrm{Aut}(C_G(A)|A)$. But $\Omega_t[C_G(A)] = A$ by Alperin [2]. Thus, by the previous theorem $\lambda[\mathrm{Im}\mu] \leq \log_2 2s/t$. Hence $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 4s/t$.

Furthermore $\mathrm{Aut}(G|A)/\mathrm{Aut}(G > C_G(A) > 1) \subset \mathrm{Aut}(C_G(A)|A)$. Therefore $\exp\mathrm{Aut}(G|A) \leqq p^{3s-t}$

(b) is proved similarly using (b) of the previous theorem.

When A is maximal subject to being normal, abelian of exponent p^t, we may obtain a finer bound on $\mathrm{Aut}(G|A)$ if we know that the exponent of G' is no greater than the exponent of A. We then have the following

Theorem II.2.4 (a) Let G be a p-group of odd order. Let $\exp G = p^s$ and $\exp G' = p^n$ where $p^n < p^s$. Let A be maximal subject to being normal abelian of exponent p^t. Then $\lambda[\mathrm{Aut}(G|A)] \leqq \log_2 2s/t$ if $t \geqq n$.

(b) Let G be a 2-group and A as in (a). Let $\exp G' = 2^n$ where $2^n < 2^s$. Then $\lambda[\mathrm{Aut}(G|A)] \leqq \log_2 2(s-1/t-1)$ if $t \geqq \max(n,2)$.

Proof (a) Let $\mu: \mathrm{Aut}(G|A) \to \mathrm{Aut}(C_G(A)|A)$ where μ is the restriction mapping. By a result of Alperin [2] $\Omega_t(C_G(A)) = A$. On the other hand, $\exp[C_G(A)]' \leqq p^n \leqq p^t$. Hence $[C_G(A)]' \leqq A$. But clearly $A \leqq Z(C_G(A))$ and therefore $C_G(A)$ is of class 2. Thus by corollary II.1.5(a), the derived length of $\mathrm{Aut}(C_G(A)|A)$ is no greater than $\log_2 s/t$. Hence, since $\ker\mu$ is abelian, $\lambda[\mathrm{Aut}(G|A)] \leqq \log_2 2s/t$.

(b) The proof is similar and we appeal to corollary II.1.5(b) to obtain $\lambda[\mathrm{Aut}(C_G(A)|A)] \leqq \log_2(s-1/t-1)$. Thus $\lambda[\mathrm{Aut}(G|A)] \leqq \log_2 2(s-1/t-1)$.

We pause to give two simple examples which will illustrate the foregoing theory.

Example 1 Consider the group generated by P and Q with defining relations $P^9 = 1$, $Q^9 = 1$, $Q^{-1}PQ = P^4$. This is a group of order 81 of class 2. Hence it is regular. A simple calculation shows that $Z(G) = \langle P^3 \rangle \otimes \langle Q^3 \rangle$. It is easy to show that $\Omega_1 = Z(G)$. We may then apply theorem II.1.4(a) to obtain $cl\mathrm{Aut}(G|\Omega_1) \leq 2\text{-}1 = 1$. Furthermore $C_G(\Omega_1) = G$ and we know that $cl[C_G(\Omega_1)] \leq 2$ by one of our results. Since $clG = 2$, we see that our bound is actually attained in this case.

Example 2 Consider the group G generated by P,Q and R with the following relations:

$$P^9 = 1,\ Q^3 = 1,\ R^3 = 1,\ [P,Q] = 1,\ [P,R] = Q \text{ and } [R,Q] = P^3.$$

This is a group of order 81 and each element of group is expressible uniquely in the form $P^iQ^jR^k$.

It is easily shown that $[P^n,R] = [P,R]^n$ for all n and hence $[P^3,R] = [P,R]^3 = 1$. Therefore $\langle P^3 \rangle \leq Z(G)$ and in fact $\langle P^3 \rangle = Z(G)$.

The class of the group is 3 and an easy calculation shows that $(P^\ell Q^m R)^3 = P^{9(m-\ell)}Q^{3(m-\ell)}R^3 = 1$ for all ℓ and m. Thus $PR \cdot R = P$, $|PR| = 3$ and $|R^2| = 3$, but order of P is 9. Hence the group is not regular and we must use the results of section 2.

Consider now the subgroup $\langle P^3,Q \rangle$. This is clearly abelian and normal of exponent 3. Suppose $P^iQ^jR^k \in C_G(\langle P^3,Q \rangle)$. Then $Q^{-1}P^iQ^jR^kQ = P^iQ^jR^k$. Hence $Q^{-1}R^kQ = R^k$. But from the relations, $Q^{-1}RQ = RP^3$. Therefore $Q^{-1}R^kQ = R^kP^{3k} = R^k$. Therefore $P^{3k} = 1$, and $k \equiv 0 \bmod 3$. Thus the only elements in $C_G(\langle P^3,Q \rangle)$ are of the form P^iQ^j and thus $\langle P^3,Q \rangle$ is maximal subject to being normal, abelian of exponent 3.

It may be shown that the mappings $P \to P^i$, $i = 1,4,7$; $Q \to Q$; $R \to P^{\ell}Q^{m}R$ for arbitrary ℓ and m are automorphisms which fix elementwise $\langle P^3,Q\rangle$ and furthermore these are the only such automorphisms. Thus $|\mathrm{Aut}(G|\langle P^3,Q\rangle)| = 81$. This is a non-abelian group of derived length 2. Applying corollary II.2.3(a), since $\exp G = 3^2$, we get $\lambda[\mathrm{Aut}(G|\langle P^3,Q\rangle)] \leq \log_2 8 = 3$. Furthermore, $C_G(\langle P^3,Q\rangle) = \langle P,Q\rangle$ which is abelian. Again the bound obtained from our results for the derived length is 3.

Section 3 In this section we obtain bounds for the derived length and exponent of $\mathrm{Aut}(G|A)$ where A is now maximal subject to being abelian and of exponent p^t. We thus drop the hypothesis that A is normal. The bounds we shall obtain will depend not only on the exponent of A and of G, but also on the class of G. We note that because of Alperin's result [2], if A is maximal subject to being normal, abelian and of exponent p^t, it is maximal subject to being abelian and of exponent p^t. In this section we are therefore extending the results of section 2 to cover a larger class of subgroups.

We start by proving the following:

Lemma II.3.1 Let G be a group and let H be a subgroup of G such that $C_G(H) \leq H$. Define $H_0 = H$ and inductively $H_i = N_G(H_{i-1})$. If $H_k = G$, then $\lambda[\mathrm{Aut}(G|H)] \leq k$. Furthermore if G is a p-group and $\exp H = p^t$, then $\exp[\mathrm{Aut}(G|H)] \leq p^{kt}$.

Proof If $H_1 = G$, i.e. $H \lhd G$, then $\mathrm{Aut}(G|H) = \mathrm{Aut}(G > H > 1)$ by lemma I.2.2, $\mathrm{Aut}(G > H > 1)$ is abelian and $\exp\mathrm{Aut}(G|H) \leq p^t$. Assume inductively the result is true if H_n is the whole group where $n < k$ and suppose G is a group with $H_k = G$. Then by induction

$\lambda[\mathrm{Aut}(H_{k-1}|H)] \leq k-1$ and $\exp\mathrm{Aut}(H_{k-1}|H) \leq p^{(k-1)t}$. Consider $\mu: \mathrm{Aut}(G|H) \to \mathrm{Aut}(H_{k-1})$. Then $\mathrm{Im}\mu \trianglelefteq \mathrm{Aut}(H_{k-1}|H)$ and $\ker\mu = \mathrm{Aut}(G|H_{k-1})$. But $H_{k-1} < G$ and $C_G(H_{k-1}) \leq H \trianglelefteq H_{k-1}$. Therefore by lemma I.2.2 $\mathrm{Aut}(G|H_{k-1})$ is abelian and $\exp\mathrm{Aut}(G|H_{k-1}) \leq p^t$ because $C_G(H_{k-1}) \leq H$.

Therefore $\lambda[\mathrm{Aut}(G|H)] \leq k$ and $\exp\mathrm{Aut}(G|H) \leq p^{kt}$.

Lemma II.3.2 Let $\mathrm{cl}G = c$ and let $H \leq G$. Let H_i be as in the previous lemma. Then $H_i \geq Z_i$, the i^{th} center of G for all i. Thus $H_c = G$.

Proof Clearly $H_1 \geq Z_1$. Assume inductively $H_i \geq Z_i$. Then since $[H_i, Z_{i+1}] \leq Z_i \leq H_i, Z_{i+1} \trianglelefteq H_{i+1}$ and by induction we are done. Since $Z_c = G$, $H_c = G$.

We note that in the above lemma, if $C_G(H) \leq H$, $Z_1 \trianglelefteq H$ and we may start the induction with $H_0 \geq Z_1$. We may therefore show that $H_{c-1} = G$.

Corollary II.3.3 Let G be a p-group and let A be maximal subject to being abelian. Suppose $\mathrm{cl}G = c$. Then $\lambda[\mathrm{Aut}(G|A)] \leq c-1$ and $\exp\mathrm{Aut}(G|A) \leq p^{(c-1)t}$ where $\exp A = p^t$.

Proof Since A is maximal abelian, $C_G(A) = A$ and the preceding two lemmas are applicable.

Theorem II.3.4 (a) Let G be a p-group of odd order. Let $\exp G = p^s$ and let class $G = c$. Let A be maximal subject to being abelian and of exponent p^t. Then $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 s/t + c$ and $\exp\mathrm{Aut}(G|A) \leq p^{(c+1)s-t}$

(b) Let G be a 2-group, $\exp G = p^s$ and class $G = c$. Let A be as in (a) with $t > 1$. Then $\lambda[\mathrm{Aut}(G|A)] \leq \log_2(s-1/t-1) + c$ and $\exp\mathrm{Aut}(G|A) \leq 2^{(3+c)s-3t}$.

Proof (a) Let μ: $\mathrm{Aut}(G|A) \to \mathrm{Aut}(C_G(A)|A)$ be the restriction mapping. Then $\ker\mu = \mathrm{Aut}(G|C_G(A))$ and $\mathrm{Im}\mu \leq \mathrm{Aut}(C_G(A)|A)$. But $\Omega_t(C_G(A)) = A$ (this time we do not have to appeal to Alperin's result as in the case A normal) and hence $\lambda[\mathrm{Aut}(C_G(A)|A)] \leq \log_2 2s/t$ and $\exp\mathrm{Aut}(C_G(A)|A) \leq p^{2s-t}$ by theorem II.2.1(a). By lemmas II.3.1 and II.3.2, $\lambda[\mathrm{Aut}(G|C_G(A)] \leq c-1$ and $\exp\mathrm{Aut}(G|C_G(A)) \leq p^{(c-1)}$. Hence $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 2s/t + c-1$ and $\exp\mathrm{Aut}(G|A) \leq p^{(c+1)s-t}$

(b) is proved similarly except that use of theorem II.2.1(b) is made.

We note that we could have obtained finer bounds for the exponents of the above automorphism groups if we had information about the exponent of $C_G(A)$.

If more is known about the group G, the bounds can be improved. For example, if G is a regular p-group of odd order and A is maximal subject to being abelian of exponent p^t, $\lambda[\mathrm{Aut}(G|A)] \leq \log_2 s/2t + c$. This is only an improvement of 1 over the general case. For the bound of the exponent of $\mathrm{Aut}(G|A)$ we do however obtain $p^{(c-2)t+s}$ which is a substantial improvement. To prove this last result we use a slightly different approach from the proof of theorem II.3.4(a). (In fact the proof given below can be carried through with minor alterations in any group in which $\langle\Omega_t\rangle = \Omega_t$, and not only in regular p-groups.)

We begin by considering μ: $\mathrm{Aut}(G|A) \to \mathrm{Aut}(\Omega_t)$ instead of $C_G(A)$. Thus A is maximal subject to being abelian in Ω_t. Hence $\exp\mathrm{Aut}(\Omega_t|A) \leq p^{(c-1)t}$ by corollary II.3.3 and $\exp\mathrm{Aut}(G|\Omega_t) \leq p^{s-t}$ by theorem II.1.7(a). Thus $\exp\mathrm{Aut}(G|A) \leq p^{(c-2)t+s}$.

We note that theorem II.3.4 proves a theorem by N. Blackburn [4] which states that $\mathrm{Aut}(G|A)$ is a p-group.

Section 4 In the previous sections we have been concerned with those subgroups of the automorphism group which fix elementwise certain subgroups of the group G. In this section we shall extend these results to include those subgroups of the automorphism group of G which do not move around "too much" the elements of certain subgroups of G. More specifically, we shall examine automorphism groups which induce power automorphisms on certain subgroups of G.

Definition II.4.1 Let $\alpha \in \mathrm{Aut}(G)$. Then α is said to be a power automorphism of G if for every $H \trianglelefteq G$, $H^{\alpha} = H$, α is said to induce a power automorphism on a subgroup K of G if α restricted to K is a power automorphism of K.

To illustrate the subsequent theory we shall work out an example in detail.

Notation Let $H \trianglelefteq G$. Then $\mathrm{Aut}(G|_{\pi}H)$ denotes the group of those automorphisms inducing power automorphisms on H.

Example Let G be the group generated by P and Q with relations $P^9 = Q^3 = 1$; $Q^{-1}PQ = P^4$. This is a group of order 27. Since $Q^{-1}P^3Q = (Q^{-1}PQ)^3 = P^{12} = P^3$, $\langle P^3 \rangle = Z(G)$ and $\mathrm{cl}G = 2$. We shall consider those automorphisms of G inducing power automorphisms on $\langle P \rangle$. Let α be any automorphism of G inducing a power automorphism on $\langle P \rangle$. Then α: $P \to P^i$, $i = 1,2,4,5,7,8$ and suppose α: $Q \to P^{\ell}Q^m$, where m is either 1 or 2. Since α is an automorphism $Q^{-m}P^iQ^m = P^{4i}$. Therefore $P^{4i^m} = P^{4i}$ and hence $4i^m \equiv 4i \bmod 9$. Thus $m = 1$. Furthermore $|P^{\ell}Q| = 3$ and

$P^{\ell} = P^{\ell}QQ^{-1}$. Therefore $|P^{\ell}| = 3$ and $\ell \equiv 0 \bmod 3$. Thus $P \to P^{i}$, $i = 1,2,4,5,7,8$ $Q \to P^{\ell}Q$, $\ell = 0,3,6$ give rise to all the automorphisms inducing power automorphisms on $\langle P \rangle$. Thus $|\mathrm{Aut}(G|_{\pi}\langle P \rangle)| = 18$. It is easily shown that $\mathrm{Aut}(G|_{\pi}\langle P \rangle)$ is non-abelian. Furthermore, $|\mathrm{Aut}(G|\langle P \rangle)| = 3$ and $[\mathrm{Aut}(G|_{\pi}\langle P \rangle)]' = \mathrm{Aut}(G|\langle P \rangle)$. Thus $\mathrm{Aut}(G|_{\pi}\langle P \rangle)/\mathrm{Aut}(G|\langle P \rangle) \approx C_6$, the cyclic group of order 6. We shall show in the theorems below that this example is typical of the general case.

We first prove some lemmas of a general nature.

Lemma II.4.1 Let $H \leqq G$. Then $[\mathrm{Aut}(G|_{\pi}H)]' \leqq \mathrm{Aut}(G|H)$.

Proof Let $\alpha,\beta \in \mathrm{Aut}(G\ H)$. Then if $x \in H$, let $x^{\alpha} = x^{i}$, $x^{\beta} = x^{j}$. Clearly $x^{\alpha^{-1}} = x^{i'}$ and $x^{\beta^{-1}} = x^{j'}$ where $ii' \equiv jj' \equiv 1 \bmod |x|$. Thus $x^{[\alpha,\beta]} = x^{i'j'ij} = x$.

Corollary II.4.2 (a) Let G be a p-group of odd order and let $\exp G = p^{s}$. Then $\lambda[\mathrm{Aut}(G|_{\pi}\Omega_{t})] \leqq \log_2 4s/t$.

(b) Let G be a 2-group, $\exp G = 2^{s}$. Then $\lambda[\mathrm{Aut}(G|_{\pi}\Omega_{t})] \leqq \log_2 4(s-1/t-1)$ where $t > 1$.

Proof Lemma II.4.1 and theorem II.2.1(a) and (b).

Lemma II.4.3 (a) Let G be a p-group of odd order and let $H \leqq G$ and $\exp H = p^{t}$. Then if $\alpha \in \mathrm{Aut}(G|_{\pi}H)$, $\alpha^{p^{t-1}(p-1)} \in \mathrm{Aut}(G|H)$.

(b) Let G be a 2-group, $\exp H = 2^{t}$. Then if $\alpha \in \mathrm{Aut}(G|_{\pi}H)$, $\alpha^{2^{t-1}} \in \mathrm{Aut}(G|H)$.

Proof (a) Let $\alpha \in \mathrm{Aut}(G|_{\pi}H)$ and let $B = G\cdot\langle\alpha\rangle$ the semi-direct product of G with $\langle\alpha\rangle$. Let $x \in H$. Then $N_B(\langle x\rangle)/C_B(x) \cong \mathrm{Aut}(\langle x\rangle) \cong C_{p^{t-1}(p-1)}$. But $\alpha \in N_B(\langle x\rangle)$. Hence $\alpha^{p^{t-1}(p-1)} \in C_B(x)$. This is true for all $x \in H$ and therefore $\alpha^{p^{t-1}(p-1)} \in \mathrm{Aut}(G|H)$.

(b) is proved similarly except that if $|x| = 2^r$, $\mathrm{Aut}(\langle x\rangle) \approx C_{2^{r-1}}$ if $r = 1$ or $r = 2$ and $\mathrm{Aut}(\langle x\rangle) \approx C_2 \wr C_{2^{r-2}}$ if $r \geq 3$.

Corollary II.4.4 (a) Let G be a p-group of odd order, $\exp G = p^s$. Then $\exp \mathrm{Aut}(G|_\pi \Omega_t) \leq p^{(2s-1)} \cdot (p-1)$.

(b) Let G be a 2-group, $\exp G = 2^s$. Then $\exp \mathrm{Aut}(G|_\pi \langle \Omega_t \rangle) \leq 2^{4s-2t-1}$.

Proof Lemma II.4.1 and theorem II.2.1(a) and (b).

Lemma II.4.5 Let G be a p-group, H a non-abelian subgroup of G. Then $\mathrm{Aut}(G|_\pi H)/\mathrm{Aut}(G|H)$ is an abelian p-group.

Proof Let $\mu: \mathrm{Aut}(G|_\pi H) \to \mathrm{Aut}(H)$ be the restriction map. Then by a result of Cooper [6], $\mathrm{Im}\mu$ is an abelian p-group.

Corollary II.4.6 Let G be a p-group of odd order. Then if Ω_t is non-abelian, $\mathrm{Aut}(G|_\pi \Omega_t)$ is a p-group of exponent at most p^{2s-1} where $\exp G = p^s$. Furthermore, whether p is odd or even, if Ω_1 is non-abelian $\mathrm{Aut}(G|_\pi \Omega_1) = \mathrm{Aut}(G|\Omega_1)$.

Proof The first statement is proved using corollary II.4.4(a) and lemma II.4.5. To prove the second statement we use a result of Cooper's, namely, that a power automorphism is central [6]. Thus, suppose $\mathrm{Aut}(G|_\pi \Omega_1) \neq \mathrm{Aut}(G|\Omega_1)$ and let $\alpha \in \mathrm{Aut}(G|_\pi \Omega_1) - \mathrm{Aut}(G|\Omega_1)$. Then there must be an element of order p, say x, such that $x^{-1}x^\alpha \neq 1$. But $x^{-1}x^\alpha \in Z(\Omega_1)$ by Cooper's result quoted above and $x^{-1}x^\alpha \in \langle x\rangle$. Hence $x \in Z(\Omega_1)$. Thus any element of order p which is moved lies in the center of Ω_1. Suppose if possible that y is an element of order p and $y \notin Z(\Omega_1)$. Then $xy \notin Z(\Omega_1)$. Thus $(xy)^\alpha = xy = x^\alpha y$ and hence $x^\alpha = x$, a contradiction. Therefore $\Omega_1 = Z(\Omega_1)$. By contraposition, the statement in the corollary follows.

We now quote a theorem due to Gaschütz [9].

Theorem II.4.7 If G is a finite abelian group and α is a power automorphism of G, then there exists a fixed positive integer m such that $g^{\alpha} = g^{m}$ for all $g \in G$.

Lemma II.4.8 (a) Let G be a p-group of odd order and H an abelian subgroup of G, $\exp H = p^{t}$. Then

$$\mathrm{Aut}(G|_{\pi}H)/\mathrm{Aut}(G|H) \cong C_{p^{t-1}(p-1)}$$

(b) Let G be a 2-group and H as in (a). Then $\mathrm{Aut}(G|_{\pi}H)/\mathrm{Aut}(G|H) \cong C_2 \otimes C_{2^{t-2}}$ if $t \geq 3$ and

$\mathrm{Aut}(G|_{\pi}H)/\mathrm{Aut}(G|H) \cong C_{2^{t-1}}$ if $t = 1$ or 2.

Proof (a) Let $\alpha \in \mathrm{Aut}(G|_{\pi}H)$. Then by theorem II.4.7, there exists a positive integer m such that $h^{\alpha} = h^{m}$ for all $h \in H$. Consider therefore the mapping $\mu: \mathrm{Aut}(G|_{\pi}H) \to J_{p^t}$, where J_{p^t} is the ring of integers $\mathrm{mod} p^{t}$, given by $\mu: \alpha \to m \ \mathrm{mod} p^{t}$. This is clearly a homomorphism into the multiplicative group of units of J_{p^t}. But the multiplicative group of units of J_{p^t} is cyclic of order $p^{t-1}(p-1)$. Furthermore, $\ker\mu = \mathrm{Aut}(G|H)$ and hence $\mathrm{Aut}(G|_{\pi}H)/\mathrm{Aut}(G|H) \cong C_{p^{t-1}(p-1)}$.

(b) is proved similarly except that the multiplicative group of units of J_{2^t} is cyclic of order 2^{t-1} if $t = 1$ or 2 and is of the form $C_2 \otimes C_{2^{t-2}}$ if $t \geq 3$.

Example This will illustrate that in general we cannot conclude that $\mathrm{Aut}(G|_{\pi}H)/\mathrm{Aut}(G|H)$ is a cyclic group if G is a 2-group.

Let $G = \langle P, Q / P^{8} = 1, Q^{2} = P^{4} \text{ and } Q^{-1}PQ = P^{7} \rangle$. G is the generalized quaternion group of order 32. We compute $\mathrm{Aut}(G|_{\pi}\langle P\rangle)$ first. If $P \to P^{i}$, $Q \to P^{\ell}Q$ is an automorphism, then $i = 1,3,5,7$. Furthermore, it is easily shown that $[P^{\ell}Q]^{2} = Q^{2}$ and that the remaining relations are satisfied. Thus $P \to P^{i}$, $Q \to P^{\ell}Q$,

$i = 1,3,5,7$, $\ell = 0,1,2,3,4,5,6,7$ constitute $\mathrm{Aut}(G|_{\pi}\langle P\rangle)$. Hence $|\mathrm{Aut}(G|_{\pi}\langle P\rangle)| = 32$. Also $P \to P$, $Q \to P^{\ell}Q$, $\ell = 0,\ldots,7$ give rise to $\mathrm{Aut}(G|\langle P\rangle)$ and so $|\mathrm{Aut}(G|\langle P\rangle)| = 8$. Therefore $\mathrm{Aut}(G|_{\pi}\langle P\rangle)/\mathrm{Aut}(G|\langle P\rangle)$ is of order 4. But if $\alpha[P \to P^{i}, Q \to P^{\ell}Q]$ is an automorphism of $\mathrm{Aut}(G|_{\pi}\langle P\rangle)$, $\alpha^2 \in \mathrm{Aut}(G|\langle P\rangle)$ since $i^2 \equiv 1 \bmod 8$ for $i = 1,3,5,7$. Hence $\mathrm{Aut}(G|_{\pi}\langle P\rangle)/\mathrm{Aut}(G|\langle P\rangle) \simeq C_2 \otimes C_2$.

Corollary II.4.9 (a) Let G be a p-group of odd order, $\exp G = p^s$ and let $\mathrm{cl}G = c$. Let A be maximal subject to being abelian and of exponent p^t. Then P, the p-sylow subgroup of $\mathrm{Aut}(G|_{\pi}A)$ is normal in $\mathrm{Aut}(G|_{\pi}A)$, $\mathrm{Aut}(G|_{\pi}A) = P.C$ where C is cyclic, $|C|\,|(p-1)$ and $\lambda[\mathrm{Aut}(G|_{\pi}A)] \leq \log_2 2s/t + c$.

(b) Let G be a 2-group, $\exp G = p^s$ and let $\mathrm{cl}G = c$. Let A be as above with $t > 1$. Then $\mathrm{Aut}(G|_{\pi}A)$ is a 2-group and $\lambda[\mathrm{Aut}(G|_{\pi}A)] \leq \log_2 2(s-1/t-1) + c$.

Proof (a) $\mathrm{Aut}(G|_{\pi}A)/\mathrm{Aut}(G|A) \hookrightarrow C_{p^{t-1}(p-1)}$ by lemma II.4.8(a). But $\mathrm{Aut}(G|A)$ is a p-group containing $[\mathrm{Aut}(G|_{\pi}A)]'$. Hence p-sylow subgroup P of $\mathrm{Aut}(G|_{\pi}A)$ is normal and therefore it possesses a p-complement C. Therefore $\mathrm{Aut}(G|_{\pi}A) = P.C$. Furthermore, $\lambda[\mathrm{Aut}(G|_{\pi}A)] \leq \log_2 2s/t + c$ by theorem II.3.4.

(b) is proved similarly using (b) of lemma II.4.8(b) and theorem II.3.4.

Section 5 In this section we investigate $\mathrm{Aut}(G|H)$ where G is a p-group and H is a maximal subgroup. We shall find application of these results in the next chapter.

Lemma II.5.1 Let G be a p-group (p-odd or even), H a maximal subgroup. Then $\lambda[\mathrm{Aut}(G|H)] \leq 2$ and $\mathrm{Aut}(G|H) = \mathrm{Aut}(G > H > 1)\cdot C$ where C is cyclic, $|C|\,|(p-1)$.

Proof Let $\mu: \mathrm{Aut}(G|H) \to \mathrm{Aut}(G/H)$ be the mapping $\mu: \alpha \to \alpha^*$ where $(gH)^{\alpha^*} = g^{\alpha}H$. Clearly μ is a homomorphism and $\ker\mu = \mathrm{Aut}(G > H > 1)$ while $\mathrm{Im}\mu \leqq \mathrm{Aut}(G/H) \approx C_{p-1}$. Thus $\mathrm{Aut}(G|H)/\mathrm{Aut}(G > H > 1) \leqq C_{p-1}$. The lemma then follows easily.

We note that if G is a 2-group, $\mathrm{Aut}(G|H) = \mathrm{Aut}(G > H > 1)$ and therefore $\mathrm{Aut}(G|H)$ is an abelian 2-group.

Corollary II.5.2 Let G be a 2-group and let $G = G_0 > \ldots > G_m$ be a chain of subgroups such that $|G_i : G_{i+1}| = 2$ for $i = 0,1,\ldots,(m-1)$. Let $A_k = \{\alpha \in \mathrm{Aut}(G) | G_i^{\alpha} = G_i,\ i = 1,\ldots,k\}$ and $B_{\ell} = \mathrm{Aut}(G|G_{\ell})$. Then $A_k \cap B_k$ is a 2-group, $k = 1,\ldots,m$ and $\lambda[A_k \cap B_k] \leqq k$.

Proof $A_1 \cap B_1 = \mathrm{Aut}(G|G_1) = \mathrm{Aut}(G > G_1 > 1)$ by the lemma and hence for $k = 1$, the corollary is true. Assume inductively that the result is true for k. Consider $\mu: A_{k+1} \cap B_{k+1} \to \mathrm{Aut}(G_k)$ where μ is the restriction map. Then

$$\ker\mu = A_{k+1} \cap B_{k+1} \cap \mathrm{Aut}(G|G_k)$$
$$= A_{k+1} \cap B_k \leqq A_k \cap B_k$$

and $\mathrm{Im}\mu \leqq \mathrm{Aut}(G_k/G_{k+1})$. By induction $A_k \cap B_k$ is a 2-group and $\lambda[A_k \cap B_k] \leqq k$. Furthermore, $\mathrm{Im}\mu$ is an abelian 2-group by the previous lemma. Therefore $A_{k+1} \cap B/A_k \cap B_k$ is an abelian 2-group and the result is proved for $k + 1$ and by induction we are done.

Theorem II.5.3 Let H be a maximal subgroup of a regular p-group G (p odd). Then either $\mathrm{Aut}(G|H)$ is an abelian p-group or $G = H \otimes \langle x \rangle$ in which case if $\alpha \in \mathrm{Aut}(G|H)$ either $\alpha^{p-1} = 1$ or α is of p-power order.

Proof There are two possibilities: either

$$\text{(a)} \quad C_{\Omega_1}(H) \leqq H$$

$$\text{or} \quad \text{(b)} \quad C_{\Omega_1}(H) \not\leqq H.$$

Suppose (a) holds. Then $\Omega_1 \cap C_G(H) \leqq H$. Let $g \in \Omega_1$ and $\alpha \in \mathrm{Aut}(G|H)$. Then by Lemma I.2.2, $g^{-1}g^{\alpha} \in C_G(H) \cap \Omega_1 \leqq \Omega_1 \cap H$. Consider therefore the mapping μ: $\mathrm{Aut}(G|H) \to \mathrm{Aut}(\Omega_1)$ where μ is the restriction. $\mathrm{Im}\mu \leqq \mathrm{Aut}(\Omega_1|\Omega_1 \cap H) = \mathrm{Aut}(\Omega_1 > \Omega_1 \cap H > 1)$ since $g^{-1}g^{\alpha} \in \Omega_1 \cap H$ for all $g \in \Omega_1$. Furthermore, $\ker\mu = \mathrm{Aut}(G|\Omega_1)$ which is a p-group by II.3.4. Hence $\mathrm{Aut}(G|H)$ is a p-group and by lemma II.5.1 we have $\mathrm{Aut}(G|H) = \mathrm{Aut}(G > H > 1)$ which is abelian.

Suppose (b) holds. Then there is an element $x \in C_{\Omega_1}(H) - H$. Therefore $G = H \otimes \langle x \rangle$. Now let $\alpha \in \mathrm{Aut}(G|H)$. Then $x^{\alpha} = x^i h_{\alpha}$, say, with $h_{\alpha} \in H$. We then get

$$x^{\alpha^{p-1}} = x^{i^{p-1}} h_{\alpha}^{1+i+i^2+\ldots+i^{p-2}}$$

$$= x h_{\alpha}^{1+i+\ldots+i^{p-2}} \qquad \text{since } i^{p-1} \equiv 1 \bmod p.$$

Now if $i = 1$, $\alpha \in \mathrm{Aut}(G > H > 1)$ and α is of p-power order. On the other hand, if $i \neq 1$, $x^{\alpha^{p-1}} = x\, h^{\frac{(1-i^{p-1})}{(1-i)}} = x$ since $|h_{\alpha}| = p$ and p divides $(1-i^{p-1})/(1-i)$. Hence in case (b) each element of $\mathrm{Aut}(G|H)$ is either a power of a prime or has order dividing (p-1).

CHAPTER III

In this chapter we investigate $\mathrm{Aut}(G)$ where G is a p-group of maximal class. This class of groups was investigated by N. Blackburn and we quote a number of results which will be found in his paper [5].

Definition III.1.1 Let G be a p-group. Then G is of maximal class if the nilpotence class of G is the greatest compatible with its order.

It is proved in [5] that if $|G| = p^m$ and G is of maximal class, then $\mathrm{cl}G = m-1$. Furthermore, it can be shown that $G/\Gamma_2 = p^2$ and G/Γ_2 is elementary abelian. This implies that $|\Gamma_i:\Gamma_{i+1}| = p$ for $i = 2,\ldots,m-1$.

Definition III.1.2 Let G be a p-group of maximal class, $|G| = p^m$, $m > 3$. Define Γ_1 by $\Gamma_1/\Gamma_4 = C_{G/\Gamma_4}(\Gamma_2/\Gamma_4)$.

Clearly Γ_1 is a characteristic subgroup and it can be shown that $|G:\Gamma_1| = p$. We thus have a chain of characteristic subgroups $G = \Gamma_0 > \Gamma_1 > \ldots > \Gamma_{m-1} > \Gamma_m = 1$, each Γ_i maximal in Γ_{i-1}. Furthermore, it is easy to show that any normal subgroup of G of index p^2 or greater is one of the Γ_i.

We first prove a few general lemmas which will be used in the sequel.

Lemma III.1.1 If G is generated by $x_1,\ldots,x_n$ and $\alpha \in \mathrm{Aut}G$ such that $x_i^{-1}x_i^{\alpha} \in H$, $H \lhd G$ for $i = 1,\ldots,n$, then

$x^{-1}x^{\alpha} \in H$ for all $x \in G$.

Proof Let $x = \prod_{i=1}^{k} y_i$ where $y_i \in \{x_1,\ldots,x_n\}$. Then if $k = 1$, $x^{-1}x^{\alpha} \in H$ by hypothesis. Assume the result is true if $k < r$ and let $x = \prod_{i=1}^{r} y_i$. Then

$$x^{-1}x^{\alpha} = (y_1\cdots y_{r-1}y_r)^{-1}(y_1\cdots y_r)^{\alpha}$$

$$= y_r^{-1}(y_1\cdots y_{r-1})^{-1}(y_1\cdots y_{r-1})^{\alpha}y_r^{\alpha} .$$

By induction $(y_1\cdots Y_{r-1})^{-1}(y_1\cdots y_{r-1})^{\alpha} = h \in H$. Therefore $x^{-1}x^{\alpha} = y_r^{-1}y_r^{\alpha}h[h,y_r^{\alpha}] \in H$ since $H \lhd G$. Hence by induction, we are done.

Lemma III.1.2 Let G be a nilpotent group. Then if $\alpha \in \mathrm{Aut}(G:\Gamma_2)$, $[x_1,\ldots,x_m]^{\alpha} = [x_1,\ldots,x_m] \bmod \Gamma_{m+1}$.

Proof When $m = 1$, the result is true by hypothesis. Assume inductively that for simple commutators of weight less than m, the result is true. Now $[x_1,\ldots,x_{m-1},x_m]^{\alpha} = [[x_1,\ldots,x_{m-1}]^{\alpha},x_m^{\alpha}]$. But $[x_1,\ldots,x_{m-1}]^{\alpha} = [x_1,\ldots,x_{m-1}]c_m$, where $c_m \in \Gamma_m$ by induction and $x_m^{\alpha} = x_m c_2$ where $c_2 \in \Gamma_2$ by hypothesis. Thus

$$[x_1,\ldots,x_m]^{\alpha} = [[x_1,\ldots,x_{m-1}]c_m,x_m c_2]$$

$$= [[x_1,\ldots,x_{m-1}]c_m,c_2][[x_1,\ldots,x_{m-1}]c_m,x_m][[x_1,\ldots,x_{m-1}]c_m,x_m,c_2]$$

by lemma I.1.2. We now use the fact that $[\Gamma_i,\Gamma_j] \leq \Gamma_{i+j}$ for all $i,j = 2,3,\ldots$ (Theorem I.1.1). Then

$$[[x_1,\ldots,x_{m-1}]c_m,c_2] \in \Gamma_{m+1} \quad \text{and} \quad [[x_1,\ldots,x_{m-1}]c_m,x_m,c_2] \in \Gamma_{m+2}.$$

Therefore $[x_1,\ldots,x_m]^{\alpha} = [[x_1,\ldots,x_{m-1}]c_m,x_m]c_{m+1},c_{m+1} \in \Gamma_{m+1}$

$$= [x_1,\ldots,x_m][x_1,\ldots,x_m,c_m][c_m,x_x]c_{m+1}$$

$$= [x_1,\ldots,x_m]c'_{m+1}, c'_{m+1} \in \Gamma_{m+1}.$$

Hence by induction the result follows.

We recall that Γ_k is generated by simple commutators of weight k. Using this fact and lemma III.1.1 we obtain the following:

Theorem III.1.3 Let G be a finite p-group. Then $\mathrm{Aut}(G:\Gamma_2) = \mathrm{Aut}(G > \Gamma_2 > \ldots > 1)$.

We prove a final lemma before considering $\mathrm{Aut}(G)$, G of maximal class.

Lemma III.1.4 Let E be an elementary abelian group of order p^2 generated by P and Q. The group A of automorphisms given by

$$P \to P^i \qquad i = 1,\ldots,(p-1)$$

$$Q \to QP^{\ell} \qquad \ell = 0,\ldots,(p-1)$$

is of order $p(p-1)$ and of derived length 2.

Proof The group A of automorphisms is clearly of order $p(p-1)$. Furthermore, it is isomorphic to the group of non-singular matrices of the form $\begin{pmatrix} i & 0 \\ \ell & 1 \end{pmatrix}$ with entries from a Galois field of p elements. Now it is easily checked that $\begin{pmatrix} i & 0 \\ \ell & 1 \end{pmatrix}^{-1} = \begin{pmatrix} i & 0 \\ -i'\ell & 1 \end{pmatrix}$ where $ii' \equiv 1 \bmod p$. Then $\begin{pmatrix} i' & 0 \\ -i'\ell & 1 \end{pmatrix}\begin{pmatrix} j' & 0 \\ -j'm & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ \ell & 1 \end{pmatrix}\begin{pmatrix} j & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ k & 1 \end{pmatrix}$ Now $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ has order p and therefore lies in the unique sylow p-subgroup which is cyclic of order p. Hence A' is abelian and A is metabelian.

We are now in a position to prove that if G is a p-group of maximal class, $|G| = p^m$, $m > 3$, then $\mathrm{Aut}(G)$ is solvable and we may

obtain a bound on the derived length together with some information about the order of $\mathrm{Aut}(G)$.

We note that if $m = 3$, $\mathrm{Aut}(G)$ is not necessarily solvable as can be seen from the following example:

Let $G = \langle P,Q,R / P^p = Q^p = R^p = 1;\ [P,Q] = [P,R] = 1;\ [Q,R] = P \rangle$ $\mathrm{Aut}(G)$ is not solvable as can be seen as follows: If $\mathrm{Aut}(G)$ were solvable, so would $\mathrm{Aut}(G|Z)$. Consider the mapping

$$P \to P$$
$$Q \to Q^{\alpha_1} R^{\beta_1}$$
$$R \to Q^{\alpha_2} R^{\beta_2}$$

Under what circumstances can this be extended to an automorphism of G? It can be extended if and only if $[Q^{\alpha_1} R^{\beta_1}, Q^{\alpha_2} R^{\beta_2}] = P$. Since $\mathrm{cl}G = 2$, the above hold if and only if

$$[Q,R]^{\alpha_1\beta_2 - \alpha_2\beta_1} = P$$

i.e. if and only if $\alpha_1\beta_2 - \alpha_2\beta_1 \equiv 1 \bmod p$. Consider now the homomorphism $\mu: \mathrm{Aut}(G|Z) \to \mathrm{Aut}(G/Z)$. Now G/Z is elementary abelian of order p^2 and hence $\mathrm{Aut}(G/Z) \approx GL_2(k)$ where k is a Galois field of p elements. Further, by what we showed above, $\mathrm{Im}\mu \approx SL_2(k)$ and $SL_2(k)/Z(SL_2(k))$ is the projective unimodular group which is simple if $p > 3$ [3]. Hence $\mathrm{Aut}(G)$ cannot be solvable.

Before proving the next theorem, we calculate $|\mathrm{Aut}(G)|$ where $|G| = p^3$ and G is non-abelian. When $p = 2$, it is well-known that $|\mathrm{Aut}(G)| = 24$ or 8. We therefore perform the calculations for p odd.

There are two non-abelian groups of order p^3, viz.

(i) The one of exponent p generated by P, Q and R with the following generating relations:

$$P^p = Q^p = R^p = 1,\ R^{-1}QR = QP,\ R^{-1}PR = P,\ Q^{-1}PQ = P \text{ and}$$

(ii) The one of exponent p generated by P and Q with generating relations $P^{p^2} = Q^p = 1,\ Q^{-1}PQ = p^{1+p}$. We calculate (i) first.

$Z(G) = \langle P \rangle$ and thus any automorphism must be of the form

$P \to P^i,\ i = 1,\ldots,)p-1)$

$Q \to P^{\ell}Q^mR^s,\ \ell = 0,\ldots,p-1$

$R \to P^{\ell'}Q^{m'}R^{s'},\ \ell = 0,\ldots,p-1$. We also require $[P^{\ell}Q^mR^s,\ P^{\ell'}Q^{m'}R^{s'}] = P^i$. Since G is of class 2, this condition is equivalent to

$$[Q,R]^{ms'}[R,Q]^{sm'} = P^i$$

Therefore $P^{ms'-sm'} = P^i$ and hence

$$ms' - sm' \equiv i \bmod p.$$

Now if neither m nor s is zero, we have p choices for the pair (m',s'). If $m = 0$ and $s \neq 0$ or if $m \neq 0$ and $s = 0$ we get p choices for (m',s') in each case and thus we have

$$(p-1)^3p^3 + 2(p-1)^2p^3 = (p-1)^2p^3(p-1+2) = (p-1)^2p^3(p+1)$$

choices and each of these gives rise to an automorphism provided $\langle Q^mR^s,\ Q^{m'}R^{s'},\ P^i \rangle \cong \langle Q,R \rangle$. Now calculating modulo $Z(G)$

$$(Q^mR^s)^{\lambda}(Q^{m'}R^{s'})^{\mu} = Q^{\lambda m+\mu m'}R^{\lambda s+\mu s'}$$

and to obtain Q(modulo $Z(G)$) we must solve the equations

$$\lambda m + \mu m' = 1$$

$$\lambda s + \mu s' = 0 \quad \text{in} \quad GF(p).$$

But $\det \begin{pmatrix} m & m' \\ s & s' \end{pmatrix} \neq 0$ and the equations have a solution. Thus $Q = (Q^m R^s)^\lambda (Q^{m'} R^{s'})^\mu \bmod Z(G)$ for some λ and μ. Similarly for R. Hence $\mathrm{Aut}(G) = (p-1)^2 p^3 (p+1)$. Now we calculate (ii).

Let $P \to P^i Q^j$

$Q \to P^{i'} Q^{j'}$ give rise to an automorphism

$(P^i Q^j)^p = P^{ip} Q^{jp} [Q^j, P^i]^{\binom{p}{2}} = P^{ip} \neq 1$. Hence $i \not\equiv 0 \bmod p$. Similarly $i' \equiv 0 \bmod p$. The following relation must also hold.

$$[P^i Q^j, P^{i'} Q^{j'}] = (P^i Q^j)^p = P^{ip}$$

Therefore $[P,Q]^{ij'-i'j} = P^{ip}$

Hence $P^{p(ij'-i'j)} = P^{ip}$ which implies that

$$ij' - i'j \equiv i \bmod p.$$

Therefore since $i' \equiv 0 \bmod p$,

$$ij' \equiv i \bmod p \quad \text{and} \quad j' \equiv 1 \bmod p.$$

Therefore $P \to P^i Q^j$ $(i,p) = 1$, $j = 0,\ldots,p-1$

$Q \to P^{i'} Q$ $i' \equiv 0 \bmod p$ gives rise to all the automorphisms of G. Hence $|\mathrm{Aut}(G)| = p^3(p-1)$. We may now prove the following

Theorem III.1.5 (a) Let G be a p-group of maximal calss, p odd, and $|G| = p^m$, $m > 3$. Then

(i) $P = \mathrm{Aut}(G > \Gamma_1 > \ldots > \Gamma_{m-1} > 1)$ is the p-sylow subgroup of $\mathrm{Aut}(G)$. Hence $P \lhd \mathrm{Aut}(G)$ and $\mathrm{cl} P \leq m-1$.

(ii) $\mathrm{Aut}(G)$ is solvable and $\lambda[\mathrm{Aut}(G)] \leq \log_2 8(m-1)$

(iii) $\mathrm{Aut}(G) = PL$ where $|L| \mid (p-1)^2$ and $p^m \leq |P| \leq p^{2m-3}$

(b) Let G be a 2-group of maximal class, $|G| = 2^m$, $m > 3$. Then $\mathrm{Aut}(G) = \mathrm{Aut}(G > \Gamma_1 > \ldots > \Gamma_{m-1} > 1)$ and hence $\mathrm{Aut}(G)$ is a 2-group of class not greater than $m-1$. Furthermore $2^m \leq |\mathrm{Aut}(G)| \leq 2^{2m-3}$.

Proof (a) (i) Let P be a p-sylow subgroup of $\mathrm{Aut}(G)$. Clearly P induces the identity automorphism on Γ_{m-1}. Assume inductively that P stabilizes $\Gamma_{m-k} > \ldots > \Gamma_{m-1} > 1$. Consider $\beta: P \to \mathrm{Aut}(\Gamma_{m-k-1}/\Gamma_{m-k}) \simeq C_{p-1}$ where β is the obvious homomorphism. Then $\mathrm{Im}\beta = 1$ and P stabilizes $\Gamma_{m-k-1} > \Gamma_{m-k} > \ldots > \Gamma_{m-1} > 1$. Hence by induction, P stabilizes $G > \Gamma_1 > \ldots > \Gamma_{m-1} > 1$. Therefore $P = \mathrm{Aut}(G > \Gamma_1 > \ldots > \Gamma_{m-1} > 1)$ and by P. Hall [13], $\mathrm{cl}P \leq m-1$.

(ii) and (iii). Let $\beta: \mathrm{Aut}(G) \to \mathrm{Aut}(G/\Gamma_1)$ be the mapping defined by $\beta: \alpha \to \alpha^*$ where $(g\Gamma_1)^{\alpha^*} = g^{\alpha}\Gamma_1 \cdot \beta$ is a homomorphism and $\mathrm{Im}\beta \simeq C_{p-1}$, $\ker\beta = \mathrm{Aut}(G;\Gamma_1)$.

Consider now $\sigma: \mathrm{Aut}(G;\Gamma_1) \to \mathrm{Aut}(G/\Gamma_2)$. G/Γ_2 is elementary abelian of order p^2. Thus $G/\Gamma_2 = \Gamma_1/\Gamma_2 \otimes M/\Gamma_2$ for some $M \leq G$. Since $\Gamma_1^{\alpha} = \Gamma_1$ for all $\alpha \in \mathrm{Aut}(G)$, $\mathrm{Im}\sigma$ is a subgroup of $\mathrm{Aut}(G/\Gamma_2)$, each element of which induces a power automorphism on Γ_1/Γ_2. Thus if $G/\Gamma_2 = \langle a\Gamma_2\rangle \otimes \langle b\Gamma_2\rangle$, $a \in \Gamma_1$, the automorphisms of $\mathrm{Im}\sigma$ are of the form

$$a\Gamma_2 \to a^i\Gamma_2 \qquad i = 1,\ldots,p-1$$

$$b\Gamma_2 \to a^{\ell}b^m\Gamma_2 \qquad \ell = 0,\ldots,p-1;\ m = 1,\ldots,p-1.$$

But we further have that $a^{\ell}b^m b^{-1} \in \Gamma_1$. Therefore $b^{m-1} \in \Gamma_1$ which implies $m = 1$. Hence the automorphisms of $\mathrm{Im}\sigma$ are of the form

$$a\Gamma_2 \to a^i\Gamma_2 \qquad i = 1,\ldots,p-1$$

$$b\Gamma_2 \to a^\ell b\Gamma_2 \qquad \ell = 0,\ldots,p-1$$

Therefore lemma III.1.4 is applicable and we have $|\mathrm{Im}\sigma| \mid p(p-1)$ and $\lambda[\mathrm{Im}\sigma] \leq 2$. Furthermore, $\ker\sigma = \mathrm{Aut}(G;\Gamma_2)$ which by theorem III.1.5 is a p-group of class at most m-2 (actually, the class is exactly m-2 since $G/Z \cong \mathrm{Aut}(G;\Gamma_2)$).

Now $|\mathrm{Aut}(G)| = |\mathrm{Aut}(G;\Gamma_1)| \cdot t$ where $t \mid (p-1)$ and $|\mathrm{Aut}(G;\Gamma_1)| = |\mathrm{Aut}(G;\Gamma_2)| \cdot kt^*$ where $k = 1$ or p and $t^* \mid (p-1)$. Therefore $|\mathrm{Aut}(G)| = p^n s$ where $s \mid (p-1)^2$.

To prove that $m \leq n \leq 2m - 3$ we observe first that since $G/Z \cong P$, $|P| \geq p^m$ as every p-group has an outer automorphism of p-power order [10]. To obtain the upper bound we argue by induction.

When $m = 3$, from the discussion preceding the theorem, $|P| \leq 2.3 - 3$. Thus assume inductively that $|P| \leq p^{2m-3}$ if $|G| = p^m$, $m > 3$. Let $|G| = p^{m+1}$. Then $|G/Z| = p^m$. Consider $\beta: P \to \mathrm{Aut}(G/Z)$ where P is the p-sylow subgroup of $\mathrm{Aut}(G)$. By induction $|\beta(P)| \leq p^{2m-3}$ and $\ker\beta = \mathrm{Aut}(G > Z > 1)$. Now since G is of maximal class, it is clearly purely non-abelian. By a result of J. Adney and T. Yen [1], $|\mathrm{Aut}(G;Z)| = |\mathrm{Hom}(G/\Gamma_2,Z)| = p^2$. Therefore $|P| \leq p^{2(m+1)-3}$ and the result follows by induction.

Finally, consider $\mathrm{Aut}(G) \geq \mathrm{Aut}(G;\Gamma_1) \geq \mathrm{Aut}(G;\Gamma_2)$. $\mathrm{Aut}(G)/\mathrm{Aut}(G;\Gamma_1)$ is abelian and $\lambda[\mathrm{Aut}(G;\Gamma_1)/\mathrm{Aut}(G;\Gamma_2)] \leq 2$. Furthermore, $\lambda[\mathrm{Aut}(G;\Gamma_2)] \leq \log_2(m-1)$ and therefore $\lambda[\mathrm{Aut}(G)] \leq \log_2 8(m-1)$. Now let L be a p-complement of $\mathrm{Aut}(G)$. We have $\mathrm{Aut}(G) = P.L.$

(b) To prove $\mathrm{Aut}(G)$ is a 2-group, we proceed in exactly the same way as above.

Corollary III.1.6 If G is a 3-group of maximal class, $G = 3^m$, $m > 3$, $\mathrm{Aut}(G)$ is a split extension of 3-group by an abelian 2-group of order 2 or 4, or a 3-group.

NOTATION

$\Omega_t(G)$	set of elements of G of order not greater than p^t
$\langle\Omega_t(G)\rangle$	group generated by $\Omega_t(G)$
$\mho_t(G)$	set of elements which are p^t-th powers of elements in G
$\mathrm{Aut}(G\|H)$	group of automorphisms of G fixing element-wise H
$\mathrm{Aut}(G;H)$	group of H-automorphisms of G (cf. Definition I.1.3(a))
$\mathrm{Aut}(G > G_1 >\ldots> G_m)$	group of automorphisms stabilizing $G > G_1 >\ldots> G_m$ (cf. Definition I.1.3(b))
$\mathrm{cl}(G)$	nilpotent class of G
$\lambda(G)$	derived length of G
$\exp(G)$	lowest common multiple of the orders of elements of G
$C(p^k), C_{p^k}$	cyclic group of order p^k
$C_G(H)$	centralizer of H in G
$N_G(H)$	normalizer of H in G
$\mathrm{Aut}(G\|_{\pi} H)$	the group of automorphisms inducing a power automorphism on H
$G \hookrightarrow G^*$	G^* contains a subgroup isomorphic to G

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