A TECHNIQUE FOR THE SPECIFICATION OF VARIABLES RELATING TO ELECTRIC POWER NETWORKS

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A TECHNIQUE FOR THE SPECIFICATION OF VARIABLES RELATING TO ELECTRIC POWER NETWORKS

by

Maurice Wolla

AN ABSTRACT

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by Maurice Wolla

To effectively plan the current operation as well as the future expansion of electric power systems requires a knowledge of the operating characteristics of the existing and/or proposed systems. Analysis of a class of power system studies utilized in determining the electrical characteristics of a power system indicates that these studies are essentially problems in the analysis of electric networks. These studies differ from problems in "conventional" network analysis primarily in two aspects: (1) the size and complexity of the network under consideration, and (2) the type of initial problem specifications. The availability of digital computers has alleviated, but not eliminated, difficulties associated with the size of the network. Problems associated with the initial specification of variables are more fundamental in nature and must be considered within the framework of the network equations since it is mandatory that inconsistencies are avoided. It is logical that a re-examination of the variable specification aspect of these network studies should originate at the level of the correlating graph and associated primary system of equations since they constitute the foundation for electric network theory.

The ω-domain graph correlates of the networks under consideration are comprised of two general types of elements: relation elements (F-elements) and no-relation elements (N-elements); the latter type is characterized by a lack of any fixed interrelation between the associated V and I variables and furthermore neither of these variables is specified initially. The resulting primary system of equations is homogeneous in form and certainly consistent. Properties of subgraphs of F-elements are subsequently utilized to define classifications of the N-elements such that either, neither, or both of their associated variables can be assigned arbitrary values with no danger of introducing inconsistencies. The investigation clearly indicates that a multiplicity of N-element classification patterns exist for a given graph and provides the basis for new and more general approaches to the analysis of electric networks--particularly those problems in which it is desired to maintain prescribed operating conditions.

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INTRODUCTION

1.1. Background

Planning the current operation and future expansion of a modern-day electric power system is a complex task and one of ever-increasing importance. These systems have grown from small, independent, local units at the turn of the century to the vast, interconnected "pools" of today - with still larger interconnections proposed for the future[1-4]. It soon became apparent that the ability to operate the small utilities successfully based primarily upon past experience was not adequate to cope with the problems associated with these rapidly expanding systems. Advances in analysis and control techniques coupled with the availability of digital and analog computers have provided the system planners of today with the opportunity to study and plan the operation of these large-scale power systems. Considerable effort has been directed toward the complete automation of system planning and operation[5-9].

The components of an electric power system can be divided into three general categories: (1) the generating stations, (2) the transmission and distribution network, and (3) the loads. The basic operating problem is to schedule the generating stations so as to supply the total load demand, plus losses within the transmission and distribution network, in a manner consistent with dependable and economical performance of the power system. The planning phase is necessary to insure that the power system has adequate generating capacity and transmitting capability to meet the constantly growing load demands.

The effective planning of the current operation and future expansion of a power system requires a knowledge of the performance characteristics of the existing and/or proposed systems. The very nature of a power system precludes any possibility of "laboratory testing" the actual system as is commonly done with many types of physical systems. Thus the system planners must devise an adequate model for the system and subsequently determine the performance characteristics of this model. The validity of the models used in such system studies is tested by correlating the performance characteristics determined from the model with those obtained from observations made on the actual system.

The investigations of this paper relate to those phases of power system studies which are primarily concerned with the steady-state electrical characteristics of the power system. These studies utilize a single-phase network representation of the power system and in general they require a complete steady-state solution for the network, i.e. voltages, currents, real and reactive powers, power factor, etc. at various points within the network. Thus, these studies are essentially problems in network analysis. However they differ from the "conventional" problems in network analysis in a number of respects. The network to be analyzed is considerably larger and more complex than one might encounter in other areas. In addition, it has generally been the case in the past that the entire network must be considered as a unit for the purposes of analysis. In contrast many communication and control systems are often analyzed on a "piece-meal" basis since signals are channeled along desired paths by the use of uni-lateral devices and/or filtering. Thus as the size of a power system increases due to expansion and interconnection the additional complexity itself presents a formidable problem and it magnifies the need for a better understanding of the fundamental concepts involved in the analysis of the associated network. Perhaps the most significant difference in this type of study is related to the manner in which the initial problem specifications are given.

In power system studies of the type under consideration the specifications are stated in terms of real and reactive powers and voltage magnitudes, whereas the foundations of network theory have been developed with voltages and currents as fundamental or primary variables[10-11]. Quantities such as real and reactive power are consequently considered as derived or secondary variables since they are defined in terms of the primary variables. Thus for any particular network element the inter-relation (complex number form) between the voltage V, current I, real power P, and reactive power Q, is given by the following relation:

$$P + jQ = VI^* \quad (1.1.1)$$

or

$$P = \text{Re}\,\{VI^*\} \quad (1.1.2)$$

$$Q = \text{Im}\,\{VI^*\} \quad (1.1.3)$$

Here one notes that on the one hand if V, I are known, then P and Q are determined; on the other hand, however, if P, Q are known, then neither V nor I is determined. This change in the specification of variables causes considerable difficulty in the attempt to determine a complete network solution. A specific example of this type of problem is considered in a later section.

Originally it was felt that the performance of a power system could be best predicted through the use of a miniaturized system commonly referred to as a network analyzer. However, in recent years the digital computer has replaced the network analyzer as the primary tool for large-scale power system studies. By its very nature the digital computer is a more versatile and flexible device. Moreover, it is capable of handling not only all of the problems which can be solved on the network analyzer, but an almost endless variety of different problems as well.

1.2. Load or Power-Flow Studies

The load or power-flow study exemplifies the general type of problem under consideration in this investigation. A study of this type requires a complete steady-state solution for a single-phase network representation of the power system. The network is made up of generator elements, load elements, and elements corresponding to the interconnecting transmission and distribution network. The general configuration of this type of network is given in Figure 1.2.1 where the details of the transmission and distribution network are not shown. For simplicity it is also assumed that each non-reference node is incident to exactly one generator or load element.

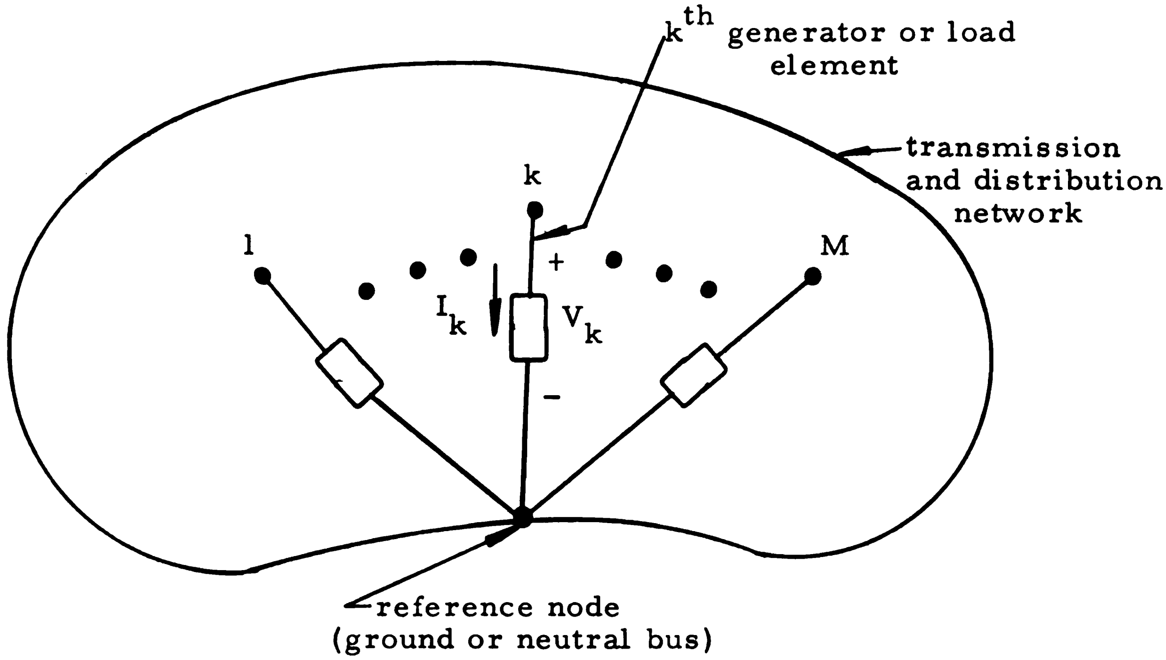


Figure 1. 2. 1. General Representation of a (M+1)-node Power Network Diagram.

The general nature of a load or power-flow study can be described as follows[2, 12-14]:

1. The complex number form of the node system of equations for the network of Figure 1. 2. 1 can be written as

$$I_k = - \sum_{n=1}^{M} Y_{kn} V_n \quad , \quad k=1, 2, \ldots, M \qquad (1.2.1)$$

where

$$I_k = |I_k| e^{j \theta_k} \qquad (1.2.2)$$

$$V_n = |V_n| e^{j \phi_n} \qquad (1.2.3)$$

$$Y_{kn} = |Y_{kn}| e^{j \sigma_{kn}} \qquad (1.2.4)$$

2. In the usual load study, three types of load and generator specifications are considered:

   A. For all load elements the real and reactive power

$$P + jQ = VI^* \qquad (1.2.5)$$

   is specified.

   B. For all generator elements except one, the real power and voltage magnitude

$$P = \mathrm{Re}\{VI^*\} \quad \text{and} \quad |V| \qquad (1.2.6)$$

   are specified.

   C. For the remaining generator element, i.e. the "slack" generator, the phasor voltage

$$V = |V|e^{j\phi} \qquad (1.2.7)$$

   is specified.

3. The problem then, is to determine a set of V_k and I_k, k=1, 2, . . . , M, such that the node system of equations, (1.2.1), is satisfied subject to the specifications,(1.2.5) through (1.2.7), for the appropriate generator and load elements. Once this has been done, then all voltages, currents, real and reactive powers, etc. within the transmission and distribution network can be calculated and the complete solution will have been determined.

Unfortunately the load and generator specifications cannot be used directly to obtain a solution to the node system of equations. Rather some form of iteration is used to determine a solution. One such approach proceeds in the following manner[12]:

1. Initial estimates are made for the V_k,
2. The corresponding I_k's are found from (1.2.1),
3. The appropriate quantities, (1.2.5) through (1.2.7) are calculated, compared to the specified values, and the errors are determined,
4. Suitable correction relations are used to determine new estimates for the V_k.

Steps 2 through 4 are repeated until (hopefully) the errors calculated in 3 are less than some prescribed precision index.

Another commonly used technique[2, 13, 14], is to modify the node equations so as to obtain a system of simultaneous nonlinear equations expressing the real and reactive power for each generator and load element in terms of the generator and load voltages and the node admittance parameters of the transmission system. Since for k=1, 2, . . . , M:

$$P_k + jQ_k = V_k I_k^* \tag{1.2.8}$$

then, from (1.2.1)

$$P_k + j\,Q_k = -\sum_{n=1}^{M} V_k\, Y_{kn}^{*}\, V_n^{*} \qquad (1.2.9)$$

Also, using (1.2.2) through (1.2.4)

$$P_k + j\,Q_k = -\sum_{n=1}^{M} |V_k|\,|Y_{kn}|\,|V_n|\, e^{j(\phi_k - \phi_n - \sigma_{kn})} \qquad (1.2.10)$$

or

$$P_k = -\sum_{n=1}^{M} |V_k|\,|Y_{kn}|\,|V_n| \cos(\phi_k - \phi_n - \sigma_{kn})$$
$$Q_k = -\sum_{n=1}^{M} |V_k|\,|Y_{kn}|\,|V_n| \sin(\phi_k - \phi_n - \sigma_{kn}) \qquad (1.2.11)$$

The initial problem specifications can now be inserted directly into any one of these last three sets of nonlinear "power equations," (1.2.9), (1.2.10), or (1.2.11). Each generator element and each load element has four associated variables in this final formulation: P_k, Q_k, $|V_k|$, ϕ_k ; for each element two of these variables are specified and the remaining two must be determined. Once again an iterative technique is used to determine a solution.

In the past the sheer size of the network and the accompanying large number of equations to be solved has been a major obstacle in load studies. However, the introduction of the digital computer, as well as specialized programs[15, 16], has reduced considerably -- although certainly not eliminated -- this problem. The major difficulties in this type of study are those associated with the iterative methods employed to determine a solution. The key considerations are those relating to existence and multiplicity of solutions, convergence, rate of convergence, the effect of initial estimates on covergence, etc. In many respects the original problem has been transformed from one in network analysis to one in numerical analysis in order to accommodate the initial problem specifications in the form given. Investigations into the problems associated with load studies, as well as other studies of this general type, have been primarily concerned with improving the iterative techniques used to obtain a solution[12, 13, 17-20]. As a result of these efforts computer programs, which are capable of handling large-scale power systems, are available and in use today[5-8, 21].

1.3. Another Viewpoint

Studies of the type considered in the preceding section play a major role in predicting the current performance and analyzing the future expansion of a power system. Thus it is essential that one be able to obtain an accurate numerical solution for a particular study. Moreover, it would be of considerable benefit to a system planner if the problem formulation and associated solution processes could also be utilized as a basis for a theoretical study of the system characteristics. Unfortunately the present problem formulation does not readily lend itself to a theoretical study of system characteristics. Rather, the resulting nonlinear equations tend to shift the emphasis to characteristics of the iterative techniques which are used to obtain a numerical solution for a particular study.

A re-examination of the basic structure of a load study, for example, indicates that this study is essentially a problem in the analysis of an electric network and that the major source of difficulty is the form in which the variables for certain elements are specified. In order to avoid nonlinearities at the outset of the problem it is necessary to reconsider the form in which the variables are specified. Since these are essentially problems in network analysis, it would seem natural to return to the basic structure of network theory and to give consideration to choosing a form for the variable specifications which is more compatible with the existing theory. Perhaps the most logical choice to consider would be that of the voltage and current variables for an element. This choice can be given initial support by noting that specifying the voltage and current variables for a network element determines the real and reactive power variables for that element - see equations (1. 1. 1) through (1. 1. 3). Thus this choice would be closely related to the original specifications. Furthermore, an extensive body of network theory already exists, in which voltages and currents are considered as fundamental or primary variables, and elements having either their voltage or current variables specified have been considered within this theory. Therefore, consideration of elements having both voltage and current variables specified would be a logical extension of this theory. Also of importance is the fact that this approach provides an orderly and precise formulation technique and thus is well-suited for use in conjunction with the digital computer.

Within the area of network analysis little consideration has been given to elements having both voltage and current variables specified. Undoubtedly this is due to the unlikelihood of finding a correlating physical device in the laboratory. However some consideration has been given to the possibility of synthesizing "pathological" elements of this type using components such as ideal transformers, ideal gyrators, and negative resistances[22, 23]. Irrespective of whether or not such devices are physically realizable, the fact remains that elements of this type can be useful in theoretical studies.

The investigations of this paper are concerned with extending the class of network elements to include those for which both voltage and current variables are specified and then utilizing these elements in the analysis of electric networks.

1.4. A Variable Specification Problem in Network Analysis

Any investigation into allowable patterns of element variable specifications must be based upon a study of the appropriate systems of network equations since it is imperative that inconsistencies be avoided. The primary system of network equations[10, 16] provides a logical starting point for such a study. This system of equations contains all of the information relative to the element voltage and current variables and ultimately it is the solution of this system of equations which is sought.

Chapter II is devoted to summarizing properties of systems of homogeneous, linear, algebraic equations with constant coefficients. These equations play a fundamental role in this investigation -- in particular those rank properties of the coefficient matrix which define partitions of the variables into dependent and independent sets. Following chapters return to a study of the electric network via the correlating oriented linear graph and associated systems of network equations. Interrelations between subgraphs and the coefficient matrix in the network equations are exploited to interpret rank properties of the coefficient matrix in terms of interconnection patterns of the linear graph and element parameters. In this manner it is possible to determine conditions on the structure of the linear graph and the element parameters such that the graph may contain elements for which both the voltage and current variables can be arbitrarily specified. In addition one finds that still another type of element is required -- one for which neither the voltage nor the current variable is specified and further, the voltage and current variables are not interrrelated in any fixed manner, as is the case for the graph elements correlating with resistors, inductors, and capacitors. Finally, consideration is given to the effect of extending the class of graph elements to include these new elements. A new approach is suggested for the analysis of large-scale electric networks by utilizing these elements in conjunction with zoning techniques.

Chapter 2

SYSTEMS OF HOMOGENEOUS, LINEAR, ALGEBRAIC, CONSTANT-COEFFICIENT EQUATIONS

2.1. Definitions and Fundamental Properties

For reference purposes and to define terminology it is convenient, at this point, to collect certain definitions and fundamental properties of homogeneous, linear, algebraic equations with constant coefficients. The proofs of the basic theorems may be found in most texts on matrix theory or linear algebra[24-28], and are not repeated here.

In the interest of brevity, and at the same time to be complete, the following abbreviation is used:

Definition 2.1.1. Holac Equations.

The abbreviation holac is used to denote homogeneous, linear, algebraic, constant-coefficient.

Consider a system of m holac equations in n variables

$$\sum_{j=1}^{n} a_{ij} x_j = 0 , \qquad i=1, 2, \ldots, m \qquad (2.1.1)$$

or, in matrix form

$$\mathcal{A}\mathcal{X} = 0 \qquad (2.1.2)$$

Definition 2.1.2. Rank of a System of Holac Equations

Let the rank of coefficient matrix, $\mathcal{A}$, in (2.1.2) be r, then r is said to be the rank of the system (2.1.2).

Definition 2. 1. 3. Characteristic of a System of Holac Equations

Let the rank of the system (2. 1. 2) be r, then the ordered triplet of integers, $\{m, n, r\}$ is said to be the characteristic of the system (2. 1. 2) and is written

$$\mathcal{A}\mathcal{X} = 0, \quad \{m, n, r,\} \quad .$$

Definition 2. 1. 4. Linearly Dependent (Independent) Equations

The holac equations, $\mathcal{A}\mathcal{X} = 0$, are said to be linearly dependent (independent) if and only if the rows of the coefficient matrix, $\mathcal{A}$, are linearly dependent (independent).

Definition 2. 1. 5. Equivalent Systems of Equations

Two systems of equations are said to be equivalent systems of equations if every solution of either system is also a solution of the other.

Consider a system of holac equations with characteristic $\{m, n, r\}$ i. e.

$$\mathcal{A}\mathcal{X} = 0, \quad \{m, n, r\} \tag{2. 1. 3}$$

Theorem 2. 1. 1. A necessary and sufficient condition that (2. 1. 3) have non-trivial solutions is that $r < n$.

Theorem 2.1.2. The equations of the system (2.1.3) are linearly dependent if and only if $r < m$.

Theorem 2.1.3. Every system of holac equations equivalent under elementary row operations to the system (2.1.3) may be represented in the form

$$(\mathcal{C}\,\mathcal{A})\mathcal{X} = 0 \qquad (2.1.4)$$

where $\mathcal{C}$ is a non-singular matrix of order m. Conversely, if $\mathcal{C}$ is any non-singular matrix of order m, then (2.1.4) is equivalent to (2.1.3) under elementary row operations.

Theorem 2.1.4. Any subset of r linearly independent equations from (2.1.3) forms an equivalent system of equations.

It should be noted that the characteristic of a system of holac equations is, in essence, a description of the coefficient matrix, i.e., $\mathcal{A}$ is of order mxn and has rank r. Thus the previously stated properties of a system of holac equations are in fact properties of the coefficient matrix-- the variables in the system play a rather minor role. In general the variables have little, if any, significance with regard to the mathematical properties of a system of equations; they serve as little more than "labels" associated with the columns of the coefficient matrix.

2.2. The Complete Solution for a System of Holac Equations

Theorem 2.1.1 states a necessary and sufficient condition on the coefficient matrix such that (2.1.3) possesses non-trivial solutions. Although the theorem itself gives no indication of what these solutions might be, the proofs of this theorem generally exhibit a complete solution of (2.1.3). It is a well-known property of a system of holac equations with characteristic $\{m, n, r\}$ that the complete solution can be obtained by solving the system of equations for some set of r variables in terms of the remaining $n-r$ variables. A necessary and sufficient condition is that the r columns of the coefficient matrix corresponding to the first set of variables be linearly independent. Before stating this result formally it is convenient to introduce the following notation for a sub-matrix. Let $\mathcal{A} = [a_{ij}]$ be a matrix of order mxn. The sub-matrix of order pxq formed from the array of entries located at the intersections of rows $i_1, i_2, \ldots, i_p$ and columns $j_1, j_2, \ldots, j_q$, where $1 \leq i_1 < i_2 < \ldots < i_p \leq m$ and $1 \leq j_1 < j_2 < \ldots < j_q \leq n$ is denoted by the symbol

$$\mathcal{A}\begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_q \end{pmatrix}.$$

i. e.,

$$\mathcal{A}\begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_q \end{pmatrix} = \begin{bmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 i_q} \\ a_{i_2 i_1} & -a_{i_2 i_2} & \cdots & a_{i_2 j_q} \\ & \cdot \quad \cdot \quad \cdot \quad \cdot & & \\ a_{i_p j_1} & a_{i_p j_2} & \cdots & a_{i_p j_q} \end{bmatrix} \qquad (2.2.1)$$

In some instances the set of row indices and the set of column indices are each denoted by a single symbol, say R and C respectively, then the above notation may be shortened to $\mathcal{A}\binom{R}{C}$.

<u>Theorem 2. 2. 1.</u> Given a system of holac equations

$$\mathcal{A}\mathcal{X} = 0 \quad \{m, n, r\} \qquad (2.2.2)$$

Let $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$, $(1 \leq j_1 < j_2 < \ldots < j_r \leq n)$, be any subset of r variables from $\mathcal{X}$. If and only if the rank of $\mathcal{A}\begin{pmatrix} 1, & 2, \ldots, m \\ j_1, & j_2, \ldots, j_r \end{pmatrix}$ is r, then the complete solution for the system (2. 2. 2) can be obtained by solving the equations for $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ in terms of the remaining n-r variables.

Consider the system of holac equations (2. 2. 2). By Theorem 2. 1. 4, any subset of r linearly independent equations from (2. 2. 2) form an equivalent system of equations. Let any such subset of equations be

$$\mathcal{A}_1 \mathcal{X} = 0, \quad \{r, n, r\} \quad . \qquad (2.2.3)$$

If r = n, then $\mathcal{A}_1$ is non-singular and the trivial solution is the only solution. If r < n, then by Theorem 2. 1. 1, the system (2. 2. 3) possesses non-trivial solutions and thus the original system (2. 2. 2) has non-trivial solutions. Since $\mathcal{A}_1$ has rank r, then there exists at least one set of column indices $1 \leq j_1 < j_2 < \dots < j_r \leq n$ such that $\mathcal{A}_1\begin{pmatrix} 1, & 2, & \dots, & r \\ j_1, & j_2, & \dots, & j_r \end{pmatrix}$ has rank r. With no loss in generality suppose that $\mathcal{A}_1\begin{pmatrix} 1, 2, \dots, r \\ 1, 2, \dots, r \end{pmatrix}$ has rank r. Then (2. 2. 3) can be written in the form

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \end{bmatrix} \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = 0 \qquad (2.2.4)$$

where $\mathcal{A}_{11}$ is rxr and non-singular. Solving for $\mathcal{X}_1$

$$\mathcal{X}_1 = -\mathcal{A}_{11}^{-1}\mathcal{A}_{12}\mathcal{X}_2 \ . \qquad (2.2.5)$$

Therefore a complete solution for (2. 2. 3) and hence (2. 2. 2) is given by

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_{11}^{-1}\mathcal{A}_{12}\mathcal{X}_2 \\ \mathcal{X}_2 \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_{11}^{-1}\mathcal{A}_{12} \\ \mathcal{U} \end{bmatrix} \mathcal{X}_2 \qquad (2.2.6)$$

where $\mathcal{U}$ is the unit matrix of order n-r.

Since (2. 2. 6) represents a trivial extension of (2. 2. 5), it is common to refer to either as representing a complete solution for (2.2.2). A complete solution (2.2.6) of the system (2.2.2) defines an infinity of particular solutions -- one particular solution of (2.2.2) for each distinct choice for $\mathcal{X}_2$.

The relation (2.2.5) indicates that the r variables in $\mathcal{X}_1$ are dependent upon the n-r variables in $\mathcal{X}_2$. It is common practice to designate the variables in $\mathcal{X}_1$ as dependent variables and those in $\mathcal{X}_2$ as independent variables. Although this is a convenient description to use, it can at times be misleading since it may lead one to view this description as a property of the variables rather than as a property of the coefficient matrix. With the appropriate interpretation it does, however, provide a useful notation. The following sequence of definitions relate to the subsequent use of this terminology in conjunction with a system of holac equations such as (2.2.2).

<u>Definition 2.2.1.</u> p-set of $\mathcal{X}$

A subset of the variables in $\mathcal{X}$: x_{j_1}, x_{j_2}, ..., x_{j_p}, where $1 \leq j_1 < j_2 < \dots < j_p \leq n$, is said to be a p-set of $\mathcal{X}$.

<u>Definition 2.2.2.</u> p-set of Dependent Variables,

Complete Set of Dependent Variables

A p-set of $\mathcal{X}$ is said to be a p-set of dependent variables for (2.2.2) if and only if the rank of $\mathcal{A}\begin{pmatrix} 1, & 2, & \ldots, & m \\ j_1, & j_2, & \ldots, & j_p \end{pmatrix}$ is p. If p = r (the rank of the system), the designation of a <u>complete</u> set of dependent variables for (2.2.2) is also used.

<u>Definition 2.2.3.</u> p-set of Independent Variables,

Complete Set of Independent Variables

A p-set of $\mathcal{X}$ is said to be a p-set of independent variables for (2.2.2) if and only if its complement in $\mathcal{X}$ contains a complete set of dependent variables for (2.2.2.) The complement in $\mathcal{X}$ of a complete set of dependent variables is also designated as the corresponding complete set of independent variables for (2.2.2).

Thus each set of r linearly independent columns of the coefficient matrix in (2.2.2) defines a partition of the n variables in $\mathcal{X}$ into two mutually exclusive, all inclusive sets -- an r-set of dependent variables and the corresponding (n-r)-set of independent variables. Since the coefficient matrix has rank r, then there exists at least one such partition, although in general there may be more. Except for the trivial case when $\mathcal{A}$ is the zero matrix (r = 0), a complete set of dependent variables always contains at least one variable; a complete set of independent variables is empty if and only if n = r.

Consider

$$\mathcal{A}\chi = 0,\ \{m, n, r\}\ ,\ 0 < r \leq n\ . \qquad (2.2.7)$$

the n variables in χ can be partitioned into an r-set and the corresponding (n-r)-set in $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ distinct ways. Any given r-set of χ may or may not be a complete set of dependent variables for (2.2.7). This fact results in the following designation:

Definition 2.2.4. Proper Partition

A partition of the n variables in (2.2.7) into an r-set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ and the corresponding (n-r)-set $\{x_{j_1}, x_{j_2}, \ldots, x_{j_{n-r}}\}$ is said to be a proper partition of the variables if and only if the rank of $\mathcal{A}\begin{pmatrix} 1, & 2, & \ldots, & m \\ i_1, & i_2, & \ldots, & i_r \end{pmatrix}$ is r; i.e., if and only if $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is a complete set of dependent variables for (2.2.7).

Each proper partition of the variables in a given system of holac equations can be used to determine a complete solution. Since any one complete solution is sufficient to generate all of the particular solutions for a system of equations, it is not necessary to determine more than one proper partition of the variables. Nevertheless, the form of the complete solution varies with the choice of proper partition and for a particular system of equations one choice may be more desirable than another.

Given a system of holac equations, such as (2. 2. 7), and one or more partitions of the variables into an r-set and the corresponding (n-r)-set the question of whether or not any particular partition is a proper partition of the variables can be answered by checking the rank of the appropriate sub-matrix of the coefficient matrix. All of the possible proper partitions could be determined by locating all mxr sub-matrices which have rank r. Unfortunately the number of sub-matrices to be tested can become large for moderate sized systems of equations. For example, to determine all of the proper partitions of the variables for a system having a characteristic of $\{10, 20, 10\}$ requires checking the rank of $\binom{20}{10} = 184,756$ sub-matrices of order 10x10. This is a formidable task and would require considerable time, even on a high-speed digital computer.

It is possible, however, to use the complete solution derived from one proper partition of the variables to readily obtain information concerning other partitions. Consider

$$\mathcal{A}\mathcal{X} = 0, \quad \{r, n, r\} \quad . \qquad (2.2.8)$$

Suppose that $\det \mathcal{A}\binom{1, 2, \ldots, r}{1, 2, \ldots, r} \neq 0$, then $\{x_1, x_2, \ldots, x_r\}$ is a complete set of dependent variables for (2.2.8). Solving for the r dependent variables in terms of the $n-r$ independent variables to obtain a complete solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_r \end{bmatrix} = \frac{-1}{\det \mathcal{A}\binom{1, 2, \ldots, r}{1, 2, \ldots, r}} \begin{bmatrix} A_{11} & \cdots & A_{r1} \\ A_{12} & \cdots & A_{r2} \\ & & \\ A_{1r} & \cdots & A_{rr} \end{bmatrix} \begin{bmatrix} a_{1, r+1} & \cdots & a_{1n} \\ a_{2, r+1} & \cdots & a_{2n} \\ & & \\ a_{r, r+1} & \cdots & a_{rn} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad (2.2.9)$$

where A_{ij} is the cofactor of a_{ij} in $\det \mathcal{A}\binom{1, 2, \ldots, r}{1, 2, \ldots, r}$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_r \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1, n-r} \\ b_{21} & b_{22} & \cdots & b_{2, n-r} \\ & & & \\ b_{r1} & b_{r2} & \cdots & b_{r, n-r} \end{bmatrix} \begin{bmatrix} x_{r+1} \\ x_{r+2} \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad (2.2.10)$$

where

$$b_{ij} = \frac{\sum_{k=1}^{r} a_{k, r+j} A_{ki}}{-\det \mathcal{A}\binom{1, 2, \ldots, r}{1, 2, \ldots, r}}, \quad \left(\begin{matrix} i=1, 2, \ldots, r \\ j=1, 2, \ldots, n-r \end{matrix}\right). \qquad (2.2.11)$$

The numerator of b_{ij} in (2.2.11) is the determinant of a sub-matrix similar to that appearing in the denominator except that column r+j of $\mathcal{A}$ has replaced column i. Thus $b_{ij} \neq 0$ (=0) if and only if the altered sub-matrix has rank = r (<r). Therefore, the r-set $\{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, x_{r+j}\}$ is a complete set of dependent variables for (2. 2. 8) if and only if $b_{ij} \neq 0$. This result can be generalized to the following:

<u>Theorem 2. 2. 2.</u> Given

$$\mathcal{A}\mathcal{X} = 0 , \quad \{ m, n, r \} \quad .$$

Let $\{x_{i1}, x_{i2}, \ldots, x_{ir}\}$ be any complete set of dependent variables for this system of equations, $\{x_{j1}, x_{j2}, \ldots, x_{jn-r}\}$ be the corresponding complete set of independent variables, and the corresponding complete solution be

$$\begin{bmatrix} x_{i1} \\ x_{i2} \\ \cdot \\ \cdot \\ \cdot \\ x_{ir} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1,n-r} \\ b_{21} & b_{22} & \cdots & b_{2,n-r} \\ & & & \\ b_{r1} & b_{r2} & \cdots & b_{r,n-r} \end{bmatrix} \begin{bmatrix} x_{j1} \\ x_{j2} \\ \cdot \\ \cdot \\ \cdot \\ x_{jn-r} \end{bmatrix} . \qquad (2.2.12)$$

The partition of the variables which results from the interchange of x_{is} and x_{jt}, (s=1, 2, . . . , r; t=1, 2, . . . , n-r), is a proper partition if and only if $b_{st} \neq 0$.

Consequently, a single complete solution such as (2.2.12) can be utilized to determine, by inspection, whether or not an additional r(n-r) partitions are proper partitions of the variables of a system of holac equations. Although Theorem 2.2.2 considers only a "singular" interchange of variables, i.e., one variable from each set, it can be used in an iterative manner to test any partition. Once a proper interchange has been found then the complete solution corresponding to the new proper partition is readily determined from the original complete solution by interchanging the two appropriate columns (with due regard to the signs of the entries) and then performing a sequence of at most r elementary row operations. A second proper interchange can be determined and the above process repeated. In this manner it is possible to determine all possible proper partitions of the variables, if desired.

2.3. The Complete Solution -- Another Viewpoint

In the discussion of Section 2.2 certain rank properties of the coefficient matrix in a system of holac equations with characteristic $\{m, n, r\}$ are used to define a proper partition of the variables. A complete solution is then obtained by solving the system for the r dependent variables in terms of the n-r independent variables. Other approaches to the problem of determining a complete solution are possible and the following discussion considers one such approach which occurs frequently in certain types of applied problems.

First, consider the sequence of steps used in Section 2.2 for determining a complete solution for the following system of holac equations:

$$\mathcal{A}\mathcal{X} = 0 , \quad \{m, n, r\} \tag{2.3.1}$$

1. An equivalent system of equations consisting of any subset of r linearly independent equations is extracted from (2.3.1):

$$\mathcal{A}_1 \mathcal{X} = 0 , \quad \{r, n, r\} \quad . \tag{2.3.2}$$

2. Let the first r column of $\mathcal{A}_1$ be linearly independent, then $\mathcal{A}_{11} = \mathcal{A}_1\binom{1, 2, \ldots, r}{1, 2, \ldots, r}$ is non-singular and a complete solution for (2.3.2) and (2.3.1) is given by

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_{11}^{-1} \mathcal{A}_{12} \\ \mathcal{U} \end{bmatrix} \mathcal{X}_2 \tag{2.3.3}$$

Since the n-r variables in $\mathcal{X}_2$ can be arbitrarily chosen let the following n-r sets of values be successively assigned to $\mathcal{X}_2$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ldots\ldots, \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

and the corresponding particular solutions determined from (2.2.3). Thus each of the n-r columns of

$$\begin{bmatrix} -\mathcal{A}_{11}^{-1}\mathcal{A}_{12} \\ \mathcal{U} \end{bmatrix}$$

is a particular solution of (2.3.1); furthermore, these n-r solutions are linearly independent. The complete solution as given in (2.3.3) may then be interpreted as any linear combination of these n-r linearly independent solutions. In general:

<u>Theorem 2.3.1.</u> If a system of holac equations has characteristic $\{m, n, r\}$, then every solution may be expressed as a linear combination of <u>any</u> n-r linearly independent solutions.

<u>Definition 2.3.1.</u> Fundamental System of Solutions, Fundamental Matrix of Solutions

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-r}$ be any n-r linearly independent solutions of (2.3.1), then $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-r}\}$ is called a fundamental system of solutions for (2.3.1) and the n x (n-r) matrix $\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-r}]$ determined by these solutions is called a fundamental matrix of solutions.

Consequently, if $\mathcal{B}$ is any fundamental matrix of solutions for the system $\mathcal{A}\mathcal{X} = 0$, then the columns of $\mathcal{B}$ are linearly independent and <u>every</u> linear combination of the columns of $\mathcal{B}$ is also a solution. From Theorem 2.3.1 it follows that a complete solution for (2.3.1) can be written in the form

$$\mathcal{X} = \mathcal{B}\mathcal{Y} \qquad (2.3.4)$$

where $\mathcal{B}$ is any fundamental matrix of solutions and $\mathcal{Y}$ is a column matrix consisting of n-r arbitrary entries. Frequently the entries in $\mathcal{Y}$ are considered as a new set of variables and (2.3.4) is subsequently considered as defining a transformation of variables. The use of the transformation of variables (2.3.4) in conjunction with the system (2.3.1) results in replacing the n variables in $\mathcal{X}$ by the n-r variables in $\mathcal{Y}$ and also reduces (2.3.1) to the matrix identity 0=0.

The interrelationship between the coefficient matrix in a system of holac equations and a fundamental matrix of solutions for that system is a characteristic of a larger class of matrices defined according to:

<u>Definition 2. 3. 2.</u> Apolar Pair [28]

Let $\mathcal{A}, \mathcal{B}$ have orders mxn and nxp , respectively. An ordered pair $(\mathcal{A}, \mathcal{B})$ of matrices is said to be an apolar pair if and only if (1) $\mathcal{A}\mathcal{B} = 0$, and (2) rank of $\mathcal{A}$ plus rank of $\mathcal{B} = n$.

Using this concept the following result can be stated:

<u>Theorem 2. 3. 2.</u> Given $\mathcal{A}\mathcal{X} = 0$, $\{m, n, r\}$ and let $\mathcal{B}$ be a matrix of order $n \times (n-r)$. Then $\mathcal{B}$ is a fundamental matrix of solutions for the given system if and only if $(\mathcal{A}, \mathcal{B})$ is an apolar pair.

The existence of a suitable $\mathcal{B}$ is assured by the following[28]:

<u>Theorem 2. 3. 3.</u> Given any matrix $\mathcal{A}$ there exists another matrix $\mathcal{B}$ such that $(\mathcal{A}, \mathcal{B})$ is an apolar pair.

For a given coefficient matrix $\mathcal{A}$ the process of constructing a matrix $\mathcal{B}$ such that $(\mathcal{A}, \mathcal{B})$ is an apolar pair closely parallels the process of the solving the system of equations themselves and hence is not considered here. However, for certain systems of holac equations, some of which will be considered in later sections, it is possible to determine the matrix $\mathcal{B}$ independently of the matrix $\mathcal{A}$

2.4. Interrelating the Two Viewpoints

Two approaches for obtaining a complete solution to a system of holac equations were considered in Sections 2.2 and 2.3. Since both techniques produce a complete solution for the same system of holac equations then they must be interrelated.

Consider the system of holac equations (2.3.1) and a complete solution given by (2.3.3). The particular solution, $\mathcal{X}_0$, corresponding to any arbitrary choice, $\mathcal{X}_{20}$, for $\mathcal{X}_2$ is, from (2.3.3):

$$\mathcal{X}_0 = \begin{bmatrix} \mathcal{X}_{10} \\ \mathcal{X}_{20} \end{bmatrix} = \begin{bmatrix} -\mathcal{A}_{11}^{-1} \mathcal{A}_{12} \\ \mathcal{U} \end{bmatrix} \mathcal{X}_{20} \tag{2.4.1}$$

Also, let $\mathcal{B}$ be any fundamental matrix of solutions for (2.3.1). Since a complete solution is given by (2.3.4), then there must be at least one set of values $\mathcal{Y}_0$, such that

$$\mathcal{X}_0 = \mathcal{B}\mathcal{Y}_0 \tag{2.4.2}$$

or

$$\begin{bmatrix} \mathcal{X}_{10} \\ \mathcal{X}_{20} \end{bmatrix} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} \mathcal{Y}_0 \tag{2.4.3}$$

From (2.4.1) and (2.4.3):

$$\mathcal{X}_{20} = \mathcal{B}_2 \mathcal{Y}_0 \qquad (2.4.4)$$

where $\mathcal{B}_2$ is a square matrix of order n-r. If $\mathcal{X}_{20} = 0$, then from (2.4.1) it follows that $\mathcal{X}_0 = 0$, and (2.4.2) yields $\mathcal{B}\mathcal{Y}_0 = 0$. Consequently $\mathcal{Y}_0 = 0$ since the columns of $\mathcal{B}$ are linearly independent. Therefore:

$$\mathcal{X}_{20} = 0 \quad \text{implies} \quad \mathcal{Y}_0 = 0 \qquad (2.4.5)$$

On the other hand, from (2.4.4) with $\mathcal{X}_{20} = 0$, one obtains a system of holac equations

$$\mathcal{B}_2 \mathcal{Y}_0 = 0, \qquad \{n-r,\ n-r,\ \rho\} \qquad (2.4.6)$$

where ρ is the rank of $\mathcal{B}_2$.

Suppose $\rho < n-r$. Then the system (2.4.6) has an infinity of non-trivial solutions for $\mathcal{Y}_0$, which contradicts (2.4.5). Since ρ cannot exceed n-r, it follows that $\rho = n-r$ and thus $\mathcal{B}_2$ is non-singular. Hence for the same particular solution, $\mathcal{X}_0$,

$$\mathcal{Y}_0 = \mathcal{B}_2^{-1} \mathcal{X}_{20} \qquad (2.4.7)$$

and there exists a one-to-one correspondence between $\mathcal{X}_{20}$ and $\mathcal{Y}_0$.

The preceeding argument also establishes a relationship between certain sub-matrices of $\mathcal{A}$ and $\mathcal{B}$.

<u>Theorem 2.4.1.</u> Given the system of holac equations

$$\mathcal{A}\,\Psi = 0\,, \qquad \{m, n, r\}$$

and any associated fundamental matrix of solutions, $\mathcal{B}$. Let $i_1, i_2, \ldots, i_r$ and $j_1, j_2, \ldots, j_{n-r}$ be sets of indices complementary with respect to the set of column indices $1, 2, \ldots, n$; i.e., $i_1, i_2, \ldots, i_r$ and $j_1, j_2, \ldots, j_{n-r}$ taken together form a complete set of indices $1, 2, \ldots, n$. Then $\mathcal{A}\binom{1, 2, \ldots, m}{i_1, i_2, \ldots, i_r}$ has rank r if and only if $\mathcal{B}\binom{j_1, j_2, \ldots, j_{n-r}}{1\ , 2\ , \ldots, n-r}$ has rank $n-r$.

Theorem 2.4.1 can be extended to the more general case of an apolar pair since if $(\mathcal{A}, \mathcal{B})$ form an apolar pair, then $\mathcal{B}$ of order nxp is rank equivalent to $[\mathcal{B}_1, O]$ where $\mathcal{B}_1$ is of order $n \times (n-r)$, has rank $n-r$, and by Theorem 2.3.2 is a fundamental matrix of solutions for the system of holac equations having $\mathcal{A}$ as a coefficient matrix. Thus:

<u>Theorem 2.4.2.</u> Let $(\mathcal{A}, \mathcal{B})$ be an apolar pair where $\mathcal{A}$ is of order mxn and rank r; $\mathcal{B}$ is a matrix of order nxp. Let $i_1, i_2, \ldots, i_r$ and $j_1, j_2, \ldots, j_{n-r}$ be sets of indices complementary with respect to the column indices $1, 2, \ldots, n$.

Then:

$$\mathcal{A}\begin{pmatrix}1\ ,2\ ,\ldots,m\\ i_1,i_2,\ldots,i_r\end{pmatrix}\quad \text{has rank } r \text{ if and only if}$$

$$\mathcal{B}\begin{pmatrix}j_1,j_2,\ldots,j_{n-r}\\ 1\ ,2\ ,\ldots,p\end{pmatrix}\quad \text{has rank } n\text{-}r\ .$$

Chapter 3

GENERAL PROPERTIES OF SYSTEMS OF NETWORK EQUATIONS

3.1. Introduction

Network equations are formulated in a number of different forms depending upon the type of solution that is sought. Formulation in the time-domain is most general and results in a system of ordinary differential and algebraic equations. However, for a large class of network problems, experience has shown that it is possible, and in fact, more convenient, to by-pass time-domain formulation in favor of frequency-domain or ω-domain formulation. With this approach the network equations are wholly algebraic in form and involve complex numbers. The solutions, in terms of complex numbers, are correlated with time-domain solutions as well as observations on the physical systems. The complex number form of the network equations, resulting from formulation in the ω-domain, is used as the basis for the investigation of this paper. Thus properties of a system of linear equations over the complex field play a fundamental role in this study.

It is possible to approach the general problem of assigning specific values to variables within a system of linear equations from two points of view. In one approach the equations are considered in a homogeneous form and rank properties of the coefficient matrix are used to define a partition of the variables into a dependent set and an independent set, i. e., a proper partition. Once a proper partition has been determined, then the variables within the independent set are assigned arbitrary values with no danger of introducing inconsistencies. As noted in the preceding chapter, the variables themselves do not enter into the partitioning process; in fact, it is possible to obtain a complete solution from the coefficient matrix alone.

The other approach is, in some respects, the reverse of the above process. As an initial step one can assign values to a subset of the variables and subsequently examine a system of non-homogeneous equations to determine whether or not it is consistent. If it is, then it is possible to determine a complete solution; it not, then the specification pattern is altered and the process repeated. With this approach, the number of variables assigned specific values is, to a certain extent, flexible; however, great care must be exercised in the choice of both the variables which are specified and the specific values which are assigned. The consistency conditions depend not only upon properties of the coefficient matrix, but also upon the specific values assigned to the variables. Insofar as this investigation is concerned, the first approach possesses a definite advantage over the second in that the nature of the process is such that it removes any doubt about consistency -- one is always assured of a solution. Because of this, it is possible to make definite decisions with regard to the question of which variables can be specified, based solely upon properties of the coefficient matrix. These decisions are not influenced or affected by any specific set of values which might have been assigned to a particular subset of the variables.

While systems of holac equations play a fundamental role in electric network theory, the theory of graphs[29, 30] also occupies a position of equal importance. The approach to network theory based upon a study of the correlating oriented linear graph[10, 11, 16, 31-35] has done much to add insight and precision into the formulation and solution of problems within this area. The mapping of significant subgraphs into matrices, which subsequently appear within the coefficient matrices of the network equations, allows one to interrelate properties of these subgraphs with associated properties of the equations and their solutions. In the present chapter consideration is given to a fundamental system of network equations and some of its general properties; the following chapter is devoted to examining, in more detail, the relationships between certain subgraphs and rank properties of the coefficient matrix.

Given a particular network for analysis, it is assumed that a correlating oriented linear graph (herafter referred to as a graph) has been established. In addition, two complex variables are associated with each element of the graph--a voltage variable, V, and a current variable, I. An extensive background of foundation material is assumed[10, 11]; the terminology and notation used here is essentially that found in these references.

3.2. Characterization of the Graph Elements

In general the graph elements used in electric network theory are classified as either relation elements (F-elements), or no-relation elements (N-elements), depending upon the presence or absence, respectively, of certain fixed mathematical equations relating the primary variables, V and I, associated with each element. Formally[16]:

Definition 3.2.1. F-element

An F-element is a graph element characterized by some fixed mathematical relation which relates V or I for that element to V or I of the same element or any other element of the graph. The mathematical relations are characteristics of the elements themselves and are independent of the manner in which the F-elements are imbedded in the graph. The corresponding equations are called F-equations.

Commonly encountered examples of F-elements are the graph element correlates of resistors, inductors, capacitors, transformers, and the like.

Definition 3. 2. 2. N-element

An N-element is a graph element characterized by: (1) either, neither, or both V and I for the element are arbitrarily specified; and (2) V and I for that element are not related in any fixed manner by an equation which is characteristic of the element iself.

N-elements are further classified into four different types depending upon the pattern of element variable specifications. Thus:

Definition 3. 2. 3. N_e-element[10]

An N-element for which the element variable V is arbitrarily specified is designated as an N_e-element. The specified variable V is designated by E, and the unspecified variable I is determined by the graph incident to this N_e-element.

<u>Definition 3.2.4.</u> N_h-element[10]

An N-element for which the element variable I is <u>arbitrarily</u> specified is designated as N_h-element. The specified variable I is designated by H, and the unspecified variable V is determined by the graph incident to this N_h-element.

<u>Definition 3.2.5.</u> N_{eh}-element

An N-element for which both of the element variables, V and I, are <u>arbitrarily</u> specified is designated as an N_{eh}-element. The specified variables V and I are designated by E and H, respectively.

<u>Definition 3.2.6.</u> N_o-element

An N-element for which neither of the element variables is <u>arbitrarily</u> specified is designated as an N_o-element. Both of the element variables, V and I, are determined by the graph incident to this N_o-element.

Of these four types of N-elements, the N_e- and N_h-elements are "familiar" since they are the graph element correlates of regulated voltage and current sources, respectively.

Applications of N_{eh}-elements were considered briefly in Section 1.3. In spite of its apparent lack of any distinguishing characteristics (other than this very lack itself), the N_o-element has considerable utility in network analysis. The complete dependence of both of its variables upon the incident graph makes the N_o-element useful for determining certain characteristics of the incident graph per se. For example, in the following pages these elements are used as "test elements" to determine allowable locations, within a graph, for the other types of N-elements.

The form in which the F equations appear is often a deciding factor in choosing a solution technique for the network equations. For a set of n_F F-elements, the F-equations can appear in three basic form in the ω-domain:

(1) the non-explicit F-equations:

$$\sum_{k=1}^{n_F} (f_{1ik}\, I_k + f_{2ik}\, V_k) = 0\ ,\quad i=1, 2, \ldots, n_F \qquad (3.2.1)$$

or, in matrix form

$$\mathcal{F}_1 \mathcal{I}_F + \mathcal{F}_2 \mathcal{V}_F = 0 \qquad (3.2.2)$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are complex matrices of order $n_F \times n_F$, and $\mathcal{I}_F$, $\mathcal{V}_F$ are column matrices of the I_k and V_k, respectively.

(2) the I-explicit F-equations:

$$\mathcal{I}_F - \mathcal{Y}_F \mathcal{V}_F = 0 \tag{3.2.3}$$

where $\mathcal{Y}_F$ is a complex matrix of order $n_F x n_F$ and is called the F-element admittance matrix;

(3) the V-explicit F-equations:

$$\mathcal{V}_F - \mathcal{Z}_F \mathcal{I}_F = 0 \tag{3.2.4}$$

where $\mathcal{Z}_F$ is a complex matrix of order $n_F x n_F$ and is called the F-element impedance matrix.

Although the non-explicit form in (3.2.2) is most general, the latter two (if they exist!) are generally more desirable forms. No matter which of the three forms is considered, the F-equations are a system of n_F holac equations in $2n_F$ variables.

3.3. Circuit and Seg Equations

The coefficient matrices for two fundamental systems of network equations -- Kirchhoff's voltage and current equations -- are established by mapping certain classes of subgraphs into matrices. As a result of mapping <u>all</u> circuits of a graph G into a matrix one obtains a <u>complete</u> circuit matrix for G; the associated system of holac equations is called a complete system of circuit equations for G:

$$\mathcal{B}_a \mathcal{V}_e = 0 \qquad (3.3.1)$$

where $\mathcal{B}_a$ is a complete circuit matrix for G, and $\mathcal{V}_e$ is a column matrix of the element V's.

Mapping all segs of G into a matrix results in a complete seg matrix for G; the associated system of holac equations -- a generalization of Kirchhoff's current equations -- is called a complete system of seg equations:

$$\mathcal{S}_a \mathcal{I}_e = 0 \qquad (3.3.2)$$

where $\mathcal{S}_a$ is a complete seg matrix for G, and $\mathcal{I}_e$ is a column matrix of the element I's.

The rank of $\mathcal{B}_a$ is e-v+1, and that of $\mathcal{S}_a$ is v-1, for an e-element, v-vertex connected graph G. Thus, from Theorem 2.1.4: any subset of e-v+1 linearly independent equations from a complete system of circuit equations constitutes an equivalent system of equations; any subset of v-1 linearly independent equations from a complete system of seg equations constitutes an equivalent system of equations. Hence it is neither necessary nor desirable to consider a complete system of circuit or seg equations in order to study their properties. Rather, one need consider only a basis system of circuit equations:

$$\mathcal{B} \mathcal{V}_e = 0, \quad \{e\text{-}v\text{+}1,\ e,\ e\text{-}v\text{+}1\} \qquad (3.3.3)$$

and, a basis system of seg equations:

$$\mathcal{S}\mathcal{Q}_e = 0, \qquad \{v-1,\ e,\ v-1\} \tag{3.3.4}$$

where

Definition 3.3.1. Basis circuit matrix $\mathcal{B}$

Any (e-v+1)-row, e-column, (e-v+1)-rank sub-matrix of a complete circuit matrix $\mathcal{B}_a$ is designated as a basis circuit matrix $\mathcal{B}$;

Definition 3.3.2. Basis seg matrix $\mathcal{S}$

Any (v-1)-row, e-column, (v-1)-rank sub-matrix of a complete seg matrix $\mathcal{S}_a$ is designated as a basis seg matrix $\mathcal{S}$.

In Section 2.2 rank properties of the coefficient matrix in a system of holac equations are used to define a proper partition of the variables. This proper partition is subsequently used to determine a complete solution for the system. The following Theorems provide a foundation on which to interrelate certain interconnection patterns of the elements of a graph and certain rank properties of the coefficient matrices which appear in basis systems of circuit and seg equations:

Theorem 3.3.1. Let G_s be an m-element subgraph of a connected graph G, $1 \leq m \leq e-v+1$; and let $\mathcal{B} = [\mathcal{B}_1 \mathcal{B}_2]$ be any basis circuit matrix for G, where the columns of $\mathcal{B}_1$ correspond to the elements of G_s. Then the columns of $\mathcal{B}_1$ are linearly independent if and only if G_s contains no seg of G.

<u>Theorem 3. 3. 2.</u> A subgraph G_s, of a connected graph G, is a subgraph of some cotree if and only if G_s contains no seg of G.

<u>Theorem 3. 3. 3.</u> Let G_s be an m-element subgraph of a connected graph G, $1 \leq m \leq v-1$; and let $\mathcal{S} = [\mathcal{S}_1 \ \mathcal{S}_2]$ be any basis seg matrix for G, where the columns of $\mathcal{S}_1$ correspond to the elements of G_s. Then the columns of $\mathcal{S}_1$ are linearly independent if and only if G_s contains no circuits.

<u>Theorem 3. 3. 4.</u> A subgraph G_s, of a connected graph G, is a subgraph of some tree of G if and only if G_s contains no circuits.

<u>Theorem 3. 3. 5.</u> Let G be a connected graph. If G_1 and G_2 are any two subgraphs of G such that: i) G_1 and G_2 have no elements in common, ii) G_1 contains no circuits, and iii) G_2 contains no seg of G; then there

exists some tree T of G such that G_1 is a subgraph of T and G_2 is a subgraph of the complement, in G, of T.

Another approach to determining a complete solution for a system of holac equations is considered in Section 2. 3. The approach there is based upon the concept of a fundamental matrix of solutions. Again the graph and certain of its associated matrices provide an effective means to obtain a fundamental matrix of solutions. The following theorem provides the needed relationships:

Theorem 3. 3. 6. Let G be a connected graph, and let $\mathcal{B}_a$ and $\mathcal{S}_a$ be, respectively, a complete circuit matrix and a complete seg matrix for G. If the columns of $\mathcal{B}_a$ and $\mathcal{S}_a$ are ordered the same with respect to the elements of G, then

$$\mathcal{S}_a \mathcal{B}_a' = 0 \quad \text{and} \quad \mathcal{B}_a \mathcal{S}_a' = 0 . \qquad (3.3.5)$$

Since

$$\text{rank of } \mathcal{B}_a + \text{rank } \mathcal{S}_a = (e-v+1) + (v-1) = e \qquad (3.3.6)$$

then, from definition 2. 3. 2, one concludes that $(\mathcal{S}_a, \mathcal{B}_a')$ and $(\mathcal{B}_a, \mathcal{S}_a')$ are apolar pairs. Furthermore, if $\mathcal{S}_a$ and/or $\mathcal{B}_a$ are replaced by any basis seg matrix $\mathcal{S}$ and any basis circuit matrix $\mathcal{B}$, respectively, then similar conclusions follow. Thus: the transpose of any basis circuit matrix $\mathcal{B}$ is a fundamental matrix of solutions for both a complete system and any basis system of seg equations; the transpose of any basis seg matrix $\mathcal{S}$ is a fundamental matrix of solutions for both a complete system and any basis system of circuit equations. Therefore, from Theorem 2.3.2,

$$\mathcal{V}_e = \mathcal{S}' \mathcal{W}_1 , \tag{3.3.7}$$

where $\mathcal{S}$ is any basis seg matrix for G, and $\mathcal{W}_1$ is an arbitrary (v-1)-rowed column matrix,

is a complete solution for (3.3.1) and (3.3.3);

$$\mathcal{I}_e = \mathcal{B}' \mathcal{W}_2 , \tag{3.3.8}$$

where $\mathcal{B}$ is any basis circuit matrix for G, and $\mathcal{W}_2$ is an arbitrary (e-v+1)-rowed column matrix,

is a complete solution for (3.3.2) and (3.3.4).

3.4. Primary Systems of Equations

Collectively, the circuit, seg, and F-equations contain all the information relative to the primary variables, V and I, associated with the elements of a graph. Due to the fundamental nature of these equations it is useful to have a single designation for this collection of equations. Accordingly:

Definition 3. 4. 1. Primary system of equations

The combination of any basis system of seg equations, any basis system of circuit equations, and the F-equations associated with a graph is designated as a primary system of equations for the graph.

A primary system of equations is also classified as non-explicit, I-explicit, or V-explicit depending upon the form of the F-equations which appear within the system. Ultimately, it is a solution for a primary system of equations that is sought, although this does not imply that one must solve these equations as they stand. It is more often the case that certain secondary or derived systems of equations, such as the mesh, branch, and node equations, are utilized to obtain numerical solutions since they generally involve fewer simultaneous equations to solve. However, these secondary systems are established from the primary system as a base.

Consider a non-explicit primary system of equations for a connected graph G containing e elements, v vertices, and n N-elements:

$$
\begin{array}{c} \\ v-1 \\ e-v+1 \\ e-n \end{array}
\begin{array}{c}
\begin{array}{cccc} n & e-n & n & e-n \end{array} \\
\begin{bmatrix} \mathcal{S}_N & \mathcal{S}_F & 0 & 0 \\ 0 & 0 & \mathcal{B}_N & \mathcal{B}_F \\ 0 & \mathcal{F}_1 & 0 & \mathcal{F}_2 \end{bmatrix}
\end{array}
\begin{bmatrix} \mathcal{I}_N \\ \mathcal{I}_F \\ \mathcal{V}_N \\ \mathcal{V}_F \end{bmatrix} = 0 \qquad (3.4.1)
$$

where the N and F subscripts refer to the N-elements and the F-elements, respectively.

If, at the outset, the N-elements are all N_o-elements, then there are no specified variables in (3. 4. 1), and this is a system of 2e-n holac equations in 2e variables. Suppose that the rank of this system is also 2e-n, as is the case with the ω-domain graph correlates of many classes of electric networks. If n > 0, then (3. 4. 1) possesses non-trivial solutions and complete sets of independent variables consist of some subsets of n of the element variables. Thus, the number of variables which can be assigned arbitrary values is equal to the number of N-elements in the graph. Consideration of the various combinations of the element variables which could occur within a complete set of independent variables indicates that if V, I, or V and I for an N_o-element appear within this set then it is possible to replace the

N_o-element by an N_e-, N_h-, or N_{eh}-element, respectively, with no danger of introducing inconsistencies. In this sense the N_o-element is used as a "test element" to check whether or not certain patterns of N-elements can be present within a given graph. Furthermore, the number of independent variables is fixed at n regardless of the type of N-elements present; thus, if both variables for one N-element occur in an independnet set then both variables for some other N-element must occur in the dependent set. In a similar manner, if an F-element variable occurs in an independent set then both variables for some N-element must occur within the dependent set.

3.5. I-Explicit Primary Systems Associated with Power Networks

The present section is devoted to a consideration of the ω-domain graph correlates of electric power networks of the type discussed in Sections 1.1 and 1.2. These graphs contain three general types of elements; namely: elements correlating with generators, loads, and the components which comprise the transmission and distribution network. The elements associated with the generators are N-elements, while those associated with the transmission and distribution network are F-elements. Unfortunately the elements corresponding to the loads cannot, in general, be categorized in a clear-cut manner. One is inclined to consider these elements as F-elements and characterize them, in the ω-domain, by a complex admittance or impedance. However, more often than not, the load admittance or impedance is either not known or, at best, only partially specified at the outset of a problem. Consequently, these elements do not readily fit into the F-element classification. The specifications for the load elements are often stated in terms of specified variables such as real and reactive powers; thus they do not immediately fit into the N-element classification, which involves V and/or I specifications. However, based upon the discussion of Section 1.3, the N-element classification seems more appropriate as a general rule. If the load admittance is given, then, as shown next, it is possible to use either classification -- F-element or N-element.

Consider an I-explicit primary system for a connected graph of e elements and v vertices. Let the graph contain n N_o-elements, and further, suppose that the system of equations has characteristic $\{2e-n, 2e, 2e-n\}$:

$$
\begin{array}{c} \\ v-1 \\ e-v+1 \\ e-n \end{array}
\begin{array}{c}
\begin{array}{cccc} n & e-n & n & e-n \end{array} \\
\left[\begin{array}{cccc}
\mathcal{S}_N & \mathcal{S}_F & 0 & 0 \\
0 & 0 & \mathcal{B}_N & \mathcal{B}_F \\
0 & \mathcal{U}_F & 0 & -\mathcal{Y}_F
\end{array}\right]
\end{array}
\begin{bmatrix} \mathcal{I}_N \\ \mathcal{I}_F \\ \mathcal{V}_N \\ \mathcal{V}_F \end{bmatrix} = 0 \qquad (3.5.1)
$$

where $\mathcal{Y}_F$ is a non-singular diagonal matrix with complex entries.

This system of equations is presented in more detail below in order to consider the most general distribution of element variables within the dependent and independent sets of a proper partition:

$$
\begin{array}{c|ccccccc|cccc|ccc|}
 & n_1 & n_2 & n_3 & n_4 & n_{F1} & n_{F2} & n_{F3} & n_1 & n_2 & n_3 & n_4 & n_{F1} & n_{F2} & n_{F3} \\
v-1 & \mathcal{S}_{N1} & \mathcal{S}_{N2} & \mathcal{S}_{N3} & \mathcal{S}_{N4} & \mathcal{S}_{F1} & \mathcal{S}_{F2} & \mathcal{S}_{F3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
e-v+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{B}_{N1} & \mathcal{B}_{N2} & \mathcal{B}_{N3} & \mathcal{B}_{N4} & \mathcal{B}_{F1} & \mathcal{B}_{F2} & \mathcal{B}_{F3} \\
\hline
n_{F1} & 0 & 0 & 0 & 0 & \mathcal{U}_{F1} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{Y}_{F1} & 0 & 0 \\
n_{F2} & 0 & 0 & 0 & 0 & 0 & \mathcal{U}_{F2} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{Y}_{F2} & 0 \\
n_{F3} & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{U}_{F3} & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{Y}_{F3}
\end{array}
\begin{bmatrix} \mathcal{I}_{N1} \\ \mathcal{I}_{N2} \\ \mathcal{I}_{N3} \\ \mathcal{I}_{N4} \\ \mathcal{I}_{F1} \\ \mathcal{I}_{F2} \\ \mathcal{I}_{F3} \\ \mathcal{V}_{N1} \\ \mathcal{V}_{N2} \\ \mathcal{V}_{N3} \\ \mathcal{V}_{N4} \\ \mathcal{V}_{F1} \\ \mathcal{V}_{F2} \\ \mathcal{V}_{F3} \end{bmatrix} = 0
$$

where $n_1 + n_2 + n_3 + n_4 = n$, and

$n_{F1} + n_{F2} + n_{F3} = e-n$.

Rearranging the columns to indicate the desired distribution of element variables

$$\left[\begin{array}{cccccc|cc}
\mathcal{S}_{N1} & \mathcal{S}_{N2} & 0 & 0 & \mathcal{S}_{F1} & 0 & \mathcal{S}_{F2} & 0 \\
0 & 0 & \mathcal{B}_{N1} & \mathcal{B}_{N3} & 0 & \mathcal{B}_{F1} & 0 & \mathcal{B}_{F3} \\
0 & 0 & 0 & 0 & \mathcal{U}_{F1} & -\mathcal{Y}_{F1} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \mathcal{U}_{F2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{Y}_{F3}
\end{array}\right]
\left[\begin{array}{c}
\mathcal{I}_{N1} \\ \mathcal{I}_{N2} \\ \mathcal{V}_{N1} \\ \mathcal{V}_{N3} \\ \mathcal{I}_{F1} \\ \mathcal{V}_{F1} \\ \hline \mathcal{I}_{F2} \\ \mathcal{V}_{F3}
\end{array}\right]$$

$$= -\left[\begin{array}{cccccc}
\mathcal{S}_{N3} & \mathcal{S}_{N4} & \mathcal{S}_{F3} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathcal{B}_{N2} & \mathcal{B}_{N4} & \mathcal{B}_{F2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -\mathcal{Y}_{F2} \\
0 & 0 & \mathcal{U}_{F3} & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\mathcal{I}_{N3} \\ \mathcal{I}_{N4} \\ \mathcal{I}_{F3} \\ \mathcal{V}_{N2} \\ \mathcal{V}_{N4} \\ \mathcal{V}_{F2}
\end{array}\right]. \qquad (3.5.3)$$

If and only if the coefficient matrix on the left-hand side of (3.5.3) is non-singular, then the above partition of the variables is a proper partition into a complete dependent set

$$\{ \mathcal{I}_{N1}, \mathcal{V}_{N1}, \mathcal{I}_{N2}, \mathcal{V}_{N3}, \mathcal{I}_{F1}, \mathcal{V}_{F1}, \mathcal{I}_{F2}, \mathcal{V}_{F3} \} \quad (3.5.4)$$

and the corresponding complete independent set

$$\{ \mathcal{V}_{N2}, \mathcal{I}_{N3}, \mathcal{I}_{N4}, \mathcal{V}_{N4}, \mathcal{V}_{F2}, \mathcal{I}_{F3} \} . \quad (3.5.5)$$

Suppose that this is the case. Then it is necessary that

$$n_1 = n_4 + n_{F2} + n_{F3} \quad ; \quad (3.5.6)$$

That is, the number of N-elements having both variables in the dependent set is equal to the number of N-elements having both variables in the independent set plus the number of F-elements having one variable, either V or I, in the independent set. It is noted that it is not possible to have an F-element of the type considered here with both variables in the independent set. However, specification of either variable for this type of F-element immediately determines the other variable. Thus specifying one variable for an F-element has the same effect as specifying both variables -- only one of which is arbitrarily chosen. In fact an F-element having one variable in the independent set can also be handled as an N-element with both variables in the independent set -- provided that one assigns values to these variables in a prescribed manner. Consider the coefficient matrix on the left-hand side of (3.5.3.):

$$\left[\begin{array}{cccccc|cc} \mathcal{S}_{N1} & \mathcal{S}_{N2} & 0 & 0 & \mathcal{S}_{F1} & 0 & \mathcal{S}_{F2} & 0 \\ 0 & 0 & \mathcal{B}_{N1} & \mathcal{B}_{N3} & 0 & \mathcal{B}_{F1} & 0 & \mathcal{B}_{F3} \\ 0 & 0 & 0 & 0 & \mathcal{U}_{F1} & -\mathcal{Y}_{F1} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{U}_{F2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{Y}_{F3} \end{array}\right]$$

$$= \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ 0 & \mathcal{M}_{22} \end{bmatrix} = \mathcal{M} \qquad (3.5.7)$$

Since, by hypothesis, the matrix $\mathcal{M}$ is non-singular then $\mathcal{M}_{11}$ and $\mathcal{M}_{22}$ are non-singular. In fact, due to the nature of $\mathcal{M}_{22}$, one has that $\mathcal{M}$ is non-singular if and only if $\mathcal{M}_{11}$ is non-singular. Suppose that the last two sets of $n_{F2} + n_{F3}$ F-equations in (3.5.2) or (3.5.3) are deleted; however, the corresponding graph elements are not deleted -- so the seg and circuit matrices remain unaltered. The only change is that these elements are now classified as N-elements. Thus both the number of equations in the system and the rank of the system are diminished by $n_{F2} + n_{F3}$, but the number of variables remains unchanged. Consequently the number of variables in a complete dependent (independent) set is decreased (increased) by $n_{F2} + n_{F3}$. If this altered system of equations is now rearranged to include the n_{F2} variables in $\mathcal{I}_{F2}$ and the n_{F3} variables in $\mathcal{V}_{F3}$ in a new set of (potential) independent variables then one obtains (retaining the notation of (3.5.2)):

$$
\begin{array}{c} \\ v-1 \\ e-v+1 \\ n_{F1} \end{array}
\begin{array}{c}
\begin{array}{cccccc} n_1 & n_2 & n_1 & n_3 & n_{F1} & n_{F1} \end{array} \\
\left[\begin{array}{cccccc}
\mathcal{A}_{N1} & \mathcal{A}_{N2} & 0 & 0 & \mathcal{A}_{F1} & 0 \\
0 & 0 & \mathcal{B}_{N1} & \mathcal{B}_{N3} & 0 & \mathcal{B}_{F1} \\
0 & 0 & 0 & 0 & \mathcal{U}_{F1} & -\mathcal{Y}_{F1}
\end{array}\right]
\end{array}
\left[\begin{array}{c}
\mathcal{I}_{N1} \\ \mathcal{I}_{N2} \\ \mathcal{V}_{N1} \\ \mathcal{V}_{N3} \\ \mathcal{I}_{F1} \\ \mathcal{V}_{F1}
\end{array}\right]
$$

$$
= -
\begin{array}{c}
\begin{array}{cccccccc} n_3 & n_4 & n_{F2} & n_{F3} & n_2 & n_4 & n_{F2} & n_{F3} \end{array} \\
\left[\begin{array}{c:ccc:c:ccc}
\mathcal{A}_{N3} & \mathcal{A}_{N4} & \mathcal{A}_{F2} & \mathcal{A}_{F3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{B}_{N2} & \mathcal{B}_{N4} & \mathcal{B}_{F2} & \mathcal{B}_{F3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c}
\mathcal{I}_{N3} \\ \hdashline \mathcal{I}_{N4} \\ \mathcal{I}_{F2} \\ \mathcal{I}_{F3} \\ \hdashline \mathcal{V}_{N2} \\ \hdashline \mathcal{V}_{N4} \\ \mathcal{V}_{F2} \\ \mathcal{V}_{F3}
\end{array}\right]
\qquad (3.5.8)
$$

The coefficient matrix on the left-hand side of (3.5.8) is precisely $\mathcal{M}_{11}$ from (3.5.7) and thus has the proper order and rank if and only if the coefficient matrix $\mathcal{M}$ on the left-hand side of (3.5.3) has the proper order and rank.

Since, by hypothesis, $\mathcal{M}$ has the proper order and rank then

$$\{\mathcal{I}_{N1}, \mathcal{V}_{N1}, \mathcal{I}_{N2}, \mathcal{V}_{N3}, \mathcal{I}_{F1}, \mathcal{V}_{F1}\} \tag{3.5.9}$$

is a complete set of dependent variables for (3.5.8) and

$$\{\mathcal{V}_{N2}, \mathcal{I}_{N3}, \mathcal{I}_{N4}, \mathcal{V}_{N4}, \mathcal{I}_{F2}, \mathcal{V}_{F2}, \mathcal{I}_{F3}, \mathcal{V}_{F3}\} \tag{3.5.10}$$

is the corresponding complete set of independent variables.

Although the systems of equations (3.5.2) and (3.5.8) are neither identical nor equivalent systems of equations they are closely related. Every solution of (3.5.2) is also a solution of (3.5.8) but the converse is not true. However, every solution of (3.5.8) for which the variables $\mathcal{I}_{F2}$, $\mathcal{V}_{F2}$, $\mathcal{I}_{F3}$, $\mathcal{V}_{F3}$ are chosen so that

$$\mathcal{I}_{F2} = \mathcal{Y}_{F2}\,\mathcal{V}_{F2} \quad \text{and} \quad \mathcal{I}_{F3} = \mathcal{Y}_{F3}\,\mathcal{V}_{F3} \tag{3.5.11}$$

is also a solution of (3.5.2).

The transition from (3.5.2) to (3.5.8) was accomplished by deleting certain F-equations from the primary system of equations; in effect changing the character of the elements involved from F-elements to N-elements, but otherwise leaving the graph unaltered. It is then possible to assign values to the variables associated with these N-elements so as to maintain any desired relationship between them. The admittance coefficients for the "transformed" F-elements no longer appear within the system of equations; thus it is possible to effectively handle situations where these coefficients are not completely known, or are variable, without introducing unknown or changing quantities into the coefficient matrix.

A distinctive feature of a "per-phase" representation of a power network is the presence of a common node or "ground bus". In particular, within the graph correlate, the N- and F-elements corresponding to the generators and loads are incident to a common vertex--the "reference" vertex. With no loss in generality it is assumed that each non-reference vertex of the graph is incident to exactly one N-element and that all N-elements are also incident to the reference vertex. For the case of a non-generator, non-load vertex, or a load vertex for which the admittance coefficient of the incident load element is known, then the incident N-element can be specified as an N_h-element for which $H \equiv 0$, i.e., an open-circuit, or possibly an N_{eh}-element for which $H \equiv 0$ and E is an arbitrarily specified complex number. With this assumption the graphs under consideration consist of the union of a v-vertex connected graph of F-elements and a Lagrangian tree of v-1 N-elements---see figure 3.5.1.

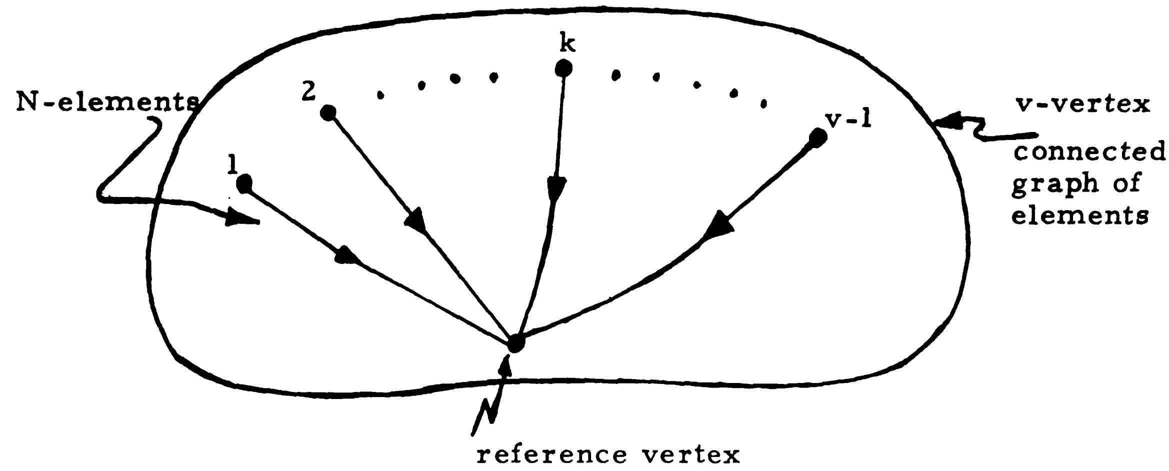


Figure 3.5.1. Graph Correlate of a Power Network.

In the formulation of a primary system of equations for the graph correlate of a network one has, in general, a variety of basis seg matrices from which to choose[10]; in the following sections an incidence-seg matrix $\mathcal{A}$ is used exclusively. There is no loss in generality in this choice, and one has an advantage in that a basis incidence-seg matrix is formed with relative ease from data which is readily available. Furthermore, this matrix is well-suited for formation by a digital computer.[15,16]

Consider an $\mathcal{I}$-explicit primary system of equations for the v-vertex graph of Figure 3.5.1--where $n=v-1$ is the number of N-elements, n_F is the number of F-elements, the F-element admittance matrix $\mathcal{Y}_F$ is a complex diagonal matrix, and the subscripts N, F refer to the N-element and F-element subgraphs respectively:

$$\begin{array}{c} \\ n \\ n_F \\ n_F \end{array}\begin{array}{c} \begin{array}{cccc} n & n_F & n & n_F \end{array} \\ \begin{bmatrix} \mathcal{U}_F & \mathcal{A}_F & 0 & 0 \\ 0 & 0 & \mathcal{B}_N & \mathcal{U}_F \\ 0 & \mathcal{U}_F & 0 & -\mathcal{Y}_F \end{bmatrix} \end{array} \begin{bmatrix} \mathcal{I}_N \\ \mathcal{I}_F \\ \mathcal{V}_N \\ \mathcal{V}_F \end{bmatrix} = 0, \; \{ 2n_F+n, \, 2(n_F+n), \, 2n_F+n \} \; . \tag{3.5.12}$$

From (3.3.5)

$$\mathcal{B}\mathcal{A}' = [\mathcal{B}_N \; \mathcal{U}_F][\mathcal{U}_N \; \mathcal{A}_F]' = \mathcal{B}_N + \mathcal{A}_F' = 0 \tag{3.5.13}$$

or

$$\mathcal{B}_N = -\mathcal{A}_F' \; . \tag{3.5.14}$$

Using the above relation to eliminate $\mathcal{B}_N$ from (3.5.12):

$$\begin{bmatrix} \mathcal{U}_N & \mathcal{A}_F & 0 & 0 \\ 0 & 0 & -\mathcal{A}_F' & \mathcal{U}_F \\ 0 & \mathcal{U}_F & 0 & -\mathcal{Y}_F \end{bmatrix} \begin{bmatrix} \mathcal{I}_N \\ \mathcal{I}_F \\ \mathcal{V}_N \\ \mathcal{V}_F \end{bmatrix} = 0, \; \{ 2n_F+n, \, 2(n_F+n), \, 2n_F+n \} \; . \tag{3.5.15}$$

Upon premultiplying the coefficient matrix in (3.5.15) by the following non-singular matrix

$$\begin{bmatrix} 0 & \mathcal{Y}_F & \mathcal{U}_F \\ 0 & \mathcal{U}_F & 0 \\ \mathcal{U}_N & -\mathcal{A}_F\mathcal{Y}_F & -\mathcal{A}_F \end{bmatrix} \tag{3.5.16}$$

one obtains, after a slight rearrangement of the variables, the following equivalent system of equations:

$$\begin{bmatrix} \mathcal{U}_F & 0 & 0 & -\mathcal{Y}_F \mathcal{A}'_F \\ 0 & \mathcal{U}_F & 0 & -\mathcal{A}'_F \\ 0 & 0 & \mathcal{U}_N & \mathcal{A}_F \mathcal{Y}_F \mathcal{A}'_F \end{bmatrix} \begin{bmatrix} \mathcal{I}_F \\ \mathcal{V}_F \\ \mathcal{I}_N \\ \mathcal{V}_N \end{bmatrix} = 0, \ \{ 2n_F+n, 2(n_F+n), 2n_F+n \} . \tag{3.5.17}$$

Thus every solution of (3.5.17) is a solution of (3.5.12) and conversely.

Consider now the last set of n equations in (3.5.17):

$$[\mathcal{U}_N \ \mathcal{Y}] \begin{bmatrix} \mathcal{I}_N \\ \mathcal{V}_N \end{bmatrix} = 0, \ \{ n, 2n, n \} \tag{3.5.18}$$

where
$$\mathcal{Y} = \mathcal{A}_F \mathcal{Y}_F \mathcal{A}'_F .$$

First, one notes that each solution of (3.5.17) certainly determines exactly one corresponding solution of (3.5.18); second, each solution of (3.5.18) determines exactly one corresponding solution of (3.5.17):

$$\begin{bmatrix} \mathcal{I}_F \\ \mathcal{V}_F \\ \mathcal{I}_N \\ \mathcal{V}_N \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{Y}_F \mathcal{A}'_F \\ 0 & \mathcal{A}'_F \\ \mathcal{U}_N & 0 \\ 0 & \mathcal{U}_N \end{bmatrix} \begin{bmatrix} \mathcal{I}_N \\ \mathcal{V}_N \end{bmatrix} . \tag{3.5.19}$$

The systems of equations (3.5.17) and (3.5.18) are, in a sense, equivalent systems of equations since each solution of either determines exactly one corresponding solution of the other. Since one of the primary concerns in this investigation is to study the properties of solutions of (3.5.12) in terms of the variables associated with the N-elements, then it is appropriate to examine the solutions of (3.5.18).

In order to consider the most general partition of the variables in (3.5.18) it is necessary to subdivide the set of N-elements into four subsets. Since each N-element is incident to exactly one non-reference vertex then the vertices of the graph are subdivided into five subsets--the reference vertex alone and four additional subsets, each corresponding to a distinct N-element subset. This necessitates a more detailed partitioning for the incidence-seg matrix:

$$\mathcal{A} = \begin{bmatrix} \mathcal{U}_{N1} & 0 & 0 & 0 & \mathcal{A}_{F1} \\ 0 & \mathcal{U}_{N2} & 0 & 0 & \mathcal{A}_{F2} \\ 0 & 0 & \mathcal{U}_{N3} & 0 & \mathcal{A}_{F3} \\ 0 & 0 & & \mathcal{U}_{N4} & \mathcal{A}_{F4} \end{bmatrix} \qquad (3.5.20)$$

and the corresponding changes in (3.5.18):

$$\begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} n_1 & n_2 & n_3 & n_4 & n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} \mathcal{U}_{N1} & 0 & 0 & 0 & \mathcal{Y}_{11} & \mathcal{Y}_{12} & \mathcal{Y}_{13} & \mathcal{Y}_{14} \\ 0 & \mathcal{U}_{N2} & 0 & 0 & \mathcal{Y}_{21} & \mathcal{Y}_{22} & \mathcal{Y}_{23} & \mathcal{Y}_{24} \\ 0 & 0 & \mathcal{U}_{N3} & 0 & \mathcal{Y}_{31} & \mathcal{Y}_{32} & \mathcal{Y}_{33} & \mathcal{Y}_{34} \\ 0 & 0 & 0 & \mathcal{U}_{N4} & \mathcal{Y}_{41} & \mathcal{Y}_{42} & \mathcal{Y}_{43} & \mathcal{Y}_{44} \end{bmatrix} \end{array} \begin{bmatrix} \mathcal{I}_{N1} \\ \mathcal{I}_{N2} \\ \mathcal{I}_{N_3} \\ \mathcal{I}_{N4} \\ \mathcal{V}_{N1} \\ \mathcal{V}_{N2} \\ \mathcal{V}_{N3} \\ \mathcal{V}_{N4} \end{bmatrix} = 0 \qquad (3.5.21)$$

where $n = v - 1 = \sum_{k=1}^{4} n_k$ and $n_k \geq 0$,

$$\mathcal{Y}_{ij} = \mathcal{A}_{Fi}\,\mathcal{Y}_F\,\mathcal{A}'_{Fj}\ , \quad i, j = 1, 2, 3, 4$$

Rearranging (3.5.21) to indicate the most general partition of the variables:

$$\begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array}\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_1 & n_3 \end{array} \\ \begin{bmatrix} \mathcal{U}_{N1} & 0 & \mathcal{Y}_{11} & \mathcal{Y}_{13} \\ 0 & \mathcal{U}_{N2} & \mathcal{Y}_{21} & \mathcal{Y}_{23} \\ 0 & 0 & \mathcal{Y}_{31} & \mathcal{Y}_{33} \\ 0 & 0 & \mathcal{Y}_{41} & \mathcal{Y}_{43} \end{bmatrix} \end{array} \begin{bmatrix} \mathcal{I}_{N1} \\ \mathcal{I}_{N2} \\ \mathcal{V}_{N1} \\ \mathcal{V}_{N3} \end{bmatrix} = - \begin{array}{c} \begin{array}{cccc} n_3 & n_4 & n_2 & n_4 \end{array} \\ \begin{bmatrix} 0 & 0 & \mathcal{Y}_{12} & \mathcal{Y}_{14} \\ 0 & 0 & \mathcal{Y}_{22} & \mathcal{Y}_{24} \\ \mathcal{U}_{N3} & 0 & \mathcal{Y}_{32} & \mathcal{Y}_{34} \\ 0 & \mathcal{U}_{N4} & \mathcal{Y}_{42} & \mathcal{Y}_{44} \end{bmatrix} \end{array} \begin{bmatrix} \mathcal{I}_{N3} \\ \mathcal{I}_{N4} \\ \mathcal{V}_{N2} \\ \mathcal{V}_{N4} \end{bmatrix} \tag{3.5.22}$$

From Definition 2.2.4 it follows that the above partition is a proper partition of the variables in (3.5.18) if and only if

$$n_1 = n_4 \tag{3.5.23}$$

and

$$\det \begin{bmatrix} \mathcal{Y}_{31} & \mathcal{Y}_{33} \\ \mathcal{Y}_{41} & \mathcal{Y}_{43} \end{bmatrix} \neq 0. \tag{3.5.24}$$

If this is the case, then a complete set of dependent variables consists of the n variables:

$$\{\mathcal{I}_{N1}, \mathcal{I}_{N2}, \mathcal{V}_{N1}, \mathcal{V}_{N3}\} \tag{3.5.25}$$

and the corresponding complete set of independent variables consists of the n variables:

$$\{ \mathscr{I}_{N3}, \mathscr{I}_{N4}, \mathscr{V}_{N2}, \mathscr{V}_{N4} \} . \qquad (3.5.26)$$

At this point the characterization of the N-elements can be specified with no danger of introducing inconsistencies; i.e., the graph can be considered to contain n_1 N_o-elements, n_2 N_e-elements, n_3 N_h-elements, and n_4 N_{eh}-elements.

As noted earlier, the condition $n_1 = n_4$ requires that for each N_{eh}-element in the graph there must also be a distinct N_o-element. Moreover, there is a stronger and more useful interrelation between the N_{eh}-elements and the N_o-elements in a graph. Since $\mathscr{Y}$ in (3.5.18) is a symmetric matrix, then in (3.5.21):

$$\mathscr{Y}_{ij} = \mathscr{Y}'_{ji} \qquad i, j = 1, 2, 3, 4. \qquad (3.5.27)$$

Hence,

$$\det \begin{bmatrix} \mathscr{Y}_{31} & \mathscr{Y}_{33} \\ \mathscr{Y}_{41} & \mathscr{Y}_{43} \end{bmatrix} = \det \begin{bmatrix} \mathscr{Y}_{13} & \mathscr{Y}_{14} \\ \mathscr{Y}_{33} & \mathscr{Y}_{34} \end{bmatrix} . \qquad (3.5.28)$$

However the necessary and sufficient conditions for

$$\{ \mathscr{I}_{N2}, \mathscr{I}_{N4}, \mathscr{V}_{N3}, \mathscr{V}_{N4} \} \qquad (3.5.29)$$

to be a complete set of dependent variables for (3.5.21) are that $n_1 = n_4$ and that the second determinant in (3.5.28) be non-zero. Therefore, the set of variables in (3.2.25) is a complete set of dependent variables for (3.5.21) if and only if the set of variables in (3.5.29) is also. Consequently, within a given allowable classification of the N-elements of a graph into the N_o-, N_e-, N_h-, and N_{eh}-classes, it is possible to interchange the N_o- and N_{eh}-classifications.

It is worthwhile to note once again that although the preceding discussion is concerned with the variables within a system of equations, it is the cofficient matrix which actually contains the necessary information and properties. In the next chapter these properties of the coefficient matrix are "reflected" back into corresponding properties of the graph.

Chapter 4

MAXIMUM TERM RANK SUBMATRICES OF $\mathcal{Y}$ AND CORRESPONDING F-ELEMENT SUBGRAPHS

4.1. Introduction

In the preceding chapters rank properties of the coefficient matrix are used to define a proper partition of the variables within a system of holac equations. In Section 3.5 this definition is applied to a general partition of the variables within a system of holac equations associated with the graph correlate of an electric network:

$$[\, \mathcal{U}_N \; \mathcal{Y}] \begin{bmatrix} \mathcal{I}_N \\ \mathcal{V}_N \end{bmatrix} = 0, \quad \{ n, 2n, n \} \tag{4.1.1}$$

where

$$\mathcal{Y} = \mathcal{A}_F \, \mathcal{Y}_F \, \mathcal{A}_F' \tag{4.1.2}$$

Of the resulting necessary and sufficient conditions for the partition of (3.5.22) to be a proper partition, the first, (3.5.23), insures that the appropriate submatrices have the proper orders, while the second condition, (3.5.24), requires that a certain minor from det $\mathcal{Y}$ does not vanish. In a particular case this minor can be evaluated and the corresponding partition checked. Although this procedure provides a definite test, it gives little insight into the reasons why a particular partition either passed or failed the test.

Now $\mathcal{Y}$, and hence all properties of $\mathcal{Y}$, are dependent solely upon the subgraph of F-elements and the associated F-element admittance coefficients; in (4.1.2), $\mathcal{A}_F$ can be construed as a basis incidence-seg matrix of order (v-1) by n_F for the connected v-vertex, n_F-element subgraph of F-elements, and $\mathcal{Y}_F$ is the diagonal matrix of the F-element admittance coefficients. The partitioning of the N-element variables, as in (3.5.22), identifies the corresponding minor from det $\mathcal{Y}$ which must be tested, but this is the extent of the effect of the N-elements in the partitioning and testing process. The following sections of this chapter are devoted to examining the composition of the matrix $\mathcal{Y}$ in (4.1.2) in order to establish interrelations between subgraphs of F-elements and non-vanishing minors of det $\mathcal{Y}$.

In general the expansion of a minor involves a summation of terms; thus a minor can vanish for one of two reasons--each term in the summation is zero, or the non-zero terms are such that their summation is zero. If the minor does not vanish then the summation of necessity contains non-vanishing terms. As shown later, the presence or absence of non-vanishing terms in the expansion of a minor of det $\mathcal{Y}$ is directly related to the existence or non-existence, respectively, of certain subgraphs of F-elements. In anticipation of this result the following terminology is introduced:[30]

Definition 4.1.1. Term Rank of a Matrix

The term rank of a matrix is the order of the greatest minor containing a non-vanishing term in its expansion.

The following theorems can be utilized to check the term rank of a matrix:[30]

Theorem 4.1.1. The expansion of the determinant of a square matrix $\mathcal{M}$ of order r contains non-vanishing terms if and only if any k rows of $\mathcal{M}$ include non-vanishing entries m_{ij} from at least k columns, $k = 1, 2, \ldots, r$.

Theorem 4.1.2. The term rank of a square matrix of order r is less than r if and only if there exists a zero submatrix of order m by n with $m + n > r$.

If a square matrix of order r has term rank r then its determinant may be non-zero for some sets of values for the entries or possibly for all sets of non-zero values for the entries. If the term rank is less than r, then the determinant vanishes for all sets of values for the entries.

Consider a minor of order r from det $\mathcal{Y}$. In either case--direct evaluation of the minor, or determination of the rank of the corresponding submatrix using Theorem 4.1.1 or Theorem 4.1.2 it is necessary to form the matrix $\mathcal{Y}$, or at least the appropriate submatrix, in order to test it; as the size and complexity of the graph increases the formation of the matrix itself becomes a significant problem. Furthermore this process provides no explicit information about the properties of the graph itself in relation to the partitioning problem although the graph of F-elements determines the properties of $\mathcal{Y}$.

4.2. The Binet-Cauchy Formula

A generalization of the Binet-Cauchy formula[25] provides a technique to establish an interrelation between subgraphs of F-elements and non-vanishing minors of det $\mathcal{Y}$:

Let $\mathcal{A}, \mathcal{B}$ be matrices of order r by n and n by q respectively, and let $\mathcal{C} = \mathcal{A}\mathcal{B}$. Consider an arbitrary minor of order m from det $\mathcal{C}$:

$$\det \mathcal{C} \begin{pmatrix} i_1, & i_2, & \ldots, & i_m \\ j_1, & j_2, & \ldots, & j_m \end{pmatrix} \tag{4.2.1}$$

where

$$m \leq \min(r, q)$$

$$1 \leq i_1 < i_2 < \ldots < i_m \leq r$$

$$1 \leq j_1 < j_2 < \ldots < j_m \leq q .$$

Then, if $1 \leq m \leq n$:

$$\det \mathcal{C} \begin{pmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{pmatrix} = \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n} \det \mathcal{A} \begin{pmatrix} i_1, i_2, \ldots, i_m \\ k_1, k_2, \ldots, k_m \end{pmatrix} \det \mathcal{B} \begin{pmatrix} k_1, k_2, \ldots, k_m \\ j_1, j_2, \ldots, j_m \end{pmatrix} \tag{4.2.2}$$

or, if $m > n$:

$$\det \mathcal{C} \begin{pmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{pmatrix} = 0 . \tag{4.2.3}$$

Now, let

$$\mathcal{D} = \mathcal{Y}_F \mathcal{A}_F' \qquad (4.2.4)$$

and consider any minor of order m, $1 \leq m \leq v-1$, from $\det \mathcal{Y}$ where $\mathcal{Y}$ is given in (4.1.2). Applying the Binet-Cauchey formula, (4.2.2), twice:

$$\det \mathcal{Y}\begin{bmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} = \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n_F} \det \mathcal{A}_F \begin{bmatrix} i_1, i_2, \ldots, i_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} \det \mathcal{D}\begin{bmatrix} k_1, k_2, \ldots, k_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} \qquad (4.2.5)$$

and

$$\det \mathcal{D}\begin{bmatrix} k_1, k_2, \ldots, k_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} = \sum_{1 \leq \ell_1 < \ell_2 < \ldots < \ell_m \leq n_F} \det \mathcal{Y}_F \begin{bmatrix} k_1, k_2, \ldots, k_m \\ \ell_1, \ell_2, \ldots, \ell_m \end{bmatrix} \det \mathcal{A}_F' \begin{bmatrix} \ell_1, \ell_2, \ldots, \ell_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} . \qquad (4.2.6)$$

Since $\mathcal{Y}_F$ is a nonsingular, diagonal matrix then the only non-vanishing minors are principal minors; thus the summation in (4.2.6) contains at most one non-vanishing term--corresponding to the case when $\ell_1 = k_1, \ell_2 = k_2, \ldots, \ell_m = k_m$. Further

$$\det \mathcal{Y}_F \begin{bmatrix} k_1, k_2, \ldots, k_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} = Y_{k_1} Y_{k_2} \cdots Y_{k_m} \qquad (4.2.7)$$

and

$$\det \mathcal{A}'_F \begin{bmatrix} k_1, k_2, \ldots, k_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} = \det \mathcal{A}_F \begin{bmatrix} j_1, j_2, \ldots, j_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} . \tag{4.2.8}$$

Combining the last four relations:

$$\det \mathcal{Y} \begin{bmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} = \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n_F} Y_{k_1} Y_{k_2} \ldots Y_{k_m} \det \mathcal{A}_F \begin{bmatrix} i_1, i_2, \ldots, i_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} \det \mathcal{A}_F \begin{bmatrix} j_1, j_2, \ldots, j_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} . \tag{4.2.9}$$

Examination of (4.2.9) indicates that the summation contains a total of

$$\begin{bmatrix} n_F \\ m \end{bmatrix} = \frac{n_F!}{m!\,(n_F - m)!} \tag{4.2.10}$$

terms, and that each term is composed of two types of factors. The first factor is a product of n of the F-element admittance coefficients; for the type of F-elements considered, each admittance coefficient is finite and non-zero, hence this factor is always finite and non-zero. In addition, this factor involves a distinct set of m F-elements, i.e., no two terms in the summation involve the same set of m F-elements. It is also noted that this factor depends solely upon the admittance coefficients and is independent of the manner in which the F-elements are interconnected. The second factor in each term is the product of two minors from the incidence-seg matrix and has a value of +1, -1, or zero since the value of any minor of an incidence-seg matrix is +1, -1, or 0.[11] This factor, in contrast with the first type, depends solely upon the manner in which the F-elements are interconnected and is independent of the F-element admittance coefficients. Thus connection patterns of F-elements determine whether or not the summation in (4.2.9) contains non-vanishing terms:

<u>Theorem 4.2.1.</u> The m by m submatrix, $1 \leq m \leq v-1$,

$$\mathcal{Y}\begin{bmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} \qquad (4.2.11)$$

in (4.2.9) has term rank m if and only if there exists at least one m-element subgraph of F-elements such that

$$\mathcal{A}_F \begin{bmatrix} i_1, i_2, \ldots, i_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} \quad \text{and} \quad \mathcal{A}_F \begin{bmatrix} j_1, j_2, \ldots, j_m \\ k_1, k_2, \ldots, k_m \end{bmatrix} \qquad (4.2.12)$$

are both nonsingular.

4.3. Nonsingular Submatrices of the Complete Incidence-seg Matrix and Corresponding Subgraphs

Consider a graph G_o which is connected, i.e. a part, and consists of e_o elements and v_o vertices. An incidence-seg matrix for G_o is formed by mapping the element-vertex incidence pattern of the graph into a matrix. The <u>complete</u> incidence-seg matrix $\mathcal{A}_a$ for G_o is a matrix of order v_o by e_o where the typical entry a_{ij} is defined as follows:[10]

$$a_{ij} = \begin{cases} +1, & \text{if element } e_j \text{ is incident to vertex } v_i \text{ and oriented away from this vertex;} \\ -1, & \text{if element } e_j \text{ is incident to vertex } v_i \text{ and oriented toward this vertex;} \\ 0, & \text{if element } e_j \text{ is not incident to vertex } v_i. \end{cases}$$

Under this mapping each row of $\mathcal{A}_a$ corresponds to a distinct vertex of G_o and each column of $\mathcal{A}_a$ corresponds to a distinct element of G_o. Since the vertex and element numbers are arbitrary it is assumed that the vertex and element numbers correspond to the row and column indices, respectively of $\mathcal{A}_a$.

Let:

V_o	denote the set of all vertices of G_o,
E_o	denote the set of all elements of G_o,
V_m	denote a proper subset of V_o consisting of m distinct vertices,
E_m	denote a proper subset of E_o consisting of m distinct elements, and
$V(E_m)$	denote the set of vertices incident to the elements in E_m.

Corresponding to any m by m submatrix of $\mathcal{A}_a$ is a set of m distinct vertices and a set of m distinct elements of G_o. This correspondence is denoted by the following notation:

$$\mathcal{A}_a \begin{bmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{bmatrix} = \mathcal{A}_a \begin{bmatrix} V_m \\ E_m \end{bmatrix} \qquad (4.3.1)$$

where

$$V_m = \{ v_{i1}, v_{i2}, \ldots, v_{im} \}$$

$$E_m = \{ e_{j1}, e_{j2}, \ldots, e_{jm} \} \quad .$$

Also associated with this submatrix is a second vertex set; namely, $V(E_m)$.

The following two theorems characterize sets of linearly independent rows and columns of $\mathcal{A}_a$:

<u>Theorem 4.3.1.</u> If V_m is any subset of m distinct vertices of G_o, $1 \leq m \leq v_o - 1$, then the m by e_o submatrix

$$\mathcal{A}_a \begin{pmatrix} V_m \\ E_o \end{pmatrix} \qquad (4.3.2)$$

has rank m.

<u>Theorem 4.3.2.</u> Let E_m be any m-element subgraph of G_o, $1 \leq m \leq v_o - 1$. The v_o by m submatrix

$$\mathcal{A}_a \begin{pmatrix} V_o \\ E_m \end{pmatrix} \qquad (4.3.3)$$

has rank m if and only if E_m contains no circuits.

Consider an arbitrary m by m submatrix of $\mathcal{A}_a$, $1 \leq m \leq v_o - 1$:

$$\mathcal{A}_a \begin{pmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{pmatrix} = \mathcal{A}_a \begin{pmatrix} V_m \\ E_m \end{pmatrix} \qquad (4.3.4)$$

where

$$V_m = \{ v_{i1}, v_{i2}, \ldots, v_{im} \} \qquad (4.3.5)$$

$$E_m = \{ e_{j1}, e_{j2}, \ldots, e_{jm} \} \ . \qquad (4.3.6)$$

Theorems 4.3.1 and 4.3.2 provide necessary conditions for the submatrix (4.3.4) to be nonsingular. Theorem 4.3.1 is certainly satisfied; if the subgraph E_m contains no circuits then the submatrix, (4.3.3), of order v_o by m has rank m. Therefore there exists at least one mxm submatrix of (4.3.3) which is nonsingular. However this nonsingular submatrix may or may not be the one in (4.3.4). The following theorem gives necessary and sufficient conditions, in terms of the subgraph E_m and the vertex set V_m, such that the submatrix (4.3.4) is nonsingular:

<u>Theorem 4.3.3.</u> If and only if: (1) E_m contains no circuits, and (2) each part of E_m contains exactly one vertex $v^* \in V(E_m)$ such $v^* \notin V_m$, then the submatrix (4.3.4) is nonsingular.

Proof: Let E_m and V_m be given, where the subgraph E_m consists of m elements, p parts, and the vertex set $V(E_m)$ contains x vertices. Let the i-th part of E_m contain m_i elements and x_i vertices, where

$$m_1 + m_2 + \ldots + m_p = m \qquad (4.3.7)$$

$$x_1 + x_2 + \ldots + x_p = x \qquad (4.3.8)$$

since a part and its complement have no elements or no vertices in common. (a) Sufficient. Let E_m contain no circuits and each part of E_m contain exactly one vertex $v^* \in V(E_m)$ such that $v^* \notin V_m$. Because E_m contains no circuits, then each part of E_m contains no circuits and:

$$m_i - x_i + 1 = 0 \qquad i = 1, 2, \ldots, p \qquad (4.3.9)$$

or

$$x_i = m_i + 1 \qquad i = 1, 2, \ldots, p \,. \qquad (4.3.10)$$

Each part of E_m contains exactly one vertex $v^* \notin V_m$, thus each part contains exactly m_i vertices from V_m. From (4.3.7) it follows that V_m is a proper subset of $V(E_m)$. Construct an incidence-seg matrix $\mathcal{A}_i$ of order m_i by m_i for the i-th part of E_m; the omitted row corresponding to that vertex of part i which is not a member of V_m. Since each part is a tree then the rank of $\mathcal{A}_i$ is m_i. By proper ordering of the elements and vertices, an incidence-seg matrix, $\mathcal{A}^*$, for E_m can be written as a block diagonal matrix:

$$\mathcal{A}^* = \begin{array}{c} \\ m_1 \\ \vdots \\ m_i \\ \vdots \\ m_p \end{array} \begin{array}{c} \begin{array}{ccccc} m_1 & \cdots & m_i & \cdots & m_p \end{array} \\ \begin{bmatrix} \mathcal{A}_1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \mathcal{A}_i & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \mathcal{A}_p \end{bmatrix} \end{array} . \qquad (4.3.11)$$

Hence, $\mathcal{A}^*$ is an mxm matrix and has rank m. The m rows of $\mathcal{A}^*$ correspond to the vertices in V_m, the m columns of $\mathcal{A}^*$

correspond to the m-elements in E_m, and

$$\mathcal{A}_a\binom{V_m}{E_m} = \mathcal{P}_1 \mathcal{A}^* \mathcal{P}_2 \tag{4.3.12}$$

where $\mathcal{P}_1$, $\mathcal{P}_2$ are conformable permutation matrices and thus nonsingular. Therefore, $\mathcal{A}_a\binom{V_m}{E_m}$ is nonsingular. (b) Necessary. Let $\mathcal{A}_a\binom{V_m}{E_m}$ be nonsingular. Since a nonsingular matrix contains no zero rows or columns, then every vertex in V_m is incident to at least one element of E_m and each element of E_m is incident to at least one vertex from V_m. Thus V_m is a subset of $V(E_m)$. Further, the columns of this submatrix are linearly independent, thus E_m contains no circuits and

$$x = m + p . \tag{4.3.13}$$

Therefore V_m is a proper subset of $V(E_m)$, and $V(E_m)$ must contain exactly p distinct vertices which are not members of V_m. There exist permutation matrices $\mathcal{P}_3$ and $\mathcal{P}_4$ such that

$$\mathcal{P}_3 \mathcal{A}_a\binom{V_m}{E_m}\mathcal{P}_4 = \begin{bmatrix} \mathcal{A}_1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \mathcal{A}_i & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \mathcal{A}_p \end{bmatrix} \tag{4.3.14}$$

where the m_i columns of $\mathcal{A}_i$ correspond to the elements of E_m in part i and the rows of $\mathcal{A}_i$ correspond to the vertices of V_m in part i -- note that at this time the number of rows in $\mathcal{A}_i$ is not known to be m_i. Suppose that all of the m_i+1 vertices of part i are members of V_m. Then $\mathcal{A}_i$ is a complete incidence-seg matrix for part i and thus the rows of $\mathcal{A}_i$ are linearly dependent. Therefore the rows of $\mathcal{A}_a \begin{pmatrix} V_m \\ E_m \end{pmatrix}$ are linearly dependent and the matrix is singular. As $\mathcal{A}_a \begin{pmatrix} V_m \\ E_m \end{pmatrix}$ is nonsingular then <u>each</u> part of E_m must contain at least one vertex which is not a member of V_m. But $V(E_m)$ contains exactly p vertices which are not members of V_m; since each of the p parts must contain at least one, then each part of E_m must contain exactly one vertex which is not a member of V_m. q. e. d.

Subgraphs which contain no circuits, such as E_m in Theorem 4.3.3, are closely related to another class of subgraphs which are defined as follows:[32]

<u>Definition 4.3.1. k-tree</u>

A k-tree, T_k, of a v_o-vertex part G_o is a subgraph of G_o which contains k parts, all v_o vertices of G_o, and no circuits. (Note: in this definition one must allow the possibility that a part may consist of an isolated vertex.)

<u>Theorem 4.3.4.</u>[32] Let G_o be a part with v_o vertices, then:

(1) A k-tree of G_o contains v_o-k elements, $1 \leq k \leq v_o$;

(2) A subgraph of G_o which contains v_o vertices, v_o-k elements, and no circuits is a k-tree of G_o.

Thus, if $V(E_m) = V_o$ then E_m is a (v_o-m)-tree of G_o; if $V(E_m)$

is a proper subset of V_o, then E_m differs from a $(v_o\text{-}m)$-tree by a set of isolated vertices--namely those vertices of V_o which are not contained in $V(E_m)$.

Definition 4.3.2. Basis for a k-tree

The subgraph of a k-tree T_k consisting of the $v_o\text{-}k$ elements is designated as the basis for that k-tree and denoted by $T_k(E_{v_o\text{-}k})$.

Theorem 4.3.5. Let G_o be a part containing v_o vertices then any subgraph of G_o containing $v_o\text{-}k$ elements, $1 \leq k \leq v_o$, and no circuits is the basis for some k-tree of G_o.

In order to characterize the subgraphs of Theorem 4.3.3, which correspond to nonsingular submatrices of the complete incidence-seg matrix, one additional concept is required:

Definition 4.3.3. k-tree pair

Let G_o be a part containing v_o vertices, and $V_{v_o\text{-}k}$ and $E_{v_o\text{-}k}$ be any subset of $v_o\text{-}k$ distinct vertices and $v_o\text{-}k$ distinct elements of G_o, respectively. $\{V_{v_o\text{-}k}, E_{v_o\text{-}k}\}$ is said to be a k-tree pair if and only if $E_{v_o\text{-}k}$ is the basis for some k-tree T_k, and each part of $T_k(E_{v_o\text{-}k})$ contains exactly one vertex v^* such that v^* is not a member of $V_{v_o\text{-}k}$.

Therefore Theorem 4.3.3 can be restated as follows:

Theorem 4.3.6.

$$\mathcal{A}_a\begin{pmatrix} V_m \\ E_m \end{pmatrix} \text{ is nonsingular if and only if } \{V_m, E_m\}$$

is a $(v_o\text{-}m)$-tree pair.

For a given e_o-element, v_o-vertex part G_o it is of interest to determine the number of $(v_o\text{-}m)$-tree pairs that exist for either a fixed set of m vertices of G_o or a fixed set of m elements of G_o, where $1 \leq m \leq v_o - 1$.

Theorem 4.3.7. Let V_m be any fixed set of m distinct vertices of G_o, $1 \leq m \leq v_o-1$, and let n_E be the number of distinct sets of m elements of G_o, E_{m_i}, for which $\{V_m, E_{m_i}\}$ is a $(v_o\text{-}m)$-tree pair, then

$$n_E = \det\left\{ \mathcal{A}_a\begin{pmatrix} V_m \\ E_o \end{pmatrix} \mathcal{A}'_a\begin{pmatrix} E_o \\ V_m \end{pmatrix} \right\}. \tag{4.3.15}$$

Proof: Using the Binet-Cauchy formula:

$$\det\left\{ \mathcal{A}_a\begin{pmatrix} V_m \\ E_o \end{pmatrix} \mathcal{A}'_a\begin{pmatrix} E_o \\ V_m \end{pmatrix} \right\} = \sum_{E_{m_i}} \det \mathcal{A}_a\begin{pmatrix} V_m \\ E_{m_i} \end{pmatrix} \det \mathcal{A}'_a\begin{pmatrix} E_{m_i} \\ V_m \end{pmatrix} \tag{4.3.16}$$

$$= \sum_{E_{m_i}} \left\{ \det \mathcal{A}_a\begin{pmatrix} V_m \\ E_{m_i} \end{pmatrix} \right\}^2 \tag{4.3.17}$$

where E_{m_i} ranges over all distinct m-element combinations of the

e_o elements of G_o. The determinant in (4.3.17) has the value +1, or -1, if and only if $\{V_m, E_{m_i}\}$ is a $(v_o - m)$-tree pair and hence the result follows. The following theorem is proved in a similar manner:

<u>Theorem 4.3.8.</u> Let E_m be any fixed set of m elements of G_o, $1 \leq m \leq v_o - 1$, and let n_V be the number of distinct sets of m distinct vertices of G_o, V_{m_i}, for which $\{V_{m_i}, E_m\}$ is a $(v_o - m)$-tree pair, then

$$n_V = \det\{\mathcal{A}'_a \binom{E_m}{V_o} \mathcal{A}_a \binom{V_o}{E_m}\} . \qquad (4.3.18)$$

As a direct consequence of theorem 4.3.2:

<u>Corollary 4.3.9.</u> The number n_V in Theorem 4.3.8 if zero if and only if E_m contains a circuit.

4.4. Maximum Term Rank Submatrices of $\mathcal{Y}$ and Corresponding Subgraphs

Necessary and sufficient conditions for an arbitrary m by m submatrix of $\mathcal{Y}$ to have term rank m are stated in Theorem 4.2.1 in terms of nonsingular submatrices of an incidence-seg matrix. If this theorem is combined with Theorem 4.3.6 these conditions can be stated in terms of properties of subgraphs.

Let G_o be a part containing n_F F-elements and v_o vertices; consider an arbitrary m by m submatrix, , $1 \leq m \leq v_o - 1$, from the matrix $\mathcal{Y}$ in (4.1.2):

$$\mathcal{Y}\begin{pmatrix} i_1, i_2, \ldots, i_m \\ j_1, j_2, \ldots, j_m \end{pmatrix} \qquad (4.4.1)$$

The relation (4.2.9) establishes a correspondence between the row and column indices in (4.4.1) and two sets of row indices from the incidence-seg matrix; thus one can associate two sets of m distinct vertices from G_o with the submatrix in (4.4.1):

$$V_{m_i} = \{ v_{i1}, v_{i2}, \ldots, v_{im} \} \qquad (4.4.2)$$

and

$$V_{m_j} = \{ v_{j1}, v_{j2}, \ldots, v_{jm} \} \quad . \qquad (4.4.3)$$

<u>Theorem 4.4.1.</u> The m by m submatrix

$$\mathcal{Y}\begin{pmatrix} V_{m_i} \\ V_{m_j} \end{pmatrix} \qquad (4.4.4)$$

in (4.4.1) has term rank m if and only if there exists at least one m-element subgraph E_m of G_o such that $\{ V_{m_i}, E_m \}$ and $\{ V_{m_j}, E_m \}$ are both $(v_o\text{-}m)$-tree pairs for G_o.

<u>Corollary 4.4.2.</u> The m by m submatrix $\mathcal{Y}\begin{pmatrix} V_{m_i} \\ V_{m_j} \end{pmatrix}$ has term rank m if and only if $\mathcal{Y}\begin{pmatrix} V_{m_j} \\ V_{m_i} \end{pmatrix}$ has term rank m.

Although Theorem 4.4.1 relates term rank to properties of subgraphs it is instructive to examine the stated conditions in more detail in order to extract additional information characterizing the type of subgraph E_m meeting these requirements.

The basis incidence-seg matrix $\mathcal{A}_F$ in (4.1.2) is derived from the complete incidence-seg matrix $\mathcal{A}_a$ by deleting the row corresponding to the reference vertex r_o; thus r_o is never contained in either V_{m_i} or V_{m_j}. Therefore, for a given V_{m_i} and V_{m_j}, the vertex set of G_o:

$$V_o = \{ v_1, v_2, \ldots, v_{v_o-1}, r_o \} , \qquad (4.4.5)$$

is partitioned into these proper subsets: V_{m_i}, V_{m_j}, and the subset consisting of all vertices of V_o which are not contained in either V_{m_i} or V_{m_j}. The latter subset always contains at least one member--the reference vertex r_o. Although V_{m_i} contains m distinct vertices, as does V_{m_j}, it is possible for the composition of these two sets to vary from that of identical sets to that of disjoint sets.

If the vertex sets V_{m_i} and V_{m_j} are identical, then it is necessary and sufficient that there exists an m-element subgraph E_m such that $\{ V_{m_i}, E_m \}$ is a $(v_o - m)$-tree pair: E_m must contain no circuits and each part of E_m must contain exactly one vertex which is not contained in V_{m_i}. For a given graph G_o, and a given vertex set V_{m_i}, these conditions can be applied and it can be determined whether or not one or more allowable subgraphs exist. Alternately, one could utilize (4.3.15) to determine n_E, the number of m-element subgraphs such that $\{ V_{m_i}, E_m \}$ is a $(v_o - m)$-tree pair. Hence the term rank of $\mathcal{Y}\binom{V_{m_i}}{V_{m_i}}$ is m if and only if $n_E \neq 0$.

If the vertex sets V_{m_i} and V_{m_j} are not identical, then G_o must be searched for allowable m-element subgraphs such that both $\{V_{m_i}, E_m\}$ and $\{V_{m_j}, E_m\}$ are $(v_o\text{-}m)$-tree pairs. Unfortunately no formula has been found from which to calculate the number of distinct m-element subgraphs that satisfy both of these conditions. A determinant:

$$\det \left\{ \mathcal{A}_a \begin{pmatrix} V_{m_i} \\ E_o \end{pmatrix} \quad \mathcal{A}'_a \begin{pmatrix} E_o \\ V_{m_j} \end{pmatrix} \right\}, \tag{4.4.6}$$

similar to that used in (4.3.15), can be formed and evaluated. Application of the Binet-Cauchy formula indicates that the non-zero terms in the expansion, +1 or -1, do correspond to allowable subgraphs; however, the heterogeneous pattern of signs which may occur allow one to conclude in general that if there are t non-vanishing terms in the expansion, then the value of the determinant might range from +t to -t and including zero. If the value of the determinant in (4.4.6) is n, then there are at least $|n|$ distinct m-element subgraphs E_{m_k} such that $\{V_{m_i}, E_{m_k}\}$ and $\{V_{m_j}, E_{m_k}\}$ are both $(v_o\text{-}m)$-tree pairs. Therefore $n \neq 0$ is sufficient for the term rank of $\mathcal{Y}\begin{pmatrix} V_{m_i} \\ V_{m_j} \end{pmatrix}$ to be m, but it is not necessary.

Consider the type of subgraph E_m for which both $\{V_{m_i}, E_m\}$ and $\{V_{m_j}, E_m\}$ are $(v_o\text{-}m)$-tree pairs when V_{m_i} and V_{m_j} are not identical. In addition to containing no circuits, each part of E_m must contain exactly one vertex which is not contained in V_{m_i} and exactly one vertex which is not contained in V_{m_j}. If V_{m_i} and

V_{m_j} are disjoint then, since each part of E_m contains at least two vertices, each part must contain exactly two vertices--one from V_{m_i} and one from V_{m_j}. The m-element subgraph E_m must consist of m parts; each part is a single element incident to one vertex from V_{m_i} and one vertex from V_{m_j}. Thus, if V_{m_i} and V_{m_j} are disjoint, the m-element, m-part subgraph can be viewed as matching the vertices of V_{m_i} onto the vertices of V_{m_j} in a one-to-one manner. Therefore, when V_{m_i} and V_{m_j} are disjoint, $\mathcal{Y}\begin{pmatrix}V_{m_i}\\V_{m_j}\end{pmatrix}$ has term rank m if and only if at least one such matching subgraph exists in G_o.

Suppose the vertex sets V_{m_i} and V_{m_j} are neither identical nor disjoint. Then the vertex set V_o in (4.4.5) is partitioned into four mutually exclusive, all inclusive, non-empty subsets: S_1, S_2, S_3, and S_4--see Figure 4.4.1.

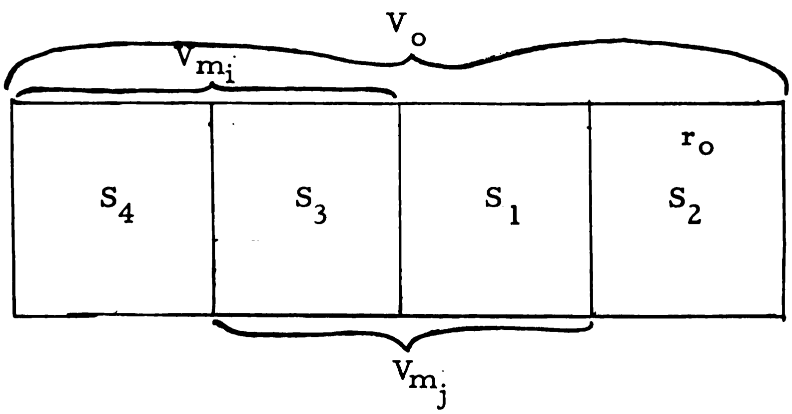


Figure 4.4.1. Vertex Partition.

Consider a subgraph E_m which contains no circuits; let P_i be any part of E_m and $V(P_i)$ be the set of vertices in P_i. If $\{V_{m_i}, E_m\}$ is a $(v_o\text{-}m)$-tree pair then there is exactly one vertex $a \in V(P_i)$ such that $a \notin V_{m_i}$; therefore

$$a \in S_1 \quad \text{or} \quad a \in S_2 \quad , \tag{4.4.7}$$

and for any vertex $c \in V(P_i)$, $c \neq a$, then $c \in V_{m_i}$:

$$c \in S_3 \quad \text{or} \quad c \in S_4 \, . \tag{4.4.8}$$

If $\{V_{m_j}, E_m\}$ is a $(v_o\text{-}m)$-tree pair then there is exactly one vertex $b \in V(P_i)$ such that $b \notin V_{m_j}$; therefore

$$b \in S_2 \quad \text{or} \quad b \in S_4, \tag{4.4.9}$$

and for any vertex $c \in V(P_i)$, $c \neq b$, then $c \in V_{m_j}$:

$$c \in S_1 \quad \text{or} \quad c \in S_3 \, . \tag{4.4.10}$$

These conditions must all be satisfied if E_m is to be an allowable subgraph. Note that a and b are particular vertices from $V(P_i)$ and that c is any vertex of $V(P_i)$, $c \neq a$, $c \neq b$; therefore: if $a = b$, then

$$a \in S_2 \quad \text{and} \quad c \in S_3; \tag{4.4.11}$$

if $a \neq b$, then

$$a \in S_1 \quad \text{and} \quad b \in S_4 \quad \text{and} \quad c \in S_3. \tag{4.4.12}$$

In the case where V_{m_i} and V_{m_j} are neither identical nor disjoint and both $\{V_{m_i}, E_m\}$, $\{V_{m_j}, E_m\}$ are $(v_o\text{-}m)$-tree pairs then E_m must contain m elements and no circuits; $V(E_m)$

must contain all of the vertices of V_{m_i}, V_{m_j}--and perhaps other vertices; each part P_i of E_m must also satisfy either one of the following criteria:

Criterion A: P_i must contain exactly one vertex which is not contained in either V_{m_i} or V_{m_j}, and all remaining vertices of P_i must be contained in both V_{m_i} and V_{m_j};

Criterion B: P_i must contain exactly one vertex which is contained in V_{m_i} but not contained in V_{m_j} and exactly one vertex which is contained in V_{m_j} but not contained in V_{m_i}; all remaining vertices of P_i must be contained in both V_{m_i} and V_{m_j}.

Further, if the number of vertices in S_4, and hence S_1, is x_1, then E_m must contain at least x_1 parts.

The subgraphs of Figure 4.4.2 illustrate, for the case $m = 2$, all possible types of two element subgraphs E_{2_k} such that both $\{V_{2i}, E_{2k}\}$ and $\{V_{2j}, E_{2k}\}$ are (v_o-2)-tree pairs. The unlabeled vertices in the Figure can be any other vertices of the graph. For subgraphs (a) - (d): $V_{2i} = V_{2j} = \{v_1, v_2\}$; (e) - (h): $V_{2i} = \{v_1, v_2\}$, $V_{2j} = \{v_2, v_3\}$; (i) and (j): $V_{2i} = \{v_1, v_2\}$, $V_{2j} = \{v_3, v_4\}$. Consequently, for a connected graph G_o containing at least four vertices: $\mathcal{Y}\begin{pmatrix}1, & 2\\ 2, & 3\end{pmatrix}$ has term rank two if and only if G_o contains at least one of the subgraphs (e) - (h) of Figure 4.4.2.

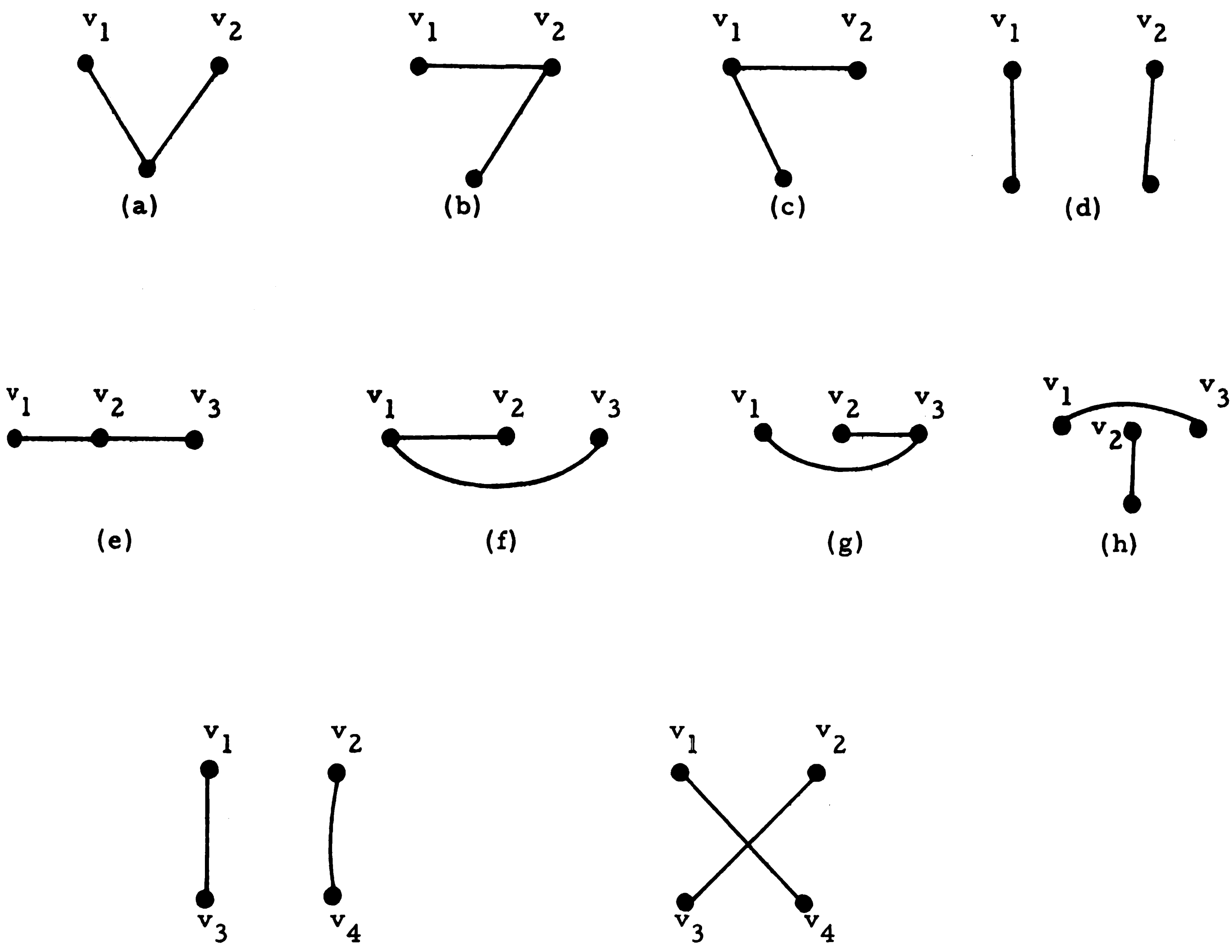


Figure 4.4.2. Allowable Subgraphs, m = 2.

Assuming that at least one allowable E_m has been found for a given V_{m_i} and V_{m_j} then, using criterion A and B in conjunction with an examination of the allocation of the vertices from V_{m_i}, V_{m_j} within the parts of E_m, it is possible to determine whether or not certain vertices can be interchanged between V_{m_i}, V_{m_j} and still maintain the $(v_o\text{-}m)$-tree pair property for E_m. If, for example, in determining the term rank of $\mathcal{Y}\binom{1,\ 2}{2,\ 3}$ one finds at least subgraph (e) of Figure 4.4.2 within G_o, then all of the following submatrices also have term rank two:

$$\mathcal{Y}\binom{1,\ 2}{2,\ 3},\ \mathcal{Y}\binom{2,\ 3}{1,\ 2},\ \mathcal{Y}\binom{1,\ 3}{2,\ 3},$$

$$\mathcal{Y}\binom{2,\ 3}{1,\ 3},\ \mathcal{Y}\binom{1,\ 2}{1,\ 3},\ \mathcal{Y}\binom{1,\ 3}{1,\ 2}. \qquad (4.4.13)$$

If, on the other hand, examination of G_o indicates that the <u>only</u> allowable subgraph contained in G_o has the form (h) of Figure 4.4.2, then only the first two submatrices in (4.4.13) have term rank two; the term rank of the last four submatrices is less than two.

The existence or non-existence of at least one allowable subgraph E_m for a given pair of vertex sets V_{m_i} and V_{m_j} can be checked by direct inspection of the graph. If the graph contains a large number of vertices and elements then this can become a time consuming process. Various algorithms have been devised and programmed to utilize a digital computer to search for numerous types of subgraphs--trees, k-trees, circuits, paths, etc.--within a given graph.[36-40] Programs of this type could be modified, or new programs devised, to implement a testing process. Consequently, this aspect is not considered further in this investigation.

In the event that all allowable subgraphs have been found--as might well be the case when a computer search program is used--then (4.2.9) can be used to evaluate the minor of $\mathcal{Y}$ without the necessity of actually forming the matrix $\mathcal{Y}$. On the assumption that $\mathcal{Y}\binom{V_{m_i}}{V_{m_j}}$ has term rank m, then two cases arise: (1) if V_{m_i} and V_{m_j} are identical; i.e. a principal minor of $\mathcal{Y}$, then the non-vanishing minors of the incidence-seg matrix in (4.2.9) have the same sign within each term and (4.2.9) becomes:

$$\det \mathcal{Y}\binom{V_{m_i}}{V_{m_i}} = \sum_{E_{m_k}} Y(E_{m_k}) \qquad (4.4.14)$$

where $Y(E_{m_k})$ is the product of the F-element admittance coefficients for the elements of E_{m_k} and the summation ranges over all E_{m_k} such that $\{V_{m_i}, E_{m_k}\}$ is a $(v_o\text{-}m)$-tree pair; (2) If V_{m_i} and V_{m_j} are not identical then the evaluation of the minor is complicated by the fact that the non-vanishing minors of the incidence-seg matrix in (4.2.9) can have either sign within a given term. A formula for the signs of the non-vanishing terms has been derived and the result is given below.

Let

$$V_{m_i} = \{ v_{i1}, v_{i2}, \ldots, v_{im} \} \tag{4.4.15}$$

and

$$V_{m_j} = \{ v_{j1}, v_{j2}, \ldots, v_{jm} \} \tag{4.4.16}$$

where

$$1 \le i_1 < i_2 < \ldots < i_m \le v_o - 1$$

$$1 \le j_1 < j_2 < \ldots < j_m \le v_o - 1$$

Since V_{m_i} and V_{m_j} are not identical, then the subsets S_1 and S_4 in Figure 4.4.1 are not empty. Let the number of vertices in S_1, and hence in S_4, be x, and

$$S_4 = \{ v_{i\alpha 1}, v_{i\alpha 2}, \ldots, V_{i\alpha x} \} \tag{4.4.17}$$

$$S_1 = \{ v_{j\beta 1}, v_{j\beta 2}, \ldots, v_{j\beta x} \} \tag{4.4.18}$$

where αi and βi are the positional indices of these vertices within V_{m_i} and V_{m_j} respectively and

$$\alpha_1 < \alpha_2 < \ldots < \alpha_x ,$$

$$\beta_1 < \beta_2 < \ldots < \beta_x .$$

Criterion B implies that any allowable E_m must contain at least x parts and further, each part containing a single vertex from S_4 must also contain a single vertex from S_1. Thus this part matches a vertex from S_4 to a corresponding vertex from S_1. If μ_k

is the number of inversions needed to rearrange the vertices in S_1 so that they appear in the same order as their corresponding matched vertices in S_4, then for an allowable E_{m_k}

$$\det \mathcal{A}_F \begin{pmatrix} V_{m_i} \\ E_{m_k} \end{pmatrix} \det \mathcal{A}_F \begin{pmatrix} V_{m_j} \\ E_{m_k} \end{pmatrix} = (-1)^{x+\Sigma \alpha_i + \Sigma \beta_i} (-1)^{\mu_k} \qquad (4.4.19)$$

Since only μ_k depends upon E_{m_k}, then (4.2.9) becomes

$$\det \mathcal{Y} \begin{pmatrix} V_{m_i} \\ V_{m_j} \end{pmatrix} = (-1)^{x+\Sigma \alpha_i + \Sigma \beta_i} \sum_{E_{m_k}} (-1)^{\mu_k} Y(E_{m_k}) \qquad (4.4.20)$$

where the summation ranges over all E_{m_k} such that $\{V_{m_i}, E_{m_k}\}$ and $\{V_{m_j}, E_{m_k}\}$ are both $(v_o\text{-}m)$-tree pairs.

4.5. From Subgraphs to Maximum Term Rank Submatrices

The preceeding discussion has assumed that the vertex sets V_{m_i} and V_{m_j} were specified initially and thus it became necessary to examine the graph G_o to determine whether or not it contained at least one subgraph E_{m_k} which satisfied both $(v_o\text{-}m)$-tree pair conditions. However, if any subgraph E_m, containing no circuits, is selected from G_o then it is a simple matter to list all the vertex sets V_{m_i} such that $\{V_{m_i}, E_m\}$ is a $(v_o\text{-}m)$-tree pair.

Associated with any two of these vertex sets is an m by m submatrix from

$$\mathcal{Y}_a = \mathcal{A}_a \mathcal{Y}_F \mathcal{A}'_a \qquad (4.5.1)$$

and this submatrix must have term rank m. Note that the complete incidence-seg matrix is used in (4.5.1) to allow for the possibility that the subgraph contains the reference vertex r_o. These results are applicable to the $\mathcal{Y}$ matrix in (4.1.2) if the vertex sets containing the reference vertex are deleted from the list.

First, consider another approach to calculating the number n_v defined in Theorem 4.3.8:

<u>Theorem 4.5.1.</u> Let G_o be a part of v_o vertices and E_m any m-element subgraph which contains no circuits. If E_m contains p parts and the k-th part contains x_k elements then the vertex set $V(E_m)$ can be decomposed into

$$n_v = \prod_{k=1}^{p} (x_k + 1) \qquad (4.5.2)$$

distinct subsets V_{m_i} of m distinct vertices such that $\{V_{m_i}, E_m\}$ is a $(v_o\text{-}m)$-tree pair, $i = 1, 2, \ldots, n_v$.

Proof: Since E_m contains no circuits, then $V(E_m)$ contains m+p vertices and each part of E_m contains x_k+1 vertices. Let V_{m_i} be any m vertex subset formed by deleting p vertices from $V(E_m)$, one vertex from each part. This can be done in any one of $n_v = \prod_{k=1}^{p} (x_k+1)$ different ways. Now, E_m contains no circuits and each part of E_m contains exactly one vertex which is not a member of V_{m_i}. Therefore $\{V_{m_i}, E_m\}$ is a $(v_o\text{-}m)$-tree pair for G_o.

Consequently, if G_o is a graph of F-elements such that the F-element admittance matrix, $\mathcal{Y}_F$, is diagonal and nonsingular, and V_{m_i}, V_{m_j} are any two of the vertex sets of Theorem 4.5.1 then

$$\mathcal{Y}_a \binom{V_{m_i}}{V_{m_j}}$$

has term rank m. Furthermore, noting that

$$\mathcal{Y}_a \binom{V_{m_i}}{V_{m_j}}$$

has term rank m if and only if

$$\mathcal{Y}_a \binom{V_{m_j}}{V_{m_i}}$$

has term rank m, and accounting for the fact that V_{m_i} and V_{m_j} can be identical, there are a total of

$$2\binom{n_v}{2} + n_v = n_v^2 \tag{4.5.3}$$

different m by m submatrices of $\mathcal{Y}_a$ associated with a single circuitless subgraph E_m, and each of these submatrices has term rank m.

Chapter 5

COMPLETE SOLUTIONS FOR THE PRIMARY SYSTEM OF EQUATIONS

5.1. Subgraphs and Feasible Proper Partitions

The preceeding chapter provides a basis for interrelating proper partitions of the variables within a system of holac equations associated with the graph correlate of an electric network and classes of subgraphs of that graph. A given partition of the variables is a proper partition if and only if a related submatrix has the proper dimensions and rank; this submatrix, in turn, has the proper <u>term</u> rank if and only if there exists within the graph at least one member of a particular class of subgraphs. An m by m submatrix has rank m only if its term rank is m; consequently the existence of a particular subgraph constitutes a necessary condition for the corresponding partition to be a proper partition of the variables. It is not in general a sufficient condition as certain subsets of values for the F-element admittance coefficients, in conjunction with certain interconnection patterns of the F-elements, can result in the rank of the submatrix being less than its term rank. In certain types of problems this characteristic is desirable--for example: balanced bridges and resonant networks, while in other types it may or may not be desirable. In recognition of these two possibilities a partition of the variables is said to be a <u>feasible</u> proper partition if and only if the related submatrix has the proper dimensions and <u>term</u> rank.

As previously noted a complete solution for a system of holac equations can be determined in many forms--for example, any proper partition of the variables determines a corresponding complete solution. Although any complete solution implies all particular solutions it is often the case that certain forms of a complete solution are more desirable than others. The approach considered here allows one to investigate the feasibility of a variety of forms of a complete solution without the necessity of actually formulating the primary system of equations. The only complete solutions which are considered explicitly here are those obtained directly from proper partitions of the variables.

The key issue in such an investigation is to determine whether or not a given partition of the variables is a feasible proper partition. If it is, then a continuation of the investigation indicates whether or not it is also a proper partition. The vertex subsets, S_1, S_2, S_3, and S_4 introduced in the preceeding chapter play a central role in these investigations. Each non-reference vertex of the graph is incident to exactly one N-element, thus a one-to-one correspondence is established between the v_o-1 N-elements and the v_o-1 non-reference vertices of the graph. Recall that

$$Vm_i = S_3 \cup S_4, \tag{5.1.1}$$

$$Vm_j = S_1 \cup S_3, \tag{5.1.2}$$

and that the set S_2 contains all other vertices of the graph--

including the reference vertex r_o. Permitting S_1 and S_4, or S_3, to be empty allows one to consider within the same framework the possibilities of Vm_i and Vm_j being identical, disjoint, or neither. Consider the system of holac equations in (3.5.21); the set

$$\{\mathscr{I}_{N1}, \mathscr{I}_{N2}, \mathscr{V}_{N1}, \mathscr{V}_{N3}\} \tag{5.1.3}$$

is a complete set of dependent variables and

$$\{\mathscr{I}_{N3}, \mathscr{I}_{N4}, \mathscr{V}_{N2}, \mathscr{V}_{N4}\} \tag{5.1.4}$$

is the corresponding complete set of independent variables for this system of holac equations if and only if

$$n_1 = n_4 \tag{5.1.5}$$

and

$$\det \mathscr{Y}\begin{pmatrix} Vm_i \\ Vm_j \end{pmatrix} = \det \begin{bmatrix} \mathscr{Y}_{31} & \mathscr{Y}_{33} \\ \mathscr{Y}_{41} & \mathscr{Y}_{43} \end{bmatrix} \neq 0. \tag{5.1.6}$$

Examination of the relationships among the allocation of the N-element variables within (5.1.3) and (5.1.4), the row and column indices of the submatrix whose determinant appears in (5.1.6), and the vertex sets S_1 through S_4 yields the following conclusions:

(1) Each of the n_1 N-elements incident to the n_1 vertices of S_1 has both associated element variables within the dependent set (5.1.3). These N-elements are classified as N_0-elements;

(2) Each of the n_2 N-elements incident to the n_2 vertices of S_2 (other than r_o) has its associated voltage variable within the independent set (5.1.4) and current variables within the dependent set (5.1.3). These N-elements are classified as N_e-elements;

(3) Each of the n_3 N-elements incident to the n_3 vertices of S_3 has its associated current variable within the independent set (5.1.4) and voltage variable within the dependent set (5.1.3). These N-elements are classified as N_h-elements;

(4) Each of the n_4 N-elements incident to the n_4 vertices of S_4 has both of its associated variables within the independent set (5.1.4). These N-elements are classified as N_{eh}-elements.

Thus a partition of the variables, such that $n_1 = n_4$, identifies the vertex sets S_1 through S_4 and conversely. This information, in conjunction with the developments of the preceeding chapter, allows one to investigate certain properties of a complete solution for the primary system of equations in terms of properties of the graph without the need for explicit formulations of the equations.

The classification of the N-elements as N_o-, N_e-, N_h-, or N_{eh}-elements is determined by properties of the system of holac equations and the graph--not by consideration of the correlating electric network. This allows one considerable freedom in the classification of the N-elements since there are often many different proper partitions of the variables for a given system of holac equations. This represents a departure from the usual practice in electric network theory where the inclusion of N_e- and N_h-elements has been generally based upon the presence of regulated voltage and current sources within the correlating electric network. Furthermore, this classification was seldom altered. The general absence of any device in the physical network having terminal characteristics which correlate with the characteristics of N_o- and N_{eh}-elements has undoubtedly been a major reason for the general lack of consideration given to these classes of N-elements.

5.2. Particular Solutions and Specification of Variables

Once a desired proper partition of the variables has been determined and the corresponding complete solution for a system of holac equations has been obtained then the variables within the independent set can be assigned arbitrary values with no danger of introducing inconsistencies. Each such set of values determines a particular solution and all particular solutions can be generated in this manner. Often it is desired to select from the multiplicity of particular solutions that solution, or solutions, which best satisfy some prescribed criteria. These criteria may be stated directly in terms of the variables within the system of holac equations and/or in terms of other secondary or derived quantities related to these variables. The form in which these criteria are stated is often a deciding factor in the choice of partitions to be checked.

The primary variables in the systems of holac equations considered in this investigation are the voltage and current variables associated with each element in the graph correlate of an electric network:

$$V_k = |V_k| e^{j\phi_k} = V_{k1} + j V_{k2}, \qquad (5.2.1)$$

and

$$I_k = |I_k| e^{j\theta_k} = I_{k1} + j I_{k2} \qquad (5.2.2)$$

for element k. The real power and reactive power variables for certain elements comprise one set of secondary variables commonly encountered in electric power network problems. Again for element k:

$$P_k + j Q_k = V_k I_k^* = |V_k| e^{j\phi_k} |I_k| e^{-j\theta_k} \qquad (5.2.3)$$

$$= |V_k| |I_k| e^{j\sigma_k} \qquad (5.2.4)$$

where

P_k is the real power,

Q_k is the reactive power, and

σ_k is the power factor angle.

Earlier developments have shown that a complete solution for a primary system of equations can be obtained in terms of certain subsets of the voltage and current variables associated with the N-elements in the graph. Thus particular solutions are obtained by assigning specific values to these variables in terms of magnitudes and angles, or in terms of real parts and imaginary parts. If one or more of the variables within the independent set is not assigned a specific value or is only specified in part--for example, the magnitude is specified but not the angle--then a family of particular solutions is obtained. These particular solutions contain one or more parameters which can be varied in an attempt to satisfy additional conditions which might have been placed upon the particular solution. Consequently, for a particular proper partition, the N-element falling into the N_e-, or N_h-element classification can have at most two parameters, while those in the N_{eh}-element classification can have at most four parameters. By the appropriate specification of the voltage and current variables for an N_{eh}-element it is possible to maintain any desired interrelation between these variables--for example, specifying the magnitudes of the voltage and current variables and the power factor angle fixes both the real and reactive power for that element yet neither the element voltage nor the element current is completely specified.

One further property of a system of holac equations is examined at this point. Consider a fixed system of holac equations; assume that at least two proper partitions of the variables exist and that the corresponding complete solution for the system has been determined for each proper partition. If the variables within the independent set of one of these proper partitions are assigned specific values than a particular solution is obtained. Now if the variables within the independent set of any other proper partition are subsequently assigned the values that they assumed in this first solution then the resulting particular solution is identical with the first one obtained. In terms of the N-element classification defined earlier this means that if a desired particular solution is obtained with a certain pattern of N_o-, N_e-, N_h-, and N_{eh}-elements then this pattern can be altered into any other allowable pattern and it is possible to maintain the same particular solution. For example, established criteria for the consistent location of regulated voltage and current sources within a physical network coincide with the conditions such that the correlating N-elements in the graph can be classified as N_e- and N_h-elements respectively.[10, 11, 41] However, within the framework of the correlating graph and primary system of equations, one is not in general restricted to this particular N-element classification. Thus if the appropriate conditions are satisfied these particular N-elements can be considered as either N_e- or N_h-elements, as well as N_o- and N_{eh}-elements. Consequently the use of the N_o- and N_{eh}-element classifications in an analysis problem need not be based upon the existence of any correlating physical device.

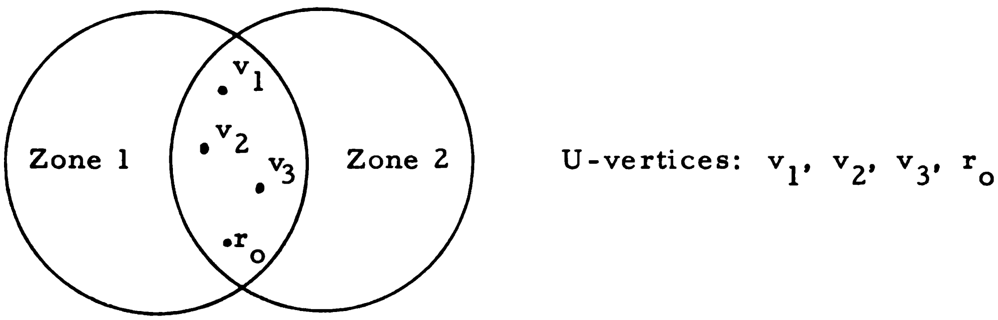
5.3. Applications in the Analysis of Electric Power Networks

Although the results of the preceeding investigation are applicable to a larger class of network problems they are particularly well-suited for use in the analysis of electric power networks. As discussed in Chapter 1, specifying variables plays an important role in many phases of electric power network analysis. The techniques developed in this investigation permit

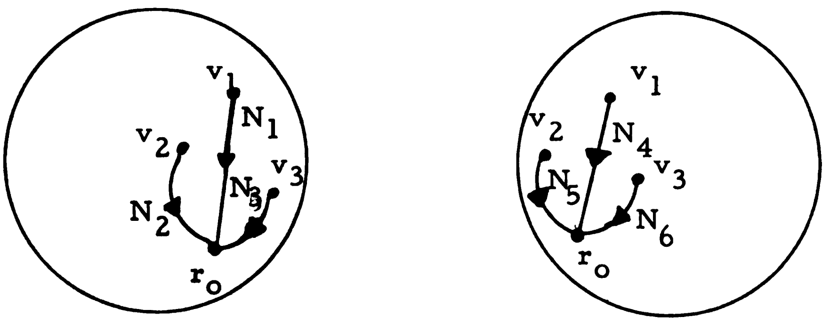
one to exercise considerably more freedom in the choice of specified variables than was possible before. At the same time one is assured that the chosen variables can be assigned arbitrary values with no danger of introducing inconsistencies into the system of network equations. These techniques are based upon properties of the primary system of equations yet the investigation can be completed without the necessity of actually formulating the equations. The required properties are stated directly in terms of the graph and thus allow one to interrelate properties of the graph and properties of solutions to the primary system of equations.

Theoretically the results of this investigation can be applied to any finite graph of the type considered; however there are practical limitations on the number of vertices and elements that can be accommodated in an effective manner. Certainly the use of a digital computer with a large storage capacity permits one to consider problems associated with larger and larger networks. However this capability, in itself, it not a panacea for all the problems in the analysis of electric networks. There are situations when it is desirable to "localize" the problem and yet not completely dissociate it from its relative place within a larger problem. The results of this investigation, coupled with a zoning concept described in the literature[15,16] provide the means for a new approach to the analysis of large-scale electric networks.

In the zoning approach[16] a large graph is decomposed into a number of subgraphs or zones such that (a) each subgraph is connected and (b) any two subgraphs are element disjoint but have certain vertices called U-vertices, in common. For the class of graphs considered in this discussion it is assumed that the reference vertex is a U-vertex for each zone and that each zone contains at least one other U-vertex distinct from the reference vertex. Once the original graph has been zoned in this manner the zones are then separated and an "external" set of N-elements is added to each zone; these N-elements are incident to the U-vertices only and each non-reference U-vertex in a zone is incident to exactly one of these added N-elements. In addition all external N-elements in a given zone are incident to the reference vertex (which is also a U-vertex for that zone). Figures 5.3.1 illustrates this for the case of two zones.



(a) Identification of zone and U-vertices.



(b) Graph after zoning and adding external N-elements.

Figure 5.3.1. Illustration of Zoning Technique.

It can be shown[16] that if the primary system of equations for the zoned graph, Figure 5.3.1(b), is augmented with the following sets of auxiliary equations:

$$\begin{bmatrix} V_{N1} \\ V_{N2} \\ V_{N3} \end{bmatrix} = \begin{bmatrix} V_{N4} \\ V_{N5} \\ V_{N6} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_{N1} \\ I_{N2} \\ I_{N3} \end{bmatrix} = - \begin{bmatrix} I_{N4} \\ I_{N5} \\ I_{N6} \end{bmatrix}, \qquad (5.3.1)$$

then each solution of this augmented primary system of equations determines exactly one solution for the primary system of equations for the original graph, Figure 5.3.1(a), and conversely. Thus the augmented primary system of equations for the zoned graph and the primary system for the original graph are essentially equivalent systems of equations.

The zoned graph consists of a number of separate graphs, or zones, and for each zone a primary system of equations can be formulated and solved. If, in addition, the auxiliary equations, such as (5.3.1), are satisfied then a solution of the primary system of equations for the original graph is obtained. This suggests a new approach to the analysis of large complex networks. If zones are so chosen that each of the external N-elements can be classified as N_{eh}-elements then, since both V and I for these N-elements can be arbitrarily specified, one is assured that the auxiliary equations can be satisfied. Once this has been established each zone can be analyzed independently--yet the composite of the zone solutions determines a corresponding solution for the original un-zoned graph.

This technique provides considerable flexibility in the analysis of electric networks. For the purpose of illustration consider the two zone case of Figure 5.3.1 and assume that the original graph has been zoned so that elements N_1, N_2, and N_3 can be classified as N_{eh}-elements for zone 1 and that elements N_4, N_5, and N_6 can be classified as N_{eh}-elements for zone 2. Suppose that one has a particular solution for the primary system of equations associated with the original unzoned graph. Using standard techniques[34] the V and I variables associated with the external N-elements are calculated so that the corresponding solution is maintained in the zoned graph. Actually the entire solution for the original graph is not needed--only that portion of the solution required to calculate the V, I variables for these N-elements is necessary. Note that this data could be obtained from measurements made in the actual network. Now consider zone 1 and its associated primary system of equations. Any solution of this primary system for which the V and I variables associated with elements N_1, N_2, and N_3 remain invariant can be combined with the existing solution for zone 2 to obtain a new solution for the primary system for the original graph. One is assured that the V and I variables for N_1, N_2, and N_3 can be maintained at the desired values since they can be classified as N_{eh}-elements. In this manner the solution for zone 1 can be varied and yet the solution for zone 2 remains invariant. Changes in the solution have been localized or confined to zone 1. When this is applied to a multi-zone problem the concept represents not only a reduction in the complexity of the problem to be analyzed but also permits one to investigate the possibilities of localizing, within prescribed zones, changes in the operating characteristics.

One further application is considered at this time. Again consider a zoned graph. The fact that the external N-elements for a zone can be classified as N_{eh}-elements implies that there exists within the zone an equal number of N_o-elements; furthermore the N_o- and N_{eh}-element classifications can always be interchanged. Thus it is possible to obtain a particular solution for the primary system associated with this zone utilizing the maximum number of specified conditions within the zone. This solution determines the V, I variables associated with the N_o-elements incident to the U-vertices of that zone. Then the auxiliary equations, such as (5.3.1), are used to transfer this condition to the N_{eh}-elements in the adjacent zones; in this process the effects of the first zone upon the adjacent zones is completely determined in terms of the V and I variables associated with the external N-elements. Depending upon the F-element subgraph with a particular zone it is also possible to interchange the N_o-, N_{eh}-element classifications for subsets of the N-elements and thus it is possible to reflect changes in the solution into selected zones.

5.4. Example

A portion of an actual electric power system--the Denver area of the Public Service Company of Colorado--is examined in this section to illustrate the preceding developments and at the same time to establish the feasibility of the zoning techniques discussed in Section 5.3. A network representation of this system contains 33 nodes plus the ground bus, 39 transmission line sections--each represented by the conventional pi equivalent, 6 generators, and 24 loads. A simplified interconnection diagram representing the transmission network is shown in Figure 5.4.1 and detailed nodal diagrams of the individual zones are shown in Figures 5.4.3 through 5.4.7.

The zones in this example were chosen in a somewhat arbitrary manner to illustrate the technique; in general the initial choice of zones would depend upon particular characteristics of the system under investigation. Factors which would influence the choice include geographical distribution of the system, service or load areas, locations of interconnections with adjacent systems, etc. Once the zones have been defined it is necessary to determine whether or not the external N-elements can be classified as N_{eh}-elements. For each zone in this example the external N-elements can be classified as N_{eh}-elements, i.e. both of the V and I variables associated with each of these N-elements can be allocated to the independent set of some proper partition of the variables within the primary system of equations for that zone.

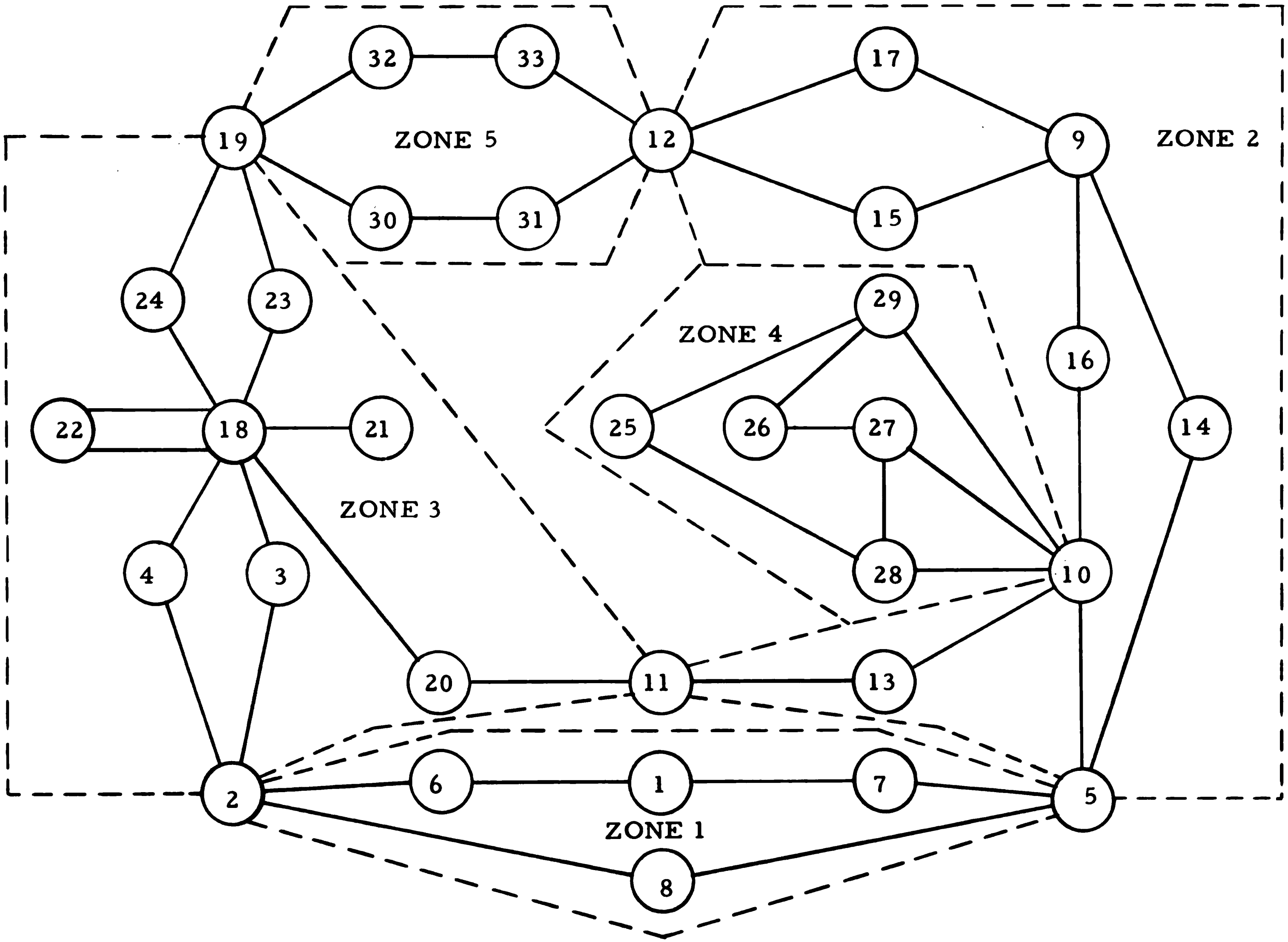


Figure 5.4.1. Interconnection Diagram for Transmission Network.

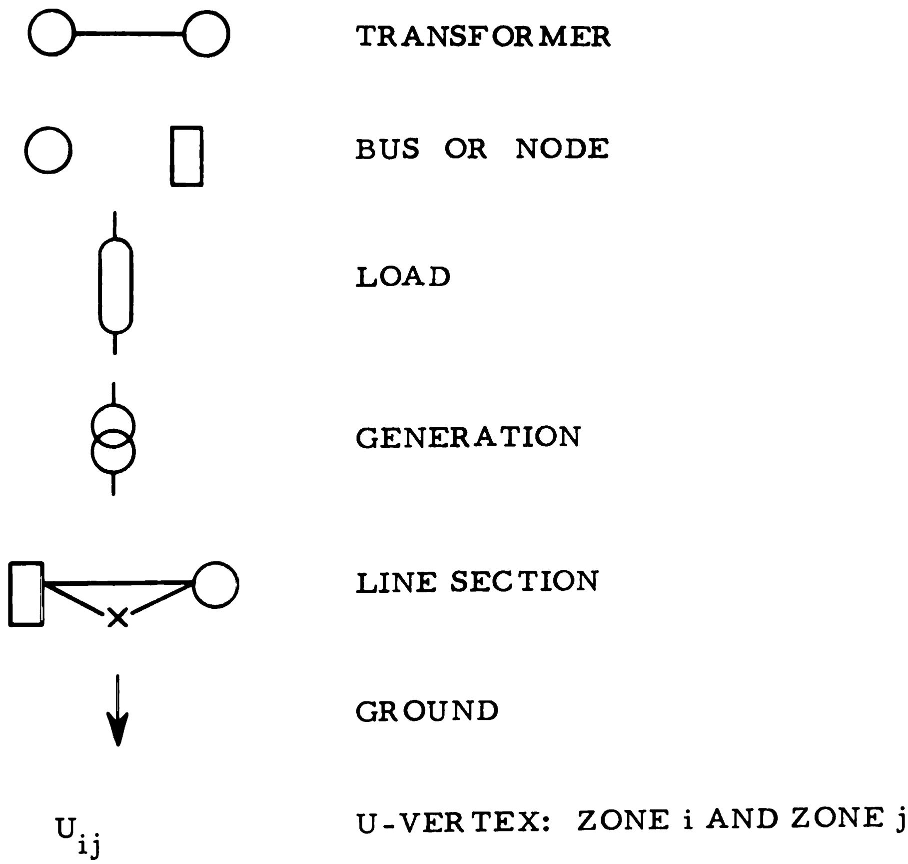


Figure 5. 4. 2. Key to Symbols.

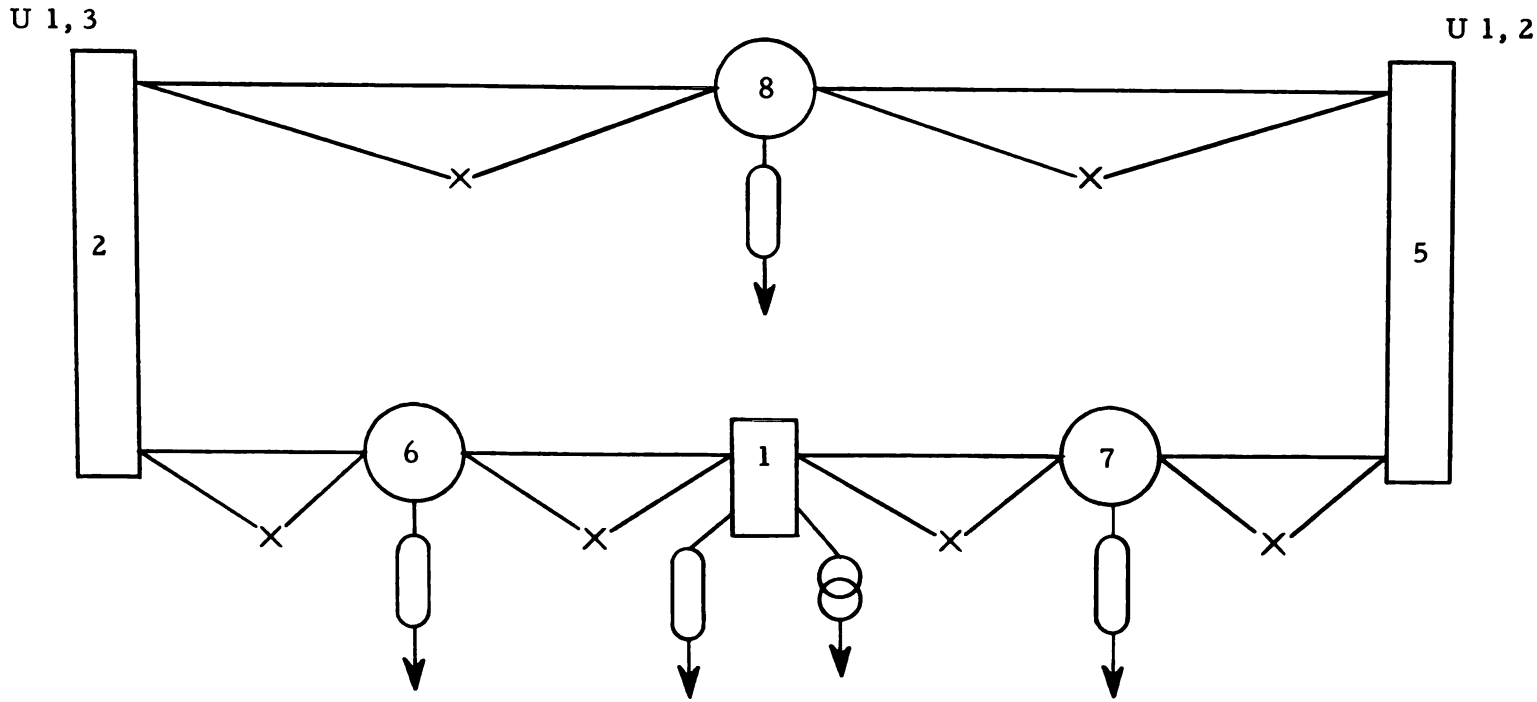


Figure 5.4.3. Nodal Diagram - Zone 1.

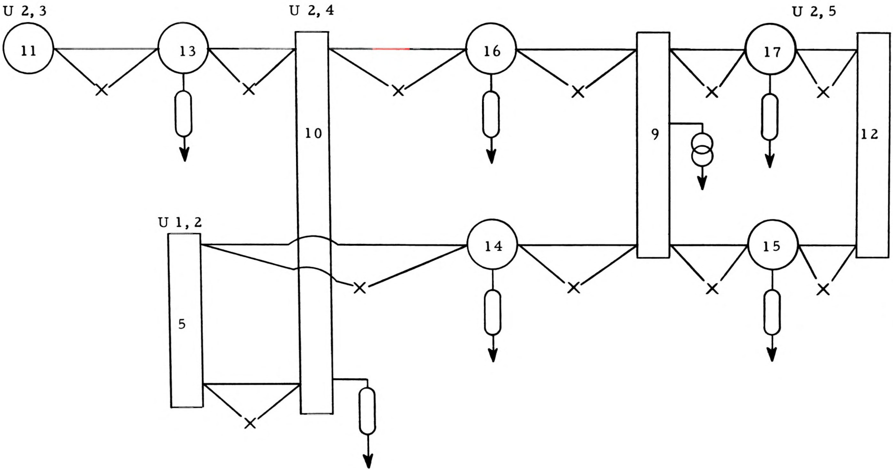


Figure 5.4.4. Nodal Diagram - Zone 2.

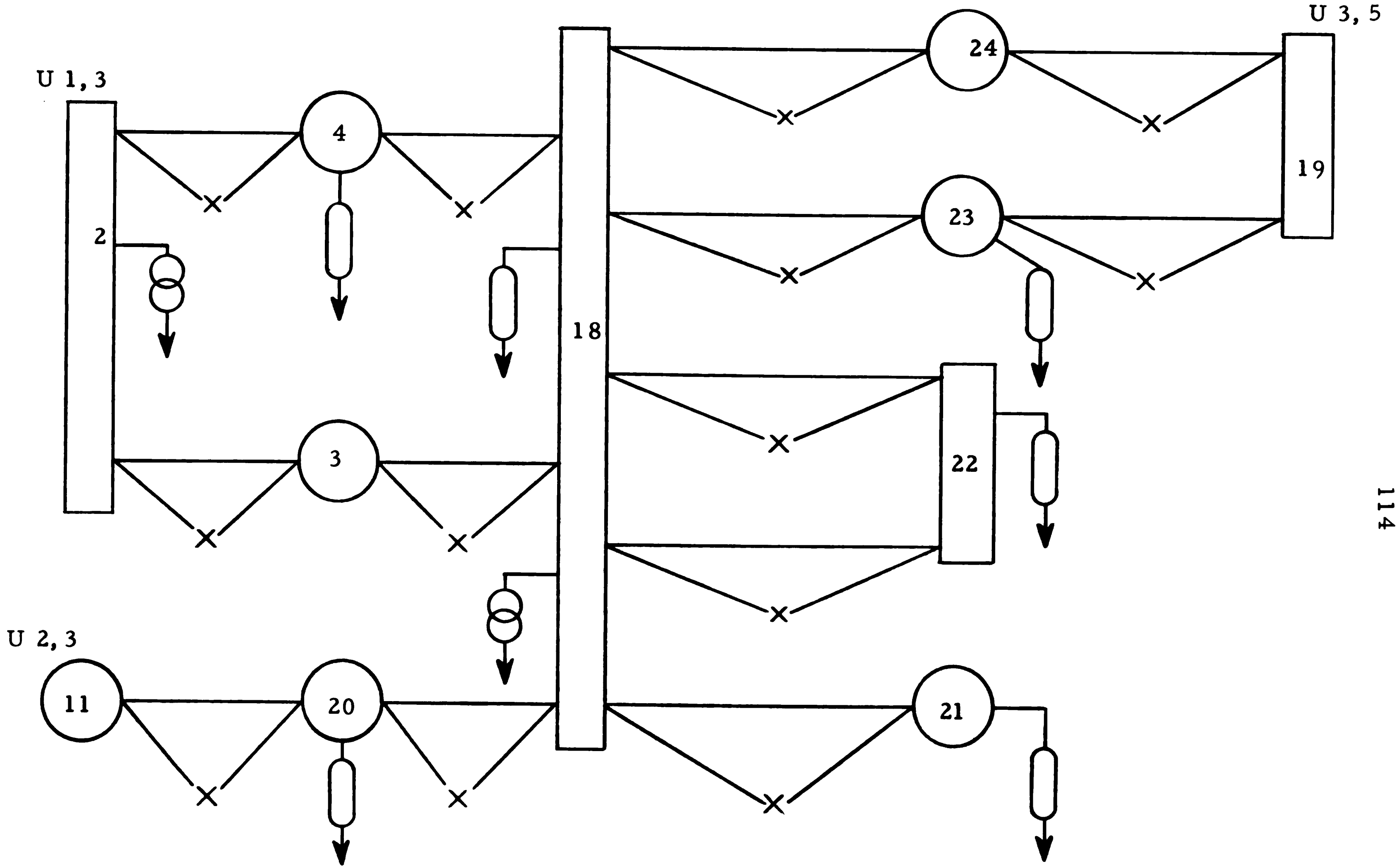


Figure 5. 4. 5. Nodal Diagram - Zone 3.

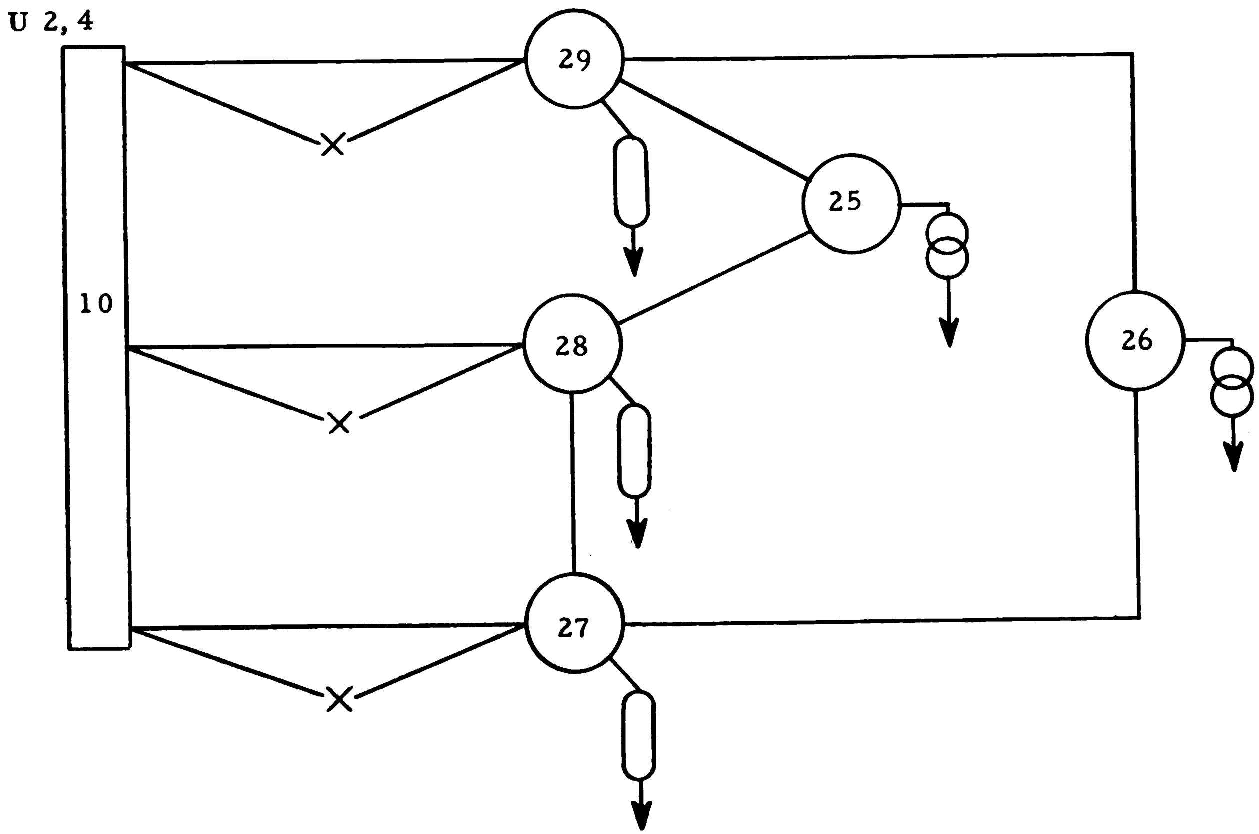


Figure 5.4.6. Nodal Diagram - Zone 4.

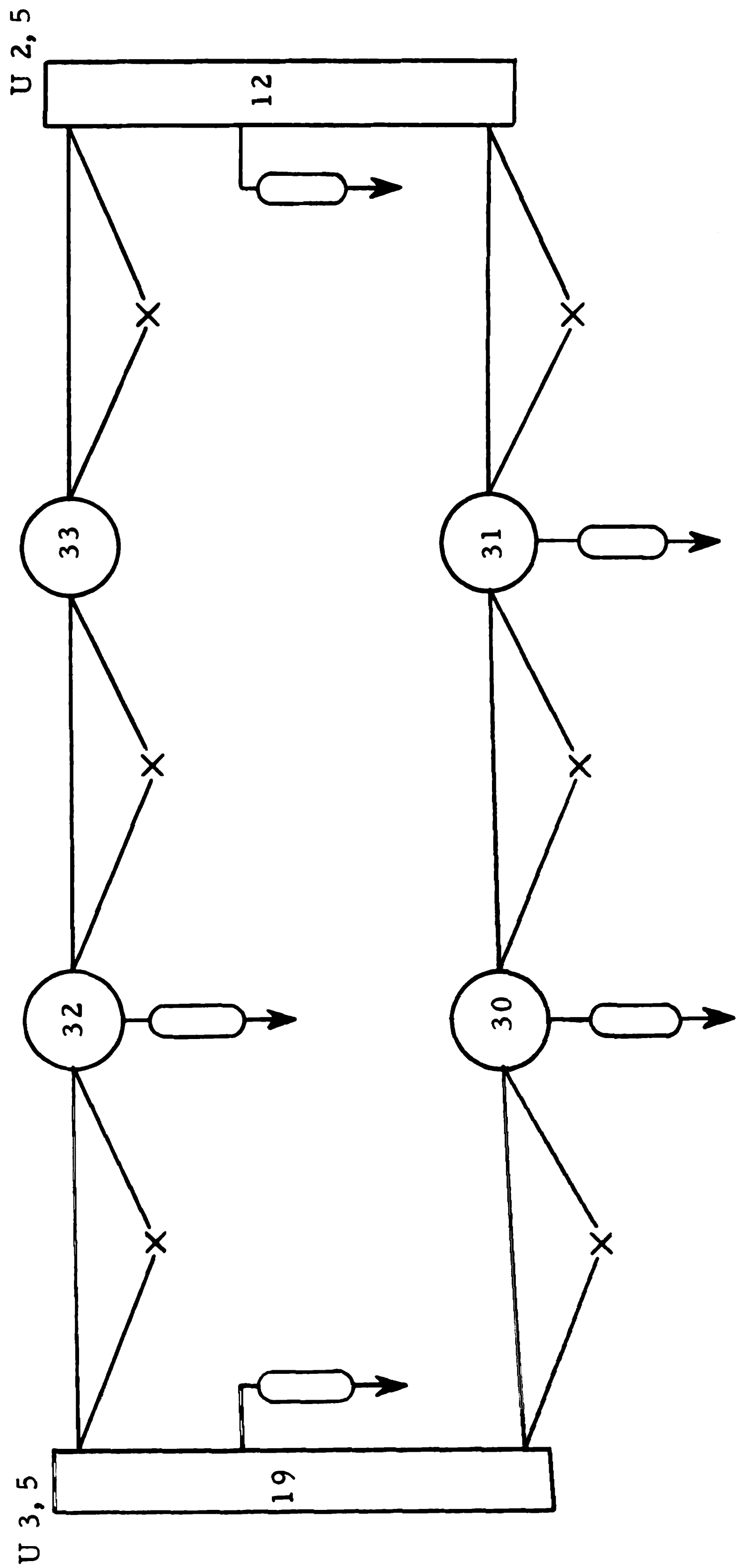


Figure 5.4.7. Nodal Diagram - Zone 5.

This fact establishes the feasibility of the zoning approach discussed in Section 5.3.

Consider the subgraph corresponding to any zone and the set of external N-elements for that zone. The graph elements corresponding to the transmission network are considered as F-elements and the graph elements corresponding to the generators are considered as N-elements. If the load admittance coefficients are not known then the corresponding graph elements are considered as N-elements; if a load admittance coefficient is known then the graph element can be handled as an F-element or as an N-element as discussed in Section 3.5. In this example it is assumed that all graph elements corresponding to loads are treated as N-elements. In addition any non-reference vertex which is not a U-vertex, or is not incident to a generator or load element, is considered to be incident to an N-element--see Section 3.5. When these steps have been completed it is possible that certain non-reference vertices are incident to more than one N-element; this is the case when a given node is incident to both a generator and a load, or when a U-vertex is incident to a generator and/or load. Any such parallel connection of N-elements is now replaced by a single N-element. There is no loss in generality in doing this since V for the equivalent N-element is the same as V for each of the parallel N-elements and I for the equivalent N-element is the sum of the I's for each of the parallel N-elements. Once this equivalent I has been determined then any choice of values for the I's of the parallel N-elements which yields this sum is satisfactory. (Note that if one of the external N-elements parallels another N-element and is subsequently replaced by a single N-element then it is necessary to determine the I associated with the external N-element before the auxiliary equations discussed in Section 5.3 are used). At this point a v_o-vertex zone contains a total of v_o-1 N-elements; each non-reference vertex is incident to exactly one N-element and all N-elements are incident to the reference vertex.

As discussed in Section 5.1 any partition of the $2(v_o-1)$ variables associated with this set of N-elements into two disjoint sets of v_o-1 variables each, i.e. a trial classification of the N-elements, defines the distribution of the vertices of the zone into the sets S_1, S_2, S_3, and S_4 which, in turn, define the two vertex sets Vm_i and Vm_j. If the graph contains at least one m-element subgraph of F-elements E_m such that both $\{Vm_i, E_m\}$ and $\{Vm_j, E_m\}$ are (v_o-m)-tree pairs then this partition is a feasible proper partition and the corresponding N-element classification is called a feasible classification. For any feasible proper partition the investigation is continued until all such subgraphs have been determined, then (4.4.20) can be utilized to determine whether or not this feasible proper partition is also a proper partition of the variables.

To illustrate this technique consider zone 2 as shown in Figure 5.4.4. The graph for this zone contains 10 vertices in addition to the reference vertex. The F-element subgraph consists of the elements corresponding to the line sections since the generator and all loads are represented by N-elements. Note that the single N-element incident to vertex 10 in the zone graph represents the parallel connection of an N-element corresponding to the load and the external N-element added to this zone by virtue of the fact that this vertex is also a U-vertex. Let the N-elements be identified by referring to the vertex to which the element is incident. The zone graph is now investigated to determine whether or not N_5, N_{10}, N_{11}, and N_{12} can be classified as N_{eh}-elements. Since a complete set of independent variables for this zone contains 10 variables this choice does not completely determine a trial classification pattern. To complete the trial classification of the N-elements let N_5, N_9, N_{10}, N_{11}, and N_{12} be considered as potential N_{eh}-elements; thus N_{13}, N_{14}, N_{15}, N_{16}, and N_{17} must be considered as potential N_o-elements. The corresponding vertex sets are

$$V_{5i} = S_4 = \{5, 9, 10, 11, 12\} \qquad (5.4.1)$$

and

$$V_{5j} = S_1 = \{13, 14, 15, 16, 17\} \qquad (5.4.2)$$

The vertex sets V_{5i} and V_{5j} are disjoint hence any subgraph of F-elements which satisfies the 6-tree pair condition for both vertex sets must consist of 5 elements which match V_{5i} onto V_{5j} in a one-to-one manner. Any such set of 5 F-elements must be included in the subgraph shown in Figure 5.4.8. Inspection of this subgraph indicates the existence of exactly two 5-element subgraphs having the desired characteristic. These subgraphs are shown in Figure 5.4.9.

The existence of either subgraph insures that the partition of variables under consideration is a feasible proper partition. Further it is known that the expansion of $\det \mathcal{Y}\begin{pmatrix} V_{5i} \\ V_{5j} \end{pmatrix}$ for this zone graph contains exactly two non-zero terms. Using (4.4.20) to evaluate this determinant:

$$\det \mathcal{Y}\begin{pmatrix} V_{5i} \\ V_{5j} \end{pmatrix} = - \{ -Y_1 Y_3 Y_7 Y_8 Y_{10} + Y_1 Y_5 Y_7 Y_8 Y_9 \} \quad (5.4.3)$$

$$= Y_1 Y_7 Y_8 (Y_5 Y_9 - Y_3 Y_{10}). \quad (5.4.4)$$

For the particular system under investigation

$$Y_3 = Y_5 \quad \text{and} \quad Y_9 = Y_{10}, \quad (5.4.5)$$

and although $\mathcal{Y}\begin{pmatrix} V_{5i} \\ V_{5j} \end{pmatrix}$ has <u>term</u> rank 5, the actual rank is less than 5. Therefore this particular partition of the variables is not a proper partition of the variables and the trial classification of the N-elements is not acceptable.

Since the original requirement was only that elements N_5, N_{10}, N_{11}, and N_{12} be classified as N_{eh}-elements it is possible that another classification of the N-elements will work. Consider the following trial classification--N_{eh}: N_5, N_{10}, N_{11}, N_{12}; N_e: N_9, N_{17}; N_o: N_{13}, N_{14}, N_{15}, N_{16}. The corresponding vertex

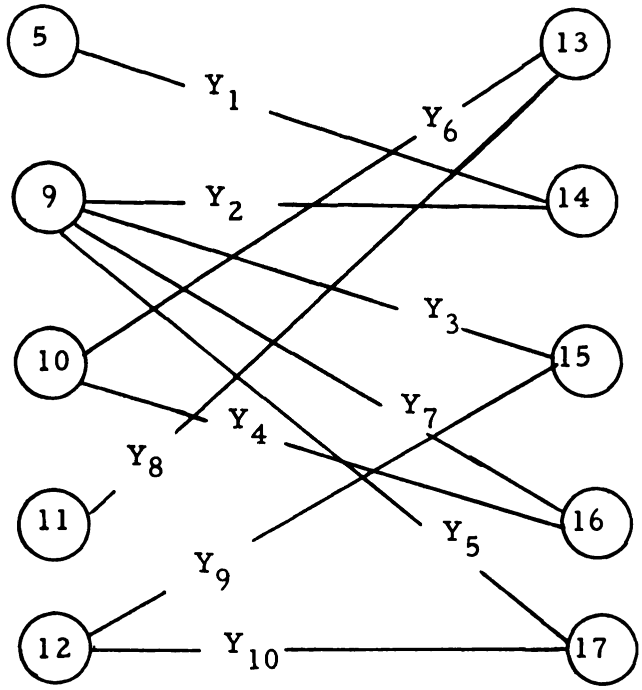


Figure 5.4.8. F-Element Subgraph

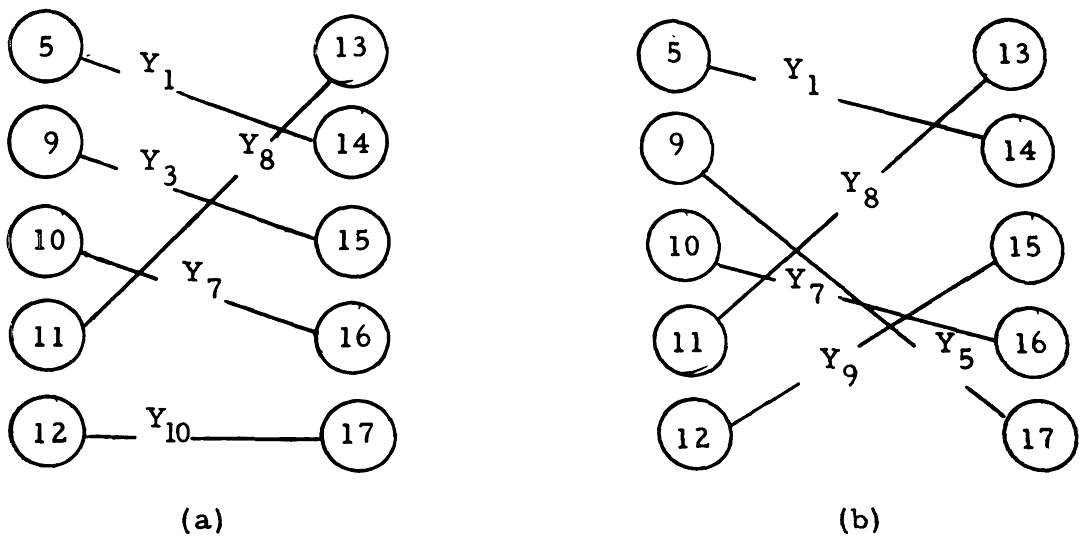


Figure 5.4.9. Subgraphs which Satisfy the 6-Tree Pair Criteria.

sets are:

$$V_{4j} = S_1 = \{13, 14, 15, 16\}, \qquad (5.4.6)$$

$$S_2 = \{9, 17, r_o\}, \qquad (5.4.7)$$

$$V_{4i} = S_4 = \{5, 10, 11, 12\}. \qquad (5.4.8)$$

Again, since V_{4i} and V_{4j} are disjoint, any 4-element subgraph E_{4k} such that $\{V_{4i}, E_{4k}\}$ and $\{V_{4j}, E_{4k}\}$ are both 7-tree pairs must match V_{4i} onto V_{4j} in a one-to-one manner and thus be contained in the F-element subgraph of Figure 5.4.8. Exactly one such subgraph exists and is shown in Figure 5.4.10. The F-element admittance coefficients are non-zero and $\det \mathcal{Y}\binom{V_{4i}}{V_{4j}}$ contains only one non-vanishing term in its expansion--therefore both the term rank and the rank are 4 and the partition under consideration is a proper partition of the variables for this zone.

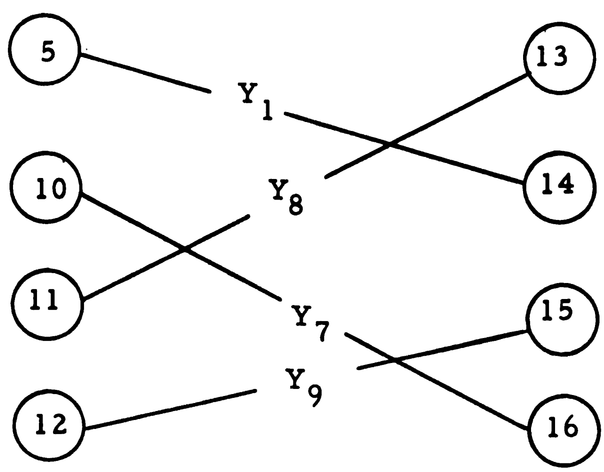


Figure 5.4.10. Subgraph which Satisfies the 7-tree Pair Criteria.

This result establishes that it is possible to classify elements N_5, N_{10}, N_{11}, and N_{12} for zone 2 as N_{eh}-elements. Other proper partitions for this zone also yield the same conclusion. In general the type of problem under consideration dictates which proper partition or partitions are apt to be more useful in a particular situation.

The preceding example illustrates the approach which proceeds from a given partition of the variables to a search for suitable subgraphs. As indicated in Section 4.5 this process can be revised in the sense that starting from a given circuitless subgraph it is possible to determine a considerable number of feasible proper partitions. This method is more direct than the first and at the same time gives a better indication of the interrelation between the interconnection patterns of the F-elements and corresponding feasible proper partitions. Of course for certain sets of values for the F-element admittance coefficients it is possible for a feasible proper partition to fail to qualify as a proper partition. To verify whether or not this is the case it becomes necessary to examine the graph and search for additional allowable subgraphs.

To illustrate this second approach consider zone 4--see Figure 5.4.6. The zone graph contains 6 N-elements and a complete solution for the primary system of equations is obtained in terms of certain subsets of 6 N-element variables. The fact that 6 variables can be selected from a set of 12 variables in any one of 924 different combinations gives at least some indication that there is potentially considerable flexibility in the choice of a set of independent variables for the primary system of equations for this zone. Consider the simple 3-element subgraph of F-elements from zone 4 as shown in Figure 5.4.11.

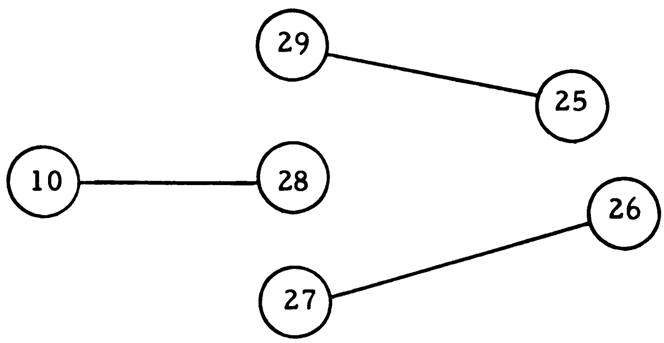


Figure 5.4.11. F-Element Subgraph from Zone 4.

There are eight distinct sets of three vertices for which the given subgraph satisfies the 3-tree pair criteria:

$$\{10, 25, 26\}, \{10, 25, 27\}, \{10, 26, 29\}, \{10, 27, 29\},$$
$$\{25, 26, 28\}, \{25, 27, 28\}, \{26, 28, 29\}, \{27, 28, 29\}. \quad (5.4.9)$$

The reference vertex is not contained in any of those vertex sets so there are then 64 different ways of choosing V_{3i} and V_{3j} from this list of vertex sets and corresponding to each choice one obtains a feasible proper partition of the variables! These results are given in Tabulation 5.4.1 in terms of feasible N-element classifications.

TABULATION 5.4.1.

A PARTIAL LISTING OF
FEASIBLE N-ELEMENT CLASSIFICATIONS FOR ZONE FOUR

	VERTEX SETS		N-ELEMENT CLASSIFICATION			
	V_{3i}	V_{3j}	N_o	N_e	N_h	N_{eh}
1	10, 25, 26	10, 25, 26	--	27, 28, 29	10, 25, 26	--
2		10, 25, 27	27	28, 29	10, 25	26
3		10, 26, 29	29	27, 28	10, 26	25
4		10, 27, 29	27, 29	28	10	25, 26
5		25, 26, 28	28	27, 29	25, 26	10
6		25, 27, 28	27, 28	29	25	10, 26
7		26, 28, 29	28, 29	27	26	10, 25
8		27, 28, 29	27, 28, 29	--	--	10, 25, 26
9	10, 25, 27	10, 25, 27	--	26, 28, 29	10, 25, 27	--
10		10, 26, 29	26, 29	28	10	25, 27
11		10, 27, 29	29	26, 28	10, 27	25
12		25, 26, 28	26, 28	29	25	10, 27
13		25, 27, 28	28	26, 29	25, 27	10
14		26, 28, 29	26, 28, 29	--	--	10, 25, 27
15		27, 28, 29	28, 29	26	27	10, 25
16	10, 26, 29	10, 26, 29	--	25, 27, 28	10, 26, 29	--
17		10, 27, 29	27	25, 28	10, 29	26
18		25, 26, 28	25, 28	27	26	10, 29
19		25, 27, 28	25, 27, 28	--	--	10, 26, 29
20		26, 28, 29	28	25, 27	26, 29	10
21		27, 28, 29	27, 28	25	29	10, 26
22	10, 27, 29	10, 27, 29	--	25, 26, 28	10, 27, 29	--
23		25, 26, 28	25, 26, 28	--	--	10, 27, 29
24		25, 27, 28	25, 28	26	27	10, 29
25		26, 28, 29	26, 28	25	29	10, 27
26		27, 28, 29	28	25, 26	27, 29	10
27	25, 26, 28	25, 26, 28	--	10, 27, 29	25, 26, 28	--
28		25, 27, 28	27	10, 29	25, 28	26
29		26, 28, 29	29	10, 27	26, 28	25
30		27, 28, 29	27, 29	10	28	25, 26
31	25, 27, 28	25, 27, 28	--	10, 26, 29	25, 27, 28	--
32		26, 28, 29	26, 29	10	28	25, 27
33		27, 28, 29	29	10, 26	27, 28	25
34	26, 28, 29	26, 28, 29	--	10, 25, 27	26, 28, 29	--
35		27, 28, 29	27	10, 25	28, 29	26
36	27, 28, 29	27, 28, 29	--	10, 25, 26	27, 28, 29	--

Since the N_o- and N_{eh}-element classifications are always interchangeable only 36 different classifications are shown.

It should be noted that once it is established that a given partition of the variables is a feasible proper partition then only specific interrelations among the admittance coefficients of the F-elements contained in the allowable subgraphs can result in this partition failing to qualify as a proper partition. Consequently there is a good possibility that most feasible partitions are also proper partitions. For example, within the partial listing of Tabulation 5.4.1 there are 16 feasible proper partitions in which N_{10} (the external N-element for this zone) is classified as an N_{eh}-element. In each case the partition is also a proper partition of the variables. Furthermore each of these proper partitions determines an additional proper partition by interchanging the N_o- and N_{eh}-element classifications. Hence of the 64 feasible proper partitions determined by the simple subgraph of Figure 5.4.10 at least 32 are also proper partitions of the variables. Consequently it is possible to select from a multiplicity of proper partitions those which are best suited for the particular study under consideration.

Chapter 6

SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

6.1. Summary

To effectively plan the current operation as well as the future expansion of electric power systems requires a knowledge of the operating characteristics of the existing and/or proposed systems. Analysis of a class of power system studies utilized in determining the electrical characteristics of a power system indicates that these studies are essentially problems in the analysis of electric networks. These studies differ from problems in "conventional" network analysis primarily in two aspects: (1) the size and complexity of the network under consideration, and (2) the type of initial problem specification. The availability of digital computers has alleviated, but not eliminated, difficulties associated with the size of the network. Problems associated with the initial specification of variables are more fundamental in nature and must be considered within the framework of the network equations since it is mandatory that inconsistencies are avoided. It is logical that a re-examination of the variable specification aspect of these network studies should originate at the level of the correlating graph and associated primary system of equations since they constitute the foundation for electric network theory.

The ω-domain graph correlates of the network under consideration are comprised of two general types of elements: relation elements (F-elements) and no-relation elements (N-elements); the latter type is characterized by a lack of any fixed interrelation between the associated V and I variables and furthermore neither of these variables is specified initially. The resulting primary system of equations is homogeneous in form and certainly consistent. Properties of subgraphs of F-elements are subsequently utilized to define classifications of the N-elements such that either, neither, or both of their associated variables can be assigned arbitrary values with no danger of introducing inconsistencies. The investigation clearly indicates that a multiplicity of N-element classification patterns exist for a given graph and provides the basis for new and more general approaches to the analysis of electric networks--particularly those problems in which it is desired to maintain prescribed operating conditions.

6.2. Suggestions for Further Research

The most apparent area for futher study, and the one in which current work is directed, is the application of these techniques to problems associated with specific power systems. As noted in the examples of Chapter 5 there exist a multiplicity of proper partitions, or N-element classification patterns, even in simple examples and in general the criteria for the selection of one or more of these allowable patterns is dictated by the system under consideration. A similar argument holds for the choice of zones within a particular system. Another important concept is implicit in these studies and this regards instrumentation. It is conceivable that studies of this nature will indicate a need for new concepts and/or techniques in instrumentation if optimal operating conditions are to be maintained as the system size increases.

Other areas of suggested research include consideration of graphs containing more general configurations of N-elements and extension of the class of N-elements to include, for example, "coupling" between N-elements. Last, but not least, is the need for continuing investigations into the general properties of the primary system of network equations.

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