TIME DOMAIN MODELS OF PHYSICAL SYSTEMS AND EXISTENCE OF SOLUTIONS

Thesis for the Degree of Ph. D.
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TIME DOMAIN MODELS OF PHYSICAL SYSTEMS AND EXISTENCE OF SOLUTIONS

by

John Lyle Wirth

AN ABSTRACT OF A THESIS

Submitted to the
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Department of Electrical Engineering

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ABSTRACT

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by John Lyle Wirth

For a number of years, attempts have been made to develop analytical techniques for the study of what are known as nonlinear systems. Such subjects as Cauchy-Taylor transforms, describing functions, etc., have been developed and used with some success.

Advancements in the area of computer technology and in the study of numerical analysis, have produced very practical methods for obtaining solutions of nonlinear equations. In addition to being able to generate solutions of a given mathematical model of a system, it is also possible to use the computer to formulate the system model starting from the mathematical models of the individual components and their interconnection pattern. Considerable progress has been realized in programming existing modeling techniques for computer execution.

In order to fully utilize the potentiality of the digital computer in the analysis of systems, however, the modeling procedures should produce a system model in a form directly suited for solution by numerical methods. These models may differ substantially from models produced by

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Almost all known numerical methods for solving simultaneous, ordinary differential equations require that the differential equations are in normal form, i.e., in the form

$$\frac{dx_i}{dt} = f_i(x_1 \ldots x_n, t) \ , \ i = 1, 2, \ldots n.$$

This thesis presents a formulation procedure for formulating a normal-form model of linear and nonlinear systems. To put the formulation procedure on a rigorous basis, a number of existence theorem pertaining to linear and nonlinear systems are presented.

In the development, the basic tools are matrix algebra and linear graph theory. Such special cases as ideal transformers, ideal levers, etc. are considered and treated in detail. Examples include such systems as compound pendulums, rocket trajectory, saturable core reactors, and systems containing ideal transformers.

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I. INTRODUCTION

Historically, much of the analysis of systems of electrical components has been carried out by means of transform techniques. These analysis techniques have been developed to a level of sophistication not paralled by other branches of engineering.[1,2,3,4]

Blackwell and Koenig[1], in their concepts of systems, have extended these orderly and precise methods of formulation to linear systems of all types of discrete components. The only prerequisite is that there exist two fundamental variables - an "across" variable which sums zero around circuits, and a "through" variable which sums zero at the vertices and suitable linear equations relating the "through" and "across" variables of each component.

Attempts have been made to extend transform techniques to include the nonlinear cases[5]. At best, these efforts have met with limited success in regard to both generality and ease of application.

The objective of this thesis is to set forth precise and orderly techniques for formulation mathematical models of nonlinear systems in general. These models are to be in a form amenable to solution by known techniques. In the pursuit of this objective, many of the concepts and developments of Reed[3], Koenig and Blackwell[1] have been used and extended.

Numerical methods of solution of systems of algebraic and/or differential equations are also considered fundamental to this development[6,7,12].

Sections II and III of the thesis deal entirely with systems composed of components described by algebraic equations. Here, such special component types as "perfect couplers" and "gyrators"[8] are treated in both the linear and nonlinear setting. Necessary conditions for the existence of a unique solution for a linear system are also present in terms of the system topology.

Section IV deals with the more general system described by both algebraic and differential equations. Theorems pertaining to the existence of solutions to nonlinear systems are presented with the linear system considered as a special case.

The proof of the existence theorems, in most cases, present practical methods for constructing the solutions to the classes of problems covered by the theorems. Section V is therefore devoted to developing algorithms for generating a mathematical model of the system in a form suitable for solution by numerical methods. These methods, in general, extend to a more general class of systems than those covered explicitly by the existence theorems. Even though the solution is not known to exist, a solution can often be constructed by these methods.

II. LINEAR ALGEBRAIC SYSTEMS

Consider first the class of systems in which the mathematical models (component equations) of all the components are algebraic. When the component models are all linear, sufficient conditions for the existence of a solution are given by the following theorem:

Theorem 2.1: Let the component equations for a system having a graph G be of the form

$$X=Z(t)Y+F(t) \tag{2.1}$$

where the matrix $Z(t) = [z_{ij}(t)]$ is positive definite* for all t on the interval I defined by $I = \left[t:t_o \leq t \leq t_1\right]$, t_o and t_1 constants, and where the vector valued function $F(t) = [f_i(t)]$ is defined for all t on I. Then there exists a unique solution for the variables X and Y of the system graph for all t on I.

Proof: For any forest T of the system graph G, the fundamental circuit and cut-set equations can be written as

$$[B_1 \;\; U] \begin{bmatrix} X_b \\ X_c \end{bmatrix} = 0 \qquad [U \;\; -B_1^t] \begin{bmatrix} Y_b \\ Y_c \end{bmatrix} = 0 \tag{2.2}$$

where the subscripts b and c refer to branch and chord variables, respectively. Let the components equations in (2.1) be rearranged

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* For the definitions of terminology and notation used throughout, see Appendix A.

and partitioned so as to show the branch and chord variables explicitly as separate vectors, thus

$$\begin{bmatrix} X_b \\ X_c \end{bmatrix} = \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} Y_b \\ Y_c \end{bmatrix} + \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} \tag{2.3}$$

Substituting (2.3) into (2.2), we have

$$[B_1 \ U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} Y_b \\ Y_c \end{bmatrix} + [B_1 \ U] \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} = 0 \tag{2.4}$$

Substituting the cut-set equations from (2.2) into (2.4) then gives

$$[B_1 \ U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_1^t \\ U \end{bmatrix} [Y_c] + [B_1 \ U] \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} = 0 \tag{2.5}$$

Since Z(t) is positive definite, the rearranged coefficient matrix shown in (2.3) is also positive definite. Therefore, by Theorem A.4, the triple product shown in (2.5) is non-singular for all t on I and we have the unique solution

$$[Y_c] = \left\{ [B_1 \ U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_1^t \\ U \end{bmatrix} \right\}^{-1} [B_1 \ U] \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} \tag{2.6}$$

The variables Y_b, X_b, and X_c can then be found uniquely from (2.2) and (2.3).

The above theorem and proof are very similar to results obtained by Reed in some unpublished works. The only difference is that Reed also included the requirement that the matrix $Z(t)$ be symetric as well as positive definite.

In one sense, Theorem 2.1 is quite general in that it places no restrictions upon the topology of the system, i.e., the components may have any mode of inter-connection. However, the requirement that $Z(t)$ be positive definite rules out cases where some of the component variables are known functions of the independent variable t. This latter case is covered by the following theorem:

Theorem 2.2: Let the component equations for a system with graph G be of the form

$$\begin{aligned} X_1 &= Z(t)Y_1 + F_1(t) \\ X_0 &= F_0(t) \\ Y_2 &= F_2(t) \end{aligned} \tag{2.7}$$

where 1) $Z(t)$ and $F_1(t)$ satisfy the hypothesis of Theorem 1.1,

2) $F_0(t)$ and $F_2(t)$ are defined for all t on I, and

3) there exists a forest T of G such that the elements corresponding to the entries of X_0 and Y_2 are branches and chords, respectively, of T.

Then there exists a unique solution for all variables of the system graph for all t on I.

Proof: As in the proof of the previous theorem, let the first set of equations in (2.7) be rearranged and partitioned according to the branch-chord classification of the elements for the forest T.

$$\begin{bmatrix} X_{1b} \\ X_{1c} \end{bmatrix} = \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} Y_{1b} \\ Y_{1c} \end{bmatrix} + \begin{bmatrix} F_{1b}(t) \\ F_{1b}(t) \end{bmatrix} \tag{2.8}$$

The fundamental circuit and cut-set equations can be written as

$$\begin{bmatrix} B_{11} & B_{12} & U & 0 \\ B_{21} & B_{22} & 0 & U \end{bmatrix} \begin{bmatrix} X_0 \\ X_{1b} \\ X_{1c} \\ X_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} U & 0 & -B_{11}^t & -B_{21}^t \\ 0 & U & -B_{12}^t & -B_{22}^t \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_{1b} \\ Y_{1c} \\ Y_2 \end{bmatrix} = 0 \tag{2.9}$$

Substituting (2.8) into the circuit equations in (2.9) and then substituting the cut-set equations in (2.9) into this result gives

$$\begin{aligned} &[B_{11}][X_0] + [B_{12} \quad U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_{12}^t \\ U \end{bmatrix} [Y_{1c}] \quad + \\ &[B_{12} \quad U] \begin{bmatrix} F_{1b}(t) \\ F_{1c}(t) \end{bmatrix} + [B_{12} \quad U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_{22}^t \\ 0 \end{bmatrix} [Y_2] = 0 \end{aligned} \tag{2.10}$$

By Theorem A.4, (2.10) can be solved uniquely for Y_{1c} in terms of X_0, Y_2, and t. However, since X_0 and Y_2 are given by (2.7), Y_{1c} is then known as a unique function of t alone. The remaining system variables can then be computed uniquely from (2.9) and (2.8).

All systems of linear algebraic two-terminal components are covered by Theorems 2.1 and/or 2.2. Since symmetry is not assumed in the matrix

$Z(t)$, many systems of multiterminal components are also covered. There are, however, many systems of multiterminal components which do not fall into the framework of the hypothesis of these theorems.

In particular, equations of the form

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 & Z_{12}(t) \\ Z_{21}(t) & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \tag{2.11}$$

occur quite often in mixed mechanical-hydraulic systems. Mechanical and electrical systems may also have component equations of the form

$$\begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 & H_{12}(t) \\ H_{21}(t) & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ X_2 \end{bmatrix} \tag{2.12}$$

Since the first equation in (2.12) is of the same form as the circuit equations, as might be suspected, certain topological configurations lead to inconsistencies in the system equations. These inconsistencies are avoided by the hypothesis of Theorem 2.3.

<u>Theorem 2.3</u>: Let the component equations for a system having a graph G be of the form given in (2.7) and (2.13).

$$\begin{bmatrix} X_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 0 & H_{12}(t) \\ -H_{12}{}^t(t) & 0 \end{bmatrix} \begin{bmatrix} Y_3 \\ X_4 \end{bmatrix} \tag{2.13}$$

where 1) (2.7) satisfies the hypothesis of Theorem 2.2,

2) the elements corresponding to X_3 and Y_4 are contained in the branch set and chord set, respectively, of the forest T of Theorem 2.2, and

3) det $[U - H_{12}(t)\, B_2] \neq 0$ for all t on I where B_2 is the coefficient matrix of the fundamental circuit equations defined by the elements corresponding to X_4, namely

$$[B_1 \;\; B_2 \;\; U] \begin{bmatrix} X^* \\ X_3 \\ X_4 \end{bmatrix} = 0$$

with the variable X^* representing all across variables except X_3 and X_4. Then there exists a unique solution for all the system variables for all t on I.

Proof: The essential parts of this proof have been considered by Blackwell(4). In particular, Blackwell has shown that under the conditions of the hypothesis, the combined rank of the circuit, cut-set and terminal equations of the form shown in (2.13) is maximum. From this result the first equation in (2.13) can be combined with the fundamental circuit equations and the second equation in (2.13) can be combined fundamental cut-set equations to yield reduced sets of equations which are independent of the variables X_3, Y_3, X_4, or Y_4. The resulting equations are

$$\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} & U & 0 \\ \underline{B}_{21} & \underline{B}_{22} & 0 & U \end{bmatrix} \begin{bmatrix} X_0 \\ X_{1b} \\ X_{1c} \\ X_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} U & 0 & -\underline{B}_{11}^t & -\underline{B}_{21}^t \\ 0 & U & -\underline{B}_{12}^t & -\underline{B}_{22}^t \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_{1b} \\ Y_{1c} \\ Y_2 \end{bmatrix} = 0 \qquad (2.14)$$

The entries in the submatrices $\underline{B}_{ij}$, however, are no longer simply +1, -1, or 0, but are now function of t depending upon the entries of $H_{12}(t)$. The proof is completed by noting that (2.14) is in the same form as that shown in (2.9). Hence, by the proof of Theorem 2.2, the solution for Y_{1c} exists and is unique. The remaining variables can be computed uniquely using (2.14), (2.13), (2.7) and the fundamental circuit and cut-set equations.

Although not as obvious as in the case of equations of the form shown in (2.12) and (2.13), systems having component models of the form shown in (2.11) may also be interconnected in such a way that a solution is not possible. Topological conditions sufficient for the existence of a solution are given by the following theorem.

<u>Theorem 2.4</u>: Let the component equations for a system having a graph G be of the form shown in (2.7) and (2.15),

$$X_3 = Z_3(t)\ Y_3 \qquad (2.15)$$

where 1) $Z_3(t)$ is a square skew-symmetric matrix for all t on I,

2) (2.7) satisfies Theorem 2.2,

3) $Z(t)$ is a non-null matrix and

4) there exists a forest T satisfying 3) of Theorem 2.2 such that the elements corresponding to X_3 are contained in the branch set.

Then there exists a unique solution for all the variables of the system graph for all t on I.

<u>Proof</u>: For the forest T, the fundamental circuit and cut-set equations are

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{bmatrix} \begin{bmatrix} X_0 \\ X_{1b} \\ X_3 \\ X_{1c} \\ X_2 \end{bmatrix} = 0 \quad \begin{bmatrix} U & 0 & 0 & -B_{11}^t & -B_{21}^t \\ 0 & U & 0 & -B_{12}^t & -B_{22}^t \\ 0 & 0 & U & -B_{13}^t & -B_{23}^t \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_{1b} \\ Y_3 \\ Y_{1c} \\ Y_2 \end{bmatrix} = 0 \qquad (2.16)$$

Substituting (2.7) and (2.15) into the circuit equations and then substituting the cut-set equations into this result, we have

$$[B_{11}]\,[X_0] + [B_{12}\ U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_{12}^t \\ U \end{bmatrix} [Y_{1c}] \ +$$

$$[B_{12}\ U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_{22}^t \\ 0 \end{bmatrix} [Y_2] + [B_{13}]\,[Z_3(t)]\,[B_{13}^t]\,[Y_{1c}] \ +$$

$$[B_{13}]\,[Z_3(t)]\,[B_{23}^t]\,[Y_2] = 0$$

In order to be able to show that there exists a unique solution for Y_{1c}, it must be shown that

$$A = [B_{12}\ U] \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_{12}^t \\ U \end{bmatrix} + [B_{13}]\,[Z_3(t)]\,[B_{13}^t]$$

is nonsingular.

Consider the quadratic form

$$X^t A X$$

where X is a nonzero vector. If it can be shown that this quadratic form is always greater than zero, then A is positive definite and, be Theorem A.2, nonsingular. By the definition of A, we have

$$X^t AX = X^t \begin{bmatrix} B_{12} & U \end{bmatrix} \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} B_{12}^t \\ U \end{bmatrix} X + X^t [B_{13}] [Z_3(t)] [B_{13}^t] X \qquad (2.18)$$

Since the coefficient matrix

$$\begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{bmatrix}$$

is non-null and positive definite, the first term in (2.18) is always greater than zero. The second term, on the other hand, is always exactly zero since $Z_3(t)$ is skew-symmetric (Theorem A.8). Therefore, A is positive definite and, by Theorem A.2, nonsingular for all t on I. It has been established that there exists a unique solution for Y_{1c}. The remaining system variables can be obtained uniquely from (2.16), (2.7), and (2.15).

For systems containing components with mathematical models of both forms (2.11) and (2.12), the hypothesis of Theorems 2.3 and 2.4 can be combined to yield sufficient conditions for the existence of a solution.

It should also be noted that although the above theorems are presented in terms of the open circuit parameters, i.e., the component equations are explicit in the across variable X, these same theorems can be stated and proved using the short circuit parameters. The proofs are almost identical to those considered above when the roles of X and Y are interchanged.

A review of the hypothesis of Theorems 2.1 through 2.4 shows that each have one point in common. In each case, if the hypothesis of the theorem is satisfied, it is possible to define a forest T of the system graph such that the component equations can be arranged in a form showing the branch across variables and the chord through variables as an explicit function of the remaining component variables. In fact, this is a necessary condition for the existence of a unique solution for all the variables of the system graph.

<u>Theorem 2.5</u>: For a system composed of components described by linear algebraic equations, a unique solution exists for all the system variables for any t on an interval I only if there exists some forest T of the system graph such that the component models can be written in the form

$$\begin{bmatrix} X_b \\ Y_c \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} Y_b \\ X_c \end{bmatrix} + \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} \tag{2.19}$$

where the subscripts b and c denote branch and chord variables, respectively.

<u>Proof</u>: Consider the set of equations consisting of an arbitrary set of component equations and the circuit and cut-set equations of an arbitrary graph G, i.e.,

$$\begin{bmatrix} B_1 & B_2 & 0 & 0 \\ 0 & 0 & A_1 & A_2 \\ G_1(t) & G_2(t) & G_3(t) & G_4(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix} \qquad (2.20)$$

where the elements of the system graph have been arbitrarily divided into sets identified by the subscripts 1, 2, 3, and 4. Note that the union of sets 1 and 2 constitutes the complete system graph as does the union of sets 3 and 4. Further, B_1 contains exactly $e - v + p$ rows and A_1 contains exactly $v - p$ rows, where e is the number of elements, v is the number of vertices, and p is the number of parts in the graph G. The proof is extablished by showing that the coefficient matrix of (2.20) is singular unless the hypothesis of the theorem is satisfied.

First, consider expanding the coefficient matrix in (2.20) by Laplace's Expansion about the first e rows. Selecting an arbitrary set of e columns in these two rows, the corresponding determinant, after some rearrangement, is

$$(\underline{+}1) \det \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix} \qquad (2.21)$$

This determinant is zero unless B and A are square, i.e., B must be of order $(e-v+p)x(e-v+p)$ and A must be of order $(v-p)x(v-p)$.

Further, by Theorem A.9, detB and detA are non-zero if, and only if, the elements corresponding to the columns of A and B constitute a forest and compliment of a forest, respectively, of the system graph.

Therefore, the determinant of the coefficient matrix in (2.20) can be written as

$$\sum_{ij} (\pm 1) \det B_i \det A_j \det [G_i, G_j] \qquad (2.22)$$

where A_j has columns corresponding to some forest T_j, B_i has columns corresponding to the compliment of some forest T_i, and $[G_i, G_j]$ represents the co-factor of the columns of B_i and A_j.

The determinant of the coefficient matrix in (2.20) is nonzero only if at least one term of the sum shown in (2.22) is nonzero. Thus, there must exist at least two forest T_i and T_j (i may equal j) such that

$$\det [G_i, G_j] \neq 0$$

Therefore, the last equation in (2.20) can be solved for X_{bi} and Y_{cj} to yield

$$\begin{bmatrix} X_{bi} \\ Y_{cj} \end{bmatrix} = \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix} \begin{bmatrix} Y_{bj} \\ X_{ci} \end{bmatrix} + \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} \qquad (2.23)$$

If i=j, the theorem is proved. If i$\neq$j, then by means of a tree transformation we have

$$\begin{bmatrix} X_{bi} \\ Y_{cj} \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} X_{bi} \\ Y_{ci} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Y_{bj} \\ X_{ci} \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} Y_{bi} \\ X_{ci} \end{bmatrix}$$

where C_1 and C_2 are nonsingular. Therefore we have

$$\begin{bmatrix} X_{bi} \\ Y_{ci} \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & C_1^{-1} \end{bmatrix} \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} Y_{bi} \\ X_{ci} \end{bmatrix} + \begin{bmatrix} U & 0 \\ 0 & C_1^{-1} \end{bmatrix} \begin{bmatrix} F_b(t) \\ F_c(t) \end{bmatrix} \tag{2.24}$$

and the proof is complete

On the basis of Theorem 2.5, many of the sufficiency requirements incorporated in previous theorems are also necessary. For example, Theorem 2.5 implies that for the existence of a unique solution it is necessary that there exist a tree which contains all of the elements having specified across variables and containing no elements having specified through variables. This result, stated earlier by Blackwell[4], is, therefore, a special case of the more general result given by Theorem 2.5. This theorem also states the restrictions on the topology of systems containing perfect couplers and gyrators.

III. NONLINEAR ALGEBRAIC SYSTEMS

The basic difficulty encountered in the analysis of nonlinear algebraic systems is that very little is known about the existence and uniqueness of a solution to a set of arbitrary nonlinear algebraic equations. Therefore, Theorem A.6 is essential to the following development. Through the use of this theorem, four existence theorems exactly parallel to Theorems 2.1 through 2.4 are presented concerning nonlinear systems. No attempt is made, however, to arrive at any necessary conditions for the existence of solutions.

Theorem 3.1: Let the component equations for a system with graph G be of the form

$$X = F\ (Y,t) \tag{3.1}$$

where 1) the Jacobian matrix $\frac{\partial F}{\partial Y}$ exists and is strictly positive definite for all t on an interval I defined by $I = \left[t{:}t_0 \leq t \leq t_1\right]$, t_0 and t_1 constants, and

2) all partial derivatives $\frac{\partial f_i}{\partial y_j}$ are bounded and satisfy a Lipschitz condition with respect to Y.

Then there exists a unique solution for all the variables of G for all t on I.

Proof: Let T be an arbitrary forest of the system graph. Then the component equations can be rewritten as

$$\begin{bmatrix} X_b \\ X_c \end{bmatrix} = \begin{bmatrix} F_1\ (Y_b,\ Y_c,\ t) \\ F_2\ (Y_b,\ Y_c,\ t) \end{bmatrix} \equiv F'\ (Y',\ t) \tag{3.2}$$

Then $F'\ (Y',\ t)$ has the same properties as $F(Y,\ t)$ in (3.1). Subsituting (3.2) into the circuit equations, (2.2), gives

$$[B_1 \quad U] \begin{bmatrix} F_1\ (Y_b,\ Y_c,\ t) \\ F_2\ (Y_b,\ Y_c,\ t) \end{bmatrix} = 0 \tag{3.3}$$

By substituting the cut-set equations from (2.2) into (3.3) we then have

$$[B_1 \quad U] \begin{bmatrix} F_1\ (B_1^t\ Y_c,\ Y_c,\ t) \\ F_2\ (B_1^t\ Y_c,\ Y_c,\ t) \end{bmatrix} = 0 \tag{3.4}$$

Since the Jacobian matrix J of (3.4) is given by

$$[B_1 \quad U] \begin{bmatrix} \dfrac{\partial F_1}{\partial Y_b} & \dfrac{\partial F_1}{\partial Y_c} \\ \dfrac{\partial F_2}{\partial Y_b} & \dfrac{\partial F_2}{\partial Y_c} \end{bmatrix} \begin{bmatrix} B_1^t \\ \\ U \end{bmatrix} \equiv J \tag{3.5}$$

J is strictly nonsingular for all t on I by Theorems A.5 and A.3. Therefore (3.4) satisfies the hypothesis of Theorem A.6 and consequently possesses a unique solution for Y_c. All other system variables can be computed uniquely from (2.2) and (3.1).

In the above proof, the method of attach is directly parallel to that used in the linear case. The only significant change is the use of Theorems A.3, A.5, and A.6 instead of their linear counterparts. Therefore, the following theorems are stated without proof.

Theorem 3.2: Let the component equations for a system having graph G be of the form

$$\begin{aligned} X_1 &= F_1(Y_1, t) \\ X_0 &= F_0(t) \\ Y_2 &= F_2(t) \end{aligned} \tag{3.6}$$

where 1) $F_1(Y_1, t)$ satisfies 1) and 2) of Theorem 3.1,

2) $F_0(t)$ and $F_2(t)$ are defined for all t on I, and

3) the topology of the system is such that there exists a forest T of the system graph containing the elements corresponding to X_0 and not containing the elements corresponding to Y_2.

Then there exists a unique solution for all the variables of the system for all t on I.

Before proceeding to the cases involving perfect couplers and gyrators, a closer examination of the hypothesis of Theorems 3.1 and 3.2 is in order. For most two terminal components, the equations relating the terminal variables of the various components are such that all partial derivatives exist and are bounded. Therefore, networks of diodes, nonlinear springs, etc., are known to have unique solutions. However, such components as the tunnel diode have characteristics such as shown in Figure 3.1.

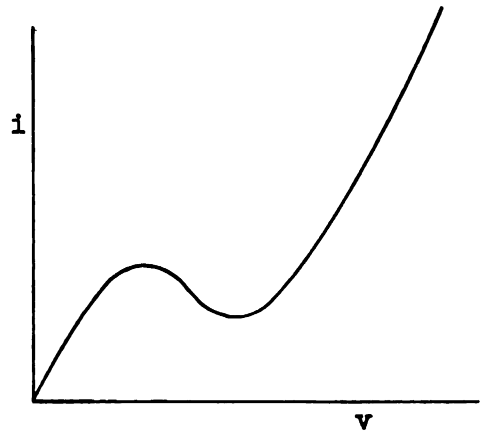


Figure 3.1 Tunnel Diode Characteristics

For such a characteristic, the partial derivative of v with respect to i is seen to vanish at two points and the hypothesis of the theorems are not satisfied. Systems involving such components may, or may not, have a unique solution - a result observed experimentally.

Turning now to perfect couplers and gyrators in nonlinear systems we have the following theorems.

<u>Theorem 3.3</u>: Let the component equations for a system with graph G be on the form shown in (3.6) and (3.7),

$$\begin{bmatrix} X_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} F_3(X_4, t) \\ F_4(Y_3, t) \end{bmatrix} \tag{3.7}$$

where, in addition to the hypothesis of Theorem 3.2,

1) the entries of $\dfrac{\partial F_3}{\partial X_4}$ and $\dfrac{\partial F_4}{\partial Y_3}$ are bounded and satisfy a Lipschitz condition for all t on I,

2) the topology of the system is such that there exists a forest T of the graph G which satisfies Theorem 3.2 and also contains the

elements of G corresponding to X_3 and contains no elements of G corresponding to Y_4,

3) the fundamental circuit equations defined by the elements of G corresponding to Y_4, i.e., the equations

$$[B_1 \quad B_2 \quad U] \begin{bmatrix} X^* \\ X_3 \\ X_4 \end{bmatrix} = 0$$

are such that

$$\det \left[U - \frac{\partial F_3}{\partial X_4} \quad B_2 \right] \neq 0$$

and

$$\det \left[U - \frac{\partial F_4}{\partial Y_3} \quad B_2^t \right] \neq 0$$

for all (X_4, Y_3) and all t on I.

Then there exists a unique solution for all the system variables for all t on I.

<u>Theorem</u> 3.4: Let the component equations be of the form shown in (3.6) and (3.8),

$$X_3 = F_3(Y_3, t) \tag{3.8}$$

where, in addition to the hypothesis of Theorem 3.2,

1) $\dfrac{\partial F_3}{\partial Y_3}$ is skew-symmetric for all Y_3 and all t on I,

2) the entries of $\dfrac{\partial F_3}{\partial Y_3}$ are bounded and satisfy a Lipschitz

condition for all t on I,

3) the topology of the system is such that there exists a forest T which satisfies 3) of Theorem 3.2 and also contains the elements corresponding to X_3 and,

4) the vector valued function $F_1(Y_1, t)$ of 3.6 is non-null.

Then there exists a unique solution for all the system variables for all t on I.

Again, as in the case of linear systems, the nonlinear theorems can be stated in terms of the short-circuit parameters. The implicit form

$$F\,(X,\,Y,\,t) \;=\; 0$$

can also be used. However, this form is not particularly amenable to theoretical discussion unless conditions sufficient for obtaining one of the explicit forms are imposed. It does have considerable merit from the standpoint of formulation and will be used to advantage in Section V.

IV. SYSTEMS DESCRIBED BY ALGEBRAIC AND DIFFERENTIAL EQUATIONS

In order to investigate the existence of solutions for systems containing components described by ordinary differential equations, the existence of solutions for systems of differential equations must be investigated. Bellman(11), and others(13) have established such existence theorems for systems of ordinary differential equations in normal form, i.e., in the form

$$\frac{dx_i}{dt} = f_i\,(x_1, x_2, \ldots x_n, t), \quad i = 1, 2, \ldots n.$$

Abian and Brown(15) have also established an existence theorem based upon the more general form

$$f\left(\frac{dx}{dt}, x, t\right) = 0$$

However, in this latter theorem, sufficient conditions are imposed to facilitate the solution for the derivative term. Therefore, for the discussion to follow, only theorems pertinent to the normal form are exploited.

Aside from the problems of existence and uniqueness of a solution, there is also the very practical problem of actually obtaining the solution for a set of simultaneous ordinary differential equations. Here again the normal form seems to be the most convenient. This form is not only necessary for solution by means of analog integration, but is also in a form easily integrated by any one of a number of numerical integration procedures[6,12]. Therefore, by formulating a normal form model of the system, both the existence and the actual solution itself can be established.

Systems Described by Differential Equations: First, consider the class of all systems made up of components which are described by ordinary differential equations only.

Theorem 4.1: Let the component equations for a system having graph G be of the form

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} F_1\,(X_1,\, X_2,\, Y_1,\, Y_2,\, t) \\ F_2\,(X_1,\, X_2,\, Y_1,\, Y_2,\, t) \end{bmatrix} \equiv F(Z) \qquad (4.1)$$

where 1) F(Z) satisfies a Lipschitz condition and

2) the system topology is such that there exists a forest T of the system graph G such that the elements corresponding to X_1 are branches and the elements corresponding to Y_2 are chords.

Then for any given set of initial conditions

$$\begin{bmatrix} X_1\,(t_0) \\ Y_2\,(t_0) \end{bmatrix} \equiv I$$

there exists a unique solution for all the variables in the system for all t such that $t_0 \leq t \leq t_0 + c$, $c > 0$.

Proof: For the forest indicated in 2) of the hypothesis, the fundamental circuit and cut-set equations are

$$[B_1 \quad U] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad [U \quad -B_1^t] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = 0 \tag{4.2}$$

Substituting (4.2) into (4.1) we have

$$\frac{d}{dt} \begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} F_1(X_1, -B_1X_1, B^tY_2, Y_2, t) \\ F_2(X_1, -B_1X_1, B^tY_2, Y_2, t) \end{bmatrix} \equiv F'(Z') \tag{4.3}$$

where $Z' \equiv (X_1, Y_2)$.

By Theorem A.10, F' satisfies a Lipschitz condition. Therefore, by Theorem A.11, (4.3) has a unique solution for the given set of initial conditions I. The remaining variables can be calculated uniquely from (4.2).

The class of systems covered by Theorem 4.1 is indeed quite small. For example, systems containing circuits of capacitors, cut-sets of inductors, etc., are not within the framework set forth in the above theorem. In an effort to extend Theorem 4.1 to a more general situation, consider the following:

<u>Theorem 4.2</u>: Let the component equations for a system with graph G be of the forms shown in (4.1) where

1) part 1) of the hypothesis of Theorem 4.1 is satisfied,

2) the matrix of functions

$$\begin{bmatrix} \frac{\partial F}{\partial Y_1} , \frac{\partial F}{\partial X_2} \end{bmatrix}$$

is defined everywhere, is strictly positive definite, and contains bounded entries which satisfy a Lipschitz condition.

Define S_1 to be the subgraph containing all elements of G having component equations which are explicit in the derivative of the across variable. Let T_1 be a forest of S_1 and let T be a forest of the complete system graph such that $T \supset T_1$, i.e., every element of T_1 is also contained in T. Then for every set of initial conditions of the form

$$\begin{bmatrix} X_{1b}(t_0) \\ Y_{2c}(t_0) \end{bmatrix} \equiv I,$$

where the subscripts b and c denote branch and chord variables, respectively, there exists a unique solution for all the variables of the system for $t_0 \leq t \leq t_0 + c$, $c \geq 0$.

Proof: The proof of this theorem is considered as a special case of Theorem 4.3 following.

Systems Described by Algebraic and Differential Equations: Up to this point, the component equations have been taken as either a set of algebraic equations or as a set of differential equations. However, in most problems presented by modern technology, neither of these two situations is likely to arise. Rather, the system under consideration is more likely to contain components which are described by both algebraic and differential equations.

A number of existence theorems pertaining to systems described by mixed algebraic-differential equations can be established by combining the results of the previous theorems. Specifically, by combining Theorem 2.2 and Theorem 4.2 we have the following:

<u>Theorem 4.3</u>: Let the component equation for a system having graph G be of the form

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} F_1(X_1, X_2, X_3, Y_1, Y_2, Y_3, t) \\ F_2(X_1, X_2, X_3, Y_1, Y_2, Y_3, t) \end{bmatrix} = F(Z) \qquad (4.5)$$

$$\begin{bmatrix} X_3 \\ X_0 \\ Y_4 \end{bmatrix} = \begin{bmatrix} F_3(Y_3, t) \\ F_0(t) \\ F_4(t) \end{bmatrix} \qquad (4.6)$$

Define the following subgraphs of the system graph G:

S_1 - The subgraph consisting of all the elements corresponding to X_0.

S_2 - The subgraph consisting of all the elements in S_1 and all elements having component equations explicit in the derivative of the X_1.

S_3 - The subgraph containing all the elements in S_2 and all the elements corresponding to X_3.

S_4 - The subgraph consisting of all the elements in S_3 and all the elements having component equations explicit in the derivative of Y_2.

S_5 - The complete system graph G.

Let T_i be a forest of S_i such that $T_i \supset T_{i-1}$ and let $T_5 = T^*$. Further, let

1) F(Z) satisfy a Lipschitz condition,

2) (4.6) satisfy the hypothesis of Theorem 3.2 for the forest T*,

3) the vector function $F_0(t)$ and $F_4(t)$ be such that $\frac{dF_0}{dt}$ and $\frac{dF_4}{dt}$ exist and satisfy a Lipschitz condition, and

4) the matrix

$$\begin{bmatrix} \frac{\partial F}{\partial Y_1}, & \frac{\partial F}{\partial X_2} \end{bmatrix}$$

is defined and strictly positive definite with entries which are bounded and satisfy a Lipschitz condition.

Then for any given set of initial conditions

$$\begin{bmatrix} X_{1b}(t_0) \\ Y_{2c}(t_0) \end{bmatrix} \equiv I$$

where the subscripts b and c denote branch and chord variables, respectively, there exists a unique solution for all the system variables for $t_0 \leq t \leq t_0 + c,\ c > 0$.

<u>Proof</u>: For the forest T^* it can be shown that the fundamental circuit and cut-set equations are of the form

$$\begin{bmatrix} B_{11} & B_{12} & 0 & 0 & U & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & B_{24} & 0 & U & 0 & 0 \\ B_{31} & B_{32} & 0 & B_{34} & 0 & 0 & U & 0 \\ B_{41} & B_{42} & B_{43} & B_{44} & 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} X_0 \\ X_{1b} \\ X_{2b} \\ X_{3b} \\ X_{1c} \\ X_{2c} \\ X_{3c} \\ X_4 \end{bmatrix} = [B\ U] \begin{bmatrix} X_b \\ X_c \end{bmatrix} = 0 \qquad (4.7)$$

and

$$\begin{bmatrix} U & -B^t \end{bmatrix} \begin{bmatrix} Y_b \\ Y_c \end{bmatrix} = 0 \tag{4.8}$$

For the forest T*, the first equation in (4.6) can be partitioned and rearranged in the form

$$\begin{bmatrix} X_{3b} \\ X_{3c} \end{bmatrix} = \begin{bmatrix} F_{3b}(Y_{3b}, Y_{3c}, t) \\ F_{3c}(Y_{3b}, Y_{3c}, t) \end{bmatrix} \tag{4.9}$$

Substituting (4.9) into the third equation in (4.7) gives

$$\begin{bmatrix} B_{31} & B_{32} \end{bmatrix} \begin{bmatrix} X_0 \\ X_{1b} \end{bmatrix} + [B_{34} \quad U] \begin{bmatrix} F_{3b}(Y_{3b}, Y_{3c}, t) \\ F_{3c}(Y_{3b}, Y_{3c}, t) \end{bmatrix} = 0 \tag{4.10}$$

From the detailed form of (4.8) we also have that

$$\begin{bmatrix} Y_{3b} \\ Y_{3c} \end{bmatrix} = \begin{bmatrix} B_{34}^t \\ U \end{bmatrix} [Y_{3c}] + \begin{bmatrix} B_{24}^t & B_{44}^t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_{2c} \\ Y_4 \end{bmatrix} \tag{4.11}$$

Substituting this result into (4.10) then yields

$$[B_{31} \quad B_{32}] \begin{bmatrix} X_0 \\ X_{1b} \end{bmatrix} + [B_{34} \quad U] \begin{bmatrix} F_{3b}(B_{34}^t\, Y_{3c} + B_{24}^t Y_{2c} + B_{44}^t Y_4,\ Y_{3c},\ t) \\ F_{3c}(B_{34}^t\, Y_{3c} + B_{24}^t Y_{2c} + B_{44}^t Y_4,\ Y_{3c},\ t) \end{bmatrix} = 0 \tag{4.12}$$

It is now claimed that for every value of X_0, X_{1b}, Y_{2c}, and Y_4, (4.12) possesses a unique solution for Y_{3c}. To show this, consider the partial derivatives of (4.12) with respect to Y_{3c}. Thus we have

$$[B_{34} \quad U] \begin{bmatrix} \frac{\partial F_{3b}}{\partial Y_{3b}} & \frac{\partial F_{3b}}{\partial Y_{3c}} \\ \frac{\partial F_{3c}}{\partial Y_{3b}} & \frac{\partial F_{3c}}{\partial Y_{3c}} \end{bmatrix} \begin{bmatrix} \frac{\partial Y_{3b}}{\partial Y_{3c}} \\ U \end{bmatrix} \tag{4.13}$$

or, from (4.11),

$$[B_{34} \quad U] \begin{bmatrix} \frac{\partial F_{3b}}{\partial Y_{3b}} & \frac{\partial F_{3b}}{\partial Y_{3c}} \\ \frac{\partial F_{3c}}{\partial Y_{3b}} & \frac{\partial F_{3c}}{\partial Y_{3c}} \end{bmatrix} \begin{bmatrix} B_{34}^t \\ U \end{bmatrix} \tag{4.14}$$

According to 2) of the hypothesis, the middle term in (4.14) is strictly positive definite - it is a row-column rearrangement of a strictly positive definite matrix and is, by Theorem A.5b, strictly positive definite. Therefore, the triple product shown in (4.14) is strictly nonsingular (Theorem A.5). Further, since F_3 satisfies the hypothesis of Theorem 3.2,

(4.12) satisfies the hypothesis of Theorem A.6. Therefore the solution to (4.12), say

$$Y_{3c} = F_5(X_0, X_{1b}, Y_{2c}, Y_4, t), \tag{4.15}$$

exists and is unique. By substituting (4.15) into (4.11) we also have that

$$Y_{3b} = F_6(X_0,\ X_{1b},\ Y_{2c},\ Y_4,\ t) \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.9) also yields

$$X_{3b} \ = \ F_7(X_0,\ X_{1b},\ Y_{2c},\ Y_4,\ t) \tag{4.17}$$

$$X_{3c} = \ F_8(X_0,\ X_{1b},\ Y_{2c},\ Y_4,\ t) \tag{4.18}$$

By applying Theorem A.7, it can be shown that F_5, F_6, F_7, and F_8 all satisfy a Lipschitz condition. Thus, by substituting (4.15), (4.16), (4.17), (4.18) and the last two equations from (4.6) into (4.5), we have the form

$$\frac{d}{dt} \begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} F_1'(X_1,\ X_2,\ Y_1,\ Y_2,\ t) \\ F_2'(X_1,\ X_2,\ Y_1,\ Y_2,\ t) \end{bmatrix} \equiv F'(Z_1) \tag{4.19}$$

From Theorem A.10, $F'(Z_1)$ satisfies a Lipschitz condition and, from 4) of the hypothesis, the matrix

$$\begin{bmatrix} \frac{\partial F'}{\partial Y_1} \ , \ \frac{\partial F'}{\partial X_2} \end{bmatrix} \tag{4.20}$$

is strictly positive definite. Note that (4.19) is of exactly the same form as the equations shown in Theorem 4.2. Hence, the

remainder of the present proof also suffices for the proof of Theorem 4.2.

By rearranging the equations in (4.19) and the variables in Z_1, (4.19) can be written in the form

$$\frac{d}{dt}\begin{bmatrix} X_{1b} \\ X_{1c} \\ Y_{2b} \\ Y_{2c} \end{bmatrix} = \begin{bmatrix} F'_{1b}(Z_2) \\ F'_{1c}(Z_2) \\ F'_{2b}(Z_2) \\ F'_{2c}(Z_2) \end{bmatrix} \tag{4.21}$$

where $Z_2 = (X_{1b}, X_{1c}, X_{2b}, X_{2c}, Y_{1b}, Y_{1c}, Y_{2b}, Y_{2c}, t)$. From (4.7) and (4.8) we have that

$$\begin{bmatrix} X_{1b} \\ X_{1c} \\ Y_{2b} \\ Y_{2c} \end{bmatrix} = \begin{bmatrix} U & 0 \\ -B_{12} & 0 \\ 0 & B^t_{23} \\ 0 & U \end{bmatrix} \begin{bmatrix} X_{1b} \\ Y_{2c} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -B_{11} & 0 \\ 0 & B^t_{43} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_4 \end{bmatrix} \tag{4.22}$$

Substituting this result into (4.21) and premultiplying by the nonsingular matrix

$$\begin{bmatrix} U & 0 & 0 & 0 \\ 0 & 0 & 0 & U \\ B_{12} & U & 0 & 0 \\ 0 & 0 & U & -B^t_{23} \end{bmatrix} \tag{4.23}$$

(4.21) becomes

$$\frac{d}{dt}\begin{bmatrix} X_{1b} \\ Y_{2c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F'_{1b}(Z_2) \\ F'_{2c}(Z_2) \\ F'_{1c}(Z_2) + B_{12}F'_{1b}(Z_2) \\ F'_{2b}(Z_2) - B^t_{23}F'_{2c}(Z_2) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ B_{11} & 0 \\ 0 & -B^t_{43} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} X_0 \\ Y_4 \end{bmatrix} \tag{4.24}$$

From (4.7) and (4.8), the variable Z_2 can be expressed as a function of the variable

$$Z_3 = (X_0, X_{1b}, X_{2b}, X_{3b}, Y_{1c}, Y_{2c}, Y_{3c})$$

However, in view of equations (4.15), (4.17), and the last two equations in (4.6), Z_3 can in turn be expressed in terms of

$$Z_4 = (X_{1b}, X_{2b}, Y_{1c}, Y_{2c}\ t)$$

Thus, recursively

$$Z_2 = G(Z_4) \tag{4.25}$$

By substituting (4.25) and the last two equations from (4.6) into (4.24) we then have the form

$$\frac{d}{dt}\begin{bmatrix} X_{1b} \\ Y_{2c} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F''_{1b}(Z_4) \\ F''_{2c}(Z_4) \\ G_1(Z_4) \\ G_2(Z_4) \end{bmatrix} \tag{4.26}$$

In order to reduce (4.26) to normal form, the last two equations must be solved for X_{2b} and Y_{1c}. This can be carried out, according to Theorem A.6, if the partial derivatives of G_1 and G_2 with respect of Y_{1c} and X_{2b} exist and satisfy certain properties. Thus, we have from (4.24),

$$\begin{bmatrix} B_{12} & U & 0 & 0 \\ 0 & 0 & U & -B^t_{23} \end{bmatrix} \begin{bmatrix} \frac{\partial F'_{1b}}{\partial Y_{1b}} & \frac{\partial F'_{1b}}{\partial Y_{1c}} & \frac{\partial F'_{1b}}{\partial X_{2b}} & \frac{\partial F'_{1b}}{\partial X_{2c}} \\ \frac{\partial F'_{1c}}{\partial Y_{1b}} & \frac{\partial F'_{1c}}{\partial Y_{1c}} & \frac{\partial F'_{1c}}{\partial X_{2b}} & \frac{\partial F'_{1c}}{\partial X_{2c}} \\ \frac{\partial F'_{2b}}{\partial Y_{1b}} & \frac{\partial F'_{2b}}{\partial Y_{1c}} & \frac{\partial F'_{2b}}{\partial X_{2b}} & \frac{\partial F'_{2b}}{\partial X_{2c}} \\ \frac{\partial F'_{2c}}{\partial Y_{1b}} & \frac{\partial F'_{2c}}{\partial Y_{1c}} & \frac{\partial F'_{2c}}{\partial X_{2b}} & \frac{\partial F'_{2c}}{\partial X_{2c}} \end{bmatrix} \begin{bmatrix} \frac{\partial Y_{1b}}{\partial Y_{1c}} & 0 \\ U & 0 \\ 0 & U \\ 0 & \frac{\partial X_{2c}}{\partial X_{2b}} \end{bmatrix} \tag{4.27}$$

Substituting (4.7) and (4.8) into (4.27) we then have

$$\begin{bmatrix} B_{12} & U & 0 & 0 \\ 0 & 0 & U & -B_{23}^t \end{bmatrix} \begin{bmatrix} \frac{\partial F'_{1b}}{\partial Y_{1b}} & \frac{\partial F'_{1b}}{\partial Y_{1c}} & \frac{\partial F'_{1b}}{\partial X_{2b}} & \frac{\partial F'_{1b}}{\partial X_{2c}} \\ \frac{\partial F'_{1c}}{\partial Y_{1b}} & \frac{\partial F'_{1c}}{\partial Y_{1c}} & \frac{\partial F'_{1c}}{\partial X_{2b}} & \frac{\partial F'_{1c}}{\partial X_{2c}} \\ \frac{\partial F'_{2b}}{\partial Y_{1b}} & \frac{\partial F'_{2b}}{\partial Y_{1c}} & \frac{\partial F'_{2b}}{\partial X_{2b}} & \frac{\partial F'_{2b}}{\partial X_{2c}} \\ \frac{\partial F'_{2c}}{\partial Y_{1b}} & \frac{\partial F'_{2c}}{\partial Y_{1c}} & \frac{\partial F'_{2c}}{\partial X_{2b}} & \frac{\partial F'_{2c}}{\partial X_{2c}} \end{bmatrix} \begin{bmatrix} B_{12}^t & 0 \\ U & 0 \\ 0 & U \\ 0 & -B_{23} \end{bmatrix} \tag{4.28}$$

But since the middle term in (4.28) is simply a partitioned form of (4.20), (4.28) is strictly nonsingular by Theorem A.5. Further, since all the partial derivatives indicated in (4.28) exist, are bounded and satisfy a Lipschitz condition, the hypothesis of Theorem A.6 is satisfied. Therefore, there exists a solution to the last two equations in (4.26) of the form

$$\begin{bmatrix} Y_{1c} \\ X_{2b} \end{bmatrix} = \begin{bmatrix} H_1(X_{1b}, Y_{2c}, t) \\ H_2(X_{1b}, Y_{2c}, t) \end{bmatrix} \tag{4.29}$$

where H_1 and H_2 satisfy a Lipschitz condition (Theorem A.7). Substituting (4.29) into the first two equations of (4.24) then yields the normal form

$$\frac{d}{dt} \begin{bmatrix} X_{1b} \\ Y_{2c} \end{bmatrix} = \begin{bmatrix} F'''_{1b}(X_{1b}, Y_{2c}, t) \\ F'''_{2c}(X_{1b}, Y_{2c}, t) \end{bmatrix} \tag{4.30}$$

where F'''_{1b} and F'''_{2c} again satisfy a Lipschitz condition. Therefore, by Theorem A.11, given the set of initial conditions I, there exists a unique solution for X_{1b} and Y_{2c} for $t_0 \leq t \leq t_0 + c$, $c > 0$. The remaining systems variables can be calculated uniquely from(4.20), (4.25), (4.6), (4.7), (4.8) and (4.22).

Other theorems may also be generated by taking various combinations of the theorems of Sections II, III, and IV. Further, additional generalizations can be included so that the theorems presented are more generally applicable. For example, in Theorem 4.3, a more general form of (4.6) might be considered, or the Lipschitz and differentiability properties might be considered over some defined region rather than every closed region. However, these added generalizations add considerably more detail to the already complicated proof and might serve only to further obscure a very important consideration - the formulation of a systematic procedure for obtaining a normal-form model of a nonlinear system.

V. A PROCEDURE FOR FORMULATING NONLINEAR MATHEMATICAL MODELS

In the previous sections, several existence theorems are presented and proved. From a practical standpoint, these theorems as such might well be considered to be of academic interest only. In practice, the solution is usually assumed to exist and the only remaining problem is that of actually obtaining it. However, the proof of Theorem 4.3 also serves as a basis for defining a formal procedure for obtaining mathematical models of nonlinear systems in normal form. Once such a model has been established, there are a number of existing numerical and analog techniques for actually constructing the solution.

<u>Systems of First Order Components</u>: First, consider the general class of systems with components described by algebraic or first order ordinary differential equations, i.e., systems having components described by equations of the form

$$\frac{dx_i}{dt} = f_i(x_1,x_2,\ldots,x_n,y_1,\ldots,y_n,t) \quad , \quad i = 1,2,\ldots,I \tag{5.1}$$

$$\frac{dy_j}{dt} = f_j(x_1,x_2,\ldots,x_n,y_1,\ldots,y_n,t) \quad , \quad j = I+1,\ldots,J \tag{5.2}$$

$$0 = f_k(x_1,x_2,\ldots,x_n,y_1,\ldots,y_n,t) \quad , \quad k = J+1,\ldots,n \tag{5.3}$$

In order to obtain the normal-form model for a representative system of this class, (5.1), (5.2), (5.3), and the circuit and cut-set equations from the system graph are combined as indicated in Section IV.

To select the variable to appear in the final system model, select a forest by defining the following subgraphs:

S_1 - The subgraph composed of all elements having across variables specified as a function of t by their respective component equations.

S_2 - The subgraph composed of all elements in S_1 and all elements having component equations of the form shown in (5.1).

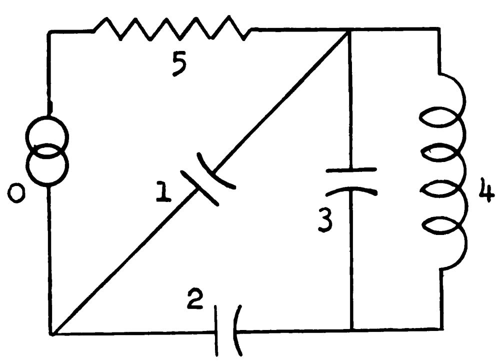
S_3 - The subgraph composed of all elements in S_2 and all elements having component equations of the form shown in (5.3).

S_4 - The subgraph composed of all elements in S_3 and all elements having component equations of the form shown in (5.2).

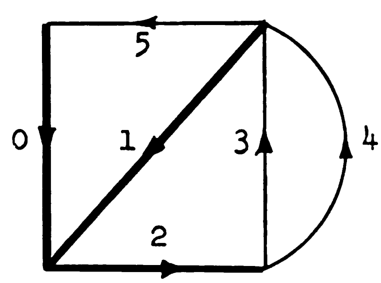
S_5 - The complete system graph.

A forest of the system graph is then choosen by defining T to be a forest of the subgraph S_i such that $T_i \supset T_{i-1}$ (every element contained in T_{i-1} is also contained in T_i). Then T_5 is the desired forest.

As an simple illustration, consider the example shown in Figure 5.1a.



(a)



(b)

Figure 5.1 A Typical R-L-C System

The component equations are of the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)y_1 \\ (1/C_2)y_2 \\ (1/C_3)y_3 \\ (1/L_4)x_4 \\ R_5y_5 - x_5 \\ x_0 - f(t) \end{bmatrix} \tag{5.4}$$

The defined subgraphs then consist of the following elements:

S_1 - element 0.

S_2 - elements 0,1,2,3.

S_3 - elements 0,1,2,3,5.

S_4 - all elements

S_5 - all elements

Thus, T_1 contains only element 0 and T_2 contains elements 0 and any two of the elements 1, 2, or 3. Since T_2 is a tree of the complete system graph, trees T_3, T_4, and T_5 are identical to T_2. One of the three possible trees T_5 is shown by heavy lines in Figure 5.1b.

Writing the fundamental circuit and cut-set equations for the graph of Figure 5.1b we have

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = 0 \tag{5.5}$$

Because of the unit matrices appearing in the fundamental circuit and cut-set equations, the chord across and branch through variables can be expressed as a function of the branch across and chord through variables. Thus, (5.5) can be rewritten in the form

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ -1 & +1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & -1 \\ -1 & -1 & +1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix} \tag{5.6}$$

This result can now be substituted into the component equations to yield, in the case of the present example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} (1/C_1)(y_3 + y_4 - y_5) \\ (1/C_2)(y_3 + y_4) \\ (1/C_3)\,(y_3) \\ (1/L_4)\,(-x_1 - x_2) \\ R_5 y_5 - x_1 + x_0 \\ x_0 - f(t) \end{bmatrix} \tag{5.7}$$

Notice that the coefficient matrix multiplying the derivative vector in (5.7) has maximum rank. This property is guaranteed by the specific choice of forest outlined earlier.

By elementary row operations, (5.7) can be reduced to the form shown in (5.8).

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ y_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\ (y_3 + y_4 - y_5) \\ (1/C_2)\ (y_3 + y_4) \\ (1/L_4)\ (-x_1 - x_2) \\ (1/C_1 + 1/C_2 + 1/C_3)\ y_3 + (1/C_1 + 1/C_2)\ y_4 - (1/C_1)\ y_5 \\ R_5 y_5 - x_1 + x_0 \\ x_0 - f(t) \end{bmatrix} \tag{5.8}$$

The last three equations in (5.8) can now be solved for the variables not appearing in the derivative vector on the left. By substituting this result into the top three differential equations we have the desired form, namely

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ y_4 \end{bmatrix} + \begin{bmatrix} j \\ k \\ l \end{bmatrix}[f(t)] \tag{5.9}$$

where

$a = -(C_1 + C_2)/(R_5)\ (C_1C_2 + C_2C_3 + C_3C_1)$

$b = 0$

$c = -(C_2)/(C_1C_2 + C_2C_3 + C_3C_1)$

$d = (C_3)/(R_5)\ (C_1C_2 + C_2C_3 + C_3C_1)$

$e = 0$

$f = (C_1)/(C_1C_2 + C_2C_3 + C_3C_1)$

$g = -(1/L_4)$

$h = g$

$i = 0$

$j = a$

$k = d$

$l = 0$

A review of the steps demonstrated by the example of Figure 5.1 shows that there are five basic steps leading to the normal form:

1). Select an appropriate forest of the system graph.

2). Write the fundamental circuit and cut-set equations in a form explicit in the chord across and branch through variables.

3). Substitute the fundamental circuit and cut-set equations into the component equations.

4). By elementary operations, eliminate the coefficient matrix multiplying the derivative vector.

5). Solve the algebraic equations for the variables not appearing in the derivative vector and substitute this result into the differential equations.

Each of the five steps listed above is well defined and is, at least in principle, easy to carry out. Further, in most situations that arise in practice, step 4 is not required. In these cases, the first four steps can often be carried out by "inspection".

The example used to demonstrate each of the above steps was linear for simplicity only. In fact, by defining $C_i = C_i(x_i)$ and $L_j = L_j(y_j)$, the notation used in the above example also suffices for the nonlinear situation. Notice, however, that if R_5 is considered as a function of y_5, it is quite likely that the fifth equation in (5.8) cannot be solved explicitly for y_5. If such is the case, the normal form cannot be obtained explicitly. Rather, the nonlinear algebraic equation must be carried along and solved simultaneously with the differential equations.

Before considering the subject of systems of components described by higher order differential equations, it should be pointed out that the formal procedure used above for selecting a forest of the system graph is simply a precise way of stating a very simple concept. That is, the indicated method of choosing a forest simply implies that

1) all the specified across drivers are made branches,
2) as many as possible of the elements having equations explicit in the derivative of the across variable are branches,
3) as many as possible of the elements having equations explicit in the derivative of the through variable are chords, and
4) all the through drivers are chords.

This specific choice of forest is made to insure that the coefficient matrix of the derivatives vector in (5.7) has maximum rank. It is therefore possible, by elementary operations, to reduce (5.7) to normal form.

<u>Higher Order Systems</u>: The procedure given for components modeled by first order differential equations is directly extendable to some of the higher order forms. The only necessary changes occur in the choice of the forest and in the form of the component equations.

Consider a component equation of the form

$$\frac{d^n x}{dt^n} = f(x, \frac{dx}{dt}, \ldots \frac{d^{n-1}x}{dt^{n-1}}, y, t) \tag{5.10}$$

This equation can be reduced to first-order form by simply defining new variables to replace the lower order derivatives. For example, the equation

$$\frac{d^3 x}{dt^3} = f\ (x, \frac{dx}{dt}, \frac{d^2 x}{dt^2}, y, t) \tag{5.11}$$

can be written in the form

$$\frac{d}{dt}\begin{bmatrix} \ddot{x} \\ \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} f\ (x, \dot{x}, \ddot{x}, y, t) \\ \ddot{x} \\ \dot{x} \end{bmatrix} \tag{5.12}$$

The last two equations in (5.12) simply define the variables used to replace the lower order derivatives. The resulting set of first order equations can then be treated as indicated in the previous example.

The extension to higher order systems also requires a slight alteration in the choice of forest. In order to obtain the desired properties, the subgraphs S_2 and S_4 must be further subdivided according to the order of the derivative of the variable associated with each element. Thus, define the following subgraphs:

$S_{2,j}$ - The subgraph consisting of the elements having component equations explicit in the j^{th} derivative of the across variable and all elements which are contained in $S_{2,j+1}$, $j \neq n$. If $j = n$, where n is the order of the highest order derivatives of an across variable, then $S_{2,n}$ also includes all elements of S_1.

$S_{4,k}$ - The subgraph consisting of elements having component equations explicit in the k^{th} derivative of a through variable and all elements which are contained in $S_{4,k-1}$, $k \neq 1$. If $k=1$, $S_{4,1}$ also contains all elements of S_3.

Now define T_i to be a forest of S_i, $T_{2,j}$ to be a forest of $S_{2,j}$, etc., such that

$$T_1 \subset T_{2,n} \subset T_{2,n-1} \subset \cdots \subset T_{2,1} \subset T_3 \subset T_{4,1} \quad \cdots \subset T_{4,m} \subset T_5$$

where n and m represent the highest order derivative of an across and through variable, respectively, that appears in the component equations. As before, T_5 is the desired forest.

It should be emphasized once again that the definition of subgraphs and their associated forests is simply a precise way of writing down a simple intuitive notion. The scheme for selecting these subgraphs is such that the elements modeled by the higher order differential equations are given priority in as far as their being selected as branches or chords, depending upon whether the component equations are explicit in the derivative of an across or a through variable, respectively.

To illustrate the procedure for component models involving higher order equations, consider the simple mechanical system shown in Figure 5.2a.

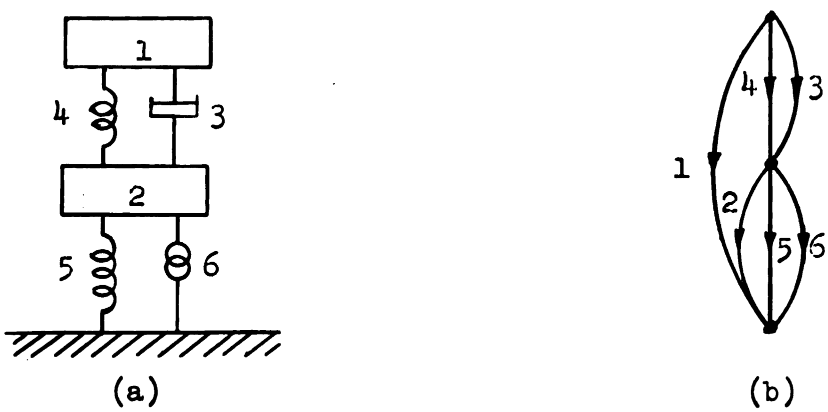


Figure 5.2 A Mechanical System

The component equations for this system are of the form

$$\begin{bmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \\ \frac{dx_3}{dt} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M_1)y_1 \\ (1/M_2)y_2 \\ (1/B_3)y_3 \\ y_4 - K_4x_4 \\ y_5 - K_5x_5 \\ y_6 - f(t) \end{bmatrix} \tag{5.13}$$

The defined subgraphs and corresponding forests are:

S_1 - null T_1 - null

$S_{2,2}$ - elements 1, 2. $T_{2,2}$ - elements 1, 2.

$S_{2,1}$ - elements 1, 2, 3. $T_{2,1}$ - elements 1, 2.

S_3 - elements 1, 2, 3, 4, 5. T_3 - elements 1, 2.

S_4 - elements 1, 2, 3, 4, 5. T_4 - elements 1, 2.

S_5 - elements 1, 2, 3, 4, 5, 6. T_5 - elements 1, 2.

Writing the component equations in first order form we have

$$\frac{d}{dt}\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M_1)y_1 \\ \dot{x}_1 \\ (1/M_2)y_2 \\ \dot{x}_2 \\ (1/B_3)y_3 \\ y_4 - K_4x_4 \\ y_5 - K_5x_5 \\ y_6 - f(t) \end{bmatrix} \tag{5.14}$$

The fundamental circuit and cut-set equations for the forest T_5 are

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} \tag{5.15}$$

Substituting (5.15) into (5.14) we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1/M_1)\,(-y_3 - y_4) \\ \dot{x}_1 \\ (1/M_2)\,(y_3 + y_4 - y_5 - y_6) \\ \dot{x}_2 \\ (1/B_3) y_3 \\ y_4 - K_4\,(x_1 - x_2) \\ y_5 - K_5\,(x_2) \\ y_6 - f(t) \end{bmatrix} \tag{5.16}$$

By elementary row operations, (5.16) reduces to

$$\frac{d}{dt} \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M_1)\,(-y_3 - y_4) \\ \dot{x}_1 \\ (1/M_2)\,(y_3 + y_4 - y_5 - y_6) \\ \dot{x}_2 \\ (1/B_3)\,y_3 - \dot{x}_1 + \dot{x}_2 \\ y_4 - K_4\,(x_1 - x_2) \\ y_5 - K_5\,(x_2) \\ y_6 - f(t) \end{bmatrix} \tag{5.17}$$

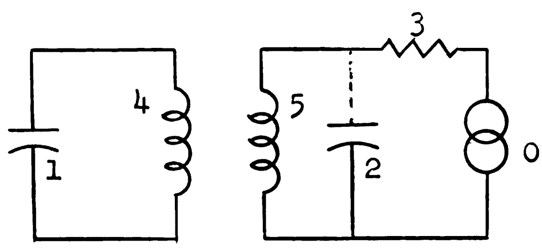
The normal form model of the system is obtained when the last four algrebraic equations in (5.17) are solved for y_3, y_4, y_5, and y_6 and the result substituted into the remaining differential equations. Thus we have

$$\frac{d}{dt}\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1/M_1)\ (B_3(\dot{x}_2 - \dot{x}_1)) + K_4(x_2 - x_1) \\ \dot{x}_1 \\ (1/M_2)\ (B_3(\dot{x}_1 - \dot{x}_2)) + K_4(x_1 - x_2) - K_4 x_2 - f(t) \\ \dot{x}_2 \end{bmatrix} \tag{5.18}$$

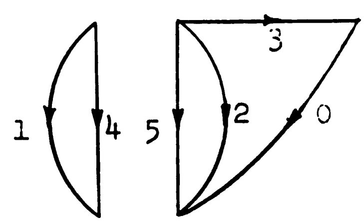
It can be seen by this example that the formulation procedure for the higher order component equations is much the same as in the first order case. It differs only in that additional steps are necessary to first reduce the higher order forms to an equivalent first order form and that a slightly different procedure is used for choosing the forest. It should also be noted that when substituting the fundamental circuit and cut-set equations into the component equations, the derivative of one or more of the circuit or cut-set equations may be required to effect the substitution.

Special Forms of Algebraic Equations: In the two previous parts of this section, a formulation procedure is outlined for systems described by first and higher order differential equations. With one exception, all of the indicated steps can be carried out in general. However, certain situations exist where the required solution of the algebraic equations either does not exist, or is nonunique. Consequently, special procedures must be employed to deal with these cases.

To demonstrate the nature of the problem, consider the system shown schematically in Figure 5.3a.



(a)



(b)

Figure 5.3 An Electrical System

The component equations for this system are

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)y_1 \\ (1/C_2)y_2 \\ R_3y_3 - x_3 \\ x_0 - f(t) \\ y_4 + Ky_5 \\ x_5 - Kx_4 \end{bmatrix} \tag{5.19}$$

Proceeding as indicated in the earlier part of this section, we have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)(-y_4) \\ (1/C_2)(-y_3 - y_5) \\ R_3 y_3 - (x_2 - x_0) \\ x_0 - f(t) \\ y_4 + Ky_5 \\ x_2 - Kx_1 \end{bmatrix} \tag{5.20}$$

Normally, the last four equations in (5.20) would be solved for y_3, y_4, y_5, and x_0 as a function of x_1, x_2, and t. But since the last equation in (5.20) is independent of y_3, y_4, y_5, and x_0, no such unique solution exists.

In an attempt to circumvent this difficulty, consider the derivative of the last equation of (5.20) with respect to t.

$$\frac{dx_2}{dt} - K\frac{dx_1}{dt} = 0 \tag{5.21}$$

If $x_1(t)$ and $x_2(t)$ are any two functions which satisfy (5.21), then

$$x_2(t) \quad -Kx_1(t) \quad = \quad c$$

for some constant c. Therefore, if it is also required that

$$x_2(t_0) \quad -Kx_1(t_0) \quad = \quad 0$$

then any two functions which satisfy (5.21) and the above boundry condition also satisfy the last equation in (5.20) and conversely. Therefore, the last equations in (5.20) can be replaced by (5.21) without loss of precision giving

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\ (-y_4) \\ (1/C_2)\ (-y_3\ -y_5) \\ R_3y_3 - (x_2 - x_0) \\ x_0 - f(t) \\ y_4 \quad Ky_5 \\ \dfrac{dx_2}{dt} - \dfrac{Kdx_1}{dt} \end{bmatrix} \tag{5.22}$$

where $x_2(t_0) - Kx_1(t_0) = 0$. At first glance it would seem that such a substitution defeats the purpose of the formulation procedure because it introduces derivatives in the right-hand side of the equations. Note, however, that these derivatives are also contained on the left-hand side of (5.22). The top two differential equations in (5.22) can therefore be substituted into the last equation to yield

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\ (-y_4) \\ (1/C_2)\ (-y_3\ -y_5) \\ R_3y_3 - (x_2 - x_0) \\ x_0 - f(t) \\ y_4 + Ky_5 \\ (K/C_1)y_4 - (1/C_1)\ (y_3 + y_5) \end{bmatrix} \tag{5.23}$$

By solving for y_3, y_4, y_5, and x_0 in the last four equations in (5.23) and substiuting this result into the first two equations, we have the normal form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \begin{bmatrix} (K/(R_3(C_2K^2 + C_1)))(f(t) - x_2) \\ \\ (K^2/(R_3(C_2K^2 + C_1)))(f(t) - x_2) \end{bmatrix} \tag{5.24}$$

where $x_2(t_0) - Kx_1(t_0) = 0$. Note that the condition imposed upon the initial conditions must be carried along to assure that the solution to (5.24) also satisfies (5.20).

The general situation typlified by this example is one in which, are performing the first four steps of the formulation procedure, the equations take on the form

$$\frac{d}{dt}\begin{bmatrix} Z_1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1(Z_1, Z_2, t) \\ F_2(Z_1, Z_2, t) \end{bmatrix} \tag{5.25}$$

where for some k,

$$f_{2k}(Z_1, Z_2, t) = f'_{2k}(Z_1, t) = 0 \tag{5.26}$$

i.e., one of the algebraic equations is a function of the entries in Z_1

alone. Since F_2 contains the same number of equations as there are entries in Z_2, it follows that from F_2, no unique solution can be found for Z_2 as a function of Z_1 and t.

Taking the derivative of (5.26) with respect to t we have

$$\sum_j \frac{\partial f'_{2k}}{\partial z_{1j}} \frac{dz_{1j}}{dt} = 0 \qquad (5.27)$$

provided that the indicated parital derivatives exist. If the added condition

$$f'_{2k} (Z_1(t_0), t_0) = 0 \qquad (5.28)$$

is imposed, then any set of functions $\left[z_{1j}(t)\right]$ which satisfy (5.27) also satisfy (5.26) and conversely.

Substituting from (5.25) for the derivatives in (5.27) yields

$$\sum_j \frac{\partial f_{2k}}{\partial z_{1j}} f_{1j} (Z_1, Z_2, t) = 0 \qquad (5.29)$$

Usually, at least for first order systems, (5.29) is dependent upon Z_2. Therefore, if (5.26) is replaced by an equivalent expression, namely (5.29) and the constraint (5.28), the difficulty is eliminated.

To illustrate further, consider the compound pendulum shown in Figure 5.4a.

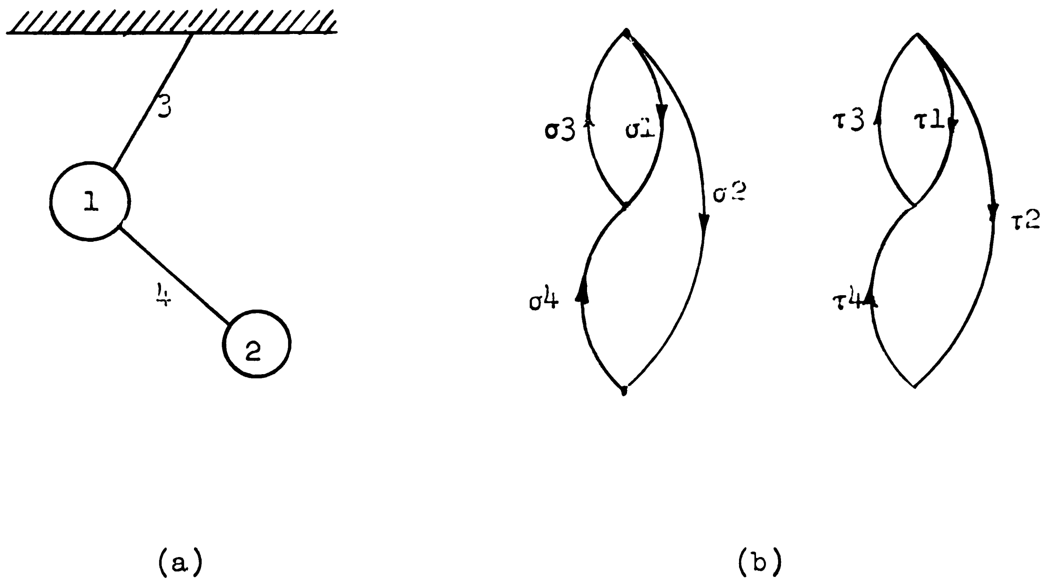


Figure 5.4 Compound Pendulum

For this example, the component equations are

$$\frac{d}{dt}\begin{bmatrix} \dot{x}_{\sigma 1} \\ x_{\sigma 1} \\ \dot{x}_{\tau 1} \\ x_{\tau 1} \\ \dot{x}_{\sigma 2} \\ x_{\sigma 2} \\ \dot{x}_{\tau 2} \\ x_{\tau 2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M_1)y_{\sigma 1} \\ \dot{x}_{\sigma 1} \\ (1/M_1)y_{\tau 1} + g \\ \dot{x}_{\tau 1} \\ (1/M_2)y_{\sigma 2} \\ \dot{x}_{\sigma 2} \\ (1/M_2)y_{\tau 2} + g \\ \dot{x}_{\tau 2} \\ x_{\sigma 3}^2 + x_{\tau 3}^2 - R_3^2 \\ x_{\sigma 3}\, y_{\tau 3} - x_{\tau 3}\, y_{\sigma 3} \\ x_{\sigma 4}^2 + x_{\tau 4}^2 - R_4^2 \\ x_{\sigma 4}y_{\tau 4} - x_{\tau 4}y_{\sigma 4} \end{bmatrix} \qquad (5.30$$

where the subscripts σ and τ denote the two co-ordinates of motion. For the forest of the system graph (Figure 5.4b) which contains elements $\sigma 1$, $\tau 1$, $\sigma 2$, and $\tau 2$, the fundamental circuit equations are

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\sigma 1} \\ x_{\sigma 2} \\ x_{\tau 1} \\ x_{\tau 2} \\ x_{\sigma 3} \\ x_{\sigma 4} \\ x_{\tau 3} \\ x_{\tau 4} \end{bmatrix} = 0 \tag{5.31}$$

Substituting (5.31) and the associated cut-set equations into (5.30) gives

$$\frac{d}{dt} \begin{bmatrix} \dot{x}_{\sigma 1} \\ x_{\sigma 1} \\ \dot{x}_{\tau 1} \\ x_{\tau 1} \\ \dot{x}_{\sigma 2} \\ x_{\sigma 2} \\ \dot{x}_{\tau 2} \\ x_{\tau 2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M_1)\,(y_{\sigma 4} - y_{\sigma 3}) \\ \dot{x}_{\sigma 1} \\ (1/M_1)\,(y_{\tau 4} - y_{\tau 3}) + g \\ \dot{x}_{\tau 1} \\ (1/M_2)\,(-y_{\sigma 4}) \\ \dot{x}_{\sigma 2} \\ (1/M_2)\,(-y_{\tau 4}) + g \\ \dot{x}_{\tau 2} \\ x_{\sigma 1}^2 + x_{\tau 1}^2 - R_3^2 \\ x_{\sigma 1}\, y_{\tau 3} - x_{\tau 1}\, y_{\sigma 3} \\ (x_{\sigma 2} - x_{\sigma 1})^2 + (x_{\tau 2} - x_{\tau 1})^2 - R_4^2 \\ (x_{\sigma 2} - x_{\sigma 1})\, y_{\tau 4} - (x_{\tau 2} - x_{\tau 1})\, y_{\sigma 4} \end{bmatrix} \tag{5.32}$$

Note that the ninth and eleventh equations in (5.32) are functions of the variables in the derivative vector only. Taking the derivative of these equations with respect to t we have

$$\begin{aligned} & x_{\sigma 1} \frac{dx_{\sigma 1}}{dt} + x_{\tau 1} \frac{dx_{\tau 1}}{dt} = 0 \\ & (x_{\sigma 2} - x_{\sigma 1}) \left(\frac{dx_{\sigma 2}}{dt} - \frac{dx_{\sigma 1}}{dt} \right) + (x_{\tau 2} - x_{\tau 1}) \left(\frac{dx_{\tau 2}}{dt} - \frac{dx_{\tau 1}}{dt} \right) = 0 \end{aligned} \tag{5.33}$$

with the constraints

$$\begin{aligned} & x_{\sigma 1} \dot{x}_{\sigma 1} + x_{\tau 1} \dot{x}_{\tau 1} = 0 \\ & (x_{\sigma 2}(t_0) - x_{\sigma 1}(t_0))^2 + (x_{\tau 2}(t_0) - x_{\tau 1}(t_0))^2 - R_4^2 = 0 \end{aligned} \tag{5.34}$$

Substituting from (5.32) for the derivatives in (5.33) we have

$$\begin{aligned} & x_{\sigma 1} \dot{x}_{\sigma 1} + x_{\tau 1} \dot{x}_{\tau 1} = 0 \\ & (x_{\sigma 2} - x_{\sigma 1}) (\dot{x}_{\sigma 2} - \dot{x}_{\sigma 2}) + (x_{\tau 2} - x_{\tau 1}) (\dot{x}_{\tau 2} - \dot{x}_{\tau 1}) = 0 \end{aligned} \tag{5.35}$$

If the ninth and eleventh equations in (5.32) are replaced by (5.35), the algebraic equations in (5.32) become

$$\begin{bmatrix} x_{\sigma 1} \dot{x}_{\sigma 1} + x_{\tau 1} \dot{x}_{\tau 1} \\ x_{\sigma 1} y_{\tau 3} - x_{\tau 1} t_{\sigma 3} \\ (x_{\sigma 2} - x_{\sigma 1}) (\dot{x}_{\sigma 2} - \dot{x}_{\sigma 1}) + (x_{\tau 2} - x_{\tau 1}) (\dot{x}_{\tau 2} - \dot{x}_{\tau 1}) \\ (x_{\sigma 2} - x_{\sigma 1}) y_{\tau 4} - (x_{\tau 2} - x_{\tau 1}) y_{\sigma 4} \end{bmatrix} = 0 \tag{5.36}$$

Once again, the same problem occurs, i.e., the first and third equations in (5.36) are not functions of the variables $y_{\sigma 3}$, $y_{\tau 3}$, $y_{\sigma 4}$, or $y_{\tau 4}$. Repeating the same procedure once again we have

$$\begin{gathered}
\frac{dx_{\sigma 1}}{dt}\, x_{\sigma 1} + x_{\sigma 1}\, \frac{dx_{\sigma 1}}{dt} + \frac{dx_{\tau 1}}{dt}\, x_{\tau 1} + x_{\tau 1}\, \frac{dx_{\tau 1}}{dt} = 0 \\
\left(\frac{d}{dt}\,(x_{\sigma 2} - x_{\sigma 1})\right)(x_{\sigma 2} - x_{\sigma 1}) + (x_{\sigma_2} - x_{\sigma 1})\, \frac{d}{dt}\,(x_{\sigma 2} - x_{\sigma 1}) + \\
\left(\frac{d}{dt}\,(x_{\tau 2} - x_{\tau 1})\right)(x_{\tau 2} - x_{\tau 1}) + (x_{\tau 2} - x_{\tau 1})\, \frac{d}{dt}\,(x_{\tau 2} - x_{\tau 1}) = 0
\end{gathered} \tag{5.37}$$

with the condition that (5.35) must be satisfied for $t = t_0$. Substituting the first eight equations of (5.30) into (5.37) and replacing the first and third equations in (5.36) with this result we have

$$\begin{bmatrix}
-x_{\sigma 1}/M_1 & -x_{\tau 1}/M_1 & x_{\sigma 1}/M_1 & x_{\tau 1}\, M_1 \\
-x_{\tau 1} & x_{\sigma 1} & 0 & 0 \\
(x_{\sigma 2} - x_{\sigma 1})/M_1 & (x_{\tau 2} - x_{\tau 1})/M_1 & -(x_{\sigma 2} - x_{\sigma 1})\,(1/M_1 + 1/M_2) & -(x_{\tau 2} - x_{\tau 1})\,(1\ M_1 + 1\ M_2) \\
0 & 0 & & -(x_{\sigma 2} - x_{\sigma 1})
\end{bmatrix}
\begin{bmatrix} y_{\sigma 3} \\ y_{\tau 3} \\ y_{\sigma 4} \\ y_{\tau 4} \end{bmatrix}
=
\begin{bmatrix}
-x_{\sigma 1}^2 - x_{\tau 1}^2 - x_{\tau 1} + g \\
0 \\
-(x_{\tau 2} - x_{\tau 1})^2 - (x_{\sigma 2} - x_{\sigma 1})^2 \\
0
\end{bmatrix} \tag{5.38}$$

By solving (5.38) for $y_{\sigma 3}$, $y_{\tau 3}$, $y_{\sigma 4}$, and $y_{\tau 4}$ and substituting the result into the first eight equations in (5.30), the normal-form model for the system is obtained. Note, however, that the initial conditions used to obtain any desired solution to the system model are not completely arbitrary. They are subject to the added conditions that they must satisfy (5.34) and (5.35) for $t = t_0$, the initial value of the independent variable.

In this latter example, it was necessary to differentiate the algebraic equations twice. In general, if some of the component equations are of n^{th} order, n differentiations of the algebraic equations may be required.

In an attempt to put order into the problem presented above, note that the equations which lead to difficulty in the three previous examples are the perfect coupler models encountered earlier in Section II, i.e., a component equation which relates an across variable to another across variable. Also notice that when attempting to obtain a solution for the algebraic equations, the variables in the derivative vector are treated essentially as if they were known functions of t. Therefore, the question of whether or not a solution exists for the algebraic equations, at least in the linear case, can be answered by determining whether or not any forest of the class outlined in the first part of this section also satisfies the hypothesis of Theorem 2.5 in-so-far as the elements of the graph having algebraic component equations are concerned.

For the example shown in Figure 5.3a, the only possible choice of forest includes elements 0, 1, and 2. This automatically classifies elements 4 and 5 as chords. Consequently, since the component equations for these elements are

$$\begin{bmatrix} x_5 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix} \begin{bmatrix} y_5 \\ x_4 \end{bmatrix}$$

the hypothesis of Theorem 2.5 is not satisfied and no unique solution exists for the algebraic equations.

In this way, at least in the linear cases, the need for special formulation procedures can be anticipated from the form of the component equations and the manner in which the components are interconnected.

As a final illustration of systems containing perfect coupler models, consider the system shown in Figure 5.5a.

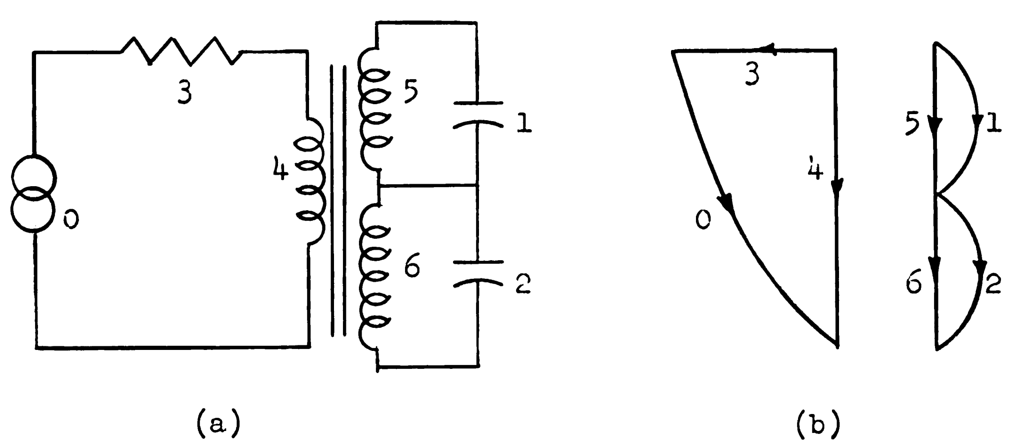


Figure 5.5 An Electrical System

Since the component equations are of the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)y_1 \\ (1/C_2)y_2 \\ x_5 - K_1x_4 \\ x_6 - K_2x_4 \\ y_4 + K_1y_5 + K_2y_6 \\ R_3y_3 - x_3 \\ x_0 - f(t) \end{bmatrix} \tag{5.39}$$

only two forests can be selected consistent with the procedure outlined in the first part of this section. Neither of the two forests, (0, 1, 2, 3) or (0, 1, 2, 4), are acceptable from the standpoint of Theorem 2.5. It is evident, therefore, that the algebraic equations are such that the normal form cannot be obtained without special procedures. Writing the fundamental circuit and cut-set equations from Figure 5.5b and substituting this result into (5.39) we have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\ (-y_5) \\ (1/C_2)\ (-y_6) \\ x_1 - K_1x_4 \\ x_2 - K_2x_4 \\ -y_3 + K_1y_5 + K_2y_6 \\ R_3y_3 - x_4 + x_0 \\ x_0 - f(t) \end{bmatrix} \tag{5.40}$$

Since the third and fourth equations are functions of x_1, x_2, and x_4 only, the five algebraic equations in (5.40) have no unique solution for y_3, y_5, y_6, x_4, and x_0 as a function of x_1, x_2, and t. But in this example, none of the equations are functions of x_1 and x_2 alone. It would appear, therefore, that the system shown in Figure 5.5 is beyond the scope of the development presented thus far.

Notice, however, that by solving the fourth equation in (5.39) for x_4, (5.39) can be rewritten in the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)y_1 \\ (1/C_2)y_2 \\ x_5 - (K_1/K_2)x_6 \\ x_4 - (1/K_2)x_6 \\ y_6 + (K_1/K_2)y_5 + (1/K_2)y_4 \\ R_3y_3 - x_3 \\ x_0 - f(t) \end{bmatrix} \tag{5.41}$$

Combining (5.41) with the circuit and cut-set equations then gives the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\ (-y_5) \\ (1/C_2)\ (-y_6) \\ x_1 - (K_1/K_2)x_2 \\ x_4 - (1/K_2)x_2 \\ y_6 + (K_1/K_2)y_5 - (1/K_2)y_4 \\ R_3y_3 - x_4 + x_0 \\ x_0 - f(t) \end{bmatrix} \tag{5.42}$$

But now the third equation does relate x_1 and x_2 and the normal-form model of the system can be obtained as in the previous examples. Thus, it is seen that for systems involving multiport perfect couplers, the component equations should be written in a certain form. Specifically, if one of the elements, say e_n, corresponding to a perfect coupler is a chord for every forest of the type indicated in the first part of this section, the perfect coupler equations should be written in the form

$$\begin{bmatrix} x_{e1} \\ x_{e2} \\ \vdots \\ x_{en-1} \\ y_{en} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & h_1 \\ 0 & 0 & \cdots & h_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h_{n-1} \\ -h_1 & -h_2 & & 0 \end{bmatrix} \begin{bmatrix} y_{e1} \\ y_{e2} \\ \vdots \\ \\ x_{en} \end{bmatrix} = 0 \tag{5.43}$$

Similarly, if one of the elements, say e_j, $j < n$, corresponding to a perfect coupler is a branch for every forest T_5 indicated earlier, the perfect coupler equations should again be in the form shown in (5.43).

In the case of the example of Figure 5.5, element 6 is a chord for every forest T_5. Therefore, a proper form for the perfect coupler equations is

$$\begin{bmatrix} x_5 \\ x_4 \\ y_6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & K_1/K_2 \\ 0 & 0 & 1/K_2 \\ -K_1/K_2 & -1/K_2 & 0 \end{bmatrix} \begin{bmatrix} y_5 \\ y_4 \\ x_6 \end{bmatrix} = 0 \tag{5.43}$$

The form

$$\begin{bmatrix} x_6 \\ x_4 \\ y_5 \end{bmatrix} - \begin{bmatrix} 0 & 0 & K_2/K_1 \\ 0 & 0 & 1/K_1 \\ -K_2/K_1 & -1/K_1 & 0 \end{bmatrix} \begin{bmatrix} y_6 \\ y_4 \\ x_5 \end{bmatrix} = 0 \tag{5.45}$$

is also acceptable because element 5 is also a chord for every forest T_5.

<u>More Examples</u>: To illustrate further the range of applicability of the formulation procedure presented in this thesis, several additional examples representing several different fields are now presented.

First, consider the problem of calculating the trajectory of a rocket on an orbit around the moon. If the center of the earth is considered as the inertial reference, the equations of the rocket are

$$\frac{d^2}{dt^2} \begin{bmatrix} x_\sigma \\ x_\tau \end{bmatrix} = \begin{bmatrix} (1/M(t))\ (T_\sigma(t) - B_\sigma\ (x_\sigma,\ x_\tau,\ \dot{x}_\sigma)) \\ (1/M(t))\ (T_\tau(t) - B_\tau\ (x_\sigma,\ x_\tau,\ \dot{x}_\tau)) \end{bmatrix}$$

where $M(t)$ is the mass of the rocket, $B_\sigma(x_\sigma,\ x_\tau,\ \dot{x}_\sigma)$ and $B_\tau(x_\sigma, x_\tau, \dot{x}_\sigma)$ are functions which determine the frictional effects of the atmosphere and $T_\sigma(t)$ and $T_\tau(t)$ represent the thrust of the rocket engine.

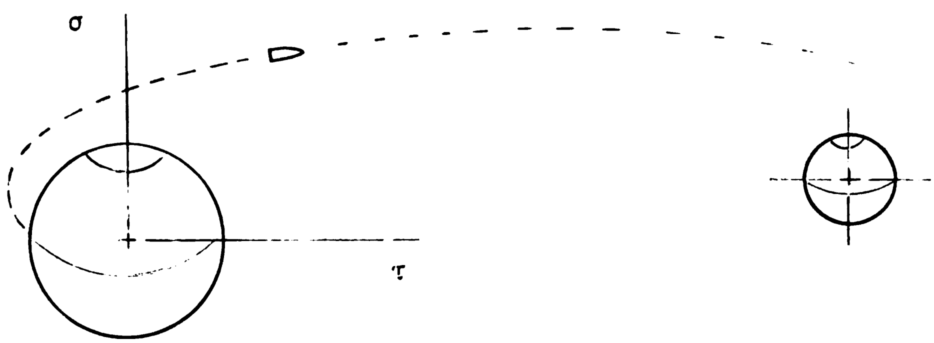
The effect of gravity upon the rocket can be considered as a component having equations of the form

$$\begin{bmatrix} y_\sigma \\ y_\tau \end{bmatrix} = -KM(t)\ (x_\sigma^2 + x_\tau^2)^{-\frac{3}{2}} \begin{bmatrix} x_\sigma \\ x_\tau \end{bmatrix}$$

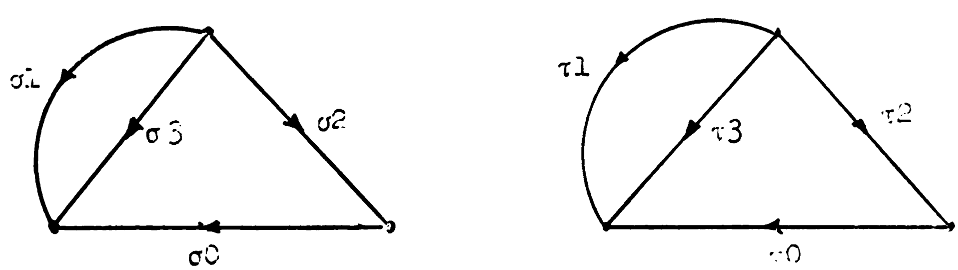
The position of the moon in its orbit around the earth is given by

$$\begin{bmatrix} x_\sigma \\ x_\tau \end{bmatrix} = K_1 \begin{bmatrix} \sin K_2 t \\ \cos K_2 t \end{bmatrix}$$

A schematic diagram of the system under consideration and the system graph are shown in Figure 5.6.



(a)



(b)

Figure 5.6 Orbital System

In the graph, element 1 pertains to the rocket - earth component, elements 2 and 3 pertain respectively to the gravitational effects of the moon and the earth upon the rocket and element 0 pertains to the relative position of the moon with respect to the earth.

Combining the various component equations into one set we have

$$\frac{d}{dt}\begin{bmatrix} \dot{x}_{\sigma 1} \\ x_{\sigma 1} \\ \dot{x}_{\tau 1} \\ x_{\tau 1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M(t))\ (T_\sigma(t) - B_\sigma(x_{\sigma 1}, x_{\tau 1}, \dot{x}_{\sigma 1}) + y_{\sigma 1}) \\ \dot{x}_{\tau 1} \\ (1/M(t))\ (T_\tau(t) - B_\tau(x_{\sigma 1}, x_{\tau 1}, \dot{x}_{\tau 1}) + y_{\tau 1}) \\ \dot{x}_{\tau 1} \\ y_{\sigma 2} + K_1 M(t)\ x_{\sigma 2}/(x_{\sigma 2}^2 + x_{\tau 2}^2)^{\frac{3}{2}} \\ y_{\tau 2} + K_1 M(t)\ x_{\tau 2}/(x_{\sigma 2}^2 + x_{\tau 2}^2)^{\frac{3}{2}} \\ y_{\sigma 3} + K_2 M(t)\ x_{\sigma 3}/(x_{\sigma 3}^2 + x_{\tau 3}^2)^{\frac{3}{2}} \\ y_{\tau 3} + K_2 M(t)\ x_{\tau 3}/(x_{\sigma 3} + x_{\tau 3}^2)^{\frac{3}{2}} \\ x_{\sigma 0} + K_3 \sin K_4 t \\ x_{\tau 0} + K_3 \cos K_4 t \end{bmatrix} \qquad (5.46)$$

For the forest indicated in Figure 5.6b, the fundamental circuit and cut-set equations can be written and substituted into (5.46) to give

$$\frac{d}{dt}\begin{bmatrix} \dot{x}_{\sigma 1} \\ x_{\sigma 1} \\ \dot{x}_{\tau 1} \\ x_{\tau 1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M(t))\ (T_\sigma(t) - B_\sigma(x_{\sigma 1}, x_{\tau 1}, \dot{x}_{\tau 1}) - (y_{\sigma 2} + y_{\sigma 3}) \\ \dot{x}_{\sigma 1} \\ (1/M(t))\ (T_\tau(t) - B_\tau(x_{\sigma 1}, x_{\tau 1}, \dot{x}_{\tau 1}) - (y_{\tau 2} + y_{\tau 3}) \\ \dot{x}_{\tau 1} \\ y_{\sigma 2} + K_1 M(t)\ (x_{\sigma 1} - x_{\sigma 0})/((x_{\sigma 1} - x_{\sigma 0})^2 + (x_{\tau 1} - x_{\tau 0})^2)^{\frac{3}{2}} \\ y_{\tau 2} + K_1 M(t)\ (x_{\tau 1} - x_{\tau 0})/((x_{\sigma 1} - x_{\sigma 0})^2 + (x_{\tau 1} - x_{\tau 0})^2)^{\frac{3}{2}} \\ y_{\sigma 3} + K_2 M(t)\ (x_{\sigma 1})/(x_{\sigma 1}^2 + x_{\tau 1}^2)^{\frac{3}{2}} \\ y_{\tau 3} + K_2 M(t)\ (x_{\tau 1})/(x_{\sigma 1}^2 + x_{\tau 1}^2)^{\frac{3}{2}} \\ x_{\sigma 0} + K_3 \sin K_4 t \\ x_{\tau 0} + K_3 \cos K_4 t \qquad (5.47) \end{bmatrix}$$

The last six algebraic equations in (5.47) can be easily solved for $y_{\sigma 2}$, $y_{\tau 2}$, $y_{\sigma 3}$, $y_{\tau 3}$, $x_{\sigma 0}$, and $x_{\tau 0}$. If this result is substituted into the first four differential equation, we have the desired normal form model of the system.

In general, the orbital problem is made somewhat more complex by virtue of a more complicated expression for the gravitational effects between the rocket and the earth. Adding a third demension to the problem also increases the complexity by adding one more differential equation and three more algebraic equations. The method of formulation would be exactly the same, however,

The formulation procedure presented in this thesis always assumes that the component equations have been solved for the highest order derivatives. Although in some cases this means that extra steps may be required, these steps may lead to very useful results over and above simply formulating a model of the system. For example, consider the equations describing a two winding transformer.

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.48}$$

It is frequently appropriate to make the approximation that

$$M_{11}M_{22} - M_{12}^2 = 0$$

and

$$M_{11}/M_{12} = N_{12}$$

where N_{12} is a constant, the turns ratio of the transformer. Then, by making the change of variable

$$y_\tau = y_1 + N_{12}y_2$$

(5.48) becomes

$$\begin{bmatrix} M_{11} & 0 \\ M_{12} & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_\tau \\ y_2 \end{bmatrix} + \begin{bmatrix} R_1 & -R_1N_{12} \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} y_\tau \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.49}$$

Premultiplying (5.49) by the nonsingular matrix

$$\begin{bmatrix} 1 & 0 \\ -N_{12} & 1 \end{bmatrix}$$

we have

$$\begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_\tau \\ y_2 \end{bmatrix} + \begin{bmatrix} R_1 & -R_1N_{12} \\ -R_1N_{12} & R_2+R_1N_{12}^2 \end{bmatrix} \begin{bmatrix} y_\tau \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - N_{12}x_1 \end{bmatrix} \tag{5.50}$$

Rewriting (5.50) and the definition of y_τ together we then have

$$\frac{d}{dt} \begin{bmatrix} y_\tau \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/M_{11})\ (-R_1y_\tau - R_1N_{12}y_2 + x_1) \\ y_\tau - y_1 - N_{12}y_2 \\ -R_1N_{12}y_\tau + (R_2 + R_1\ N_{12}^2)y_2 - x_2 + N_{12}x_1 \end{bmatrix} \tag{5.51}$$

The variable y_τ is, by its definition, proportional to the total ampere-turns on the transformer core. Thus, by using (5.51) as the component equations, a model of the system can be formulated using the ampere-turns as a variable. A problem of this kind is considered in the next example.

The coefficient M_{11} can also be considered to be a function of y_τ as long as

$$\begin{aligned} M_{11}(y_\tau)\, M_{22}(y_\tau) - M_{12}^2\,(y_\tau) &= 0 \\ M_{11}(y_\tau)/M_{12}(y_\tau) &= N_{12} \end{aligned} \qquad (5.52)$$

where N_{12} is still a constant. For many core materials having the so called "square loop" hysteresis properties, (5.52) is very nearly satisfied. Consequently, (5.51) is a good model for such components.

For the system shown in Figure 5.7, the component equations are of the form shown in (5.53).

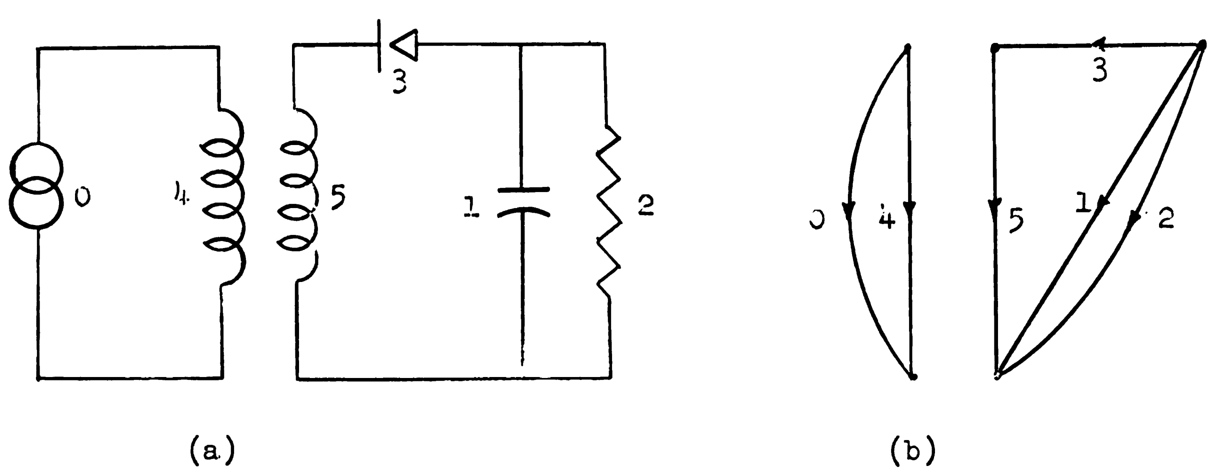


Figure 5.7 Saturable Reactor

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ y_\tau \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)y_1 \\ (1/M_{44}(y_\tau))\ (-R_4y_\tau + R_4N_{45}y_5 + x_4) \\ y_\tau - y_4 - N_{45}y_5 \\ -R_4N_{45}y_\tau + (R_5 + R_4N_{45}^2)\ y_5 - x_5 + N_{45}x_4 \\ x_2 - R_2y_2 \\ x_3 - f(y_3) \\ x_0 - f_1(t) \end{bmatrix} \tag{5.53}$$

The variable y_τ is actually a "dummy" variable and is not related to any of the other system variables except through the second equation in (5.53). Therefore, when the circuit and cut-set equations are substituted into (5.53), no substitutions are made for y_τ. For the tree indicated in Figure 5.7b, we then have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ y_\tau \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\ (y_3 - y_2) \\ (1/M_{44}(y_\tau))\ (-R_4y_\tau - R_4N_{45}y_3 + x_0) \\ y_\tau - y_4 + N_{45}y_3 \\ -R_4N_{45}y_\tau - (R_5 + R_4N_{45}^2)y_3 - x_5 + N_{45}x_0 \\ x_1 - R_2y_2 \\ x_5 - x_1 - f(y_3) \\ x_0 - f_1(t) \end{bmatrix} \tag{5.54}$$

Four of the last five algebraic equations can be eliminated from (5.54) to obtain

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ y_\tau \\ 0 \end{bmatrix} = \begin{bmatrix} (1/C_1)\,(y_3 - x_1/R_2) \\ (1/M_{44}(y_\tau))\,(-R_4 y_\tau - R_4 N_{45} y_3 + f_1(t)) \\ -R_4 N_{45} y_\tau - (R_5 + R_4 N_{45}^2)\,y_3 - x_1 - f(y_3) + N_{45} f_1(t) \end{bmatrix} \quad (5.55)$$

Note that because of the nonlinear function $f(y_3)$, no explicit solution for y_3 is possible in the last equation in (5.55). The three nonlinear equations shown in (5.55) must be solved simultaneously in order to obtain a solution.

VI CONCLUSION

In the first sections of this thesis, a number of existence theorems are presented and proved. The second section is devoted entirely to systems having components modeled by linear algebraic equations and extended existing theorems to cover the more general case where the coefficient matrix is not symmetric. A particularly interesting result of this section is presented in Theorem 2.5. Here, necessary conditions are established for the existence of a solution for a completely general linear algebraic system. This theorem covers, as special cases, the restrictions on the topology of regulated sources, perfect couplers, etc.

To establish results similar to those of Section II for nonlinear systems, the mathematical structure presented in Appendix A is found useful. This development introduces the definitions of strictly positive definite matrices and strictly nonsingular matrices. These definitions are in turn used in the presentation of theorems and proofs which are patterned after the linear cases. Probably the most significant result of this development is given by Theorem A.6. Here, sufficient conditions are presented for the existence of a solution to the vector equation $F(X) = 0$. The proof, in addition to establishing the validity of the theorem, also provides new numerical techniques for obtaining the solutions for such equations. The solution is shown to exist by demonstrating that a certain differential equation has a solution. This solution then provided the solution to $F(X) = 0$. Therefore, all the techniques currently available for solving differential equations are now applicable to the problem of solving algebraic equations. In fact, it can be shown that if the first order Euler method with step size equal to one is used, the resulting numerical method is identical to the familiar Newton-Raphson method. The important point is, however, that higher order methods, such as the fourth order Runge-Kutta method, can be used with small step size thereby eliminating the problem of convergence.

In Section IV, differential equation models of system components are considered. The major result, presented in Theorem 4.3, pertains to mixed algebraic and differential equation models. The proof of this result is used as a basis for defining a formal procedure for generating normal-form models of systems.

The formulation procedure, presented in Section V, is applicable to first and higher order systems although more development is needed in the latter case. Systems containing perfect couplers, gyrators, etc., also fit into the framework presented. Here, however, special forms of the component equations are required for systematic treatment.

By combining the results of Section V with existing numerical methods for solving differential equations, it is possible to write general purpose digital computer programs to obtain solutions to a broad class of system problems. In this way, any of the systems presented as examples can be solved automatically by a single computer program.

APPENDIX A

THEOREMS AND DEFINITIONS

In this development, it is convenient to refer to large numbers of equations or symbols by simplified notation. Therefore, matrix notation and matrix algebra[9] are used throughout. A matrix, or a vector, is denoted by upper case letters such as Z and G, while subscripted lower case letters donote individual entries in these matrices, e.g., z_i, g_{ij}, etc. The symbols X and Y are used exclusively for vectors.

Vector valued functions of real vector variables are also useful and are denoted in the customary manner. For example, the symbols F(X) serve to denote the vector function

$$F(X) = \begin{bmatrix} f_1(x_1, x_2, \dots x_n) \\ f_2(x_1, x_2, \dots x_n) \\ \vdots \\ f_m(x_1, x_2, \dots x_n) \end{bmatrix}$$

Functions of more than one vector variable are written in the form

$$F(X,Y) = \begin{bmatrix} f_1(x_1, \dots x_n, y_1, \dots y_m) \\ f_2(x_1, \dots x_n, y_1, \dots y_m) \\ \vdots \\ f_k(x_1, \dots x_n, y_1, \dots y_m) \end{bmatrix}$$

Differentiation in matrix notation has the following meaning:

$$\frac{\partial F(X,Y)}{\partial X} = \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix}$$

As illustrated above, the arguments of functions are occasionally dropped when there is no danger of misunderstanding.

Two measures of a vector, the norm and bound, are also convenient. The norm of an n-dimensional vector X is defined as

$$N(X) = \left(\sum_i^n x_i^2 \right)^{\frac{1}{2}}$$

and the bound of X is defined as

$$\|X\| = \underset{i}{\mathrm{Max}} \left| x_i \right|$$

In order to establish some of the properties required in the proofs presented in the main body of the thesis, a number of definitions and

Theorems are needed.

<u>Definition</u> <u>A.1</u>: A real, square, constant matrix B is said to be positive definite if for every nonzero vector Y we have

$$Y^t BY > 0$$

<u>Definition</u> <u>A.2</u>: A matrix B(X) whose entries are bounded, continuous, real valued functions of the vector X is said to be strictly positive definite if, for all X, there exists a constant c, c > 0, such that for any vector Y,

$$Y^t B(X)Y > cN^2(Y)$$

<u>Theorem</u> <u>A.1</u>: If and only if B is a positive definite matrix, then B is strictly positive definite.

<u>Proof</u>: Sufficient: Consider the quadratic form

$$Y^t BY \tag{1a}$$

Let (1a) be rewritten in the form

$$Y^t BY = N^2(Y)\ Z^t BZ \tag{2a}$$

where N(Z) = 1. It must now be shown that there exists a constant

c, $c > 0$, such that

$$Z^t BZ \geq c$$

for every Z such that $N(Z) = 1$. Suppose no such c exists. Then for every element of a sequence $[c_i]$, $c_i > 0$, there exists a Z_i with $N(Z_i) = 1$ such that

$$Z_i^t BZ_i < c_i$$

Now, choose a sequence $[c_i]$ having zero as a limit. The corresponding sequence $[Z_i]$ represents an infinite, bounded set and, therefore, has a limit point, say Z. Now choose a subsequence of the sequence $[Z_i]$, say $[Z_k]$, which has Z as a limit. Then given $\epsilon > 0$, there exists a constant m such that for all $k > m$,

$$Z_k^t BZ_k < \epsilon$$

By the continuity of the quadratic form, we then have

$$Z^t BZ = 0.$$

Thus, since $N(Z) = 1$, B is not positive definite and the contradiction is complete.

Necessity: If B is strictly positive definite,

$$Y^t BY \geq cN^2(Y), \quad c > 0$$

Then, if $Y \neq 0$, $N(Y) > 0$ and $Y^t BY > 0$.

Theorem A.2:* If a matrix A is positive definite, then A is nonsingular.

Proof: Assume to the contrary, i.e., that A is singular. Then there exists a nonsingular matrix K such that

$$B \equiv K^{-1}A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$$

where A_1 is an rxn matrix with r equal to the rank of A. Then we have

$$A = KB = KB(K^t)^{-1} K^t = KCK^t$$

where $C = B(K^t)^{-1}$.

Therefore

$$X^t AX = (K^t X)^t C(K^t X) = Y^t CY$$

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* The proof of this theorem when A is symmetric can be found in reference 14.

where $Y = K^t X$.

But from the definitions of C and B, the last n-r rows of C are identically zero. Thus, by choosing

$$Y^t = (\underbrace{0\ 0\ 0\ \ldots\ 0}_{r}\ \underbrace{1\ 1\ \ldots\ 1}_{n-r})$$

we have that

$$Y^t C Y = X^t A X = 0$$

Thus, since

$$X = (K^t)^{-1} Y,$$

a nonzero X has been produced for which

$$X^t A X = 0$$

Therefore, A is not positive definite and the theorem is proved.

<u>Theorem A.3</u>: If an nxn matrix A(X), with entries which are real, bounded, continuous functions of the vector X, is strictly positive definite, then A(X) is strictly nonsingular, i.e., there exists a constant n, n > 0, such that

$$\det A(X) \geq n$$

for every X.

<u>Proof</u>: For any value of X, say $X=X_1$, $A(X_1)$ is a positive definite matrix. Therefore, there exists a matrix K such that

$$KA(X_1) = T$$

where T is upper triangular, t_{nn} is equal to the determinant of $A(X_1)$, K is lower triangular, and $k_{nn} = 1$.
Now consider the quadratic form

$$Y^t A(X_1) Y = (Y^t K^{-1})\, KA\,(X_1)\, K^t\, (Y^t K^{-1})^t$$

Since $KA(X_1) = T$, the n,n entry in $KA(X_1)K^t$ is equal to the determinant of $A(X_1)$. Thus, if

$$Z^t = (\ 0\ 0\ 0\ \ldots\ 0\ 1),$$

then

$$Z^t KA(X_1) K^t Z = \det A(X_1)$$

Now let $(K^t)^{-1}\ Y = Z$. Then

$$Y = K^t Z$$

and, from the properties of K, $y_n = 1$. Therefore, for any value of X, there exists a Y such that $N(Y) \geq 1$ and

$$Y^t A(X) Y = \det A(X)$$

Now suppose, contrary to the conclusion of the theorem, that for every $n > 0$ there exists an X such that

$$\det A(X) < n$$

Then for $n = c$ we have

$$Y^t A(X) Y < c \leq c N^2(Y)$$

for some X and Y and any c, $c > 0$. But since $A(X)$ is strictly positive definite,

$$Y^t A(X) Y \geq c N^2(Y)$$

for every X and Y and some c. Hence the contradiction, and the theorem is proved.

Theorem A.4: If A is a positive definite matrix of order $m \times m$ and B is a $m \times n$ matrix with rank n, $n \leq m$, then

$$B^t A B$$

is positive definite and, consequently, nonsingular.

Proof: After Tokad and Kesavan[14].

Theorem A.5: If $A(X)$ is an $n \times n$ strictly positive definite matrix and B

is a constant matrix of the form

$$B = \begin{bmatrix} B_1 & U \end{bmatrix}$$

then

$$BA(X)B^t$$

is strictly positive definite.

<u>Proof</u>: Consider the quadratic form

$$Y^t BA(X)B^t Y = Z^t A(X)Z$$

where $Z = B^t Y$.

Since

$$Z = B^t Y = \begin{bmatrix} B_1 \\ U \end{bmatrix} [Y]$$

it follows that $N(Z) \geq N(Y)$. Further, since $A(X)$ is strictly positive definite, we also have that

$$Z^t A(X)Z \geq cN^2(Z) \geq cN^2(Y)$$

for some c, $c > 0$.

Therefore

$$Y^t BA(X)B^t Y \geq cN^2(Y)$$

and $BA(X)B^t$ is strictly positive definite.

Corollary A.5a: If $A(X)$ is an nxn matrix of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & A_{22}(X) \end{bmatrix} = A(X)$$

where $A_{22}(X)$ is strictly positive definite and of order mxm, and if B is a pxn matrix of the form

$$B = \begin{bmatrix} B_1 & U \end{bmatrix}$$

with $p \leq m$, then

$$BA(X)B^t$$

is strictly positive definite.

Proof: The matrix B can be partitioned in the form

$$B = \begin{bmatrix} B_2 & B_3 \end{bmatrix}$$

where B_3 is a pxm matrix of the form

$$B_3 = \begin{bmatrix} B_4 & U \end{bmatrix}$$

Then $$BA(X)B^t = B_3 A_{22}(X) B_3^t$$

which, according to Theorem A.5, is strictly positive definite.

Theorem A.5b: If $A(X)$ is strictly positive definite, and if $B(X)$ is a row-column rearrangement of $A(X)$, i.e.,

$$B(X) = DA(X)D^t$$

where D is a matrix with one and only one nonzero entry, +1, in each column and row, then $B(X)$ is also strictly positive definite.

Proof: Since $A(X)$ is strictly positive definite, we have that

$$Z^t A(X)Z \geq cN^2(Z)$$

From the structure of the matrix D, it can be seen that D is nonsingular and if

$$D^t Y = Z$$

then $N(Y)=N(Z)$. Therefore, for every Y there exists a Z such that we have

$$Y^t DA(X)D^t Y = Z^t A(X)Z \geq cN^2(Z) = cN^2(Y)$$

Therefore $DA(X)D^t$ is strictly positive definite.

Definition A.3: A vector valued function $F(Z)$ is said to satisfy a Lipschitz condition if for every closed region R defined by $\|Z - C\| \leq c_1$ we have, for some c_2 depending on R,

$$\|F(Z_1) - F(Z_2)\| < c_2 \ \|Z_1 - Z_2\|$$

where $Z_1 \epsilon R$ and $Z_2 \epsilon R$.

Note that the above definition refers to every closed region R. If, for any reason, it is desired to consider a specific region, this region is to be stated explicitly.

Theorem A.6: Let F(X,Y) be an arbitrary q-dimensional vector function of the q-dimensional variable X and the p-dimensional parameter Y.

For every Y such that

1) the entries of $\frac{\partial F}{\partial X}$ exist everywhere, are bounded, and satisfy Lipschitz condition with respect to X and
2) det $\frac{\partial F}{\partial X} \geq k > 0$ for all X, k a constant,

there exists a unique X such that $F(X,Y) = 0$.

Proof: Existence: Let $\underline{X}$ be an arbitrary q-dimensional constant vector and cosider

$$H(X,Y,z) \equiv F(X,Y) - zF(\underline{X},Y) \tag{3a}$$

Differentiating (3a) with respect to z we have

$$\frac{dH}{dz} = \frac{\partial F}{\partial X}\frac{dX}{dz} - F(\underline{X},Y) \tag{4a}$$

Note that the above partial derivatives exist by hypothesis. It is now claimed that there exists an X(z) such that (4a) vanishes. If such an X(z) exists, it must satisfy

$$\frac{dX}{dz} = \frac{\partial F}{\partial X}^{-1} F(\underline{X},Y) \tag{5a}$$

where the entries in $\frac{\partial F^{-1}}{\delta X}$ are again bounded and satisfy a Lipschitz condition. By Theorem A.11, (5a) does have a solution X(z) on the interval $0 \leq z \leq 1$ for any set of initial conditions X(1). Substituting this solution into (5a) we have

$$\frac{dH}{dz} \quad (X(z),Y,z) = 0 \tag{6a}$$

Integrating (6a) with respect to z then yields, by the Fundamental Theorem of Calculus,

$$\int_1^0 \frac{dH}{dz} \quad (X(z),\ Y,\ z)\ dz = \int_1^0 0\ dz = 0 \tag{7a}$$

$$= H(X(0),Y,)) - H(X(1),Y,1)$$
$$=F(X(0),Y) - F(X(1),Y) + F(\underline{X},Y)$$

Therefore if the initial condition is $X(1) = \underline{X}$, the solution to (5a) evaluated at z=0, namely X(0), is a solution to F(X,Y) = 0.

<u>Uniqueness</u>: Let $\underline{X}'$ be an arbitrary vector and let X'(z) be the resulting solution to (5a). Then it has been shown that X'(0) is a root of F(X,Y), i.e., F(X'(0),Y) = 0. Now assume, contrary to the conclusing of the theorem, that there exists a second root, say Q, $Q \neq X'(0)$.

Define L to be the "line segment" joining $\underline{X}'$ and Q, i.e.,

$$L = \left[Z\colon\ Z = \alpha\underline{X}' - (\alpha - 1)\ Q,\ 0 \leq \alpha \leq 1 \right]$$

It is now claimed that there exists a δ such that if $\underline{X}_1 \in L$, $\underline{X}_2 \in L$, and $\|\underline{X}_1 - \underline{X}_2\| < \delta$, then the solutions to the equations

$$\frac{dX_1}{dz} = \frac{\partial F}{\partial X_1}(X_1,Y)^{-1} F(\underline{X}_1,Y) \quad ; \quad X_1(1) = \underline{X}_1 \tag{8a}$$

$$\frac{dX_2}{dz} = \frac{\partial F}{\partial X_2}(X_2,Y)^{-1} F(\underline{X}_2,Y) \quad ; \quad X_2(1) = \underline{X}_2 \tag{9a}$$

namely $X_1(z)$ and $X_2(z)$, are such that $X_1(0) = X_2(0)$.

To show this result, it is first shown that the solution to (8a) for arbitrary $\underline{X}_1 \in L$ lies in some closed region.

Since a bound on the solution (8a) is given by the product of the length of the interval of integration and the maximum value of the derivative, for $0 \leq z \leq 1$ we have

$$\left\| X_1(z) - \underline{X}_1 \right\| \leq qm \left\| F(\underline{X}_1,Y) \right\|$$

where m is an absolute bound on the entries in $\frac{\partial F}{\partial X}^{-1}$. From the fact that $\frac{\partial F}{\partial X}$ exists and satisfies a Lipschitz condition, F(X,Y) satisfies a Lipschitz condition over the region L. Therefore, there exists a constant m_1 such that

$$\left\| F(\underline{X}_1,Y) \right\| \leq m_1$$

and we have

$$\left\| X_1(z) - \underline{X}_1 \right\| \leq qmm_1$$

Thus,

$$\begin{aligned} \left| X_1(z) - \underline{X}' \right| &= \left| X_1(z) - \underline{X}_1 - \underline{X}' + \underline{X}_1 \right\| \\ &\leq \left| X_1(z) - \underline{X}_1 \right\| + \left| \underline{X}_1 - \underline{X}' \right| \\ &\leq qmm_1 + \left\| \underline{X}' - Q \right\| \equiv m_2 \end{aligned}$$

Therefore, for an arbitrary vector $\underline{X}_1 \in L$, the solution to (8a) is such that for $0 \leq z \leq 1$,

$$\left\| X_1(z) - \underline{X}' \right\| \leq m_2 \tag{10a}$$

Next, it is shown that given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\underline{X}_1 \in L$, $\underline{X}_2 \in L$ and $\left\| \underline{X}_1 - \underline{X}_2 \right\| < \delta$, then $\left\| X_1(0) - X_2(0) \right\| < \epsilon$. Substracting (9a) from (8a) we have

$$\begin{aligned}
\left\| \frac{d}{dx} (X_1 - X_2) \right\| &= \left\| \frac{\partial F}{\partial X_1} (X_1, Y)^{-1} F(\underline{X}_1, Y) - \frac{\partial F}{\partial X_2} (X_2, Y)^{-1} F(\underline{X}_2, Y) \right\| \\
&\leq \left\| \frac{\partial F}{\partial X_1} (X_1, Y)^{-1} \right\| \left\| F(\underline{X}_1, Y) - F(\underline{X}_2, Y) \right\| \\
&+ \left\| \frac{\partial F}{\partial X_1} (X_1, Y)^{-1} - \frac{\partial F}{\partial X_2} (X_2, Y)^{-1} \right\| \left\| F(\underline{X}_2, Y) \right\| \\
&\leq qm \left\| F(\underline{X}_1, Y) - F(\underline{X}_2, Y) \right\| \\
&+ qc_2 \left\| F(\underline{X}_2, Y) \right\| \left\| X_1 - X_2 \right\|
\end{aligned}$$

where $c_2 = \underset{i}{\text{Max}}\ c_i$ and c_i are the Lipschitz constants for the entries in $\frac{\partial F}{\partial X}^{-1}$ over the closed region defined by (10a).

Since $F(X,Y)$ satisfies a Lipschitz condition, given $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that if $\left\| \underline{X}_1 - \underline{X}_2 \right\| < \delta_1$, then

$$\left\| F(\underline{X}_1, Y) - F(\underline{X}_2, Y) \right\| < \epsilon_1$$

Under these conditions, we have

$$\left\| \frac{d}{dz} (X_1 - X_2) \right\| < qm\, \epsilon_1 + qc_2 m_1 \left\| X_1 - X_2 \right\| \tag{11a}$$

for $0 \leq z \leq 1$.

Then, since the boundries of the region

$$\|X_1(z) - X_2(z)\| \leq \|\underline{X}_1 - \underline{X}_2\| \exp (qc_2m_1(1-z)) + \frac{m\epsilon_1}{c_2m_1} (\exp (qc_2m_1(1-z)) - 1)$$

are such that if $(Z_1 - Z_2)$ is any function contained in this boundry,

$$\left\|\frac{d}{dz} (Z_1 - Z_2)\right\| = qm \ \epsilon_1 + qc_2m_1 \|Z_1 - Z_2\|$$

all solutions to (8a) and (9a) subject to the conditions imposed in the earlier parts of the proof must be contained in this region. Thus, for z = 0 we have that

$$\|X_1(0) - X_2(0)\| \leq \|\underline{X}_1 - \underline{X}_2\| \exp (qc_2m_1 + \frac{m\epsilon_1}{c_2m_1} (\exp (qc_2m_1) - 1)$$

Now let $\epsilon_1 = \epsilon\, c_2m_1/(2m\ (\exp(qc_2m_1) - 1))$ and let $\delta_2 = \epsilon/2\exp(qc_2m_1)$. Thus, given $\epsilon > 0$ there exists a $\delta > 0$, namely $\delta = \min\ (\delta_1, \delta_2)$, such that if $\underline{X}_1 \in L$, $\underline{X}_2 \in L$, and $\|\underline{X}_1 - \underline{X}_2\| < \delta$ then

$$\|X_1(0) - X_2(0)\| \leq \epsilon$$

By Theorem 7.5 of Apostol(10), under the conditions of the hypothesis, there exists an ϵ-neighborhood of a root of $F(X,Y)$ which contains only that root. Therefore, since both $X_1(0)$ and $X_2(0)$ are roots, it follows that

$$X_1(0) = X_2(0)$$

To complete the proof, choose a finite number of points from L, say $\underline{X}_k$, k = 0, 1, ... σ, such that

$$\left\|\underline{X}_k - \underline{X}_{k-1}\right\| < \delta$$
$$\underline{X}_0 = \underline{X}'$$
$$\underline{X}_\sigma = Q.$$

Then we have that

$$X_0(0) = X'(0)$$
$$X_1(0) = X_0(0)$$
$$\vdots$$
$$X_\sigma(0) = X_{\sigma-1}(0)$$

Thus, $X'(0) = X_\sigma(0)$.

But, since $\underline{X}_\sigma = Q$, and since Q is a root of F(X,Y), we have that

$$X_\sigma(z) = X_\sigma(1) = \underline{X}_\sigma = Q = X_\sigma(0) = X'(0).$$

Therefore, Q = X'(0) and the two roots are identical. This completes the proof.*

<u>Theorem A.7</u>: If, in addition to the hypothesis of Theorem A.6, F(X,Y) satisfies a Lipschitz condition with respect to (X,Y), then for any closed region R of the form $\|Y - C\| \leq c_1$ there exists a c_2 depending on R such that if $Y_1 \in R$ and $Y_2 \in R$, then

* This proof was suggested by Mr. J. T. Olsztyn of General Motors Research Laboratories.

$$\|X_1 - X_2\| \leq C_2 \|Y_1 - Y_2\|$$

where $F(X_1, Y_1) = 0$ and $F(X_2, Y_2) = 0$.

Proof: Let $Y_1 \in R$ and $Y_2 \in R$ and consider the method of solution used in the proof of Theorem A.6. The solution to

$$\frac{dX}{dz} = \frac{\partial F}{\partial X}(X, Y_2)^{-1} \quad F(X_1, Y_2),\ X(1) = X_1,$$

evaluated at $z = 0$ is equal to X_2. However, since $F(X,Y)$ satisfies a Lipschitz condition with respect to (X,Y) for some $c > 0$ we have

$$\|F(X_1,Y_2) - F(X_1,Y_1)\| \leq c \|Y_2 - Y_1\|$$

Therefore, since $F(X_1,Y_1) = 0$,

$$\|F(X_1,Y_2)\| \leq c \|Y_2 - Y_1\|$$

Since the solution cannot exceed the product of the maximum value of the derivative times the length of the interval we have

$$\|X_2 - X_1\| \leq n\, m\, c \|Y_2 - Y_1\|$$

where, as in the proof of Theorem A.6, m is an absolute bound on the entries of $\frac{\partial F}{\partial X}(X,Y)^{-1}$.

Definition A.4: A matrix B(X) is said to be skew-symmetric if, for all

X, $b_{ij}(X) = -b_{ji}(X)$ for $i \neq j$, and $b_{ij} = 0$ for $i = j$.

Theorem A.8: If $B(X)$ is a skew-symmetric matrix, then for every vector X and every vector Y,

$$Y^t B(X) Y = 0$$

Proof: Consider the quadratic form

$$Y^t B(X) Y$$

Then

$$Y^t B(X) Y = \sum_{i,j} y_i b_{ij}(X) y_j$$

$$= \sum_{i} b_{ii}(X) y_i^2 + \sum_{i,j \quad i \neq j} y_i b_{ij}(X) y_j$$

But since $b_{ij}(X) = -b_{ji}(X)$,

$$\sum_{i,j \quad i \neq j} y_i b_{ij}(X)\, y_j = 0$$

Further, since $b_{ii}(X) = 0$,

$$\sum_{i} b_{ii}(X) y_i^2 = 0$$

Thus,

$$Y^t B(X) Y = 0 \quad \text{for all } X.$$

Theorem A.9: Given a S matrix (the coefficient matrix of the cut-set equations) for a graph of p parts, any q columns, $q \leq (v - p)$, are independent if, and only if, the columns correspond to elements which form a subgraph of some forest. Given a B matrix (the coefficient matrix of the circuit equations) for a graph of p parts, any q columns, $q \leq (e - v + p)$, are independent if, and only if, the columns correspond to elements which form a subgraph of the compliment of some forest.

Proof: After Blackwell and Reed.

Theorem A.10: If a vector valued function F(Z) satisfies a Lipschitz condition, and if

$$Z = (Z_1, Z_2)$$

and

$$Z_2 = F_1(Z_1)$$

where F_1 also satisfies a Lipschitz condition, then

$$F'(Z_1) \equiv F(Z_1, F_1(Z_1))$$

also satisfies a Lipschitz condition.

Proof: Consider a closed region R defined by

$$R = \left[Z_1 : \quad \| Z_1 - C \| \leq c_1 \right]$$

for arbitrary C and c_1. Since F_1 satisfies a Lipschitz condition, if Z_1' and Z_1'' belong to R, then

$$\|F_1(Z_1') - F_1(Z_1'')\| \leq c_2 \|Z_1' - Z_1''\| \leq 2c_1c_2$$

for some c_2.

Therefore, since Z_1'' & C belongs to R,

$$\|F_1(Z_1') - F_1(C)\| \leq 2c_1c_2$$

and

$$\|F_1(Z_1'') - F_1(C)\| \leq 2c_1c_2$$

Let R_1 be the region defined by

$$R_1 = [Z: \|Z - C_3\| \leq c_4]$$

where

$$C_3 = \begin{bmatrix} C \\ F_1(C) \end{bmatrix}$$

and

$$c_4 = \mathrm{Max}(C_1, 2C_1C_2)$$

Since F satisfies a Lipschitz condition, there exists a constant c_5 such that

$$\|F(Z') - F(Z'')\| \leq c_5 \|Z' - Z''\|$$

if Z' and Z" belong to R_1. Thus, if Z_1' and Z_1'' belong to R, then

$$Z' = \begin{bmatrix} Z_1' \\ F_1 \; (Z_1') \end{bmatrix} \quad \text{and} \quad Z'' = \begin{bmatrix} Z_1'' \\ F_1 \; (Z_1'') \end{bmatrix}$$

belong to R_1 and

$$\left\| F'(Z_1') - F'(Z_1'') \right\| = \left\| F(Z_1', F_1(Z_1')) - F(Z_1'', F_1(Z_1'')) \right\|$$

$$\leq c_5 \left\| \begin{bmatrix} Z_1' \\ F_1(Z_1') \end{bmatrix} - \begin{bmatrix} Z_1'' \\ F_1(Z_2'') \end{bmatrix} \right\|$$

$$\leq c_5 \operatorname{Max}(1, C_2) \left\| Z_1' - Z_1'' \right\|$$

This completes the proof.

<u>Theorem</u> <u>A.11</u>: If, for any two vectors X and Y in the region R defined by $\|Z - C\| \leq c_1$ we have

$$\left\| F(X) - F(Y) \right\| \leq c_3 \left\| X - Y \right\|$$

where c_3 is a constant depending only upon R, thenthere exists a unique solution to

$$\frac{dZ}{dt} = F(Z) \quad , \quad Z(0) = C$$

for $0 \leq t \leq c_1/c_2$ where

$$c_2 = \operatorname*{Max}_{R} \quad F(Z)$$

<u>Proof</u>: After Bellman.

Corollary A.11: If the function $F'(Z',t)$ satisfies a Lipschitz condition with respect to (Z',t) then the differential equation

$$\frac{dZ'}{dt} = F'(Z',t), \quad Z(t_0) = C' \tag{12a}$$

has a unique solution for $t_0 \leq t \leq t_0 + C''$, $C'' > 0$.

Proof: If the change of variable $t_1 = t - t_0$ is made in (12a) we have

$$\frac{dZ''}{dt_1}(t_0 + t_1) = F'(Z'(t_0 + t_1), t_0 + t_1) \; ; \; Z'(t_0 + 0) = C'$$

or

$$\frac{dZ''(t_1)}{dt_1} = F''(Z''(t_1), t_1) \; ; \quad Z''(0) = C' \tag{13a}$$

where $Z''(t_1) \equiv Z'(t_0 + t_1)$.

Since the equation

$$\frac{d\tau}{dt_1} = 1 \quad ; \quad \tau(0) = 0$$

has the solution $\tau = t_1$, (13a) can be written as

$$\begin{bmatrix} Z'' \\ \tau \end{bmatrix} = \begin{bmatrix} F''(Z'',\tau_1) \\ 1 \end{bmatrix} \quad ; \quad \begin{bmatrix} Z''(0) \\ \tau(0) \end{bmatrix} = \begin{bmatrix} C' \\ 0 \end{bmatrix} \tag{14a}$$

If $Z(t_1) \equiv (Z''(t_1), \tau(t_1))$, then by Theorem A.11, (14a) has a unique solution for $0 \leq t_1 \leq c$, $c > 0$.

Therefore, since

$$Z'(t) = Z'(t_0 + t_1) = Z''(t_1),$$

a solution for $Z'(t)$ is established for $t_0 \leq t \leq t_0 + c$, $c > 0$. To show that the solution is unique, assume that a second solution, $\underline{Z}'(t)$, exists. Then a second solution, say $\underline{Z}''(t_1)$, must exist for (14a), contradicting the conclusion of Theorem A.11.

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