OPTIMAL APERIODIC SAMPLING FOR SIGNAL REPRESENTATION AND SYSTEM CONTROL

Dissertation for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
RONALD LEE VAN WIEREN
1975

This is to certify that the

thesis entitled

OPTIMAL APERIODIC SAMPLING FOR SIGNAL REPRESENTATION AND SYSTEM CONTROL

presented by

RONALD LEE VAN WIEREN

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in SYSTEM SCIENCE

Robert Schlueter
Major professor

Date Oct 31, 1975

O-7639

ABSTRACT

OPTIMAL APERIODIC SAMPLING FOR
SIGNAL REPRESENTATION AND SYSTEM CONTROL

By

Ronald Lee Van Wieren

There were two investigations conducted in this thesis. First, signal representation through use of non-periodic sampling was investigated in an effort to reduce the sampling errors inherent in periodically sampled piecewise constant approximations of continuous time signals. Second, non-periodic sampling was applied to the feedback control problem by sampling of the continuous time feedback control error signal.

A performance index for the signal representation problem was proposed which measured the errors caused by passing a continuous-time signal through a sample and hold mechanism. A cost for implementation was developed and included to form a system performance measure. The cost of implementation measured the costs for computer utilization, software and data storage, and data communication for each of the sampling methods (periodic, aperiodic, and adaptive) used.

The optimal aperiodic sampling criterion was determined by first obtaining a derived performance index as a function of the sampling interval sequence, (i. e. a sequence of sampling interval lengths) and

the number of sampling intervals being used. This derived performance index was then minimized with respect to the sampling intervals for a particular number of sampling intervals. The optimal sampling interval sequence was then determined to be the optimal number of samples and the resulting optimal sampling interval sequence which minimized the derived system performance index.

A remote display problem was presented as an example for comparing periodic, adaptive, and sub-optimal aperiodic sampling techniques used to obtain piecewise constant representations of continuous time signal records. The continuous time signal record used in the display problem was generated by a signal model, then stored, and later sampled for transmission via a specified communications network to the remote display.

Optimal aperiodic sampling was extended from the signal representation problem and applied to a control implementation problem. The sampling process for measuring the outputs and actuating the control inputs was to be designed. Two system configurations employing control signal sampling were considered. The first sampling configuration samples both the input and feedback signals while the second samples only the feedback with the input signal being continuously applied to the system plant. The control law was assumed to be designed previously using either classical or modern techniques. Thus, the only design parameters are the number of samples and the sampling interval lengths.

A quadratic control performance index is assumed which measures control energy, tracking error and terminal error. A cost for implementation is developed and included to form a system performance index. This cost of implementation measured the computer utilization, data and program storage, and data communication costs for each method of sampling used.

The optimal aperiodic sampling criterion was again computed by determining a derived system performance index which was a function of the number of samples and the length of each sampling interval. The derived system performance index was minimized with respect to the lengths of sampling intervals for a particular number of sampling intervals. The optimal aperiodic sampling criterion was then determined to be the optimal number of samples and the resulting optimal sampling interval sequence which minimized the system performance index.

The results of the investigations conducted on optimal sampling for signal representation of continuous time signals resulted in the following findings. The derived performance costs for sub-optimal aperiodic sampling in all cases considered yielded lower performance costs than comparable periodic or adaptive sampling methods applied to the same continuous time signal record with an equal number of sampling intervals. The costs for implementation of sub-optimal aperiodic sampling were in general higher than periodic or adaptive sampling as the number of sampling intervals. This increase was due to the increased computational requirements of aperiodic sampling.

Investigations of non-periodic sampling for system control indicated that through use of sub-optimal aperiodic sampling control performance costs could be reduced below periodic sampling costs using an equal number of sampling intervals in each case considered. In several cases control performance costs using aperiodic sampling of slow responding control systems were reduced below continuous time performance costs over a fixed control interval. Again obtaining the optimal sampling interval sequence proved costly with respect to computer processing time except if the number of sampling intervals was small.

OPTIMAL APERIODIC SAMPLING FOR SIGNAL REPRESENTATION AND SYSTEM CONTROL

By

Ronald Lee Van Wieren

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering and Systems Science

1975

ACKNOWLEDGEMENTS

I am deeply grateful to my advisor, Dr. Robert A. Schlueter, for his friendship, constant encouragement, and professional criticisms made during the preparation of this dissertation.

For their interest and advice, I also thank the other members of my committee: Drs. G. L. Park, J. S. Frame, J. Preminger, and K. Y. Lee.

TABLE OF CONTENTS

LIST OF FIGURES

Figure | Page

Figure		Page

LIST OF TABLES

I. INTRODUCTION

The subject of this thesis deals with the development of a theory for optimal aperiodic sampling[1] for both representation of signals and system control.

The design of sampling criteria has always been based on control performance, signal sampling efficiency, cost of implementation, and ease of design and analysis. Periodic sampling criteria have often been used because periodic sampled-data single-input single-output control systems can be easily designed using transform techniques. For periodic sampling the sampling rate is generally constrained to satisfy the sampling theorem. As a result, the sampling rate becomes greater than twice the system bandwidth in order to minimize aliasing and thus provide reasonable control performance. The actual choice of a sampling rate is chosen by a tradeoff between control performance and the economic cost for instrumentation and communications hardware. The continuous-time controller is implemented if the cost for communicating data is low and computer processing is not required to implement the system design.

The design of sampling criteria for multiple-input multiple-output systems is more involved since the bandwidths are generally quite different for each element of the transfer function. Because of this,

[1] See Appendix A for a definition of sampling as used in the context of this thesis. See Appendix B for discussion of periodic, adaptive, and aperiodic sampling.

multi-rate and asynchronous sampling criteria [28] have been introduced. Adaptive sampling criteria [4-11] have also been developed which vary sampling rate with respect to changes in output or error signals. If the signals used in an adaptive or aperiodic sampling criterion accurately represent the system dynamics, where system dynamics depend on the system state equation, control inputs, disturbance noise, and initial conditions, the sampling criterion can be considered tuned to the system. Tuning the sampling rule to the system dynamics can improve both control performance and sampling efficiency.

Most adaptive sampling criteria have been derived using either an integral-absolute-error performance index or an integral-squared-error performance index which measures the error caused by the sample and hold mechanism over one interval. This performance index can be shown to explicitly depend on the system state equations, control inputs, and initial conditions. To insure a non-trivial sampling rule the integral performance index is augmented by a functionwhich is inversely proportional to the length of the sampling intervals. This additional term is implied to represent the costs associated with the sample and hold mechanism on a cost per sample per unit time basis.

The performance index is frequently simplified by approximating continuous time signals by a Taylor series expansion. The number of terms used to approximate the continuous signals will determine how well the performance index is tuned to the particular signal. Retaining only the first two terms of the Taylor series approximation will degrade the representation of the continuous signals in the sampling criteria. [24] As a result the cost function no longer explicitly depends on the system state equation, disturbance noise, and control inputs, but instead depends on the signal and its derivatives. Through use of these approximations, integration of the performance index can be performed analytically yielding a performance measure dependent on the length of the sampling interval, the signal, and the derivatives of the signal. Having the analytical solution of the performance index, the adaptive sampling rule can be obtained by setting the derivative of the performance index with respect to the sampling interval equal to zero and solving for the sampling interval length.

As an alternative to minimizing an integral performance measure, Tomovic and Bekey [6] and Tait [11] proposed that sampling occur only when the sampling error exceeds a predetermined level. This approach did not require signal derivatives and insured an error bound [14].

In a paper by Smith [1], an evaluation of adaptive sampling compared to periodic sampling was undertaken. The adaptive techniques were shown to have a tendency to sample at the sampling interval extremes if constraints on the minimum and maximum sampling interval were imposed. In general, the result of his study indicated that if the input was known a priori, any adaptive sampling rule could out perform periodic sampling. If the input was not known a priori, periodic sampling was generally more desirable.

Aperiodic sampling for signal representation is proposed as an alternative to adaptive sampling. In aperiodic sampling the number of sampling intervals and the lengths of each sampling interval are chosen to minimize the performance index defined over the entire time interval of interest. The choice of any particular interval length is dependent on the fixed final time and the length of the other sampling intervals. The continuous time system dynamics and associated implementation costs[1] are incorporated into the performance index thereby influencing the number of sampling intervals used and the length of each interval. A poor choice of sampling intervals will increase the performance cost while a good choice of interval lengths will yield a lower performance cost for a given number of samples. Therefore the aperiodic sampling performance index is derived as a function of the sampling interval sequence. This derived performance index can then be minimized with consideration being given to the system dynamics and costs for implementation to yield the optimal sampling interval sequence.

Optimal aperiodic sampling can also provide better control performance than a similar non-optimal sampling criterion. In this case, the performance index used must reflect the performance objectives for the control system as well as indicate control implementation costs with respect to computer utilization and data communications. Hsia [24] and Smith [1] treated the adaptive sampling problem for control as a special case of the signal representation problem. Therefore a signal representation type performance index was used for system control without regard to system control objectives such as: tracking and final state errors and control energy expenditures found in most control type performance indices. The assumption made is that if the sampling process minimizes the errors between the signal and its sampled and hold approximation, good control performance will result. This assumption is clearly invalid and

[1] Implementation costs will be discussed in more detail in Chapter II.

therefore the performance index used for the control problem should reflect specific control objectives and the relative implementation costs required to achieve the control objectives.

The approximations made to obtain simple analytic expressions for the adaptive sampling performance indices are not used in the evaluation of the optimal aperiodic sampling for control problem. If these approximations were made, the performance of the resulting sampling rule would depend on the accuracy of the approximation to a particular signal and the sampling rule would thus become dependent on the signal and its derivative and independent of the a priori information available on the system state equation, inputs, disturbances, initial conditions, and control law.

The first optimal sampling problem for control [45, 46] was formulated to obtain an optimal sampled data control where both the level of control over a sampling interval and the length of the sampling interval were chosen together to minimize the performance index. The optimal control law was thus specified by an optimal control sequence-optimal sampling interval sequence combination. Necessary conditions were derived, but were not used to obtain an efficient computational algorithm for the optimal control. An algorithm was developed [47] by discretizing the cost functional and state equations and adjoining constraints on the sampling intervals. The resulting nonlinear programming problem was solved using a sequential unconstrained minimization technique. The optimal sampled-data control problem can be solved directly if the optimal control sequence can be determined as a unique function of the sampling interval sequence chosen. In this case, the performance index can also be derived as a unique function of the sampling interval sequence. The resulting derived performance index can be minimized using a search algorithm to obtain the optimal sampling interval sequence. The optimal control sequence can then be determined by specifying the optimal sampling interval sequence. This algorithm solves the optimal sampling problem directly and is more efficient than solving the discrete time problem using the sequential unconstrained minimization technique. Moreover, the errors incurred by a priori discretization are eliminated. This later algorithm was used to determine optimal aperiodic [19], state dependent [20], and adaptive [24] control laws for the regulator problem. These problem formulations neglected system disturbances and costs for implementation and used a fixed number of sampling intervals for all investigations.

Another approach to obtaining optimal sampling processes for control systems was proposed [3, 40] for the case of periodic sampling. This control problem included an additional performance cost proportional to the number of uniformly spaced samples used during the particular control interval. An integer programming algorithm was then used to compute the optimal number of sampling intervals and thus the optimal periodic sampling rate.

It should be noted that in order to optimize the performance index for aperiodic sampling, it is desirable that the system input be known for the time in which the system is to be controlled. This restriction is not as severe as it would seem since in general, applications such as numerical control [41] or linear regulator problems [19-21] the input to the system is known a priori or is constant.

Optimal aperiodic sampling of continuous time feedback error signals for system control is proposed. A predetermined continuous time optimal or sub-optimal feedback control law is used to generate the feedback error signals. The performance of the control system is determined by a quadratic performance index which is a function of the tracking errors, control energy, and thus implicitly depends on the disturbance noise, system state equation, and initial conditions. The performance index is augmented by a cost for implementation through sampling which approximates computational costs, data storage requirements, and communications costs. The performance index is reformulated as a derived function of the sampling interval sequence and then minimized with respect to the number of sampling intervals and the length of each sampling interval. The optimal sampling interval sequence obtained is then used in the sampling of the continuous time feedback error signals in an effort to improve overall system performance while yielding acceptable computational and communications costs.

In summary, the hypothesis of this research is that signal representation performance indices are appropriate for data acquisition and signal sampling where control using the sampled signal is not an immediate objective. A control type performance index should be used for designing sampling processes for system control. This is why there are two separate problem formulations found in this work. In addition, each problem formulation will use a cost for implementation model to determine the feasibility of system implementation. Heuristic implementation models have always been used by system designers to determine implementation requirements for various system designs. This research will formulate a generalized cost for implementation model based on computer and communications system utilization. Thus in Chapter II a theory for optimal stochastic aperiodic sampling for signal representation is formulated. The cost functional is derived and used in conjunction with a cost for implementation based on system computational and communication requirements. In Chapter III the optimal stochastic aperiodic sampled-data tracking problem is formulated and derived for various feedback configurations. In Chapter IV various tests are performed and comparisons made in order to evaluate the performance of several sampling rules for signal representation and system control. Chapter V summarizes the results, reviews the thesis, and suggests further research.

II. SAMPLING FOR SIGNAL REPRESENTATION

Communications and control very often use sample and hold mechanisms[1] in system design. These systems are usually designed around a minicomputer or microprocessor thus enabling the sample and hold devices to be under direct computer control. Therefore with the availability of the computer, the sample and hold devices can be utilized more efficiently through optimal sampling techniques.

The function of a sample and hold device is to approximate a continuous signal over discrete intervals. The accuracy of the signal approximation for periodic sampling is dependent on the sampling rate while for adaptive and aperiodic sampling the accuracy is dependent on the length of each sample interval and the number of intervals. Increasing the number of samples taken over any given time interval for periodic, aperiodic, or adaptive sampling will generally decrease the error in the signal approximation but will increase computational and data communications expense. The signal representation performance index should measure errors in signal approximations and indicate the relative costs for implementation for each method of sampling under consideration.

This structure for the design of optimal sampling criteria also provides a framework for understanding the analytic design of non-optimal adaptive sampling criteria [24]. A large class of adaptive

[1] See Appendix A.

sampling criteria can be derived by minimizing a Taylor series approximation of the cost functionals

$$\underline{J}_i = \frac{1}{T_i^{k_1}} \int_{t_i}^{t_i + T_i} [\,|\,\underline{s}(t) - \underline{s}(t_i)\,|\,]^{k_2} dt + a_1 e^{-a_2 T_i} \qquad (2.1)$$

with respect to the sampling intervals T_i for $i = 0, 1, \ldots, N-1$ where $\underline{s}(t)$ is any continuous time output or error signal. The first term in each cost functional is a weighted integral of a function of the error $\underline{s}(t) - \underline{s}(t_i)$ caused by the sample and hold operation over $(t_i, t_i + T_i)$. The second term represents the so called "cost for sampling." The parameters k_1, k_2, a_1, a_2 are permitted to take on the following values

$$k_1 \in \{-1, 0, 1\}\ ;\ k_2 \in \{1, 2\}\ ;\ a_1,\ a_2 \in (0, \infty)$$

This functional $\underline{J}_i$ penalizes sampling interval lengths with respect to the mean of the integral if $k_1 = 1$, the time-weighed integral if $k_1 = -1$, or the unweighted integral if $k_1 = 0$.

The signal s(t) can be approximated by the truncated Taylor series over each sampling interval

$$s(t) \cong s(t_i) + \dot{s}(t_i)(t - t_i) \qquad t \geq t_i$$

and the "cost for sampling" can be approximated by either

$$a_i e^{-a_2 T_i} \approx \begin{cases} a_1 [1 - a_2 T_i] \\ a_1 [1 - a_2 T_i + (a_2 T_i)^2/2] \end{cases}$$

The sampling rules can then be determined by minimizing the resulting approximations to the cost functionals $\underline{J}_i$ with respect to T_i for $i = 0, 1, \ldots, N-1$.

The adaptive sampling rules obtained by Hsia determine the length of only one sampling interval for a particular set of signal measurements. This one step interval determination was not dependent on system dynamics, control laws, or the statistical description of the initial states and system disturbances. The adaptive sampling rules depended only on the values of the signal and its derivative and provided an analytic rule which minimized an approximation to the initial performance index as indicated in (2.1).

The derived adaptive sampling rules were shown to outperform periodic sampling if the parameters of the sampling rules were adjusted appropriately and the number of samples to be taken along with the system input were known a priori. The criterion used to compare sampling rule performance was an integral-absolute-error and thus all sampling rules derived from the generalized performance index (2.1) could be compared [1].

In contrast to adaptive sampling, optimal periodic sampling with fixed costs for sampling and a variable number of samples was investigated [3]. The results indicate that optimization with respect to the number of samples was indeed possible and yielded improved performance for the various system dynamics considered.

These previous developments in signal sampling have been considered in the formulation of the optimal aperiodic signal representation problem. The performance index used in adaptive sampling measured sampling errors under the assumption that good signal representation of the feedback error signals implied good system control performance. This assumption was obviously incorrect because the adaptive sampling performance index measured sampling errors rather than optimizing system performance objectives (e.g. minimization of final state errors, tracking errors, etc.). Therefore two problems are formulated in this thesis; the sampling for control problem found in Chapter III and the signal representation problem found in this chapter.

A performance index for optimal aperiodic sampling for signal representation will be developed as a function of system dynamics, system inputs, disturbance noise, and costs for implementation. The system dynamics are obtained from the continuous time signal model used to generate the signal record which is to be sampled. The system dynamics will not be discretized a priori thus reducing system modeling errors. In addition the length of each sampling interval will not be optimized on an individual basis but instead the entire sampling interval sequence will be optimized with respect to sampling interval lengths for a fixed number of intervals to obtain a sub-optimal aperiodic sampling criterion. If the performance is then optimized over the number of sampling intervals an optimal sampling criterion for signal representation is obtained.

The computational costs required to obtain the optimal aperiodic signal representation as well as communications costs resulting in the transmission of the data which represents the sampled signal will be considered in a cost for implementation model. This model represents a new approach to implementation cost modeling not considered in any of the previous references. Therefore, the signal representation performance index will measure the errors in signal representation and indicate the relative costs for implementation for each method of sampling under consideration.

The work which is to follow in this chapter states the signal representation problem and defines periodic, optimal periodic, aperiodic, sub-optimal aperiodic, and optimal aperiodic sampling critera. A general cost function is formulated which can be used to evaluate each sampling criteria with respect to signal representation quality and costs for implementation.

2. 1. Signal Representation Problem Statement

Consider the linear time invariant system,

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}_e(t) + \underline{w}(t) \tag{2.2}$$

$$\underline{s}(t) = \underline{C}\underline{x}(t) + \underline{D}\underline{z}(t) \tag{2.3}$$

where

$\underline{x}(t)$ - n dimensional state vector of the system producing $\underline{y}(t)$;

$\underline{u}_e(t)$ - r dimensional error signal vector;

$\underline{z}(t)$ - r dimensional input signal vector;

$\underline{s}(t)$ - m dimensional signal vector; and

$\underline{w}(t)$ - n dimensional disturbance vector.

Matrices $\underline{A}$, $\underline{B}$, and $\underline{C}$, $\underline{D}$ are constant and compatible with the above vectors. The initial time t_o is known, with $t_f > t_o$ being the fixed terminal time. The initial state $\underline{x}(t_o)$ is assumed to be a Gaussian random vector with mean and variance

$$E\{\underline{x}(t_o)\} = \underline{m}_o, \quad Cov\{\underline{x}(t_o), \underline{x}(t_o)\} = \underline{V}(t_o) \tag{2.4}$$

The plant noise process $\underline{w}(t)$ is assumed to be non-zero mean Gaussian white noise

$$E\{\underline{w}(t)\} = \underline{w}_o, \quad t \in [t_o, t_f] \tag{2.5}$$

$$Cov\{\underline{w}(t), \underline{w}(\tau)\} = \underline{\Psi}(t)\delta(t-\tau), \quad t, \tau \in [t_o, t_f] \tag{2.6}$$

where $\delta(t-\tau)$ is a delta function. In addition, $\underline{x}(t_o)$ and $\underline{w}(t)$ are assumed independent.

This model was chosen to generate a general signal vector which could be produced by either a communications system or a control system. The actual model dynamics are either known or identified offline from previous output and control histories.

With the system specified a signal vector generated by this system is to be sampled at the sampling times t_i; such that,

$$t_{initial} = t_o < t_1 < t_2 < \ldots < t_{N-1} < t_N = t_{final} \tag{2.7}$$

with a sample interval being defined as T_i where

$$T_i = t_{i+1} - t_i \tag{2.8}$$

for $i = 0, 1, \ldots, N-1$ and

$$\underline{T} = [T_o, T_1, \ldots, T_{N-1}] \tag{2.9}$$

An optimal periodic sampling (OPS) criterion can be specified by selecting the number of sample intervals, N, to satisfy

$$N_{min} \leq N \leq N_{max} \tag{2.10}$$

and imposing the equality constraint knowing t_f and t_o

$$\underline{g}(\underline{T}) = \left\{ \frac{t_f - t_o}{N} - T_i = 0, \quad i = 0, 1, \ldots, N-1 \text{ for any } N \right. \tag{2.11}$$

where N_{max} is determined by sampler performance specifications and N_{min} by signal representation quality. Optimal periodic sampling is obtained through the minimization of a signal representation performance index with respect to N satisfying (2.10) and (2.11) thus yielding the optimal number of samples N^*. Periodic sampling (PS) is specified by a fixed N such that

$$N_{min} = N = N_{max} \tag{2.12}$$

and requiring (2.11) be satisfied for the N chosen.

An optimal aperiodic sampling (OAS) criterion can be specified by sampling interval constraints of the form

$$0 < T_{min} \leq T_i \leq T_{max} \tag{2.13}$$

and given t_f and t_o

$$\underset{\sim}{g}(\underline{T}) = \sum_{i=0}^{N-1} T_i - [t_f - t_o] = 0 \tag{2.14}$$

where T_{min} is determined by the sampler performance specifications and T_{max} by system stability or worst case signal representation quality. The number of sample intervals, N, is chosen to satisfy (2.10). Optimal aperiodic sampling is obtained through minimization of a signal representation performance index with respect to both N and $\underline{T}$ satisfying (2.10), (2.13), and (2.14) to yield the optimal number of sample intervals, N*, and the optimal sample interval sequence, $\underline{T}^*$. Sub-optimal aperiodic sampling (SAS) specifies a fixed N satisfying (2.12) with the sample interval sequence free but satisfying (2.13) and (2.14). Sub-optimal aperiodic sampling criterion is obtained through minimization of the signal representation performance index with respect to $\underline{T}$ with N satisfying (2.12). Finally, aperiodic sampling (AS) is specified when N and $\underline{T}$ are chosen to satisfy (2.12), (2.13), and (2.14) with no optimization on N or $\underline{T}$.

Five sampling criteria have been specified. Each of the optimal criteria require the minimization of a signal representation performance index with respect to certain variables. The form of the performance index is based on various periodic and adaptive sampling cost functionals similar to (2.1). A quadratic rather than an absolute error function is chosen since large sampling errors, either positive or negative, should be penalized more than small errors.

The measure of the errors introduced by a sample and hold mechanism does not provide an adequate basis for determining the particular sampling method to be implemented for a particular data acquisition system. The costs for implementation; including computer utilization time, computer program and data storage, and communications of output data, must also be considered. Therefore, the performance index for a signal representation problem must measure both the errors caused by the sample and hold operation and the costs for implementation. Thus, the signal representation performance index becomes

$$J = J_o + J_f \tag{2.15}$$

where

$$J_o = E\left\{\sum_{i=0}^{N-1} \frac{1}{T_i^a} \int_{t_i}^{t_{i+1}} (\underline{s}(t) - \underline{s}(t_i))' \underline{S} (\underline{s}(t) - \underline{s}(t_i)) dt\right\} \tag{2.16}$$

measures the performance cost over N sampling intervals due to the piecewise constant approximation of the continuous time signal $\underline{s}(t)$. The matrix $\underline{S}$ is assumed to be a positive semi-definite m x m symmetric matrix, T_i is assumed to satisfy the appropriate sampling interval constraints and $E\{\cdot\}$ is the expectation operator taken over the random variables found in the system which generates the continuous time signal vector $\underline{s}(t)$. The functional J_o penalizes sampling errors with respect to the mean of the integral if $a = 1$, the time-weighted integral if $a = -1$, or the unweighted integral if $a = 0$.

The reason for including the J_f portion of (2.15) is to represent the computational costs necessary to obtain the sampling interval sequence, $\underline{T}$, and the communications costs required for transmission of sampling interval sequence and signal approximation, $\underline{s}(t_i)$, information. The sum of the computational and communications costs will be termed costs for implementation, J_f, throughout this thesis.

Implementation costs have always been considered to some extent in data acquisition or control system design. The amount of attention paid to these costs range from complete neglect for systems where data storage or retrieval is not a concern to extensive analysis and redesign for systems where realization would be impossible due to excessive implementation costs. Therefore modeling of implementation costs are justified since it will indicate to the system designer the feasibility of implementation for the system design being considered.

The costs for computation, required on-line or off-line, can often determine the feasibility of a particular design approach. Explicit expressions for the computational costs have been determined for some simple computational tasks [51]. The processing time required to perform the task has been expressed in terms of the execution time for a particular operation and the number of times this operation was performed. The memory requirements for both data storage and computation are also enumerated. The determination of explicit expressions for processing time and computer memory for more difficult computational tasks may be impossible or not worth the effort required. For example, the computational costs for computing the optimal sampling interval sequence will depend on the type of optimization routine chosen, the convergence criterion imposed, the number of sampling intervals to be optimized, and the complexity of the performance index to be minimized. The computational costs will also depend on the dimensions of the system matrices and vectors, the number of signals which will require processing, and the number of data points used to represent each signal. In addition any added subroutines, input/output routines, and auxiliary storage accesses [51] will also increase computer utilization expenses. Thus, improved models for computation costs must be determined which are realistic and intuitively appealing to the system designer.

Data communications costs can also be extremely important for data acquisition and remote sensing problems where retrieval of the signal representation data is vital for system operation. Therefore a communications costs model is needed to inform the system designer of possible tradeoffs between such communication system variables as channel capacities, data record lengths, data transmission time delays, and channel reliability. Thus a measure of the communications costs must also be developed.

Implementation costs, either computational or communications, are highly problem specific with dependency on each detail of the system being considered and the economic constraints placed on the system being designed. The models for implementation costs also depend on whether the system hardware (e.g. computers, samplers, etc.) is to be purchased based on system design or whether a particular system design is to be implemented through use of existing computer/communications equipment. Also of major importance are the design philosophies, previous system design experience, and technological awareness of the system designer.

The particular implementation cost model which will be developed in this section is one of many possible. In this work it is assumed that the computer and communications system for the signal representation problem are specified and that utilization of the computer system and data acquisition system should be minimized with respect to a given level of performance cost as expressed in (2.16). Thus the J_f portion of (2.15) will have the form

$$J_f = J_{cu} + J_{cc} \tag{2.17}$$

where J_{cu} is the computer utilization/processing costs and J_{cc} is the communications costs. The assumption made here is that the computer utilization and communications utilization are separable.

The computer utilization/processing costs is assumed to have the form

$$J_{cu} = w_1 \times M \times U \qquad (2.18)$$

where

U - computer utilization/processing time (seconds)

M - program/data storage requirements (words of main memory)

w_1 - utilization/processing weighting factor ($(\text{word-second})^{-1}$)

The computer utilization/processing cost model expressed in (2.18) weights the product of computer utilization time measured in seconds, and the software/data storage requirements commonly expressed in words of main memory. This form of the computer utilization/processing cost model was selected based on common data processing practices for determining computer charges. (e.g. The CDC 6500 computer system found at Michigan State University uses this formula, along with others, to determine computer user charges.) Having written all necessary computer software, the program/data storage requirements become fixed for the particular version of the main program, optimization routine, etc. being considered. It is a common practice of most computer assemblers to indicate the number of words of main memory required by the assembled computer program. This makes M, program/data storage requirements, readily available to the system designer for the system software he is considering for implementation. If the storage requirements for the system software requires a large portion of the available computer memory w_1 will have to be increased appropriately. Thus U, computer utilization/processing time, becomes the only real variable in the particular design problem. U will increase as the processing time required to compute the optimal sampling interval sequence increases. Thus a 10% increase in U will result in a 10% increase in J_{cu} holding M and w_1 fixed for the particular problem begin considered. The weighting factor, w_1, is chosen by the system designer based upon the relative importance of computer utilization costs.

After the sampling interval sequence has been used in sampling of the desired continuous time signal record to obtain the piecewise constant signal representation, it is sometimes necessary to transmit the resulting output data to a remote site via a specific communications network. As with the computer utilization costs as found in (2.18), communications costs have to be considered in system implementation. The importance or the relative weight of data communications to overall system design will determine its contribution to the total implementation costs found in (2.17).

Communications costs are in general, dependent on the speed, reliability, and relative cost of the data transmitter and receiver to be used as well as the bandwidth of the communications channel over which the data will be transmitted. A communications cost model should reflect data transmission rates, the amount of data to be transmitted, and the relative importance ascribed to the communications process. Thus the communications cost, J_{cc}, portion of (2.17) is assumed to have the form

$$J_{cc} = \frac{w_2 \times K \times W}{R} \qquad (2.19)$$

where

W - output data word bit length (bits/word)

K - amount of periodically or aperiodically sampled data to be transmitted (words)

R - data transmission rate of the communications channel (bits/second)

w_2 - communications cost weighting factor $(\text{second})^{-1}$

The communications cost model expressed in (2.19) weights the product of output word lengths expressed in bits/word times the number of words to be transmitted over the communications channel divided by the channel capacity in bits/second. This form for the communications cost model was also selected based on commonly used data communications accounting practices. This form of communications cost is based on "connect time" and is the formula used on the CDC 6500 computer installation at Michigan State University. Any changes in the communications process (i.e. more data, lower channel capacity, etc.) will be reflected directly in the communication cost model shown in (2.19). In applications of remote sensing or data acquisition the communications costs are considerably more important than for a control problem application where a computer is usually near the system under control.

Insight into the particular application, available tradeoffs in hardware and computer software, and the importance in the overall system design will determine the relative weighting of computational and communications costs as opposed to performance improvements. In Chapter IV hypothetical signal representation and system control design problems will be formulated. The problems will evaluate performances of various methods of sampling and yield relative computational and communications costs for implementation of each method of sampling under consideration.

Thus, the optimal signal representation problem using the desired sampling criteria and appropriate constraints can be stated as follows:

Given the linear dynamical system (2. 2) and (2. 3) with arbitrary input $\underline{z}(t)$, determine the optimal sampling interval sequence, $\underline{T}^*$ and the optimal number of samples, N* such that J ($\underline{T}$, N), is minimized for a given output or error signal subject to (2. 9), and the system disturbances and initial conditions found in (2. 4) through (2. 6).

2. 2. The Derived Signal Representation Performance Index

Given the J_o cost functional as found in (2. 16), consider the special case where $\underline{s}(t) = \underline{u}_e(t)$[1], since $u_e(t)$ was used in the original adaptive sampling studies [24, 1]:

$$\underline{s}(t) = \underline{u}_e(t) = \underline{H}\,\underline{z}(t) - \underline{G}\,\underline{y}(t) \tag{2.20}$$

$\underline{G}$ is an r x m dimensional constant feedback gain matrix, $\underline{H}$ is an r x r dimensional constant input signal gain matrix, and $\underline{z}(t)$ is an arbitrary deterministic input signal vector. Therefore using (2. 3) with $\underline{D} = 0$, (2. 16) becomes

$$J_o = E\left\{\sum_{i=0}^{N-1} \frac{1}{T_i^{\,a}} \int_{t_i}^{t_{i+1}} \hat{\underline{e}}'(t)\,\underline{S}\,\hat{\underline{e}}(t)\,dt\right\} \tag{2.21}$$

$$\text{where } \hat{\underline{e}}(t) = \underline{H}\,\underline{z}(t) - \underline{G}\,\underline{C}\,\underline{x}(t) - \underline{H}\,\underline{z}(t_i) + \underline{G}\,\underline{C}\,\underline{x}(t_i) \tag{2.22}$$

with $\hat{\underline{e}}'(t)$ indicating the transpose of (2. 22). Making the substitutions

$$\hat{\underline{x}}(t) = \underline{x}(t) - \underline{x}(t_i) \tag{2.23}$$

$$\hat{\underline{z}}(t) = \underline{z}(t) - \underline{z}(t_i) \tag{2.24}$$

into (2. 22) and then into (2. 21) yields

$$J_o = E\left\{\sum_{i=0}^{N-1} \frac{1}{T_i^{\,a}} \int_{t_i}^{t_{i+1}} (\underline{H}\,\hat{\underline{z}}(t) - \underline{G}\,\underline{C}\,\hat{\underline{x}}(t))'\,\underline{S}\,(\underline{H}\,\hat{\underline{z}}(t) - \underline{G}\,\underline{C}\,\hat{\underline{x}}(t))dt\right\} \tag{2.25}$$

---

[1] $\underline{s}(t)$ can be any desired signal vector, either generated by a signal model or supplied from an external source.

Expanding the integrand in (2.25) and taking the expectation of each of the three resulting terms using the relationships that

$$E\{\hat{\underline{x}}'(t)\underline{T}\hat{\underline{x}}(t)\} = \hat{\underline{m}}'(t)\underline{T}\hat{\underline{m}}(t) + tr\{\underline{T}\,\underline{V}(t)\}$$

where $\underline{V}(t)$ is found in (2.4) and $tr\{\cdot\}$ is the trace operation,

$\underline{T} = \underline{C}'\underline{G}'\,\underline{S}\,\underline{G}\,\underline{C}$ with

$$E\{\underline{x}(t)\} - E\{\underline{x}(t_i)\} = E\{\underline{x}(t) - \underline{x}(t_i)\} = E\{\hat{\underline{x}}(t)\} = \hat{\underline{m}}(t). \qquad (2.26)$$

Therefore the original cost functional found in (2.25) becomes

$$J_o = \sum_{i=0}^{N-1} \frac{1}{T_i^{\,a}} \int_{t_i}^{t_{i+1}} [(\underline{H}\hat{\underline{z}}(t) - \underline{G}\,\underline{C}\hat{\underline{m}}(t))'\,\underline{S}(\underline{H}\hat{\underline{z}}(t) - \underline{G}\,\underline{C}\hat{\underline{m}}(t) + tr(\underline{T}\,\underline{V}(t))]\,dt \qquad (2.27)$$

Now computing the variance as required in (2.27)

$$\hat{\underline{x}}(t) = \underline{x}(t) - \underline{x}(t_i) = [\underline{\Phi}(t, t_i) - \underline{I}]\,\underline{x}(t_i) + \int_{t_i}^{t} \underline{\Phi}(t, \tau)[\underline{B}\,\underline{u}_e(t_i) + \underline{w}(\tau)]\,d\tau \qquad (2.28)$$

where

$\underline{x}(t_i)$ is an n x 1 state vector at time t_i;

$\underline{u}_e(t)$ is an r x 1 error input to the system;

$\underline{w}(\tau)$ is noise as in (2.2), a n x 1 vector;

$\underline{\Phi}(t, t_i)$ is an n x n state transition matrix at time $(t - t_i)$.

The mean of $\hat{\underline{x}}(t)$ becomes

$$\hat{\underline{m}}(t) = \underline{m}(t) - \underline{m}(t_i) = [\underline{\Phi}(t, t_i) - \underline{I}]\,\underline{m}(t_i) + \int_{t_i}^{t} \underline{\Phi}(t, \tau)[\underline{B}\,\underline{u}_e(t) + \underline{w}_o]\,d\tau \qquad (2.29)$$

where

$\underline{m}(t_i)$ is the mean of the process at t_i, and

$\underline{w}_o$ is the mean of the noise process, n x 1 vector.

Subtracting (2.29) from (2.28):

$$[\hat{\underline{x}}(t) - \hat{\underline{m}}(t)] = [\underline{x}(t) - \underline{x}(t_i) - \underline{m}(t) + \underline{m}(t_i)]$$

$$= [\underline{\Phi}(t, t_i) - \underline{I}]\,[\underline{x}(t_i) - \underline{m}(t_i)] + \int_{t_i}^{t_{i+1}} \underline{\Phi}(t, \tau)[\underline{B}\,\underline{u}_e(t) + \underline{w}(\tau) - \underline{B}\,\underline{u}_e(t) - \underline{w}_o]\, d\tau \qquad (2.30)$$

using (2. 30), and taking the expectation,

$$E\left\{(\hat{\underline{x}}(t) - \hat{\underline{m}}(t))(\hat{\underline{x}}(t) - \hat{\underline{m}}(t))'\right\}$$

$$= [\underline{\Phi}(t, t_i) - \underline{I}]\,[\underline{x}(t_i) - \underline{m}(t_i)]\,[\underline{x}(t_i) - \underline{m}(t_i)]'[\underline{\Phi}(t, t_i) - \underline{I}]' + \int_{t_i}^{t}\int_{t_i}^{\tau} \underline{\Phi}(t, \tau)[\underline{w}(\tau) - \underline{w}_o]\,[\underline{w}(a) - \underline{w}_o]'\underline{\Phi}(t, a)'\; d\tau\, da \qquad (2.31)$$

yields

$$\underline{V}(t) = [\underline{\Phi}(t, t_i) - \underline{I}]\,\underline{V}(t_i)[\underline{\Phi}(t, t_i) - \underline{I}]' + \int_{t_i}^{t} \underline{\Phi}(t, \tau)\underline{\Psi}(\tau)\underline{\Phi}'(t, \tau)\, d\tau \qquad (2.32)$$

Therefore (2.32) is the variance equation required in (2. 27).

The derived cost as found in (2. 27) will be added to the cost for implementation, to be given in detail in Chapter IV, to yield the total derived performance index as found in (2.15) for the signal representation problem.

III. OPTIMAL APERIODIC SAMPLING FOR CONTROL

Periodic sampling criteria have often been used in sampled-data control systems to simplify design, analysis, and implementation. Aperiodic and adaptive sampling criteria have become quite practical due to the introduction of digital computers into system design and control implementation. As a result, various aperiodic and adaptive sampling criteria have been formulated to improve system performance and reduce sampling costs with respect to periodic sampling.

A general framework for the analytical design of non-periodic sampling criteria was first proposed by Hsia [24]. A class of adaptive sampling rules were derived from a continuous time integral performance index which measured the squared error introduced by sampling the error signals of a feedback control system. The performance index was augmented by a "cost for sampling," as introduced by Hsia, which was inversely proportional to the sampling interval length in order to insure that the sampling intervals were greater than zero. The performance index was defined over just one sampling interval and thus the sampling rules obtained determined the length of only one sampling interval for each set of measurements taken of the signal being sampled. The sampling rules were derived by approximating the error signal and the cost for sampling by a Taylor series. These approximations were made in order to derive an analytic expression for the sampling rule. The resulting sampling rules were explicitly dependent on the signal and its derivatives and were not explicitly dependent on the system dynamics, system inputs, system disturbances, and costs of implementation. The dependence of the sampling rule on the a priori knowledge of the system was destroyed by the approximations made to the cost functional.

Each sampling rule was tested by using it to adaptively sample a continuous time feedback error signal. The results of the tests revealed that each of the adaptive sampling rules had a tendency to sample at the sampling extremes if constraints on the minimum and maximum sampling intervals were imposed. The underlying assumption of this work was that good signal representation by adaptive sampling of the continuous time error feedback signals implied good system performance over the desired control interval.

In general, the performance of a feedback control system implemented through use of periodic sampling is dependent on the sampling rate. Periodic sampling requires the sampling rate to be greater than twice the system bandwidth in order to minimize aliasing and thus provide reasonable control performance. The actual sampling rate used is a compromise between control performance and the costs for computation, data storage, data communications, and sampling hardware.

Optimal periodic sampling of discrete time control systems was first formulated by Kushner [40] and later extended by Aoki [3] in order to obtain an analytic formulation for the design of periodic sampling criteria. A discrete-time performance index was used which consisted of the terminal errors squared, a summation of the squared piecewise constant closed loop controls, and a "cost for sampling" term proportional to the number of samples taken over the control interval. The system state equation and control law were also modeled in discrete time with explicit dependence on the sampling rate by substitution of the system state equation and control law into the performance index. The derived performance index was then minimized with respect to the sampling rate through use of an integer programming algorithm. The result was an optimal periodically sampled discrete time feedback control system with consideration being given to sampling costs. These papers [3, 40] were the first attempts at using a control performance index and "costs for sampling" to obtain optimal sampling for control.

A general optimal aperiodic sampled-data control problem was first formulated by Jordan and Polak [46]. A continuous-time cost functional was used with the control being a piecewise constant vector function over each sampling interval. Both the control level over each interval and the length of each sampling interval were selected to minimize the performance index. As a result, the optimal control consisted of an optimal piecewise constant vector function and an optimal sampling interval sequence. The necessary conditions were established for the control problem but a computational algorithm was not developed to obtain the optimal control. The performance index which was considered was not a function of the number of sampling intervals and did not have provisions for incorporating implementation costs or system disturbances. Tabak and Kuo [47] solved the optimal control problem introduced by Jordan and Palak [46] by discretizing the performance index and state equations a priori thus enabling them to solve the non-linear programming problem through use of a sequential unconstrained minimization technique.

The optimal aperiodic sampled-data regulator problem was investigated for state dependent sampling by Schlueter and Levis [21] and for constrained sampling times by Schlueter [19]. Schlueter developed an efficient computational algorithm for the problem formulated by Jordan and Polak [46] for the special case where the optimal control could be determined uniquely as a function of the sampling interval sequence. This optimal control sequence was determined by solving the Kuhn Tucker conditions. The control sequence was then substituted into the performance index to obtain a derived cost functional which was dependent on the sampling interval sequence. The optimal sampling interval sequence was then determined by minimizing the derived cost functional subject to the sampling constraints. The optimal sampling interval sequence specified the optimal aperiodic sampled-data control law. The optimal control-optimal sampling interval combination was shown to outperform an optimal control law with any periodic or arbitrary aperiodic sampling criteria.

All of the previous references were considered in the formulation of the optimal aperiodic sampling for control problem to be presented in this chapter. A continuous time performance index and system state equation were chosen similar to those found in Schlueter [19] and Jordan and Polak [45] and unlike the performance indices and system equations found in Aoki [3], Kushner [40], and Tabuk and Kuo [48]. The performance index is augmented by a cost for implementation term proportional to computer computational time requirements, data and software storage requirements, and output data communication costs. Implementation costs were implied by the "cost for sampling" terms as found in Aoki [3] and Hsia [24] but neglected in the other references. The performance index will consider system noise disturbances, which were neglected in earlier references and will be dependent on the sampling interval sequences as in Jordan and Polak [46], Tabak and Kuo [47], and Schlueter [19]. The number of aperiodic sampling intervals used during the control interval will be free and selected optimally as in Aoki [3] for the special case of periodic sampling. All other references fix the number of samples used during the control interval. The control law will be specified a priori as found in Aoki [3] and Hsia [24].

The design of optimal aperiodic sampling for control implementation, with a specified control law is proposed. The performance index will be formulated with dependence on the system state equation, control inputs, tracking errors, initial conditions, disturbance noise, and augmented by costs for implementation as discussed in Chapter II. The performance index will then be derived as a function of the sampling interval sequence and implicitly as a function of the number of sampling intervals. Using non-linear programming the derived performance index will be minimized with respect to the sampling interval sequence for a fixed number of sampling intervals to yield the sub-optimal aperiodic sampling criteria. By selecting the number of samples, and through use of repeated optimizations the optimal aperiodic sampling criteria for control is obtained.

3.1. Aperiodic Sampling for Control Problem Statement

Consider the linear time invariant system,

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}_e(t) + \underline{w}(t) \qquad (3.1)$$

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) \qquad (3.2)$$

where

$\underline{x}(t)$ - n dimensional state vector of the system producing $\underline{y}(t)$;

$\underline{u}_e(t)$ - r dimensional error signal vector;

$\underline{z}(t)$ - r dimensional system input signal vector;

$\underline{y}(t)$ - m dimensional output signal vecotr; and

$\underline{w}(t)$ - n dimensional disturbance noise vector.

Matrices $\underline{A}$, $\underline{B}$, and $\underline{C}$ are constant and compatible with the above vectors. The initial time t_o is known, with $t_f > t_o$ being the fixed terminal time. The initial state $\underline{x}(t_o)$ is assumed to be a Gaussian random vector with mean and variance

$$E\{\underline{x}(t_o)\} = \underline{m}(t_o), \quad Cov\{\underline{x}(t_o), \underline{x}(t_o)\} = \underline{V}(t_o) \tag{3.3}$$

The plant noise process $\underline{w}(t)$ is assumed to be non-zero mean Gaussian white noise

$$E\{\underline{w}(t)\} = \underline{w}_o, \qquad t \in [t_o, t_f] \tag{3.4}$$

$$Cov\{\underline{w}(t), \underline{w}(\tau)\} = \underline{\Psi}(t)\,\delta(t-\tau), \; t, \tau \in [t_o, t_f] \tag{3.5}$$

where $\delta(t-\tau)$ is a delta function. In addition, $\underline{x}(t_o)$ and $\underline{w}(t)$ are assumed independent.

The performance of the system found in equations (3.1) and (3.2) will be investigated for two cases of the feedback control error signal vector, $\underline{u}_e(t)$. Case 1 will investigate system control by using

$$\underline{u}_e(t) = \underline{u}_e(t_i) = \underline{H}\,\underline{z}(t_i) - \underline{G}\,\underline{y}(t_i) \tag{3.6}$$

where $\underline{H}$ and $\underline{G}$ are r x r and r x m dimensional constant input and feedback gain matrices respectively and $\underline{z}(t)$ is an arbitary continuous time input signal vector. Figure 3.1 shows the location of the sample and hold device used in the sampled-data control system being considered in Case 1. This is the only case which will be investigated in Chapter IV since it is the most generally implemented.

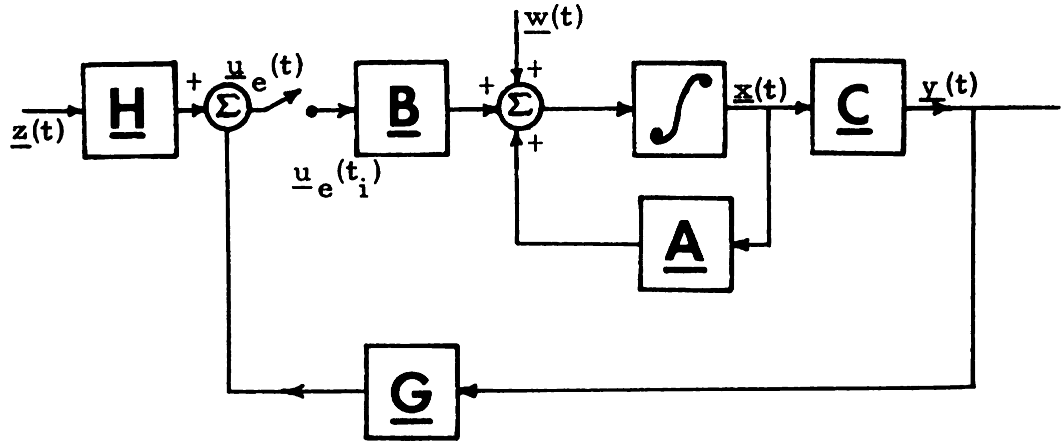


Figure 3.1 - Control System Configuration Using a Sampled Feedback Error Signal

$\underline{H}$ and $\underline{G}$ are constant r x r and r x m dimensional matrices respectively with the other system matrices being defined in equations (3.1) and (3.2).

Case 2 will consider system control using

$$\underline{u}_e(t) = \underline{Hz}(t) - \underline{Gy}(t_i) \tag{3.7}$$

where $\underline{H}$, $\underline{G}$, and $\underline{z}(t)$ are as defined earlier. Figure 3.2 shows the location of the sample and hold device used in the sampled-data control system being considered in Case 2

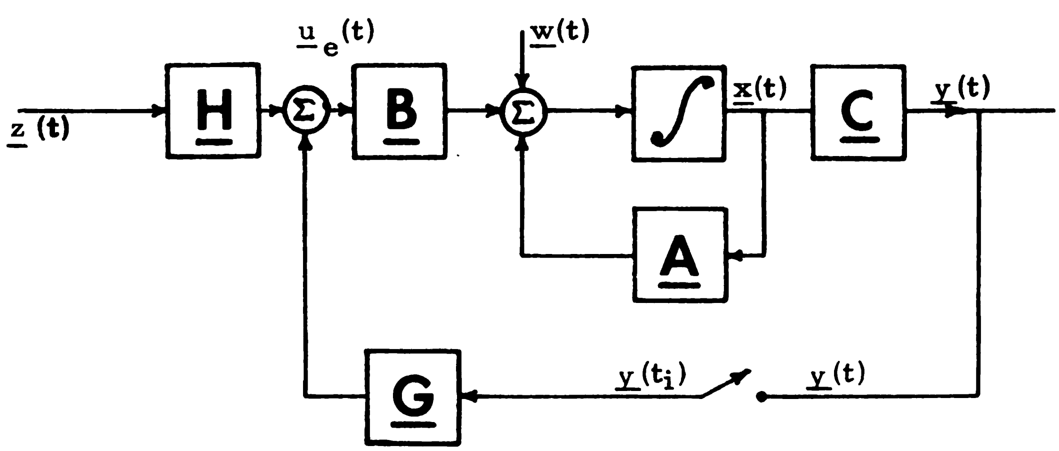


Figure 3.2 - Control System Configuration Using a Continuous Input Signal and Sampled Feedback

where the above sampled-data control system is defined in equations (3.1) and (3.2) with $\underline{H}$, $\underline{C}$, and $\underline{G}$ being r x r, m x n, and r x m dimensional constant matrices.

With the system specified, the feedback control signal generated during the control interval is to be sampled at the sampling times t_i; such that,

$$t_{initial} = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = t_{final} \tag{3.8}$$

with a sampling interval being defined as T_i where

$$T_i = t_{i+1} - t_i \tag{3.9}$$

for i = 0, 1, . . . , N - 1 with the sampling interval sequence being

$$\underline{T} = [T_0, T_1, \dots, T_{N-1}] \tag{3.10}$$

An optimal periodic sampling criterion for control can be specified by having the number of sampling intervals, N, satisfy

$$N_{min} \leq N \leq N_{max} \tag{3.11}$$

and the equality constraint knowing t_f and t_o

$$\underline{g}(\underline{T}) = \left\{ \frac{t_f - t_o}{N} - T_i = 0 \qquad i = 0, 1, \dots, N-1, \text{ for any } N \right. \tag{3.12}$$

where N_{max} is determined by sampler performance specifications and N_{min} by system stability. Optimal periodic sampling is obtained through minimization of the derived control implementation performance index with respect to N satisfying (3.11) and (3.12) thus obtaining the optimal number of samples N^*. Periodic sampling for control is specified by a fixed N such that

$$N_{min} = N = N_{max} \tag{3.13}$$

and having (3.12) satisfied for the N chosen.

An optimal aperiodic sampling criterion for control can be specified by having the number of sampling intervals, N, satisfy (3.11). The sampling intervals satisfy constraints of the form

$$0 < T_{min} \leq T_i \leq T_{max} \qquad (3.14)$$

where T_{min} is determined by the sampler performance specifications and T_{max} by system stability. Thus given t_f and t_o, the following sampling constraint is to be satisfied,

$$g(\underline{T}) = \sum_{i=o}^{N-1} T_i - [t_f - t_o] = 0 \qquad (3.15)$$

Optimal aperiodic sampling for control is obtained through minimization of the derived implementation performance index with respect to both N and $\underline{T}$ satisfying (3.11), (3.14) and (3.15) to yield the optimal number of sampling intervals N^* and the optimal sampling interval sequence, $\underline{T}^*$. Sub-optimal aperiodic sampling for control specifies a fixed N such that

$$N_{min} = N = N_{max} \qquad (3.16)$$

with the sampling interval sequence, $\underline{T}$, free but satisfying (3.14) and (3.15). The optimal sampling interval sequence, $\underline{T}^*$, is obtained through minimization of the derived aperiodic sampling for control implementation performance index with respect to $\underline{T}$ only.

Each of the optimal sampling criteria specified require minimization of a performance index with respect to certain variables. The exact form of the control implementation performance index is based on various references. A quadratic rather than an absolute error performance index is chosen since large tracking errors either positive or negative and large control energy expenditures should be penalized more heavily. Thus the sampling for control performance index is

$$J = J_o + J_f \qquad (3.17)$$

where

$$J_o = E\{ (\underline{y}(t_N) - \underline{z}(t_N))' \underline{Y} (\underline{y}(t_N) - \underline{z}(t_N))$$

$$+ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} [(\underline{y}(t) - \underline{z}(t))' \underline{Q} (\underline{y}(t) - \underline{z}(t)) + \underline{u}_e'(t) \underline{R}\, \underline{u}_e(t)]\, dt \} \tag{3.18}$$

Matrices $\underline{Y}$, $\underline{Q}$, and $\underline{R}$ are assumed to be positive semi-definite symmetric m, m, and r dimensional square matrices respectively and $E\{\cdot\}$ is the expectation operator taken over the random variable $\underline{x}(t)$.

The J_f portion of (3.17) represents the implementation costs such as, computer computation time, computer program and data storage, and costs for output data communications if necessary as discussed in Chapter II.

Thus, the optimal aperiodic sampling for control problem using the desired sampling criteria and appropriate sampling constraints can be stated as follows:

Given the linear dynamical system (3.1) and (3.2) with arbitrary system input $\underline{z}(t)$, determine the optimal sampling interval sequence, $\underline{T}^*$, and the optimal number of sampling intervals, $\underline{N}^*$, such that the performance index (3.17) is minimized for the particular control system specified in (3.3) through (3.7).

3.2. The Derived Control Implementation Performance Index

The feedback control configurations shown in Figures 3.1 and 3.2 will now be used to obtain the derived control performance index. In each case sampling will be involved in the determination of the feedback control signal $\underline{u}_e(t)$. Sampling of the feedback error signal, $\underline{u}_e(t)$, will occur at the sampling times t_i for $i = 0, 1, \ldots, N-1$ as determined by the derived control performance index and the optimization routine being used.

The performance of the Case 1 feedback control system found in Figure 3.1 will be determined through use of the performance index as found in (3.17) with $\underline{u}_e(t)$ being defined in (3.6). The performance index found in (3.18) plus the implementation costs as discussed in Chapter II yields the total control performance index.

The control performance indices for Case 1 and Case 2 will now be derived in terms of the sampling interval sequence, $\underline{T}$, defined in (3.10), and the number of sampling intervals, N, satisfying (3.11) or (3.13).

Derived Performance Index for Case 1

Substitution of equations (3.2) and (3.6) with $\underline{H} = \underline{I}$, where $\underline{I}$ is the identity matrix, into (3.18) yields

$$J_o = E\{(\underline{C}\,\underline{x}(t_N) - \underline{z}(t_N))'\,\underline{Y}\,(\underline{C}\,\underline{x}(t_N) - \underline{z}(t_N)) + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} [(\underline{C}\,\underline{x}(t) - \underline{z}(t))'\,\underline{Q}\,(\underline{C}\,\underline{x}(t) - \underline{z}(t)) + (\underline{z}(t_i) - \underline{G}\,\underline{C}\,\underline{x}(t_i))'\,\underline{R}\,(\underline{z}(t_i) - \underline{G}\,\underline{C}\,\underline{x}(t_i))]\,dt\} \qquad (3.19)$$

Taking the expectation of (3.19) and using the relationship

$$E\{\underline{x}'(t)\,\underline{M}\,\underline{x}(t)\} = \underline{m}'(t)\,\underline{M}\,\underline{m}(t) + tr[\underline{M}\,\underline{V}(t)] \qquad (3.20)$$

where $\underline{M}$ is an appropriately dimensioned matrix and

$$E\{\underline{x}(t)\} = \underline{m}(t),\quad E\{\underline{x}(t)\,\underline{x}'(t)\} = \underline{V}(t) \qquad (3.21)$$

yields the derived performance index for Case 1;

$$J_o(\underline{T}, N) = ((\underline{C}\,\underline{m}(t_N) - \underline{z}(t_N))'\,\underline{Y}\,(\underline{C}\,\underline{m}(t_N) - \underline{z}(t_N)) + tr[\underline{C}'\,\underline{Y}\,\underline{C}\,\underline{V}(t_N)]$$

$$+ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} [(\underline{C}\,\underline{m}(t) - \underline{z}(t))'\,\underline{Q}\,(\underline{C}\,\underline{m}(t) - \underline{z}(t)) + tr[\underline{C}'\,\underline{Q}\,\underline{C}\,\underline{V}(t)]$$

$$+ (\underline{z}(t_i) - \underline{G}\,\underline{C}\,\underline{m}(t_i))'\,\underline{R}\,(\underline{z}(t_i) - \underline{G}\,\underline{C}\,\underline{m}(t_i))$$

$$+ tr[\underline{C}'\,\underline{G}'\,\underline{R}\,\underline{G}\,\underline{C}\,\underline{V}(t_i)]\,]\ dt \qquad (3.22)$$

where

$$\underline{T} = [T_0, T_1, \ldots, T_i, \ldots, T_{N-1}] \text{ and } T_i = t_{i+1} - t_i.$$

The variance term $\underline{V}(t)$ found in (3.22) is obtained from the following equations. The system state is represented by

$$\underline{x}(t) = \underline{\Phi}(t, t_i)\,\underline{x}(t_i) + \int_{t_i}^{t} \underline{\Phi}(t, \tau)\ \underline{B}[\underline{z}(t_i) - \underline{G}\,\underline{C}\,\underline{x}(t_i)] + \underline{w}(\tau)\ d\tau \qquad (3.23)$$

where

$$\underline{\Phi}(t, t_i) = e^{\underline{A}(t - t_i)}, \qquad \underline{\Phi}(t, \tau) = e^{\underline{A}(t - \tau)}$$

both being state transition matrices of the system found in (3.1). Taking the expectation of (3.23) yields

$$\underline{m}(t) = \underline{\Phi}(t, t_i)\,\underline{m}(t_i) + \int_{t_i}^{t} \underline{\Phi}(t, \tau)\ \underline{B}[z(t_i) - \underline{G}\,\underline{C}\,\underline{m}(t_i)] + \underline{w}_o\ dt \qquad (3.24)$$

Subtracting (3.24) from (3.23) yields

$$[\underline{x}(t) - \underline{m}(t)] = \underline{\Phi}(t, t_i)[\underline{x}(t_i) - \underline{m}(t_i)]$$

$$+ \int_{t_i}^{t} \underline{\Phi}(t, \tau) \left(-\underline{B}\,\underline{G}\,\underline{C}[\underline{x}(t_i) - \underline{m}(t_i)] + \underline{w}(\tau) - \underline{w}_o\right) d\tau \qquad (3.25)$$

or

$$[\underline{x}(t) - \underline{m}(t)] = \underline{\theta}(t, t_i)[\underline{x}(t_i) - \underline{m}(t_i)]$$

$$+ \int_{t_i}^{t} \underline{\Phi}(t, \tau)[\underline{w}(\tau) - \underline{w}_o]\, d\tau \qquad (3.26)$$

where

$$\underline{\theta}(t, t_i) = \underline{\Phi}(t, t_i) - \underline{D}(t, t_i) \tag{3.27}$$

$$\underline{D}(t, t_i) = \int_{t_i}^{t} \underline{\Phi}(t, \tau)\, d\tau\, \underline{B}\,\underline{G}\,\underline{C} \tag{3.28}$$

Therefore since the noise $\underline{w}(\tau)$ for $\tau > t_0$ is uncorrelated with the state $\underline{x}(t_0)$, the final expression for the variance becomes

$$\underline{V}(t) = E\{(\underline{x}(t) - \underline{m}(t))(\underline{x}(t) - \underline{m}(t))'\}$$

$$= \underline{\theta}(t, t_i)\, E\{[\underline{x}(t_i) - \underline{m}(t_i)]\,[\underline{x}(t_i) - \underline{m}(t_i)]\}'\,\underline{\theta}'(t, t_i)$$

$$+ \int_{t_i}^{t_o}\int_{t_i}^{t_o} \underline{\Phi}(t,\tau)\, E\{[\underline{w}(\tau)-\underline{w}_o][\underline{w}(\alpha)-\underline{w}_o]'\}\,\underline{\Phi}'(t, \alpha)\, d\tau\, d\alpha \tag{3.29}$$

where $E\{[\underline{x}(t_i) - \underline{m}(t_i)]\,[\underline{x}(t_i) - \underline{m}(t_i)]\} = \underline{V}(t_i)$ and

$E\{[\underline{w}(\tau) - \underline{w}_o]\,[\underline{w}(\alpha) - \underline{w}_o]'\} = \underline{\psi}(\tau)\,\delta(\tau - \alpha)$, and the integration with respect to α yields

$$\underline{V}(t) = \underline{\theta}(t, t_i)\,\underline{V}(t_i)\,\underline{\theta}'(t, t_i) + \int_{t_i}^{t} [\underline{\Phi}(t,\tau)\,\underline{\psi}(\tau)\,\underline{\Phi}'(t, \tau)]\, d\tau \tag{3.30}$$

which is the required expression for the variance to be used in (3.22)

Derived Performance Index for Case 2. (Figure 3.2)

The derived performance index for Case 2 is similar to the derived performance obtained for Case 1. The difference is in the fact that $\underline{u}_e(t_i)$ is replaced by $\underline{u}_e(t)$ as seen in Figure 3.2 and equation (3.7). Therefore substitution of equation (3.2) and (3.7), with $\underline{H} = \underline{I}$, into (3.18) yields a performance index equation identical to (3.19) with the exception that $\underline{z}(t_i)$ is replaced by $\underline{z}(t)$. Thus using equations (3.21) and (3.22) the derived performance index for Case 2 becomes

$$J_o(\underline{T}, N) = (\underline{C}\,\underline{m}(t_N) - \underline{z}(t_N))'\,\underline{Y}\,(\underline{C}\,\underline{m}(t_N) - \underline{z}(t_N)) + \mathrm{tr}\,[\underline{C}'\,\underline{Y}\,\underline{C}\,\underline{V}(t_N)]$$
$$+ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (\underline{C}\,\underline{m}(t) - \underline{z}(t))'\,\underline{Q}\,(\underline{C}\,\underline{m}(t) - \underline{z}(t)) + \mathrm{tr}[\underline{C}'\,\underline{Q}\,\underline{C}\,\underline{V}(t)]$$
$$+ (\underline{z}(t) - \underline{G}\,\underline{C}\,\underline{m}(t_i))'\,\underline{R}\,(\underline{z}(t) - \underline{G}\,\underline{C}\,\underline{m}(t_i))$$
$$+ \mathrm{tr}[\underline{C}'\,\underline{G}\,\underline{R}\,\underline{G}\,\underline{C}\,\underline{V}(t_i)] \quad dt \qquad (3.31)$$

where $\underline{T} = [T_0, T_1, \ldots, T_i, \ldots, T_{N-1}]$ and $T_i = t_{i+1} - t_i$. The variance term $\underline{V}(t)$ is derived in equations (3.23) through (3.29).

IV. EVALUATION OF SAMPLING METHODS

In any signal representation or control performance evaluation program, suitable simulation models have to be developed from which the proposed methods of sampling can be investigated under various signal and system conditions. In particular, the performance of periodic, adaptive, and aperiodic methods of sampling for signal representation and system control will be investigated. The signal which is to be sampled will be generated by a known signal model for the signal representation problem and by a known feedback control system for the sampling for system control implementation problem. The initial conditions, system matrices, weighting coefficients, and final time will be specified for each investigation. A step, ramp, parabola, and noise will be used as system inputs with all optimizations being performed using the non-linear programming optimization subroutine ZXPOWL as found of the CDC 6500 computer system. [See reference [53] for a detailed description of the ZXPOWL subroutine.]

4.1. Sampling for Signal Representation

The sampling for signal representation problem is restated from Chapter II:

> Determine the optimal sampling interval sequence $(N^*, \underline{T}^*)$ that specifies the optimal sample and hold approximation
>
> $\hat{\underline{s}}(t) = \underline{s}(t_i), \quad t \in [t_i , t_{i+1}]; \quad i = 0, 1, \ldots , N-1$
>
> that minimizes a signal representation performance index (2.15) subject to the sampling constraints (2.10), (2.13), and (2.14).

The signal representation performance index (2.15) not only measures the error

$$s(t) - \hat{s}(t), \; t \in [t_i, t_{i+1}] ; \quad i = 0, 1, \ldots, N-1$$

caused by the sample and hold mechanism but also measures the costs for implementation which includes a cost for computer utilization, program and data storage requirements, and data communications.

The following signal representation problem is presented to illustrate the theory and provide insight into the performance of optimal aperiodic sampling for signal representation.

Objective: Investigate the tradeoffs in periodic, adaptive and aperiodic sampling of a data signal record (to be supplied from auxiliary storage upon request) for communications to a remote signal level display panel.

General Specifications[1]:

1. The data signal record will consist of 101 data points spanning one unit of time.
2. A minimum of 2 and a maximum of 8 piecewise constant approximations will be allowed for the representation of each signal data record.
3. Each data point inputted from auxiliary storage consists of 3 ASCII[2] characters. This will also be true for the sampling time and piecewise constant signal level output data as obtained through use of periodic, adaptive, or aperiodic sampling.

---

[1] Details not specifically stated in "General Specifications" are left to the discretion of the System Designer.

[2] ASCII is the commonly used abbreviation for American Standard Code for Information Interchange.

4. Output data (time and level information) will be transmitted to the remote display via serial data transmission techniques.

5. A start, stop, and parity bit will be added to each ASCII output character before transmission.

6. Any parity errors occuring during data transmission and detected at the remote display will be indicated by a "parity error" light. The error indicator will be automatically reset after the next signal is received for display. No other means of error detection or correction are necessary.

Figure 4.1 presents the Remote Display System in block diagram form. The most important block shown is the "Main Signal Record Processor and P/S Converter Controller" block. This block represents the computer portion of the display system. The computer will be required to perform the following tasks:

a) request continuous time signal records from the auxiliary storage facility

b) load the signal record into the proper portion of its main memory for purposes of signal representation through sampling

c) execute the predetermined optimization program stored in main memory with respect to the minimization of the derived signal representation performance index

d) control the transmission of the resultant signal level and sampling time information to the remote display. This entails the control of the parallel to serial converter via the P/S control line as indicated. (The S/P converter on the display end of the data transmission line is assumed a passive device.)

The computer is assumed to be entirely committed to either signal processing through sampling or data transmission via the P/S converter and communications line. The Signal Record Processor is assumed too small with respect to memory size and too slow with respect to computational speed to allow signal multiplexing and therefore optimal aperiodic sampling will be performed for data compression. The rest of the blocks shown in Figure 4.1 are explained in the figure or are self-explanatory.

A simplified flowchart of the computer program used in the evaluation of the three methods of sampling, viz. periodic, adaptive, and sub-optimal aperiodic, for the remote display problem is shown in Figure 4.2.

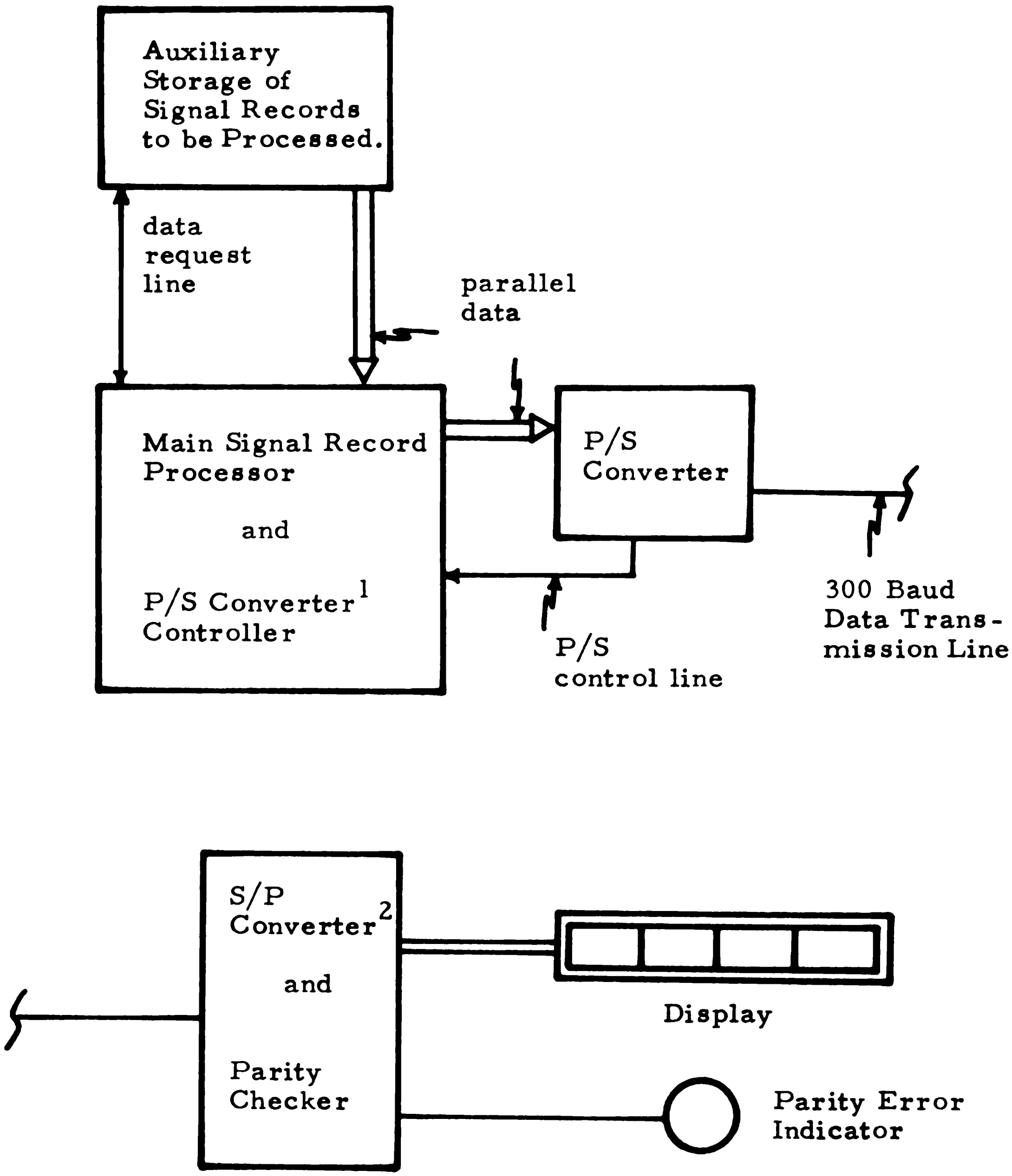


Figure 4.1 - Block Diagram of the Remote Display System

---

[1] P/S Converter converts data from parallel data inputted on several information lines to serial data which is transmitted to the display using only one transmission line.

[2] Reverse of P/S Converter.

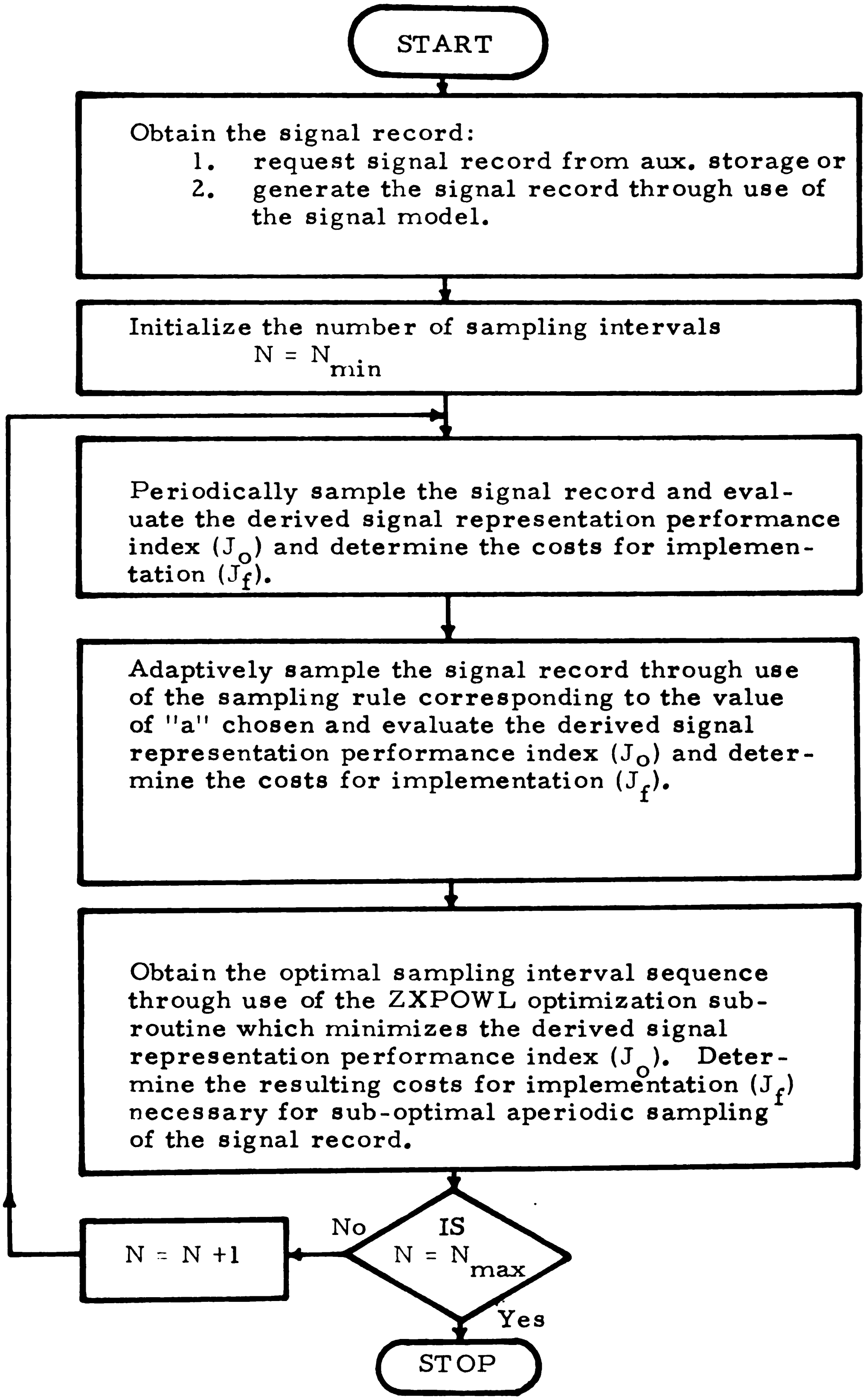


Figure 4.2 - Computer Program Flowchart for Signal Representation Investigation

4.2. Signal Representation Performance Results

The sampling evaluation computer program as found in the simplified flowchart form in Figure 4. 2 will be used to investigate periodic, adaptive, and aperiodic sampling for signal representation as formulated in Chapter II. The following system model and derived signal representation performance index will be used

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 10 \\ 100 \end{bmatrix} u_e(t) , \qquad \underline{x}(t_o) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{4.1}$$

$$y(t) = [1\ 0]\ \underline{x}(t) \quad + [\ 0]\ z(t) \tag{4.2}$$

with

$$s(t) = u_e(t) = z(t) - y(t) \tag{4.3}$$

where $\underline{z}(t)$ can be one of the following input signals

a) 1.0 (Step)

b) t (Ramp)

c) t^2 (Parabola)

for $t \epsilon [t_o, t_f]$ where $t_o = 0.0$ and $t_f = 1.0$. The control signal $u_e(t)$ will be sampled for signal representation at the sampling times t_i such that,

$$t_{initial} = t_o < t_1 < t_2 < \dots < t_{N-1} < t_N = t_{final}$$

with a sampling interval being defined as T_i where

$$T_i = t_{i+1} - t_i$$

The derived signal representation performance index is

$$J = \sum_{i=0}^{N-1} \frac{1}{T_i^a} E\left\{ \int_{t_i}^{t_{i+1}} [u_e(t) - u_e(t_i)]^2 dt \right\} + J_f \tag{4.4}$$

The cost for implementation, J_f, for periodic sampling is

$$J_f = w_1 \times M \times U + \frac{w_2 \times K \times W}{R} \tag{4.5}$$

where

M = 800 words of memory

w_1 = .0001 (word-second)$^{-1}$

W = 30 bits/output word (record) to be transmitted

K = (N+4) words (records)[1]

R = 300 bits/second

w_2 = .75 seconds^{-1}

The cost for implementation for adaptive and aperiodic sampling is

$$J_f = w_1 \times M \times U + \frac{w_2 \times K \times W}{R} \qquad (4.6)$$

where

M = 1000 words of memory

w_1 = .0001 (word-second)$^{-1}$

W = 30 bits/output word (record) to be transmitted

K = 2 x (N+1) words (records)[2]

R = 300 bits/second

w_2 = .75 seconds^{-1}

---

[1](N+4) is obtained from the fact that specifying periodic data requires

1. 1 data word to indicate the initial time.
2. 1 data word to indicate the time between data points.
3. N + 1 data points.
4. 1 data word to indicate the number of data points sent.

Thus the communications costs for periodic sampling will be based on (N + 4) data words to be sent for each sampled signal record which is processed.

[2]2 (N +1) is obtained from the fact that specifying aperiodic or adaptive sampling requires

1. (N + 1) data words to specify the N + 1 signal magnitudes.
2. (N + 1) data words to indicate the times of each signal magnitude change.

Thus the communications costs for aperiodic/adaptive sampling depends on 2 (N +1) data words to be sent for each sampled signal record which is processed.

In the computer utilization portion of (4.5) or (4.6) the computer memory requirements (M) are determined by the number of words required to store the sampling criteria software in the main memory of the computer being used to obtain the signal representation data. The value U, which depends on the number of sampling intervals; N, the complexity of the system software, (i.e., cost functional subroutine, optimization routines, etc.) and the length of the signal record, is found by determined the computer processing time required, in this case through use of the CDC 6500 computer, to obtain the optimal sampling interval sequence of the input signal record using any particular value of N. The weighting coefficient w_1 is the cost per word-second for the particular computer being used.

In the communications cost portion of (4.5) and (4.6) the values of R and W are determined respectively by specifying the minimum communications channel capacity to be 300 Baud and that each signal level measurement and sampling time be transmitted by using 3 ASCII characters with start, stop, and parity bits being added to each character. The number of words (records) (K) to be transmitted via the communications net work depends on the number of sampling intervals and the type of sampling being used. The weighting coefficient w_2 is the cost per second for use of the communications channel for data transmission. In general both weighting factors (i.e., w_1 and w_2) are highly problem specific and will change with system specifications or advances in technology.

A comparison of optimal aperiodic sampling criteria determined for three different performance indices will be made first. The parameter "a" will take on the values of -1, 0, and 1 resulting in a time-weighted, unweighted, or average error type performance indices. The sub-optimal aperiodic sampling criterion for each N will be compared with periodic and the appropriate adaptive sampling rule for the value of "a" being used. These adaptive sampling rules were derived in Hsia [24] using the three values of "a" as indicated.

Adaptive Rule 1 (AR1) for a = 0:

$$T_i = c_1 / |\dot{u}_e(t_i)|, \text{ where } c_1 \text{ is adjusted to yield the desired number of sampling intervals.} \tag{4.7}$$

Adaptive Rule 2 (AR2) for a = -1:

$$T_i = c_2 / \left(\dot{u}_e(t_i)^2\right)^{1/3}, \text{ where } c_2 \text{ is adjusted to yield the desired number of sampling intervals.} \tag{4.8}$$

Adaptive Rule 3 (AR3) for a = 1:

$$T_i = T_{max} / \left(\alpha \dot{u}_e(t_i) + 1.0\right), \text{ where } \alpha \text{ is adjusted to yield the desired number of sampling intervals. } T_{max} = T_{final} \tag{4.9}$$

Therefore given the system, performance index, and cost for implementation function, the figures which follow will graphically illustrate performance cost changes, J_o, as a function of the number of sampling intervals used to represent the signal being sampled. In addition the total or system performance costs, J, will also be shown in each figure, (i.e. $J = J_o + J_f$). Both performance costs will be investigated using the three "a" values for the periodic, optimal aperiodic and the appropriate adaptive sampling criteria. The performance indices will be evaluated for the case of a step input, $(z(t) = 1)$ where $\underline{x}(t_o)' = [\,-1 \;\; 0\,]$ is specified, and the disturbance noise is zero. The performance costs for sub-optimal aperiodic sampling represent the minimized performance cost obtained through use of the ZXPOWL optimization subroutine for the specified number of sampling intervals. (The exact numerical value of each performance cost is listed by figure number in Appendix C.)

The performance of optimal aperiodic sampling for signal representation will be compared with adaptive and periodic sampling. The control performance costs for aperiodic, adaptive, and periodic sampling will be designated as J_1, J_2, J_3, respectively in this chapter. The optimal performance costs for each sampling criterion, denoted as J_i*, i = 1, 2, 3, are defined as the minimum value of J_i over the set of N satisfying the constraint $2 \leq N \leq 8$, (2.10). The optimal system performance ratio between adaptive and aperiodic sampling, denoted by $(J_2/J_1)^*$, is defined as the maximum ratio of J_2/J_1 over the set (2.11) of feasible N. The optimal system performance ratio between periodic and aperiodic sampling criteria, denoted by $(J_3/J_1)^*$, is the maximum ratio J_3/J_1 over the set of feasible N, (2.11).

Sub-optimal aperiodic sampling reduces signal representation errors compared to periodic or adaptive sampling for each value of N for the three different cases a = -1, a = 0, and a = 1 as shown in Figures 4.3(a), 4.4(a), and 4.5(a) respectively. The optimal aperiodic (OA) cost, J_1^*, is significantly smaller than the optimal adaptive sampling (OAD) cost, J_2^*, or optimal periodic sampling (OPS) cost, J_3^*, for each case (a = -1, 0, 1) shown in Table 4.1. Moreover, the maximum ratio of the performance costs between adaptive and sub-optimal aperiodic sampling $(J_2/J_1)^*$ and the maximum ratio of performance costs between periodic and sub-optimal aperiodic sampling $(J_3/J_1)^*$ are quite large. These large performance ratios indicate that selecting the sequence of sampling intervals together rather than individually as is done in adaptive sampling or using equal length sampling intervals as in periodic sampling, can significantly reduce the errors caused by the sample and hold operation. It is also apparent from Table 4.1 that the optimal performance J_i^* for optimal aperiodic (i=1), optimal adaptive (i=2), and optimal periodic (i=3) increases as "a" increases. These increases can be explained by noting that the performance index is proportional to $(1/T_i)^a$ which increases with "a" since in all cases T_i is less than one (t_f is equal to one).

	OA		OAD		OPS		AD/OA		PS/OA	
"a"	J_1^*	N_1^*	J_2^*	N_2^*	J_3^*	N_3^*	$\frac{J_2}{J_1}^*$	$\frac{N_2}{1}$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}$
-1	.003	8	.036	8	.036	8	17.9	2	15.6	2
0	.040	6	.073	5	.276	8	12.8	5	6.7	5
1	.522	6	.638	6	2.12	8	5.9	5	1.6	4

Table 4.1. Data summary for Figures 4.3(a) to 4.5(a).

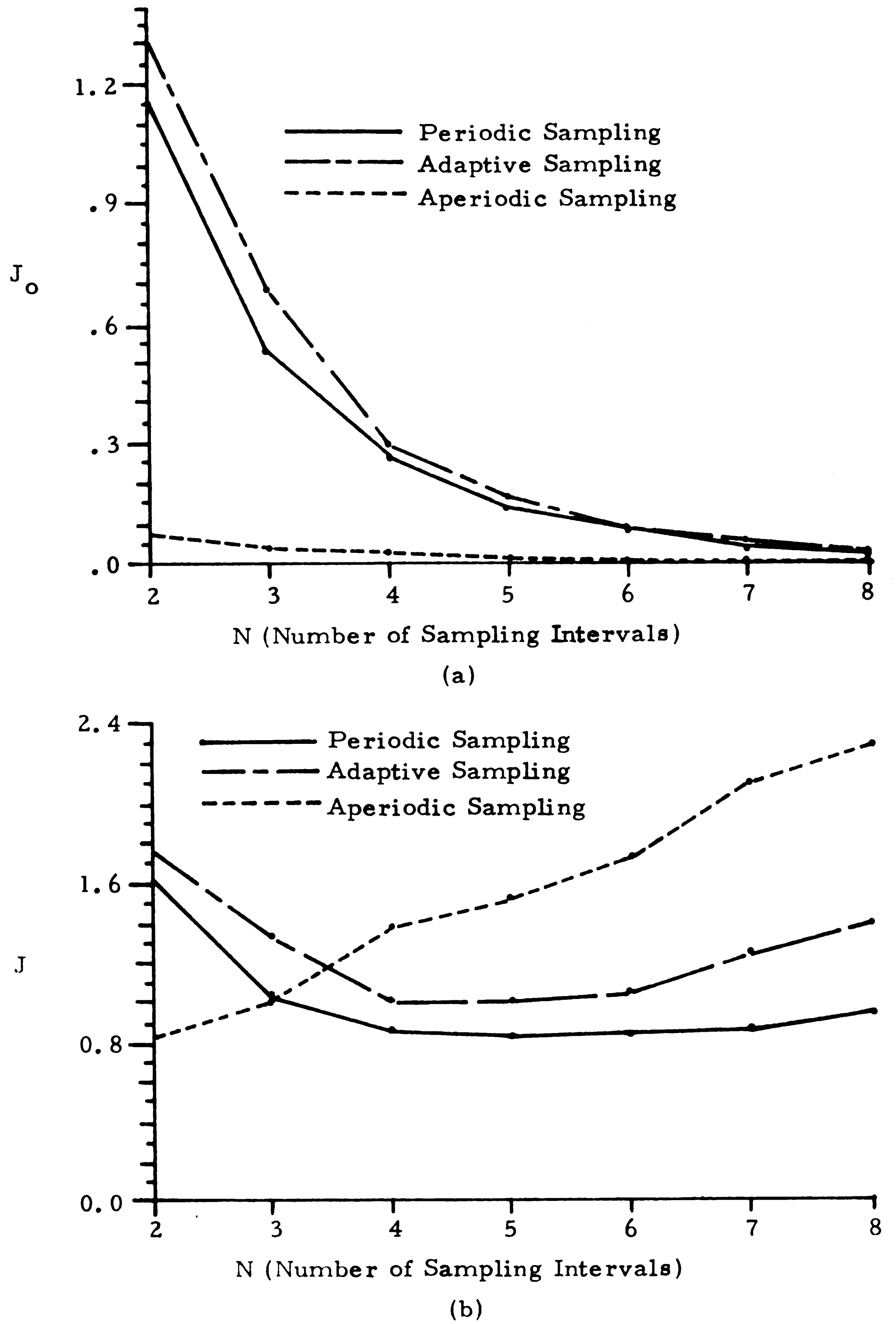


Figure 4.3 - Signal Representation Costs Using
a = -1 and Step Input

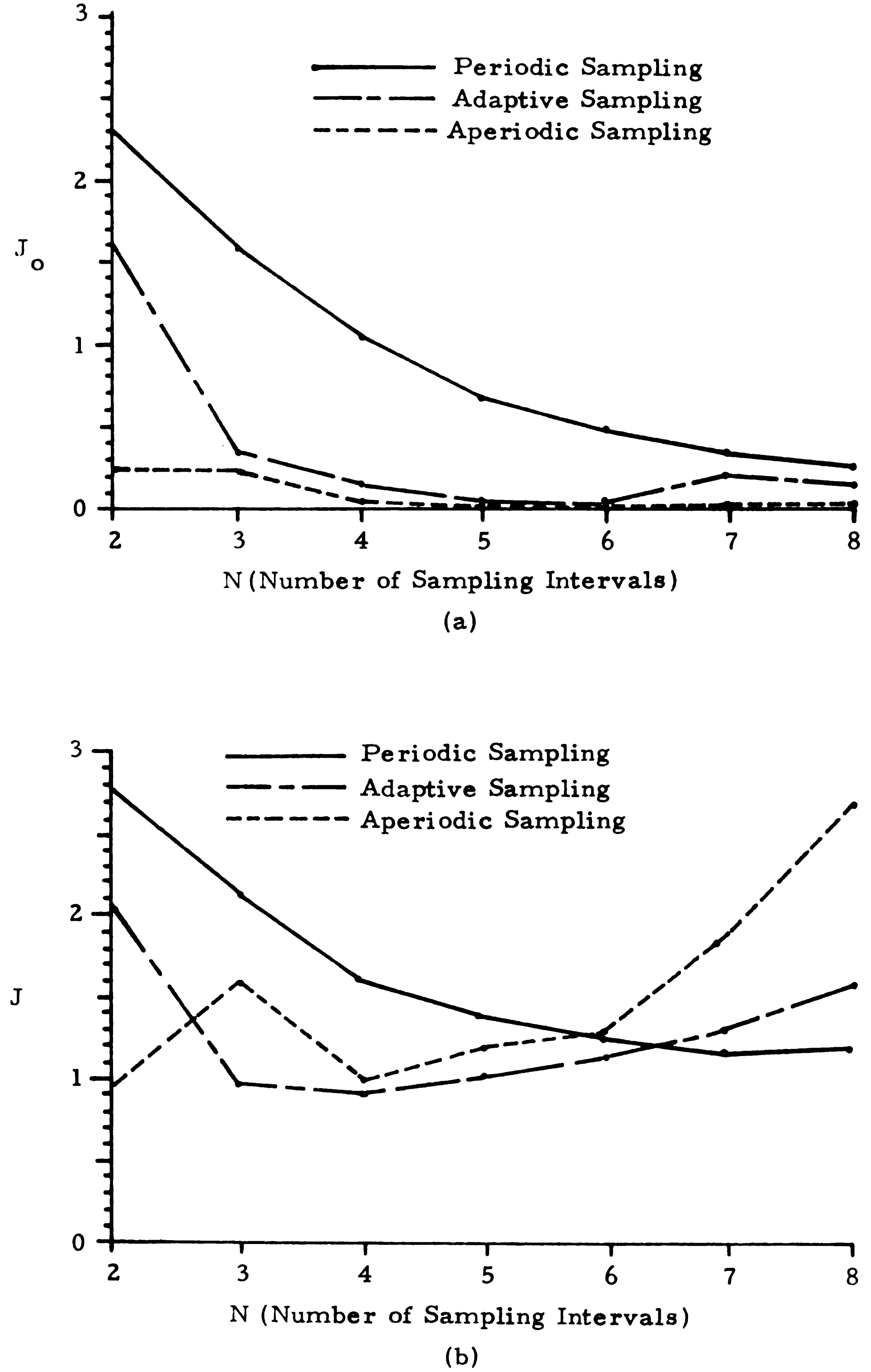


Figure 4.4 - Signal Representation Costs Using
a = 0 and Step Input

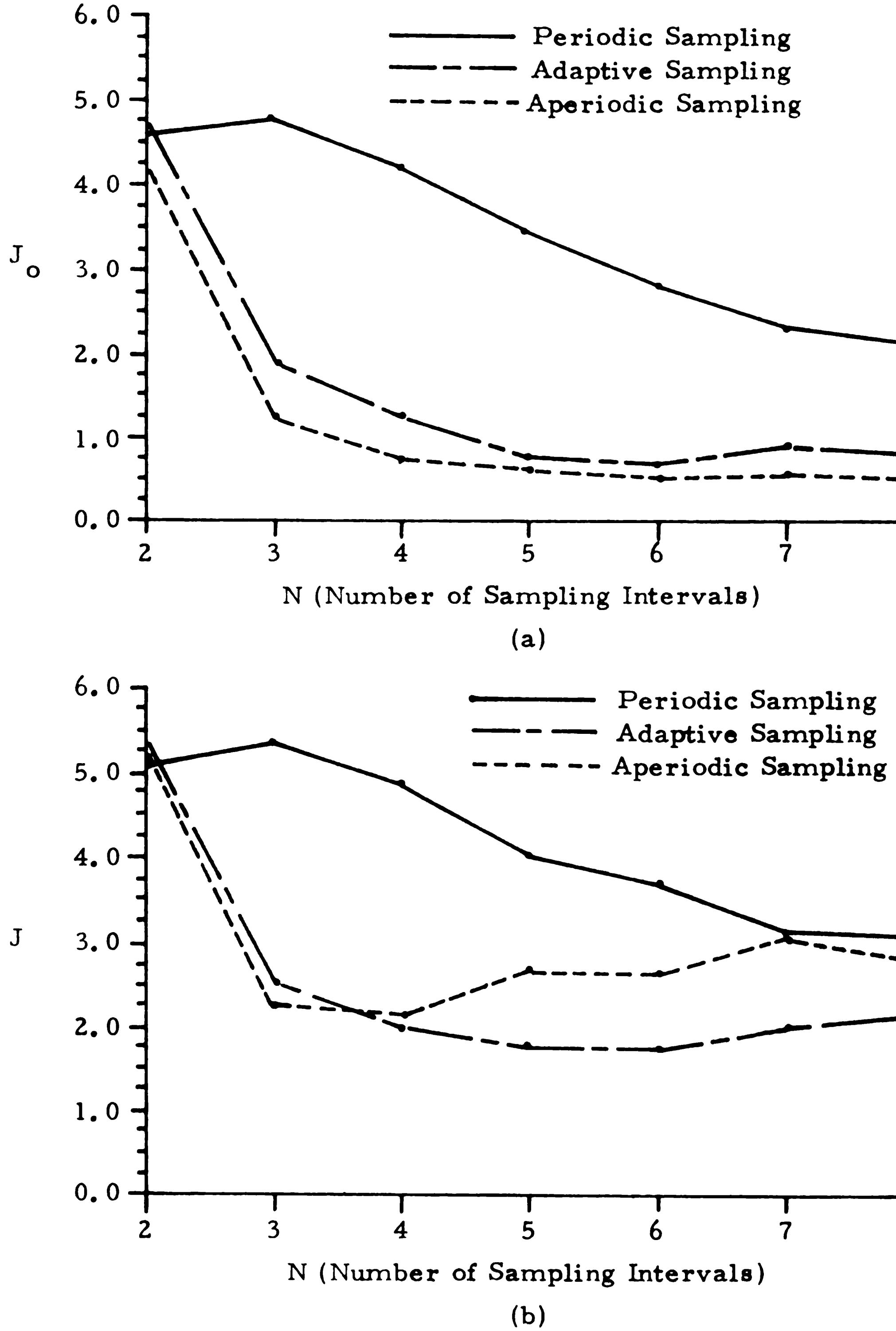


Figure 4.5 - Signal Representation Costs Using
a = 1 and Step Input

The maximum performance improvement ratios between adaptive or periodic sampling and sub-optimal aperiodic sampling, $(J_2/J_1)^*$ and $(J_3/J_1)^*$ respectively, decrease with increases in "a". Therefore, the tradeoffs possible in selecting the N sampling intervals together apparently provide greater performance improvement in the case when "a" is small.

The length of the sampling intervals for both adaptive and sub-optimal aperiodic sampling criteria are shown in Figure 4.6 for a = -1, a = 0, and a = 1. The length of the sampling intervals becomes less periodic as "a" increases for both adaptive and sub-optimal aperiodic sampling criteria. Thus, the sampling rate is better tuned to the rate of change of the signal when "a" increases.

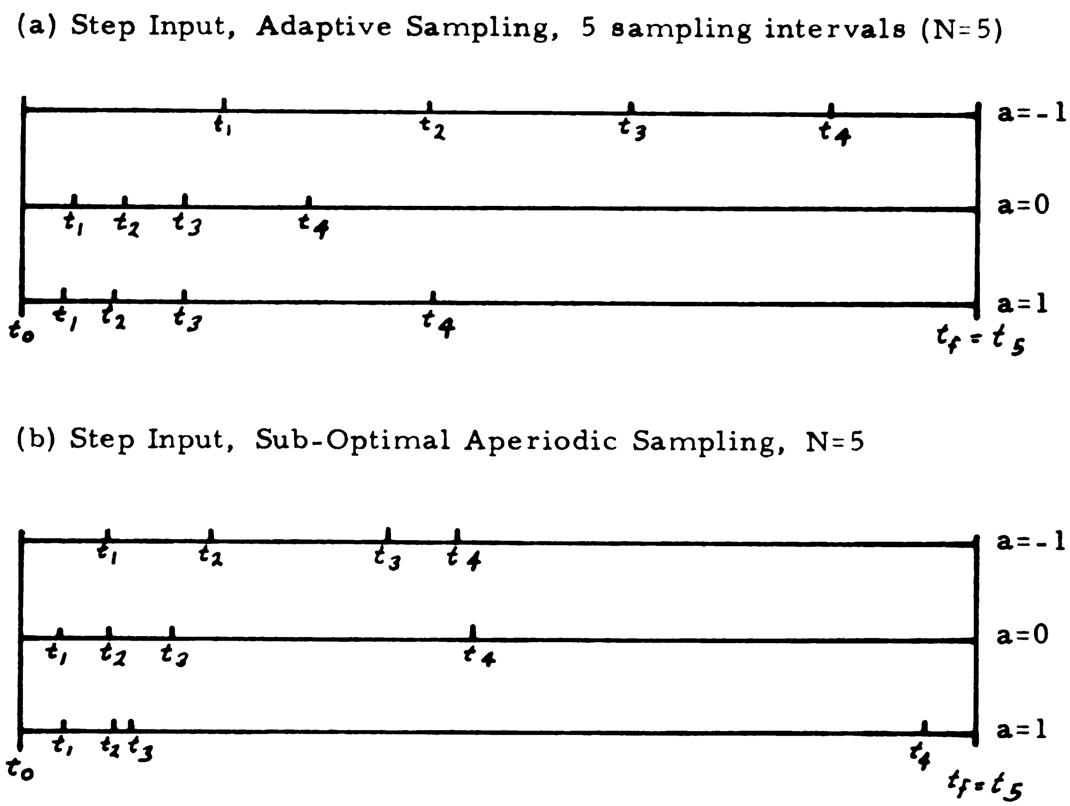


Figure 4.6. Representative Sampling Interval Lengths for Figures 4.3(a) to 4.5(a) using N=5.

Thus the selection of the proper performance index depends on the particular application and the smallest sampling interval length interval allowable. For this particular problem, a = 1 or a = 0 would be preferred since the sampling criterion would be better tuned to the changes in the signal and since no lower bound exists for the length of the sampling interval.

Clearly as the number of sampling intervals are increased the sampling errors are reduced. It should be noted that through use of as few as two sampling intervals performance improvements of as much as 15.6 : 1 can be realized through the use of optimal length sampling intervals. The remote display problem indicates that through use of optimal aperiodic sampling, reductions in the amount of data necessary to represent the sampled input signal record while maintaining a desired signal representation performance index is possible.

The system performance costs are graphically presented in Figures 4.3(b), 4.4(b), and 4.5(b) as a function of the number of sampling intervals, N, used to represent the input signal record. As can be seen, an increase in the number of sampling intervals results in increased costs for implementation in most cases. At points where system performance costs for sub-optimal aperiodic sampling exceeds that for either periodic or adaptive sampling, the computer utilization/processing costs necessary to obtain the optimal sampling interval sequence, $\underline{T}^*$, and the communications costs necessary to transmit the resultant signal level and sampling time data exceed any performance improvements resulting from aperiodically sampling the desired signal record. These implementation costs become apparent at the 4th sampling interval in Figure 4.3(b) and at the 6th sampling interval in Figure 4.4(b). Figure 4.5(b) shows that sub-optimal aperiodic sampling costs exceed those of adaptive sampling after 4 sampling intervals. If the values of the weighting factors w_1 and w_2 were reduced, or if a faster computer, more efficient optimization algorithm, or higher data transmission rate communications channel were used, the costs for implementation could be reduced proportionally. These reductions would make optimal aperiodic sampling more desirable at higher values of N. In Figures 4.3 and 4.4 good performance improvements can be realized with respect to adaptive and periodic sampling by using a = -1 or a =0 with N being either 2 or 3. The ultimate choice of the sampling criterion and the value of N will depend on the performance level required for J_o.

The performance of the optimal aperiodic sampling for signal representation will now be investigated with an unweighted integral performance index (a = 0) for the following four inputs:

a) step, $z(t) = 1.0$

b) ramp, $z(t) = t$

c) parabolic, $z(t) = t^2$

with initial state covariance and noise covariance being

$$E\{\underline{x}(t_o)\} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \underline{V}_x(t_o) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\Psi}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The derived signal representation performance index found in equation (4.4) and the costs for implementation shown in either equation (4.5) or (4.6) will be used. The fourth input is

d) noise, $z(t) = 0$

where the noise and randomly distributed initial states are assumed gaussian with

$$E\{\underline{x}(t_o)\} = \underline{O}, \qquad \underline{V}_x(t_o) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.10}$$

and

$$E\{\underline{w}(t)\} = \underline{O}, \qquad \underline{\Psi}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.11}$$

The derived performance index for the case of noise disturbances with $z(t) = 0$ becomes

$$J = \sum_{i=0}^{N-1} \frac{1}{T_i^a} \int_{t_i}^{t_{i+1}} \left\{ z(t) - z(t_i) - [\ 1 \quad 0]\ \underline{m}(t) + [\ 1 \quad 0]\ \underline{m}(t_i)\ ^2 + \mathrm{tr}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V(t) \right] \right\} dt + J_f \tag{4.12}$$

where the costs for implementation, J_f, are found in (4.5) or (4.6).

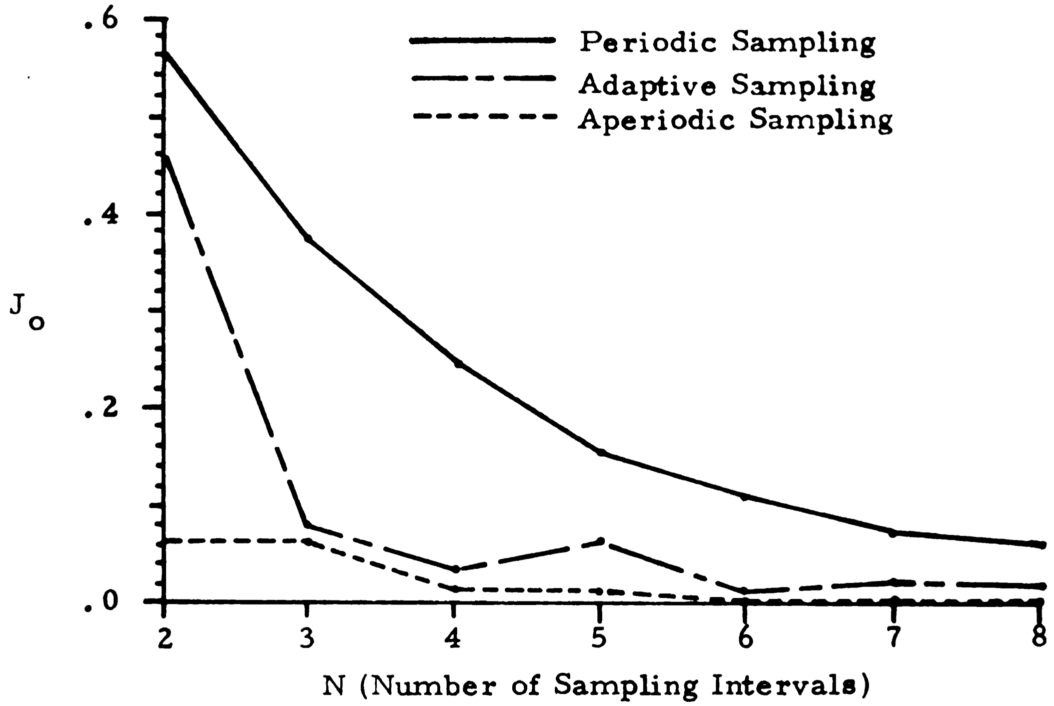
The values of the performance index for periodic, adaptive, and sub-optimal aperiodic sampling criteria using $a = 0$ are plotted in Figures 4.4(a), 4.7(a), 4.8(a), 4.9(a) as a function of the number of sampling intervals for the step, ramp, parabolic, and noise inputs respectively. The values of the performance index, J_i^*, for the optimal aperiodic ($i = 1$), optimal adaptive ($i = 2$), and optimal periodic criteria ($i = 3$) are shown in Table 4.2 for each of the four input types. The maximum performance improvement ratios, $(J_2/J_1)^*$ and $(J_3/J_1)^*$, for sub-optimal aperiodic criteria over adaptive and periodic criteria respectively are also given in Table 4.2.

The results indicate that sub-optimal aperiodic sampling produces smaller signal sampling errors as compared to periodic or adaptive sampling for each input and value of N shown in Figures 4.4(a), 4.7(a), 4.8(a), and 4.9(a). The optimal aperiodic (OA) cost, J_1^*, is lower than either the optimal adaptive (OAD) cost, J_2^*, or the optimal periodic sampling (OPS) cost, J_3^*, for each input considered. The maximum ratio of the performance costs between adaptive and sub-optimal aperiodic sampling $(J_2/J_1)^*$ and the maximum ratio between periodic and sub-optimal aperiodic sampling $(J_3/J_1)^*$ are large for a step, ramp, and parabolic input. These performance ratios indicate that optimal selection of the sampling interval sequence is preferable to individual sampling interval selection as found in adaptive or use of equal length sampling intervals as in periodic sampling. Thus using the optimal sampling interval sequence, errors caused by the sample and hold operation can be significantly reduced for these input types.

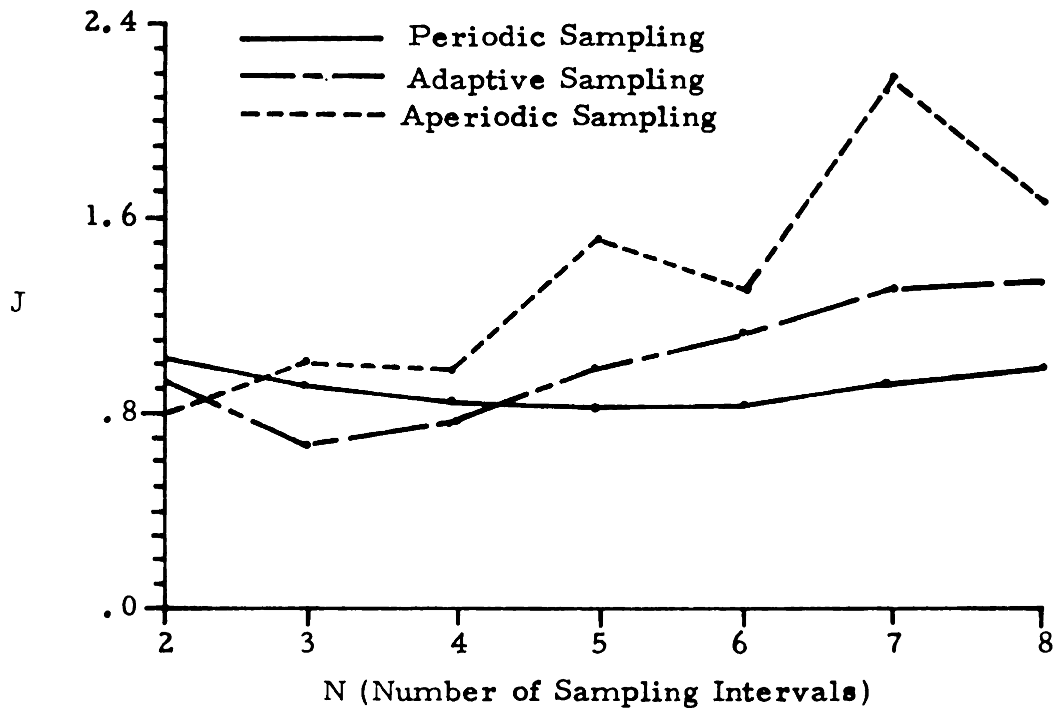
	OA		OAD		OPS		$(AD/A)^*$		$(PS/A)^*$	
INPUT (a=0)	J_1^*	N_1^*	J_2^*	N_2^*	J_3^*	N_3^*	$\frac{J_2}{J_1}^*$	$\frac{N_2}{1}^*$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}$
STEP	.040	6	.073	5	.276	8	12.8	5	6.7	5
RAMP	.008	8	.013	6	.062	8	7.6	2	11.0	6
PARABOLA	.007	8	.011	8	.069	8	7.4	2	11.6	4
NOISE	.028	8	-	-	.029	8	-	-	1.3	5

Table 4.2. Data Summary for Figures 4.6(a) to 4.8(a).

The optimal aperiodic cost J_1^* for a noise input is only slightly lower than the optimal periodic cost J_3^*. In addition the maximum performance improvement ratio $(J_3/J_1)^*$ is considerably lower than the corresponding performance ratios for the other inputs. This indicates that periodic sampling is in fact near optimal if noisy signals are to be sampled.



(a)



(b)

Figure 4.7 - Signal Representation Costs Using
a = 0 and Ramp Input

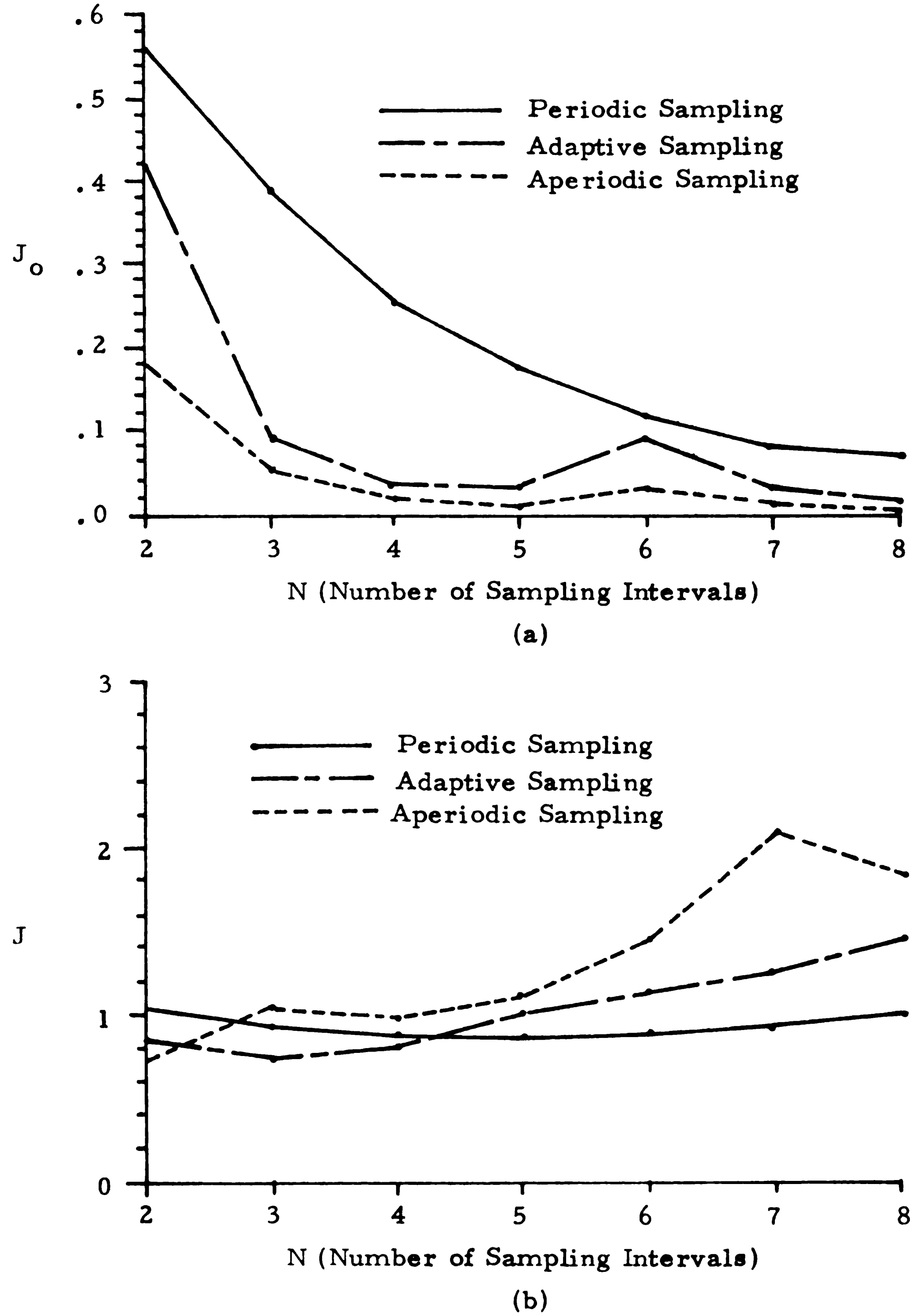


Figure 4.8 - Signal Representation Costs Using
a = 0 and Parabolic Input

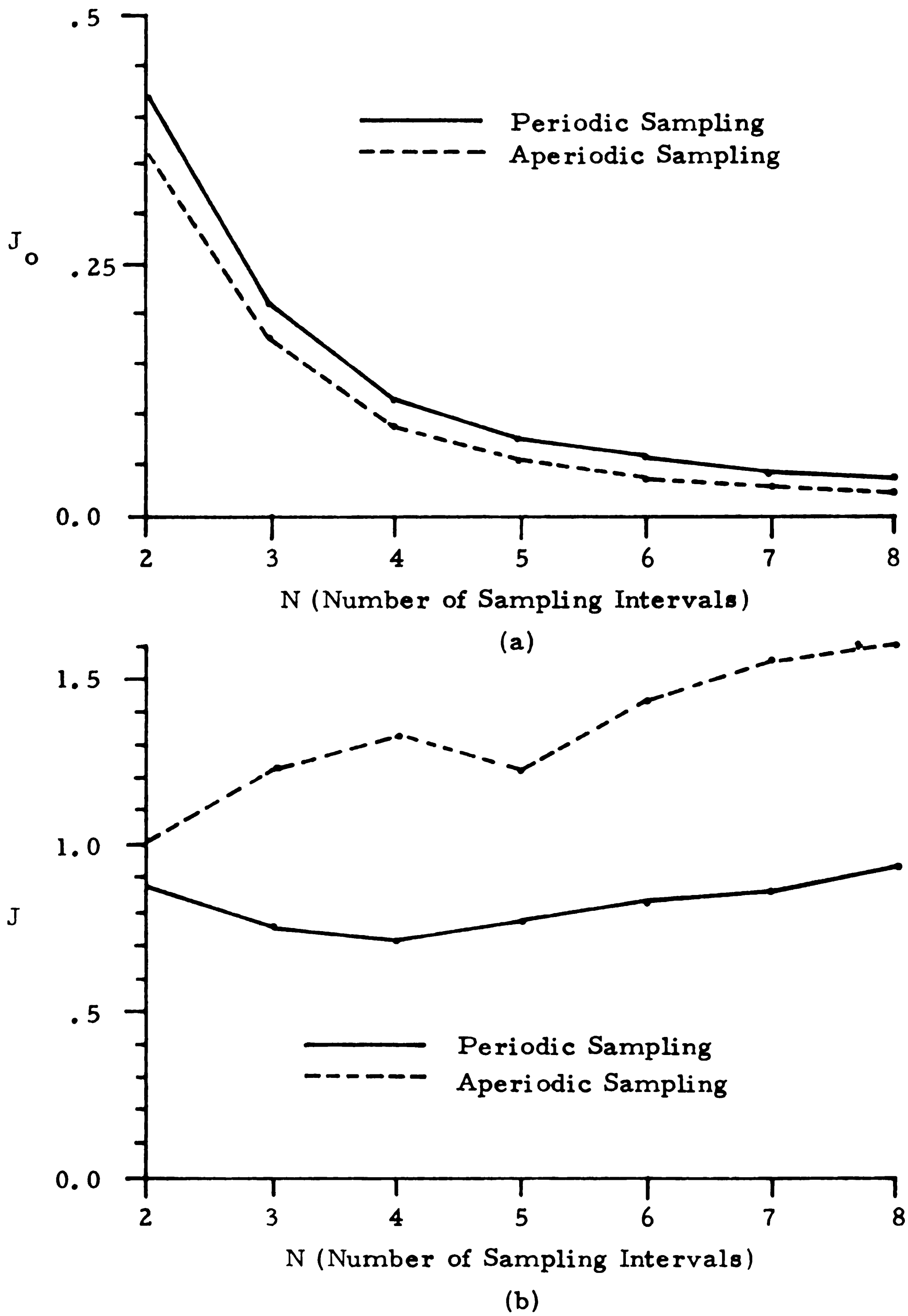
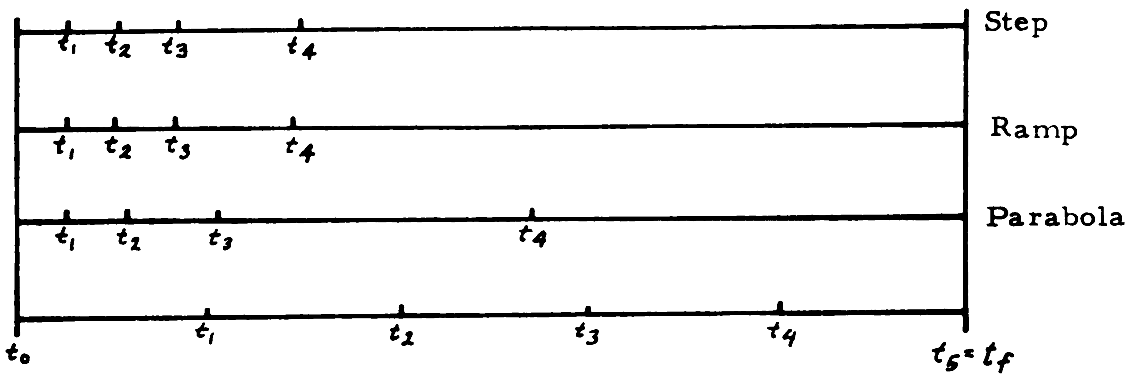


Figure 4.9 - Signal Representation Costs Using
a = 0 and Noise Input

The sampling interval sequences for both adaptive and sub-optimal aperiodic sampling using a = 0 and N = 5 are shown in Figure 4.10 for each of the four input types. (See Appendix C for the exact numerical length of each sampling interval.) In general, the sampling interval lengths for step, ramp, and parabolic inputs are longer for sub-optimal aperiodic sampling. The sampling interval sequence for the noise input using sub-optimal aperiodic sampling is very close to periodic as was indicated by the performance cost for the noise input found in Table 4.2.

(a) Adaptive Sampling (a=0, N=5)



(b) Sub-Optimal Aperiodic Sampling (a=0, N=5)

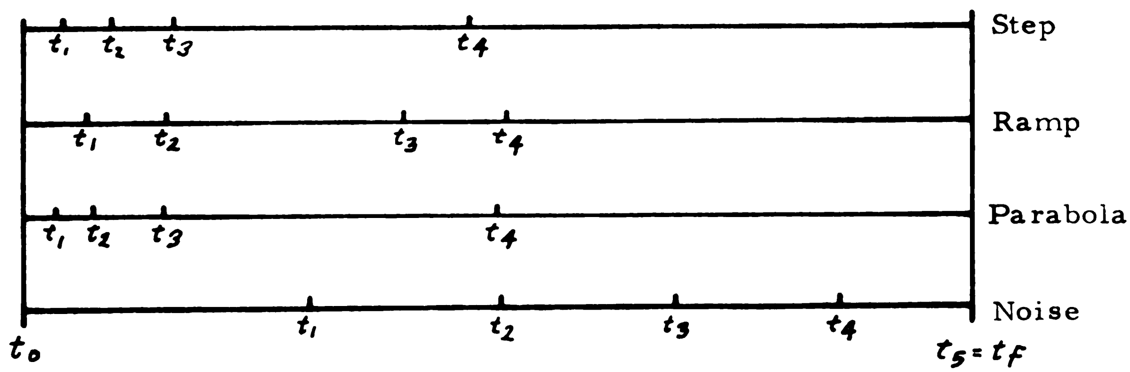


Figure 4.10 Representative Sampling Interval Lengths for Figures 4.7(a) to 4.9(a) using N=5.

As can be seen in Figure 4. 7(b) the costs for implementation added to the performance costs increased the system performance costs for sub-optimal aperiodic sampling above those for periodic and adaptive sampling after 3 sampling intervals. The shape of the cost curve for a ramp input, found in Figure 4. 7(b) for sub-optimal aperiodic sampling, is far from a "smooth" curve as compared to the curves for periodic and adaptive sampling. The shape of the aperiodic performance curve is due to the fact that an increase in the number of sampling intervals does not necessarily imply a corresponding increase in the computational costs (J_{cu}) required to obtain the optimal sampling interval sequence $\underline{T}^*$. As can be seen in Figure 4. 7(b) the shift in the system performance for sub-optimal aperiodic sampling is due mainly to the increased costs for implementation.

In Figure 4. 8(b) the system performance cost for periodic sampling remains relatively constant while the costs for adaptive and sub-optimal aperiodic sampling increase above the periodic sampling costs after 4 sampling intervals. The increases in system performance costs for adaptive and aperiodic sampling using a parabolic input are due primarily to the increased computational expenses required to obtain the optimal sampling interval sequences $\underline{T}^*$ as indicated in Figure 4. 10 for N = 5.

In Figure 4. 9(b) the costs for implementation required for sub-optimal aperiodic sampling were so high that periodic sampling had the lowest system costs for each N considered. This implies in conjunction with Figure 4. 10 that if "noisy" signals are to be sampled for signal representation, based on signal variances as in (2. 32), periodic sampling will provide good signal representation with lower implementation costs than optimal aperiodic sampling for each N considered.

In summary, aperiodic sampling as specified in Chapter II yielded improved signal representation performances with respect to periodic or adaptive sampling for all values of "a" and input signals considered. If signal representation performance costs are to be considered alone in the choice of the sampling method to be used, clearly all results indicate that sub-optimal aperiodic sampling should be chosen.

In any design problem, in particular the one shown in Figure 4.1, signal representation performance costs can not be used as the only basis for choosing the method of sampling to implement. As can be seen in the figures which have been presented, the costs for implementation when considered jointly with the performance costs can outweigh any reductions in sampling errors obtained through use of more sophisticated sampling techniques such as optimal aperiodic sampling. Thus if the costs for implementation as introduced in Chapter II and discussed in detail for the remote display problem are low, aperiodic sampling should be employed; if the costs for implementation are high, periodic or possibly adaptive sampling should be employed. The greater the knowledge the system designer has about the problem and the constraints he is required to work within the better the system design will become.

The procedure for selecting the particular sampling criterion is to first determine the maximum value of the signal representation costs permitted and then determine the minimum number of sampling intervals required to obtain this value of performance using each sampling criteria. Then selecting the sampling criterion with the lowest total costs assuming that each criterion is only considered in the region above its minimum number of sampling intervals. Another approach would be to select the sampling criteria with the lowest implementation costs over the same set of acceptable number of sampling intervals.

EXAMPLE DESIGN PROBLEM

Consider the signal representation performance index cost curves for a ramp input signal using a = 0 as shown in Figure 4.7. Two possible methods of selecting the best sampling criterion given the system design specifications will be presented.

METHOD 1. (See Figure 4.7(a))

STEP 1. Determine the maximum allowable signal representation performance cost, J_o. For this example, the performance cost will be arbitrarily chosen, thus let $J_o = .04$.

STEP 2. Select the feasible values of N, where $2 \leq N \leq 8$ as seen in Figure 4.7(a), such that STEP 1 is satisfied.

Sampling Type	Feasible N	Minimum N (N_i^*)
Periodic Sampling	none	none
Adaptive Sampling	4, 6, 7, 8	4
Sub-Optimal Aperiodic Sampling	4, 5, 6, 7, 8	4

STEP 3. Choose the method of sampling with the lowest system performance costs. (See Figure 4.7(b)) For this example, adaptive sampling using 4 sampling intervals yields the lowest total performance cost, J. (The exact numerical values of the performance costs are found in Appendix C.)

Thus in this example the use of METHOD 1 indicates that adaptive sampling should be chosen for implementation based on the performance results indicated in Figure 4.7.

METHOD 2. (See Figure 4. 7(a))

STEP 1 and STEP 2 are identical to those in METHOD 1.

STEP 3. Choose the method of sampling with the lowest implementation cost. Implementation costs J_f are determined by subtracting the performance cost J_o from the system performance cost J. Using this procedure, adaptive sampling with N = 4 has the lowest implementation costs, J_f.

Thus using this method of sampling criterion selection, adaptive sampling would again be chosen for implementation. As can be seen in Figure 4. 7, a different choice of the value of J_o would lead to different sampling criterion selection.

The two methods of sampling type selection are similar for STEP 1 and STEP 2 but different in STEP 3. STEP 3 causes METHOD 1 to emphasize total or system performance costs whereas METHOD 2 emphasizes implementation costs alone. The method used in the selection of the sampling criteria for implementation depends on the system design objectives and allowable system performance.

4.3. Sampling for Control Implementation

The performance of the optimal sampling for control implementation problem will now be investigated. Performance will be investigated as a function of system response times (bandwidth), the system type (the number of poles at the origin in the system plant), and the inputs forcing the system during the control interval. The performance index will be augmented by costs for implementation as discussed in Chapter II. The cost for data communications will be essentially neglected since the controlling computer is assumed to be very near the system under control.

The sampling for control problem, as restated from Chapter III, is:

> Determine the optimal sampling interval sequence (N^*, $\underline{T}^*$) that specifies the optimal sample and hold approximations
>
> $\hat{u}_e(t) = u_e(t_i)$, $t \epsilon [t_i , t_{i+1}]$; $i = 0, 1, \ldots , N-1$
>
> that minimize the control implementation performance index (3.17) subject to the sampling constraints (3.11), (3.12), and (3.13).

The control implementation performance index (3.17) measures the terminal errors, the system tracking errors, the control energy expenditures and implementation costs. The implementation costs include computer utilization, software and data storage costs for computing the optimal sampling intervals, and data communications costs for transmitting the control information from the computer to the system under control.

The primary objective of investigating this problem is to evaluate the trade-offs between performance and implementation costs for implementating a digital control system through use of a periodic or an optimal aperiodic sampling process. A comparison of performance and costs for implementation will be made for the following cases:

[1] a Type 2 (two poles at the origin) system with FAST, MEDIUM, and SLOW speeds of response to a step input.

[2] a Type 2 MEDIUM response time system for a step, ramp, and a random noise input.

[3] a Type 1 (one pole at the origin) MEDIUM response time system for a step, ramp, and a random noise input.

[4] a Type 0 (no poles at the origin) MEDIUM response time system for a step, ramp, and a random noise input.

A comparison of the performance and implementation costs for these two sampling criteria will also be compared based on the system type for a particular input signal.

The control performance of the feedback control system implemented with a continuous time (analog) control is evaluated to provide a basis for evaluating the control performance of the two sampled-data systems. It is assumed that the implementation costs for the continuous time (analog) control system is prohibitive.

The basic assumption made in evaluating and comparing the performances of the control laws implemented with the periodic and optimal aperiodic sampling criterion are:

[1] a second order system will be used in all cases:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_e(t) \tag{4.13}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(t) \tag{4.14}$$

[2] the control law is specified by

$$u_e(t) = z(t) - y(t) \tag{4.15}$$

over a control interval $t \in [0, \ 1]$ where z(t) is the programmed control.

[3] measurements of the control law are transmitted at the sampling times $\{t_i\}_{i=0}^{N-1}$ where

$$t_0 = t_{initial} = 0 < t_1 < \ldots < t_{N-1} < t_N = t_{final} = 1 \tag{4.16}$$

where N is restricted to satisfy

$$2 \leq N \leq 8 \tag{4.17}$$

based on computer software constraints.

[4] the control error is generated only at the sampling times and held over each sampling interval such that

$u_e(t) = e(t_i)$, t $[t_i , t_{i+1}]$

and i = 0, 1, . . . , N-1.

[5] the performance objectives for this system are assumed to be met by selecting a performance index which penalizes terminal errors, tracking errors, and control energy expenditures as found in (3.17). The computational costs are the off-line costs for computing the optimal sampling criterion and the communications costs are the on-line costs for transmitting data between the computer and the system being controlled. The sum of the computational and communications costs result in the costs for control system implementation as discussed in Chapter II.

The derived control performance index used is

$$J = E\left\{(y(t_N) - z(t_N))' \; 5 \; (y(t_N) - z(t_N)) + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} [(y(t) - z(t))' \; .1 \; (y(t) - z(t)) + .02u_e(t)^2]\, dt\right\} + J_f \tag{4.18}$$

The cost for implementation, J_f, for periodic sampling is

$$J_f = w_1 \times M \times U + \frac{w_2 \times K \times W}{R} \tag{4.19}$$

where

M = 1700 words of memory

w_1 = .0001 (word-second)$^{-1}$

W = 30 bits/output word (record)

K = (N+4) words (records)[1]

R = 300 bits/second

w_2 = .00005 seconds^{-1}

The cost for implementation for aperiodic sampling is

$$J_f = w_1 \times M \times U + \frac{w_2 \times K \times W}{R} \qquad (4.20)$$

where

M = 1700 words of memory

w_1 = .0001 $(\text{word-second})^{-1}$

W = 30 bits/output word (record)

K = 2 x (N+1) words (records)[2]

R = 300 bits/second

w_2 = .00005 seconds^{-1}

In the computer utilization portion of (4.19) or (4.20) the computer memory requirements (M) are determined by the number of words required to store the control system software in the main memory of the computer being used for system control. The value of U, which depends on the number of sampling intervals; N, the complexity of the control system software, (i.e. cost functional subroutine, optimization routines, etc.) and the control time interval, is found by determining the computer processing time required, in this case through use of the CDC 6500 computer, to obtain the optimal sampling interval sequence, $\underline{T}^*$, using any particular value of N. The weighting coefficient w_1 is the cost per word-second for the particular computer being used.

---

[1] See Page 46 , Section 4.2.

[2] See Page 46 , Section 4.2.

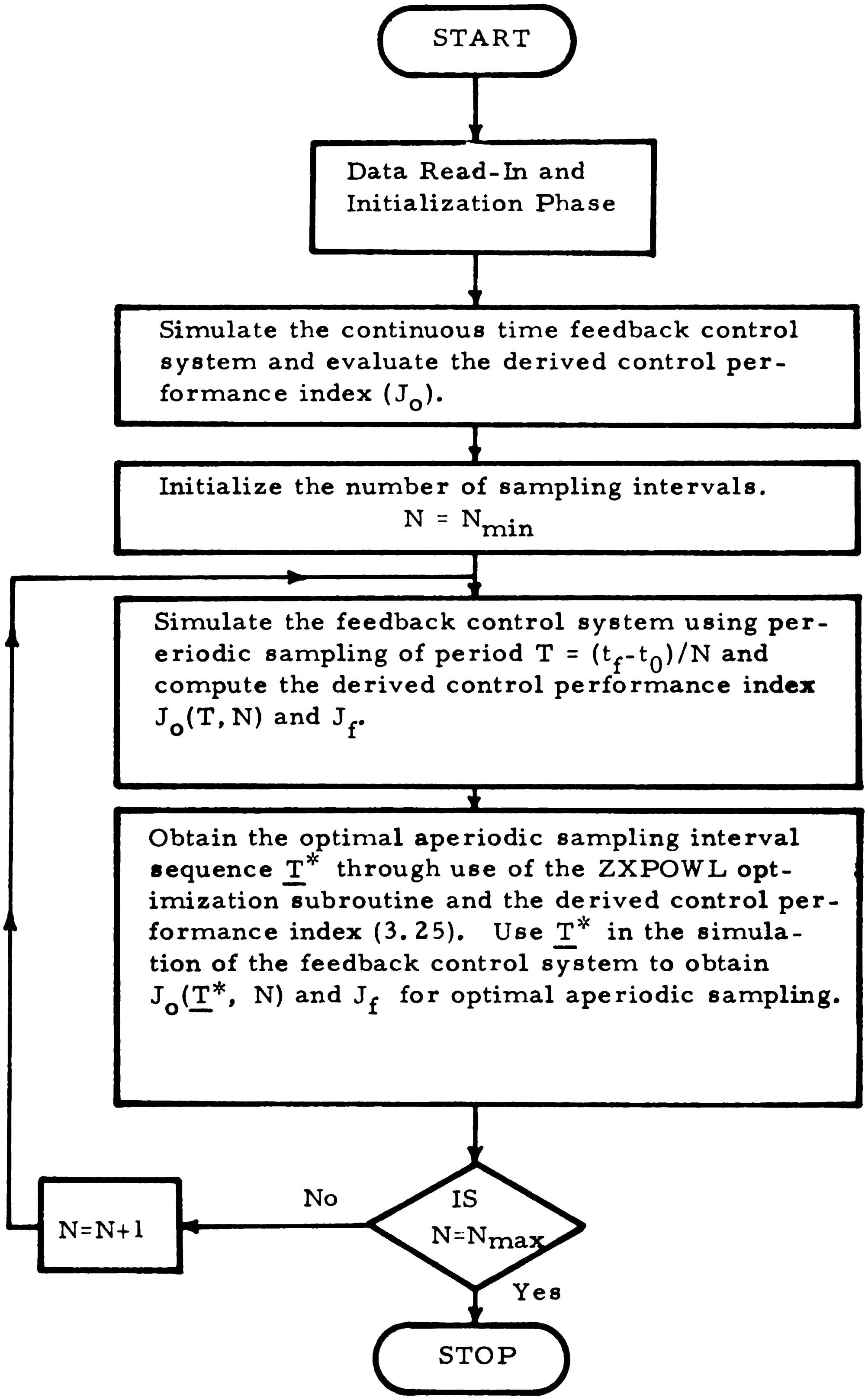


Figure 4.11. Computer Program Flowchart for Control Performance Evaluation.

In the communications cost portion of (4.19) or (4.20) the values of R and W are determined respectively by specifying the minimum communications channel capacity to be 300 Baud and that each signal level measurement and sampling time be transmitted by using 3 ASCII characters with start, stop, and parity bits being added to each character. The number of words (records) (K) to be transmitted via the communications network depends on the number of sampling intervals and the type of sampling being used. The weighting coefficient w_2 is the cost per second for use of the communications channel for data transmission. In general both weighting factors (i.e. w_1 and w_2) are highly problem specific and will change with system specifications or advances in technology.

Section 4.4 will investigate three Type 2 control system models having FAST, MEDIUM, and SLOW response times. Performance costs for each system and input being investigated will be found using the computer program shown in simplified flowhcart form in Figure 4.11.

4.4. Control Implementation Performance Results

The trade-offs in performance and implementation costs for a sampled-data control implementated with periodic and optimal aperiodic sampling will be investigated for a Type 2 control system which has been used extensively in research [1, 19, 20, 24, 52] on comparison and evaluation of adaptive sampling criteria. This system will be investigated for gain settings which result in FAST, MEDIUM, and SLOW response times. The three system models have the form

a) FAST response time system :

$$\underline{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 10 \\ 100 \end{bmatrix} u_e(t) \tag{4.21}$$

$$y(t) = [\ 1 \quad 0\ ]\ \underline{x}(t) \tag{4.22}$$

b) MEDIUM response time system :

$$\underline{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 5 \\ 25 \end{bmatrix} u_e(t) \tag{4.23}$$

$$y(t) = [\ 1 \quad 0\ ]\ \underline{x}(t)$$

c) SLOW response time system :

$$\underline{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 2.5 \\ 6.25 \end{bmatrix} u_e(t) \tag{4.24}$$

$$y(t) = [\ 1 \quad 0\ ]\ \underline{x}(t)$$

These three systems will be investigated for a step input

$$z(t) = 1\ , \qquad t \in [\ 0\ ,\ 1\ ]$$

with initial conditions

$$\underline{m}(t_o) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad , \qquad \underline{V}_x(t_o) = \underline{0}$$

and noise covariance

$$\underline{\Psi}(t) = \underline{0}$$

Therefore given the three response time systems, the performance index, and the cost for implementation function, the figures which follow show the control performance costs and system performance costs. Each value of the performance index for the suboptimal aperiodic sampling represents the minimized performance cost obtained through use of the ZXPOWL optimization subroutine for the specified number of sampling intervals. (The exact numerical values are listed by figure number in Appendix C.) The value of the continuous time feedback control performance costs are obtained by applying the feedback control signal continuously as the system input control signal. The performance of this continuous time (analog) control will be denoted J_4 and the performance of the periodic and aperiodic sampled data control will be denoted as J_3 and J_1 respectively. The optimal performance costs for periodic and aperiodic, denoted by J_i^* for i = 3 and i = 1 respectively, are defined as the minimum value of J_i over the set of feasible N (3.11). The optimal system performance ratio between periodic and aperiodic sampling $(J_3/J_1)^*$ is defined as the maximum value of J_3/J_1 over the set of feasible N (3.11). The optimal system performance ratio between continuous and aperiodic sampled-data control, $(J_4/J_1)^*$, is defined as the maximum ratio J_4/J_1 over the set of feasible N (3.11).

Figures 4.12(a), 4.13(a), and 4.14(a) graphically represent the values of the derived control performance index using the indicated number of sampling intervals for control of the three Type 2 unity feedback control systems presented. In each figure sub-optimal aperiodic sampling performance costs will be compared to both periodic sampling and continuous time control. As can be seen in Table 4.3, continuous time feedback control performance costs (C) increase as the control response time of the system decreases. The main reason for the increase in performance costs is due to the fact that continuous time control is unable to significantly reduce the terminal errors as system response times decrease and is thus penalized by the control performance index. The optimal aperiodic (OAS) cost J_1^* is lower than the optimal periodic sampling (OPS) costs J_3^* for each system considered but outperforms continuous time control performance only for the

	C	OAS		OPS		(C/OA)*		(PS/OA)*	
Control Response	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	N_1	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}$
FAST	2.2	4.9	8	8.4	8	.44	8	10^4	2
MEDIUM	5.2	2.3	8	5.3	5	2.3	8	10^4	2
SLOW	11.6	4.5	6	6.8	2	2.6	6	2.3	7

Table 4.3. Data Summary for Figures 4.12(a) to 4.14(a).

MEDIUM and SLOW systems. The maximum ratio of the performance costs between continuous and sub-optimal aperiodic sampling $(J_4/J_1)^*$ increases as system control response decreases. This implies that optimal aperiodic sampling can be more effective than continuous time feedback control in reducing final state errors for a fixed length control interval for slower responding systems. The maximum ratio of performance costs between periodic sampling and suboptimal aperiodic sampling $(J_3/J_1)^*$ is large for the FAST and MEDIUM control response systems indicating that the periodic sampled-data system is unstable for N = 2. This conclusion is verified from Figures 4.12 and 4.13 respectively. It should be noted that system stability improved and performance cost reductions were made through use of periodic sampling for the SLOW response time system.

In the figures which have been presented, performance costs of a continuous time control system, having feedback error signals sampled either periodically or aperiodically and applied as a system

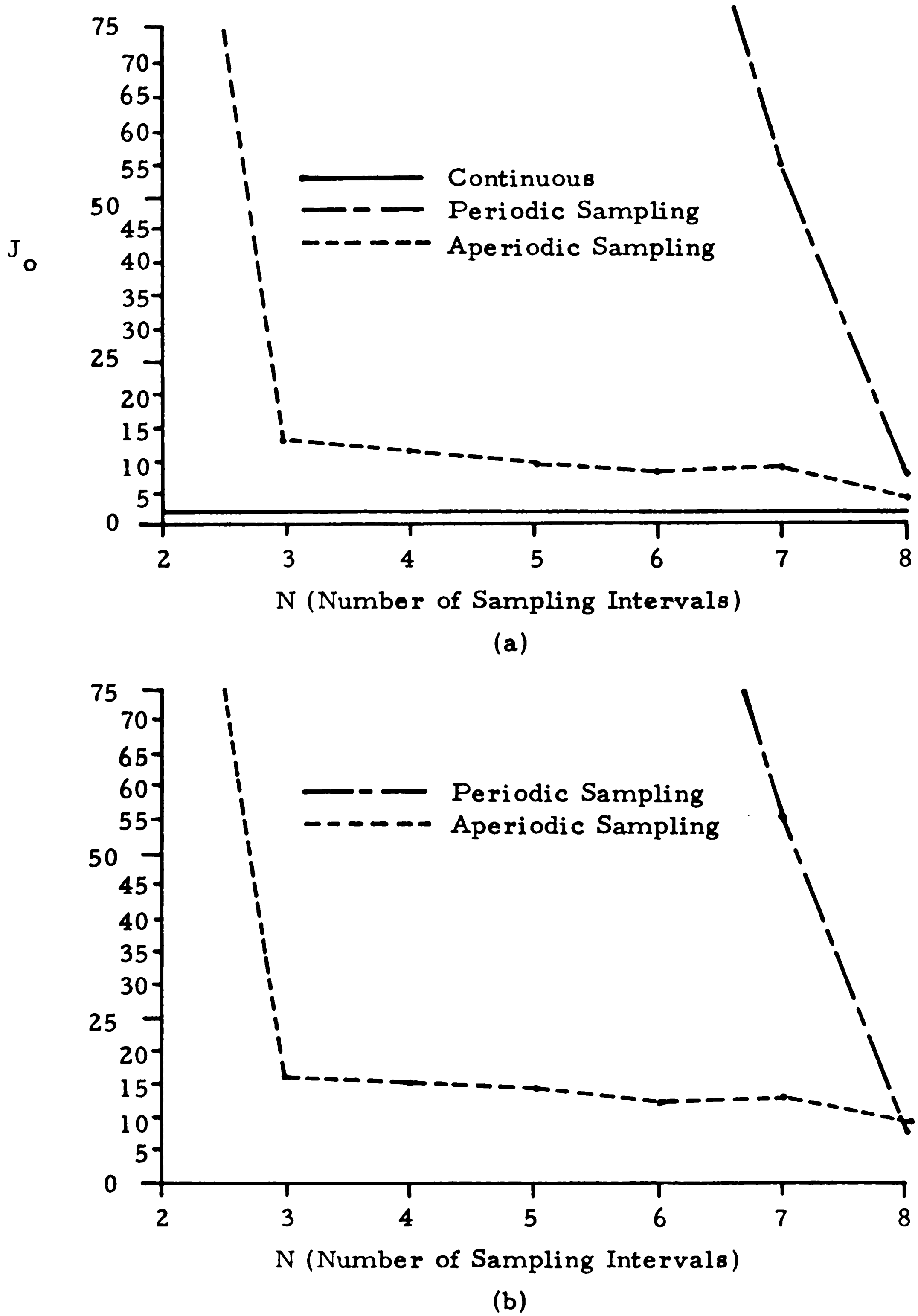
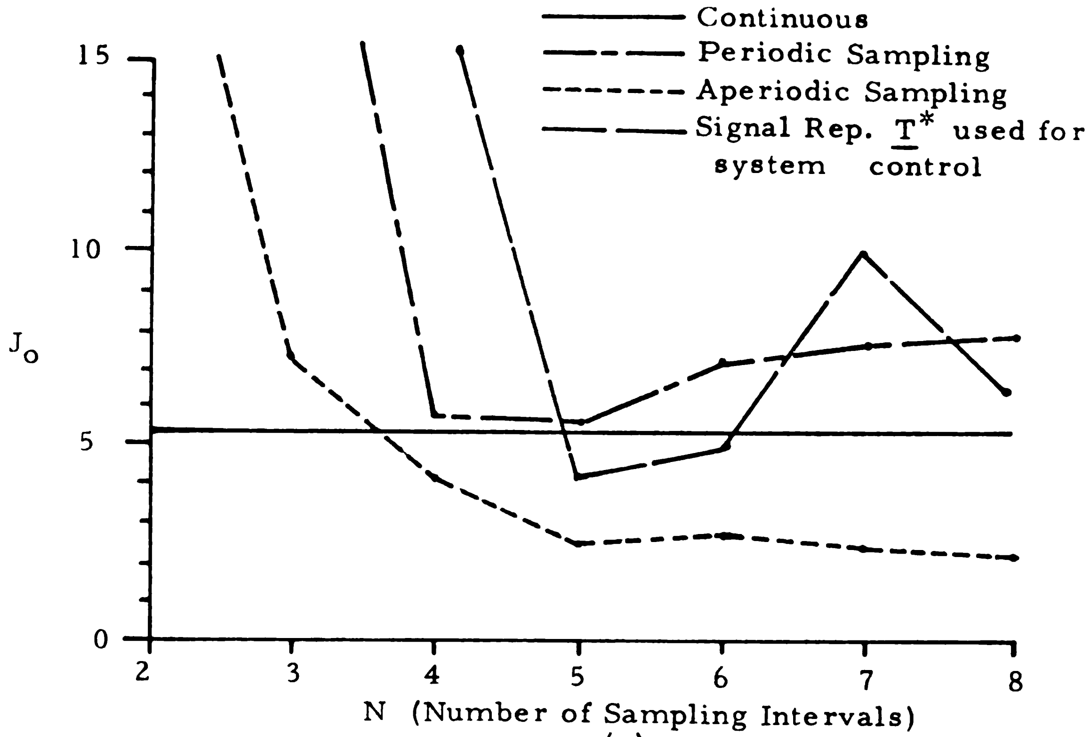
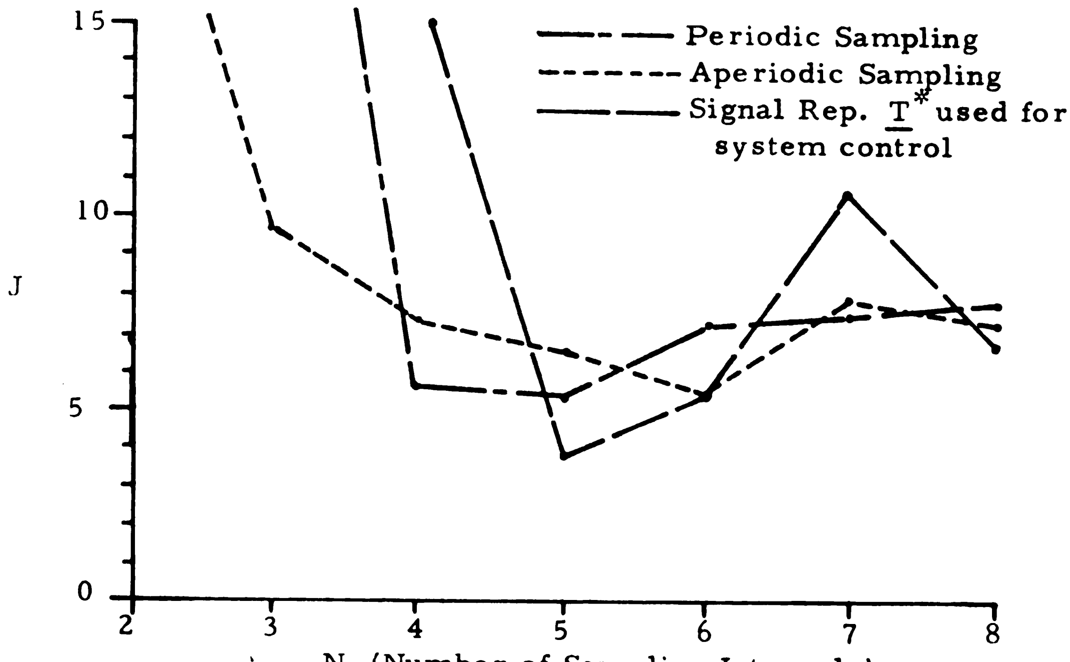


Figure 4.12 - Performance Costs for a Type 2 FAST Response Time Control System Using a Step Input



(a)



(b)

Figure 4.13. Performance Costs for a Type 2 MEDIUM Response Time Control System Using a Step Input.

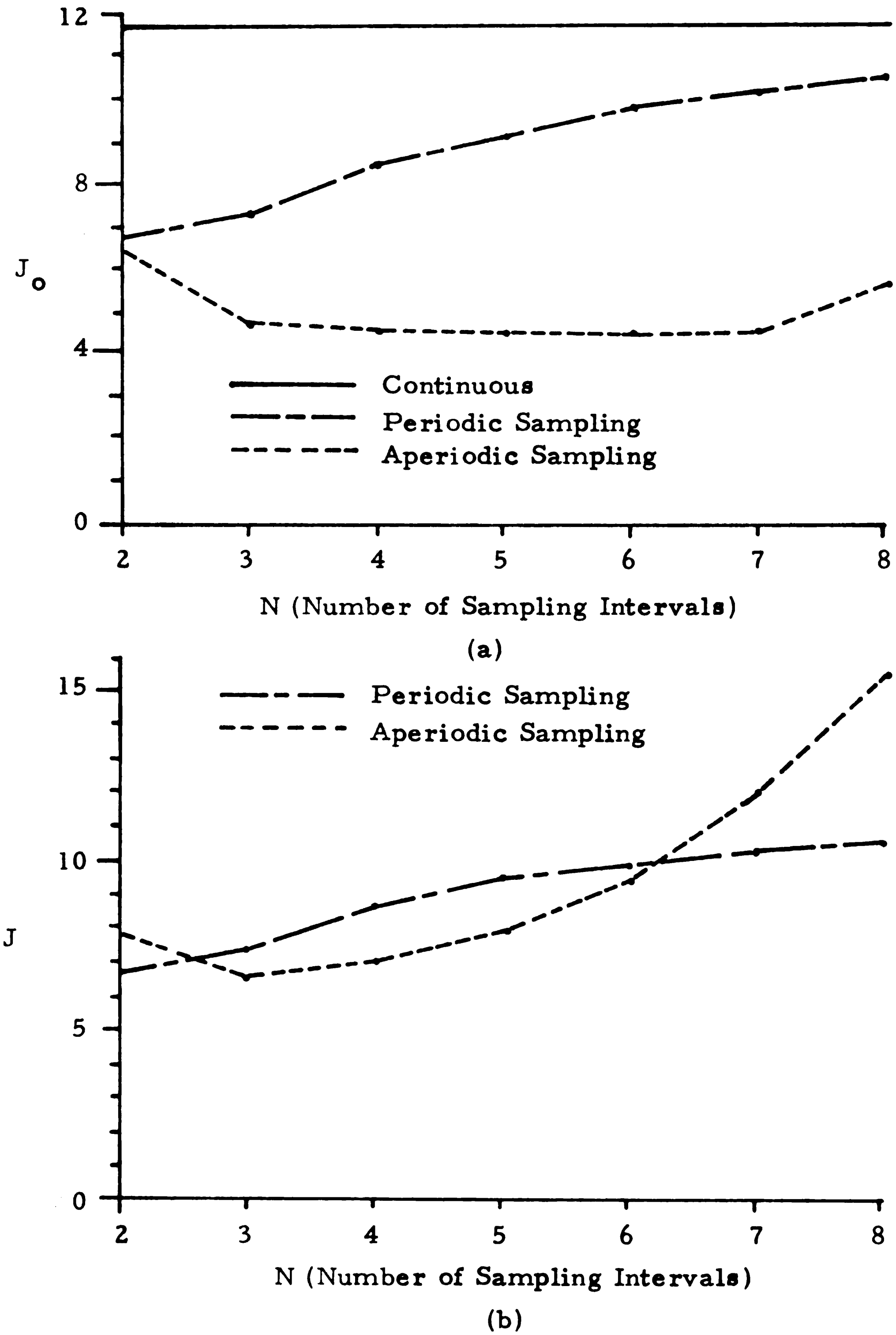


Figure 4.14 - Performance Costs for a Type 2 SLOW Response Time Control System Using a Step Input

control input, nearly equal or in the case of sub-optimal aperiodic sampling out-perform continuous time feedback control. These performance results are explained by the fact that through use of sampling, either periodic or sub-optimal aperiodic, delays are introduced into the feedback error signal. These delays introduce a misrepresentation of the actual input error control signal by indicating to the system under control that the feedback error is greater (smaller) than it actually is. As a result, the system is made to respond faster (slower) than it normally would through use of a continuous time feedback error signal which has no delays. For example, consider the control error signal $u_e(t)$ found in (4.15). At the initial time, t_o, $u_e(t_o)$ will have an initial value dependent on $z(t_o)$ and $x_1(t_o)$. Since the systems being investigated are assumed stable, $u_e(t)$ will eventually be reduced depending on the system type and the input signal. If $u_e(t)$ is sampled and held until time t_1, such that $t_1 > t_o$, any changes in $u_e(t)$ between times t_o and t_1 will not become apparent until time t_1 when $u_e(t)$ is again sampled and held. This "delay" in the error feedback signal if used properly (e.g. adjusted through use of sub-optimal aperiodic sampling) will improve system response as indicated in the figures being presented. If the "delay" in the error signal is not adjusted severe system instability can result as shown in Fig.4.12 for periodic sampling.

It should be noted that in all cases the performance costs for both periodic and sub-optimal aperiodic sampling approach the value of performance for the continuous time control as the number of sampling intervals increase. This result stems from the fact that using more sampling intervals the feedback control signal becomes a better approximation to the continuous time control.

Figure 4.13 also indicates the performance costs resulting from use of the optimal signal representation sampling interval sequence being applied to the control implementation problem. The results, as seen in Figure 4.13(a), show that control performance can be improved in three cases over periodic sampling and in two cases over continuous feedback control. The control performance using sub-optimal aperiodic sampling determined by the control performance index, out-performed the optimal signal representation sampling interval sequence in all cases as was expected. These results indicate that use of the optimal sampling interval sequence for signal representation for system control does not necessarily yield good system control performance as was implied in earlier references [1,24].

The results as seen in Figure 4.12(b) indicates that for 3 to 7 sampling intervals sub-optimal aperiodic sampling would have to be employed since the control system is extremely unstable using periodic sampling for fewer than 8 sampling intervals. In Figure 4.13(b) sub-optimal aperiodic sampling would again have to be used if 4 or less sampling intervals were required. If more than 4 sampling intervals could be used either periodic or sub-optimal aperiodic sampling could be used. Figure 4.14(b) indicates that the total performance costs for periodic and sub-optimal aperiodic sampling are approximately the same until 7 sampling intervals and than the costs for implementation for the aperiodic sampling criterion, in particular the processing costs, become large.

The lengths of the sampling intervals which yield the performance costs indicated at 5 sampling intervals in Figures 4.12 through 4.14 are shown in Figure 4.15. The initial time t_0 is 0.0 with the final time t_f being 1.0 for each sampling interval sequence. Each sampling time t_i, for i = 0, 1, 2, 3, 4, 5, is indicated with the length of a sampling interval being $t_{i+1} - t_i$. The exact length of each sampling interval is found in Appendix C.

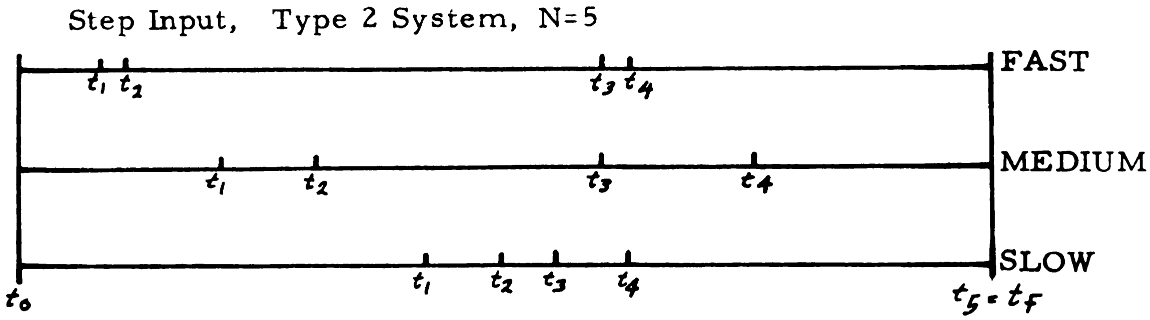


Figure 4.15. Representative Sampling Interval Lengths for Figures 4.12(a) to 4.14(a) using N=5.

In Figure 4.15 the shift in the sampling intervals as the bandwidth of the Type 2 control system decreases can be attributed to the fact that since the same performance index is used for each system, sampling has to yield large piecewise control signals and delay times to compensate for the "slowness" in the control system response. It should be noted that during the first sampling interval ($T_0 = t_1 - t_0$) the control input error signal will reflect the initial conditions of the control system. In this case using the initial conditions of -1.0 for the "position" state variable and using a step input having an amplitude of 1.0 at time t_o results in an initial input error control signal of 2.0 (e.g. $u_e(t_o) = z(t_o) - x_1(t_o)$ or $u_e(t_o) = 1.0-(-1.0) = 2.0$) during the first sampling interval, T_1. The longer sampling interval T_1 and the corresponding error feedback delay becomes, the longer the control system is subjected to the value of 2.0 as the input control error signal. Thus for the SLOW responding system the first sampling interval becomes large with respect to the first sampling interval for the FAST responding system in order to better use the delay with the large initial error to help decrease the terminal error which is penalized severely in the performance index. Controlling of error feedback delay was used so effectively that for MEDIUM and SLOW response time systems the use of optimal aperiodic sampling and in some cases optimal periodic sampling were able to out-perform continuous time control.

The conclusions which can be drawn from Figures 4.12 through 4.14 are that the slower the system dynamics the more the control performance of the system can be improved through use of periodic or sub-optimal aperiodic sampling. For either of the two "slower" systems considered, periodic and sub-optimal aperiodic sampling yield comparable performance plus costs for implementation. Thus the sub-optimal aperiodic criterion would be implemented since the control performance is significantly lower than the control performance for periodic sampling.

The number of sampling intervals used in the implementation process would depend on the maximum value of the control performance which is acceptable. The number of sampling intervals selected for any particular control implementation would generally be the minimum number such that the control performance remains below this maximum.

4.5 Control Performance Investigation by System Type

In the previous section, control performance of a Type 2 feedback control system having various response times, namely SLOW, MEDIUM, and FAST, were investigated for a step input. In this section the control performance of Type 2, Type 1, and Type 0 feedback control systems will be investigated using a step, ramp, and noise input signals. Each of the three systems will have a bandwidth of approximately 7 radians per second which corresponds to that of the MEDIUM response time system in the previous section. The performance of the Type 2 MEDIUM response time system found in (4.13), having been previously investigated for a step input will now be investigated using a ramp and noise input. The ramp input will have the form

$$z(t) = t \quad \text{for} \quad 0 \leq t \leq t_f$$

with the initial system state and noise statistics being

$$E\{\underline{x}(t_o)\} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \underline{V}_x(t_o) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\Psi}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The resultant control signal $u_e(t)$ found in (4.15) will be sampled to satisfy (4.16) and (4.17). The derived control performance index and cost for implementation model are identical to those found in (4.18), (4.19), and (4.20) for the step input investigation. The noise input will have the form

$$z(t) = 0 \quad \text{for} \quad 0 \leq t \leq t_f$$

where the noise and randomly distributed initial states are assumed gaussian with

$$E\{x(t_o)\} = \underline{0}\ , \qquad \underline{V}_x(t_o) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$E\{x(t_o)\} = \underline{0}\ , \qquad \underline{\Psi}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The derived performance index for the case of system noise disturbances has the form

$$J = [[\ 1\ 0\ ]\underline{m}(t_N) - z(t_N)]'\ .01\ [[\ 1\ 0\ ]\underline{m}(t_N) - z(t_N)] + tr[\begin{bmatrix} .01 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{V}(t_N)] + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} [.1\ [[\ 1\ 0\ ]\underline{m}(t) - z(t)]^2 + tr[\begin{bmatrix} .1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{V}(t)\ ] + .02\ [\ z(t_i) - [\ 1\ 0\ ]\underline{m}(t_i)\ ]^2 + tr[\begin{bmatrix} .02 & 0 \\ 0 & 0 \end{bmatrix}\underline{V}(t_i)\ ]]\ dt + J_f \tag{4.25}$$

with J_f being defined in (4.19) and (4.20). The added trace terms are a result of the specified system statistics.

Therefore given the Type 2 system, performance index, and cost for implementation function, the control performance levels for periodic and sub-optimal aperiodic sampling criteria are plotted in Figures 4.10, 4.12, and 4.18 for the step, ramp, and noise inputs respectively. As can be seen in Table 4.4, continuous time control (C), optimal periodic sampling (OPS), and optimal aperiodic sampling (OA) performance costs decrease using a ramp input rather than a step input and increase sharply with a noise input. The decrease in performance costs is primarily due to differences in the terminal errors caused by the step and ramp inputs. The large performance cost for the disturbance noise input is attributable to the fact that the performance index penalizes large variances in the terminal error on the same level as it penalizes large tracking errors. (see performance index in (4.21) Thus it is apparent that with the specified initial state and noise covariances for the Type 2 system being considered state variances increase significantly over the control interval.

As can be seen in Table 4.4, optimal aperiodic (OA) sampling J_1^* out-performs either continuous time control (C) or optimal periodic sampling (OPS) for each input type considered. The maximum ratio of performance costs between continuous and sub-optimal aperiodic sampling $(C/OA)^*$ remains the same for the step and ramp inputs. The maximum ratio of performance costs between periodic and sub-optimal aperiodic sampling $(PS/A)^*$ shows vast improvements

	C	OA		OPS		$(C/OA)^*$		$(PS/OA)^*$	
Type 2 INPUT	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	$\frac{N_4}{1}$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}$
STEP	5.2	2.3	8	5.3	5	2.3	8	10^4	2
RAMP	.931	.39	8	1.0	4	2.4	8	10^3	2
NOISE	11.0	8.7	2	8.9	2	1.3	2	1.03	2

Table 4.4. Data Summary for Figures 4.13(a), 4.16(a), and 4.17(a).

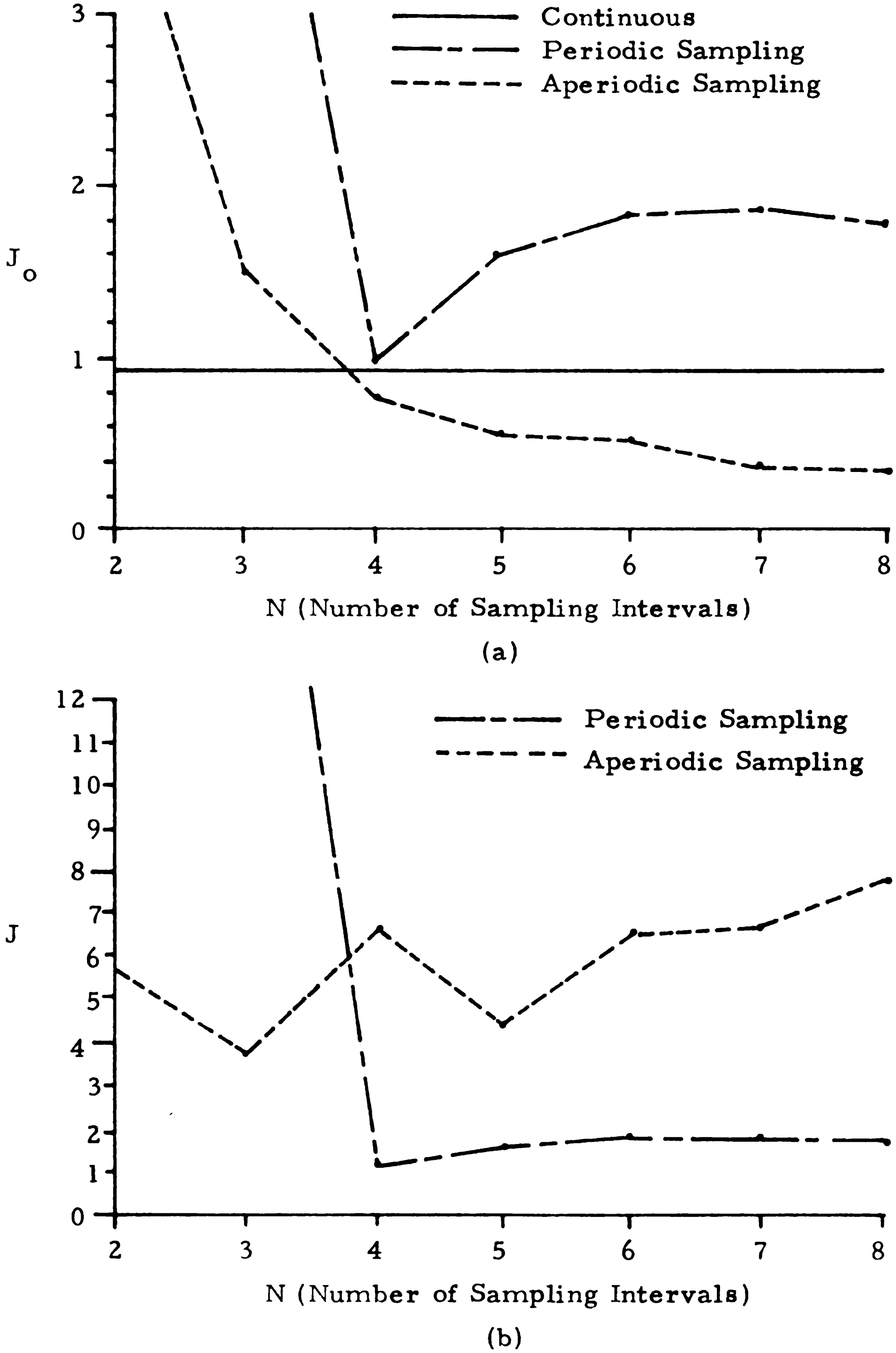
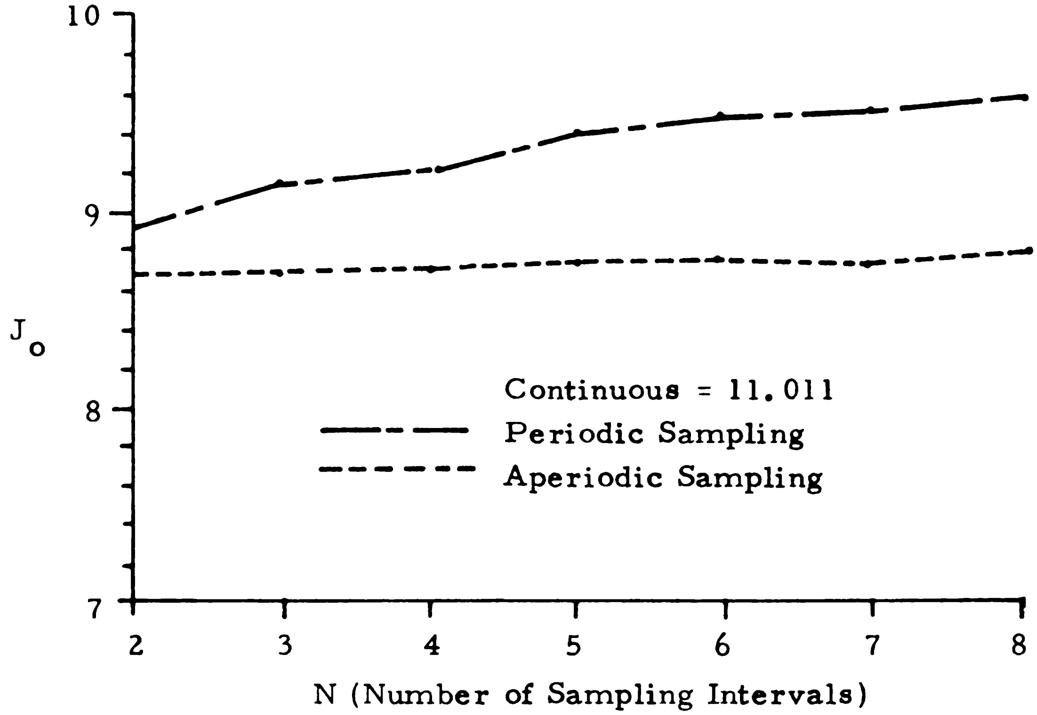
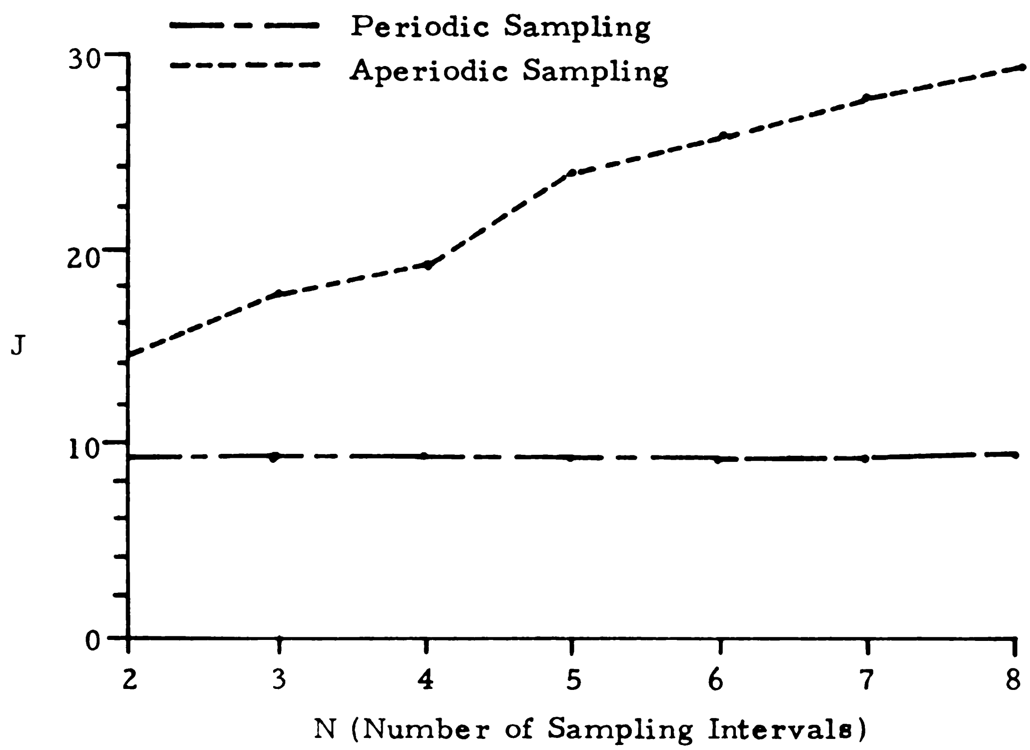


Figure 4.16 - Performance Costs for a Type 2 MEDIUM Response Time Control System Using a Ramp Input



(a)



(b)

Figure 4.17-- Performance Costs for a Type 2 MEDIUM Response Time Control System Using a Noise Input

because the periodic sampled-data system was unstable using only two sampling intervals. The maximum performance ratios between periodic and sub-optimal aperiodic sampling (PS/OA)* for noise inputs indicate periodic sampling is near optimal when the system is to be controlled for noise disturbances only.

Optimal aperiodic sampling yielded lower performance costs, as seen in Figures 4.13, 4.16, and 4.17, than either periodic or continuous time control for each input and number of sampling intervals considered. These performance results are again attributed to the effective use of feedback error signal delay to improve control performance (See page 80 for additional discussion on delay.) As the number of sampling intervals increase, the delays are reduced yielding a sampled feedback error signal which appears more continuous to the system under control. As is indicated in Figure 4.12(a) for 8 sampling intervals, the performance costs for periodic sampling would decrease and approach continuous time control as a limit as the number of sampling intervals became large. [52] Likewise for sub-optimal aperiodic sampling as the number of sampling intervals were increased the values of the derived performance index would approach that of the continuous time control. [52]

In Figure 4.16(b) the control performance costs plus costs for implementation are clearly lower for periodic sampling using 4 or more sampling intervals. The high costs for sub-optimal aperiodic sampling are attributed to the processing involved in obtaining the optimal sampling interval sequence. Using less than 4 sampling intervals, periodic sampling results in system instability and thus sub-optimal aperiodic sampling would have to be used.

Figure 4.17(b) indicates that periodic sampling yields almost constant performance costs compared to sub-optimal aperiodic sampling. The increases in costs for aperiodic sampling are due in all cases to increased computational expenses resulting from optimization of the sampling interval sequence. Since sub-optimal aperiodic and periodic sampling yield almost the same performance and since optimal aperiodic yields almost periodic samples, periodic sampling should be used for noise inputs.

The performance of a Type 1 MEDIUM response time unity feedback control system will now be investigated for a step, ramp, and noise input. The following control system model will be used:

$$\underline{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 25 \end{bmatrix} u_e(t)$$

$$y(t) = [\ 1 \quad 0\ ]\ \underline{x}(t)$$

with the control error signal for the above system being

$$u_e(t) = z(t) - y(t)$$

where z(t) is a specified input signal for $0 \leq t \leq t_f$.

The first two input signals being considered have the form :

$$z(t) = 1 \quad \text{(step input)}$$

and

$$z(t) = t \quad \text{(ramp input)}$$

with the initial system state and noise statistics being

$$E\left\{\underline{x}(t_o)\right\} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} , \quad \underline{V}_x(t_o) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \quad \underline{\Psi}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The resultant control signal $u_e(t)$ found above, will be sampled to satisfy (4.16) and (4.17). The derived control performance index and cost for implementation model are identical to those found in (4.18) and (4.19), and (4.20) for the step input investigation.

The third input, the noise, will have the following form:

$$z(t) = 0$$

where the noise and randomly distributed initial states are assumed gaussian with

$$E\left\{\underline{x}(t_o)\right\} = \underline{0} , \quad \underline{V}_x(t_o) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$E\left\{\underline{w}(t)\right\} = \underline{0} , \quad \underline{\Psi}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The derived control performance index and cost for implementation model are identical to those found in (4.25), (4.18), and (4.19).

It should be noted that for a ramp input a Type 1 unity feedback control system has a steady-state error. Therefore, errors in the final state values using a ramp input will always be present with continuous time feedback control.

As can be seen in Table 4.5, optimal aperiodic (OA) sampling J_1^* outperforms either continuous time control (C) or optimal periodic sampling (OPS) for each input considered. The maximum ratio of performance costs between continuous and sub-optimal aperiodic sampling $(C/OA)^*$ occurs for a ramp input. The maximum ratio of performance costs between periodic and sub-optimal aperiodic sampling $(PS/OA)^*$ also occurs for a ramp input. These optimal improvement ratios can be attributed to the fact that optimal aperiodic sampling is able to reduce final state errors to a greater extent for a ramp input than a step input compared to continuous time or periodically sampled control. The maximum performance ratio between periodic and sub-optimal aperiodic sampling $(PS/OA)^*$ for noise inputs indicate periodic sampling is near optimal when the system is to be controlled for noise disturbances only.

	C	OA		OPS		$(C/OA)^*$		$(PS/OA)^*$	
Type 1 INPUT	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	$N_{\frac{4}{1}}$	$\frac{J_3}{J_1}^*$	$N_{\frac{3}{1}}$
STEP	5.7	3.8	8	22.6	8	1.5	8	8.5	2
RAMP	1.0	.29	8	3.9	8	3.4	8	24.0	4
NOISE	.45	.37	2	.38	2	1.2	2	1.02	3

Table 4.5. Data Summary for Figures 4.18(a) to 4.20(a).

Figure 4.18(a) and 4.19(a) indicate that delay caused by periodic sampling degraded control performance for the number of sampling intervals considered. Sub-optimal aperiodic sampling with its "controlled delay" was able to outperform continuous time control for N greater than six sampling intervals for a step input and three sampling intervals

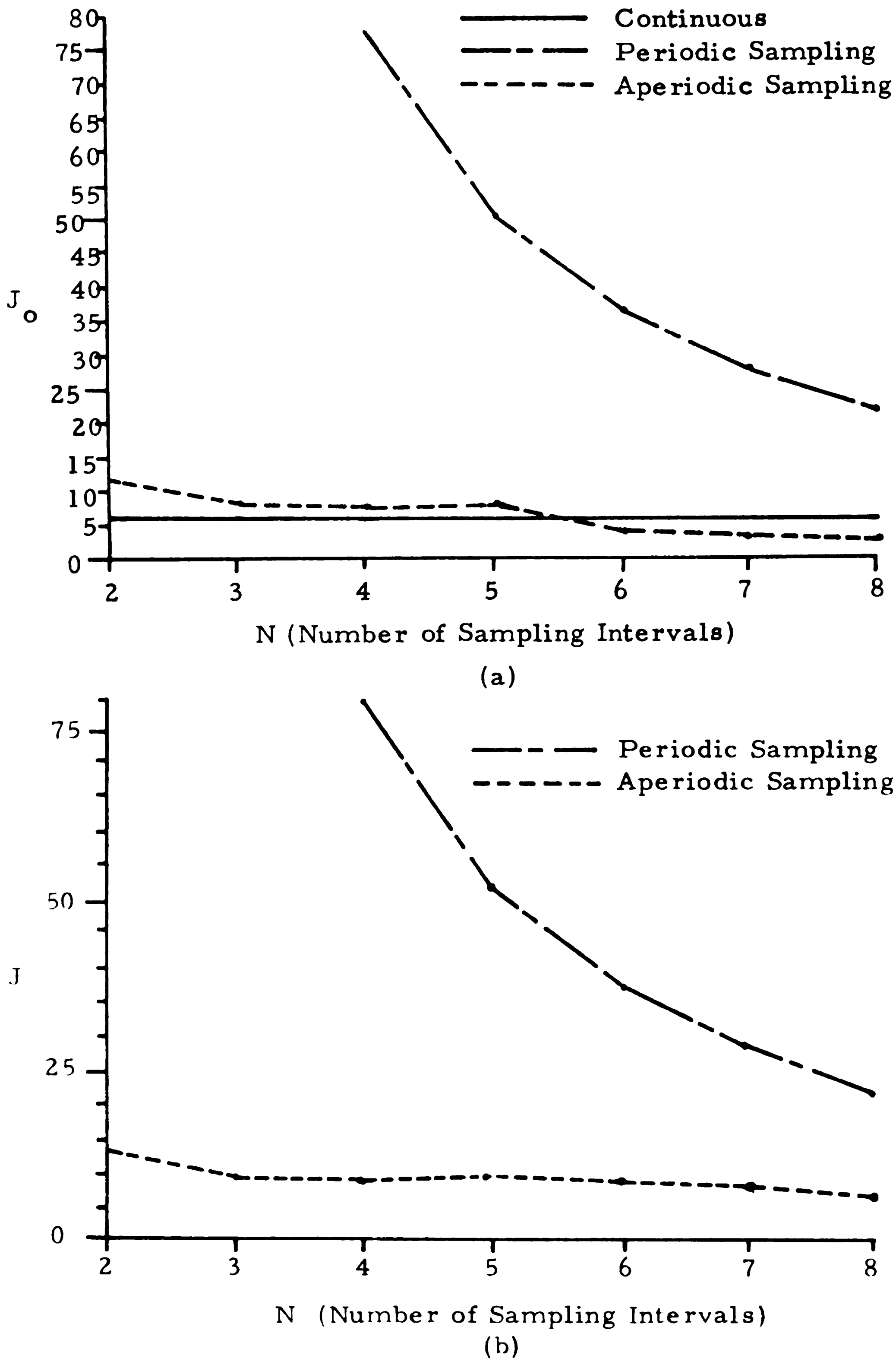


Figure 4.18. Performance Costs for a Type 1 MEDIUM Response Time Control System using a Step Input.

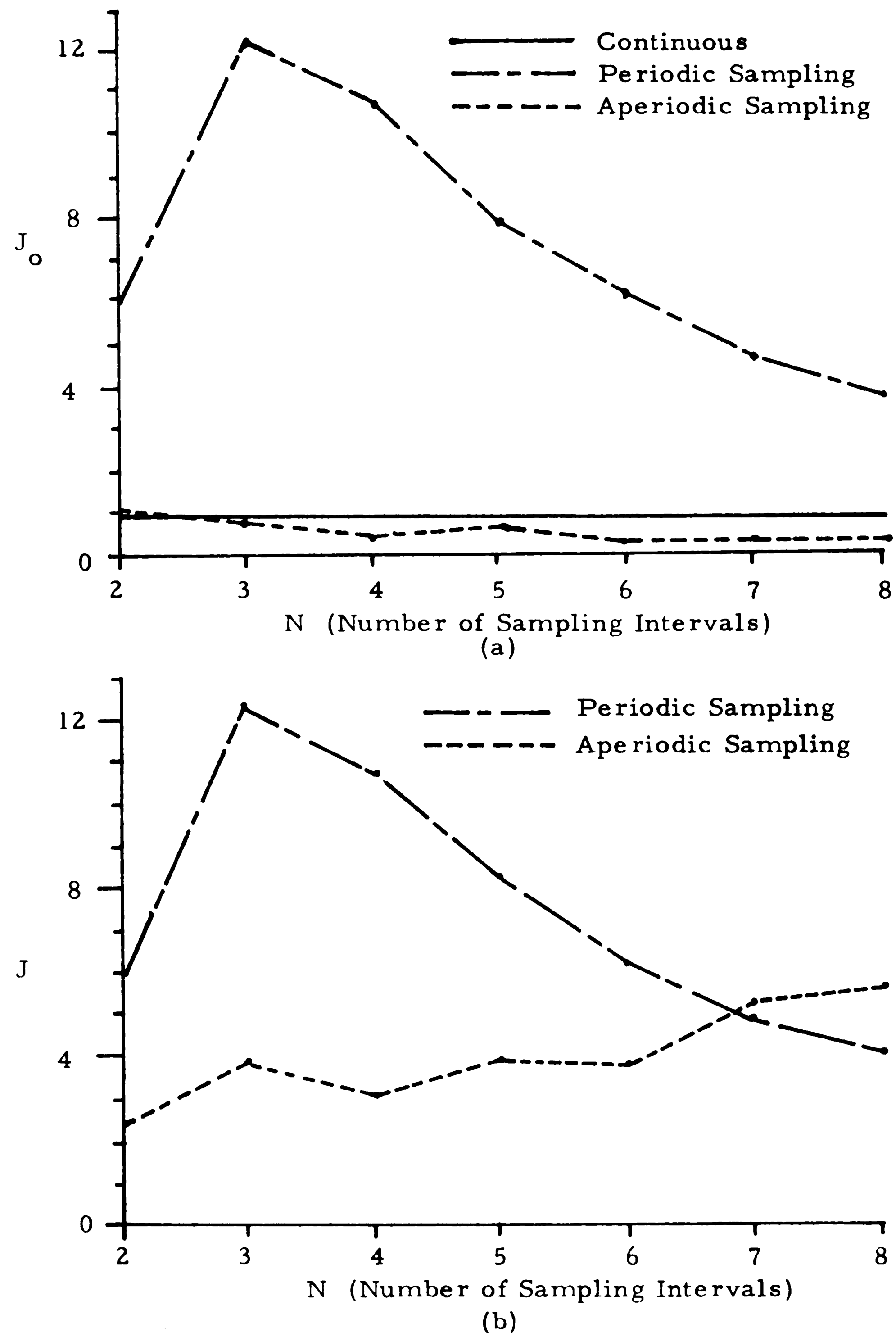
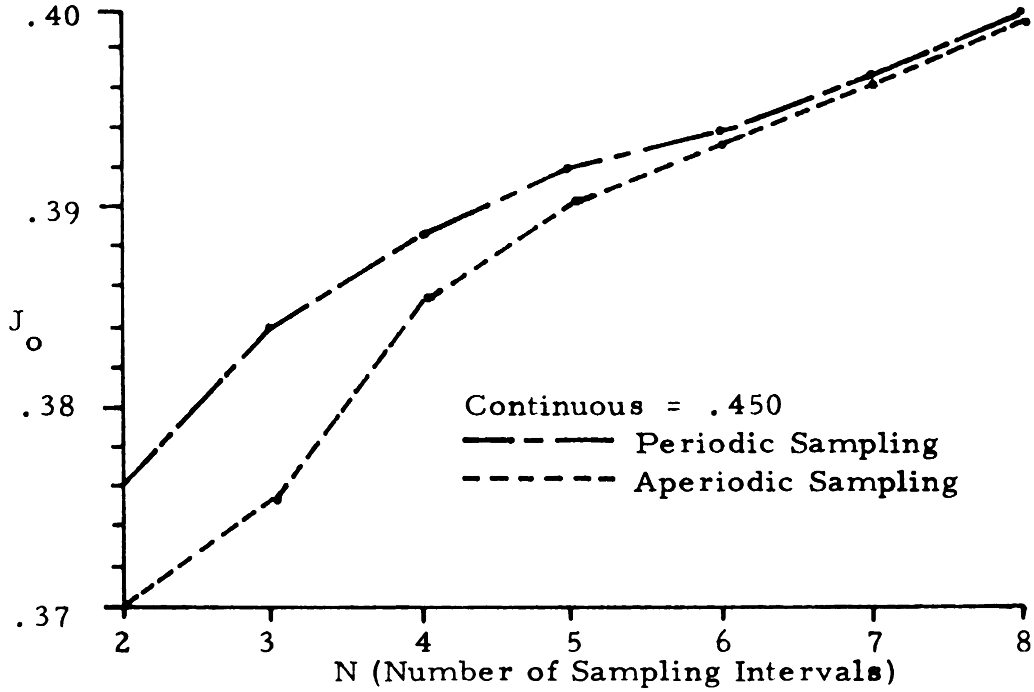


Figure 4.19. Performance Costs for a Type 1 MEDIUM Response Time Control System using a Ramp Input.



(a)

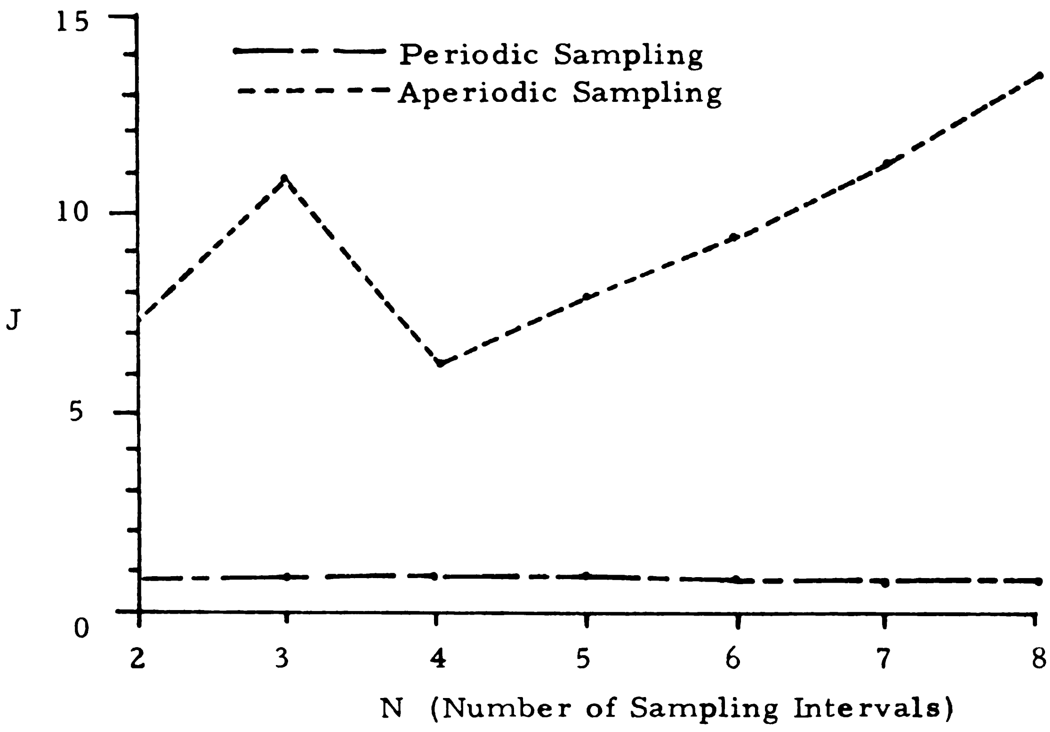


Figure 4.20. Performance Costs for a Type 1 MEDIUM Response Time Control System using a Noise Input.

for a ramp input since it was more effective in reducing final state errors. In the case of sub-optimal aperiodic sampling where the sampling delays can be "tuned to the system" through variation of the sampling interval lengths, the performance costs have been reduced. Periodic sampling cannot adjust sampling interval lengths, and thus the delays caused by sampling, as a result performance costs are increased. As was pointed out previously, and as can be seen in Figures 4.18(a) and 4.19(a), the performance cost in periodic and optimal aperiodic sampling approach the continuous time performance cost as N increases.

Figure 4.19(a) indicates the performance costs for the Type 1 control system using the specified noise input. As can be seen the performance costs for periodic and aperiodic sampling almost identical. This indicates that periodic sampling is near optimal when used to control for noise disturbances.

Figure 4.18(b) indicates sub-optimal aperiodic sampling to be the only means of implementation using the number of sampling intervals investigated.

Figure 4.19(b) indicates sub-optimal aperiodic sampling yields lower performance index plus costs for implementation than periodic sampling for up to 6 sampling intervals. Since the processing costs are large for sub-optimal aperiodic sampling, periodic sampling yields lower overall costs for 7 and 8 sampling intervals even though a performance improvement of 13.5 was realized through the use of sub-optimal aperiodic sampling at 8 sampling intervals.

Figure 4.20(b) indicate periodic sampling yields almost constant performance costs compared to sub-optimal aperiodic sampling. The increases in costs for aperiodic sampling are due in all cases to increased computational expenses resulting from optimization of the sampling interval sequence.

The performance of a Type 0 MEDIUM response time unity feedback control system will now be investigated for a step, ramp, and noise input. The following control system model will be used:

$$\underline{x}(t) = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 19 \end{bmatrix} u_e(t)$$

$$y(t) = [\ 1 \quad 0\ ]\ \underline{x}(t)$$

with the control error signal for the above system being

$$u_e(t) = z(t) - y(t)$$

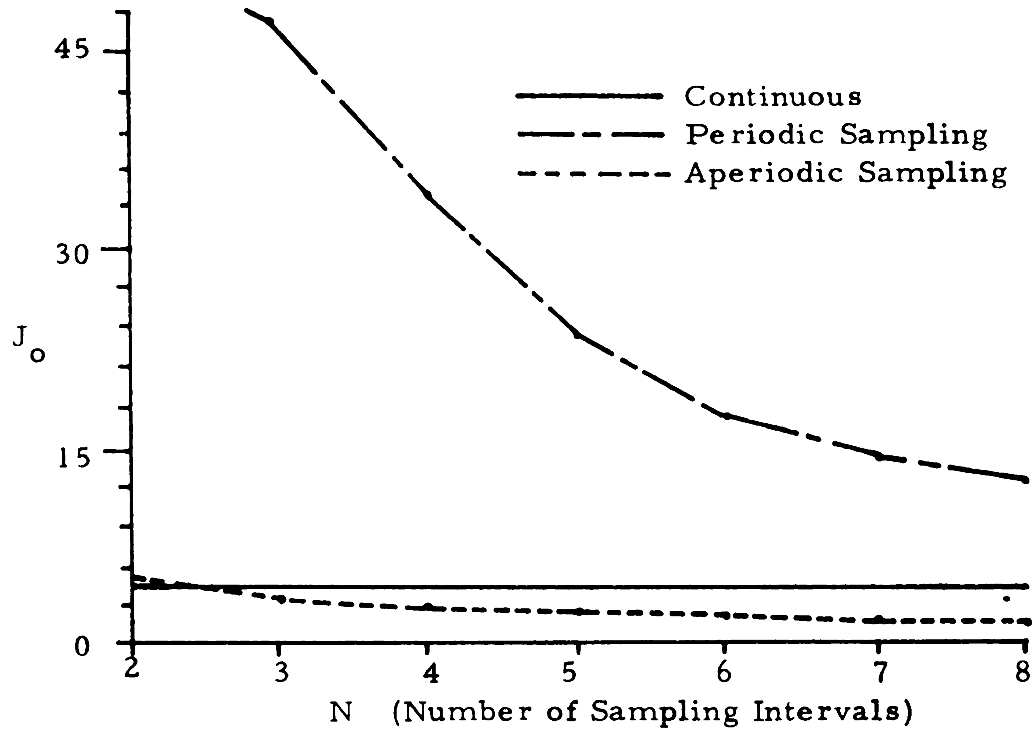
where z(t) is a specified input signal for $0 \leq t \leq t_f$. The resulting control signal $u_e(t)$, will be sampled to satisfy (4.16) and (4.17). The three input signals being considered for this system, viz. step, ramp, and noise, have been specified in the investigation of the Type 1 system just completed. The derived performance index and cost for implementation model to be used for the noise input are found in (4.25), (4.18), and (4.19).

It should be noted that for a step input to a Type 0 unity feedback control system there occurs a steady state error in the output signal with respect to the input signal which can not be eliminated by conventional feedback techniques. A ramp used as an input to a Type 0 unity feedback control system yields errors between the input and output signals which increase with time.

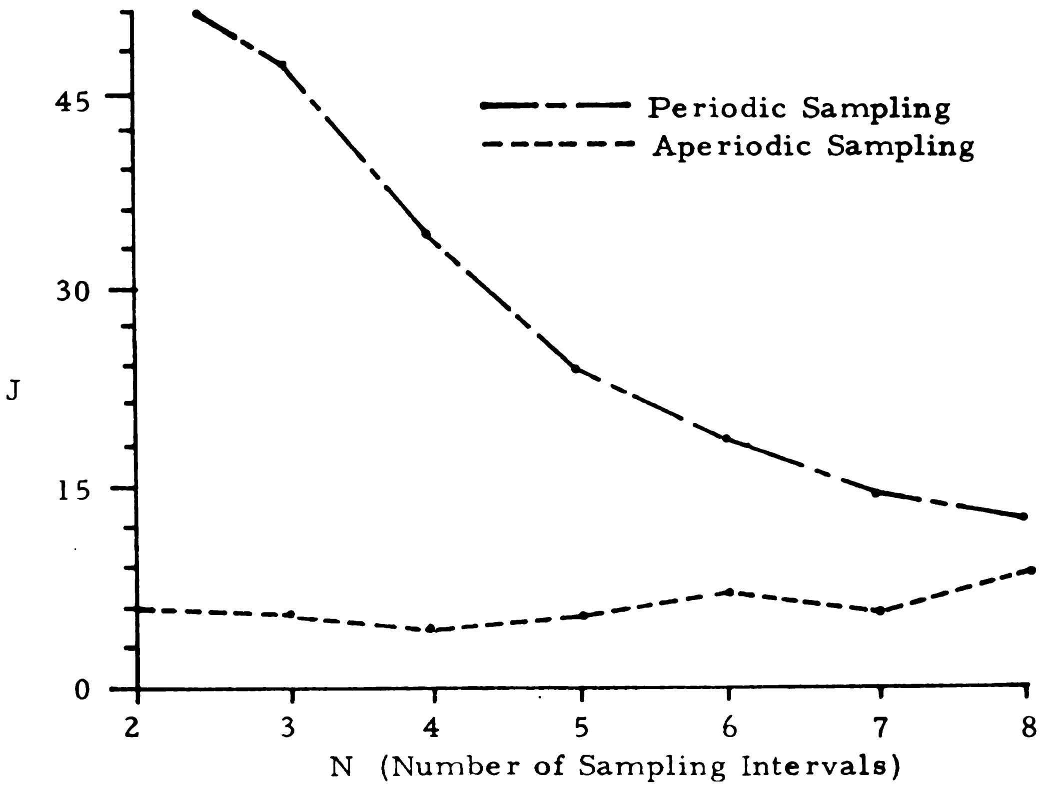
As can be seen in Table 4.6, optimal aperiodic (OA) sampling J_1^* outperforms either continuous time control (C) or optimal periodic sampling (OPS) for each input considered. The maximum ratio of performance costs between continuous and optimal aperiodic sampling $(C/OA)^*$ occurs for a ramp input. The maximum ratio of performance costs between periodic and sub-optimal aperiodic sampling $(PS/OA)^*$ also occurs for a ramp input. These optimal improvement ratios can be attributed to the fact that optimal aperiodic sampling is able to reduce final state errors to a greater extent for a ramp input than a step input compared to continuous time or periodically sampled control. The maximum performance ratio between periodic and sub-optimal aperiodic sampling $(PS/OA)^*$ for noise inputs indicate periodic sampling is near optimal when the

	C	OA		OPS		$(C/OA)^*$		$(PS/OA)^*$	
Type 0 INPUT	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	$\frac{N_4}{1}^*$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}^*$
STEP	4.3	1.6	8	12.2	8	2.7	8	11.5	2
RAMP	2.7	.22	8	5.4	8	12.2	8	33.0	3
NOISE	.23	.19	2	.19	2	1.2	2	1.02	2

Table 4.6. Data Summary for Figures 4.21(a) to 4.23(a).



(a)



(b)

Figure 4.21. Performance Costs for a Type 0 MEDIUM Response Time Control System using a Step Input.

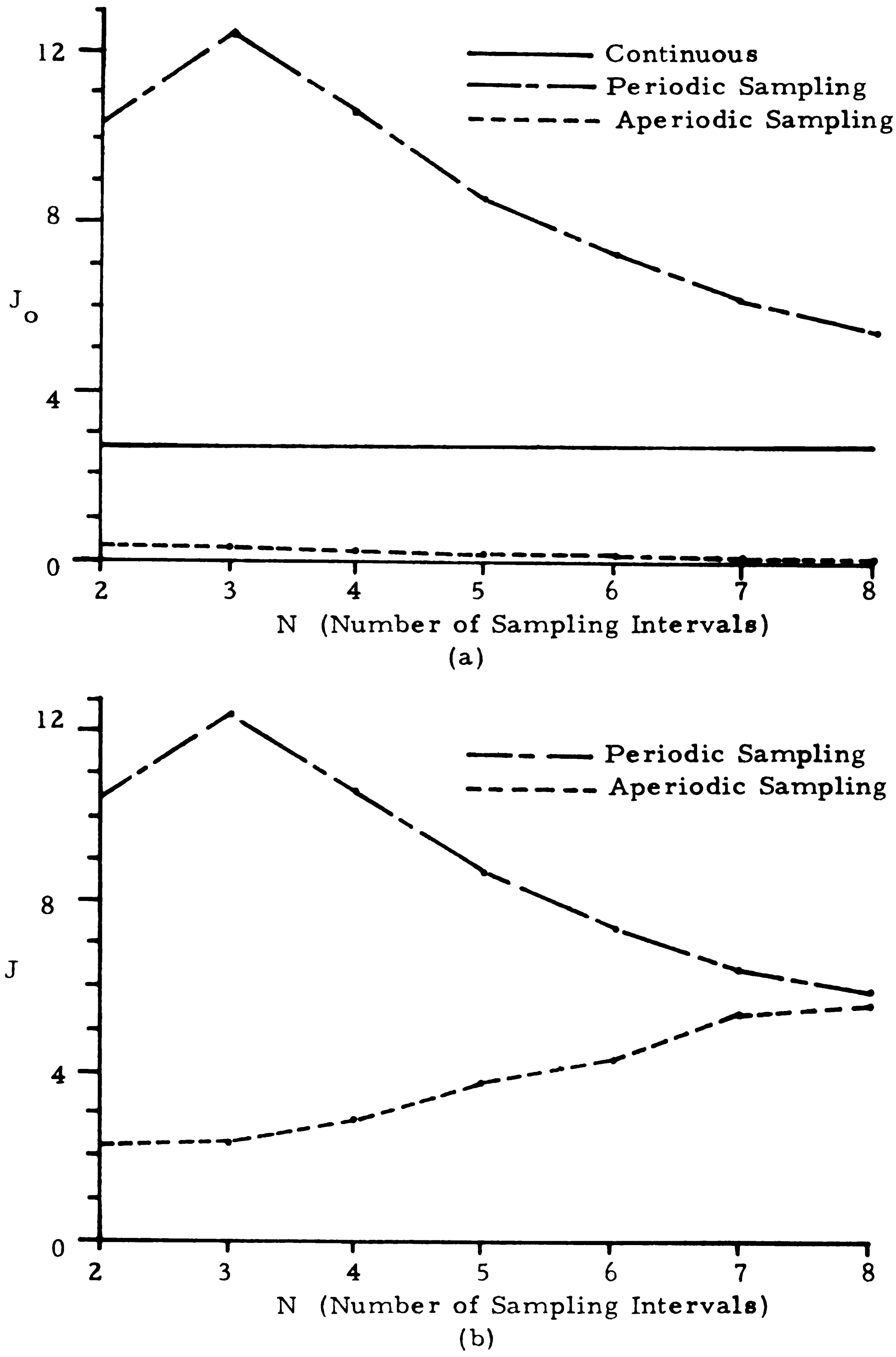
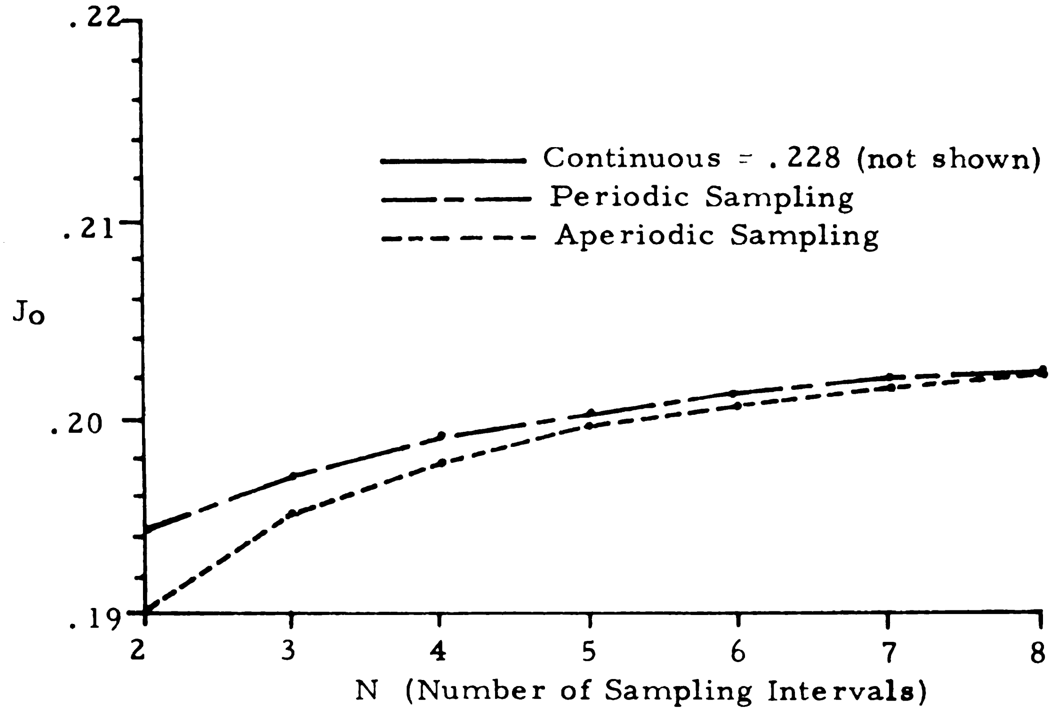
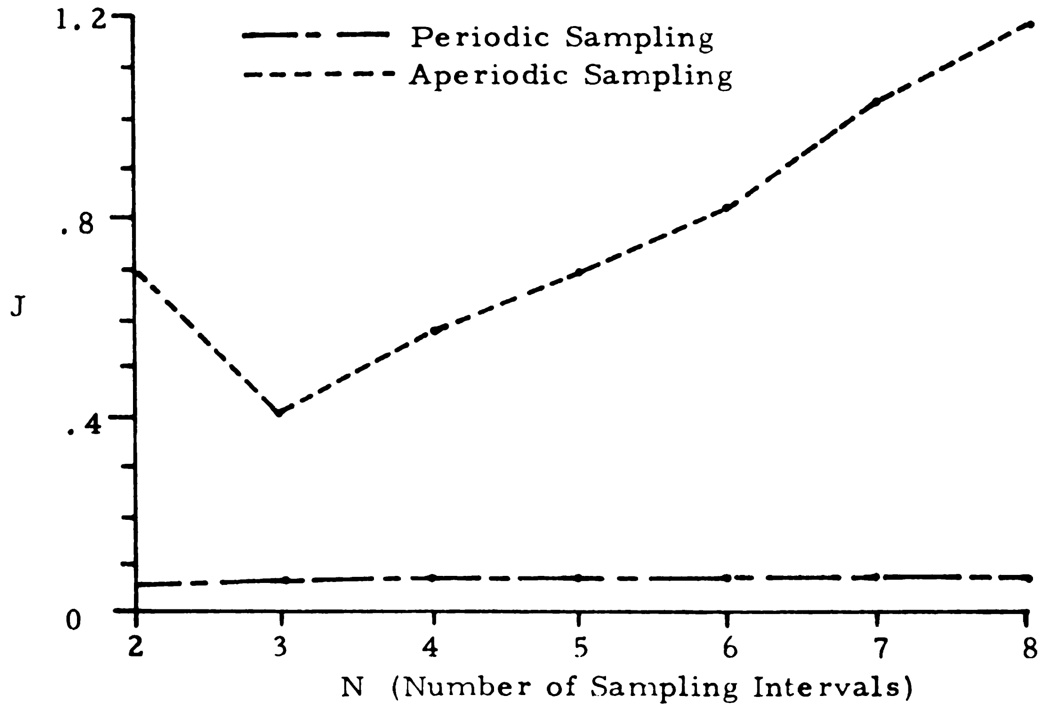


Figure 4.22. Performance Costs for a Type 0 MEDIUM Response Time Control System using a Ramp Input.



(a)



(b)

Figure 4.23. Performance Costs for a Type 0 MEDIUM Response Time Control System using a Noise Input.

the system is to be controlled for noise disturbances only.

Performance improvements in sub-optimal aperiodic sampling are again attributed to the ability of aperiodic sampling to "tune sampling delays" to the system and thus minimize the derived performance index through use of the optimal sampling interval sequence.

Using the ramp input this example was able to show major improvements in control performance through use of sub-optimal aperiodic sampling. Again as in the past explanations, tuning of the sampling intervals, and thus the feedback control error delays, through use of the sub-optimal aperiodic sampling techniques as presented in Chapter III, vastly improve control response for, in this example, MEDIUM speed Type 0 unity feedback control system with ramp inputs.

Figure 4.23(a) indicates the performance costs for the Type 0 control system using the specified noise input. As can be seen the performance costs for periodic and sub-optimal aperiodic sampling yielded almost identical performance costs. This indicates that periodic sampling is near optimal when used to control for noise disturbances.

Figure 4.21(b) indicates the total sub-optimal aperiodic sampling costs to be lower than periodic sampling for all the sampling intervals considered. As can be seen in the sub-optimal aperiodic sampling case, the precesssing costs increase with an increasing number of sampling intervals and could easily result in the total sub-optimal aperiodic sampling costs exceeding those for periodic sampling using 9 or 10 sampling intervals.

Figure 4.22(b) indicates improved performance plus costs for implementation for sub-optimal aperiodic sampling for all sampling intervals. As can be seen for 7 and 8 sampling intervals, processing costs offset improvements in control performance shown in Figure 4.22(a).

Figure 4.23(b) indicates periodic sampling yields almost constant performance costs compared to sub-optimal aperiodic sampling. The increases in costs for aperiodic sampling are due in all cases to increased computational expenses resulting from optimization of the sampling interval sequence.

4.6 Analysis by System Type.

The lengths of the sampling intervals using N=5 for the Type 2, Type 1, and Type 0 feedback control systems are shown in Figure 4.25(a), 4.25(b), and 4.25(c) for a step, ramp, and noise inputs respectively. As can be seen the lengths of the sampling intervals decrease as system type number decreases for both step and ramp inputs. The length of the sampling intervals increase as the type number decreases for the noise input. The changes in the sampling interval lengths with respect to system type are a result of the differences in the open loop plants being considered for each system type.

Consider the open loop responses of the three plants to a unit step input for $t \in [0, 1]$ as shown in Figure 4.24(a) with the plant transfer function for a step input being shown in Figure 4.24(b). As can be seen, the Type 2 system responds quickly to the step input, having a terminal value of 17.5 at t = 1. The response of the Type 0 and Type 1 systems are very similar, each being nearly linear over the time interval being considered with the Type 0 system having approximately twice the slope of the Type 1 system. (See time domain responses in Figure 4.24(b))

As a result of the sample and hold operation used in system control, piecewise constant signals are applied as inputs to the various open loop plants. (See Chapter III) The responses of each open loop plant, as seen in Figure 4.24, will proceed as shown

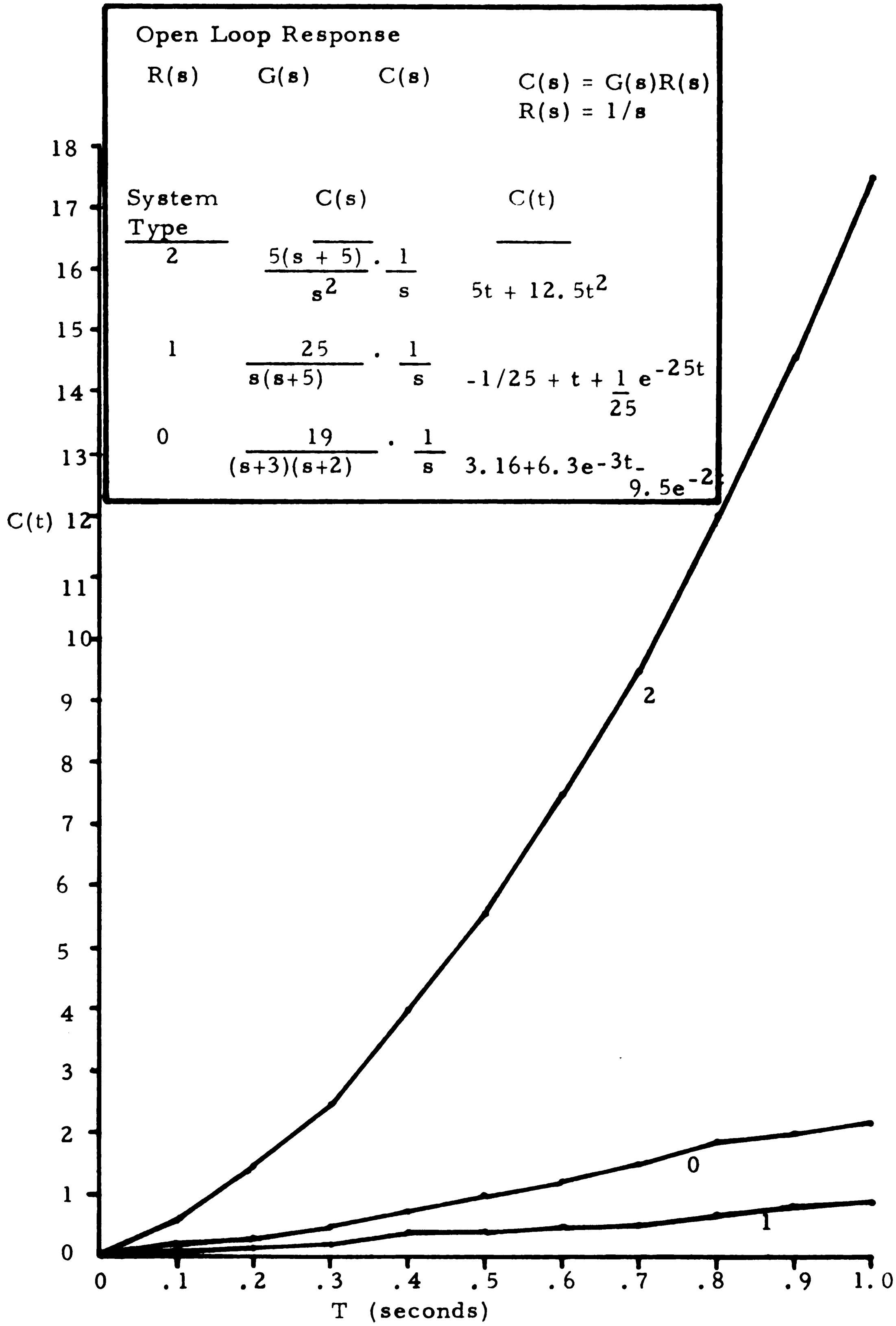


Figure 4.24. Open Loop Plant Responses of the Type 2, 1, and 0 Plants for a Unit Step Input.

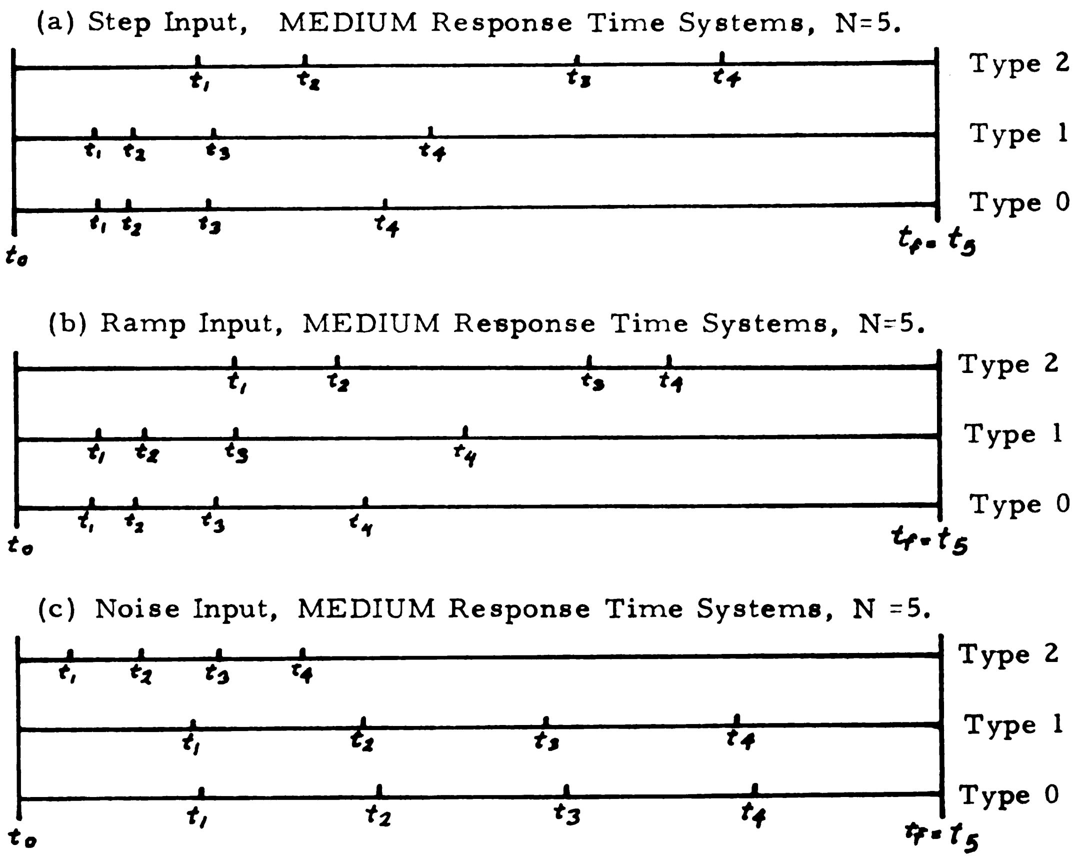


Figure 4.25. Sub-Optimal Sampling Interval Sequences, N=5, for Type 2, 1, and 0 Control Systems.

until a different piecewise constant input is applied. The length of time each piecewise constant signal is held is used as a control variable for the system. Therefore using the sub-optimal sampling interval sequences as control variables, the resulting system responses are shown in Figure 4.26 for a step input.

The closed loop step responses as seen in Figure 4.26 indicate an underdamped response to a step input for the Type 2 system compared to the rather linear overdamped responses of the Type 1 and Type 0 systems. Longer sampling intervals for the Type 2 control system are due to the fact that the control performance index severely penalizes terminal errors making it necessary to reduce the levels of the piecewise constant input error signals as quickly as possible thereby reducing changes in the system state due to non-zero input control levels during the final sampling interval.

The reduction in the sampling interval lengths for the Type 1 and Type 0 systems are attributable to the fact that each of these systems respond slowly to the step input, thus large feedback control errors have to be used effectively to reduce terminal errors. Effective use of the feedback control errors to improve system response is accomplished by sampling earlier in the control interval and thus

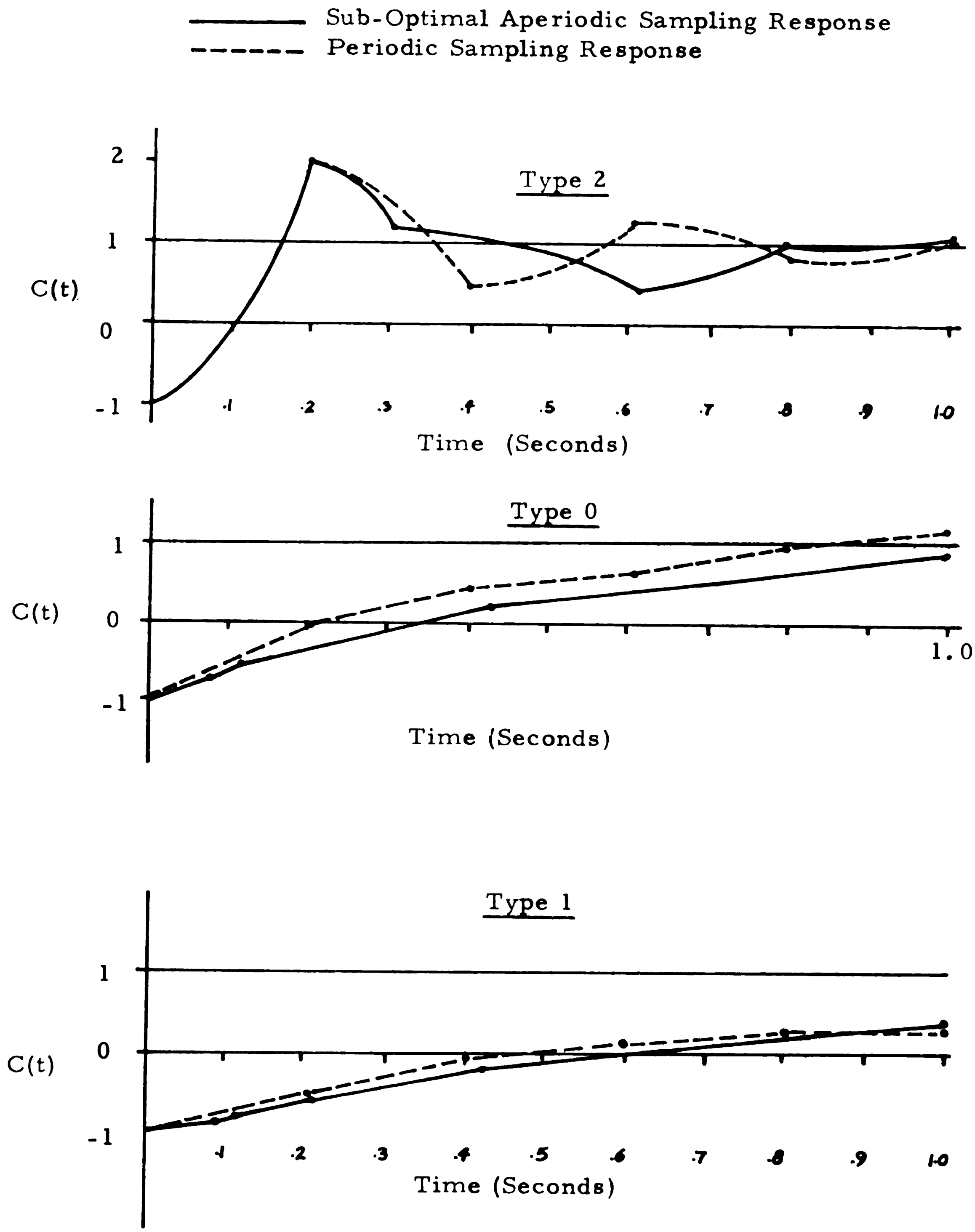


Figure 4.26. Type 2, 1, and 0 Feedback Control System Response to a Unity Step Input using Periodic and Sub-Optimal Aperiodic Sampling.

taking advantage of the large initial errors between the system state and the step input signal. A similar explanation is used to explain the differences in the sampling interval lengths, shown in Figure 4.25(b), between the Type 2 and Type 0 and 1 systems for a ramp input signal. It should be noted that the terminal errors are larger, as seen in Figure 4.26, for the Type 1 system than either the Type 2 or Type 0 systems. These large terminal errors for the Type 1 system are due to the relatively slow response of the Type 1 system and the short control interval which does not allow significant changes to occur to reduce terminal errors. (See Figure 4.24(b))

The third input considered, the noise input, yields a sampling interval sequences which are short and periodically spaced at the beginning of the control interval for the Type 2 system compared to the periodically spaced sampling intervals of the Type 0 and Type 1 systems. (See Figure 4.25(c)) The noise which occurs early in the control interval has a more severe effect on the terminal errors as the number of integrators in the plant increase. (See Figure 4.26(b) for the plant transfer functions.) Thus, the sampling intervals must be chosen smaller as the type number increases to offset the effect of the increase in noise during the initial part of the control interval.

Table 4.7 summarizes optimal control performance data found in Chapter IV for the Type 2, 1, and 0 unity feedback control systems with respect to step, ramp, and noise inputs for various methods of control. The optimal performance results present in Table 4.7 will be compared by method of sampling versus sytem Type for each input considered. Figure 4.27 is presented to show trends and generalities not readily seen in tabular data presentations.

Ramp Input (Figure 4.27(a))

In the unity feedback control systems being considered, a decrease in system type increases the steady state terminal errors. [50] For a ramp input the Type 2 system has no inherent steady state terminal errors, while the Type 1 system has constant steady state terminal errors, and the Type 0 system has terminal errors which increase with time. As can be seen in Figure 4.27(a), the continuous time control (C) and optimal periodic sampling (OPS) control performance costs increase with a decrease in system type. These increases in performance costs are attributed to the increased terminal errors which result from changes in system type. Since the control performance index heavily penalizes terminal errors, increases in performance costs with decreases in system type were expected. It should be noted that through use of optimal aperiodic sampling control performance costs were reduced compared to the other methods of control being investigated. This reduction in performance cost indicates that controlled feedback error delays, as discussed in previous sections, are effective in reducing the inherent terminal errors present in the Type 0 and Type 1 control systems having a ramp input.

Type 2 INPUT	C	OA		OPS		$(C/OA)^*$		$(PS/OA)^*$	
	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	$\frac{N_4}{1}^*$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}^*$
STEP	5.2	2.3	8	5.3	5	2.3	8	10^4	2
RAMP	.921	.39	8	1.0	4	2.4	8	10^3	2
NOISE	11.0	8.7	2	8.9	2	1.3	2	1.03	2

Type 1 INPUT	C	OA		OPS		$(C/OA)^*$		$(PS/OA)^*$	
	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	$\frac{N_4}{1}^*$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}^*$
STEP	5.7	3.8	8	22.6	8	1.5	8	8.5	2
RAMP	1.0	.29	8	3.9	8	3.4	8	24.0	4
NOISE	.45	.37	2	.38	2	1.2	2	1.02	3

Type 0 INPUT	C	OA		OPS		$(C/OA)^*$		$(PS/OA)^*$	
	J_4	J_1^*	N_1^*	J_3^*	N_3^*	$\frac{J_4}{J_1}^*$	$\frac{N_4}{1}^*$	$\frac{J_3}{J_1}^*$	$\frac{N_3}{1}^*$
STEP	4.3	1.6	8	12.2	8	2.7	8	11.5	2
RAMP	2.7	.22	8	5.4	8	12.2	8	33.0	3
NOISE	.23	.19	2	.19	2	1.2	2	1.02	2

Table 4.7. Numerical Data for Figure 4.25.

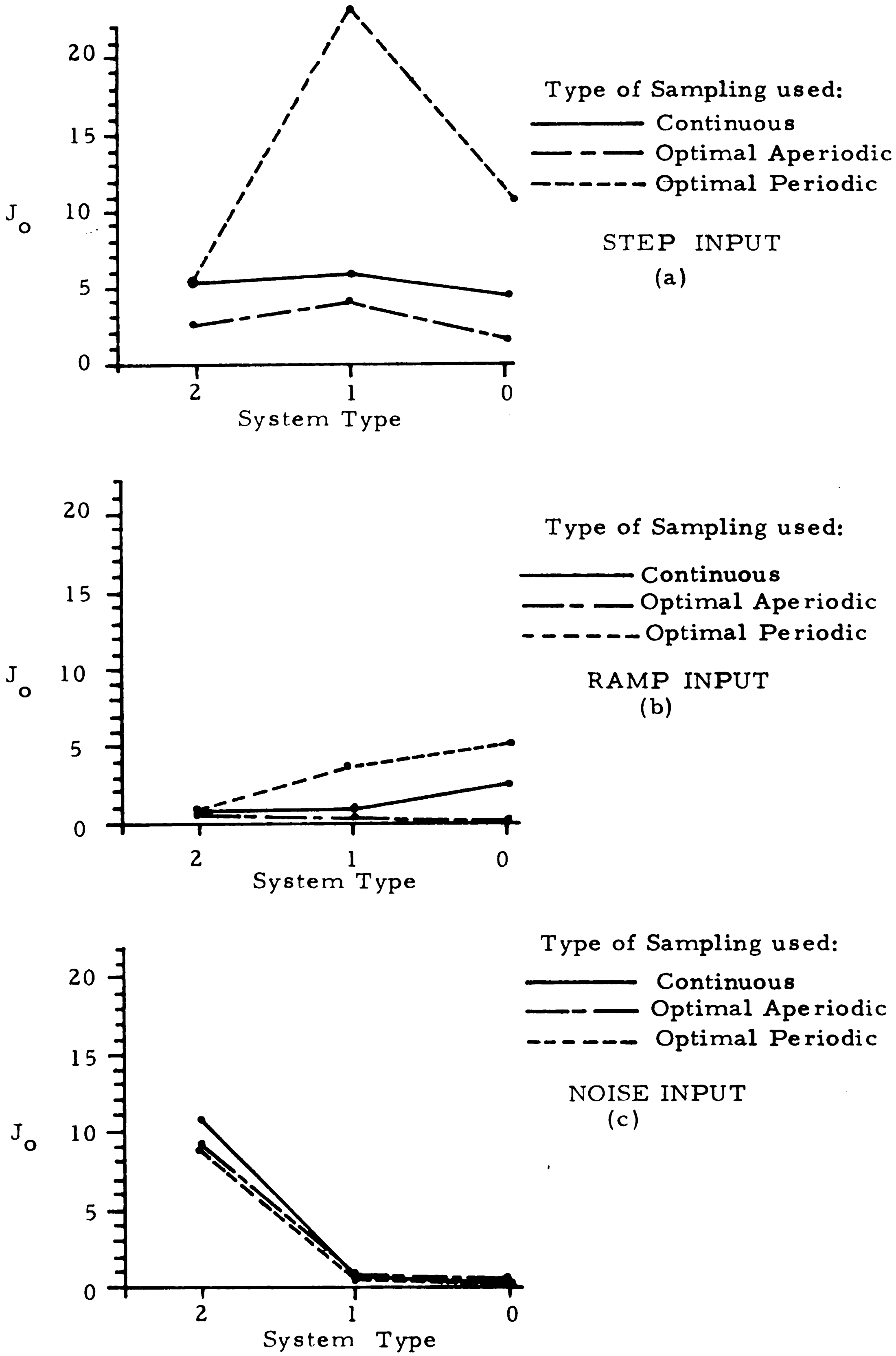


Figure 4.27. Optimal Performance Costs by System Type and Input Signal.

Step Input (Figure 4.27(b))

For a step input the Type 1 and Type 2 control system have no steady state errors compared to an inherent constant steady state error for the Type 0 system. Thus based on the performance cost results for the ramp input it was expected that performance costs would increase with a decrease in system type for the step input. Instead performance costs for the Type 1 control system exceeded those for the Type 2 and Type 0 systems for each method of sampling being considered. Therefore since the steady state errors for the Type 1 system are zero, the plant response time must be considered in explaining the large performance costs for the Type 1 system. Clearly from Figure 4.24 the Type 1 system responded slower over the chosen control interval to a step input than did the Type 2 and Type 0 systems. As can be seen in Figures 4.26 the response of the Type 1 system yields large terminal errors. Thus with the terminal errors being penalized heavily by the control performance index, costs increased as seen in Figure 4.27(b). The Type 0 system out-performed the Type 1 system even with its inherent steady state errors for a step input. The control performance cost reductions for the Type 0 system are due to the fact that the larger feedback control errors coupled with a faster system response over the control interval resulted in a reduction in terminal errors. For optimal aperiodic sampling the faster system response time for the Type 0 system along with optimal adjustment of the feedback error delays improved control performance for each input considered.

Noise Input (Figure 4.27(c))

The performance costs with respect to system disturbance noise decreases as system type decreases for each method of sampling investigated. This implies that control performance costs with respect to noise are dependent on system structure, and thus system type. (See the control performance index in equation 4.21)) Apparently the integration of noise drastically effects the value of the control performance index since performance costs decrease rapidly as the system type decreases. This is consistent with the analysis of sampling times for the different system types since it was found that sampling must be performed faster initially in the control interval for systems with integrators in order to control for the integration effect of the noise that occurs early in the control interval.

In summary, this section considered the open loop response times of the Type 2, 1, and 0 system plants subject to a unit step input signal. The lengths of the sampling intervals for N=5 were justified using the open loop plant response times for a step input in an effort to better explain system behaviour using the various methods of sampling under consideration. A summary of the optimal performance costs for each system type, as shown in Figure 4.27, were used in comparing optimal performance costs for various type systems for a given input.

In conclusion this chapter revealed that optimal aperiodic sampling, as specified in Chapter III, yielded improved control performance with respect to optimal periodic sampling and in several cases of continuous time control. If control performance costs alone are to be considered, especially for the slow responding Type 0 control systems with inherent terminal errors, optimal aperiodic sampling appears very promising. As in the case of signal representation, performance costs alone cannot be considered in the choice of the sampling method to employ. Therefore, the greater the knowledge the system designer has about the control problem at hand and the constraints he is required to work within the better the control system design will become.

V. CONCLUSIONS

This chapter summarizes the main results of this thesis and suggests areas of future research.

5.1. Summary

This thesis has developed a theory of optimal aperiodic sampling for signal representation and system control. The theory encompasses previous work in adaptive sampling, optimal periodic sampling, and aperiodic sampling in addition to presenting a cost for implementation model.

The theory of optimal aperiodic sampling considers the signal representation and system control problems to be different and are thus considered separately. Past studies have considered the signal representation and system control problems together under the assumption that good signal representation through sampling leads to good system control. This theory also develops a cost for implementation model which indicates the relative cost required to implement a particular signal representation or system control problem via the selected method of sampling. Previous studies of sampling methods have neglected or misrepresented implementation costs and thus have reduced the practical significance of the method of sampling being considered.

The theory which has been developed optimizes both the sampling interval sequence and the number of sampling intervals. Three methods of sampling were considered in the development of the optimal aperiodic sampling theory. Each sampling method studied optimized either the sampling interval lengths or the number of sampling intervals used but never considered joint optimization. Adaptive sampling was studied since it was the first non-periodic method of sampling which yielded improved performance over periodic sampling. Adaptive sampling techniques optimized the length of each sampling interval on an individual basis through use of a sampling rule obtained by using various approximations to the original integral performance index. Optimal periodic sampling was next studied since it optimized the number of periodically spaced sampling intervals. Finally work in aperiodic sampling which optimized the sampling interval sequence rather than optimizing the lengths of the individual sampling intervals (as was done in adaptive sampling) was investigated. Concepts from each of these methods of sampling were used in the development of the theory of optimal aperiodic sampling for signal representation and system control.

Therefore the theory of optimal aperiodic sampling provides a general framework for investigating various methods of sampling applied to either the signal representation or system control problem. Through use of the proper constraints on the number of sampling intervals or the length of each sampling interval, each method of sampling, viz. adaptive, optimal periodic, and aperiodic, can be investigated using the theory which has been developed for optimal aperiodic sampling. In addition a cost for implementation model is presented which indicates the relative implementation costs associated with the method of sampling being investigated.

For the signal representation problem the approach taken was to sample a given continuous time signal record non-periodically adjusting the length of each sampling interval to minimize a derived signal representation performance index. The derived signal representation performance index was made dependent on the signal model, initial conditions, input driving signals, and model disturbances. Adjustment of the sampling intervals were made via the ZXPOWL non-linear programming optimization subroutine found on the CDC 6500 computer system. For each adjustment made in the length of a sampling interval(s), a re-evaluation of the derived performance index was made until the minimization of the derived performance index was accomplished. The resulting sampling interval sequence yielded minimum sampling errors with respect to the performance index using the number of sampling intervals allowed and thus was used with the corresponding signal magnitudes to yield a piecewise constant representation of the original continuous time signal. The piecewise constant signal representation required approximately 10% of the storage needed for the continuous time signal record with sampling errors being reduced as much as 95% with respect to an equal number of periodically spaced sampling intervals.

A simplified remote display problem was proposed. The display problem investigation was used as a basis for the introduction of the processing and communications costs as they apply to implementation of various methods of sampling. These costs for implementation were added to the performance costs obtained through minimization of the derived performance index to yield a total performance level reflecting improvements in signal representation through aperiodic sampling and at the same time indicating the relative costs for these improvements in the quality of the signal representation.

Two main conclusions can be made as a result of the sampling for signal representation investigations conducted in Chapter IV, Section 4.2. First, minimization of the derived signal representation performance index, as found in (2.24), through adjustment of the length of each sampling interval can substantially reduce the sampling errors present in the piecewise constant representation of the continuous time signal record with respect to periodic or adaptive sampling. Second, obtaining of the optimal sampling interval lengths, and hence the best piecewise constant representation using the specified number of sampling intervals, results in increased computational expense and complexity compared to adaptive or periodic sampling. Thus in each situation where continuous time signal records are to be sampled, the system designer should consider each sampling alternative at his disposal.

The control implementation problem was based on the premise that periodic sampling need not be used to implement feedback control systems designed through use of continuous time techniques. As a result, sampling for signal representation was extended to the sampling of the continuous time feedback error signal generated by the system control law. Again as in the case of signal representation, a performance index was proposed (see (3.16)). The control implementation performance index resembled conventional optimal control performance indices in that it penalized for control energy expenditures, tracking deviations, and final state errors. However, the control implementation index also included a cost of implementation term. The performance index was then derived in terms of the sampling interval sequence, $\underline{T}$ and the number of sampling intervals, N. The derived performance index was minimized through use of the ZXPOWL non-linear programming optimization subroutine found on the CDC 6500 computer system. Each attempt at minimization required evaluation of the derived control implementation performance index through simulation of the control system using a specified sampling interval sequence. Thus given the required system information as indicated in Chapter IV, Section 4.3, control of various feedback systems were investigated with respect to sampling of the feedback control error signal.

The results of Chapter IV, Section 4.3, indicate that sub-optimal aperiodic sampling of the continuous time feedback control error signal can improve control performance, based on the derived control implementation performance index, compared to periodic sampling and some cases of continuous time control. Control performance improvements were more apparent as the bandwidth of the closed loop unity feedback system decreased and the system type numbers were reduced. Computational cost requirements necessary to obtain improved control performance through use of sub-optimal aperiodic sampling increased with the number of sampling intervals being optimized such that sampling interval optimizations for a large number of intervals become computationally prohibitive. Using as few sampling intervals as possible yielded totally unacceptable periodic sampling performance while sub-optimal aperiodic sampling was able to maintain reasonable system stability through adjustment of sampling delays as discussed in detail in Chapter IV, Section 4.3.

5.2. Future Research

In both the signal representation and control implementation problems the signal which is to be sampled needs to be known as in the case of signal representation or generated through control system simulation for the control implementation problem. For on-line applications where the continuous time signal record or feedback control error signal are not known a priori further research into on-line signal prediction, estimation, and system modelling is necessary.

As indicated throughout this thesis optimization of the lengths of each sampling interval is necessary to maximize signal representation or system control performance through use of aperiodic sampling. As was shown in Chapter IV, aperiodic sampling improves performance but the computational expenditures necessary for optimization made actual implementation unlikely. Therefore investigation into more computationally efficient optimization routines, or possible adaptive/ aperiodic hybrid sampling could be further studied.

LIST OF REFERENCES

LIST OF REFERENCES

[1] M. L. Smith, Jr., "An Evaluation of Adaptive Sampling," IEEE Trans. on Automatic Control, Vol. AC-16, pp. 282-284, June 1971.

[2] M. Athans, W. S. Levine, and A. H. Levis, "A System for the Optimal and Suboptimal Position and Velocity Control for a String of High-Speed Vehicles," 5th International Analogue Computation meetings, 1966.

[3] M. Aoki and M. T. Li, "Optimal Discrete-Time Control System with Cost for Observation," IEEE Trans. on Automatic Control, Vol. AC-14, pp. 165-175, April 1969.

[4] R. A. Volz and L. F. Kazda, "Design of a Digital Controller for a Tracking Telescope," IEEE Trans. on Automatic Control, Vol. AC-12, pp. 359-367, April 1967.

[5] R. C. Dorf, M. C. Farren, and C. A. Phillips, "Adaptive Sampling for Sampled-Data Control Systems," IRE Trans. on Automatic Control, Vol. AC-7, pp. 38-47, January 1962.

[6] R. Tomovic and G. A. Bekey, "Adaptive Sampling Based on Amplitude Sensitivity," IEEE Trans. on Automatic Control, Vol. AC-11, pp. 282-284, April 1966.

[7] D. Tabak, "Optimization of Nuclear Reactor Fuel Recycle Via Linear and Quadratic Programming," IEEE Trans. on Nuclear Science, Vol. NS-15, pp. 60-64, 1968.

[8] D. Ciscator and L. Mariani, "On Increasing Sampling Efficiency by Adaptive Sampling," IEEE Trans. on Automatic Control, Vol. AC-12, p. 318, June 1967.

[9] A. W. Bennett and A. P. Sage, "Discrete System Sensitivity and Variable Increment Sampling," Proceedings Joint Automatic Control Conference, pp. 603-612, June 1967.

[10] K. E. Tait, "Evaluation of Signal Dependent Sampling," International Journal on Control, Vol. 4, No. 4, pp. 201-239, 1966.

[11] K. E. Tait, "Further Sampling Interval Criteria and their Evaluation in Discrete-Continuous Feedback Control Systems," International Journal on Control, Vol. 4, pp. 297-324, 1966.

[12] A. Sano and M. Terao, "Measurement Optimization in Optimal Process Control," Automatica, Vol. 6, pp. 705-714, Pergamon Press, 1970, Printed in Great Britain.

[13] I. Troch, "Sampling with Arbitrary Choice of the Sampling Instants," Automatica, Vol. 9, pp. 117-124, Pergamon Press, 1973, Printed in Great Britain.

[14] J. R. Mitchell and W. L. Mc Daniel, Jr., "Adaptive Sampling Technique," IEEE Trans. on Automatic Control, Vol. AC-14, pp. 200-201, April 1969.

[15] S. C. Gupta, "Increasing the Sampling Efficiency for a Control System," IEEE Trans. on Automatic Control, (Correspondence), Vol. AC-8, pp. 263-264, July 1963.

[16] R. Tamovic and G. A. Bekey, "Adaptive Sampling Based on Amplitude Sensitivity," IEEE Trans. on Automatic Control, (Short Paper), Vol. AC-11, pp. 282-284, April 1966.

[17] G. A. Bekey and R. Tamovic, "Sensitivity of Discrete Systems to Variation of Sampling Intervals," IEEE Trans. on Automatic Control, (Short Paper), Vol. AC-11, pp. 284-287, April 1966.

[18] E. Tse, "On the Optimal Control of Stochastic Linear Systems," IEEE Trans. on Automatic Control, Vol. AC-16, No. 6, December 1971.

[19] R. A. Schlueter, "The Optimal Linear Regulator with Constrained Sampling Times," IEEE Trans. on Automatic Control, Vol. AC-18, pp. 515-518, October 1973.

[20] A. H. Levis, M. Athans, and R. A. Schlueter, "On the Behavior of Optimal Linear Sampled-Data Regulators," International Journal on Control, Vol. 13, pp. 343-361, February 1971.

[21] R. A. Schlueter and A. H. Levis, "The Optimal Linear Regulator with State-Dependent Sampling," IEEE Trans. on Automatic Control, Vol. AC-18, pp. 512-515, October 1973.

[22] L. Meier III, R. E. Larson, and A. J. Tether, "Dynamic Programming for Stochastic Control of Discrete Systems," IEEE Trans. on Automatic Control, Vol. AC-16, pp. 767-775, December 1971.

[24] T. C. Hsia, "Analytic Design of Adaptive Sampling Control Law in Sample-Data Systems," IEEE Trans. on Automatic Control, Vol. AC-19, pp. 39-42, February 1974.

[25] P. D. Joseph and J. T. Tou, "On Linear Control Theory," AIEE Transactions, pp. 193-196, September 1961.

[26] G. L. Gunckel and G. F. Franklin, "A General Solution for Linear, Sampled-Data Control," ASME Journal of Basic Engineering, pp. 197-203, June 1963.

[27] C. M. Ermer and D. V. Vandelinde, "Output Feedback Gains for a Linear-Discrete Stochastic Control Problem," IEEE Trans. on Automatic Control, Vol. AC-18, pp. 154-157, April 1973.

[28] R. E. Kalman and J. E. Bertram, "A Unified Approach to the Theory of Sampling Systems," Proceedings of the IRE, Vol. 46, 1958.

[29] H. D. Young, Statistical Treatment of Experimental Data, McGraw-Hill Book Co., Inc., New York, 1962.

[30] C. E. Shannon and W. Weaver, The Mathematical Theory of Communications, The University of Illinois Press, Urbana, 1949.

[31] D. Middleton, Introduction to Statistical Communications Theory, McGraw-Hill Book Company, Inc., New York, 1960.

[32] W. D. Hershberger, Principles of Communication Systems, Prentice-Hall, Inc., New York, 1955.

[33] R. M. Fano, Transmission of Information, A Statistical Theory of Communications, The M.I.T. Press and John Wiley and Sons, New York, 1961.

[34] 1964 Joint Automatic Control Conference, Preprints of papers, Stanford University, Stanford, California.

[35] D. J. Sakrison, Communications Theory: Transmissions of Waveforms and Digital Information, John Wiley and Sons, Inc., New York, 1968.

[36] J. L. Yen, "On Nonuniform Sampling of Bandwidth-Limited Signals," IRE Trans. on Circuit Theory, CT-3, p. 252, 1956.

[37] J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering, John Wiley and Sons, Inc., New York, 1965.

[38] M. Schwartz, R. Bennett, and S. Stein, Communication Systems and Techniques, McGraw-Hill Book Co., New York, 1966.

[39] W. J. Hemmerte, Statistical Computations on a Digital Computer, Blaisdell Publishing Co., Waltham, Mass., 1967.

[40] H. J. Kushner, "On the Optimal Timing of Linear Control Systems with Unknown Initial State," IEEE Trans. on Automatic Control, Vol. AC-9, pp. 144-150, April 1964.

[41] P. H. Ellis, "Extension of Phase Plane Analysis to Quantized Data Systems," IRE Trans. on Automatic Control, No. 2, pp. 43-55, November 1959.

[42] B. D. Anderson and J. B. Moore, Linear Optimal Control, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[43] D. E. Kirk, Optimal Control Theory, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1970.

[44] R. J. Schwartz and B. Friedland, Linear Systems, McGraw-Hill, Inc., New York, 1965.

[45] R. A. Volz, "Optimal Control of Discrete Systems with Constrained Sample Times," Proceedings Joint Automatic Control Conference, pp. 16-21, June 1967.

[46] B. W. Jorden and E. Polak, "Optimal Control of Aperiodic Discrete-Time Systems," J. SIAM Control, Ser. A, Vol. 2, No. 3, pp. 332-345, 1965.

[47] B. C. Kuo and D. Tabak, "A Nonlinear Programming Approach to Optimal Control of Discrete Systems with Constrained Sampling Times," Proceedings Joint Automatic Control Conference, pp. 918-922, 1968.

[48] P. Dorato and A. H. Levis, "Optimal Linear Regulators: The Discrete-Time Case," IEEE Trans. on Automatic Control, Vol. AC-16, pp. 613-620, December 1971.

[49] J. M. Mendel, "Computational Requirements for a Discrete Kalman Filter," IEEE Trans. on Automatic Control, Vol. AC-16, pp. 748-758, December 1971.

[50] R. Saucedo and E. E. Schiring, Introduction to Continuous and Digital Control Systems, Macmillian Co., New York, 1968.

[51] R. H. Caron, "On the Computational Complexity of Certain L-Relation Operators," Ph.D. Dissertation, Department of Electrical Engineering and Systems Science, East Lansing, Mich., 1973.

[52] K. T. Ma, "On the Optimal Sample-Data Tracking Problem," Ph. D. Dissertation, Dept. of Electrical Engineering and Systems Science, East Lansing, MI, 1975.

[53] International Mathematical and Statistical, Inc., "The IMSL Library 3 Reference Manual," Houston, Texas, First Edition, May, 1972.

APPENDICES

APPENDIX A

THE SAMPLE AND HOLD OPERATION

The operation of a sample and hold mechanism involves two phases. The first phase is the measurement of the continuous time signal magnitude at the desired sampling instant. The figure below shows a continuous signal e(t), with values indicated at various times, t_i.

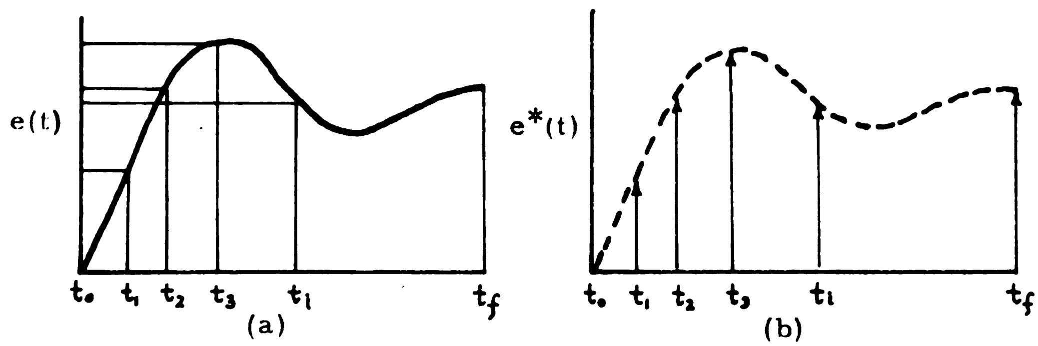


Figure A-1. Signal Sampling using Impulses.

The sequence of signal values at the sampling times as shown in Figure A-1 (a) may be represented by a series of impulses as shown in Figure A-1 (b) with the strengths of the impulses being equal to the magnitude of e(t) at the corresponding sampling time. Thus the series of impulses as shown in Figure A-1 discretize the original signal e(t).

$$e(t) \approx e^{*}(t) = \sum_{i=0}^{N-1} e(t_i)\, \delta\, (t - t_i) \qquad 0 \le t \le t_{final}\,,\ N < \infty$$

The term $e^*(t)$ is called the sampled signal equaling the continuous time signal at instants $t = t_i$. Whether $e^*(t)$ represents values of the continuous signal e(t) at each sampling instant, or if $e^*(t)$ is actually a series of impulses at $t_{i+1} - t_i$ length intervals, the impulse representation may be thought of as a switch which closes instantaneously at the beginning of each interval. If an infinite number of samples were taken the continuous time signal could be represented exactly at any given point by the magnitude of the appropriate impulse function. In all practical applications this is clearly impossible and thus a finite number of samples are used. The magnitude of each impulse can then be quantized, stored, and later used for various calculations as required.

After each sample (impulse) is obtained the second, or holding phase of the sample and hold operation occurs. The "hold" operation maintains a constant signal level equal to $e(t_i)$ until $t = t_{i+1}$ at which time sampling again occurs. Figure A-2 (b) indicates the piecewise constant signal used to approximate the continuous time signal e(t) found in Figure A-1 (a). The holding operation as depicted in Figure A-2(b) is

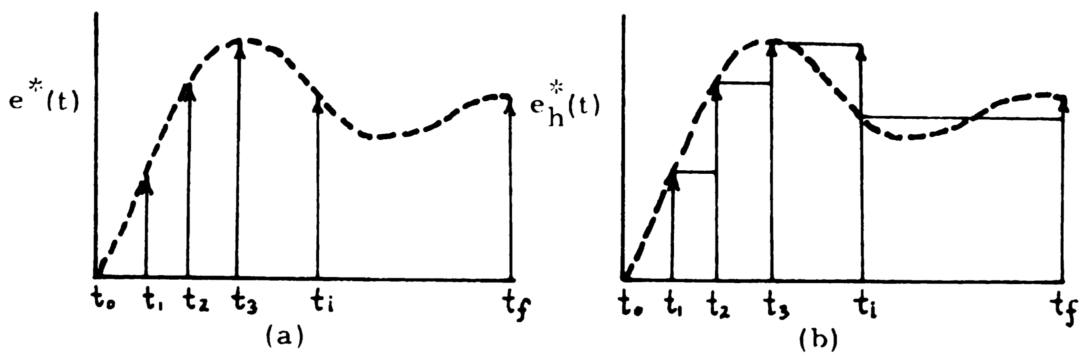


Figure A-2. The Impulse Holding Operation.

more formally called a zero-order-hold as opposed to a nth order hold which approximates the signal using a polynomial curve fit between the data points where the degree of the polynomial is proportional to n.

Thus the terms "sampling" or "sample and hold" will indicate the representation of signal, as in Figure A-1(a), by piecewise constant levels as indicated in Figure A-2(b).

APPENDIX B

PERIODIC, ADAPTIVE, AND APERIODIC SAMPLING

The "sampling" of a continuous time signal, as described in Appendix A, using equally spaced sampling times is known as periodic sampling as shown in Figure B-1.

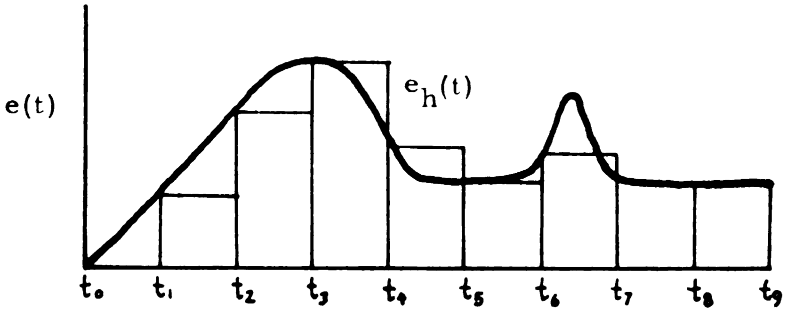


Figure B-1. Periodic Sampling

The sampling interval is defined for Figure B-1 as

$$T_i = t_{i+1} - t_i \qquad i = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

where for periodic sampling

$$T_o = T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = T_8 .$$

Periodic sampling as shown in Figure B-1 is entirely independent of signal changes within a sampling interval. The piecewise constant signal $e_h(t)$ is determined only by the magnitude of the original continuous time signal at the sampling instants, t_i.

The "sampling" of a signal on a non-periodic basis can be a result of using adaptive or aperiodic sampling rules. Adaptive sampling is a method of sampling where the length of the sampling interval changes as a result of the information received about the signal (e.g., signal magnitude, signal derivative) at the end of the previous sampling interval.

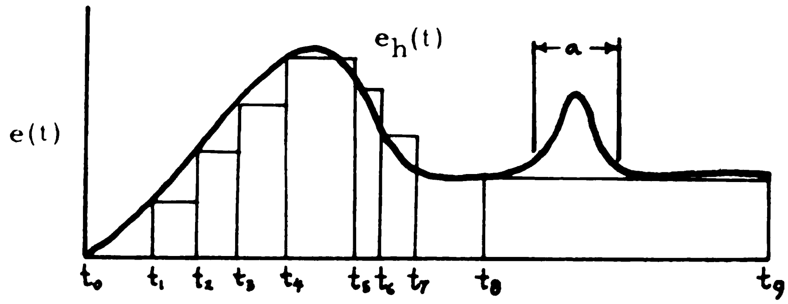


Figure B-2. Adaptive Sampling

Consider a typical adaptive sampling rule,

$$T_i = 1.0/|\dot{e}(t_i)| \qquad i = 0, 1, 2, 3, 4, 5, 6, 7, 8.$$

As can be seen in Figure B-2 the lengths of the sampling intervals will vary according to the inverse of the absolute value of the continuous time derivative at the sampling time, t_i. For the example in Figure B-2 no minimum or maximum sampling interval length constraints were imposed, thus with a signal derivative approaching zero the sampling interval becomes very large as seen in Figure B-2. During this interval the signal variation (i.e., during interval "a") was entirely neglected. Imposing sampling interval constraints with T_{max} being very small would reduce sampling errors of the kind found during interval "a", but could drastically increase the number of samples required during the time interval of interest.

Aperiodic sampling, in general, also yields non-periodic sampling intervals as does adaptive sampling. In contrast to adaptive sampling where each sampling interval is determined on an individual basis, aperiodic sampling considers a specified number of variable length sampling intervals having a combined length equal to $t_{final} - t_{initial}$.

Therefore given an initial sampling interval sequence, aperiodic sampling varies the length of each sampling interval to minimize a predetermined sampling error performance index while maintaining the sum of the sampling interval lengths equal to $t_{final} - t_{initial}$. Adjusting each sampling interval based on minimization of sampling errors rather than signal derivatives will hopefully decrease the possibility of large sampling errors from going undetected as seen in Figure B-2. Figure B-3 shows a possible sampling interval sequence which would detect the signal variation during the interval "a". This sampling interval sequence could be obtained through use of aperiodic sampling assuming that the sampling interval sequence shown in Figure B-3 yielded lower sampling errors than the sampling intervals in Figure B-2.

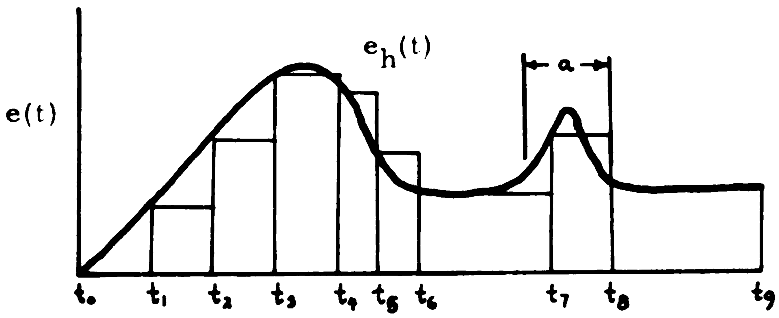


Figure B-3. Aperiodic Sampling

APPENDIX C

NUMERICAL DATA FOR FIGURES FOUND IN CHAPTER IV

The data presented in this Appendix accurately presents the various performance cost values as graphically presented in Chapter IV. The data found in this Appendix is arranged by Figure number.

Figure 4.3(a)

N	Periodic	Adaptive	Aperiodic
2	1.158	1.330	.074
3	.525	.696	.042
4	.264	.294	.041
5	.136	.158	.013
6	.082	.082	.008
7	.045	.056	.004
8	.036	.036	.003

Figure 4.3(b)

N	Periodic	Adaptive	Aperiodic
2	1.623	1.797	.828
3	1.066	1.313	1.040
4	.881	1.061	1.392
5	.828	1.075	1.521
6	.850	1.150	1.739
7	.887	1.273	2.100
8	.953	1.403	2.280

Figure 4.4(a)

N	Periodic	Adaptive	Aperiodic
2	2.315	1.610	.240
3	1.590	.326	.239
4	1.055	.134	.092
5	.681	.073	.053
6	.485	.088	.040
7	.323	.236	.060
8	.276	.182	.056

Figure 4.4(b)

N	Periodic	Adaptive	Aperiodic
2	2.780	2.079	.993
3	2.131	.942	1.613
4	1.671	.902	1.037
5	1.372	1.016	1.201
6	1.251	1.156	1.360
7	1.164	1.489	1.943
8	1.193	1.590	2.735

Figure 4.5(a)

N	Periodic	Adaptive	Aperiodic
2	4.631	4.822	4.246
3	4.818	1.909	1.272
4	4.220	1.258	.764
5	3.403	.788	.580
6	2.852	.638	.522
7	2.304	.814	.538
8	2.124	.767	.522

Figure 4.5(b)

N	Periodic	Adaptive	Aperiodic
2	5.097	5.289	5.131
3	5.359	2.526	2.349
4	4.837	2.025	2.178
5	4.095	1.705	2.590
6	3.619	1.702	2.542
7	3.146	2.030	3.057
8	3.040	2.134	2.810

Figure 4.6

(a) Step Input, Sub-Optimal Aperiodic,

a=0 [.0346, .0530, .0617, .3161, .5345]
a=-1 [.0830, .1037, .1888, .0776, .5469]
a=1 [.0400, .0428, .0251, .8283, .0635]

(b) Step Input, Adaptive,

a=0 [.0500, .0500, .0600, .1200, .7200]
a=-1 [.2100, .2100, .2100, .2100, .1600]
a=1 [.0400, .0500, .0700, .2500, .5900]

Figure 4.7(a)

N	Periodic	Adaptive	Aperiodic
2	.563	.463	.061
3	.376	.081	.061
4	.243	.029	.024
5	.155	.060	.022
6	.110	.013	.010
7	.073	.022	.012
8	.062	.017	.008

Figure 4. 7(b)

N	Periodic	Adaptive	Aperiodic
2	1.029	.930	.801
3	.917	.698	1.050
4	.859	.796	.989
5	.846	.977	1.589
6	.876	1.111	1.347
7	.915	1.323	2.161
8	.979	1.384	1.700

Figure 4. 8(a)

N	Periodic	Adaptive	Aperiodic
2	.563	.430	.058
3	.388	.089	.058
4	.255	.033	.022
5	.167	.028	.015
6	.119	.069	.033
7	.079	.029	.012
8	.068	.011	.007

Figure 4. 8(b)

N	Periodic	Adaptive	Aperiodic
2	1.028	.897	.737
3	.929	.706	1.063
4	.876	.801	.972
5	.859	.973	1.116
6	.886	1.137	1.458
7	.921	1.283	2.107
8	.985	1.380	1.810

Figure 4.9(a)

N	Periodic	Aperiodic
2	.426	.367
3	.207	.172
4	.118	.099
5	.077	.061
6	.053	.046
7	.042	.035
8	.030	.028

Figure 4.9(b)

N	Periodic	Aperiodic
2	.892	1.061
3	.748	1.225
4	.735	1.335
5	.768	1.220
6	.820	1.442
7	.885	1.566
8	.948	1.819

Figure 4.10

(a) Ramp Input, a=0,

Adaptive	[.0500, .0500, .0600, .1200, .7200]
Sub-Opt. Aper.	[.0648, .0796, .2539, .1029, .4986]

(b) Parabola Input, a=0,

Adaptive	[.0500, .0600, .0900, .3300, .4700]
Sub-Opt. Aper.	[.0296, .0383, .0661, .3702, .4959]

(c) Noise Input, a=0,

Periodic	[.2000, .2000, .2000, .2000, .2000]
Sub-Opt. Aper.	[.3048, .2000, .1828, .1696, .1428]

Figure 4.12(a)

N	Periodic	Aperiodic
2	$1.3x10^7$	140.4
3	$4.5x10^7$	13.54
4	$1.6x10^7$	11.93
5	$2.0x10^5$	10.72
6	1082.0	7.10
7	55.4	7.89
8	8.4	4.95

Figure 4.12(b)

N	Periodic	Aperiodic
2	$1.3x10^7$	143.43
3	$4.5x10^7$	16.51
4	$1.6x10^7$	15.08
5	$2.0x10^5$	14.97
6	1082.20	12.72
7	55.50	13.53
8	8.51	8.14

Figure 4.13(a)

N	Periodic	Aperiodic	Sig. Rep.
2	45779.	21.87	1352.8
3	223.0	7.29	301.1
4	5.59	4.19	16.32
5	5.30	2.50	4.11
6	7.01	2.74	5.31
7	7.70	2.51	10.36
8	7.80	2.34	6.47

Figure 4.13(b)

N	Periodic	Aperiodic	Sig. Rep.
2	49779.4	23.57	1353.2
3	223.2	9.70	301.3
4	5.68	7.44	16.41
5	5.39	6.68	4.21
6	7.11	5.40	5.41
7	7.80	7.89	10.46
8	7.97	7.38	6.64

Figure 4.14(a)

N	Periodic	Aperiodic
2	6.78	6.59
3	7.20	4.68
4	8.55	4.55
5	9.35	4.54
6	9.84	4.50
7	10.17	4.52
8	10.40	5.74

Figure 4.14(b)

N	Periodic	Aperiodic
2	6.86	7.90
3	7.28	6.52
4	8.64	7.14
5	9.44	8.02
6	9.94	9.28
7	10.27	12.53
8	10.50	15.32

Figure 4.15

System Response Time, Step Input, Tyep 2 Systems:

FAST	[.0852, .0080, .5010, .0184, .3881]
MEDIUM	[.1992, .1139, .2962, .1469, .2437]
SLOW	[.4192, .0845, .0516, .0704, .3742]

Figure 4.16(a)

N	Periodic	Aperiodic
2	8938.5	3.91
3	24.00	1.50
4	.990	.764
5	1.58	.519
6	1.83	.510
7	1.85	.397
8	1.79	.395

Figure 4.16(b)

N	Periodic	Aperiodic
2	8938.7	5.72
3	24.09	3.95
4	1.08	6.53
5	1.68	4.31
6	1.93	6.42
7	1.94	6.58
8	1.89	7.80

Figure 4.17(a)

N	Periodic	Aperiodic
2	8.92	8.70
3	9.15	8.71
4	9.30	8.71
5	9.39	8.73
6	9.47	8.73
7	9.52	8.73
8	9.56	8.76

Figure 4.17(b)

N	Periodic	Aperiodic
2	9.20	14.69
3	9.43	17.89
4	9.58	19.12
5	9.68	24.46
6	9.75	26.64
7	9.81	28.21
8	9.87	29.36

Figure 4.18(a)

N	Periodic	Aperiodic
2	113.70	13.45
3	112.66	7.69
4	77.24	6.83
5	51.50	7.39
6	36.65	5.13
7	27.99	4.10
8	22.62	3.79

Figure 4.18(b)

N	Periodic	Aperiodic
2	113.79	14.92
3	112.76	9.50
4	77.34	9.10
5	51.60	10.71
6	36.75	9.04
7	28.08	8.45
8	22.72	6.48

Figure 4.19(a) Continuous = 1.051

N	Periodic	Aperiodic
2	5.88	1.18
3	12.10	.613
4	10.70	.441
5	8.00	.656
6	6.04	.412
7	4.77	.288
8	3.94	.292

Figure 4.19(b)

N	Periodic	Aperiodic
2	5.97	2.37
3	12.20	1.55
4	10.79	3.02
5	8.09	3.99
6	6.14	3.89
7	4.87	5.31
8	4.04	5.68

Figure 4.20(a)

N	Periodic	Aperiodic
2	.3760	.3696
3	.3837	.3752
4	.3888	.3853
5	.3922	.3906
6	.3947	.3938
7	.3965	.3961
8	.3978	.3974

Figure 4.20(b)

N	Periodic	Aperiodic
2	.8031	7.268
3	.8373	10.978
4	.8271	6.186
5	.8429	7.906
6	.8467	9.544
7	.8548	11.329
8	.8628	13.565

Figure 4.21(a) Continuous = 4.261

N	Periodic	Aperiodic
2	52.08	4.57
3	47.87	3.86
4	33.47	2.13
5	23.65	2.57
6	17.91	2.08
7	14.46	1.75
8	12.24	1.64

Figure 4.21(b)

N	Periodic	Aperiodic
2	52.17	6.05
3	47.96	5.80
4	33.57	4.57
5	23.75	5.84
6	18.02	6.30
7	14.56	3.89
8	12.35	7.13

Figure 4.22(a) Continuous = 2.755

N	Periodic	Aperiodic
2	10.37	.379
3	12.69	.384
4	10.63	.300
5	8.51	.280
6	7.06	.253
7	6.10	.232
8	5.45	.225

Figure 4.22(b)

N	Periodic	Aperiodic
2	10.46	2.118
3	12.78	2.271
4	10.73	2.864
5	8.60	2.711
6	7.16	4.358
7	6.21	5.205
8	5.56	5.767

Figure 4.23(a)

N	Periodic	Aperiodic
2	.1944	.1903
3	.1971	.1951
4	.1991	.1986
5	.2003	.2001
6	.2012	.2010
7	.2019	.2018
8	.2023	.2023

Figure 4.23(b)

N	Periodic	Aperiodic
2	.5801	7.093
3	.5914	4.126
4	.5981	5.776
5	.6006	6.901
6	.6164	8.079
7	.6330	10.372
8	.6355	11.927

Figure 4.25

MEDIUM Response Time System, Step Input.

Type 2 [.1992, .1139, .2962, .1469, .2437]
Type 1 [.0841, .0345, .0863, .2293, .5658]
Type 0 [.0896, .0338, .0818, .1951, .5995]

MEDIUM Response Time System, Ramp Input.

Type 2 [.2339, .1046, .2763, .1409, .2443]
Type 1 [.0884, .0439, .0950, .2430, .5268]
Type 0 [.0806, .0327, .0828, .1573, .6466]

MEDIUM Response Time System, Noise Input.

Type 2 [.0542, .0709, .0819, .0891, .7038]
Type 1 [.1823, .1879, .1975, .2025, .2297]
Type 0 [.1924, .1947, .1995, .2056, .2076]