PERFORMANCE CHARACTERISTICS OF A SYSTEM AS A FUNCTION OF ITS STRUCTURE

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ABSTRACT

PERFORMANCE CHARACTERISTICS OF A SYSTEM
AS A FUNCTION OF ITS STRUCTURE

by Henry F. Williams

A system is a collection of component equations and constraint (or graph) equations describing component interconnections. These two classes of equations together make up what is called the system structure. A solution to the system is a function that satisfies the system equations. This thesis examines the solution characteristics of existence, uniqueness, and stability, as they relate to the system structure.

Grassman Algebra and Matroid Theory provide the principal mathematical tools by which the results are obtained.

The principal results are:

1. A theorem providing necessary and sufficient conditions for a linear monotonic system of n-port components to have a unique solution;
2. A generalization of system theory to Hilbert spaces with two existence theorems;

3. The most general existence and uniqueness theorem yet published for non-linear n-port component systems;
4. Explicit relations between system structure and stability of its solutions for a wide variety of systems;
5. The determination of algorithms for the system determinants of linear time-invariant systems. These algorithms show explicitly in one equation the relation between the system determinant and structure;
6. The algorithms in (5) also show the usefulness of Grassman Algebra in a new, simplified, and generalized topological analysis technique.

PERFORMANCE CHARACTERISTICS OF A SYSTEM AS A FUNCTION OF ITS STRUCTURE

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CHAPTER I

INTRODUCTION

Modern system theory can be described as the discipline which seeks to determine the performance characteristics of interconnected components from a knowledge of their patterns of interconnection and component characteristics. In lumped parameter system theory where a finite number of components are interconnected, the circuit and cutset equations obtained from a linear graph, adequately describe the interconnections of components. The components themselves are described by a set of equations relating the thru and across variables [FR-1]. These component equations may be algebraic, integral, differential, or of the functional analysis type.

Some performance characteristics that are frequently examined are existence and uniqueness of a solution, stability, optimization with respect to some criteria, or for linear systems, location of characteristic frequencies.

This thesis examines the performance characteristics of existence, uniqueness and stability of a solution as a direct function of the system structure - namely the component equations and their pattern of interconnection.

Various authors have examined system existence and uniqueness, but in a much more restrictive sense than that used in this thesis. (See [DU-1], [DU-2], [DU-3], [DU-4], [BI-1], [MI-1], [WI-1], [DE-1].) For instance, no one, to the author's knowledge, has given a theorem similar to Theorem 3-1-3 which essentially reduces the problem of existence and uniqueness to that of a subset of the component equations.

Previous stability studies have been mostly restricted to the use of energy functions or analysis of state models which give little specific information regarding system structure. However this thesis emphasizes this aspect of the system's problem as it relates to both the system analysis and synthesis.

In linear systems, the algorithms of Chapters III and V provide very useful equations relating system structure to network functions and determinants. These results provide a useful basis for realizing a best fit to a desired system response in terms of either the parameters in the component equations or alteration of the component interconnections.

The techniques are also novel in as much as they, for the first time, make extensive use of Grassman algebra and matroid theory in examining performance characteristics.

In Chapter II, various preliminary concepts regarding matroid theory, Grassman algebra, component equations and graph theory are explained and some original theoretical results for use in the remainder are derived.

Chapter III examines how the property of existence and uniqueness of a solution affects the interconnections of a given set of components. First the linear time-stationary systems are examined. Methods are devised which give the class of all interconnections that yield a unique solution. Several necessary and sufficient conditions are given for restricted classes of components to have a unique solution. The same problem is then formulated for the non-linear systems, including mappings in Hilbert space. Some sufficient conditions on the topology for a unique solution are given.

Chapter IV contains results on stability. Here various types of stability of systems of components are examined from the standpoint of the central problem: namely, given a collection of components, determine the class of interconnections that yield a stable solution. Again both linear and non-linear equations are examined, and the results given in the theorems shed light on the stability problem.

Chapter V applies the results of Chapters III and IV, to several examples and shows the usefulness and generality of the Grassman algebra technique over classical topological analysis techniques.

CHAPTER II

MATHEMATICAL PRELIMINARIES

The purpose of Chapter II is to define the pertinent notation for linear graphs and components,and to provide a characterization of the cutset and circuit space of a graph. The unstarred theorems and lemmas herein proved are the author's original results. However, Theorem 2-2-12 and the techniques of Grassman algebra are not to determine whether a given vector subspace is graphic. The construction method described in [TU-5] is much simpler and straightforward for this purpose. These are derived and used for the purpose of describing graphic vector subspaces. They are also used in Chapter III.

Theorem 2-1-6 is a new result in the theory of Grassman algebra and useful in the algorithm of Chapter III. Grassman algebra has a direct practical application in topological analysis as described in Chapter V.

The rather extensive discussion on matroid theory is given to provide mathematical foundation for the application of Tuttes matroid results to real vector spaces. To the author's knowledge this foundation is missing from the literature and is accomplished here for the first time. Tuttes concepts of chains of integers have been extended to chains of real numbers and the concept of a chain group to a chain vector space. However, many of Tuttes results carried over intact in this extension.

In passing, it should be mentioned that theorems on graphic (mod 2) vector spaces which follow directly from Tuttes results, are given without proof in [SE-1] and [KI-1]. A more precise and practical statement of the conditions on Kuratowski matroid minors is given in this thesis.

Part I. Grassman Algebra

Definition 2-1-1: A linear algebra over a field F is a set which is a finite dimensional vector space over F and which admits an associative and bi-linear multiplication.

Definition 2-1-2: A linear algebra G over a field F which contains the finite dimensional vector space V over F is a Grassman Algebra over V if

1. G contains a multiplicative identity element, e_o
2. G is generated by e_o and V
3. If x is in V, $x^2 = 0$
4. The dimension of G (as a vector space) is 2^n. (n = dimension V).

The associative multiplication of the algebra will be called progressive multiplication.

Lemma 2-1-1: If $g, x \in V$, then $xg = -gx$.

Proof: $(x + g) \in V$, $(x + g)^2 = x^2 + gx + xg + g^2 = 0$
But $x^2 = g^2 = 0$, therefore $gx = -xg$.

---

*Lemma 2-1-2: Any two Grassman algebras G and G^1 over the same vector space V are isomorphic.

Proof: See [MAC-1].

Let G be any Grassman algebra over V. G contains a unique identity element e_o and all its scalar multiples ae_o. Identify each scalar $a \in F$ with the corresponding multiples ae_o, therefore $e_o = 1$.

(Throughout the rest of this thesis, assume F is the real numbers.)

Select any ordered basis $e_1, \ldots, e_n$ for V. Then G contains all products of the various e_i's. If $P = (i_1, \ldots i_p)$ is a set of indices (a subset of $(1, \ldots, n)$, arranged in increasing order), write

$$e_P = \left[e_{i_1} e_{i_2} \cdots e_{i_p} \right] \qquad (2\text{-}1\text{-}1)$$

Since $e_i \in V$, $e_i^2 = 0$, $e_i e_j = -e_j e_i$. Using these rules any product of e's can be arranged so that it either has the form (2-1-1) with increasing subscripts or is zero. Since any vector of V is a linear combination of the basis elements, it follows by the distributive law that any product of vectors of V is a linear combination of the elements e_P. Since G is generated by e_o and V, it follows by distributivity that G is spanned by the elements e_P, for P a subset of $(1,\ldots,n)$. But this has 2^n subsets and G has exactly the dimension 2^n so these elements are linearly independent and are a basis for G.

*<u>Theorem 2-1-1</u>: The vectors $u_1,\ldots,u_t$ in V are linearly independent if and only if their product $[u_1 \ldots u_t]$ in the Grassman algebra G over V is not zero.

<u>Proof</u>: After [MAC-1]. If the vectors are independent, they may be used as part of a basis

$$u_1 = e_1,\ldots,u_t = e_t, \quad e_{t+1},\ldots,e_n \quad \text{of} \quad V.$$

The product $\left[u_1 \ldots u_t\right] = \left[e_1 \ldots e_t\right]$ is then one of the vectors in the basis e_P of G, hence is not zero.

Conversely, if $u_1,\ldots,u_t$ are dependent, then some u_i is a linear combination of the others so the product consists of t-1 terms each with a repeated factor, hence is zero.

<u>Definition 2-1-3</u>: A form of <u>degree 0</u> is a scalar multiple of the identity. A form of degree p (expressed $d(g)=p$) is an element $g \in G$ which can be expressed as a sum of products $\left[u_1 \ldots u_p\right]$ with factors u_i in V. A form of degree p that is a product of p of the chosen basis vectors of V, is called a <u>basic form</u> of degree p. A basic form of degree

1 is called a basic unity. It was shown above that the basic forms of degree p are independent. A linear combination of a set of basic forms is called a canonical form, if each basic form is ordered as in equation (2-1-1).

Example: Let e_1, e_2, and e_3 be a basis for R_3 (the real Euclidean space of dimension 3). Then 1, e_1, e_2, e_3, e_1e_3, e_2e_3, e_1e_2, $e_1e_2e_3$, are a basis for G, e_1e_3 is a basic form of degree 2, and e_1 is a basic unity, $e_1e_2+e_1e_3+e_1e_2e_3$ is a canonical form, $e_1e_2+e_1e_3$ is a canonical form of degree 2.

By the distributive law, it follows that any form of degree p can be written as a linear combination of basic forms of degree p. Notice that 0 is a form of degree p for any p.

Definition 2-1-4: A simple form is a form that can be expressed as a product of vectors in V. A simple form is sometimes called an outer product.

Definition 2-1-5: Two forms A and B of degree p are equivalent if, and only if, there exists a scalar $c \neq 0$ such that $A = cB$.

*<u>Theorem 2-1-2</u>: Two sets of p independent vectors $s_1, \ldots, s_p$ and $t_1, \ldots, t_p$ span the same subspace of V, if and only if there exists a scalar $c \neq 0$ such that for $T = [t_1 \ldots t_p]$, $S = [s_1 \ldots s_p]$, $T = cS$, i.e. they are equivalent.

<u>Proof</u>: If the factors of T and of S span the same subspace then

$$t_i = \sum_{j=1}^{p} a_j^{\,i}\, s_j \qquad \text{for} \quad 1 \leq i \leq p.$$

Therefore,

$$T = \left[(\sum a_j^{\,1} \; s_j) \;\; (\sum a_j^{\,2} \; s_j) \; \ldots \; (\sum a_j^{\,p} \; s_j) \right] = cS$$

since every nonvanishing term in the product will be a permutation of S. By Theorem 2-1-1,

$$c \neq 0 \quad \text{since} \quad T \neq 0.$$

Conversely suppose $T = cS$, $c \neq 0$. If some t_i is not dependent on $s_1 \ldots s_p$, then $0 = [T\ t_i] = c\ S\ t_i \neq 0$ by Theorem 2-1-1. Therefore all t_i are linear combinations of the s_i.

By Theorem 2-1-2, any subspace of V is defined by an equivalent simple form in G. By the distributive law, any simple form in G can be written as the sum of a set of basic forms. Let

$$0 \neq \pi = \left[q_1 \ \ldots \ q_m \right] = \sum_i d_i \ e_{P_i}$$

where the e_{P_i} are each of the $\binom{n}{m}$ basic forms of degree m in G. By definition $\sum_i d_i e_{P_i}$ is a canonical form. The relation between the q_i and the e_i in V can be written in matrix notation as $Q = KE$, where Q is the column vector with components q_i, E is the column vector with components e_i (the distinguished basis for V), and K, mxn, is a change of basis matrix (see [BI-1], P. 244). Let $K^{(i)}$ be the square matrix consisting of the $(i_1, \ldots, i_m)$ columns of K in order. If $e_{P_i} = \left[e_{i_1} \ \ldots \ e_{i_m}\right]$, then

$$d_i = \det K^{(i)}. \qquad (2\text{-}1\text{-}2)$$

This follows directly from the definition of determinant and outer product [FO-1], [MAC-1].

From the above, the co-ordinates d_i of a simple canonical form π are the determinants of maximum rank submatrices of the matrix K taking the chosen canonical basis E into a basis Q for the subspace represented by the form. Therefore, the determinants of the maximum rank submatrices of the matrix K uniquely determine the subspace spanned by Q, since they determine an equivalent simple form.

<u>Definition 2-1-6</u>: A subspace in V is <u>just</u> if it has a canonical product whose co-ordinates d_i are restricted to the values

$$d_i = +d, -d, 0 \quad \text{all } i. \qquad (d = \text{real number}).$$

The rule for going from an outer product (in G) of an arbitrary subspace to the outer product of its orthogonal complement is extremely simple and is stated below.

Here assume that the chosen canonical basis for V is an orthogonal set.

<u>Theorem 2-1-3</u>: If a subspace V_1 of V has a canonical outer product $\pi = \sum_i d_i \left[e_{i_1} \ldots e_{i_m}\right]$, then an outer product of V_1' (the orthogonal complement of V_1) is

$$\sum_i d_i \, (-1)^{\sum_{j=1}^{m} i_j} \left[e_{i_{m+1}} \ldots e_{i_n}\right]$$

with $\left[e_{i_{m+1}} \ldots e_{i_n}\right]$ arranged in ascending order.

<u>Proof</u>: Consider the matrix:

$$\begin{pmatrix} U & A \\ -A^T & U \end{pmatrix} \qquad (1)$$

which is the echelon matrix of a subspace and its orthogonal complement. The first set of rows represents a basis for the subspace. Premultiplying (1) by

$$\begin{pmatrix} U & 0 \\ A^T & U \end{pmatrix} \qquad (2)$$

obtain

$$\begin{pmatrix} U & A \\ O & U + A^T A \end{pmatrix} \tag{3}$$

Therefore, the determinant of (1) is the determinant of $(U + A^T A)$. But:

$$(-A^T \;\; U) \begin{pmatrix} -A \\ U \end{pmatrix} = (U + A^T A) \tag{4}$$

Expanding (4) by the Cauchy-Binet Theorem [HO-1] shows that the determinant of (1) is equal to the sum of the squared maximum rank minors $M_{(k)\,(\ell)}$ of

$$(-A^T \;\; U)$$

$$\det.\ (U + A^T A) = \sum M^2_{(k)\,(\ell)} \tag{5}$$

where (k) represents a sequence of rows, (ℓ) a sequence of columns of (5). Expanding (1) by the Laplace expansion [HO-1] about the first partitioned set of rows shows that

$$\det.\ (U + A^T A) = \sum (-1)^{\sum (i) + (j)} M_{(i)(j)} \; M_{(k)(\ell)} \tag{6}$$

where $M_{(i)(j)}$ is a minor of $(U\ A)$ and $M_{(k)(\ell)}$ is the complementary minor of $(-A^T\ U)$.

Similarly,

$$\sum M^2_{(i)(j)} = \det.\ (U + A\ A^T) = \det.\ (U + A^T A) \tag{7}$$

and

$$\det.\ (U+A\ A^T) = \sum(-1)^{\sum(k)+(\ell)}\ M_{(k)(\ell)}\ M_{(i)(j)} \tag{8}$$

and

$$(-1)^{\sum(k)+(\ell)} = (-1)^{\sum(i)+(j)} \tag{9}$$

Let x be the p-tuple of the $M_{(i)(j)}$

Let y be the p-tuple of the complementary $M_{(k)(\ell)}$

Let (x,y) be their Euclidean inner product

Let z be the p-tuple of the $(-1)^{\sum(k)+(\ell)}\ M_{(k)(\ell)}$

Obviously, $(z,z) = (y,y)$.

From (5), (6), (7), and (8)

$$(x,x) = (y,y) = \sum(-1)^{\sum(k)+(\ell)}\ M_{(k)(\ell)}\ M_{(i)(j)} = \\ = (z,x) = (z,z). \tag{10}$$

Therefore, $(x-z,\ x-z) = (x,x) + (z,z) - 2(x,z) = 0$ (11)

Therefore, $x = z$

and

$$M_{(k)(\ell)} = (-1)^{\sum(i)+(j)}\ M_{(i)(j)} \tag{12}$$

Since we do not wish to choose particular bases for V_1 and V_1', we have for arbitrary bases

$$(-1)^{\sum(i)} c\, M_{(k)(\ell)} = (-1)^{\sum(j)} M_{(i)(j)} \tag{13}$$

but $c(-1)^{\sum(i)}$ is a constant independent of the columns of $\begin{bmatrix} U & A \end{bmatrix}$.

Choose the representative matrix of the orthogonal complement of V_1 to have the minors

$$M_{(k)(\ell)} = (-1)^{\sum(j)} M_{(i)(j)} \tag{14}$$

Since the coefficients of an outer product of V_1' are $M_{(k)(\ell)}$ and those of V_1 are $M_{(i)(j)}$ we have for an outer product of V_1':

$$\sum_i d_i\, (-1)^{\sum_{j=1}^{m} i_j} \begin{bmatrix} e_{i_{m+1}} & \cdots & e_{i_n} \end{bmatrix} \tag{15}$$

Example: Consider the example given after definition (2-1-3). The subspace represented by the change of basis matrix
$$K = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$
with $E = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$ has e_1+e_2, e_1+e_3 as a basis. Hence, its canonical outer product is $e_2\, e_3 - e_1\, e_2 + e_1\, e_3$. The coefficient 1, of $e_2 e_3$ is the determinant of the second and

third columns. The coefficients of other columns are their respective determinants. The outer product of the orthogonal complement of the subspace is $-e_1 + e_3 + e_2$, where E is taken to be an orthogonal basis.

Definition 2-1-7: If $E_n = e_1 \ldots e_n$ is a basic form of degree n on the n-dimensional vector space V, then $[\ \ ]_o$ is a mapping from the basic forms of degree n into the real numbers, such that

1. $[E_n]_o = 1$

2. If $E_{n1} = P(E_n)$ is a basic form that is a permutation of the factors of E_n, then E_{n1} is an outer product and $[E_{n1}]_o = (\text{sgn } P)\,[E_n]_o$ where (sgn P) is the sign of the permutation P. (See [MAC-1], [BI-1]).

If E_m is a basic form of degree m then $\{E_m\}$ is used to denote the set of basic unities in the outer product E_m.

If A is any simple form of degree n then the mapping $[\ \ ]_o$ is defined as follows:

1. if $A = 0$,

   $[A]_o = 0$

2. if $A \neq 0$, $A = c\,E_n$ where $c \neq 0$ and c is real, since E_n is the only linearly independent form of degree n (See [MAC-1]) and

$\left[A\right]_o = c.$

<u>Definition 2-1-8</u>: If E is any basic form, the <u>supplement</u> of E (denoted by $/E$) is:

$$/E = \left[E\ E'\right]_o E',$$

where the factors of E' are all of the basis vectors, $e_1, \ldots, e_n$ which do not appear in E, (arranged in <u>any order</u>), each taken just once.

<u>Definition 2-1-9:</u> If $A = k_1 E_1 + \ldots + k_r E_r$ where $E_1, \ldots, E_r$ are basic forms of degree m and k_i is a real number, for each i, then

$$/A = k_1 /E_1 + \ldots + k_r /E_r.$$

From Theorem 2-1-3 , above, and the definition of supplement it can be shown that if A is the outer product of a subspace, then $/A$ is an outer product of the orthogonal complement of A. In fact, by the distributive law for supplements and Theorem 2-1-3 , it suffices to show this for the basic forms. If E is a basic form $/E = \left[E\ E'\right]_o E'$ where E' is the complement of E. It suffices to take both E and E' in increasing order since all other orders differ from this by the sign of permutation. By Theorem 2-1-3 , if $E = \left[e_{i_1} \ldots e_{i_m}\right]$, $E' = \left[e_{i_{m+1}} \ldots e_{i_n}\right]$ arranged in ascending order the orthogonal complement of E is

$$(-1)^{\sum_{j=1}^{m} i_j} \left[e_{i_{m+1}} \ldots e_{i_n}\right] = (-1)^{\sum_j i_j} E'.$$

The assertion is established when it is shown that $\left[E\ E'\right]_o$ $= c\ (-1)^{\sum_{j=1}^{m} i_j}$ (where c is an arbitrary constant). Now the number of transpositions required to put e_{i_m} into its proper position in E' is $i_m - m$. The number of transpositions to put $e_{i_{m-1}}$ into its proper position, is $i_{m-1} - m+1$. By induction $\left[E\ E'\right]_o = (-1)^{(i_m - m)+(i_{m-1}-m+1)} \ldots \left[e_1 \ldots e_n\right]_o$ so $\left[E\ E'\right]_o = (-1)^{\sum_{j=1}^{m} i_j - m^2 + \frac{m(m-1)}{2}} = c\ (-1)^{\sum_{j=1}^{m} i_j}$.

(where ($c = +1$ or -1) and is a function only of m.) Therefore, $/E$ is an outer product of an orthogonal complement of E. By the distributive law for supplements and Theorem 2-1-3, $/A$ is the orthogonal complement of A.

Now in addition to the 3 operations of algebra given in definition 2-1-2, we add a fourth operation, called a regressive multiplication (as opposed to the "progressive" multiplication). From now on, let E be the unique basic form of degree n, with the basis elements of V arranged in increasing order. Progressive multiplication is denoted by simple brackets with no subscript, (i.e.: $\left[\quad\right]$), and regressive multiplication is denoted by simple brackets with the subscript 1, (i.e.: $\left[\quad\right]_1$).

Definition 2-1-10: The regressive outer product, $[\cdot\ ]_1$, of two basic forms E_1 and E_o of degree m and k respectively, where $m + k > n$, is:

(1) If $\{E_1\} \cup \{E_o\} = \{E\}$,

$$[E_1\ E_o]_1 \equiv (\text{sgn } P)\ [(E_1)(E_3E_2)]_1$$

$$\equiv (\text{sgn } P)\ [E_1\ E_2]_o\ E_3$$

where $d(E_1) + d(E_2) = d(E)$ and $\{E_1\} \cup \{E_2\} = \{E\}$, and (sgn P) is the sign of the permutation P, taking E_o into $(E_3\ E_2)$.

(2) If $\{E_1\} \cup \{E_o\} \neq \{E\}$, $[E_1\ E_o]_1 \equiv 0$

the regressive outer product is defined for arbitrary forms by the distributive law.

Example: Consider the example given after definition (2-1-3). The supplement of $e_1\ e_2$ is, by definition (2-1-8), $[e_1\ e_2\ e_3]_o\ e_3 = e_3$. The supplement of $5\ e_1\ e_2 + e_1\ e_3$ is, by definition (2-1-9), $5\ [e_1\ e_2\ e_3]_o\ e_3 + [e_1\ e_3\ e_2]_o\ e_2 = 5\ e_3 - [e_1\ e_2\ e_3]_o\ e_2 = 5\ e_3 - e_2$. The regressive outer product of $[e_1\ e_3]$ and $[e_2\ e_1]$ is, by definition (2-1-10), $[[e_1\ e_3]\ [e_2\ e_1]]_1 = (-1)\ [[e_1\ e_3]\ [e_1\ e_2]]_1 =$
$= (-1)\ [e_1\ e_3\ e_2]_o\ e_1 = (+1)\ [e_1\ e_2\ e_3]_o\ e_1 = e_1$.

Forder [FO-1] defines a regressive outer product (denoted $[\ \ ]_2$) as follows:

If $d(E_1) + d(E_o) > n$, $[E_1\ E_o]_2$ is such that

$$/\ [E_1\ E_o]_2 = [/E_1\ /E_o]\ . \tag{3}$$

It is now shown that this definition is equivalent to (2-1-10) so that results given by Forder can be used in this thesis.

The right hand side of (eq. 3) is the usual progressive outer product. Therefore, the regressive outer product is well defined by (eq. 3). By (eq. 3) and the distributive law for progressive products and supplements, regressive products are distributive. By definition (2-1-10) if $[E_1\ E_2]_1 \neq 0$, then $[E_1\ E_o]_1 \equiv (\mathrm{sgn}\ P)\ [E_1\ E_2]_o\ \ E_3$

where

$$E_o = (\mathrm{sgn}\ P)\ [E_3\ E_2]\ .$$

It follows therefore that

$$/\ [E_1\ E_o]_1 = (\mathrm{sgn}\ P)\ /\Big[[E_1\ E_2]_o\ \ E_3\Big]$$

$$= (\mathrm{sgn}\ P)\ \ [E_1\ E_2]_o\ [E_3\ E_3']_o\ \ E_3'$$

and

$$[/E_1\ \ /E_o] = \Big[\big[[E_1\ E_2]_o\ \ E_2\big]\ \big[/(\mathrm{sgn}\ P)\ [E_3\ E_2]\big]\Big] =$$

$$\Big[[E_1\ E_2]_o\ \ E_2\ (\mathrm{sgn}\ P)\ [E_3\ E_2\ (E_3\ E_2)']_o\ (E_3\ E_2)'\Big] =$$

$$(\mathrm{sgn}\ P)\ \ [E_1\ E_2]_o\ \ [E_3\ E_2\ (E_3\ E_2)']_o\ \ [E_2\ (E_3\ E_2)']$$

Now $\left\{E_2 \quad [E_3 \; E_2]'\right\} = \left\{E_3'\right\}$, since $\left[\left\{E_2\right\} \cup \left\{(E_3 \; E_2)'\right\}\right]' =$

$\left\{E_2'\right\} \cap \left\{(E_3 \; E_2)\right\} = \left\{E_3\right\}$.

Since the sets are the same and the order of the complement is arbitrary, let $\left[E_2 \; (E_3 \; E_2)'\right] = E_3'$. It follows that

$$/ \left[E_1 \; E_o\right]_1 = \left[/E_1 \; /E_o\right] .$$

If $\left[E_1 \; E_o\right]_1 = 0$, then $\left\{E_1\right\} \cup \left\{E_o\right\} \neq \left\{E\right\}$, so

$$/ \left[E_1 \; E_o\right]_1 = 0$$

and

$$\left[/E_1 \; /E_o\right] = \left[E_1 \; E_1'\right]_o \left[E_o \; E_o'\right]_o \left[E_1' \; E_o'\right] = 0 .$$

This last result follows from the fact that

$$\left\{E_1'\right\} \cap \left\{E_o'\right\} = \left(\left\{E_1\right\} \cup \left\{E_o\right\} \right)' \neq \left\{E'\right\} = \phi,$$

where ϕ is the null set.

Since E_1' and E_o' have a common factor, $\left[E_1' \; E_o'\right] = 0$ and

$$/ \left[E_1 \; E_o\right]_1 = \left[/E_1 \; /E_o\right] .$$

Thus it is established that the definition (2-1-10) of a regressive outer product is equivalent to the definition of Forder [FO-1] . However definition (2-1-10) is computationally easier.

By convention, whenever referring to the product of two basic forms (the sum of whose degrees is greater than n) regressive multiplication is implied.

*<u>Theorem 2-1-4</u>: Rule of a repeated factor. If A, B, C are any simple forms, the sum of whose degrees is n, then

$$\left[[A \quad B] \cdot [A \quad C] \right]_1 = [A\ B\ C]_0 \quad A$$

<u>Proof</u>: Page 231 of [FO-1].

This theorem shows that a regressive outer product either is a constant times the factor that is common to the two multiplicands or is zero. The second case occurs when the factors of A, B, and C are not independent.

<u>Definition 2-1-11</u>: A set of echelon forms for a canonical form A of degree m is any set of m basic forms (E_i) $(i=1,\ldots,m)$ of degree $(n - m + 1)$ where each has just one factor of a distinguished non-zero term of A.

<u>Theorem 2-1-5</u>: An echelon basis for a subspace of dimension m can be obtained from a set of echelon forms by the regressive multiplication

$$[A\ E_i]_1 \qquad i = 1,\ldots,m.$$

where A is the outer product of the subspace of dimension m and (E_i) are a set of echelon forms for A.

<u>Proof</u>: By Theorem 2-1-4, $\left[A\ E_i\right]_1$ yields a vector in the intersection of the subspace represented by A and that represented by E_i. Therefore the set $\left\{\left[A\ E_i\right]_1\right\}$ is a set of m vectors in A.

By the definition of a set of echelon forms and of the regressive product, the distinguished term in A, when multiplied by E_i, gives the one common factor which appears in no other term of the product. Since the E_i differ only in this factor, when the distinguished term in A is multiplied by E_i the result is a common factor other than $\left[A\ E_j\right]_1$, $j \neq i$. From the definition of regressive product it follows that this common factor appears in no other term of any of the $\left[A\ E_i\right]_1$. This is precisely an echelon basis.

Example: Consider the example given after definition (2-1-3). The form $e_1\ e_2 + 5\ e_2\ e_3$ is simple and has the set $\left\{e_1\ e_3, e_2\ e_3\right\}$ as a set of echelon forms for the distinguished term $\left[e_1\ e_2\right]$. Applying Theorem 2-1-5

$$\left[\left[e_1\ e_2 + 5\ e_2\ e_3\right]\ \left[e_1\ e_3\right]\right]_1 = \left[e_1\ e_2\ e_3\right]_0\ e_1 - 5\ \left[e_2\ e_3\ e_1\right]_0\ e_3$$

$$= e_1 - 5\ e_3;\ \left[\left[e_1\ e_2 + 5\ e_2\ e_3\right]\ \left[e_2\ e_3\right]\right]_1 = \left[e_1\ e_2\ e_3\right]_0\ e_2 + 0 = e_2.$$

Therefore the vectors, $e_1 - 5\,e_3$, and e_2, are an echelon basis for the subspace represented by the simple form $e_1\,e_2 + 5\,e_2\,e_3$.

Theorem 2-1-6: Let A be a canonical form of degree m. Then A is simple if, and only if, $\left[\left[A\ E_i\right]_1\ A\right] = 0$ for $i=1 \ldots m-1$, where (E_i), $i=1 \ldots m-1$, is some $(m-1)$ of the forms in any one set of echelon forms.

Example: To illustrate Theorem 2-1-6 , consider the space R^4 (the four dimensional real Euclidean space) and let e_1, e_2, e_3, and e_4 be a basis for it. Is the form $\left[e_1\,e_2 + e_3\,e_4\right] = A$ simple? The set of echelon forms for the distinguished term $e_1\,e_2$ is $\left\{e_1\,e_3\,e_4,\ e_2\,e_3\,e_4\right\}$. The products indicated in Theorem 2-1-6 are

$$\left[\left[e_1\,e_2 + e_3\,e_4\right]\left[e_1\,e_3\,e_4\right]\right]_1 = \left[e_1\,e_2\,e_3\,e_4\right]_0\ e_1 = e_1.$$

and

$$\left[\left[A\ \left[e_1\,e_3\,e_4\right]\right]_1\ A\right] = \left[e_1\ \left[e_1\,e_2 + e_3\,e_4\right]\right] = \left[e_1\,e_3\,e_4\right] \neq 0.$$

Therefore $\left[e_1\,e_2 + e_3\,e_4\right]$ is not simple.

Proof of Theorem 2-1-6: Suppose $\left[\left[A\ E_i\right]_1\ A\right] = 0$, $i=1,\ldots,m-1$, for a set of echelon forms, (E_i).

Since the regressive product $\left[A\ E_i\right]_1$ is of degree 1 it is simple. By definition of a regressive product each

$\left[A\ E_i\right]_1 \neq 0$. By an argument identical to the proof of Theorem 2-1-5, $\left[AE_i\right]_1$ is in echelon form, so the set of vectors $\left[AE_i\right]_1$ are independent. Let $\left[AE_i\right]_1$ $(i=1,\ldots,m-1)$, be made part of a basis for a vector space and expand A in the basic forms corresponding to this basis; i.e.,

$$A = \sum_{i=1}^{b} c_i \left[\left[AE_1\right]_1 \ B_i\right] + \sum_{i=b+1}^{\ell} d_i\ D_i$$

where $\left[\left[AE_1\right]_1 \ B_i\right]$ and D_i are the independent basic forms of degree m and D_i do not contain $\left[AE_1\right]_1$. Since $\left[\left[AE_1\right]_1 \ A\right] = 0$,

$$\sum_{i=b+1}^{\ell} d_i \left[\left[AE_1\right]_1 \ D_i\right] = 0.$$

But since the $\left[\left[AE_1\right]_1 \ D_i\right]$ $(i=b+1,\ldots,\ell)$ are independent basic forms of degree $m+1$. It follows that

$$d_i = 0,\ (i=b+1,\ldots,\ell)$$

and

$$A = \sum_{i=1}^{b} c_i \left[\left[AE_1\right]_1 \ B_i\right]$$

Setting

$$P_1 = \sum_{i=1}^{b} c_i\ B_i$$

gives

$$A = \left[\left[AE_1\right]_1 \ P_1\right]$$

Assume

$$A = \left[\left[AE_1\right]_1 \cdots \left[AE_k\right]_1 \quad P_k \right]$$

then P_k is a form of degree $(m-k)$ which does not contain any of the factors $\left[AE_i\right]_1$, $i=1,\ldots,k$.

Let

$$P_k = \sum_{i=1}^{f} d_i \left[\left[AE_{k+1}\right]_1 \quad F_i \right] + \sum_{i=f+1}^{j} d_i \, H_i$$

where

$\left[\left[AE_{k+1}\right]_1 \; F_i \right]$ and H_i are the independent basic forms of degree $(m-k)$ and H_i do not contain $\left[AE_{k+1}\right]_1$.

Since

$$\left[\left[AE_{k+1}\right]_1 \quad A \right] = 0$$

it follows that

$$\sum_{i=f+1}^{j} d_i \left[\left[AE_{k+1}\right]_1 \left[AE_1\right]_1 \cdots \left[AE_k\right]_1 \quad H_i \right] = 0$$

But since the forms

$$\left[\left[AE_{k+1}\right]_1 \left[AE_1\right]_1 \cdots \left[AE_k\right]_1 \; H_i \right] \quad (i=f+1,\ldots,j)$$

are independent of degree m, it follows that

$$d_i = 0 \quad (i=f+1,\ldots,j)$$

and

$$P_k = \sum_{i=1}^{f} d_i \left[\left[AE_{k+1}\right]_1 \quad F_i \right]$$

Setting

$$P_{k+1} = \sum_{i=1}^{f} d_i F_i$$

gives

$$A = \Big[[AE_1]_1 \cdots [AE_k]_1 \; [AE_{k+1}]_1 \quad P_{k+1} \Big]$$

By induction, it follows that

$$A = \Big[[AE_1]_1 \cdots [AE_{m-1}]_1 \quad P_{m-1} \Big]$$

where P_{m-1} is a form of degree 1.

Therefore A is simple.

Conversely, if A is simple, then $[AE_i]_1$ is a factor of A by Theorem 2-1-4, so $\Big[[AE_i]_1 \quad A \Big] = 0$ for all E_i.

Theorem 2-1-6 provides a method of determining whether a canonical form, A, is simple. If A is given, a set of echelon forms can be chosen and the products

$$\Big[[A\ E_i]_1 \quad A \Big]$$

formed for $i = 1, \ldots m - 1$. This procedure gives one condition on the coefficients of A for each term in the above products. The total number of such terms is

$$(m-1) \left[\sum_{i=1}^{m} \binom{n-i}{m} \right] = (m-1) \left[\binom{n}{m+1} - \binom{n-m}{m+1} \right]$$

where $\binom{i}{j} = \frac{i!}{(i-j)!j!}$

An examination of the determinants of an echelon basis for a subspace of dimension m in an n-dimensional vector space, shows that there are

$$\left[\binom{n}{m} - 1 - m\,(n-m)\right]$$

conditions on the coefficients of its outer product. Since it can be shown that for $n > m$

$$(m-1)\left[\binom{n}{m+1} - \binom{n-m}{m+1}\right] > \left[\binom{n}{m} -1 - m\,(n-m)\right] \text{ for } m > 1,$$

there is much redundancy if Theorem 2-1-6 is used to determine whether a canonical form is simple. The next two Corollaries and the remark following them give a simplified criterion which eliminates all the above redundancy.

Definition 2-1-12: A set of echelon unities for the canonical form A of degree m is a set of m-basic unities of a distinguished non-zero term of A_1. Therefore, each echelon unity corresponds in a (1-1) fashion to the echelon form that contains it.

Corollary 2-1-1: Let A be a canonical form of degree m, and (E_i) $(i=1,\ldots m)$ a set of echelon forms of A, for a distinguished non-zero term, P, of A. Let each E_i correspond to the echelon unity e_i of P. $(i=1,\ldots m)$. Let

$A_o = A$, and A_i be the subsum of A_{i-1} that multiplies the echelon unity e_i of P. $(i=1,\dots m)$. (A subsum is the sum of all forms that multiply a common basic unity in A. See definition 2-2-5.) Then A is simple if, and only if,

$$\left[\left[A\ E_i\right]_1 A_{i-1}\right] = 0 \qquad (\text{for } i = 1,\dots m-1) \qquad (2\text{-}1\text{-}3)$$

where (E_i) $(i=1,\dots m-1)$ are some $m-1$ of the set of echelon forms.

Proof: By Theorem 2-1-1, A is simple if, and only if,

$$\left[\left[A\ E_i\right]_1 A\right] = 0 \qquad (\text{for } i=1,\dots m-1)$$

Thus the Corollary is proven when it is shown that

$$\left[\left[A\ E_i\right]_1 A\right] = 0 \text{ if, and only if, } \left[\left[A\ E_i\right]_1 A_{i-1}\right] = 0 \quad (2\text{-}1\text{-}4)$$

for $i=1,\dots m-1$.

For $i=1$, statement (2-1-4) is an identity.

Suppose statement (2-1-4) is true for $i \leq k$.

We have

$$A_{i-1} = A_i\ e_i + B_i \qquad (i=1,\dots k) \qquad (2\text{-}1\text{-}5)$$

where no term of B_i contains e_i. Let

$$C_i = \left[e_1\ e_2 \dots e_i\right], \quad C_o = 1$$

Then

$$A = \left[A_k\ C_k + \sum_{i=1}^{k} B_i\ C_{i-1}\right] \qquad (2\text{-}1\text{-}6)$$

First suppose

$$\left[\left[A\ E_{i+1}\right]_1\ A\right] = 0 \qquad (i=1,\ldots k)$$

By (2-1-6),

$$\left[\left[A\ E_{k+1}\right]_1\ A_k\ C_k\right] + \sum_{i=1}^{k} \left[\left[A\ E_{k+1}\right]_1\ B_i\ C_{i-1}\right] = 0$$

But

$$\left[\left[A\ E_{k+1}\right]_1\ B_i\ C_{i-1}\right] \text{ has no } e_i, \qquad (i=1,\ldots k).$$

Therefore

$$\left[\left[A\ E_{i+1}\right]_1\ A_i\ C_i\right] = 0 \qquad (i=1,\ldots k).$$

Since no factor of C_i is contained in $\left[\left[A\ E_{i+1}\right]_1\ A_i\right]$,

$$\left[\left[A\ E_{i+1}\right]_1\ A_i\right] = 0.$$

Conversely suppose

$$\left[\left[A\ E_{i+1}\right]_1\ A_i\right] = 0. \qquad (i=1,\ldots k).$$

By (2-1-6),

$$\left[\left[A\ E_{k+1}\right]_1\ A\right] = \left[\left[A\ E_{k+1}\right]_1 \left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right]. \qquad (2\text{-}1\text{-}7)$$

But

$$\left[\left[A\ E_k\right]_1\ A\right] = 0$$

so

$$\left[\left[A\ E_k\right]_1\ A_k\ C_k\right] + \left[\left[A\ E_k\right]_1 \left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right] = 0\,. \qquad (2\text{-}1\text{-}8)$$

Also every term of

$$\left[\left[A\ E_k\right]_1\ A_k\ C_k\right] + d\left[e_k \left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right] \qquad (2\text{-}1\text{-}9)$$

contains e_k as a factor (where d is the coefficient of e_k in $\left[A\ E_k\right]_1$), and no term of

$$\left[\left(\left[A\ E_k\right]_1 - d\ e_k\right)\left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right] \qquad (2\text{-}1\text{-}10)$$

contains an e_k. Therefore (2-1-9) and (2-1-10) are linearly independent and are both zero.

From (2-1-9)

$$\left[\left[A\ E_{k+1}\right]_1 d\ e_k \left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right] = \left[\left[A\ E_k\right]_1 \left[A\ E_{k+1}\right]_1 A_k\ C_k\right] = 0$$

Since e_k is not a factor in any term of

$$\left[\left[A\ E_{k+1}\right]_1 \left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right],$$

$$\left[\left[A\ E_{k+1}\right]_1 \left(\sum_{i=1}^{k} B_i\ C_{i-1}\right)\right] = 0 \qquad (2\text{-}1\text{-}11)$$

Therefore, by (2-1-7), and (2-1-11)

$$\left[\left[A\ E_{i+1}\right]_1 A\right] = 0 \qquad (i=1,\ldots k).$$

It follows that statement (2-1-4) is true for $i=k+1$. Consequently, by induction, statement (2-1-4) is true for $i=1,\ldots m-1$, and the Corollary follows.

The following Corollary gives in (2-1-12), an equivalent and shortened form of (2-1-3).

<u>Corollary 2-1-2</u>: Let all notation be the same as in Corollary (2-1-1), and let

$$\left[A\ E_i\right]_1 = \left[d_i\ e_i + D_i\right], \qquad \text{where } D_i \text{ does not contain } e_i \text{ and } d_i \neq 0.$$

Then

$$\left[\left[A\ E_i\right]_1\ A_{i-1}\right] = 0 \quad \text{if, and only if,}$$

$$\left[d_i\ e_i\ A_{i-1} + D_i\ A_i\ e_i\right] = 0. \quad (i=1,\dots m-1). \qquad (2\text{-}1\text{-}12)$$

<u>Proof</u>: By (2-1-5)

$$A_{i-1} = A_i\ e_i + B_i$$

where no term of B_i contains e_i.

Suppose

$$\left[\left[A\ E_i\right]_1\ A_{i-1}\right] = 0$$

Then

$$\left[\left[d_i\ e_i + D_i\right]\ \left[A_i\ e_i + B_i\right]\right] = \left[d_i\ e_i\ B_i + D_i\ A_i\ e_i + D_i\ B_i\right] = 0.$$

Hence

$$\left[d_i\ e_i\ B_i + D_i\ A_i\ e_i\right] = 0 \quad \text{and} \quad \left[D_i\ B_i\right] = 0.$$

Therefore

$$\left[d_i\ e_i\ A_{i-1} + D_i\ A_i\ e_i\right] = \left[d_i\ e_i\ B_i + D_i\ A_i\ e_i\right] = 0$$

Conversely suppose

$$\left[d_i \; e_i \; A_{i-1} + D_i \; A_i \; e_i\right] = 0$$

Then

$$\left[d_i \; e_i \; B_i + D_i \; A_i \; e_i\right] = 0$$

and

$$\left[D_i \; d_i \; e_i \; B_i\right] = 0$$

Since $\left[D_i \; B_i\right]$ does not contain e_i

$$D_i \; B_i = 0$$

and

$$\left[\left[A \; E_i\right]_1 \; A_{i-1}\right] = \left[d_i \; e_i \; B_i + D_i \; A_i \; e_i + D_i \; B_i\right] = 0$$

<u>Remark</u>: In Corollary 2-1-1 , $A_i \; C_i$ can be written as

$$A_i \; C_i = \pm\left[\left[A \; E_{i+1}\right]_1 (/E_{i+1})\right] + D_i$$

where D_i does not contain any term of

$$\left[\left[A \; E_{i+1}\right]_1 (/E_{i+1})\right]$$

Therefore

$$\left[\left[A \; E_{i+1}\right]_1 \; A_i\right] = \left[\left[A \; E_{i+1}\right]_1 \; D_i\right]$$

Thus the computation of

$$\left[\left[A\ E_{i+1}\right]_1\ A_i\right]$$

produces some redundant terms that are always zero. These terms can be removed if

$$\left[\left[A\ E_{i+1}\right]_1\ (/E_{i+1})\right]$$

is removed from $\left[A_i\ C_i\right]$ prior to performing the multiplication. If

$$\left[A_i^!\ C_i\right] = \left[A_i\ C_i\right] \mp \left[\left[A\ E_{i+1}\right]_1\ (/E_{i+1})\right] \quad \text{, then}$$

$$\left[\left[A\ E_{i+1}\right]_1\ A_i^!\right] = \left[\left[A\ E_{i+1}\right]_1\ A_i\right]$$

Let A_i'' be the subsum of $A_{i-1}^!$ that multiplies the echelon unity e_i.

Then (2-1-12) and (2-1-3) can be replaced by the equivalent condition:

A is simple if, and only if,

$$\left[d_i\ e_i\ A_{i-1}' + D_i\ A_i''\ e_i\right] = 0 \quad (i=1,\ldots,m-1) \qquad (2\text{-}1\text{-}13)$$

Exactly

$$\sum_{i=1}^{m-1} \binom{n-i}{m+1-i} - (m-1)\ (n-m) = \left[\binom{n}{m} - 1 - m(n-m)\right]$$

different terms must be zero for (2-1-13) to be satisfied. Thus there is not redundancy of computation in this instance, since each independent condition is checked only once.

Example: To illustrate Corollaries 2-1-1, 2-1-2, and the above remark, consider the space R_5. Is A simple?

$$A = A_o = e_1e_2e_3 + 3e_1e_2e_4 - e_1e_3e_5 - e_2e_3e_4.$$

Choose $e_1e_2e_3$ as the distinguished term of A. In the notation of the two corollaries and remark, we obtain

$$\begin{array}{lll} A_1 = e_2e_3 + 3e_2e_4 - e_3e_5 & B_1 = -e_2e_3e_4, & E_1 = e_1e_4e_5 \\ A_2 = e_3 + 3e_4 & B_2 = -e_3e_5, & E_2 = e_2e_4e_5 \\ A_3 = 1 & B_3 = 3e_4, & E_3 = e_3e_4e_5 \end{array}$$

$$\left[A\ E_1\right]_1 = (e_1 - e_4),\ d_1 = 1,\ D_1 = -e_4$$
$$\left[A\ E_2\right]_1 = (e_2 + e_5),\ d_2 = 1,\ D_2 = +e_5$$
$$\left[A\ E_3\right]_1 = (e_3 + 3e_4),\ d_3 = 1,\ D_3 = +3e_4$$

$$\left[\left[A\ E_1\right]_1 (/E_1)\right] = e_1e_2e_3 - e_4e_2e_3,\quad A_o' = 3e_1e_2e_4 - e_1e_3e_5$$
$$A_1'' = 3e_2e_4 - e_3e_5$$
$$\left[\left[A\ E_2\right]_1 (/E_2)\right] = e_2e_1e_3 + e_5e_1e_3,\quad A_1' = 3e_2e_4$$
$$A_2'' = -3e_4$$
$$\left[\left[A\ E_3\right]_1 (/E_3)\right] = e_3e_1e_2 + 3e_4e_1e_2,\ A_2' = 0.$$

Then

$$\left[d_1\ e_1\ A_o' + D_1\ A_1''\ e_1\right] = e_4e_3e_5e_1 \neq 0$$

$$\left[d_2\ e_2\ A_1' + D_2\ A_2''\ e_2\right] = -3e_5e_4e_2.$$

Therefore A is not simple.

Part II

Linear Graphs and Matroids

Preliminary Definitions:

A finite graph G is a finite set $E(G)$ of edges, a finite set $V(G)$ of vertices, and a ternary relation of incidence which associates with each edge an ordered pair of vertices, called its ends.

A sequence $P = (a_o, A_1, a_1, \ldots, A_n, a_n)$, having at least one term, is a path from a_o to a_n if the following conditions are satisfied:

1. The terms of P are alternately vertices a_1 and edges A_j of G
2. If $1 \leq j \leq n$ then a_{j-1} and a_j are the two ends in G of A_j.

If $x, y \in V(G)$, we say x and y are connected in G if there is a path in G from x to y. The relation of connection is an equivalence relation partitioning $V(G)$ into disjoint equivalence classes $(V_1, \ldots, V_k)$ [TU-4] [OR-1] [SE-1]. The subgraph whose vertices are members of V_i and whose edges have both ends in V_i will be called a component of G. This is a graphic component, to be distinguished from physical component equations. The context always will clarify what definition of component is used.

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All the definitions given in [FR-1] will be used except the following:

1. Cutset is a minimal non-null set of edges whose removal separates a component into two disjoint components.
2. A circuit is a set of edges that form a simple closed curve.

A forest is a collection of trees, each taken from a distinct component. It can be shown that a forest is a maximum set of edges that contains no circuits. A co-forest is a complement of a forest. It can be shown that a co-forest is a maximum set of edges that contains no cutsets. Theorem (3.2.2) of [FR-1] is true when tree is replaced by forest, and when co-tree is replaced by co-forest.

A matroid on a finite set M_1, is a class M of non-null subsets of M_1 which satisfies the following axioms:

Axiom 1: No member of M contains another as a proper subset.

Axiom 2: If $X, Y \in M$, $a \in X \cap Y$, and $b \in (X - Y)$, then there exists $Z \in M$ such that

$$b \in Z \subseteq (X \cup Y) - \{a\}.$$

The elements of M are called the points of M.

Let $C(G)$ be the set of circuits of a finite graph G. Then $C(G)$ is a matroid. In fact the edges of the graph are the set M_1. Since a circuit is a simple closed curve formed from a subset of $E(G)$, [FR-1], it has no crossover vertices, therefore, contains no other closed curves, so satisfies axiom 1. If X and Y are two circuits with a common edge a and some edge $b \in (X - Y)$, let γ and β denote the two vertices of a, then the curve Z' from β along X to γ, then along Y to β is closed and does not contain a. Now Z' contains b but may not be simple, however, since $b \in (X - Y)$ there exists a simple closed curve Z contained in Z' such that $b \in Z$. Z is the required circuit that satisfies axiom 2. Henceforth $C(G)$ is called the circuit matroid of the finite graph G.

Let $B(G)$ be the set of cutsets of a finite graph G. Then $B(G)$ is a matroid. To see this, let the edges of the graph be the set M_1. A cutset is defined as a minimal non-null set of edges whose removal separates a component into two disjoint components. Since cutset is defined as minimal it satisfies axiom 1. If X and Y are two cutsets, $a \in (X \cap Y)$, and $b \in (X - Y)$ then let Z' be the set of edges $(X \cup Y) - \{a\}$. Let the disjoint sets of vertices of the components formed by the removal of X be denoted by C_1 and C_2, and those for Y be D_1 and D_2, the two vertices of a by α and γ. Let the set of vertices of the component be C. Therefore $C_1 \cup C_2 = D_1 \cup D_2 = C$. Since without loss of generality,

$$\alpha \in C_1 \cap D_1, \gamma \in C_2 \cap D_2,$$

then either $C_1 \cap D_2$ or $C_2 \cap D_1$ is non-empty, since otherwise $X = Y$. Assume $C_1 \cap D_2$ is non-empty. Then Z' separates the vertices C into the disjoint sets $C_1 \cap D_2$ and $C_1 \cap D_1$. Therefore, Z' separates the component whose vertices are C into at least two parts. Since $b \in (X - Y) \subset Z'$, let Z be the minimal subset of Z' containing b which separates the component into exactly two disjoint components. Therefore $B(G)$ satisfies axiom 2. $B(G)$ will be called the cutset matroid of the finite graph G.

A matroid is graphic if it is the cutset matroid of a finite graph, and co-graphic if it is the circuit matroid of a finite graph. The points of a matroid M on M_1 are the elements of M.

To describe any finite dimensional vector space by a matroid let R denote the real numbers. If M_1 is any finite set define a chain on M_1 over R as a mapping f of M_1 into R. If $a \in M_1$ then $f(a)$ is the coefficient of a in f. The set of all $a \in M_1$, such that $f(a) \neq 0$ is the domain $|f|$ of f. If $f(a) = 0$ for all a then f is the zero chain on M_1 over R.

The sum $f + g$ of two chains f and g on M_1 over R is a chain on M_1 over R defined by the following rule:

$$(f + g)(a) = f(a) + g(a) \qquad a \in M_1$$

With this definition of addition, the chains on M_1 over R are the elements of an additive Abelian group where the zero chain is the zero element, and the negative of a chain f is the chain $(-f)$ where the coefficients of f are multiplied by (-1).

Scalar multiplication is defined by the following rule: If

$$r \in R, \quad f \text{ is a chain} \quad a \in M_1, \quad \text{then} \quad rf(a) = r \times f(a).$$

With this definition of addition and scalar multiplication the set of all chains form a vector space N_o, isomorphic to the vector space spanned by a corresponding set of n-tuples.

If M_1 is a finite set of n elements we thus have a 1-1 correspondence between the elements of M_1 and a distinguished set of orthonormal basis vectors E, for the n-dimensional chain vector space.

Throughout the remainder of this thesis a canonical outer product refers to the expansion of the outer product in terms of this set E taken as the basic unities. A matrix for this set of basis vectors is defined as follows:

Let $M_1 = \left\{a_1, \ldots, a_n\right\}$.
Then $\left[f(a_1), \ldots, f(a_n)\right]$

is a row vector which is called the representative vector of the chain f with respect to the chosen enumeration of M_1. Let A be a matrix of r rows and n columns whose elements are elements of R and r rows are linearly independent over R. Then the set of chains on M_1 whose representative vectors are linear combinations of rows of A with coefficients from R are elements of a chain subspace N, of N_o on M_1. A is called the representative matrix of N. For the above basis, E, A is also the change of basis matrix defining the subspace, N. (See (2-1-1).)

Now any n dimensional vector space is isomorphic to the chain vector space N_o on n elements since the set of n-tuples is isomorphic to any n-dimensional vector space [BI-1].

By the following theorem and the above paragraph we have a matroid associated with every finite dimensional vector subspace. A chain f of N is elementary if it is non-zero and there is no non-zero $g \in N$ such that $|g|$ is a proper subset of $|f|$.

*<u>Theorem 2-2-1</u>: The Class M(N) of domains of elementary chains of N is a matroid on M_1.

<u>Proof</u>: See [TU-3].

The following lemma gives an important property of every chain vector subspace.

***Lemma 2-2-1:** The domain of every non-zero chain of N is a union of points of $M(N)$.

Proof: [TU-3].

A subspace N is called graphic (cographic) if its matroid $M(N)$ is graphic (cographic).

A primitive chain of N is an elementary chain f of N in which the coefficients $f(a)$ are restricted to the values 0, 1, -1.

The subspace, N, is regular if to each elementary chain there corresponds a primitive chain with the same domain. A matroid is called regular if it is the matroid of a regular chain vector subspace. By the above definition every regular matroid has a representative matrix.

A condition equivalent to the regularity of a chain vector subspace is given in the following theorem. In its proof the following definition is used.

Definition 2-2-1: A dendroid is a minimal non-null subset of M_1 such that it meets the domain of every non-zero chain of N.

Theorem 2-2-2: Let A be a representative matrix of a vector space, N, of order $r \times n$. Then a necessary and sufficient condition that A be the representative matrix of a regular chain vector space on M_1 is that the determinants of its square submatrices of order r are 0, d, and

-d, where d is a real number $\neq 0$.

Proof: Suppose A is the representative matrix of a regular vector space. Let P be a set of r columns of A that has a non-vanishing determinant. Without loss of generality assume P is in the first r columns of A and write

$$P^{-1} A = P^{-1} [P\ B] = [U\ A_o].$$

Since $[U\ A_o]$ is another representative matrix of a regular chain vector space and the chains of $[U\ A_o]$ are elementary, there exists a set of primitive chains in the vector space, with the same domain as those of $[U\ A_o]$. Therefore, another representative matrix for the regular chain vector space is the matrix $[U\ A_1]$ in which every row vector of the matrix $[U\ A_1]$ is a primitive chain.

Since A and $[U\ A_1]$ both have maximum rank and represent the same vector space, there exists a non-singular real matrix M such that

$$M\ A = [U\ A_1]$$

Let P_1 be any r columns of $[U\ A_1]$ that have a non-vanishing determinant and without loss of generality write

$$[U\ A_1] = [A_2\ P_1\ A_3]$$

and

$$P_1^{-1} [A_2 \; P_1 \; A_3] = [A_4 U \;\; A_5]$$

Since $[A_4 \; U \; A_5]$ is another representative matrix of a regular chain vector space and the chains of $[A_4 \; U \; A_5]$ are elementary, there exists a set of primitive chains in the vector space with the same domain as the row chains of $[A_4 \; U \; A_5]$. Therefore, another representative matrix for the regular chain vector space is the matrix $[A_6 \; U \; A_7]$ in which every row vector is a primitive chain with the same domain as its correspondent in $[A_4 \; U \; A_5]$ and there exists some matrix P_2 such that

$$P_2 \; [U \; A_1] = P_2 \;\; [A_2 \; P_1 \; A_3] = [A_6 \; U \;\; A_7]$$

But by definition of a primitive chain the entries of $[A_6 \; U]$ are either ± 1, or 0, consequently the columns of $[A_6 \; U]$ corresponding to the unit matrix in $[U \; A_1]$ are a linear combination (with ± 1, or 0) of the rows of the unit matrix of $[U \; A_1]$. It follows that all entries of P_2 are ± 1, or 0. But since

$$P_2 \; P_1 = U$$

and both P_2 and P_1 are matrices of integers, it follows that (det. P_2) (det. P_1) = 1 and (det. P_2) = ± 1.

It has been established that the determinant of any nonsingular submatrix of $[U\ A_1]$ equals ± 1. Therefore the determinant of any nonsingular submatrix of A is $(\det. M^{-1})\ (\pm 1) = \pm d$.

Conversely, suppose the determinants of the square submatrices of the representative matrix A are 0, d, and $-d$. Let N be the vector space spanned by the rows of A. Let f be any elementary chain of N. Let $\{a\}$ be any member of $|f|$ and C any dendroid of $N \cdot (M_1 - |f|)$; (i.e., if $S \subseteq M_1$, $N \cdot S$ is the class of restrictions to S of the chains of N.) Take f such that $f(a) = 1$. Then if a chain h of N has a domain not meeting $C \cup \{a\}$ its domain must be a subset of $|f| - \{a\}$. Since f is elementary, this is possible only if h is zero. Therefore some subset D, of $C \cup \{a\}$ is a dendroid of N. Since D must meet $|f|$, $D \cap |f| = \{a\}$. The following lemma is needed.

---

*<u>Lemma 2-2-2</u>: Let A be an r-rowed representative matrix of N. Then a subset S of M_1 is a dendroid of N if, and only if, it has just r elements and is such that $\det. A(S) \neq 0$. $A(S)$ is the matrix whose columns correspond to elements of S.

<u>Proof</u>: [TU-1].

Returning to the proof of Theorem 2-2-2, since A has all determinants equal to $\pm d$ or 0, $\det. A(D) = \pm d$.

Let $A' = (A(D))^{-1} A$. The matrix A' is a representative matrix for N. The matrix $A'(D)$ is a unit matrix. Therefore there exists a chain g of N such that $g(a) = 1$ and $|g| \cap D = \{a\}$. Then $f - f(a)\, g$ is a zero chain since its domain does not meet D. Accordingly $f = f(a)\, g$, and $f(a) = 1$, so $f = g$. Also since $(A(D))^{-1}$ has a determinant equal to $\pm \frac{1}{d}$ and every $(r \times r)$ submatrix of A has determinant $\pm d$, every $(r \times r)$ submatrix of A' has determinant ± 1.

Now consider the $(r \times r)$ submatrix $A'(D')$ formed from the columns corresponding to $D' = (D - \{a\}) \cup \{b\}$ where $b \in |f|$ such that $b \neq a$. Then det. $A'(D')$ equals $\pm f(b)$ since the columns corresponding to $D - \{a\}$ are distinct columns of the unit $(r \times r)$ matrix. But det $A'(D') = \pm 1$ from above, so $f(b) = \pm 1$. The same is true for every element $b \in |f|$. Therefore f is primitive.

<u>Lemma 2-2-3</u>: N is a regular vector subspace if, and only if, its outer product is just.

<u>Proof</u>: Follows immediately from definition 2-1-6 and Theorem 2-2-2.

Let R_1 be the field of integers (mod 2). By the same way a matroid was defined on a vector space, N_o, over the field R of reals, a matroid can be defined on a vector space N_o' over the field R_1 of integers (mod 2).

N'_o is called a binary chain vector space. N_o and N'_o define the trivial matroids where each element of M is an element of M_1. In the same way that the subspaces N of N_o define the non-trivial matroids, the subspaces N' of N'_o define the non-trivial matroids. N' is called a binary chain vector subspace.

A matroid will be called binary if it is the matroid of a binary chain vector subspace.

*<u>Theorem 2-2-4</u>: Every regular matroid is binary.

<u>Proof</u>: [TU-3].

<u>Theorem 2-2-5</u>: Every regular matroid corresponds to a unique binary vector subspace of N'_o.

<u>Proof</u>: Let M be a regular matroid. By Theorem 2-2-4 M is binary so is the matroid of a binary vector subspace N'. Suppose M is also the matroid of another binary chain vector subspace N'_1. By Theorem 2-2-1, the elementary chains of N' and N'_1 have the same domain and since their coefficients are both (mod. 2), the chains are equal.

Take any echelon representative matrix A' of N' for the distinguished set of orthonormal basis vectors of N'_o. Then each row of A' corresponds to an elementary chain of N'. Therefore A' is also a representative matrix for N'_1. Since N' and N'_1 have the same representative matrix, $N' = N'_1$.

The unique binary vector subspace associated with a regular matroid M can be determined from any real vector subspace N which has M as its matroid. First obtain an echelon representative matrix A for N. Since every row of A corresponds to an elementary chain of N, the rows of A can be replaced by a set of rows having ± 1 or 0 only as coefficients since N is regular. The new matrix B thus obtained is in echelon form. Consequently, its rows are linearly independent and have all elements ± 1 or 0. Replace all non-zero elements of B by the corresponding residue class (mod. 2), to obtain B'. All rows of B' are linearly independent (mod. 2). Also, the rows of B' are elementary chains of N' and their domains are the same as the elementary chains of N.

The determinant of every maximum rank submatrix of the matrix B above is equal to ± 1 or 0 by Theorem 2-2-2, since B is in echelon form and B is a representative matrix of a regular vector subspace. By the way in which the matrix B' is obtained, it follows that the determinants of maximum rank submatrices of B' are equal to the determinants of B (mod 2). Let the basis vectors of N_o and N'_o be put in 1-1 correspondence with M_1. Now by equation (2-1-1) the corresponding terms of the canonical outer products of N and N' have for their coefficients the determinants of their corresponding columns in B and B'.

Therefore:

<u>Lemma 2-2-4</u>: If $M(N) = M(N')$ where N is regular and N' is binary, then a coefficient of a term of the outer product of N' is non-zero if, and only if, the coefficient of the corresponding term of the canonical outer product of N is non-zero. Here the distinguished basis vectors of N_o and N'_o are in a 1-1 correspondence.

Given a chain vector subspace N, on the elements of the finite set M_1, and $S \subseteq M_1$, let $N \cdot S$ be defined as the class of restrictions to S of the chains of N, and $N \times S$ as the class of restrictions to S of those chains f of N for which $|f| \subseteq S$.

Let

$$N_1 = \left\{ f \; : \; f \in N, \quad |f| \subseteq S \subseteq M_1 \right\}.$$

N_1 is a subspace of N, since if $f \in N_1$, $|a\, f| = |f| \subseteq S$. If $f \in N_1$, $g \in N_1$, then

$$|f + g| \subseteq |f| \cup |g| \subseteq S.$$

Therefore, by Theorem 1, p. 164, of [BI-1], N_1 is a subspace of N.

There is a 1-1 correspondence between the basis vectors E of N_o and the elements of M_1. Let E_1 be the subset of E corresponding to the elements of S. It follows that N_1 is the subspace of vectors of N that are linear combinations of the chains of E_1.

If S has m elements, E_1 has m elements. Consider the m-dimensional space, N_2, spanned by the restrictions to S of the chains of E_1. The chain f', formed from chain $f \in N$ by restricting f to S, is an element of N_2. The class $N \times S$ is the subspace of N_2 formed from the chains $g \in N_1$ by restricting g to S. Thus, $N \times S$ and $N \cdot S$ are subspaces of N_2.

Note that N_2 is an m-dimensional space and each of its vectors have m coefficients, one corresponding to each element of its basis,whereas N and N_1 are subspaces of N_o which is n-dimensional and therefore, each vector has n coefficients. Now consider the subspace $(N \times S) \cdot T$. If $E_2 \subseteq E$ are the basis elements of N_o corresponding to the set $(S \cap T)$, and $S \cap T$ has t elements, E_2 has t elements. Let N_3 be the t-dimensional vector space spanned by the restrictions to S, of the chains of E_2. Therefore $(N \times S) \cdot T$ is a subspace of N_3.

Let M be a matroid on the set M_1, and the matroid $M(N)$ on the subspace N defined by Theorem 2-2-1 .

Let $(M \times S)$ be the class of all sets of M which are subsets of S. Then $(M \times S)$ satisfies axioms I and II and is therefore a matroid on S.

Let M_S be the class of intersections with S of members of M, and let $(M \cdot S)$ be the class of all minimal non-null members of M_S. In [TU-3], it is shown that $(M \cdot S)$ is a matroid on S.

From: [TU-3]

$$M(N \times S) = M(N) \times S$$

$$M(N \cdot S) = M(N) \cdot S$$

The matroids of the form $(M \times S) \cdot T$ (where T, $S \subseteq M_1$) are called minors of M.

If M is the matroid of a chain vector space N, then

$$(M(N) \times S) \cdot T = M(N \times S) \cdot T = M((N \times S) \cdot T) \qquad (2\text{-}2\text{-}1)$$

Matroid Minors and Outer Products

In the notation above, $(N \times S) \cdot T$ can be interpreted as the class of restrictions to $(T \cap S)$ of the chains of N_1. Let E_3 be the restrictions to $(S \cap T)$ of the chains of E_2. If a representative matrix A of N is chosen such that a maximum set B_1 of linearly independent vectors of N_1 are represented by rows of A, then $(N \times S) \cdot T$ is spanned by those linear combinations of elements of E_3 represented by the vectors of B_1. Let A_o be the submatrix of A composed of the m rows corresponding to B_1 and n columns

corresponding to their coefficients in E_1. Therefore a submatrix A_1 of A_o whose rows represent vectors that span $(N \times S) \cdot T$ is the subset of columns of A_o corresponding to E_2, and the rows of A_o. A representative matrix for $(N \times S) \cdot T$ could be obtained from A_1 by choosing a maximal linearly independent set D of rows of A_1. Thus D is some submatrix of A. Also since the rows of A corresponding to the vectors of B_1 are linearly independent and all coefficients of vectors of B_1 corresponding to $(E - E_1)$ are zero, it follows that the rows of the submatrix A_o are linearly independent. By performing linear combinations on the rows of A_o it can be assumed that the complementary set of rows to matrix D in A_1 is O. Let A_2 denote the matrix composed of the rows of A_o that were deleted to form D and the columns of A_o corresponding to the coefficients of $(E_1 - E_2)$. Then the rows of A_2 are linearly independent since if they were not, A_o would not have linearly independent rows. Let A_3 be a submatrix of A_2 with linearly independent columns. By elementary row operations the entries of the submatrix of A_o corresponding to the rows of D and the columns of A_3 can be made O.

The remaining rows of A can be split into two submatrices. Let A_4 be those columns corresponding to E_1 and let A_5 be those columns corresponding to $(E - E_1)$. It is claimed that the rows of A_5 are linearly independent.

If they were not, there would be some row combination of the matrix $[A_4 \ \ A_5]$ with coefficients not all zero such that the columns corresponding to A_5 are zero. But the same combination of the rows of A_4 would be non-zero since A is a set of linearly independent rows. By definition the resulting row is a vector of N_1. But it was already assumed A_o contains all such rows. Therefore, the rows of A are not all linearly independent, contrary to assumption on A. This contradiction establishes that the rows of A_5 are linearly independent. Thus the following has been established:

Lemma 2-2-5: A representative matrix D of any subspace of the form $(N \times S) \cdot T$ can be made a submatrix of a representative matrix A for N in which the submatrix D_1 of A composed of the set of rows complementary to D and the set of columns complementary to D has linearly independent rows, and the submatrix D_2 of A composed of the set of rows of D and a set of columns of maximum rank in D_1 is zero.

Lemma 2-2-5 and equation (2-2-1) give:

Theorem 2-2-6: A matroid $M(N)$ on M_1 contains a given minor K if, and only if,

1. There exists a representative matrix A, of N that possesses a submatrix D that is a representative matrix of the minor K,

2. The submatrix D_1 of A composed of the set of rows complementary to D and the set of columns complementary to D has linearly independent rows, and

3. The submatrix D_2 of A composed of a set of columns of maximum rank in D_1 and the set of rows of D has zero for each entry.

Using Theorem 2-2-6 , one can determine from the canonical outer product F of N, whether $M(N)$ contains a given minor K. Suppose $M(N)$ possesses a given minor K, and dimension of N is n_1. From A the outer product of N can be expanded in terms of the elements of E since A is a change of basis matrix. By equation (2-1-1), the coefficient of term E_{P_i} of F is the determinant of the $n_1 \times n_1$ submatrix of A taken from the n_1 columns corresponding to the factors of E_{P_i}. Let D_3 be a square submatrix of D_1 which is nonsingular. Since D has linearly independent rows and since the sum of all rows of D_3 and D is n_1 and D_2 is zero, the determinant of every square submatrix of D multiplied by (det. D_3) is a coefficient of a term of F. The terms corresponding to these coefficients are the terms of E corresponding to the columns of D_3 and the columns of D. Thus:

<u>Theorem 2-2-7</u>: A matroid $M(N)$ on M_1 contains a given minor K if and only if the canonical Grassman outer product F of N, contains the outer product F_1 (of the vector space corresponding to K) multiplied by a basic form E_3.

The above basic form E_3 is the subset of E corresponding to the columns of any nonsingular D_3.

Since any regular matroid $M(N)$ is the matroid of a unique binary vector space N' by Theorem 2-2-5 , and since any minor of a regular matroid is regular [TU-1 (3.5)], for regular matroids, we may substitute N' for N in the above theorem and the outer product F' of N' for F and the outer product F'_1 of the binary vector space corresponding to the minor K, for F_1.

Using Lemma 2-2-4 , the problem of determining the existence of a given minor of a regular matroid is reduced to the determination of the existence of a collection of sets (called residue sets).

<u>Definition 2-2-2</u>: A residue set of an echelon representative matrix A of N is the unordered subset of E (the basis vectors of the space N_o) which have non-zero coefficients in some chosen row of A.

If an echelon representative matrix A has n_1 rows, it has n_1 residue sets, one for each row of A.

Definition 2-2-3: A reduced residue set of an echelon representative matrix A of N is a residue set where the basis element corresponding to the diagonal coefficient has been removed.

Definition 2-2-4: The residue sets (reduced residue sets) of a non-vanishing term of an outer product are the residue sets (reduced residue sets) of the echelon representative matrix associated with the set of echelon forms of the non-vanishing term (by Theorem 2-1-5).

The reduced residue sets or the residue sets are a way of describing an echelon basis within a column permutation in a mod. 2 space. If M is regular Theorem 2-2-6 can be applied to its unique binary vector space of Theorem 2-2-5, and determine the minors present in some echelon representative matrix. Therefore, to determine the minor's presence the reduced residue sets need only be examined for their size and intersections.

The above discussion is summarized in this lemma:

Lemma 2-2-6: Let N' be a unique binary m-dimensional vector subspace derived from a regular matroid M on a finite set M_1. Let the set M_1 have n elements which correspond 1-1 to a basis set E of a binary vector space N'_o. Of course $N' \subseteq N'_o$. Then M contains a given minor K if and only if there exist reduced residue sets of an echelon representative matrix of N' which when intersected with a subset of E, form the reduced residue sets of the unique binary vector space N'_1 which represents K.

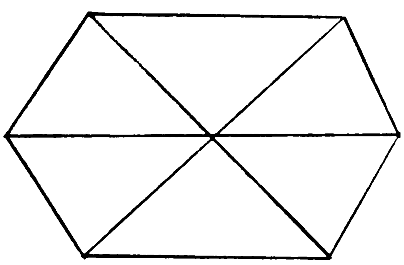
If all possible echelon representative matrices of N'_1 were known, every set of k columns need be examined only once. This is the technique used in Theorem 2-2-12 .

Graphic Matroids and Outer Products

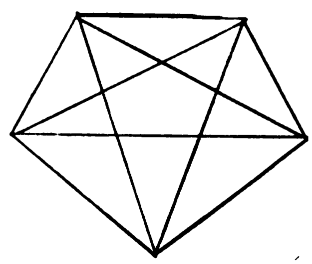
Theorem 2-2-8: A matroid, M, is graphic (cographic) if and only if it is regular and has no minor which is the circuit-matroid (cutset-matroid) of a Kuratowski graph.

Proof: [TU-4] (Main Theorem)

The two Kuratowski graphs are the Thompson and the complete - 5. They are shown below:



Thompson Graph



Complete - 5 Graph

Theorem 2-2-9: A real vector subspace, $N \subseteq N_0$, is graphic if and only if it has a just canonical outer product F which does not contain the outer product F_1 (of a circuit vector space corresponding to a Kuratowski graph) multiplied by a basic form E_3.

Proof: The proof is an immediate application of the definition of regular matroid, Lemma 2-2-3 , Theorem 2-2-7 , and Theorem 2-2-8 .

Theorem 2-2-10: A real vector subspace, $N \subseteq N_o$, is graphic if, and only if, it is:

1. regular,
2. has no representative matrix, A, that possesses a submatrix D that is a representative matrix of a Kuratowski circuit minor,
3. the submatrix D_1 of A composed of the set of rows complementary to D and the set of columns complementary to D has linearly independent rows, and
4. the submatrix D_2 of A composed of a set of columns of maximum rank in D_1 and the set of rows of D has zero in each position.

Proof: Follows immediately from the definition of regular matroid, Lemma 2-2-3 , Theorem 2-2-6 , and Theorem 2-2-8 .

Theorem 2-2-11: A real vector subspace N is graphic if, and only if, it has a just canonical outer product F that has no non-vanishing term with reduced residue sets such that their intersection with some subset of E forms the reduced residue sets of a Kuratowski circuit subspace.

<u>Proof</u>: By the definition of regular and Lemma 2-2-2, we see that N is regular if and only if N has a just outer product. By Lemma 2-2-6, M(N) contains a Kuratowski circuit minor if and only if there exist reduced residue sets of some echelon representative matrix of N' (where N' is the unique binary vector space such that M(N) = M(N')) whose intersection with some subset of E forms the reduced residue sets of a Kuratowski circuit subspace. But by the construction process described after Theorem 2-2-5, the reduced residue sets of echelon forms of N and N' are identical.

This theorem provides the simplest interpretation of a graphic vector space since it means that once we know a vector space is just, we perform the search for non-graphic subspaces in set algebra, or in its equivalent, (mod. 2) algebra.

The following analysis of the circuit subspaces of Kuratowski graphs provides the final theorem which is useful in the algorithm of Chapter 3. An m-form is a form generated by m of the distinguished basis elements.

Analysis of the Thompson graph shows that there are only 2 distinct structures of co-trees,

1. those that form a path and,
2. those where three edges form a path and the fourth edge is not connected to the other three.

Within a suitable permutation of the columns the two echelon representative matrices for these structures are unique because of the symmetry of the graph; and can be put into the forms below:

For structure 1:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For structure 2:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where, of course, the order of the columns between structures 1 and 2 has been permuted. The orientation of the elements is neglected since by Lemma 2-2-6, it is unimportant.

Let A be an arbitrary simple just nine form of degree 4. Denoting the first five elements of both structure 1 and structure 2, by a, b, c, d, e, it is obvious that a non-vanishing term of A corresponds to the unit matrix in structure 1, if and only if the residue sets of the term are:

$$\{a, b, c\}, \{a, c, e\}, \{b, d, e\}, \{c, d, e\} \quad \text{(T1)}$$

for some suitable permutation of the first five columns. Likewise, a non-vanishing term of A corresponds to the unit matrix in structure 2 if, and only if, the residue sets of the term are:

$$\left\{a, b, c, d, e\right\}, \left\{a, b, c\right\}, \left\{a, b, d\right\}, \left\{a, d, e\right\} \qquad \text{(T2)}$$

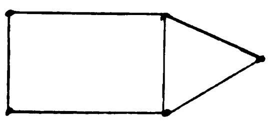
for some suitable permutation of the first five columns.

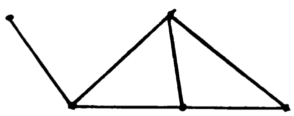
Residue sets (T1) and (T2), are called the Thompson residue sets. Thus the following has been shown:

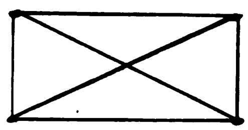
Lemma 2-2-7: Let A be a simple just nine-form of degree 4. A corresponds to a circuit space of Thompson graph if and only if any non-vanishing term has a Thompson residue set. (Only one term need be examined since A is simple, and the echelon set uniquely determines an echelon basis by Theorem 2-1-5).

An analysis, similar to the above for a complete-5 graph shows that there are 3 distinct structures of co-trees:

1. those corresponding to a union of two circuits with one edge in common,
2. those corresponding to a union of two 3-edge circuits with one edge having a free vertex, and
3. those corresponding to a complete-4 graph.







Structure (1) Structure (2) Structure (3)

Again, as in the Thompson graph, because of the symmetry of the graph, the two echelon representative matrices for these structures are unique except for a suitable permutation of the columns. Allowing for this, the echelon matrices are as follows:

For structure 1:

$$\begin{bmatrix} 1&1&1&1&1&0&0&0&0&0 \\ 1&1&1&0&0&1&0&0&0&0 \\ 0&1&1&1&0&0&1&0&0&0 \\ 1&1&0&0&0&0&0&1&0&0 \\ 0&1&1&0&0&0&0&0&1&0 \\ 0&0&1&1&0&0&0&0&0&1 \end{bmatrix}$$

For structure 2:

$$\begin{bmatrix} 1&1&1&0&1&0&0&0&0&0 \\ 1&1&0&1&0&1&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0 \\ 0&1&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&0&1&0 \\ 0&0&1&1&0&0&0&0&0&1 \end{bmatrix}$$

For structure 3:

$$\begin{bmatrix} 1&1&0&0&1&0&0&0&0&0 \\ 1&0&1&0&0&1&0&0&0&0 \\ 1&0&0&1&0&0&1&0&0&0 \\ 0&1&1&0&0&0&0&1&0&0 \\ 0&1&0&1&0&0&0&0&1&0 \\ 0&0&1&1&0&0&0&0&0&1 \end{bmatrix}$$

Let B be an arbitrary simple just ten-form of degree 6. Let the first four columns in the matrices above be denoted by a, b, c, d.

Similarly to the above analysis for the Thompson graph, a non-vanishing term of B corresponds to the unit matrix in structure 1 if and only if the residue sets of the term are:

$$\{a,b,c,d\},\ \{a,b,c\},\ \{b,c,d\},\ \{a,b\},\ \{b,c\},\ \{c,d\} \quad \text{(K1)}$$

for some permutation of the first four columns.

Similarly a non-vanishing term of B corresponds to the unit matrix of structures 2 or 3, if and only if, residue sets of the term are:

$$\{a,\ b,\ c\},\ \{a,\ b,\ d\},\ \{a,\ b\},\ \{b,\ c\},\ \{b,\ d\},\ \{c,\ d\} \quad \text{(K2)}$$

or

$$\{a,\ b\},\ \{a,\ c\},\ \{a,\ d\},\ \{b,\ c\},\ \{b,\ d\},\ \{c,\ d\} \quad \text{(K3)}$$

respectively, for some permutation of the first four columns.

Residue sets (K1), (K2), and (K3), are called the complete-5 residue sets. It has been established that:

Lemma 2-2-8: Let B be a simple just ten-form of degree 6. B corresponds to a circuit space of a complete-5 graph if and only if any non-vanishing term has a complete-5 residue set. (As for the Thompson graph, one non-vanishing term need be examined since B is simple and the echelon set uniquely determines a basis.)

Definition 2-2-5: A sub-sum of an outer product is the sum of all forms that multiply a common basic form. Example: Let $\{x_1,\ x_2,\ x_3\}$ be the basic unities. Then

$$a\ x_1\ x_2 + b\ x_1\ x_3 + c\ x_2\ x_3$$

has $(b\ x_1 + c\ x_2)$ as the sub-sum multiplying the common basic form, x_3.

Theorem 2-2-12: Let E be the set of basic unities of N_o. Let N_o be a real vector space of dimension n. A real vector sub-space, N, of N_o, is graphic if and only if the just canonic outer product F of N has no sub-sum which has either Thompson or complete-5 reduced residue sets for a non-vanishing term.

Proof: The process of forming an intersection with the residue sets is equivalent to forming a sub-sum and then taking the residue sets of the sub-sum. The rest follows from Theorem 2-2-11 , and Lemma's 2-2-7 , and 2-2-8 .

Note here that there is no need to form the complete sub-sum but only to look for the presence of certain terms in the sub-sum.

The following two important graphic operations used in the next chapter are taken from [TU-1].

Definition 2-2-6: Let G be a finite graph, and S a subset of edges in G (i.e., S $E(G)$). Let $G \cdot S$ be the subgraph of G whose edges are members of S and whose vertices are the ends of members of S. $G : S$ is the subgraph of G whose edges are members of S and whose vertices , are all the vertices of G.

<u>Definition 2-2-7</u>: Let G be a finite graph, S a subset of $E(G)$. G ctr. S is the subgraph of G whose vertices are the components of G: $(E(G) - S)$ and whose edges are the members of S; the ends in G ctr. S of an edge A are those components of G: $(E(G) - S)$ which contain as vertices the ends of A in G. We may regard G ctr. S as obtained from G by contracting each component of G: $(E(G) - S)$ to a single point. $G \times S$ is the graph obtained from G ctr. S by suppressing its isolated vertices. These vertices are clearly those components of G whose edges all belong to $E(G) - S$.

If $C(G)$ is the circuit matroid of the graph G, and $B(G)$ is the cutset matroid of G,

$$B(G \cdot S) = B(G) \cdot S$$

$$B(G \times S) = B(G) \times S$$

$$C(G \cdot S) = C(G) \times S$$

$$C(G \times S) = C(G) \cdot S$$

For proof, see [TU-4].

By the definitions of $B(G) \cdot S$ and the others before (2-2-1) we have

$$\begin{aligned} B(G \cdot S) &= M(N_1 \cdot S) \\ B(G \times S) &= M(N_1 \times S) \\ C(G \cdot S) &= M(N \ \ \times S) \\ C(G \times S) &= M(N \ \ \cdot S) \end{aligned} \qquad (2\text{-}2\text{-}2)$$

where N_1 is the graphic vector space of G, N is the cographic vector space of G, and M is the unique matroid corresponding to each vector space by Theorem (2-2-1). Likewise

$$\begin{aligned} B[(G \times S) \cdot T] &= M\,[(N_1 \times S) \cdot T] \\ B[(G \cdot S) \times T] &= M\,[(N_1 \cdot S) \times T] \\ C[(G \cdot S) \times T] &= M[(N \times S) \cdot T] \\ C[(G \times S) \cdot T] &= M[(N \cdot S) \times T] \end{aligned} \tag{2-2-3}$$

More complex minors are computed similarly to these.

Part III

Components and Systems

Let CE be a set of equations called component equations, perhaps parametrized by an independent variable t, relating the coefficients of a distinguished set K, of orthonormal basis elements of a finite dimensional vector space V of dimension $2e$. Let all the basis elements of V be put into exactly e ordered pairs, and let P be the vector space generated by the first elements of the ordered pairs. Let P' be the vector space generated by the second elements of the ordered pairs. The distinguished basis elements of P are called across variables, and the distinguished basis elements of P' are called through variables.

Let G be a finite graph of e edges and let F be a 1-1 correspondence between the set of edges of G and the ordered pairs of V. Let N be the co-graphic vector subspace of P and N_1 the graphic vector subspace of P' corresponding to the graph G by F.

A system is defined as the ordered triple $\{CE, G, F\}$. A system can be described by a set of equations, CE together with the equations formed by equating all vectors of N and N_1 to 0. (These latter equations are the generalized Kirchoff's laws of [FR-1].)

A graph, G_1, is a subgraph of a graph, G, if $E(G_1) \subset E(G)$ and $V(G_1) \subset V(G)$. Let N' be the co-graphic vector subspace of P corresponding to the subgraph G_1 by F and N'_1 the graphic vector subspace of P' corresponding to the graph G_1 by F. Then a subsystem of $\{CE, G, F\}$ is the ordered triple $\{CE_1, G_1, F\}$. The equations describing the subsystem are:

1. the equations CE_1 relating the coefficients of the distinguished basis ordered pairs of V that correspond to the edges of G_1 (it is assumed that G_1 is chosen so that all other coefficients are in the remaining equations of CE)
2. the equations formed by equating all vectors of N' and N'_1 to 0.

It is assumed that all component equations are written with respect to the distinguished basis of V. The representative matrix of N or of its orthogonal complement is assumed to be defined with respect to the distinguished basis of V.

Throughout this thesis systems are examined whose component equations are: ordinary differential, integral, algebraic, or combinations of the above three; i.e., a single valued mapping or operator between abstract spaces or function spaces, as described in works on functional analysis, and system theory, [ZAH-1],[ZAM-1],[WIE-1],[ZAH-2],[BOS-1],[BAR-1],[MC-1]. (An algebraic equation means a relation between coefficients of basis elements that does not relate differentials, integrals, or limits of the variables.)

The component equations considered are identified as the following types:

Type (1): Differential and algebraic equations of form:

$$\frac{d}{dt} \Psi_k = F_k(\Psi_k, Z_{i_k}, t)$$

$$Z_{o_k} = G_k(\Psi_k, Z_{i_k}, t)$$

where Z_{i_k} and Z_{o_k} are each vectors of a finite dimensional space V of order N_k-1 and each contains exactly one coefficient x_j or y_j corresponding to each edge $j=(1,2,\dots N_k-1)$ of the corresponding graph (defined in [FR-1]). The vector Ψ_k is called the state vector of the multi-terminal component and t is an independent real parameter, usually time. Z_{o_k}, Z_{i_k}, Ψ_k are vectors in E^n which vary with the parameter t.

Type (2): Integral and algebraic equations of the general form:

$$\Psi_k(t) = \int_{t_o}^{t_1} F_k(Z_{i_k}(s), s, t)\, ds$$

$$Z_{o_k} = G_k(\Psi_k, Z_{i_k}, t)$$

Type (3): If the component is described by an operator between abstract Hilbert spaces, equations of the following form:

$$Z_{o_k} = F_k(Z_{i_k}, t)$$

where the quantities are as defined above, except that the vectors are elements of the Hilbert space L_{2m_k}, [See Part II, Chapter III] or the real Euclidean space of dimension m_k.

The vector subspace N generates, for a $2e$ dimensional vector space V, e linear algebraic constraint equations. As shown in [FR-1] and [KO-1], these constraint equations represent a generalization of Kirchoff's laws and they can be written without loss of generality in the forms

$$A\ Y = 0$$

and (2-3-1)

$$B\ X = 0$$

where A is a representative matrix of the orthogonal complement of N, B is a representative matrix of N, Y is a vector of thru variables, one coefficient corresponding to each edge of the graph and therefore to each basis element of the space, and X is the corresponding vector of across variables. This notation is all defined in [FR-1].

If a system has m components, the equations describing the system behavior can be written as the direct sum of m component equations of Type (1), (2), or (3), (which is CE), and the set of e algebraic constraint equations (2-3-1).

CHAPTER III

UNIQUENESS AND EXISTENCE OF A SOLUTION

A system, $\left\{E, G, F\right\}$, is said to have a unique solution if all system variables are determined uniquely from the component and graph equations.

In the first part of this chapter, uniqueness and existence results are given for linear, constant coefficient systems of algebraic, integral, and differential equations. Since the linear operators of differentiation, integration and algebra are commutative, the determinant of these such linear operators has meaning and is useful in uniqueness studies. Perhaps the most novel theorem is 3-1-3 where uniqueness problems involving positive semi-definite components are reduced to the linear independence of a subset of the component and graph equations.

In the second section the conditions for existence and uniqueness of non-linear systems are examined, and some new results derived which are a generalization of those previously known. [WI-1].

Part I. Linear Systems

Introduction

Consider first the linear time-stationary components of type (1); i.e., components governed by differential and algebraic equations of the form:

$$\frac{d}{dt} \Psi_k = P_o \Psi_k + P_1 Z_{i_k} + F_{o_k}(t),$$

$$Z_{o_k} = P_2 \Psi_k + P_3 Z_{i_k} + F_{1_k}(t) \qquad (3\text{-}1\text{-}1)$$

where $F_{o_k}(t)$ and $F_{1_k}(t)$ are almost everywhere continuous real-valued functions of t on some interval I.

Now let $(s = \frac{d}{dt})$, and write (3-1-1) in the form

$$(s\, D_{1_k} + D_{2_k})\, \Psi_k + D_{3_k} Z_{i_k} + D_{4_k} Z_{o_k} = F_{3_k}(t) \quad (3\text{-}1\text{-}2)$$

where $D_{1_k} = \begin{bmatrix} U \\ 0 \end{bmatrix}$, $D_{2_k} = \begin{bmatrix} P_o \\ P_2 \end{bmatrix}$, $D_{3_k} = \begin{bmatrix} P_1 \\ P_3 \end{bmatrix}$, $D_{4_k} = \begin{bmatrix} 0 \\ U \end{bmatrix}$,

$$F_{3_k} = \begin{bmatrix} F_{o_k} \\ F_{1_k} \end{bmatrix}$$

and U is the unit matrix.

Solving the first set in (3-1-1) for Ψ_k gives:

$$\Psi_k = (s\,U - P_o)^{-1} P_1 Z_{i_k} + (s\,U - P_o)^{-1} F_{o_k}(t) \qquad (3\text{-}1\text{-}3)$$

Substitute (3-1-3) into the second expression of (3-1-1) to obtain:

$$Z_{o_k} = P_2(s\,U - P_o)^{-1} P_1\, Z_{i_k} + P_3\, Z_{i_k} + P_2(s\,U - P_o)^{-1}$$

$$F_{o_k}(t) + F_{1_k}(t) \tag{3-1-4}$$

Setting

$$(P_2 \text{ Adj. } (s\,U - P_o)\;\; P_1 + P_3) = -C_{1_k}(s)$$

$$[\text{det. } (s\,U - P_o)]\;\; U = C_{o_k}(s)$$

and

$$C_{o_k}(s)\, P_2\, (s\,U - P_o)^{-1}\;\; F_{o_k}(t) + F_{1_k}(t) = F_{3_k}(t)$$

(3-1-4) becomes

$$C_{o_k}(s)\, Z_{o_k} + C_{1_k}(s)\, Z_{i_k} = F_{3_k}(t) \tag{3-1-5}$$

where $\left[C_{o_k}(s)\;\; C_{1_k}(s)\right]$ is a square matrix of polynomials in s and each row is assumed to be a vector polynomial of minimum degree.

Consider an arbitrary system S of components of type (3-1-5). Let the direct sum of the component equations be written as

$$C_o(s)\, Z_o + C_1(s)\, Z_i = F_o(t) \tag{3-1-6}$$

Repartitioning the Z_o and Z_1 into the thru variables Y and across variables X, (3-1-6) takes on the form

$$E_1(s)\,X + E_2(s)\,Y = F_1(t) \qquad (3\text{-}1\text{-}7)$$

where $E_1(s)$ and $E_2(s)$ are square matrices of polynomials in s, and the entries of X and Y are ordered so that complementary variables corresponding to the same edge are in corresponding positions.

Combining (3-1-7) with the circuit and cutset equations (2-3-1), a general form of the system equations for a linear system is given as follows:

$$\begin{bmatrix} E_1(s) & E_2(s) \\ B & O \\ O & A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(t) \\ O \\ O \end{bmatrix} \qquad (3\text{-}1\text{-}8)$$

The following analysis is valid for any equations in form (3-1-8). Some Type 2 components can also be put in this form as well as any equations with derivatives on the input variables.

A necessary and sufficient condition for a unique solution to (3-1-8), on some interval on t, with incompatible boundary values [IN-1], is that the determinant of the matrix on the left of (3-1-8) is not identically zero. [IN-1] [CR-1] [GA-1].

An alternate form of (3-1-8) for equations of Type 1 is obtained if the direct sum of the components of a system S_1 of form (3-1-2) is written as

$$(s\ D_1 + D_2)\ \Psi + D_3\ Z_i + D_4\ Z_o = F_3(t) \qquad (3\text{-}1\text{-}9)$$

The alternate for equation (3-1-8) is:

$$\begin{bmatrix} s\ D_1 + D_2 & E_1 & E_2 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix} \begin{bmatrix} \Psi \\ X \\ Y \end{bmatrix} = \begin{bmatrix} F_3(t) \\ 0 \\ 0 \end{bmatrix} \qquad (3\text{-}1\text{-}10)$$

where E_1 and E_2 are the properly partitioned columns of D_3 and D_4.

It is well known that for systems having driver type components (i.e., $X_o = F_a(t)$, $Y_1 = F_b(t)$) a unique solution exists only if all edges corresponding to variables X_o can be put in a forest T of the system graph G, and all edges corresponding to variables Y_1 can be put in the co-forest of T in the system graph G. [KO-1].

Within the context of the notation used here a driver type component is defined as:

Definition 3-1-1: A driver-type component is a component of form (3-1-1) where P_1 and P_3 are zero matrices.

The following theorem and its corollary shows how the driver type components can be removed from further consideration in existence and uniqueness studies.

<u>Theorem 3-1-1</u>: Let the direct sum of a set of component equations, CE, of system $\{CE, G, F\}$ be of the form:

$$(s D_1 + D_2) \Psi + E_{11} X_1 + E_{21} Y_1 = F_1(t) \qquad (3\text{-}1\text{-}11(a))$$

$$\begin{aligned} X_o &= F_o(t) \\ Y_2 &= F_2(t) \end{aligned} \qquad (3\text{-}1\text{-}11(b))$$

where the set of edges corresponding to X_o and Y_2 contain all the driver-type components. Let S^{X_i} be the edges corresponding to the variables X_i, i = 0, 1, 2. Let G' be the subgraph $\left[[G \times (E(G) - S^{X_o})] \cdot S^{X_1}\right]$.

(a) If the set of edges corresponding to X_o can be made part of a forest T of the graph G and the edges corresponding to Y_2 can be made part of the co-forest of T, then the system $\{CE, G, F\}$ has a unique solution if, and only if, the subsystem $\{CE_1, G', F\}$ has a unique solution.

(b) If the conditional part of (a) is not satisfied, the system $\{CE, G, F\}$ has no unique solution on I.

<u>Proof</u>: Assume the hypothesis of (a). Using the notation of [FR-1], take the circuit and cutset matrices in echelon form. Under the partitioning given in (3-1-11) (a) and (b), (3-1-10) for system $\{CE, G, F\}$, can be written as

$$\begin{bmatrix} (sD_1+D_2) & O & E_{111} & E_{112} & O & O & E_{211} & E_{212} & O \\ O & U & O & O & O & O & O & O & O \\ O & O & O & O & O & O & O & O & U \\ O & B_1 & B_2 & U & O & O & O & O & O \\ O & B_3 & B_4 & O & U & O & O & O & O \\ O & O & O & O & O & U & O & -B_1^T & -B_3^T \\ O & O & O & O & O & O & U & -B_2^T & -B_4^T \end{bmatrix} \begin{bmatrix} \Psi \\ X_o \\ X_{1b} \\ X_{1c} \\ X_2 \\ Y_o \\ Y_{1b} \\ Y_{1c} \\ Y_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_o(t) \\ F_2(t) \\ O \\ O \\ O \\ O \end{bmatrix} \quad (3\text{-}1\text{-}12)$$

(where X_{1b} and X_{1c} is a suitable partitioning of X_1 and X_i and Y_i are complementary pairs corresponding to the same edges).

System $\left\{CE, G, F\right\}$ has a unique solution if, and only if, the matrix of (3-1-12) is nonsingular. By elementary row operations transform (3-1-12) into

$$\begin{bmatrix} (sD_1+D_2) & O & E_{111} & E_{112} & O & O & E_{211} & E_{212} & O \\ O & U & O & O & O & O & O & O & O \\ O & O & O & O & O & O & O & O & U \\ O & O & B_2 & U & O & O & O & O & O \\ O & O & B_4 & O & U & O & O & O & O \\ O & O & O & O & O & U & O & -B_1^T & O \\ O & O & O & O & O & O & U & -B_2^T & O \end{bmatrix} \begin{bmatrix} \Psi \\ X_o \\ X_{1b} \\ X_{1c} \\ X_2 \\ Y_o \\ Y_{1b} \\ Y_{1c} \\ Y_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_o(t) \\ F_2(t) \\ -B_1F_o(t) \\ -B_3F_o(t) \\ B_3^TF_2(t) \\ B_4^TF_2(t) \end{bmatrix} \quad (3\text{-}1\text{-}13)$$

By (2-2-3), (3-1-10) for system $\left\{CE_1, G_1, F\right\}$ can be written as

$$\begin{bmatrix} (s\,D_1+D_2) & E_{111} & E_{112} & E_{211} & E_{212} \\ 0 & B_2 & U & 0 & 0 \\ 0 & 0 & 0 & U & -B_2^T \end{bmatrix} \begin{bmatrix} \psi \\ X_{1b} \\ X_{1c} \\ Y_{1b} \\ Y_{1c} \end{bmatrix} = \begin{bmatrix} F_1(t) \\ 0 \\ 0 \end{bmatrix} \tag{3-1-14}$$

Comparing (3-1-13) and (3-1-14), it follows that (3-1-13) has a unique solution if, and only if, (3-1-14) has a unique solution, and the conclusion (a) follows.

If the condition in (a) is not satisfied then by Theorem 3.3.1 in [FR-1] and an analogous theorem in [KO-1], the system $\left\{CE, G, F\right\}$ has no unique solution, and Theorem 3-1-1 is proved.

Definition 3-1-2: The operation of breaking a vertex into two vertices and then connecting an edge between them is defined as a vertex splitting operation.

Corollary 3-1-1: Let CE be the component equations of (3-1-11, (a) and (b)). Let the system $\left\{CE_1, G', F\right\}$ have a unique solution. Let G be the graph formed when each across driver in (3-1-11(b)), is added by a succession of vertex splitting operations, and each thru driver in (3-1-11(b)) is added between any two connected vertices in G'. Then the system $\left\{CE, G, F\right\}$ has a unique solution on I.

Proof: Since each vertex splitting operation is performed on a single vertex by a succession of splitting operations the graph G_1 thus formed contains no circuits of across drivers. Therefore, by Theorem 2-12 of [SE-1] and its immediate extension to disconnected graphs, the edges of the across drivers are part of some forest T of the graph G_1.

Adding the edges E_1 corresponding to thru drivers in succession between two existing connected vertices of G_1, generates the graph G having the same vertices as G_1. The forest T is also a forest of the graph G and therefore contains no thru driver edge, and the corollary follows.

Remark: In reducing graph G to G' by the operations in Theorem 3-1-1, 1-edge circuits (called loops) and 1-edge cut-sets may be formed. This is equivalent to short circuiting or open circuiting component terminals.

In view of Theorem 3-1-1 and Corollary 3-1-1, throughout the remainder of Part I, it is assumed that no driver type components are present in the systems examined.

Existence and Uniqueness Theorems

The following theorem gives a sufficient condition for a unique solution for an important class of passive components.

<u>Theorem 3-1-2</u>: Consider component equations of the form (3-1-7). If for some real constant s the quadratic form $X^T E_1(s) E_2^T(s) X \neq 0$ for all real vectors $X \neq 0$, then any system having the component equations (3-1-7) has a unique solution.

<u>Proof</u>: It will be shown that the matrix on the left of (3-1-8) is non-singular when the conditions of this theorem are satisfied.

<u>Lemma 3-1-1</u>: Let E and K_o be the following square matrices:

$$E = \begin{bmatrix} R & E_1 & E_2 \end{bmatrix} \begin{bmatrix} U & 0 & 0 \\ 0 & A^T & 0 \\ 0 & 0 & B^T \end{bmatrix}$$

$$K_o = \begin{bmatrix} R & E_1 & E_2 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix}$$

where B and A are given in (2-3-1). Then if A has t rows and e columns,

$$\det. E = (-1)^{e(e-t)} \det. K_o.$$

<u>Proof</u>: Suppose the columns of K_o are permuted so that there is a nonsingular matrix in the first $(e-t)$ columns of B. Let the columns containing A be permuted similarly. Then if P is the permutation matrix, by Theorem 3.2.1 of [FR-1]:

$$K_oP = \begin{bmatrix} R & E_3 & E_4 & E_5 & E_6 \\ 0 & C_1 & C_1B_1 & 0 & 0 \\ 0 & 0 & 0 & -C_2B_1^T & C_2 \end{bmatrix}$$

or

$$K_oP = C_o\, K_1$$

where

$$K_1 = \begin{bmatrix} R & E_3 & E_4 & E_5 & E_6 \\ 0 & U & B_1 & 0 & 0 \\ 0 & 0 & 0 & -B_1^T & U \end{bmatrix}, \quad \text{and } C_o = \begin{bmatrix} U & & \\ & C_1 & \\ & & C_2 \end{bmatrix}$$

$[E_3\ E_4]$ and $[E_5\ E_6]$ are matrices formed by suitable permutations of the columns of E_1 and E_2, C_1 and C_2 are the nonsingular matrices obtained by the permutation.

Since P does the same permutation on both the columns of E_1 and the columns of E_2, P is an even permutation so

$$\det.\ P = 1.$$

If $\det. \begin{bmatrix} U & & \\ & C_1 & \\ & & C_2 \end{bmatrix} = c \neq 0$, then

$$\det.\ K_o = c \det.\ K_1. \tag{3-1-15}$$

Premultiply K_1 by

$$\begin{bmatrix} U & -E_3 & -E_6 \\ O & U & O \\ O & O & U \end{bmatrix}$$

to obtain

$$K_2 = \begin{bmatrix} R & O & E_4-E_3B_1 & E_5+E_6B_1^T & O \\ O & U & B_1 & O & O \\ O & O & O & -B_1^T & U \end{bmatrix}$$

and $\det.\ K_2 = \det.\ K_1$.

Now

$$c \det.\ K_2 = c(-1)^{e(e-t)} \det. \begin{bmatrix} R & E_4-E_3B_1 & E_5+E_6B_1^T \end{bmatrix}$$

$$= (-1)^{e(e-t)}\det.\begin{bmatrix} R & E_3 & E_4 & E_5 & E_6 \end{bmatrix}\begin{bmatrix} U & O & O \\ O & -B_1 & O \\ O & U & O \\ O & O & U \\ O & O & B_1^T \end{bmatrix}\begin{bmatrix} U & & \\ & C_2^T & \\ & & C_1^T \end{bmatrix}$$

$$= (-1)^{e(e-t)}\det.\begin{bmatrix} \begin{bmatrix} R & E_1 & E_2 \end{bmatrix} & \begin{bmatrix} U & O & O \\ O & A^T & O \\ O & O & B^T \end{bmatrix} \end{bmatrix}$$

From (3-1-15)

$$\det.\ K_o = (-1)^{e(e-t)}\det.\ E, \text{ and the Lemma follows.}$$

Returning to the proof of Theorem 3-1-2, suppose the matrix of (3-1-8) is singular. By Lemma 3-1-1, there is a non-zero row vector $Z_1^T(s)$ with polynomial entries in powers of s such that the vector

$$Z_1^T(s)\ \begin{bmatrix} E_1(s) & E_2(s) \end{bmatrix} \qquad (3\text{-}1\text{-}16)$$

is orthogonal to the row space of $\begin{bmatrix} A & O \\ O & B \end{bmatrix}$; i.e., for some row vectors $Z_2^T(s)$ and $Z_3^T(s)$ with polynomial elements.

$$Z_1^T(s)\ E_1(s) = Z_2^T(s)\ B$$

and (3-1-17)

$$Z_1^T(s)\ E_2^T(s) = Z_3^T(s)\ A\ .$$

Therefore, by Theorem 3.2.1 of [FR-1],

$$Z_1^T(s)E_1(s)E_2^T(s)Z_1(s) = Z_2^T(s)\ B\ A^T\ Z_3(s) = 0 \qquad (3\text{-}1\text{-}18)$$

Equation (3-1-18) is true for every s, and at least one co-ordinate of $Z_1(s)$ is a non-zero polynomial.

If $Z_1(s)$ were zero for some real s (say $s = a$), then every coefficient of $Z_1(s)$ would have a factor $(s - a)$ and could be written as $Z_1(s) = (s - a)\ Z_1'(s)$, where $Z_1'(s)$ satisfies (3-1-16), (3-1-17), and (3-1-18), for some $Z_2'(s)$ and $Z_3'(s)$. Proceeding in this fashion obtain a vector $Z_1''(s)$ of minimum degree that vanishes for no real s and satisfies (3-1-16), (3-1-17), and (3-1-18), for some $Z_2''(s)$ and some $Z_3''(s)$. It suffices, therefore, to suppose that $Z_1(s)$ is a vector of minimum degree, not equal to zero for any s.

If follows, that (3-1-18) is a contradiction to the hypothesis of Theorem 3-1-2 , so, the system has a unique solution for all interconnections.

<u>Corollary 3-1-2</u>: Let the component equations be given in the form (3-1-9). Let $(s\ D_1+D_2) = s \begin{bmatrix} U \\ O \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix}$, $D = \text{adj.}\ (s\ U + A)$,

and $d = \det.(s\,U + A)$. If for s equal to some real constant the quadratic form

$$X^T \begin{bmatrix} -CD & dU \end{bmatrix} E_1\, E_2^T \begin{bmatrix} -(CD)^T \\ dU \end{bmatrix} \qquad X \neq O$$

for all real $X \neq O$, then any system having these component equations has a unique solution.

<u>Proof</u>: It will be shown that the hypothesis implies that the matrix (3-1-10) is nonsingular.

If the matrix in (3-1-10) is singular, then there is a non-zero vector $Z_1^T(s)$ such that

$$Z_1^T(s) \begin{bmatrix} (s\,D_1 + D_2) & E_1 & E_2 \end{bmatrix} \tag{3-1-19}$$

is orthogonal to

$$\begin{bmatrix} U & O & O \\ O & A & O \\ O & O & B \end{bmatrix} \tag{3-1-20}$$

Therefore, $Z_1^T(s)\,(s\,D_1 + D_2) = 0$

and

$$Z_1^T(s) = y_o^T(s) \quad \begin{bmatrix} -CD & dU \end{bmatrix}$$

since the rows of $[-CD \quad dU]$ span the orthogonal complement of the columns of $(s\,D_1 + D_2)$.

From relations analogous to (3-1-17) it follows that

$$y_o^T(s)\ [-CD \quad dU]\ E_1\ E_2^T \begin{bmatrix} -(CD)^T \\ dU \end{bmatrix} y_o(s) = Z_2^T(s)\ BA^T\ Z_3(s) = 0$$

(3-1-21)

If $Z_1(s)$ is chosen so that $y_o(s)$ is a minimal degree polynomial vector in s, then $y_o(s)$ is non-zero for all s, which contradicts the hypothesis. Therefore, the system has a unique solution for any graph.

If, in particular, $\overline{\left[E_1(s)\ E_2^T(s)\right]}$ is negative definite for some real s, by Theorem 3-1-2 the system has a unique solution. When a system contains some <u>semi-definite components</u>, (those where $E_1\ E_2^T$ is a semi-definite matrix), Theorem 3-1-2 can be extended to yield a new sufficient condition for a unique solution.

<u>Corollary 3-1-3</u>: Let the component equations be given in form (3-1-7), and let the components be subdivided into two classes (for $s = c$ (a real constant)),

1. Those where

   $X^T\ E_1'\ (c)\ E_2'^T\ (c)\ X < 0$, for real $X \neq 0$, and

2. Those where

   $X_1^T\ E_1''\ (c)\ E_2''^T\ (c)\ X_1 \leq 0$, for real X_1, and

$\begin{bmatrix} E_1''(c) & E_2''(c) \end{bmatrix}$ has maximum row rank. If there exists no circuit or cutset in G composed entirely of edges corresponding to the components of $E_1''(s)$, the system $\{C, G, F\}$ has a unique solution.

Proof: Write the component equations in a direct sum as follows

$$\begin{bmatrix} E_1'(s) & 0 \\ 0 & E_1''(s) \end{bmatrix} X + \begin{bmatrix} E_2'(s) & 0 \\ 0 & E_2''(s) \end{bmatrix} Y = F_1(t) \qquad (3\text{-}1\text{-}22)$$

The matrix of equation (3-1-8) must be shown to be non-singular. If it is singular, then by Theorem 3-1-2, for $s = c$, there exists a non-zero real vector $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ such that

$$Z^T \begin{bmatrix} E_1'(c) \; E_2'^T(c) & 0 \\ 0 & E_1''(c) \; E_2''^T(c) \end{bmatrix} Z = 0 \qquad (3\text{-}1\text{-}23)$$

Expanding (3-1-23):

$$Z_1^T \; E_1'(c) \, E_2'^T(c) \; Z_1 + Z_2^T \; E_1''(c) \, E_2''^T(c) \; Z_2 = 0 \; .$$

If Z_1 is non-zero, equation (3-1-23) is less than zero. Therefore Z_1 is zero and Z_2 is non-zero. By the proof for Theorem 3-1-2, and 3-1-17, Z can be chosen such that

$$Z^T \begin{bmatrix} E_1'(c) & 0 \\ 0 & E_1''(c) \end{bmatrix} = Z_3^T B$$

and

$$Z^T \begin{bmatrix} E_2'(c) & 0 \\ 0 & E_2''(c) \end{bmatrix} = Z_4^T A$$

for some real Z_3 and Z_4.

Since $Z_1 = 0$ it follows that

$$Z_2^T \begin{bmatrix} 0 & E_1''(c) \end{bmatrix} = Z_3^T B$$

and (3-1-24)

$$Z_2^T \begin{bmatrix} 0 & E_2''(c) \end{bmatrix} = Z_4^T A$$

But since Z_2 is non-zero and the component equations have linearly independent rows, both $Z_2^T E_1''(c)$ and $Z_2^T E_2''(c)$ cannot vanish simultaneously. If Z_2^T $[0\ E_1''(c)]$ $= Z_5$ is not zero, then in the terminology (of Chapter II) it is a non-zero chain in the co-graphic chain vector space N, spanned by the rows of the representative matrix B. Hence there exists an elementary chain in N whose domain is contained in $|Z_5|$. But by Theorem 2-2-1 and the fact that N is a co-graphic vector space, the domain of each elementary chain of N is a circuit of the graph. Therefore, there exists a circuit of the graph whose edges correspond to semi-definite components only, contrary to hypothesis. This contradiction establishes the Corollary.

<u>Remark</u>: The non-existence of a circuit or cutset of negative semi-definite components is equivalent to the existence of a forest and a co-forest of negative definite components by Lemma 2-2-2 and definition 2-2-1.

A necessary and sufficient condition for a unique solution for systems containing negative definite and semi-definite components only, is given next.

<u>Theorem 3-1-3</u>: Let the direct sum of component equations, CE, be written in form (3-1-22), where for all real s on some interval I_1, (I_1 is not a point interval)

1. $X^T E_1'(s) E_2'^T(s) X < 0$ for all real $X \neq 0$
2. $X_1^T E_1''(s) E_2''^T(s) X_1 \leq 0$ for all real X_1.

Let S denote the set of edges corresponding to the components of $E_1''(s)$ and $E_2''(s)$. For a given graph G, let N be the co-graphic subspace with representative matrix B, and N_1 be the graphic subspace with representative matrix A. Let B_1 be a representative matrix of the subspace $N \times S$, (as defined in Chapter II) (i.e., the subspace spanned by the circuits composed of edges of components corresponding to $E_1''(s)$ and $E_2''(s)$). Let A_1 be a representative matrix of the subspace $N_1 \times S$ (i.e., the subspace spanned by the cutsets composed of edges of components corresponding to $E_1''(s)$ and $E_2''(s)$). Then the system $\{CE, G, F\}$ has a unique solution if, and only if, the matrix

$$\begin{bmatrix} E_1''(s) & E_2''(s) \\ B_1 & O \\ O & A_1 \end{bmatrix} \tag{3-1-25}$$

has linearly independent rows.

<u>Proof</u>: Tutte, [TU-4], has shown that the orthogonal complement of $(N \times S)$ is $(N_1 \cdot S)$. By its definition, $(N_1 \times S)$ is contained in the subspace $(N_1 \cdot S)$. Therefore, the last two rows of (3-1-25) are of maximum rank and have linearly independent rows. Also B_1 can be obtained directly from A by considering the submatrix of A made up of the columns of S and taking its orthogonal complement. Similarly A_1 can be obtained directly from B.

When E_1 is the direct sum of E_1' and E_1'' and E_2 the direct sum of E_2' and E_2'' the system $\{CE, G, F\}$ has a unique solution if, and only if, the coefficient matrix of (3-1-8) has a determinant that does not vanish identically.

Suppose the system has no unique solution. Then, following the proof of Corollary 3-1-3, for every s on I_1, there exists some row vector $Z_2^T(s)$ such that

$$Z_2^T(s)\, E_1''(s)\, E_2''^T(s)\, Z_2(s) = O,$$

$Z_2(s)$ is non-zero for all but a finite number of points on I_1, and

$$Z_2^T(s) \begin{bmatrix} 0 & E_1''(s) \end{bmatrix} = Z_3^T(s)\, B$$
$$Z_2^T(s) \begin{bmatrix} 0 & E_2''(s) \end{bmatrix} = Z_4^T(s)\, A \qquad (3\text{-}1\text{-}26)$$

By (3-1-26), the non-zero entries of $Z_3^T(s)\, B$ correspond to edges of the components $E_1''(s)$. Therefore, $Z_3^T(s)\, B$ is a chain of $N \times S$, and there exists some row vector $Z_5^T(s)$ such that

$$Z_3^T(s)\, B = Z_5^T(s) \begin{bmatrix} 0 & B_1 \end{bmatrix}$$

and

$$Z_2^T(s)\, E_1''(s) = Z_5^T(s)\, B_1.$$

Similarly, there exists a $Z_6^T(s)$ such that

$$Z_4^T(s)\, A = Z_6^T(s)\, A_1$$

and

$Z_2^T(s)\, E_2''(s) = Z_6^T(s)\, A_1$ for some $Z_6(s)$, and $Z_2(s)$ does not vanish identically on I.

It follows that the rows of (3-1-25) are linearly dependent for s on I_1. Since the determinants of all maximum order matrices of (3-1-25) are polynomials, which vanish for an infinite number of points of I_1, they are zero.

Conversely, suppose the rows of (3-1-25) are linearly dependent. Then there exists a non-zero row vector

$\left[Z_7^T(s)\ Z_8^T(s)\ Z_9^T(s)\right]$ such that

$$Z_7^T(s)\ E_1''(s) = -Z_8^T(s)\ B_1$$

and (3-1-27)

$$Z_7^T(s)\ E_2''(s) = -Z_9^T(s)\ A_1$$

Since $\left[E_1''(s)\ E_2''(s)\right]$, B_1, and A_1, each have linearly independent rows, $Z_8^T(s)\ B_1$ or $Z_9^T(s)\ A_1$ or both, is non-zero, and $Z_7^T(s)$ is non-zero. Without loss of generality, suppose $Z_8^T(s)\ B_1$ is non-zero.

Now every vector of B_1 is a vector of $N \times S$, so $Z_8^T(s)\left[0\ \ B_1\right] = Z_{10}^T(s)\ B$ for some non-zero row vector $Z_{10}^T(s)$.

Therefore

$$Z_7^T(s)\left[0\quad E_1''(s)\right] = Z_{10}^T(s)\ B$$

Similarly

$$Z_7^T(s)\left[0\quad E_2''(s)\right] = Z_{11}^T(s)\ A$$

where $Z_{11}^T(s)$ is some row vector, possibly zero.

Rewriting (3-1-8) for this system, and premultiplying by $\left[0\ Z_7^T(s)\quad -Z_{10}^T(s)\quad -Z_{11}^T(s)\right]$ gives

$$\begin{bmatrix} 0 & Z_7^T(s) & -Z_{10}^T(s) & -Z_{11}^T(s) \end{bmatrix} \begin{bmatrix} \begin{bmatrix} E_1'(s) & 0 \\ 0 & E_1''(s) \end{bmatrix} & \begin{bmatrix} E_2'(s) & 0 \\ 0 & E_2''(s) \end{bmatrix} \\ B & 0 \\ 0 & A \end{bmatrix} = 0 \qquad (3\text{-}1\text{-}27)$$

from which it follows that (3-1-8) is singular and the system $\{CE, G, F\}$ has no unique solution.

<u>Remark</u>: By (2-2-2) the graphs of $N \times S$ and $N_1 \times S$ are $G \cdot S$ and $G \times S$ respectively.

Notice in the second part of the proof of Theorem 3-1-3, no use is made of conditions (1) and (2) of the hypothesis. The second part of this theorem is, therefore, valid for all systems and is worth rephrasing as a separate corollary.

<u>Corollary 3-1-4</u>: Let the direct sum of the component equations be written as in form (3-1-22). Let S, N, N_1, B, B_1, A, and A_1 be as defined in Theorem 3-1-3. If the matrix (3-1-25) has linearly dependent rows, the system has no unique solution.

Corollary 3-1-4 is important to the synthesis problem, and indicates that if a given subassembly has no unique solution, the circuits and cutsets involved in the dependent set (3-1-25) must be altered.

Corollary 3-1-5: Let the direct sum of the component equations be written in form (3-1-23), where E_1'' and E_2'' are constant matrices. Suppose for some real number c

1. $X^T E_1'(c) E_2'^T(c) X < 0$ for all real $X \neq 0$,
2. $X^T E_1'' E_2''^T X \leq 0$ for all real X.

Let S denote the set of edges corresponding to the components of E_1'' and E_2''. Let N, N_1, B, B_1, A, and A_1 be defined as in Theorem 3-1-3. Then the system has a unique solution if, and only if, the matrix

$$\begin{bmatrix} E_1'' & E_2'' \\ B_1 & 0 \\ 0 & A_1 \end{bmatrix} \tag{3-1-28}$$

has linearly independent rows.

Proof: Identical to Theorem 3-1-3 where the constant c is substituted for the interval I_1.

Corollary 3-1-6: Suppose, just for this corollary, driver type elements only are allowed to compose $E_1''(s)$ and $E_2''(s)$. Let the direct sum of the component equations be given in form (3-1-22), where for some real number c, $X^T E_1'(c) E_2'^T(c) X \neq 0$ for all $X \neq 0$. Then any system containing component equations (3-1-22) has a unique solution if, and only if, there exists no circuit of across drivers nor cutset of thru drivers.

<u>Proof</u>: The conditions of Corollary 3-1-5 are met. Therefore, the system has a unique solution if, and only if, (3-1-28) has linearly independent rows. For these component equations rewrite (3-1-28) as follows:

$$\begin{bmatrix} \begin{bmatrix} U & O \\ O & O \end{bmatrix} & \begin{bmatrix} O & O \\ O & U \end{bmatrix} \\ B_1 & O \\ O & A_1 \end{bmatrix} \tag{3-1-29}$$

Let P be the edges corresponding to the across drivers. Obviously, the rows of (3-1-29) are linearly independent if, and only if, there exists no non-zero vector in $(N \times S) \times P$ or $(N_1 \times S) \times (S-P)$. There exists a non-zero vector in $(N \times S) \times P$ or in $(N_1 \times S) \times (S-P)$ if, and only if, there exists an elementary vector in $(N \times S) \times P$ or in $(N_1 \times S) \times (S-P)$. There exists an elementary vector in $(N \times S) \times P$ or in $(N_1 \times S) \times (S-P)$ if, and only if, there exists a circuit of elements of P (or a cutset of elements of $(S-P)$).

Suppose the system is composed of linear components (3-1-1) but which are entirely algebraic and time-varying. After suitable manipulation the system can be described by (3-1-8) and, of course, the necessary and sufficient condition for a unique solution for $t = c$ is that the determinant not vanish for $t = c$. These systems were studied in [WI-1]. The following corollary is a useful addition to the results of [WI-1].

Corollary 3-1-7: Suppose the component equations, CE, are algebraic and time-varying linear, and $E_1(t)$ and $E_2(t)$ are composed of the direct sum of $E_1'(t)$ and $E_1''(t)$ and $E_2'(t)$ and $E_2''(t)$ respectively. Let G, S, N, N_1, B, B_1, A, and A_1 be as in Theorem 3-1-3. Then, for each t such that:

1. $X^T E_1'(t)\, E_2'^T(t)\, X < 0$ for all real $X \neq 0$,
2. $X_1^T E_1''(t)\, E_2''^T(t)\, X_1 \leq 0$ for all real X_1,

the system $\{CE, G, F\}$ has a unique solution if, and only if, the matrix

$$\begin{bmatrix} E_1''(t) & E_2''(t) \\ B_1 & 0 \\ 0 & A_1 \end{bmatrix}$$

has linearly independent rows.

Proof: Similar to Theorem 3-1-3 with obvious changes of notation.

By the following Lemma and remark, the matrix of (3-1-25) has linearly independent rows if, and only if, a matrix with fewer rows has linearly independent rows.

Lemma 3-1-2: Let

$$E = \begin{bmatrix} E_1(s) & E_2(s) \\ B_1 & O \\ O & A_1 \end{bmatrix} \quad \text{and } K_o = \begin{bmatrix} E_1(s) & E_2(s) \end{bmatrix} \begin{bmatrix} A_2^T & O \\ O & B_2^T \end{bmatrix}$$

where A_1 and B_1 have linearly independent columns and A_2 spans the orthogonal complement of the rows of B_1, B_2 spans the orthogonal complement of the rows of A_1. Then E has linearly independent rows if, and only if, K_o has linearly independent rows.

Proof: If E has linearly dependent rows, there exists a non-zero column vector, $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, with polynomial entries such that

$$Z_1^T \begin{bmatrix} E_1(s) & E_2(s) \end{bmatrix} = Z_2^T \begin{bmatrix} B_1 & O \\ O & A_1 \end{bmatrix} \tag{3-1-30}$$

The vector Z_1 is non-zero since the matrix on the right of (3-1-30) has linearly independent rows and $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ is non-zero.

Also

$$Z_2^T \begin{bmatrix} B_1 & O \\ O & A_1 \end{bmatrix} \begin{bmatrix} A_2^T & O \\ O & B_2^T \end{bmatrix} = O$$

by hypothesis. Therefore,

$$Z_1^T K_o = 0$$

and K_o has linearly dependent rows.

Conversely, if K_o has linearly dependent rows, there exists some non-zero column vector Z_1 such that

$$Z_1^T K_o = 0$$

$$\text{or} \quad Z_1^T \begin{bmatrix} E_1(s) & E_2(s) \end{bmatrix} = Z_2^T \begin{bmatrix} B_1 & 0 \\ 0 & A_1 \end{bmatrix}$$

for some row vector Z_2^T. Therefore,

$$\begin{bmatrix} Z_1^T & -Z_2^T \end{bmatrix} E = 0$$

and E has linearly dependent rows.

<u>Remark</u>: Let the component equations be given in the form (3-1-22) and let G, S, N, N_1, B_1, and A_1 be defined as in Theorem 3-1-3. Let A_2 be the representative matrix of $(N_1 \cdot S)$ and B_2 be the representative matrix of $(N \cdot S)$. By [TU-4], $(N_1 \cdot S)$ is the orthogonal complement of $(N \times S)$, and $(N \cdot S)$ is the orthogonal complement of $(N_1 \times S)$. Then by Lemma 3-1-2 it follows that (3-1-25) has linearly independent rows if, and only if, $\begin{bmatrix} E_1''(s)A_2^T, E_2''(s)B_2^T \end{bmatrix}$ has linearly independent rows.

<u>Definition 3-1-3</u>: A set of component equations in the form

$$\begin{bmatrix} E_1(s) & 0 \\ 0 & E_2(s) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} ,$$

where $E_2(s)$ is the orthogonal complement to $E_1(s)$ for every s, is called a perfect coupler component.

If the components represented by $E_1''(s)$ and $E_2''(s)$ in Theorem 3-1-3 are perfect coupler components, then by a procedure analogous to that used in Lemma 3-1-2 it can be shown that the system has a unique solution if, and only if, the matrix

$$\begin{bmatrix} B_1 \left[E_2''(s)\right]^T & 0 \\ 0 & A_1 \left[E_1''(s)\right]^T \end{bmatrix} \qquad (3\text{-}1\text{-}31)$$

has linearly independent rows. Since this matrix usually has very few rows, the independence of the rows is easily checked.

<u>Corollary 3-1-8</u>: Let CE, G, S, N, N_1, B, B_1, A, and A_1 be as in Theorem 3-1-3. Let E be a non-singular submatrix (i.e., a matrix with a non-zero determinant) of $\left[E_1''(s)\ E_2''(s)\right]$. Let S_1 be the edges of G corresponding to the columns of E_1'' in E. Let S_2 be the edges of G corresponding to the columns of E_2'' in E. If no edge of S_1 is in a circuit of $G \cdot S$ and no edge of S_2 is in a cutset of $G \times S$, the system $\{CE, G, F\}$, has a unique solution.

<u>Proof</u>: By Theorem 3-1-3, $\{CE, G, F\}$ has a complete and unique solution if, and only if, (3-1-25) has linearly independent rows. Under the hypothesis and by (2-2-2), upon permutation of its columns, (3-1-25) becomes

$$\begin{bmatrix} D_1 & E_3 & D_2 & E_6 \\ D_3 & E_4 & D_4 & E_7 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \qquad (3\text{-}1\text{-}32)$$

where D_1 and D_2 correspond to the edges of S_1 and S_2 respectively and B_2 and A_2 are suitable submatrices of B_1 and A_1 respectively.

The matrix $\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$ has linearly independent rows by hypothesis. The matrix $\begin{bmatrix} B_2 & 0 \\ 0 & A_2 \end{bmatrix}$ has linearly independent rows by definition. Therefore (3-1-32) has linearly independent rows and $\{CE, G, F\}$ has a unique solution.

Theorem 2.4 of [WI-1] is a special case of Corollary 3-1-8.

<u>Corollary 3-1-9</u>: Let the component equations, CE be given in the form (3-1-22), where $E_1'(s)$, $E_2'(s)$, $E_1''(s)$, $E_2''(s)$ are as in Theorem 3-1-3. Let the semi-definite component equations* of CE be called C_2 and the edges corresponding to the C_2 variables be called S. If either the subsystem

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*See page 85.

$\left\{C_2,\ G \times S,\ F\right\}$ or $\left\{C_2,\ G \cdot S,\ F\right\}$ has a unique solution, the system $\{C,\ G,\ F\}$ has a unique solution.

<u>Proof</u>: By (2-2-2), the subsystem $\left\{C_2,\ G \times S,\ F\right\}$ corresponds to the vector spaces $(N \cdot S)$ and $(N_1 \times S)$, and the subsystem $\left\{C_2,\ G \cdot S,\ F\right\}$ corresponds to the vector spaces $(N \times S)$ and $(N_1 \cdot S)$. (Here the notation of the paragraph preceding definition 3-1-3 is used.)

By Theorem 3-1-3, the system has a unique solution if, and only if, (3-1-25), has linearly independent rows. But $(N \times S)$ is contained in $(N \cdot S)$ and $(N_1 \times S)$ is contained in $(N_1 \cdot S)$. The theorem follows immediately by noting that if the rows of the representative matrix of $\left\{C_2,\ G \times S,\ F\right\}$ or $\left\{C_2,\ G \cdot S,\ F\right\}$ are linearly independent, the rows of (3-1-25) are linearly independent, since these are a subset of the rows of the above.

<u>Remark</u>: Indefinite components (i.e., those where $E_1\ E_2^T$ is an indefinite matrix) can be included in each of the above theorems and corollaries if it is assumed that

a. each edge corresponding to an indefinite component is either

   1) in a circuit of across drivers, (call all these edges S_1) or,

   2) in a cutset of thru drivers, and (call these edges S_2),

b. the indefinite component equations are such that they can be written explicit in the thru variables corresponding to the edges of S_1 and the across variables corresponding to the edges of S_2,

In this case, the circuit and cutset equations together with the driver components uniquely determine the across variables corresponding to the edges of S_1 and the thru variables corresponding to the edges of S_2. From the component equations the remaining variables are given explicitly. Therefore the thru and across variables for these edges can be treated as known, and the edges removed from discussion.

Algorithms

The following definitions and theorems, besides being useful for analysis, are the foundations of an algorithm for determining the class of all graphs that yield a unique solution for a given set of component equations.

In the following, again assume there are no driver type components, (3-1-1), since all graphs yielding a unique solution can be obtained from the graphs of the system without drivers by the techniques of Theorem 3-1-1 and Corollary 3-1-1.

Let A be a representative matrix for the graphic vector space N_1.

<u>Definition 3-1-4</u>: Let T be the set of all forests of a finite graph G. Let every $t_i \in T$ be assigned a distinct positive integer. Let $S = \{+1, -1\}$.

The function sgn T $[i, j]$: $(T \times T) \longrightarrow S$, is defined as follows:

sgn T $[i, j] = +1$ if forest t_i and forest t_j have the same sign determinant in the incidence matrix.

sgn T $[i, j] = -1$ otherwise.

Therefore, sgn T $[i, j]$ defines a partition of T into two disjoint classes.

<u>Definition 3-1-5</u>: Let the columns of A be numbered in the natural order. The function sgn $[i]$: $T \longrightarrow S$, is defined as follows:

sgn $[i] = +1$ if the sum of the column numbers of A corresponding to the edges of tree t_i is even.

sgn $[i] = -1$ otherwise.

By Theorem 2-1-3, definition 3-1-5 corresponds to determining the relative determinantal sign between a forest and its co-forest.

<u>Definition 3-1-6</u>: The signed summation of all tree products is defined when the admittance matrix, $Y_{adm.}$, exists

(i.e., when in (3-1-7), $E_2 = U$, and $Y_{adm.} = E_1$), and is

$$\sum_{i>j} (\text{sgn } T[i, j]) \left[(\text{det. } Y_{adm.}^{(i)(j)}) + (\text{det. } Y_{adm.}^{(j)(i)})\right]$$

$$+ \sum_{i} (\text{det. } Y_{adm.}^{(i)(j)})$$

where $Y_{adm.}^{(i)(j)}$ is the submatrix of $Y_{adm.}$ composed of the rows corresponding to the edges of the forest i and of the columns corresponding to the edges of forest j, both taken in their natural order.

Definition 3-1-7: The signed summation of all co-tree products is defined when the impedence matrix $Z_{imp.}$, exists (i.e., when in (3-1-7), $E_1 = U$, and $Z_{imp.} = E_2$), and is

$$\sum_{i>j} (\text{sgn } T[i, j]) \; (\text{sgn } [i]) \; (\text{sgn } [j]) \left[(\text{det. } Z_{imp.}^{(i)(j)}) + (\text{det. } Z_{imp.}^{(j)(i)})\right] + \sum_{i} (\text{det. } Z_{imp.}^{(i)(i)})$$

where $Z_{imp.}^{(i)(j)}$ is the submatrix of $Z_{imp.}$ composed of the rows corresponding to the edges of the co-forest of forest i, and of the columns corresponding to the edges of the co-forest of forest j, both taken in their natural order.

Definition 3-1-8: The signed summation of all [tree - co-tree] products is defined when the component equations are written in the form:

$$R(s)\,\psi + E_1(s)\,X + E_2(s)\,Y = F_4(t) \qquad (3\text{-}1\text{-}32)$$

(This form includes both (3-1-9) and (3-1-7)), and is

$$\sum_{i,j} \text{sgn } T\,[i,\ j]\ \text{sgn }[j]\ \left(\det.\begin{bmatrix} R & E_1^{(i)} & E_2^{(j)} \end{bmatrix}\right)$$

where $E_1^{(i)}$ are columns of E_1 corresponding to tree (i), and $E_2^{(j)}$ are the columns of E_2 corresponding to co-tree (j), both taken in their natural order.

Theorem 3-1-4: Let

$$K_o = \begin{bmatrix} R & E_1 & E_2 \\ O & B & O \\ O & O & A \end{bmatrix}$$

where B and A are given in (2-3-1). Then det. K_o is equal to (± 1) times the signed summation of all [tree - co-tree] products of the graph corresponding to B and A.

Proof: By Lemma 3-1-1,

$$\det.\ K_o = (\pm\ 1)\ \det.\ E$$

where

$$E = \begin{bmatrix} R & E_1 & E_2 \end{bmatrix} \begin{bmatrix} U & O & O \\ O & A^T & O \\ O & O & B^T \end{bmatrix} \qquad (3\text{-}1\text{-}33)$$

But by the Cauchy-Binet determinant expansion,

$$\det.\ E = \sum_{i,j} (\det.\ A^{(i)})\ (\det.\ B^{(j)})\ (\det.\ R\ E_1^{(i)} E_2^{(j)})$$

(3-1-34)

where $A^{(i)}$ and $E_1^{(i)}$ are a set of columns of A and E_1 (resp.) corresponding to the sequence (i), and $B^{(j)}$ and $E_2^{(j)}$ are a set of columns of B and E_2 (resp.) corresponding to the sequence (j). All columns are taken in their natural order. The sequences (i) and (j) are both strictly monotonic. Now A and B are regular so they can be chosen so that every square submatrix of maximum order must have a determinant ± 1 or 0. By Theorem 3.2.2 of [FR-1] (see also [SE-1]), every determinant of maximum rank minor of A corresponds to a forest, and every determinant a maximum rank minor of B corresponds to a co-forest. It follows from Definitions 3-1-4 and 3-1-5 that

$$(\det.\ A^{(i)})(\det.\ B^{(j)}) = (\mathrm{sgn.}\ T[i,j])(\mathrm{sgn.}[j]) \qquad (3\text{-}1\text{-}35)$$

Substituting (3-1-35) into (3-1-34) and using definition 3-1-8, the theorem follows.

<u>Corollary 3-1-10</u>: If the component equations are given in form (3-1-32), then there exists a unique solution for all system variables if, and only if, the signed summation of all [tree - co-tree] products of the graph is not identically zero.

<u>Theorem 3-1-5</u>: If the component equations are given in form (3-1-7), where $E_1(s) = U$ (unit matrix), and the entries of E_2 are rational functions of s, then there exists a unique solution for all system variables if, and only if, the signed summation of all co-tree products of the graph is not identically zero.

<u>Proof</u>: By Lemma 3-1-1, the system has a unique solution if, and only if, the matrix

$$\begin{bmatrix} U & E_2(s)\ B^T \\ B & O \end{bmatrix} \tag{3-1-36}$$

is non-singular. But (3-1-36) is non-singular if, and only if,

$$F = B\ E_2(s)\ B^T$$

is non-singular. By the Cauchy-Binet expansion,

$$\det.F = \sum_{i,j} (\det.B^{(i)})(\det.E_2{}^{(i)(j)})(\det.B^{(j)}) \neq 0, \tag{3-1-37}$$

where $B^{(i)}$ is a set of columns of B corresponding to sequence (i), $E^{(i)(j)}$ is the submatrix of E with rows corresponding to sequence (i) and columns corresponding to sequence (j). All columns are taken in their natural order. (i) and (j) are both strictly monotonic sequences of positive integers. Choose B so that every square submatrix of maximum order must have a determinant ± 1 or 0. (Theorem

2-2-2.) By [FR-1], [SE-1], every determinant of a maximum rank minor of B corresponds to a co-forest. Therefore from Definitions 3-1-3 and 3-1-4,

$$(\det. B^{(i)})(\det. B^{(j)}) = (\mathrm{sgn}.\, T[i,j])(\mathrm{sgn}.(i))(\mathrm{sgn}.(j)) \tag{3-1-38}$$

Substituting (3-1-38) into 3-1-37), making use of the fact that $(\mathrm{sgn}.\ T\ [i,i]) = 1$, and applying Definition 3-1-7, the theorem follows.

Corollary 3-1-11: If the component equations are given in form (3-1-7) where $E_2(s) = U$ and the entries in $E_1(s)$ are rational functions in s, then there exists a unique solution for all system variables if, and only if, the signed summation of all tree products of the graph is not identically zero.

Proof: Identical to Theorem 3-1-5, with obvious changes of notation.

Remark: By Lemma 3-1-1, det. F in (3-1-37), is equal to $(\pm 1)\det. K_o$.

Theorem 3-1-4 has an immediate practical application in the analysis of general linear systems. In topological analysis, (see [SE-1]), formulas for network determinants are confined to the components where the impedance or admittance matrix exist. Theorem 3-1-4 on the other hand, applies to network determinants with no restriction on the types of components.

Theorem 3-1-5, and Corollaries 3-1-10 and 3-1-11, provide the basis for two algorithms that generate the graphs of all systems that have a unique solution.

Corollary 3-1-10 immediately suggests the following algorithm.

Algorithm-1

Let the component equations be given in form (3-1-32). Consider the matrix of order $(\ell + n)x(\ell + 2n)$

$$[R(s)\ E_1(s)\ E_2(s)] \tag{3-1-39}$$

obtained from the matrices of (3-1-32). Suppose E_1 and E_2 each have n-columns and K is the set of columns from E_1. Let the columns of E_1 and E_2 be each numbered from 1 to n in their natural order. Determine the values of all $\binom{2n}{n}\left(= \frac{(2n)!}{(n!)^2}\right)$ determinants of the set L of maximum order square submatrices of (3-1-39), where each submatrix in L contains all columns of R.

Let those submatrices of L having i columns from K be designated L_i. Each L_i has $\binom{n}{i}^2$ elements and $\bigcup_{i=0}^{n} L_i = L$. For each matrix, B, of L_i, let (k) represent the sequence of columns of B from E_1, and (j) the sequence of columns of E_2 not in B.

Now proceed as follows for each i:

1. For a square matrix J_i made up of the determinants of each element of L_i, where the rows of J_i correspond to the sequences (k), and the columns of J_i correspond to the sequences (j), and if $(k) = (j)$, the corresponding entry is on the diagonal of J_i. Therefore, the (k,j) entry of J_i corresponds to the sequences (k) and to the sequence (j).
2. For each column of J_i, evaluate the sum $\sum(j) = \sum_{h=1}^{i} j_h$ where $(j) = (j_1, \ldots j_i)$. If the sum is odd, change the sign of all entries in this column of J_i. (By definition 3-1-5, this corresponds to evaluating the function sgn. [j].)
3. Find all solutions to the equation

$$X \; J_i \; X^T = 0 \qquad (3\text{-}1\text{-}40)$$

   where X is a row vector, with all entries $+1$, -1, or 0 of dimension $\binom{n}{i}$.
4. The r^{th} entry of X corresponds to the sequence representing the r^{th} row of J_i. Each such sequence corresponds to a unique canonical basic form where the columns of K are the basic

unities. Therefore, every vector X corresponds to a unique homogeneous multilinear form, A_X, of degree i. Thus for each solution, X, to (3-1-40), form A_X.

5. By the techniques of Corollaries 2-1-1, and 2-1-2, examine each A_X obtained above to see whether it is simple.
6. Examine each simple A_X to determine whether it contains a sub-sum with a complete-5 or Thompson reduced residue set. If A_X does not contain a sub-sum with either of these, A_X is graphic, by Theorem 2-2-12.

   Conversely if a simple A_X does contain such a sub-sum, then A_X is not graphic.
7. The set of all graphic simple A_X represents the set of all graphs that yield no unique solution to the linear system.

Remark 1: Since (3-1-40) is a quadratic form, the solutions of (3-1-40) may sometimes be more easily found by making J_i upper triangular. This can be accomplished by simply adding the (k,j) entry to the (j,k) entry, $k>j$, and then substituting zero for each entry below the diagonal of J_i. Also since the component equations are a direct sum, most of the entries of J_i are zero.

Remark 2: An analysis of the negative definite, semi-definite, and indefinite portions of J_i in (3-1-40) provides a fundamental insight into the uniqueness problem for linear systems. Possible system behavior characteristics are also found in the matrix J_i as (3-1-40) represents the system determinant by Theorem 3-1-4; i.e.,

$$X \, J_i \, X^T = (\pm) \text{ det. } K_o. \qquad (3\text{-}1\text{-}41)$$

The matrix J_i is a function of the component equations only. The vector X is a function of the graph only. Consequently, it is believed that (3-1-40), when thought of as a system determinant, is a fundamental structural tool in the synthesis of linear systems.

Remark 3: The solution of equations such as (3-1-40) with integer constraints has been studied by many authors in various facets of quadratic and nonlinear programming (see [GR-1]).

Step 3 can be changed into a programming problem since (3-1-40) can be squared and the resulting function minimized, subject to the constraints on X.

Remark 4: Equation 3-1-41 has deeper significance than the algorithm mentions. Suppose that a given system performance is desired (as reflected in the system determinant, for instance, certain eigenvalues may be wanted). Then given a set of components with parameter and unrestrained interconnections, J_i can be determined as a function of the parameters from the component outer products for each i.

By solving (3-1-41) with the desired determinant either an exact solution or a "best fit" in the squared sense of Remark 3 can be obtained.

Remark 5: Step 5 can be altered if a given entry of X is assumed to be non-zero. Then the conditions given in Corollaries 2-1-1 and 2-1-2 can be stated in terms of $\left[\binom{n}{i} - 1 - i(n-i)\right]$ quadratic equations. The resulting equations can be solved simultaneously with (3-1-40). However, because of the large number of equations and the assumption of a non-zero entry of X it seems that the method given in step 5 is preferable.

Remark 6: The entire algorithm can be fitted to machine computation since Grassman algebra (used in steps 5 and 6) can easily be performed by a computer.

Remark 7: The work of evaluating the $\binom{2n}{n}$ determinants can be performed by Grassman algebra, since by (2-1-2), the $\binom{2n}{n}$ determinants are simply part of the Grassman outer product of the subspace spanned by the rows of (3-1-39), with the columns as the basic unities.

<u>Remark 8</u>: Equation 3-1-41 has profound applications to topological analysis since in this case the graph is given so X can be found as the outer product of the graphic vector space N_1, and J_i can be determined immediately from the outer product of the component equations.

<u>Remark 9</u>: In step 3 of the algorithm, if all primitive vectors, X, are found that satisfy (3-1-40), then all unions of disjoint primitive vectors are also solutions.

In cases where an impedance or admittance matrix exists, $R(s)$ in (3-1-39) has no columns, and the determinants involved can be evaluated with smaller submatrices. The following algorithm is the adaption of Algorithm 1 to these cases.

<u>Algorithm-2</u>

Let the component equations be given in form (3-1-7), and assume the impedance (admittance) matrix exists. (See definitions 3-1-6 and 3-1-7.) Suppose E_1 and E_2 each have n-columns and number them in both E_1 and E_2 from 1 to n in their natural order. Evaluate the $\binom{2n}{n}$ determinants of all square submatrices, L, of the impedance matrix E_2 (or of the admittance matrix E_1, if the admittance matrix exists).

For the admittance matrix let those submatrices of L of order i be designated L_i. For the impedance matrix, let those submatrices of L of order $(n-i)$ be designated L_i. Each L_i has $\binom{n}{i}^2$ elements and $\bigcup_{i=0}^{n} L_i = L$. For the admittance matrix, for each matrix, B, of L_i, let (k) represent the sequence of rows of B, and (j) the sequence of columns of B. For the impedance matrix, and for each matrix, B, of L_i, let (k) represent the sequence of rows not in B, and (j) the sequence of columns not in B.

Now proceed as follows for each i:

1. Form a square matrix J_i made up of the determinants of each element of L_i, where the rows of J_i correspond to the sequences (k), and the columns of J_i correspond to the sequences (j), and if $(k) = (j)$, the corresponding entry is on the diagonal of J_i.
2. If this algorithm is being carried out on the admittance matrix, skip this step. If this algorithm is being carried out on the impedance matrix, for each column of J_i, evaluate the sum $\Sigma(j)$ as in (2) of Algorithm-1. If the sum is odd, change the sign of all entries in this column and in its corresponding row of J_i. (This corresponds to the evaluation of sgn. [j] in Definitions 3-1-5 and 3-1-7.)

3. Find all solutions to equation (3-1-40) where X is a row vector with all entries $+1$, -1, or 0.
4. Same as (4) of Algorithm-1.
5. Same as (5) of Algorithm-1.
6. Same as (6) of Algorithm-1.
7. Same as (7) of Algorithm-1.

Remark 1: By the remark after Theorem 3-1-5, and Corollary 3-1-11, (3-1-41) is the system determinant when the impedance or admittance matrix exist.

Remark 2: The solution of (3-1-40) and the work of step 5 are simplified when all components are two terminal. Then J_i is diagonal for all i, so signs of the entries of X have no effect on (3-1-40) and step 5 can be performed in (mod. 2) arithmetic.

Suppose the component equations are given in form (3-1-7). Each of the above algorithms for a graph of n-edges requires the evaluation of $\binom{2n}{n}$ determinants of order less than or equal to n. An alternate method examines the determinant of a matrix of form (3-1-8), or of the form of K_o in Theorem 3-1-4, for every graphic subspace N_1. (Each of these latter matrices has order $2n$) and the number of different graphic subspaces for $n > 4$ is far greater than $\binom{2n}{n}$ as the following table shows.

TABLE (3-1-1)

n = no. of edges	no. of different graphic subspaces for non-separable graphs	$\binom{2n}{n}$
2	2	6
3	8	20
4	64	70
5	832	252
6	10,336	924
7	139,904	3,432

Even this table is not a complete comparison since the number of non-separable graphs on n-edges are only a portion (about one-third in the above examples) of the total number of all graphs on n-edges.

<u>Example</u>: Of Algorithm-2:

Suppose the direct sum of the impedance matrices is:

$$\begin{bmatrix} \frac{-6}{s} & & & & & \\ & -5s^2+1 & & & & \\ & & -1 & 5 & & \\ & & 4 & -1 & & \\ & & & & -2 & \\ & & & & & 1 \end{bmatrix}$$

Since this matrix is non-singular, if each edge of the graph is a self-loop the system has a unique solution.

Also, if all edges of the graph form a forest, the system has a unique solution. The matrices J_1, J_2, J_3, J_4, and J_5, defined by the algorithm are:

$$J_5 = \left\{ \begin{array}{cccccc} (23456) & (13456) & (12456) & (12356) & (12346) & (12345) \\ \frac{-6}{s}, & \underline{-5s^2+1}, & -1, & -1, & -2, & 1, \end{array} \right.$$

$$\left. \begin{array}{c} (12456)\ (12356) \\ -9 \end{array} \right\}$$

$$J_4 = \left\{ \begin{array}{cccccc} (3456) & (2456) & (2356) & (2346) & (2345) & (1456) \\ \frac{-6(-5s^2+1)}{s}, & \frac{6}{s}, & \frac{6}{s}, & \frac{12}{s}, & \frac{-6}{s}, & -(-5s^2+1), \end{array} \right.$$

$$\begin{array}{ccccc} (1356) & (1346) & (1345) & (1256) & (1246) \\ -(-5s^2+1), & -2(-5s^2+1), & +(-5s^2+1), & -19, & 2, \end{array}$$

$$\begin{array}{ccccc} (1245) & (1236) & (1235) & (1234) & (2456)(2356) \\ -1, & 2, & -1, & -2, & \frac{+54}{s}, \end{array}$$

$$\left. \begin{array}{ccc} (1456)\ (1356) & (1246)(1236) & (1245)(1235) \\ -9(-5s^2+1), & +18, & -9 \end{array} \right\}$$

$$J_3 = \left\{ \begin{array}{cccc} (456) & (356) & (346) & (345) \\ \frac{+6(-5s^2+1)}{s}, & \frac{6}{s}(-5s^2+1), & \frac{12}{s}(-5s^2+1), & \frac{-6}{s}(-5s^2+1), \end{array}\right.$$

$$\begin{array}{ccccccc} (256) & (246) & (245) & (236) & (235) & (234) & (156) \\ \frac{114}{s}, & \frac{-12}{s}, & \frac{6}{s}, & \frac{-12}{s}, & \frac{6}{s}, & \frac{12}{s}, & -19(-5s^2+1), \end{array}$$

$$\begin{array}{cccc} (146) & (145) & (136) & (135) \\ 2(-5s^2+1), & -1(-5s^2+1), & 2(-5s^2+1), & -1(-5s^2+1), \end{array}$$

$$\begin{array}{cccccc} (134) & (124) & (126) & (125) & (123) & (456)(356) \\ -2(-5s^2+1), & 2, & 38, & -19, & 2, & \frac{+54}{s}(-5s^2+1), \end{array}$$

$$\begin{array}{cccc} (246)(236) & (245)(235) & (146)(136) & (145)(135) \\ \frac{-108}{s}, & \frac{+54}{s}, & +18(-5s^2+1), & -9(-5s^2+1), \end{array}$$

$$\left. \begin{array}{c} (124)(123) \\ 18 \end{array} \right\}$$

$$J_2 = \left\{ \begin{array}{cccc} (56) & (46) & (45) & (36) \\ \frac{114}{s}(-5s^2+1), & \frac{-12}{s}(-5s^2+1), & \frac{6}{s}(-5s^2+1), & \frac{-12}{s}(-5s^2+1), \end{array}\right.$$

$$\begin{array}{cccccc} (35) & (34) & (26) & (25) & (24) & (23) \\ \frac{6}{s}(-5s^2+1), & \frac{12}{s}(-5s^2+1), & \frac{-228}{s}, & \frac{114}{s}, & \frac{-12}{s}, & \frac{-12}{s}, \end{array}$$

$$\begin{array}{cccc} (16) & (15) & (14) & (13) \\ 38(-5s^2+1), & -19(-5s^2+1), & 2(-5s^2+1), & 2(-5s^2+1), \end{array}$$

$$\left.\begin{array}{ccccc} (12) & (46)(36) & (45)(35) & (24)(23) & (14)(13) \\ 38, & \frac{-108}{s}(-5s^2+1), & \frac{+54}{s}(-5s^2+1), & \frac{-108}{s}, & +18(-5s^2+1) \end{array}\right\}$$

$$J_1 = \left\{\begin{array}{cccc} (6) & (5) & (4) & (3) \\ \frac{-228}{s}(-5s^2+1), & \frac{114}{s}(-5s^2+1), & \frac{-12}{s}(-5s^2+1), & \frac{-12}{s}(-5s^2+1), \end{array}\right.$$

$$\left.\begin{array}{ccc} (2) & (1) & (4)(3) \\ \frac{-228}{s}, & 38(-5s^2+1), & \frac{-108}{s}(-5s^2+1) \end{array}\right\}$$

where each of the above J_i's has had its sign changed as in step 2, and made upper triangular. For shortness of notation, J_i is not written in matrix form and only the non-zero elements of J_i have been given together with their corresponding sequences. The diagonal elements of J_i refer to only one sequence.

The work to obtain J_i can be shortened by using the results of J_{i-1}.

Substituting J_5 into (3-1-40), solving for X, and forming A_X gives the four multilinear forms:

$$+1\ z_{12456} \pm 1\ z_{12345},\ +1\ z_{12356} \pm 1\ z_{12345}$$

where $Z_{ij} = Z_i Z_j$. This means that edge 6 and edge 3 or 4 cannot form both a cutset and a circuit. Proceeding analogously, by steps 3 and 4, the remaining solutions to (3-1-40) yield:

from J_4:

$(+z_{2456} \pm z_{2345}), (+z_{2356} \pm z_{2345}), (+z_{1456} \pm z_{1345}), (+z_{1356}$
$\pm z_{1345}), (+z_{1246} \pm z_{1234}), (+z_{1236} \pm z_{1234}), (+z_{2456} \pm z_{2345}$
$\pm z_{1456} \pm z_{1345}), (z_{2456} \pm z_{2345} \pm z_{1356} \pm z_{1345}), (z_{2456} \pm z_{2345}$
$\pm z_{1246} \pm z_{1234}), (z_{2456} \pm z_{2345} \pm z_{1236} \pm z_{1234}), (z_{2456} \pm z_{2345}$
$\pm z_{1456} \pm z_{1345} \pm z_{1246} \pm z_{1234}), (z_{2456} \pm z_{2345} \pm z_{1456} \pm z_{1345}$
$\pm z_{1236} \pm z_{1234}), (z_{2456} \pm z_{2345} \pm z_{1356} \pm z_{1345} \pm z_{1246} \pm z_{1234}),$
$(z_{2456} \pm z_{2345} \pm z_{1356} \pm z_{1345} \pm z_{1236} \pm z_{1234}), (z_{2356} \pm z_{2345}$
$\pm z_{1456} \pm z_{1345}), (z_{2356} \pm z_{2345} \pm z_{1356} \pm z_{1345}), (z_{2356} \pm z_{2345}$
$\pm z_{1246} \pm z_{1234}), (z_{2356} \pm z_{2345} \pm z_{1236} \pm z_{1234}), (z_{2356} \pm z_{2345}$
$\pm z_{1456} \pm z_{1345} \pm z_{1246} \pm z_{1234}), (z_{2356} \pm z_{2345} \pm z_{1456} \pm z_{1345}$
$\pm z_{1236} \pm z_{1234}), (z_{2356} \pm z_{2345} \pm z_{1356} \pm z_{1345} \pm z_{1246} \pm z_{1234}),$
$(z_{2356} \pm z_{2345} \pm z_{1356} \pm z_{1345} \pm z_{1236} \pm z_{1234}), (z_{1456} \pm z_{1345}$
$\pm z_{1246} \pm z_{1234}), (z_{1456} \pm z_{1345} \pm z_{1236} \pm z_{1234}), (z_{1356} \pm z_{1345}$
$\pm z_{1246} \pm z_{1234}), (z_{1356} \pm z_{1345} \pm z_{1236} \pm z_{1234}),$

From J_3: $(+z_{456} \pm z_{345}), (+z_{356} \pm z_{345}), (+z_{246} \pm z_{234}), (+z_{236} \pm z_{234}),$
$(+z_{146} \pm z_{134}), (+z_{136} \pm z_{134}), (z_{456} \pm z_{345} \pm z_{246} \pm z_{234}),$

$(z_{456} \pm z_{345} \pm z_{236} \pm z_{234})$, $(z_{456} \pm z_{345} \pm z_{146} \pm z_{134})$, $(z_{456} \pm z_{345} \pm z_{136} \pm z_{134})$, $(z_{456} \pm z_{345} \pm z_{246} \pm z_{234} \pm z_{146} \pm z_{134})$, $(z_{456} \pm z_{345} \pm z_{246} \pm z_{234} \pm z_{136} \pm z_{134})$, $(z_{456} \pm z_{345} \pm z_{236} \pm z_{234} \pm z_{146} \pm z_{134})$, $(z_{456} \pm z_{345} \pm z_{236} \pm z_{234} \pm z_{136} \pm z_{134})$, $(z_{356} \pm z_{345} \pm z_{246} \pm z_{234} \pm z_{146} \pm z_{134})$, $(z_{356} \pm z_{345} \pm z_{246} \pm z_{234} \pm z_{136} \pm z_{134})$, $(z_{256} \pm z_{345} \pm z_{236} \pm z_{234} \pm z_{146} \pm z_{134})$, $(z_{356} \pm z_{345} \pm z_{236} \pm z_{234} \pm z_{136} \pm z_{134})$, $(z_{356} \pm z_{345} \pm z_{246} \pm z_{234})$, $(z_{356} \pm z_{345} \pm z_{236} \pm z_{234})$, $(z_{356} \pm z_{345} \pm z_{146} \pm z_{134})$, $(z_{356} \pm z_{345} \pm z_{136} \pm z_{134})$, $(z_{246} \pm z_{234} \pm z_{146} \pm z_{134})$, $(z_{246} \pm z_{234} \pm z_{136} \pm z_{134})$, $(z_{236} \pm z_{234} \pm z_{146} \pm z_{134})$, $(z_{236} \pm z_{234} \pm z_{136} \pm z_{134})$

from J_2: $(+z_{46} \pm z_{34})$, $(+z_{36} \pm z_{34})$

from J_1: There are none.

Application of step 5 of the Algorithm, shows that the following of the above forms are simple:

from J_5: $(z_{12456} \pm z_{12345})$, $(z_{12356} \pm z_{12345})$

from J_4: $(z_{2456} \pm z_{2345}), (z_{2356} \pm z_{2345}), (z_{1456} \pm z_{1345}), (z_{1356} \pm z_{1345}), (z_{1246} \pm z_{1234}), (z_{1236} \pm z_{1234}), (z_{2456} \pm z_{2345} \pm z_{1456} + (\mathrm{sfn})\ z_{1345}), (z_{2456} \pm z_{2345} \pm z_{1246} - (\mathrm{sfn})\ z_{1234}), (z_{2356} \pm z_{2345} \pm z_{1356} + (\mathrm{sfn})\ z_{1345}), (z_{2356} \pm z_{2345} \pm z_{1236} - (\mathrm{sfn})\ z_{1234}), (z_{1456} \pm z_{1345} \pm z_{1246} - (\mathrm{sfn})\ z_{1234}), (z_{1356} \pm z_{1345} \pm z_{1246} - (\mathrm{sfn})\ z_{1234})$

From J_3: $(z_{456} \pm z_{345}), (z_{356} \pm z_{345}), (z_{246} \pm z_{234}), (z_{236} \pm z_{234}), (z_{146} \pm z_{134}), (z_{136} \pm z_{134}), (z_{456} \pm z_{345} \pm z_{246} - (\mathrm{sfn})\ z_{234}), (z_{456} \pm z_{345} \pm z_{146} - (\mathrm{sfn})\ z_{134}), (z_{456} \pm z_{345} \pm z_{246} - (\mathrm{sfn})\ z_{234} \pm z_{146} - (\mathrm{sfn})\ z_{134}), (z_{356} \pm z_{345} \pm z_{236} - (\mathrm{sfn})\ z_{234} \pm z_{136} - (\mathrm{sfn})\ z_{134}), (z_{356} \pm z_{345} \pm z_{236} - z_{234}), (z_{356} \pm z_{345} \pm z_{136} - (\mathrm{sfn})\ z_{134}), (z_{246} \pm z_{234} \pm z_{146} + (\mathrm{sfn})\ z_{134}), (z_{236} \pm z_{234} \pm z_{136} + (\mathrm{sfn})\ z_{134})$,

from J_2: $(z_{46} \pm z_{34}), (z_{36} \pm z_{34})$

where (sfn.) is either equal to (+1), or (-1) and is evaluated as follows:

Consider the (sfn) coefficient of the basic form $z_{i'jkl'}$ (or z_{ijk}). Take all the residue sets of this basic form, ignoring the elements of the residue sets which have an (sfn) as a coefficient. Multiply the coefficients of the two remaining elements of the residue sets to obtain (sfn).

Since there are only six edges in the graph, the multilinear forms contain no forbidden residue set, and are therefore, graphic. They represent the set of graphs in which there is no unique solution. All other graphs have a unique solution.

Part II. Non-Linear Systems

The first two theorems, (3-2-1), and (3-2-2), of this section apply to system components of Type 3 (see Chapter II, Part 3). As will be shown in Chapter V, these results apply also to many components of Type 2. These theorems are an extension of the results of [MI-1] and [DU-2], to general positive semi-definite components.

Theorem 3-2-3 and Theorem 3-2-4 are restricted to components of Type 1. Theorem 3-2-3 and Theorem 3-2-4 are a generalization of most known results on monotonic mappings as found in [DU-2], [DU-3], [DU-4], [WI-1], [DE-1]. Many of the results published there are special cases of these two theorems.

Most of the mathematical details upon which the results of Part II are based, are contained in Appendix B.

The conditions called for in the theorems are given in the appendix.

Before giving these theorems, some pertinent definitions for systems with Type 3 components are introduced. The reader is assumed to be familiar with the definitions of a vector space, norm, and Lebesgue integral.

Formally, a real Hilbert space, H, is a complete normed real vector space with an inner product defined on it. A complete space is one in which every Cauchy sequence converges to an element of the space. An inner product (denoted $\langle\ ,\rangle$) on a real vector space, H, is a symmetric bilinear mapping from H X H into the real numbers such that $\langle x,x\rangle \geq 0$, and $\|x\|^2 = \langle x,x\rangle$, ($\|\ \|$ denotes the norm that is defined on the space). A Banach space is defined as a complete normed vector space. Therefore a Hilbert space is simply a Banach space with an inner product defined on it.

A few examples of real Hilbert spaces are:

1. the space ℓ_2, of sequences $\{y_i\}$ of real numbers, such that $\sum_i y_i^2 < \infty$, with inner product

$$\langle\{x_i\}, \{y_i\}\rangle = \sum_i x_i\, y_i .$$

2. a real finite dimensional Euclidean space in which the inner product is the sum of the products of the coordinates for an orthonormal basis set.

3. the space L_2 (a,b); i.e., the set of real valued functions $y(s)$ of the real variable s such that the Lebesgue integral $\int_a^b y^2(s)\, ds < \infty$, with inner product

$$\langle y(s), x(s) \rangle = \int_a^b y(s)\, x(s)\, ds.$$

4. the space used in describing components of Type 3 is the space $L_{2,m}(a,b)$; i.e., the set of real m-tuple valued functions $Y(s)$ of the real variable s, such that

$$\int_a^b \sum_{i=1}^{m} y_i^2 (s)\, ds < \infty$$

where the integral is again Lebesgue and $y_i(s)$ are the real coordinates of the m-tuple $Y(s)$, with inner product

$$\langle Y(s), X(s) \rangle = \int_a^b \sum_{i=1}^{m} y_i(s)\, x_i(s)\, ds.$$

It can be shown that ℓ_2 and L_2 are isomorphic and, in fact, that every separable infinite dimensional real Hilbert space is isomorphic to these. Also, every finite dimensional real Hilbert space is isomorphic to the real Euclidean space of the same dimension.

Lemma's B-11 and B-12 use the concept of a direct product of Hilbert spaces. As shown on (p. 303), of [RI-1], the direct product space, $H \times H$, of any Hilbert space, H, is also a Hilbert space defined as the set of all ordered pairs (x,y) where $x \in H$ and $y \in H$. The inner product on, $H \times H$, is defined as

$$\langle (x_1, y_2), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle .$$

Scalar multiplication is defined as $c(x,y)=(cx,cy)$. Addition is defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Higher order product spaces are defined similar to the above.

Components of Type 3 are either mappings from the real Euclidean space of dimension m into itself or mappings from the space $L_{2,m}(a,b)$ into itself, where m may vary from one component to another and where a and b are finite.

Algebraic component equations of Types 1 and 2 are examples of a mapping from E_m into itself. (E_m is the Euclidean space of dimension m.) An example of mapping from $L_{2,m}$ into itself, is a mixed algebraic and integral operator of Type 2; i.e.,

$$Z_o(t) = \int_a^b F(Z_i(s), s, t)\, ds + G(Z_i(t), t).$$

<u>Definition 3-2-1</u>: A mapping F of a Hilbert space H into itself is called <u>monotonic</u> if $\langle x_1 - x_2, F(x_1) - F(x_2) \rangle \geq 0$ for all x_1 and x_2 in H. If

$$\langle x_1 - x_2, F(x_1) - F(x_2) \rangle \geq c \, \| x_1 - x_2 \|^2$$

for some constant $c > 0$, and for $x_1, x_2 \in H$, F is called <u>strongly monotonic</u>.

<u>Definition 3-2-2</u>: A mapping F from the Hilbert space H into itself is said to satisfy a Lipschitz condition on D, if

$$\|F(x_1) - F(x_2)\| \leq c \|x_1 - x_2\| \qquad (c = \text{constant})$$

for all x_1 and x_2 in D. If $D=H$, F is said simply to satisfy a Lipschitz condition.

Lemmas B-7 and B-8 provide a basis for examining operators on a Hilbert space for monotonicity. Lemma B-9 provides a practical method for examining algebraic equations, and Lemmas B-10 and B-11, together with the remark after Lemma B-8 provide a basis for checking integral equations for monotonicity. In general, Lemmas B-9, B-10, and B-11, can be used to examine all points $x_o \in H$ that satisfy these lemmas in a neighborhood of x_o. The remaining points must be checked by some other means.

For mappings in $L_{2,m}$, the following notation is used in this thesis. Since a component equation is, by definition, a set of relations between a distinguished basis set on a finite dimensional vector space, the set of m-tuples that are the range of $y(s) \in L_{2,m}$ are considered to be the finite dimensional vector space. The distinguished basis set for this space is chosen as the orthonormal set $\rho_1=(1,0,\ldots,0)$, $\rho_2=(0,1,0,\ldots, 0)$, etc. The constraint (graph) equations are defined in terms of this distinguished basis set.

The component equations of Type 3 for component j are:

$$Z_{o_j}(s,t) = F(Z_{i_j}(s,t),\ t) \qquad (3\text{-}2\text{-}1)$$

where

$$Z_{o_j}(s,t) = \begin{bmatrix} X_{o_j}(s,t) \\ Y_{o_j}(s,t) \end{bmatrix}, \quad Z_{i_j}(s,t) = \begin{bmatrix} Y_{i_j}(s,t) \\ X_{i_j}(s,t) \end{bmatrix}$$

and corresponding entries of the Z_{o_j} and Z_{i_j} m_j-tuples are the paired distinguished basis elements which correspond to to the m_j edges of a graph G_j and $Z_{o_j}(s,t)$ and $Z_{i_j}(s,t)$ are assumed to be elements of $L_{2,m_j}(a,b)$. The variable t is a parameter of the mapping (not necessarily time).

The constraint equations for mappings in the Hilbert space, $L_{2,m}$ are of the form given in (2-3-1), since they only relate values in the range of $Z_o(s,t)$ and $Z_i(s,t)$, where Z_o and Z_i are the respective direct sums of Z_{o_j} and Z_{i_j}.

In this section and in Chapter IV it is more convenient to rewrite (2-3-1), in the form of the primary and secondary variables of Frame and Koenig, (see [FR-1]).

Existence Theorems for Systems of Type 3 Components

Suppose all components in the system are of form (3-2-1). Let the topology (or interconnection pattern) of the system graph be such that there exists a forest T for which:

1. The direct sum of the terminal equations for all m components in the system can be written in the form:

$$\begin{aligned}
Z_{s1} &= F_1\,(Z_{p1}, Z_{s2}, t) \\
Z_{p2} &= F_2\,(Z_{p1}, Z_{s2}, t) \\
Z_{s3} &= F_3\,(Z_{p3}, Z_{s4}, t) \\
Z_{p4} &= F_4\,(Z_{p3}, Z_{s4}, t) \\
Z_{p5} &= F_5\,(Z_{s5}, t) \\
Z_{p6} &= F_6\,(t)
\end{aligned} \qquad (3\text{-}2\text{-}2)$$

where the mapping

$$\begin{bmatrix} F_1\,(Z_{p1}, Z_{s2}, t) \\ F_2\,(Z_{p1}, Z_{s2}, t) \end{bmatrix}$$

satisfies condition (C1) of Appendix-B for all t on set I, and the mapping

$$\begin{bmatrix} F_3\,(Z_{p3}, Z_{s4}, t) \\ F_4\,(Z_{p3}, Z_{s4}, t) \end{bmatrix}$$

is continuous in all variables except t, and monotonic for all t on I.

2. The constraint equations of the system graph, G, for correspondence F, are of the form

$$\begin{bmatrix} Z_{s1} \\ Z_{s2} \\ Z_{s3} \\ Z_{s4} \\ Z_{s5} \\ Z_{s6} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & Q_{16} \\ -Q_{12}^T & 0 & 0 & 0 & 0 & Q_{26} \\ -Q_{13}^T & 0 & Q_{33} & Q_{34} & 0 & Q_{36} \\ -Q_{14}^T & 0 & -Q_{34}^T & Q_{44} & 0 & Q_{46} \\ 0 & 0 & 0 & 0 & 0 & Q_{56} \\ -Q_{16}^T & -Q_{26}^T & -Q_{36}^T & -Q_{46}^T & -Q_{56}^T & Q_{66} \end{bmatrix} \begin{bmatrix} Z_{p1} \\ Z_{p2} \\ Z_{p3} \\ Z_{p4} \\ Z_{p5} \\ Z_{p6} \end{bmatrix} \quad (3\text{-}2\text{-}3)$$

where Q_{ii} is skew for $i=1, 3, 4, 6$.

Let S_{pi} denote the subset of edges of $E(G)$ corresponding to the variables Z_{pi} for $(i=1, 2, 3, 4, 5, 6)$. Let S^x_{pi} and S^y_{pi} denote the subset of edges of $E(G)$ corresponding to the variables X_{pi} and Y_{pi} respectively for $(i=1, 2, 3, 4, 5, 6)$.

Then (3-2-3) restricts G in the following four ways:

1. $C(G \cdot S^x_{p6}) = \emptyset$ (the empty set);

2. $B(G \times S^y_{p6}) = \emptyset$;

3. $B([G \cdot (S^y_{p5} \cup S^x_{p6})] \times S^y_{p5}) = \emptyset$;

4. $C([G \times (S^x_{p5} \cup S^y_{p6})] \cdot S^x_{p5}) = \emptyset$

This statement follows from (2-2-2) and (2-2-3), and the fact that the remaining zeros can be obtained by suitable choice of forest (see [WI-1]).

Note: Throughout this thesis, $C(G_1)$ and $B(G_1)$ represent the circuit and cutset matroids respectively on the graph G_1.

Lemma 3-2-1: Let the component and constraint equations for $\{CE, G, F\}$ be given in (3-2-2) and (3-2-3) respectively.

Then there exists a forest T_2 with constraint equations (3-2-3) in which $S_{p3} = 0$ and Q_{44} is a zero matrix if, and only if, the following two conditions are satisfied:

1. $B([G \times (S^y_{p1} \cup S^x_{p2} \cup S_{p3} \cup S_{p4} \cup S^y_{p6})] \cdot (S^x_{p3} \cup S^y_{p4})) = \emptyset$,

2. $C([G \cdot (S^x_{p1} \cup S^y_{p2} \cup S_{p3} \cup S_{p4} \cup S^x_{p6})] \times (S^y_{p3} \cup S^x_{p4})) = \emptyset$.

Proof: Conditions (1) and (2) are satisfied if, and only if, the following two conditions are satisfied:

3. $B([G \times (S^y_{p1} \cup S_{p3} \cup S_{p4} \cup S^y_{p6})] \cdot (S^x_{p3} \cup S^y_{p4})) = \emptyset$,

4. $C([G \cdot (S^x_{p1} \cup S_{p3} \cup S_{p4} \cup S^x_{p6})] \times (S^y_{p3} \cup S^x_{p4})) = \emptyset$.

To show that (1) is true if, and only if, (3) is true, let

$$G_{10} = [G \times (S^y_{p1} \cup S^x_{p2} \cup S_{p3} \cup S_{p4} \cup S^y_{p6})] \cdot (S^x_{p3} \cup S^y_{p4})$$

$$G_{20} = [G \times (S^y_{p1} \cup S_{p3} \cup S_{p4} \cup S^y_{p6})] \cdot (S^x_{p3} \cup S^y_{p4}),$$

$$G_{30} = G \times (S^y_{p1} \cup S_{p3} \cup S_{p4} \cup S^y_{p6}) \text{ and}$$

$$G_{40} = G \times (S^y_{p1} \cup S^x_{p2} \cup S_{p3} \cup S_{p4} \cup S^y_{p6}).$$

Then $B(G_{20}) \subset B(G_{10})$, so if $B(G_{10}) = \emptyset$, $B(G_{20}) = \emptyset$. If $B(G_{10}) \neq \emptyset$, let $e_1 \neq \emptyset$, and $e_1 \in B_1 \in B(G_{10})$. Then by (2-2-2), $(B_1 \cup B_2) \in B(G_{40})$ for some set $B_2 \subset E(G)$. If $e_2 \in S^x_{p2} \cap (B_1 \cup B_2)$, then by definition of S^x_{p2}, there is another $B_3 \in B(G_{40})$ such that $B_3 \subset (S^y_{p1} \cup S^y_{p6} \cup B_1 \cup B_2) - \{e_2\}$ and $e_1 \in B_3$. If $e_3 \in S^x_{p2} \cap B_3$, for the same reason as above there exists another $B_4 \in B(G_{40})$ such that $e_1 \in B_4$ and $B_4 \subset (S^y_{p1} \cup S^y_{p6} \cup B_3) - \{e_3\}$. Proceeding in this fashion if $e_i \in (B_i \cap S^x_{p2})$, there exists another $B_{i+1} \in B(G_{40})$ such that $e_1 \in B_{i+1}$, $B_{i+1} \subset (S^y_{p1} \cup S^y_{p6} \cup B_i) - \{e_i\}$. Since S^x_{p2} is finite, there exists some $B_n \in B(G_{40})$ such that $e_1 \in B_n$ and $B_n \subset (S^y_{p1} \cup S^y_{p6} \cup S_{p3} \cup S_{p4})$. Therefore $e_1 \in B_n \in B(G_{30})$ and $e_i \in B_n \cap (S^x_{p3} \cup S^y_{p4}) \in B(G_{20}) \neq \emptyset$.

It follows that (1) is true, if, and only if, (3) is true.

The proof that (2) is true if, and only if, (4) is true, and is identical to the above with obvious changes of notation.

Now assume that (3) and (4) are true. By the definition of a forest and co-forest in Chapter II,

$(S^x_{p1} \cup S^y_{p3} \cup S^x_{p4} \cup S^x_{p6})$ can be contained in a forest, T_2,

and $(S^y_{p1} \cup S^x_{p3} \cup S^y_{p4} \cup S^y_{p6})$ can be contained in the co-forest

of T_2, since the two sets are disjoint. (See Theorem 6-10 of [SE-1].) For forest $T_2, S_{p3} = \emptyset$, and T_2 has constraint equations (3-2-3).

By (2-2-3) and (3) and (4) above, the matrix $Q_{44} = 0$.

Conversely suppose $Q_{44} = 0$ and $S_{p3} = \emptyset$ in (3-2-3) for forest T_2. Then by (2-2-3), (3) and (4) are true.

<u>Theorem 3-2-1</u>: Let the component and constraint equations for $\{CE, G, F\}$ be given in (3-2-2) and (3-2-3) respectively. Suppose the following two conditions are satisfied:

1. $B([G \times (S^y_{p1} \cup S^x_{p2} \cup S_{p3} \cup S_{p4} \cup S^y_{p6})] \cdot (S^x_{p3} \cup S^y_{p4})) = \emptyset$

2. $C([G \cdot (S^x_{p1} \cup S^y_{p2} \cup S_{p3} \cup S_{p4} \cup S^x_{p6})] \times (S^y_{p3} \cup S^x_{p4})) = \emptyset$.

Then $\{CE, G, F\}$ has a unique solution for all t on I.

Proof: By Lemma 3-2-1, there exists a forest T_2 with constraint equations (3-2-3) in which $S_{p3} = \emptyset$ and Q_{44} is a zero matrix.

Substituting (3-2-3) into (3-2-2) and reducing,

$$F_1(Z_{p1}, -Q_{12}^T Z_{p1} + Q_{26} F_6(t), t) - Q_{11} Z_{p1} - Q_{12} F_2(Z_{p1}, -Q_{12}^T Z_{p1} + Q_{26} F_6(t), t) - Q_{14} F_4(-Q_{14}^T Z_{p1} + Q_{46} F_6(t), t)$$

$$= Q_{16} F_6(t). \qquad (3\text{-}2\text{-}4)$$

By Lemma B6 the left hand side of (3-2-4) satisfies condition (C1) for all t on I. Therefore by Theorem B1, there exists a unique solution for Z_{p1} for all t. All other variables can be obtained uniquely from these.

The following Corollary is an important special case of Theorem 3-2-1.

Corollary 3-2-1: Let the component and constraint equations for $\{CE, G, F\}$ be given in (3-2-2) and (3-2-3) respectively. If $(S_{p3} \cup S_{p4}) = \emptyset$, the system $\{CE, G, F\}$ has a unique solution for all t on I.

Lemma 3-2-2: Let the component and constraint equations for $\{CE, G, F\}$ be given in (3-2-2) and (3-2-3) respectively. Then there exists a forest T_2 with constraint equations (3-2-3) in which $S_{p4} = \emptyset$ and Q_{13}^T is maximum row rank if, and only if, the following four conditions are satisfied:

1. $C(G \cdot (S^x_{p1} \cup S^y_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p6})) - C(G \cdot (S^x_{p1} \cup S^y_{p2} \cup S^x_{p6})) = \emptyset$

2. $B(G \times (S^y_{p1} \cup S^x_{p2} \cup S^y_{p3} \cup S^x_{p4} \cup S^y_{p6})) - B(G \times (S^y_{p1} \cup S^x_{p2} \cup S^y_{p6})) = \emptyset$

3. $C(G \cdot (S_{p3} \cup S_{p4} \cup S^x_{p6})) = \emptyset$

4. $B(G \times (S_{p3} \cup S_{p4} \cup S^y_{p6})) = \emptyset$.

Proof: Conditions (1) and (2) are satisfied if, and only if, the following two conditions are satisfied:

5. $C(G \cdot (S^x_{p1} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p6})) = \emptyset$;

6. $B(G \times (S^y_{p1} \cup S^y_{p3} \cup S^x_{p4} \cup S^y_{p6})) = \emptyset$.

To show this let

$$G_1 = G \cdot (S^x_{p1} \cup S^y_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p6})$$

$$G_o = G \cdot (S^x_{p1} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p6}) \text{ and}$$

$$G_2 = G \cdot (S^x_{p1} \cup S^y_{p2} \cup S^x_{p6}).$$

If $C_1 \in C(G_o)$, then $C_1 \notin C(G_2)$ since $(S^x_{p1} \cup S^x_{p6})$ contains no circuits by its definition. Therefore, $C_1 \in C(G_1) - C(G_2)$.

Conversely if $C_1 \in C(G_1) - C(G_2)$ and $e_1 \in C_1 \cap S^y_{p2}$, by definition of S^y_{p2} there exists another circuit $C_2 \in C(G_1) - C(G_2)$ such that $C_2 \subset (S^x_{p1} \cup S^x_{p6} \cup C_1) - \{e_1\}$.

If $e_2 \in C_2 \cap S^y_{p2}$, for the same reason as above there exists another circuit $C_3 \subset (S^x_{p1} \cup S^x_{p6} \cup C_2) - \{e_2\}$. Proceeding in this fashion, if $e_i \in C_i \cap S^y_{p2}$, there exists another circuit $C_{i+1} \subset (S^x_{p1} \cup S^x_{p6} \cup C_i) - \{e_i\}$. Since S^y_{p2} is finite, there exists some circuit $C_n \subset (S^x_{p1} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p6})$. Therefore $C(G_o) \neq \emptyset$. This shows that (5) is true, if, and only if, (1) is true.

The proof that (6) is true, if, and only if, (2) is true is identical to the above with obvious changes of notations.

Assume (3), (4), (5), and (6), are true. Since (5) and (6) are true, by the proof for Lemma 3-2-1, there exists a forest T_2 with constraint equations (3-2-3), such that $S_{p4} = \emptyset$.

Since there is no circuit in $(S_{p3} \cup S_{p4} \cup S^x_{p6})$, by (2-2-2) and definition (2-2-1), S^x_{p1} must contain a dendroid of $C(G \cdot (S^x_{p1} \cup S_{p3} \cup S_{p4} \cup S^x_{p6})$.

Also since there is no cutset contained in $(S_{p3} \cup S_{p4} \cup S^y_{p6})$, by (2-2-2) and definition (2-2-1), S^y_{p1} must contain a dendroid of $B(G \times (S^y_{p1} \cup S_{p3} \cup S_{p4} \cup S^y_{p6}))$. Together these two conditions imply, by Lemma 2-2-2, that the matrix Q^T_{13} has maximum row rank.

Conversely suppose there exists a forest T_2 with constraint equations (3-2-3) in which $S_{p4} = \emptyset$ and Q^T_{13} has maximum row rank. By Lemma 3-2-1, (5) and (6) are true.

By Lemma 2-2-2, S^y_{p1} and S^x_{p1} contain a dendroid of $B(G \times (S^y_{p1} \cup S_{p3} \cup S_{p4} \cup S^y_{p6}))$ and $C(G \cdot (S^x_{p1} \cup S_{p3} \cup S_{p4} \cup S^x_{p6}))$ respectively, so $B(G \times (S_{p3} \cup S_{p4} \cup S^y_{p6})) = \emptyset$ and $C(G \cdot (S_{p3} \cup S_{p4} \cup S^x_{p6})) = \emptyset$.

<u>Theorem 3-2-2</u>: Let the component equations, CE, of Type 3, be given in form (3-2-2) and the constraint equations in form (3-2-3). Suppose:

1. the mapping $\begin{bmatrix} F_1 \ (Z_{p1}, Z_{s2}, t) \\ F_2 \ (Z_{p1}, Z_{s2}, t) \end{bmatrix}$ satisfies condition (L1) for all t on I;
2. $C(G \cdot (S^x_{p1} \cup S^y_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p6})) - C(G \cdot (S^x_{p1} \cup S^y_{p2} \cup S^x_{p6})) = \emptyset$;
3. $B(G \times (S^y_{p1} \cup S^x_{p2} \cup S^y_{p3} \cup S^x_{p4} \cup S^y_{p6})) - B(G \times (S^y_{p1} \cup S^x_{p2} \cup S^y_{p6})) = \emptyset$;
4. $C(G \cdot (S_{p3} \cup S_{p4} \cup S^x_{p6})) = \emptyset$;
5. $B(G \times (S_{p3} \cup S_{p4} \cup S^y_{p6})) = \emptyset$.

Then the system $\{CE, G, F\}$ has a unique solution for all t on I.

<u>Proof</u>: By Lemma 3-2-2, there exists a forest T_2 with constraint equations (3-2-3) in which $S_{p4} = \emptyset$ and Q^T_{13} has maximum row rank.

Substituting (3-2-3) into (3-2-2) yields

$$F(Z_{p1},-Q_{12}^{T} Z_{p1} + Q_{26} F_6(t),t)-Q_{11}Z_{p1}-Q_{12}F_2(Z_{p1},-Q_{12}^{T} Z_{p1} + Q_{26} F_6(t),t)$$

$$= Q_{13} Z_{p3} + Q_{16} F_6(t) \qquad (3\text{-}2\text{-}5)$$

and

$$F_3(Z_{p3}, t) + Q_{13}^{T} Z_{p1} - Q_{33} Z_{p3} = Q_{36} F_6(t). \qquad (3\text{-}2\text{-}6)$$

The left hand side of (3-2-5) satisfies condition (L1). By Theorem B1 and Lemma B6,

$$Z_{p1} = F_7(Q_{13} Z_{p3} + Q_{16} F_6(t), t) \qquad (3\text{-}2\text{-}7)$$

where $F_6(t)$ is specified and F_7 satisfies condition (C1) and is continuous in all variables except t.

Substituting (3-2-7) into (3-2-6) gives

$$F_3(Z_{p3}, t) + Q_{13}^{T}F_7(Q_{13}Z_{p3} + F_8(t)) - Q_{33}Z_{p3} = Q_{36}F_6(t). \quad (3\text{-}2\text{-}8)$$

By Lemmas B5 and B6, the left hand side of (3-2-8) satisfies condition (C1) so by Theorem B1, Z_{p3} is uniquely determined. From Z_{p3}, all other variables can be uniquely determined.

Corollary 3-2-2: Let the component and constraint equations for $\{CE, G, F\}$ be given in (3-2-2) and (3-2-3) respectively. Suppose:

1. the mapping $\begin{bmatrix} F_1\ (Z_{p1},\ Z_{s2},\ t) \\ F_2\ (Z_{p1},\ Z_{s2},\ t) \end{bmatrix}$ is strongly monotonic and satisfies a Lipschitz condition for all t on I;

2. $C(G \cdot (S_{p3} \cup S_{p4} \cup S^x_{p6})) = \emptyset$;

3. $B(G\ X\ (S_{p3} \cup S_{p4} \cup S^y_{p6})) = \emptyset$.

Then the system $\{CE,\ G,\ F\}$ has a unique solution for all t on I.

<u>Proof</u>: By (2) and (3), $(S^x_{p3} \cup S^y_{p4} \cup S^x_{p6})$ can be contained in a forest T_2 and $(S^y_{p3} \cup S^x_{p4} \cup S^y_{p6})$ can be contained in the co-forest of T_2. By Lemma B4, the mapping $\begin{bmatrix} F_1\ (Z_{p1},\ Z_{s2},\ t) \\ F_2\ (Z_{p1},\ Z_{s2},\ t) \end{bmatrix}$ can be solved explicitly for the primary variables of T_2, and the resulting mapping is again (STRMLC) for all t on I. Consequently, $S_{p2} = \emptyset$ and conditions (1) and (2) of Theorem 3-2-2 are satisfied. The assertions of the Corollary follow by application of Theorem 3-2-2.

Existence Theorems for Systems of Type 1 Components

Suppose all components in the system are of Type 1. Initial value problems of such systems will be examined. The initial value problems will be called consistent when the initial values of all variables which are differentiated in the equations are given and such values satisfy the system algebraic and constraint equations.

Let the topology of the system graph be such that there exists a forest T for which:

1. the direct sum of the terminal equations for all m components in the system can be written in the form:

$$
\begin{aligned}
\dot{\psi}_1 &= F_1\,(\psi_1, Z_{s1}, t)\\
Z_{p1} &= G_1(\psi_1, Z_{s1}, t)\\
\dot{\psi}_2 &= F_2(\psi_2, Z_{s2}, t)\\
Z_{p2} &= G_2(\psi_2, t)\\
\dot{\psi}_3 &= F_{31}(\psi_3, Z_{p3}, Z_{s4}, Z_{s3}, Z_{p4}, t)\\
\dot{Z}_{p3} &= F_{32}(\psi_3, Z_{p3}, Z_{s4}, Z_{s3}, Z_{p4}, t)\\
\dot{Z}_{s4} &= F_{33}(\psi_3, Z_{p3}, Z_{s4}, Z_{s3}, Z_{p4}, t) \qquad (3\text{-}2\text{-}9)\\
Z_{s5} &= F_5\,(Z_{p5}, Z_{s6}, t)\\
Z_{p6} &= F_6\,(Z_{p5}, Z_{s6}, t)
\end{aligned}
$$

$Z_{s7} = F_7\ (Z_{p7},\ Z_{s8},\ t)$

$Z_{p8} = F_8\ (Z_{p7},\ Z_{s8},\ t)$

$Z_{p9} = F_9\ (t)$

where the mapping $\begin{bmatrix} F_{32}(\psi_3,\ Z_{p3},\ Z_{s4},\ Z_{s3},\ Z_{p4},\ t) \\ F_{33}(\psi_3,\ Z_{p3},\ Z_{s4},\ Z_{s3},\ Z_{p4},\ t) \end{bmatrix}$

is strongly monotonic in the variables Z_{s3}, Z_{p4}, for every ψ_3, Z_{p3}, and Z_{s4} and every t on an open set I of reals, the mapping $\begin{bmatrix} F_5\ (Z_{p5},\ Z_{s6},\ t) \\ F_6\ (Z_{p5},\ Z_{s6},\ t) \end{bmatrix}$

is strongly monotonic for all t on I, the mapping

$\begin{bmatrix} F_7\ (Z_{p7},\ Z_{s8},\ t) \\ F_8\ (Z_{p7},\ Z_{s8},\ t) \end{bmatrix}$ is monotonic for all t on I,

F_1, F_2, F_{31}, F_{32}, F_{33}, F_5, F_6, F_7, F_8, G_1 and G_2

satisfy a Lipschitz condition in all variables except t, and all mappings of (3-2-9) are continuous in t on I.

2. The constraint equations for system graph, G, for correspondence F, are of the form

$$\begin{bmatrix} Z_{s1} \\ Z_{s2} \\ Z_{s3} \\ Z_{s4} \\ Z_{s5} \\ Z_{s6} \\ Z_{s7} \\ Z_{s8} \\ Z_{s9} \end{bmatrix} = \begin{bmatrix} 0 & Q_{12} & Q_{13} & 0 & 0 & 0 & 0 & 0 & Q_{19} \\ -Q_{12}^T & Q_{22} & Q_{23} & 0 & Q_{25} & Q_{26} & Q_{27} & Q_{28} & Q_{29} \\ -Q_{13}^T & -Q_{23}^T & Q_{33} & Q_{34} & Q_{35} & Q_{36} & Q_{37} & Q_{38} & Q_{39} \\ 0 & 0 & -Q_{34}^T & 0 & 0 & 0 & 0 & 0 & Q_{49} \\ 0 & -Q_{25}^T & -Q_{35}^T & 0 & Q_{55} & Q_{56} & Q_{57} & Q_{58} & Q_{59} \\ 0 & -Q_{26}^T & -Q_{36}^T & 0 & -Q_{56}^T & 0 & 0 & 0 & Q_{69} \\ 0 & -Q_{27}^T & -Q_{37}^T & 0 & -Q_{57}^T & 0 & Q_{77} & Q_{78} & Q_{79} \\ 0 & -Q_{28}^T & -Q_{38}^T & 0 & -Q_{58}^T & 0 & -Q_{78}^T & Q_{88} & Q_{89} \\ -Q_{19}^T & -Q_{29}^T & -Q_{39}^T & -Q_{49}^T & -Q_{59}^T & -Q_{69}^T & -Q_{79}^T & -Q_{89}^T & Q_{99} \end{bmatrix} \begin{bmatrix} Z_{p1} \\ Z_{p2} \\ Z_{p3} \\ Z_{p4} \\ Z_{p5} \\ Z_{p6} \\ Z_{p7} \\ Z_{p8} \\ Z_{p9} \end{bmatrix} \quad (3\text{-}2\text{-}10)$$

where Q_{ii} is skew for $i = 2, 3, 5, 7, 8, 9$.

Let S_{pi} denote the subset of edges of $E(G)$ corresponding by F to the variables Z_{pi}, for $i = 1, \ldots, 9$. Let S^x_{pi} and S^y_{pi} denote the subset of edges of $E(G)$ corresponding to the variables X_{pi} and Y_{pi} respectively for $i = 1, \ldots, 9$.

Then (3-2-10) restricts G in the following six ways:

1. $C(G \cdot S^x_{p9}) = \emptyset$ (the empty set);

2. $B(G \times S^y_{p9}) = \emptyset$;

3. $B([G \cdot (S^y_{p1} \cup S^x_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p9})] \times S^y_{p1}) = \emptyset$;

4. $C([G \times (S^x_{p1} \cup S^y_{p2} \cup S^y_{p3} \cup S^x_{p4} \cup S^y_{p9})] \cdot S^x_{p1}) = \emptyset$;

5. $B([G \times (S^y_{p1} \cup S^y_{p2} \cup S^y_{p3} \cup S^x_{p4} \cup S^y_{p9})] \cdot (S^y_{p1} \cup S^y_{p2})) = \emptyset$;

6. $C([G \cdot (S^x_{p1} \cup S^x_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p9})] \times (S^x_{p1} \cup S^x_{p2})) = \emptyset$.

This statement follows from (2-2-2) and (2-2-3), and the fact that the remaining zeros of (3-2-10) can be obtained by suitable choice of forest.

Theorem 3-2-3: Let the component and constraint equations for the system $\{CE, G, F\}$ be given in (3-2-9) and (3-2-10) respectively. Suppose the following conditions are satisfied:

1. $B([G \times (S^y_{p2} \cup S^y_{p3} \cup S^x_{p4} \cup S^y_{p5} \cup S^x_{p6} \cup S_{p7} \cup S_{p8} \cup S^y_{p9})]$

   $\cdot (S^x_{p7} \cup S^y_{p8})) = \emptyset$;

2. $C([G \cdot (S^x_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S^x_{p5} \cup S^y_{p6} \cup S_{p7} \cup S_{p8} \cup S^x_{p9})]$

   $\times (S^y_{p7} \cup S^x_{p8})) = \emptyset$.

Then the system $\{CE, G, F\}$ has a unique solution for a consistent initial value problem in a neighborhood of any t on I.

Proof: By a technique identical to the proof of Lemma 3-2-1, conditions (1) and (2) above are true if, and only if, the following two conditions are true:

3. $B([G \times (S^y_{p2} \cup S^y_{p3} \cup S^y_{p5} \cup S_{p7} \cup S_{p8} \cup S^y_{p9})] \cdot (S^x_{p7} \cup S^y_{p8})) = \emptyset$;

4. $B([G \cdot (S^x_{p2} \cup S^x_{p3} \cup S^x_{p5} \cup S_{p7} \cup S_{p8} \cup S^x_{p9})] \cdot (S^y_{p7} \quad S^x_{p8})) = \emptyset$.

Again following the proof of Lemma 3-2-1, there exists a forest T_2 with constraint equations (3-2-10) such that $S_{p7} = \emptyset$ and $Q_{88} = 0$.

Substituting the appropriate rows of (3-2-10) into the equations for Z_{s5}, Z_{s6} and Z_{p8} in (3-2-9) gives:

$$F_5\,(Z_{p5},\ -Q_{26}^T\,Z_{p2}\ -Q_{36}^T\,Z_{p3}\ -Q_{56}^T\,Z_{p5} + Q_{69}\,Z_{p9},\ t) - Q_{55}\,Z_{p5}$$
$$-Q_{56}\,F_6\,(Z_{p5},\ -Q_{26}^T\,Z_{p2}\ -Q_{36}^T\,Z_{p3}\ -Q_{56}^T\,Z_{p5} + Q_{69}\,Z_{p9},\ t)$$
$$-Q_{58}\,F_8\,(-Q_{28}^T\,Z_{p2}\ -Q_{38}^T\,Z_{p3}\ -Q_{58}^T\,Z_{p5} + Q_{89}\,Z_{p9},\ t) =$$
$$-Q_{25}^T\,Z_{p2}\ -Q_{35}^T\,Z_{p3} + Q_{59}\,Z_{p9} \qquad (3\text{-}2\text{-}11)$$

By Lemma B6, the left hand side of (3-2-11) is strongly monotonic in Z_{p5}. It also satisfies a Lipschitz condition in all variables except t since it is a composite mapping of functions satisfying Lipschitz conditions.

By Lemma B4, the inverse exists and

$$Z_{p5} = F_{10}\,(Z_{p2},\ Z_{p3},\ t) \qquad (3\text{-}3\text{-}12)$$

By Lemma B4 and the Corollary to Lemma B3, F_{10} satisfies a Lipschitz condition in all variables except t and by these Lemmas is continuous in t.

Substituting the appropriate rows of (3-2-10) into the equations for Z_{p3} and Z_{s4} in (3-2-9) gives:

$$Q_{34}^T\ F_{32}\ (\psi_3,\ Z_{p3},\ -Q_{34}^T\ Z_{p3} + Q_{49}\ Z_{p9},\ -Q_{13}^T\ Z_{p1}\ -Q_{23}^T\ Z_{p2}$$

$$+ Q_{33}\ Z_{p3} + Q_{34}\ Z_{p4} + Q_{35}\ Z_{p5} + Q_{36}\ Z_{p6} + Q_{38}\ Z_{p8} + Q_{39}\ Z_{p9},\ Z_{p4},\ t)$$

$$F_{33}\ (\psi_3,\ Z_{p3},\ -Q_{34}^T\ Z_{p3} + Q_{49}\ Z_{p9},\ -Q_{13}^T\ Z_{p1}\ -Q_{23}^T\ Z_{p2} + Q_{33}\ Z_{p3}$$

$$+ Q_{34}\ Z_{p4} + Q_{35}\ Z_{p5} + Q_{36}\ Z_{p6} + Q_{38}\ Z_{p8} + Q_{39}\ Z_{p9},\ Z_{p4},\ t)$$

$$= Q_{49}\ \dot{F}_9\ (t) \qquad (3\text{-}2\text{-}13)$$

By Lemma B6, the left hand side of (3-2-13) is strongly monotonic in Z_{p4}. It also satisfies a Lipschitz condition in all variables except t, and it is continuous in t.

By Lemma B4, the inverse exists and

$$Z_{p4} = F_{11}(Z_{p1},\ Z_{p2},\ Z_{p3},\ Z_{p5},\ Z_{p6},\ Z_{p8},\ t) \qquad (3\text{-}2\text{-}14)$$

By Lemma B4 and the Corollary to Lemma B3, F_{11} satisfies a Lipschitz condition in all variables except t and by these Lemmas is continuous in t.

By (3-2-9), (3-2-10), (3-2-12), and (3-2-14), all terminal variables are known explicitly as a function of ψ_1, ψ_2, ψ_3, Z_{p3}, and t which satisfies a Lipschitz condition and is continuous in t. Substituting these into the differential equations gives a normal form model of the system which satisfies a Lipschitz condition in all variables except t, and is continuous for t on I.

By Theorem 2.1 of [LE-1], the normal form model has a unique solution locally from which all other variables can be determined uniquely.

A different hypothesis on the algebraic equations yields the following.

Theorem 3-2-4: Let the component and constraint equations for the system $\{CE, G, F\}$ be given in (3-2-9) and (3-2-10) respectively. Suppose the following two conditions are satisfied:

1. $C(G \cdot (S^x_{p2} \cup S^x_{p3} \cup S^y_{p4} \cup S_{p7} \cup S_{p8} \cup S^x_{p9}))$

   $-C(G \times (S^x_{p3} \cup S^y_{p4} \cup S^x_{p9})) = \emptyset;$

2. $B(G \times (S^y_{p2} \cup S^y_{p3} \cup S^x_{p4} \cup S_{p7} \cup S_{p8} \cup S^y_{p9}))$

   $-B(G \times (S^y_{p3} \cup S^x_{p4} \cup S^y_{p9})) = \emptyset.$

Then the system $\{CE, G, F\}$ has a unique solution for a consistent initial value problem in a neighborhood of any t on I.

Proof: By the proof for Lemma (3-2-2), (1) and (2) are true if, and only if, the following two conditions are true:

3. $C(G \cdot (S^x_{p2} \cup S^x_{p3} \cup S_{p7} \cup S_{p8} \cup S^x_{p9})) = \emptyset;$

4. $B(G \times (S^y_{p2} \cup S^y_{p3} \cup S_{p7} \cup S_{p8} \cup S^y_{p9})) = \emptyset.$

By (3), (4), and (3-2-10), $(S^x_{p1} \cup S^x_{p2} \cup S^x_{p3} \cup S^x_{p4} \cup S^x_{p7} \cup S^y_{p8} \cup S^x_{p9})$ can be contained in a forest T_2 and $(S^y_{p1} \cup S^y_{p2} \cup S^y_{p3} \cup S^y_{p4} \cup S^y_{p7} \cup S^x_{p8} \cup S^y_{p9})$ can be contained in the co-forest of T_2. By Lemma B4, the mapping $\begin{bmatrix} F_5\,(Z_{p5}, Z_{s6}, t) \\ F_6\,(Z_{p5}, Z_{s6}, t) \end{bmatrix}$

can be solved explicitly for the primary variables of T_2 and the resulting mapping is again (STRMLC) for all t on I. Consequently, the constraint equations for T_2 are in form (3-2-10) with $(S_{p6} \cup S_{p8}) = \emptyset$.

Following the proof for Lemma (3-2-2), Q^T_{57} is maximum row rank.

Substituting the appropriate rows of (3-2-10) into the equations for Z_{s5} and Z_{s7} in (3-2-9) yields

$$F_5(Z_{p5}, t) - Q_{55}Z_{p5} = -Q^T_{25}\,Z_{p2} - Q^T_{35}\,Z_{p3} + Q_{57}\,Z_{p7} + Q_{59}\,Z_{p9}. \tag{3-2-15}$$

The left hand side of (3-2-15) is (STRMLC) for all t on I, so by Lemma B4, it has an inverse

$$Z_{p5} = F_{13}(-Q^T_{25}\,Z_{p2} - Q^T_{35}\,Z_{p3} + Q_{57}\,Z_{p7} + Q_{59}\,F_9\,(t), t).$$

Also by Lemma B4 and the Corollary to Lemma B3, F_{10} is (STRMLC) for all t on I and is continuous in t.

Substituting (3-2-16) and the appropriate row of (3-2-10) into (3-2-9) gives:

$$F_7 (Z_{p7}, t) - Q_{77} Z_{p7} + Q_{57}^T F_{10} (-Q_{25}^T Z_{p2} - Q_{35}^T Z_{p3}$$
$$+ Q_{57} Z_{p7} + Q_{59} F_9 (t), t) = -Q_{27}^T Z_{p2} - Q_{37}^T Z_{p3} + Q_{79} F_9 (t). \tag{3-2-17}$$

By Lemmas B5 and B6, the left side of (3-2-17) is strongly monotonic in Z_{p7} and also satisfies a Lipschitz condition in all variables for all t on I. By Lemma B4, the left hand side of (3-2-17) has an inverse with respect to Z_{p7}. Thus

$$Z_{p7} = F_{14} (Z_{p2}, Z_{p3}, t). \tag{3-2-18}$$

By Lemma B4 and the Corollary to Lemma B3, F_{14} satisfies a Lipschitz condition in all variables except t for all t on I, and also is continuous in t on I.

Substituting suitable rows of (3-2-10) into (3-2-9) yields an equation similar to (3-2-13) and by the same reasons given in Theorem 3-2-3 ,

$$Z_{p4} = F_{15} (Z_{p1}, Z_{p2}, Z_{p3}, Z_{p5}, Z_{p7}, t), \tag{3-2-19}$$

and for all t on I, F_{15} satisfies a Lipschitz condition in all variables except t and is continuous in t.

By (3-2-9), (3-2-10), (3-2-18), and (3-2-19), all terminal variables are known explicitly as a function of $\psi_1, \psi_2, \psi_3, Z_{p3}$ and t, which satisfies a Lipschitz condition for all variables and is continuous in t on I.

Substituting these into the differential equations gives a normal form model of the system which satisfies a Lipschitz condition in all variables except t, and is continuous in t on I.

By Theorem 2.1 of [LE-1], the normal form model has a unique solution locally from which all other variables can be determined uniquely.

Conclusion

Theorem 3-1-3 and its corollaries provide the complete background for the examination of linear semi-definite component systems. By these theorems, the problem of uniqueness is reduced to the examination of the interconnections of only semi-definite components. These theorems provide a fundamental tool in the examination of such systems.

The algorithms of Part I provide the second basic contribution of the thesis. The material here culminates in (3-1-41) which is an expression for the determinant of an arbitrary linear time invariant system in terms of its component equations (reflected in J_i) and its graph (reflected in X). This is the simplest equation yet given in the literature which shows the relation between the system structure and the graph for any general linear time invariant system.

Theorems 3-2-1 and 3-2-2 are the first theorems known to the author on systems in a Hilbert space. These theorems are a generalization to Hilbert space of the monotonic properties of mappings as utilized in Theorems 3-2-3 and 3-2-4.

Theorems 3-2-3 and 3-2-4 are the most extensive existence theorems for algebraic and differential equations yet seen by the author. Practically every other theorem on uniqueness of systems published is a special case of these or a slight modification of a special case. For example, the theorems of [DU-1], [DU-2], [DU-3], [DU-4], [SE-1], [BI-2], [DE-1], and [WI-1], all fall into this category.

CHAPTER IV

STABILITY STUDIES OF SYSTEM SOLUTIONS

Most contemporary stability studies are based on the so-called Second Method of Lyapunov as applied to a set of first-order differential equations characterizing the system. Almost nothing has been said about stability as it relates to the two fundamental structural features of the system; namely, the characteristics of the system components and their topology; i.e., their pattern of interconnection. A given set of system components, for example, may be stable when connected in one manner but unstable when the connections are altered.

This chapter examines several classes of systems, containing both linear and nonlinear components, and establishes sufficient conditions on the topology of the system for stability of a solution. A set of necessary and sufficient conditions for system stability are also given on the component characteristics of a system having a given topology.

In this study components of Type (1) only, are examined for stability of a solution subject to a perturbation in initial conditions.

Stability, as it relates to the structure of the system is concerned, then, with a study of the stability characteristics of the system of e equations of Type (1) when the vector, (Z_0, Z_i) of order 2e is subjected to the e linear constraint equations in (2-3-1). The stability characteristics discussed in this chapter are limited to the class of systems for which the equations characterizing each of the m components in the system are of the form

$$\frac{d}{dt}\psi = F(Z_i, t)$$

$$Z_0 = \psi \qquad (4\text{-}1\text{-}1)$$

or

$$Z_0 = G(Z_i)$$

The most general linear forms considered are

$$\frac{d}{dt}\psi = P(t)\ Z_i + F(t)$$

$$Z_0 = \psi \qquad (4\text{-}1\text{-}2)$$

and

$$Z_0 = C(t)\ Z_i + G(t)$$

The definition of stability considered is that given originally by Lyapunov.

<u>Definition 4-0-1</u>: A solution $\psi_0(t)$ of the system of equations

$$\dot{\psi} = F(\psi, t)$$

is stable for the initial point $t = t_0$ if, and only if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that $||\psi_0(t) - \psi(t)|| < \epsilon$ for all $t \geq t_0$ if $||\psi_0(t_0) - \psi(t_0)|| < \delta$, where $\psi_0(t)$ is an n-tuple of time functions, $\psi(t)$ is an arbitrary solution and the indicated norm is the euclidean norm.

The development is based on the so-called direct method of Lyapunov [HA-1], using a positive definite, real valued, continuous function v on the n+1 dimensional Euclidean space (ψ, t). The system is stable if the total time derivative of v along the trajectories $\psi(t)$ is not positive in H_{h,t_0}: $||\psi - \psi_0|| \leq h$, $t \geq t_0$. Following Hahn [HA-1], if R is the set of real numbers, then a function ϕ: $R \longrightarrow R$ belongs to class K, means that ϕ is a continuous real function defined on the closed interval $0 \leq r \leq h$ and $\phi(r)$ vanishes at $r = 0$ and increases strictly monotonically with r.

A positive definite function v of radius h at ψ_0 is a real valued function from the (ψ, t) space which vanishes at one point ψ_0 for $t \geq t_0$ and in the half-cylindrical neighborhood

$$H_{h,t_0}: ||\psi - \psi_0|| \leq h, \ t \geq t_0$$

and

$$v(\psi, t) \geq \phi(\|\psi - \psi_0\|)$$

A matrix A, whose entries are continuous functions of (Y, t) is said to be positive definite if for any vector $\psi \neq 0$, $\psi^T A \psi > 0$ for all $t \geq t_0$, and all Y, and positive semi-definite if $\psi^T A \psi \geq 0$ for all Y, $t \geq t_0$. It can be shown that the quadratic form associated with a positive definite matrix is a positive definite function of some radius h at $\psi_0 = 0$. Whereas most applications in the literature of positive definite and semidefinite matrices are restricted to symmetric matrices, the applications in this chapter require no such restriction.

Part I. Linear Systems

Let the mathematical models of each of the m multi-terminal components of a system having a graph G be given in one of the three following forms:

1. Dynamic Components

$$\dot{Z}_0 = P(t)\, Z_i + F(t)$$

2. Algebraic Components

$$Z_0 = C(t)\, Z_i + G(t)$$

3. Excitation Components

$$Z_0 = E(t)$$

where $E(t)$, $G(t)$ and $F(t)$ are known continuous vector functions of t, the entries of the matrices $P(t)$ and $C(t)$ are continuous functions of t, and Z_0 and Z_i are complementary terminal vectors, i.e., the direct sum (Z_0, Z_i) contains exactly one component x_j and one component y_j corresponding to each edge in the terminal graph of the component.

Let the topology of the system be such that there exists a forest T_o in G for which:

1. the direct sum of the terminal equations for all m components in the system can be written in the form

$$\begin{bmatrix} \dot{Z}_{p1} \\ \\ \dot{Z}_{s4} \end{bmatrix} = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ \\ P_{21}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} Z_{s1} \\ \\ Z_{p4} \end{bmatrix} + \begin{bmatrix} F_1(t) \\ \\ F_4(t) \end{bmatrix}$$

$$\begin{bmatrix} Z_{p2} \\ \\ Z_{s3} \end{bmatrix} = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ \\ C_{21}(t) & C_{22}(t) \end{bmatrix} \begin{bmatrix} Z_{s2} \\ \\ Z_{p3} \end{bmatrix} + \begin{bmatrix} G_2(t) \\ \\ G_3(t) \end{bmatrix} \tag{4-1-3}$$

$$Z_{p5} = E(t)$$

where $Z_{pj} = (X_{pj}, Y_{pj})$ $(j=1, 2, 3, 4, 5)$ is a vector of primary variables, i.e., all components of X_{pj} correspond to edges in forest T_o (branches) and all components of Y_{pj}

correspond to edges in the complement of T_o (chords). The vector $Z_{sj} = (Y_{sj}, X_{sj})$ (j=1, 2, 3, 4, 5) represents the complement of Z_{pj}, i.e., all components of Y_{sj} correspond to edges in T_o and all components of X_{sj} correspond to edges in the complement of T_o.

2. the constraint equations of the system graph are of the form:

$$\begin{bmatrix} Z_{s1} \\ Z_{s2} \\ Z_{s3} \\ Z_{s4} \\ Z_{s5} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ -Q_{12}^T & Q_{22} & Q_{23} & 0 & Q_{25} \\ -Q_{13}^T & -Q_{23}^T & Q_{33} & 0 & Q_{35} \\ -Q_{14}^T & 0 & 0 & 0 & Q_{45} \\ -Q_{15}^T & -Q_{25}^T & -Q_{35}^T & -Q_{45}^T & Q_{55} \end{bmatrix} \begin{bmatrix} Z_{p1} \\ Z_{p2} \\ Z_{p3} \\ Z_{p4} \\ Z_{p5} \end{bmatrix} \qquad (4\text{-}1\text{-}4)$$

The zeros in (4-1-4) are obtained by selecting the forest T_o such that the vector Z_{s4} is of lowest possible order. Since (4-1-4) represents the constraint equations for one forest T_o, the matrices Q_{11}, Q_{22}, Q_{33}, and Q_{55} are skew. The only restraint that (4-1-4) has placed on the graph G is that there be no cut-sets of thru drivers or circuits of across drivers which is a necessary condition on the system for the existence of a unique solution. [KO-1].

Let S_{pi} denote the subset of edges of $E(G)$ corresponding to the variables Z_{pi} for $(i=1, 2, 3, 4, 5)$. Let S^x_{pi} and S^y_{pi} denote the subset of edges of $E(G)$ corresponding to the variables X_{pi} and Y_{pi} respectively for $(i=1, 2, 3, 4, 5)$.

Let G_1 be the graph $\left[[G \times (E(G) - (S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})] \cdot (S_{p2} \cup S_{p3})\right]$. Let CE represent the component equations (4-1-3), and let F be the correspondence between the component equations and G as described in Chapter II. Then the subsystem $\{CE, G_1, F\}$ has a unique solution for any t if, and only if,

$$\det.\left[\begin{bmatrix} U & O \\ -Q^T_{23} & Q_{33} \end{bmatrix} - \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} \begin{bmatrix} Q_{22} & Q_{23} \\ O & U \end{bmatrix}\right] \neq 0 \quad (4\text{-}1\text{-}5)$$

In this case the component equations are:

$$\begin{bmatrix} Z_{p2} \\ Z_{s3} \end{bmatrix} = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} \begin{bmatrix} Z_{s2} \\ Z_{p3} \end{bmatrix} + \begin{bmatrix} G_2(t) \\ G_3(t) \end{bmatrix}$$

By (2-2-3), the graph equations for G_1 are given by the submatrix in the second and third rows and second and third columns of (4-1-4). Substituting the graph equations from (4-1-4) into the component equations gives

$$\left[\begin{bmatrix} U & O \\ -Q_{23}^T & Q_{33} \end{bmatrix} - \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix} \begin{bmatrix} Q_{22} & Q_{23} \\ O & U \end{bmatrix}\right] \begin{bmatrix} Z_{p2} \\ Z_{p3} \end{bmatrix} = \begin{bmatrix} G_4(t) \\ G_5(t) \end{bmatrix}$$

which yields (4-1-5) immediately.

Suppose (4-1-5) is satisfied. If, in addition the matrix

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$$

in (4-1-3) is positive definite and symmetric, then it is a simple algebraic exercise to show that when the linear constraint equations in (4-1-4) are substituted into (4-1-3), the system model can be reduced to the form

$$\dot{Z}_{p1} = K(t)\ J(t)\ Z_{p1} + F(t) \qquad (4\text{-}1\text{-}6)$$

where $F(t)$ is a continuous vector function of time and

$$J(t)=Q_{11}- \begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} \left[\begin{bmatrix} U & O \\ -Q_{23}^T & Q_{33} \end{bmatrix} - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} Q_{22} & Q_{23} \\ O & U \end{bmatrix}\right]^{-1} \begin{bmatrix} C_{11} & O \\ C_{21}-U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ Q_{13}^T \end{bmatrix}$$

$$K(t) = \left[\xi_{11} + \xi_{12}Q_{14}^T + Q_{14}\xi_{21} + Q_{14}^T\xi_{22}Q_{14}^T\right]^{-1}$$

with

$$\begin{bmatrix} \xi_{11}(t) & \xi_{12}(t) \\ \xi_{21}(t) & \xi_{22}(t) \end{bmatrix} = P(t)^{-1}$$

Since $P(t)$ is symmetric, $P(t)^{-1}$ is symmetric and by Lemma A2, is also positive definite. Since the matrix

$$\begin{bmatrix} U & O \\ Q_{14} & U \end{bmatrix}^T \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} U & O \\ Q_{14}^T & U \end{bmatrix} = \begin{bmatrix} K^{-1} & \xi_{12} + Q_{14}\xi_{22} \\ \xi_{21} + \xi_{22}Q_{14}^T & \xi_{22} \end{bmatrix} \quad (4\text{-}1\text{-}7)$$

is symmetric and positive definite, it follows that K is positive definite and symmetric.

In one of the stability theorems following, $P(t)$ is not necessarily positive definite, only nonsingular and symmetric. If Z_{s4} is of zero order and condition (4-1-5) is satisfied, then the system model in (4-1-6) reduces to

$$\dot{Z}_{p1} = P(t)\, J(t)\, Z_{p1} + F'(t) \quad (4\text{-}1\text{-}8)$$

where $F'(t)$ is a continuous vector function of time.

By definition 4-0-1, stability of any solution depends only on a perturbation of the initial conditions. The perturbation equations for investigating the stability of an arbitrary solution of the linear system

$$\dot{\psi} = A(t)\, \psi + F(t) \quad (4\text{-}1\text{-}9)$$

are determined as follows:

Let $\psi_0(t)$ represent a solution for which stability is to be investigated and let $\psi'(t)$ be the perturbed solution derived from a perturbation of the initial conditions. Let

$$Z(t) = \psi'(t) - \psi_0(t)$$

then (4-1-9) becomes

$$\dot{Z}(t) = A(t)\ Z(t)$$

and it follows that the stability of any solution to the system under consideration is determined by the stability of the homogeneous parts of (4-1-6) and (4-1-8).

In the remainder of Part I, let

$$C(t) = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix}$$

where

$$\begin{bmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{bmatrix}$$

is the matrix of (4-1-3).

The following definition and lemmas are needed for Theorem 4-1-1.

Definition 4-1-1: Let G be a graph and S and T be complimentary non-null subsets of $E(G)$. Then G is said to be separable into the two parts, $G \cdot S$ and $G \cdot T$ if G has no cutsets or circuits with edges in both $G \cdot S$ and $G \cdot T$.

Lemma 4-1-1: There exists a forest T_1 of the system graph, G, that has constraint equations (4-1-4) with Q_{11}, Q_{22}, Q_{23}, and Q_{33} zero if and only if the subgraph

$$\left[[G \times (E(G) - S^x_{p5})] \cdot (E(G) - S_{p5})\right] = G_1$$

is separable into two parts such that S^x_{p1}, S^y_{p4}, S^y_{p2} and S^y_{p3} are in one part and S^y_{p1}, S^x_{p4}, S^x_{p2}, and S^x_{p3} are in the other part.

Proof: By (2-2-3), the first 4 rows and columns of (4-1-4) is a representative matrix for the graphic and cographic vector spaces corresponding to G_1.

Suppose G_1 is separable as above. Let

$$S = (S^x_{p1} \cup S^y_{p4} \cup S^y_{p2} \cup S^y_{p3}).$$

The representative matrices, A, and B, of $B(G)$ and of $C(G)$ respectively can, after rearrangement of rows and columns, be partitioned as $A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix}$, where the columns of A_1 and B_1 correspond to the edges of S. The zero entries of A and B contain Q_{11}, Q_{22}, Q_{33} and Q_{23}.

Conversely suppose Q_{11}, Q_{22}, Q_{33}, and Q_{23} are zero. Then the two representative matrices A, and B, of G_1 can be partitioned as above, so G is separable into $G \cdot S$ and $G \cdot (E(G) - S)$.

Lemma 4-1-2: The submatrix $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ of (4-1-4) has maximum column rank if, and only if, the following two conditions are satisfied:

1. There is no circuit contained in $(S_{p2} \cup S_{p3} \cup S^x_{p5})$,

and

2. There is no cutset contained in $(S_{p2} \cup S_{p3} \cup S^y_{p5})$.

Proof: Assume 1 and 2. Since there is no circuit contained in $(S_{p2} \cup S_{p3} \cup S^x_{p5})$, by (2-2-2) and definition 2-2-1, S^x_{p1} must contain a dendroid of $C(G \cdot (S^x_{p1} \cup S_{p2} \cup S_{p3} \cup S^x_{p5})$. Also since there is no cutset contained in $(S_{p2} \cup S_{p3} \cup S^y_{p5})$, by (2-2-2) and definition 2-2-1, S^y_{p1} must contain a dendroid of $B(G \times (S^y_{p1} \cup S_{p2} \cup S_{p3} \cup S^y_{p5}))$. Together these two conditions imply by Lemma 2-2-2, that the matrix $\begin{bmatrix} Q^T_{12} \\ Q^T_{13} \end{bmatrix}$ has maximum row rank, so its transpose has maximum column rank.

Conversely, assume $[Q_{12}\ Q_{13}]$ is maximum column rank. Then by Lemma 2-2-2, S^y_{p1} and S^x_{p1} contain a dendroid of $B(G \times (S^y_{p1} \cup S_{p2} \cup S_{p3} \cup S^y_{p5})$ and $C(G \cdot (S^x_{p1} \cup S_{p2} \cup S_{p3} \cup S^x_{p5}))$ respectively, so there is no cutset contained in $(S_{p2} \cup S_{p3} \cup S^y_{p5})$ and no circuits contained in $(S_{p2} \cup S_{p3} \cup S^x_{p5})$.

Theorem 4-1-1: Consider a system for which the model is given in the form of component equations (4-1-3) and constraint equations (4-1-4), where $P(t)$ and $C(t)$ are constant matrices and P is symmetric and positive definite. If the graph of the system is such that:

1. The subgraph $\left[[G \times (E(G) - S^x_{p5})] \cdot (E(G) - S_{p5})\right] = G_1$ is separable into two parts, $G_1 \cdot S$ and $G_1 \cdot T$, such that S^x_{p1} and S^y_{p4} are in $G_1 \cdot S$ and S^y_{p1} and S^x_{p4} are in the $G_1 \cdot T$.

2. There is no circuit of G contained in $(S_{p2} \cup S_{p3} \cup S^x_{p5})$.

3. There is no cutset of G contained in $(S_{p2} \cup S_{p3} \cup S^y_{p5})$.

4. The output variables of each of the algebraic components can be partitioned into those corresponding to

edges in $G_1 \cdot S$ or $G_1 \cdot T$. Thus let X_1^i and Y_2^i be the output variables of component i corresponding to the edges of component i in $G_1 \cdot T$. Similarly let X_3^i and Y_4^i be the output variables of component i corresponding to the edges of component i in $G_1 \cdot S$.

Suppose each of the algebraic components of (4-1-3) are as follows:

for component i:

$$\begin{bmatrix} X_1^i \\ Y_2^i \\ X_3^i \\ Y_4^i \end{bmatrix} = \begin{bmatrix} C_{11}^i & C_{12}^i & C_{13}^i & C_{14}^i \\ -C_{12}^{iT} & C_{22}^i & C_{23}^i & C_{24}^i \\ -C_{13}^{iT} & C_{23}^{iT} & C_{33}^i & C_{34}^i \\ C_{14}^{iT} & -C_{24}^{iT} & -C_{34}^{iT} & C_{44}^i \end{bmatrix} \begin{bmatrix} Y_1^i \\ X_2^i \\ Y_3^i \\ X^i \end{bmatrix} + G^i(t) \qquad (4\text{-}1\text{-}10)$$

where C_{11}^i, C_{22}^i, C_{33}^i and C_{44}^i are symmetric, and the

matrix $\begin{bmatrix} C_{22}^i & C_{23}^i \\ C_{23}^{iT} & C_{33}^i \end{bmatrix}$ is nonsingular.

Then any solution to the system is stable if, and only if, C(t) is positive semidefinite.

<u>Proof</u>: Since there is no circuit contained in $(S_{p2} \cup S_{p3} \cup S^{x}_{p5})$, and no cutset contained in $(S_{p2} \cup S_{p3} \cup S^{y}_{p5})$, $(S_{p2} \cup S_{p3})$ contains no cutset nor circuit of G_1. The edges of $S - (S^{x}_{p1} \cup S^{y}_{p4})$ are a subset of $(S_{p2} \cup S_{p3})$ so they contain no cutsets. Therefore, $(S^{x}_{p1} \cup S^{y}_{p4})$ contains a dendroid, D_1, of $B(G_1 \cdot S)$. (See definition (2-2-1).) By (2-2-2) and the definition of G_1, $S^{x}_{p1} = D_1$. Also, the edges of $T-(S^{y}_{p1} \cup S^{x}_{p4})$ are a subset of $(S_{p2} \cup S_{p3})$ so they contain no circuits. Therefore $(S^{y}_{p1} \cup S^{x}_{p4})$ contains a dendroid, D_2, of $C(G \cdot T)$. Also by (2-2-2) and the definition of G_1, $S^{y}_{p1} = D_2$.

Since $B\ (G_1 \cdot T) \quad B\ (G_1 \cdot S) = B\ (G_1)$, $D_1 \cup D_2$ is a dendroid of $B\ (G_1)$ so are the edges of a forest, T_o, of G_1. Let (4-1-4) be written for forest T_o. Then S^{y}_{p2} and S^{y}_{p3} are in S and S^{x}_{p2} and S^{x}_{p3} are in T.

By Lemma 4-1-1, Q_{11}, Q_{22}, Q_{23}, and Q_{33} of (4-1-4) are zero. By Lemma 4-1-2, the matrix $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ of (4-1-4) has maximum column rank.

Let X_j, Y_j represent the direct sum of all component X^{i}_{j}, Y^{i}_{j} respectively for $j=1, 2, 3, 4$. It is clear that for forest T_o,

$$Z_{p2} = \begin{bmatrix} X_1 \\ \\ Y_4 \end{bmatrix}, \qquad Z_{p3} = \begin{bmatrix} X_2 \\ \\ Y_3 \end{bmatrix}$$

and the matrix

$$C = \begin{bmatrix} C_{11} & C_{12} \\ \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ \\ -C_{12}^T & C_{22} \end{bmatrix}$$

where C_{11} and C_{22} are symmetric, since C is the direct sum of the component equations (4-1-10).

Also C_{22} is nonsingular since it is the direct sum of nonsingular matrices.

Then J of (4-1-6) becomes

$$J = -\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} \begin{bmatrix} U & -C_{12} \\ \\ O & -C_{22} \end{bmatrix}^{-1} \begin{bmatrix} C_{11} & O \\ \\ -C_{12}^T & -U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ \\ Q_{13}^T \end{bmatrix}$$

$$-\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} \begin{bmatrix} U & C_{12} \\ \\ O & U \end{bmatrix} \begin{bmatrix} C_{11} & O \\ \\ O & C_{22}^{-1} \end{bmatrix} \begin{bmatrix} U & O \\ \\ C_{12}^T & U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ \\ Q_{13}^T \end{bmatrix}$$

so J is symmetric.

Since P is positive definite, K is positive definite and can be written as the product

$$K = K_1 \ K_1^T$$

where K_1 is nonsingular [BE-2]. Setting $Z_{p1} = K_1 R$ in the homogeneous part of (4-1-6) gives

$$\dot{R} = K_1^T J K_1 R$$

Since the eigenvalues of KJ of (4-1-6) are the same as the eigenvalues of the symmetric matrix $K_1^T J K_1$, it follows that the system is stable if, and only if, the eigenvalues of $K_1^T J K_1$ are not positive. Consequently, the system is stable if, and only if, J is negative semidefinite. But since $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ has maximum column rank, it follows from Lemma A8 that J is negative semidefinite if, and only if, C is positive semidefinite, and the theorem is proved.

Corollary 4-1-1: Assume all the hypotheses of Theorem 4-1-1 except that $P(t)$ can be time varying. Suppose

1. $\dot{P}(t)^{-1} = \frac{d}{dt} P^{-1}(t)$ exists and is negative semidefinite for $t \geq t_o$.

2. Each algebraic component in (4-1-10) can have C_{11}^i, C_{22}^i, C_{33}^i, C_{44}^i, C_{23}^i, and C_{14}^i as time varying matrices.

3. For each component i, the matrix

$$\frac{d}{dt}\begin{bmatrix} \begin{bmatrix} C^i_{11} & C^i_{14} \\ C^{iT}_{14} & C^i_{44} \end{bmatrix} & \\ & \begin{bmatrix} C^i_{22} & C^i_{23} \\ C^{iT}_{23} & C^i_{33} \end{bmatrix}^{-1} \end{bmatrix}$$

is negative semidefinite for all $t \geq t_o$.

4. All time varying entries of each component in (4-1-10) are bounded and $C_{22}(t)$ and $C_{11}(t)$ are nonsingular for all $t \geq t_o$. Then any solution to the system for initial time t_o is stable if, and only if, $C(t)$ is positive semidefinite for all $t \geq t_o$.

<u>Proof</u>: In this case Q_{11}, Q_{22}, Q_{23}, and Q_{33} are 0 and

$$C = \begin{bmatrix} C_{11} & C_{12} \\ -C^T_{12} & C_{22} \end{bmatrix}$$

with C_{11} and C_{22} symmetric and nonsingular for all t.

As in Theorem 4-1-1, J is symmetric. J is also nonsingular since C_{11} and C_{22} are nonsingular for all $t \geq t_o$.

Consider the scalar function

$$v = \psi^T J(t) \psi$$

Along the trajectories

$$\dot{v} = 2\psi^T J(t) \dot{\psi} + \psi^T \dot{J}(t)$$

$$= 2\psi^T J(t) K(t) J(t)\psi + \psi^T \dot{J}(t)$$

$$J(t) = -\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} \begin{bmatrix} U & C_{12} \\ O & U \end{bmatrix} \begin{bmatrix} C_{11} & O \\ O & C_{22}^{-1} \end{bmatrix} \begin{bmatrix} U & O \\ C_{12}^T & U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ Q_{13}^T \end{bmatrix}$$

where C_{12} is a constant matrix so

$$\dot{J}(t) = -\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} \begin{bmatrix} U & C_{12} \\ O & U \end{bmatrix} \left[\frac{d}{dt} \begin{bmatrix} C_{11} & O \\ O & C_{22}^{-1} \end{bmatrix} \right] \begin{bmatrix} U & O \\ C_{12}^T & U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ Q_{13}^T \end{bmatrix}$$

which is positive semidefinite since

$$\frac{d}{dt} \begin{bmatrix} C_{11} & O \\ O & C_{22}^{-1} \end{bmatrix}$$

is a direct sum of negative semidefinite matrices. Since $K(t)$ is positive definite, $\dot{v}$ is positive definite. Since $C(t)$ is bounded for $t \geq t_o$, v is decrescent, by Theorem 1.2 of [HA-1].

If $J(t)$ is not positive semidefinite for all $t \geq t_o$, then v has a domain < 0 and by Theorem 5.2 of [HA-1], any solution is unstable. If $J(t)$ is positive semidefinite for all $t \geq t_o$, then take the scalar function

$$v = \psi^T K^{-1}(t)\psi$$

Along the trajectories

$$\begin{aligned} \dot{v} &= 2\psi^T K^{-1}(t)\ \dot{\psi} + \psi^T \dot{K}^{-1}(t)\psi \\ &= 2\psi^T J(t)\psi + \psi^T \dot{K}^{-1}(t)\psi \end{aligned}$$

The matrix $\dot{K}(t)^{-1} = \frac{d}{dt} K^{-1}(t)$ is negative semidefinite by Lemma A10. Since $J(t)$ is negative semidefinite, $\dot{v}$ is negative semidefinite. Also, since $K(t)$ is positive definite, by Lemma A2, $K^{-1}(t)$ is positive definite and any solution is stable.

Therefore $J(t)$ is negative semidefinite if and only if the system is stable.

As in Theorem 4-1-1, $J(t)$ is negative semidefinite if, and only if, $C(t)$ is positive semidefinite, and the Corollary is proved.

Theorem 4-1-2: If in (4-1-3), $P(t)$ is positive definite and symmetric, if $\dot{P}(t)^{-1} = \frac{d}{dt} P^{-1}(t)$ exists and is negative semidefinite and if $C(t)$ is positive definite, then the system is stable for all graphs for which the constraint equations for some forest T in the system graph can be written in the form (4-1-4).

Proof: Condition (4-1-5) is satisfied by Lemma A9. Therefore, the system model is (4-1-6). The matrix $\dot{K}(t)^{-1} = \frac{d}{dt} K^{-1}(t)$ is negative semidefinite by Lemma A10.

Consider the scalar function

$$v = \psi^T K(t)^{-1} \psi \tag{4-1-11}$$

Along the trajectories

$$\begin{aligned} \dot{v} &= 2\psi^T K(t)^{-1} \dot{\psi} + \psi^T \dot{K}(t)^{-1} \psi \\ &= 2\psi^T J(t) \psi + \psi^T \dot{K}(t)^{-1} \psi \end{aligned}$$

But since $J(t)$ is negative semidefinite, $\dot{v}$ is negative semidefinite and the theorem is proved.

By use of the following lemma, a condition for asymptotic stability of the system can be obtained.

Lemma 4-1-3: The submatrix $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ has maximum row rank if and only if the following two combinations are satisfied:

1. There is no circuit contained in $(S_{p1} \cup S_{p4} \cup S^x_{p5})$ other than those circuits contained in $(S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})$; i.e., $C(G \cdot (S_{p1} \cup S_{p4} \cup S^x_{p5})) - C(G \cdot (S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})) = \emptyset$ and where $\emptyset$ is the empty set.

2. There is no cutset contained in $(S_{p1} \cup S_{p4} \cup S^y_{p5})$ other than those cutsets contained in $(S^y_{p1} \cup S^x_{p4} \cup S^y_{p5})$; i.e.,

$B(G \times (S_{p1} \cup S_{p4} \cup S^y_{p5})) - B(G \times (S^y_{p1} \cup S^x_{p4} \cup S^y_{p5})) = \emptyset$.

<u>Proof</u>: Assume 1 and 2.

Conditions (1) and (2) are satisfied if and only if the following two conditions are satisfied:

3. There is no circuit contained in $(S_{p1} \cup S^x_{p4} \cup S^x_{p5})$;

i.e., $C(G \cdot (S_{p1} \cup S^x_{p4} \cup S^x_{p5})) = 0$,

and

4. There is no cutset contained in $(S_{p1} \cup S^y_{p4} \cup S^y_{p5})$;

i.e., $B(G \times (S_{p1} \cup S^y_{p4} \cup S^y_{p5})) = 0$. To show this let

$G_o = [G \cdot (S_{p1} \cup S^x_{p4} \cup S^x_{p5})]$, $G_1 = [G \cdot (S_{p1} \cup S_{p4} \cup S^x_{p5})]$

and $G_2 = [G \cdot (S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})]$.

Let $C_1 \in C(G_o)$, then $C_1 \notin C(G_2)$ since $S^x_{p1} \cup S^x_{p5}$ contain no circuits by their definition. Therefore, $C_1 \in C(G_1) - C(G_2)$.

Conversely let $C_1 \in C(G_1) - C(G_2)$. If C_1 contains an edge, e_1, of S^y_{p4}, by definition of S^y_{p4}, there exists another circuit $C_2 \in C(G_1) - C(G_2)$ such that

$C_2 \subset (S^x_{p1} \cup S^x_{p5} \cup C_1 - \{e_1\})$.

If $e_2 \in (C_2 \cap S^y_{p4})$, for the same reason as above there exists another circuit $C_3 \subset (S^x_{p1} \cup S^x_{p5} \cup C_2 - \{e_2\})$. Proceeding in this fashion if $e_i \in (C_i \cap S^y_{p4})$, there exists another circuit $C_{i+1} \subset (S^x_{p1} \cup S^x_{p5} \cup C_i - \{e_i\})$. Since S^y_{p4} is finite, there exists some circuit $C_n \subset (S_{p1} \cup S^x_{p4} \cup S^x_{p5})$. Therefore, $C\ (G_o) \neq \emptyset$. This shows (3) is true if and only if (1) is true.

The proof that (2) is true if and only if (4) is true is identical to the above with obvious changes in notation.

Since there is no circuit contained in $(S_{p1} \cup S^x_{p4} \cup S^x_{p5})$, by (2-2-2) and definition 2-2-1 , $(S^x_{p2} \cup S^x_{p3})$ must contain a dendroid of $C(G \cdot (S_{p1} \cup S^x_{p2} \cup S^x_{p3} \cup S^x_{p4} \cup S^x_{p5}))$.

Also since there is no cutset contained in $(S_{p1} \cup S^y_{p4} \cup S^y_{p5})$, by (2-2-2) and definition 2-2-1 , $(S^y_{p2} \cup S^y_{p3})$ must contain a dendroid of $B(G \times (S_{p1} \cup S^y_{p2} \cup S^y_{p3} \cup S^y_{p4} \cup S^y_{p5}))$.

Together these two conditions imply, by Lemma 2-2-2, that the matrix $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ is maximum row rank.

Conversely, assume $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ is maximum row rank. Then by Lemma 2-2-2, $(S^y_{p2} \cup S^y_{p3})$ and $(S^x_{p2} \cup S^x_{p3})$ contain a dendroid of B $(G \times (S_{p1} \cup S^y_{p2} \cup S^y_{p3} \cup S^y_{p4} \cup S^y_{p5}))$ and C $(G \cdot (S_{p1} \cup S^x_{p2} \cup S^x_{p3} \cup S^x_{p4} \cup S^x_{p5}))$ respectively so there is no cutset contained in $(S_{p1} \cup S^y_{p4} \cup S^y_{p5})$ and no circuit contained in $(S_{p1} \cup S^x_{p4} \cup S^x_{p5})$.

Corollary 4-1-2: If in addition to the hypothesis of Theorem 4-1-2, $P^{-1}(t)$ in (4-1-3) is bounded for all $t \geq t_o$ and the following two conditions are satisfied:

1. There is no circuit contained in $(S_{p1} \cup S_{p4} \cup S^x_{p5})$ other than those circuits contained in $(S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})$.

2. There is no cutset contained in $(S_{p1} \cup S_{p4} \cup S^y_{p5})$ other than those cutsets contained in $(S^y_{p1} \cup S^x_{p4} \cup S^y_{p5})$. Then any solution to the system is asymptotically stable.

Proof: The submatrix $\begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix}$ is maximum row rank by Lemma 4-1-3. Therefore by the Corollary of Lemma A4, $J(t)$ is negative definite.

It follows from (4-1-7) that $K^{-1}(t)$ is bounded for all $t \geq t_o$.

The scalar function of (4-1-11) is positive definite. It is also decrescent by Theorem 1.2 of [HA-1]. Along the trajectories, $\dot{v}$ is negative definite so for initial instant t_o, any solution to the system is asymptotically stable.

Theorem 4-1-3: If in (4-1-3) $P(t)$ is positive definite and symmetric, $\dot{P}(t)^{-1}$ exists and is negative semidefinite, $C(t)$ is positive semidefinite and condition (4-1-5) is satisfied for all $t \geq t_o$, then any solution to the system is stable for a topology corresponding to (4-1-4).

Proof: The system model is given in (4-1-6). By Lemmas A3 and A4, $J(t)$ is negative semidefinite. Therefore, the Lyapunov function (4-1-11) shows that any solution to the system is stable.

The following lemma and corollary provide a sufficient condition for condition (4-1-5) to be satisfied.

Lemma 4-1-4: Let the submatrix

$$\begin{bmatrix} Q_{22} & Q_{23} \\ -Q_{23}^T & Q_{33} \end{bmatrix},$$

of (4-1-4) be rearranged and partitioned according to positive definite or positive semidefinite components so that

$$\begin{bmatrix} Z_{s3}^{PD} \\ Z_{s2}^{PD} \\ Z_{s3}^{PS} \\ Z_{s2}^{PS} \end{bmatrix} = \begin{bmatrix} Q_{111} & Q_{112} & Q_{211} & Q_{212} \\ -Q_{112}^{T} & Q_{113} & Q_{213} & Q_{214} \\ -Q_{211}^{T} & -Q_{213}^{T} & Q_{311} & Q_{312} \\ -Q_{212}^{T} & -Q_{214}^{T} & -Q_{312}^{T} & Q_{313} \end{bmatrix} \begin{bmatrix} Z_{p3}^{PD} \\ Z_{p2}^{PD} \\ Z_{p3}^{PS} \\ Z_{p2}^{PS} \end{bmatrix}$$

where PD refers to positive definite components and PS refers to positive semidefinite components.

Let S^{PD} and S^{PS} refer to the edges corresponding to the positive definite components and positive semidefinite components respectively. Then the submatrix

$$\begin{bmatrix} Q_{211}^{T} & Q_{213}^{T} \\ Q_{212}^{T} & Q_{214}^{T} \end{bmatrix}$$

is maximum row rank if and only if the following two conditions are satisfied:

1. There is no circuit contained in $(S^{PS} \cup S_{p1}^{x} \cup S_{p4}^{y} \cup S_{p5}^{x})$ other than those circuits contained in $(S_{p1}^{x} \cup S_{p4}^{y} \cup S_{p5}^{x})$; i.e.,

$$C(G \cdot (S_{p1}^{x} \cup S_{p4}^{y} \cup S^{PS} \cup S_{p5}^{x})) - C(G \cdot (S_{p1}^{x} \cup S_{p4}^{y} \cup S_{p5}^{x})) = \emptyset$$

and

2. There is no cutset contained in $(S^{PS} \cup S^{y}_{p1} \cup S^{x}_{p4} \cup S^{y}_{p5})$ other than those cutsets contained in $(S^{y}_{p1} \cup S^{x}_{p4} \cup S^{y}_{p5})$; i.e.,

$B\ (G \times (S^{y}_{p1} \cup S^{PS} \cup S^{x}_{p4} \cup S^{y}_{p5})) - B\ (G \times (S^{y}_{p1} \cup S^{x}_{p4} \cup S^{y}_{p5})) = \emptyset.$

<u>Proof</u>: Assume 1 and 2.

Conditions (1) and (2) are satisfied if and only if the following two conditions are satisfied:

3. There is no circuit contained in $(S^{x}_{p1} \cup S^{PS} \cup S^{x}_{p5})$; i.e., $C\ (G \cdot (S^{x}_{p1} \cup S^{PS} \cup S^{x}_{p5})) = \emptyset$

and

4. There is no cutset contained in $(S^{y}_{p1} \cup S^{PS} \cup S^{y}_{p5})$; i.e., $B\ (G \times (S^{y}_{p1} \cup S^{PS} \cup S^{y}_{p5})) = \emptyset$.

To show that (2) is true if and only if (4) is true, let $G_o = [G \times (S^{y}_{p1} \cup S^{PS} \cup S^{y}_{p5})]$, $G_1 = [G \times (S^{y}_{p1} \cup S^{x}_{p4} \cup S^{PS} \cup S^{y}_{p5})]$ and $G_2 = [G \times (S^{y}_{p1} \cup S^{x}_{p4} \cup S^{y}_{p5})]$.

Suppose $B_1 \in B\ (G_o)$. Then $B_1 \notin B\ (G_2)$ since $(S^{y}_{p1} \cup S^{y}_{p5})$ contains no cutsets by definition. Therefore

$B_1 \in B\ (G_1) - B\ (G_2)$.

Conversely, suppose $B_1 \in B\ (G_1) - B\ (G_2)$. If B_1 contains an edge, e_1, of S^x_{p4}, by definition of S^x_{p4} there exists another cutset $B_2 \in B\ (G_1) - B\ (G_2)$ such that $B_2 \subset (S^y_{p1} \cup S^y_{p5} \cup B_1 - \{e_1\})$.

If $e_2 \in (B_2 \cap S^x_{p4})$, for the same reason as above there exists another cutset $B_2 \subset (S^y_{p1} \cup S^y_{p5} \cup B_2 - \{e_2\})$. Proceeding in this fashion if $e_1 \in (B_i \cap S^x_{p4})$, there exists another circuit $B_{i+1} \subset (S^y_{p1} \cup S^y_{p5} \cup B_i - \{e_i\})$. Since S^x_{p4} is finite, there exists some cutset $B_n \subset (S^y_{p1} \cup S^y_{p5} \cup S^{PS})$. Therefore $B\ (G_o) \neq \emptyset$.

Therefore (4) is true if and only if (2) is true.

The proof that (1) is true if and only if (3) is true is identical to the above with obvious change of notation.

Let $\dot{S}^{PDx}$ refer to the edges of S^{PD} in the chosen forest of (4-1-4) and S^{PDy} refer to the edges of S^{PD} in the co-forest corresponding to (4-1-4).

Since there is no circuit contained in $(S^{PS} \cup S^x_{p1} \cup S^x_{p5})$, by (2-2-2) and definition 2-2-1, S^{PDx} must contain a dendroid of $C\ (G \cdot (S^x_{p1} \cup S^{PDx} \cup S^{PS} \cup S^x_{p5}))$.

Also since there is no cutset contained in $(S^{PS} \cup S^y_{p1} \cup S^y_{p5})$, by (2-2-2) and definition 2-2-1, S^{PDy}

must contain a dendroid of $B\ (G \times (S^{y}_{p1} \cup S^{PDy} \cup S^{PS} \cup S^{y}_{p5}))$.

Together these two conditions imply, by Lemma 2-2-2, that the matrix

$$\begin{bmatrix} Q^{T}_{211} & Q^{T}_{213} \\ Q^{T}_{212} & Q^{T}_{214} \end{bmatrix}$$

is maximum row rank.

Conversely, assume the matrix in question is maximum row rank. Then by Lemma 2-2-2, S^{PDx} and S^{PDy} contain a dendroid of $C\ (G \cdot (S^{x}_{p1} \cup S^{PDx} \cup S^{PS} \cup S^{y}_{p5}))$ and $B\ (G \times (S^{y}_{p1} \cup S^{PDy} \cup S^{PS} \cup S^{y}_{p5}))$ respectively so there is no circuit contained in $(S^{y}_{p1} \cup S^{PS} \cup S^{y}_{p5})$ and no cutset contained in $(S^{x}_{p1} \cup S^{PS} \cup S^{x}_{p5})$.

Corollary 4-1-3: Let all conditions on the component equations in Theorem 4-1-3 apply except that it is not known that condition (4-1-5) is satisfied. Let the direct sum of the algebraic equations in (4-1-3) be rearranged if necessary so that $C(t)$ is partitioned into the direct sum of positive definite components and positive semidefinite components.

Let S^{PD} and S^{PS} refer to the edges corresponding to the positive definite and positive semidefinite components respectively. If:

1. There is no circuit contained in $(S^{PS} \cup S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})$

other than those circuits contained in $(S^x_{p1} \cup S^y_{p4} \cup S^x_{p5})$.

and

2. There is no cutset contained in $(S^{PS} \cup S^y_{p1} \cup S^x_{p4} \cup S^y_{p5})$

other than those cutsets contained in $(S^y_{p1} \cup S^x_{p4} \cup S^y_{p5})$.

Then any solution to the system is stable.

<u>Proof</u>: Let the following be the above mentioned partition of C(t).

$$\begin{bmatrix} z^{PD}_{s3} \\ z^{PD}_{p2} \\ z^{PS}_{s3} \\ z^{PS}_{p2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} & \\ & \begin{bmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{bmatrix} \end{bmatrix} \begin{bmatrix} z^{PD}_{p3} \\ z^{PD}_{s2} \\ z^{PS}_{p3} \\ z^{PS}_{s2} \end{bmatrix} \qquad (4\text{-}1\text{-}12)$$

The matrix H of Lemma A7 is a rearrangement of the rows and columns of the matrix of (4-1-5), so is nonsingular if and only if condition (4-1-5)is satisfied.

By Lemma 4-1-4 and Lemma A7, condition (4-1-5) is satisfied, and the Corollary follows from Theorem 4-1-3.

Theorem 4-1-4: In (4-1-3), let $P(t)$ be nonsingular, symmetric and $P^{-1}(t)$ bounded for $t \geq t_o$, let $\dot{P}(t)^{-1}$ be negative semidefinite and let Z_{p4} be of zero order. If $C(t)$ is positive definite and the conditions on the topology stated in Corollary 4-1-2 are satisfied, the system is stable only if $P(t)$ is positive semidefinite for all $t \geq t_o$.

Proof: By Lemma A9, condition (4-1-5) is satisfied, and the system model is given in (4-1-6). Consider the scalar function

$$v = \psi^T P^{-1} \psi$$

Since $P^{-1}(t)$ is bounded for $t \geq t_o$, v is decrescent. Along the trajectories

$$\dot{v} = 2\psi^T J(t) \dot{\psi} + \psi^T \dot{P}(t)^{-1} \psi$$

But $J(t)$ is negative definite by Corollary 4-1-2 . Consequently, $\dot{v}$ is negative definite, and by Lyapunov's instability theorem [HA-1] the system will be unstable if $P(t)$ is not positive semidefinite for all $t \geq t_o$.

Part II. Nonlinear Systems

Let the topology of the system graph be such that there exists a tree T for which:

1. The direct sum of the terminal equations for all m components in the system can be written in the form

$$\dot{Z}_{p1} = F\,(Z_{s1},\ t) \tag{4-2-1}$$

$$\begin{bmatrix} Z_{p2} \\ \\ Z_{s3} \end{bmatrix} = G\,(Z_{s2},\ Z_{p3})$$

where F and G have continuous first partials in all their arguments.

$$\text{Let } \frac{\delta G}{\delta(Z_{s2},\ Z_{p3})} = \begin{bmatrix} G_{11} & G_{12} \\ \\ G_{21} & G_{22} \end{bmatrix} \tag{4-2-2}$$

where the columns of G_{11} and G_{21} correspond to the partial derivatives with respect to the variables of Z_{s2}, and the columns of G_{12} and G_{22} correspond to the partial derivatives with respect to the variables of Z_{p3}.

2. The constraint equations of the system graph are of the form

$$\begin{bmatrix} Z_{s1} \\ Z_{s2} \\ Z_{s3} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ -Q_{12}^T & Q_{22} & Q_{23} \\ -Q_{13}^T & -Q_{23}^T & Q_{33} \end{bmatrix} \begin{bmatrix} Z_{p1} \\ Z_{p2} \\ Z_{p3} \end{bmatrix}$$

where Q_{11}, Q_{22}, and Q_{33} are skew.

If

$$\det . \left[\begin{bmatrix} U & O \\ -Q_{23}^T & Q_{33} \end{bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} Q_{22} & Q_{23} \\ O & U \end{bmatrix} \right] \neq 0 \qquad (4\text{-}2\text{-}4)$$

for all values of (Z_{s2}, Z_{p3}) and $t \geq t_o$, then upon substituting the constraint equations in (4-2-3) into (4-2-1) there results

$$\dot{Z}_{s1} = J(Z_{s2}, Z_{p3})\, \dot{Z}_{p1} \qquad (4\text{-}2\text{-}5)$$

where

$$J(Z_{s2}, Z_{p3}) = Q_{11} - \begin{bmatrix} Q_{12} & Q_{13} \end{bmatrix} \left[\begin{bmatrix} U & O \\ -Q_{23}^T & Q_{33} \end{bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} Q_{22} & Q_{23} \\ O & U \end{bmatrix} \right]^{-1} \begin{bmatrix} G_{11} & O \\ G_{21} & -U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ Q_{13}^T \end{bmatrix}$$

It can be shown that if G is differentiable in t, (4-2-4) is precisely the condition that the order of the system (i.e., the order of the state vector) be maximum. In fact, suppose (4-2-4) is not satisfied. Then let the functionally independent algebraic equations in (4-2-1) be differentiated with respect to t. Consequently, by the chain rule,

$$\begin{bmatrix} \dot{z}_{p2} \\ \dot{z}_{s3} \end{bmatrix} = \begin{bmatrix} \dfrac{\delta G}{\delta (z_{s2}, z_{p3})} \end{bmatrix} \begin{bmatrix} \dot{z}_{s2} \\ \dot{z}_{p3} \end{bmatrix} \qquad (4\text{-}2\text{-}6)$$

Substituting the constraint equations (4-2-3), into (4-2-5) and (4-2-1) gives

$$\dot{z}_{p1} = F\,(Q_{11}\, z_{p1} + Q_{12}\, z_{p2} + Q_{13}\, z_{p3}, t)$$

$$\left[\begin{bmatrix} U & O \\ -Q_{23}^T & Q_{33} \end{bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} Q_{22} & Q_{23} \\ O & U \end{bmatrix} \right] \begin{bmatrix} \dot{z}_{p2} \\ \dot{z}_{p3} \end{bmatrix} = \begin{bmatrix} G_{11} & O \\ G_{21} & -U \end{bmatrix} \begin{bmatrix} Q_{12}^T \\ Q_{13}^T \end{bmatrix} F \qquad (4\text{-}2\text{-}7)$$

The system of differential equations, (4-2-7), is of maximum order if and only if the matrix of (4-2-4) is non-singular. But the system (4-2-7) is obtained from (4-2-1) and (4-2-3), by differentiating a set of algebraic equations. Thus the order of (4-2-7) is equal to the number of functionally independent algebraic equations differentiated plus the order of the system (4-2-1) and (4-2-3). Since (4-2-4) is assumed not satisfied the order of (4-2-7) is not maximum. Thus the order of the system (4-2-1) and (4-2-3) is not maximum.

If the indefinite line integral

$$\int_{Z_{s1}^{o}}^{Z_{s1}} F_1(Z_{s1},t) \cdot d\, Z_{s1}$$

is a positive definite function in some half-cylindrical neighborhood H_{h,t_o} of some point Z_{s1}^{o} in the Z_{s1} space, where $F_1(Z_{s1}^{o},t) \equiv 0$ for all $t \geq t_o$, then the system is said to satisfy condition N_1.

If the indefinite line integral

$$\int_{Z_{s1}^{o}}^{Z_{s1}} \frac{\delta}{\delta t} F_1\,(Z_{s1},\, t) \cdot d\, Z_{s1}$$

is a negative semidefinite function in some half-cylindrical neighborhood H_{h,t_o} for some point Z_{s1}^{o} in the Z_{s1} space, where $F_1(Z_{s1}^{o},\, t) \equiv 0$ for all $t \geq t_o$, then the system is said to satisfy condition N_2.

In this thesis, only stability or equilibrium points are considered with $t = t_o$ representing the initial time. It is assumed that $F_1(Z_{s1},\, t)$ vanish only at isolated points of the Z_{s1} space.

By Theorem 1.11 of [NE-1], the conditions imposed above on the functions F and G assure the existence of a solution locally if the initial conditions satisfy the algebraic set in (4-2-1) and (4-2-3).

<u>Theorem 4-2-1</u>: In (4-2-1), let the matrix $\frac{\delta F}{\delta Z_{s1}}$ be symmetric for all $t \geq t_o$. If $\frac{\delta G}{\delta (Z_{s2}, Z_{p3})}$ is positive definite for all Z_{s2} and Z_{p3}, $t \geq t_o$, then each equilibrium solution where conditions N_1 and N_2 are satisfied is stable.

<u>Proof</u>: Since $\frac{\delta G}{\delta (Z_{s2}, Z_{p3})}$ is positive definite, (4-2-4) is satisfied, by Lemma A9. Consequently, $J(Z_{s2}, Z_{p3})$ in (4-2-5) is negative semidefinite by Lemma A4, and (4-2-5) applies.

Consider the scalar function, for the equilibrium point, θ,

$$v = \int_{\theta}^{Z_{s1}} F\,(Z_{s1}, t) \cdot d\,Z_{s1} \qquad (4\text{-}2\text{-}8)$$

Since $\frac{\delta F}{\delta Z_{s1}}$ is symmetric, this line integral is independent of path. [KA-1] [CI-1]. When evaluated along the trajectories, $\dot{v}$ gives

$$\dot{v} = F^T \dot{Z}_{s1} + \int_{\theta}^{Z_{s1}} \frac{\delta F}{\delta t} \cdot d\, Z_{s1}$$

$$= F^T J F_1 + \int_{\theta}^{Z_{s1}} \frac{\delta F}{\delta t} \cdot d\, Z_{s1} \tag{4-2-9}$$

Therefore, $\dot{v}$ is non-positive in a cylindrical neighborhood H_{h,t_o} of each equilibrium point, θ, where conditions N_1 and N_2 are satisfied, and the theorem is proved.

Corollary 4-2-1: Suppose, in addition, to the hypothesis of Theorem 4-2-1, F is bounded in a neighborhood H_{h,t_o} of an equilibrium point where conditions N_1 and N_2 are satisfied, and the following two conditions are also satisfied:

1. There is no circuit contained in S_{p1};

and

2. There is no cutset contained in S_{p1}.

Then the equilibrium point is asymptotically stable.

Proof: The scalar function of (4-2-8) is a function only of Z_{s1} and t, and v is decrescent by Theorem 1.2 of [HA-1], since F is bounded for all $t \geq t_o$.

In this case J in (4-2-5) is negative definite by Lemma 4-1-3, the Corollary to Lemma A4, and Lemma A5.

Then $\dot{v}$ is negative definite by (4-2-9) and then each such equilibrium point is asymptotically stable.

Theorem 4-2-2: In (4-2-1), if $\frac{\delta F}{\delta Z_{s1}}$ is symmetric, if $\frac{\delta G}{\delta (Z_{s2}, Z_{p3})}$ is positive semidefinite and if (4-2-4) is satisfied for all Z_{s2} and Z_{p3}, then each equilibrium point satisfying conditions N_1 and N_2 is stable.

Proof: Since (4-2-4) is satisfied for all Z_{s2} and Z_{p3} (4-2-5) applies.

By Lemma A4, $J\ (Z_{s2}, Z_{p3})$ in (4-2-5) is negative semidefinite.

The scalar function of (4-2-8) is, as in Theorem 4-2-1, only a function of Z_{s1} and t, and $\dot{v}$ in (4-2-9) is non-positive in a cylindrical neighborhood H_{h,t_o} of each equilibrium point where conditions N_1 and N_2 are satisfied. Therefore, each such equilibrium point is stable.

Corollary 4-2-2: Suppose all the hypotheses of Theorem 4-2-2 are satisfied except that it is not known whether condition (4-2-4) is satisfied.

Let the direct sum of the algebraic equations in (4-2-1) be rearranged if necessary so that G is partitioned into the direct sum of positive definite components and positive semidefinite components. Let S^{PD} and S^{PS} be as in Corollary 4-1-3, and S^x_{p1} and S^y_{p1} as in Part 1 of this chapter.

If:

1. There is no circuit contained in $(S^{PS} \cup S^x_{p1})$,

and

2. There is no cutset contained in $(S^{PS} \cup S^y_{p1})$, then

any equilibrium point satisfying conditions N_1 and N_2 is stable.

Proof: Let $\dfrac{\delta G}{\delta(Z_{s2}, Z_{p3})}$ be partitioned as in (4-1-12).

Then the matrix H of Lemma A7 is a rearrangement of the rows and columns of the matrix of (4-2-4) so is nonsingular if and only if condition (4-2-4) is satisfied. Lemma 4-1-4 applies since S_{p4} and S_{p5} are null.

By Lemma 4-1-4 and Lemma A7, condition (4-2-4) is satisfied, and the Corollary follows from Theorem 4-2-2.

Theorem 4-2-3: In (4-2-1) let $\dfrac{\delta F_1}{\delta Z_{s1}}$ be symmetric. Let $F_1(Z_{s1}, t)$ be bounded in a neighborhood H_{h,t_o} of an equilibrium point, θ. Consider the line integral

$$v = \int_{\theta}^{Z_{s1}} F_1\,(Z_{s1}, t) \cdot d\,Z_{s1} \tag{4-2-10}$$

If $\dfrac{\delta G}{\delta(Z_{s2}, Z_{p3})}$ is positive definite, if condition N_2 is satisfied in H_{h,t_o} and if there is no circuit or cutset contained in S_{p1}, then the equilibrium point, θ, is stable only if the line integral in (4-2-10) is positive semidefinite for $t \geq t_o$ in a neighborhood of the equilibrium point.

<u>Proof</u>: Since $\dfrac{\delta F_1}{\delta Z_{s1}}$ is continuous and symmetric, v is a function only of Z_{s1} and t, and v is decrescent in H_{h,t_o} by Theorem 1.2 of [HA-1] since $F_1\,(Z_{s1}, t)$ is bounded.

As in Corollary 4-2-1, J in (4-2-5) is negative definite by Lemma 4-1-3, the Corollary to Lemma A4, and Lemma A5.

Then $\dot{v}$ is negative definite by (4-2-9).

If v in (4-2-10) is not positive semidefinite in some neighborhood of the equilibrium point for $t \geq t_o$, the equilibrium point is unstable by Theorem 5.2 of [HA-1]. The Theorem follows.

<u>Conclusion</u>

Although the stability theorems given in this thesis are restricted as to component type and system topology, they apply to a useful class of systems. In general, most systems of purely dynamic or pure algebraic components are covered.

Theorem 1 is particularly useful when the system contains only constant coefficient dynamic components. The theorems on linear systems cover certain classes of systems containing n-terminal perfect couplers, gyrators, and transistors and vacuum tubes having sufficient forward conductance or negative feedback to make the coefficient matrix in the component model positive semidefinite.

The primary limitation of the theorems on stability of nonlinear systems is the restriction to the stability of equilibrium points and the inability to include excitation functions. The theorems, however, apply to systems including such components as nonlinear two-terminal semidefinite components, to which there has been some literature devoted, nonlinear perfect couplers, gyrators and vacuum-tube type elements.

In addition to their application to particular stability studies the theorems also show that, given certain topological configurations, it is possible to go from a stable to an unstable system only by altering the component parameters so as to introduce indefinite coefficient matrices in the component models.

CHAPTER V

APPLICATIONS AND EXAMPLES

The purpose of this chapter is to show some applications of the theorems developed in the preceeding chapters. During the discussion an attempt is made to show the advantage of these methods over classical methods.

Part I. Application of Grassman Algebra to Topological Analysis

Topological analysis is a means of computing linear time-invariant network functions, such as transfer and driving point admittances or impedances, by inspection of the network without actually expanding various determinants and cofactors. The topological methods provide a short-cut in evaluating network determinants and cofactors. All these methods use the Cauchy-Binet determinant expansion and the unimodular property of cut-set and circuit matrices. All methods so far devised have assumed the impedance (or admittance) matrix exists for all components [SE-1].

The theorems of this thesis and the theory of Grassman Algebra provide an essentially new technique of topological analysis. As opposed to older methods, this technique is simpler both computationally and conceptually and applies to all linear time-invariant components. This method applies universally no matter whether the impedance, admittance, or neither matrix exists.

Before explaining this new technique, some definitions and discussion about the algorithm of Chapter III are needed.

<u>Definition 5-1-1</u>: Let a system, $\{CE, G, F\}$, have component equations in form (3-1-32) with possibly some driver type components included in (3-1-32). (See Def. 3-1-1.) Then the <u>system matrix</u> of $\{CE, G, F\}$ is given by K_o in Theorem 3-1-4.

<u>Definition 5-1-2</u>: The <u>system determinant</u> of system $\{CE, G, F\}$ is the determinant of the system matrix of $\{CE, G, F\}$.

Suppose the system, $\{CE, G, F\}$, contains some driver type components. Let $\{CE_1, G', F\}$ be the subsystem of $\{CE_1, G, F\}$ that is described in Theorem 3-1-1. In view of Theorem 3-1-1 and Corollary 3-1-1, to determine whether the system $\{CE, G, F\}$ has a unique solution, it suffices to examine the subsystem $\{CE_1, G', F\}$ for a non-zero system determinant. Therefore in the algorithms of Chapter III, the system determinant for $\{CE_1, G', F\}$ is examined. The system determinant of $\{CE, G, F\}$ is different from the system determinant of $\{CE_1, G', F\}$. However, Theorem 3-1-4 holds for any system matrix of form K_o so can be used to evaluate the system determinant of $\{CE, G, F\}$. It follows that if J_i in (3-1-41) is formed for component equations CE (instead of CE_1) and X is formed for graph G (instead of graph G'), (3-1-41) gives the system determinant of $\{CE, G, F\}$.

Suppose G has t edges in a forest. An outer product, P, of the graphic vector space of G is

$$\left[\rho_1 \ \rho_2 \dots \rho_t\right] \tag{5-1-1}$$

where ρ_i, i=1,...,t, is an incidence vector [FR-1], and $\rho_i \neq \rho_j$ for $i \neq j$. By [KI-1], the incidence matrix, [FR-1], is unimodular so every coefficient in the expansion of (5-1-1) in terms of the basic unities is $\pm$ 1, or 0.

The vector, X, to be substituted in (3-1-41) for the graph G, can be taken directly from the outer product (5-1-1). It is simply the vector of coefficients in the expansion of (5-1-1) in terms of the basic unities.

Algorithm 2 is simply another method of obtaining the matrix J_i of Algorithm 1. Therefore in this chapter the results will be given only for systems with components of the general form (3-1-32).

The General Transfer Function

Suppose one wants to know the ratio between the response of one system variable and an excitation. Such a ratio is called a transfer function. To determine a transfer function, all internal sources in the system are set equal to zero and a specified non-zero driver, (either thru or across), is inserted between two chosen vertices as an excitation. The response of the other system variable is then measured. Fortunately for linear time-invariant systems, the entire process can be done analytically.

Let the system $\{CE, G, F\}$ of Definition (5-1-1) be given. Let the given excitation variable be x_j (or y_j), the response variable x_i (or y_i), and place all other specified functions equal to zero. The resulting system equations for $\{CE, G, F\}$ can be written as either:

1. if x_j is the excitation

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ R_1 & 0 & E_{11} & E_{12} & 0 & E_{21} & E_{22} \\ 0 & B_1 & B_2 & B_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \psi \\ x_j \\ X \\ x_i \\ y_j \\ Y \\ y_i \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5-1-2}$$

or

2. if y_j is the excitation,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ R_1 & 0 & E_{11} & E_{12} & 0 & E_{21} & E_{22} \\ 0 & B_1 & B_2 & B_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & A_2 & A_3 \end{bmatrix} \begin{bmatrix} \psi \\ x_j \\ X \\ x_i \\ y_j \\ Y \\ y_i \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5-1-3}$$

where x_j, x_i, y_j, y_i are variables corresponding to their respective edges, X and Y are vectors containing the remaining system variables, R_1 is the appropriate submatrix of R in (3-1-32), and A_i and B_i for i=1, 2, 3, are suitable partitions of A and B of (2-3-1).

There are four possible transfer functions, two with x_j as excitation, and two with y_j as excitation. From (5-1-2) and (5-1-3), they can be computed as:

$$\frac{x_i}{x_j} = \frac{\Delta_{i1}}{\Delta_x}$$

$$\frac{y_i}{x_j} = \frac{\Delta_{1i}}{\Delta_x}$$

$$\frac{x_i}{y_j} = \frac{\Delta_{i1}}{\Delta_y} \tag{5-1-4}$$

$$\frac{y_i}{y_j} = \frac{\Delta_{1i}}{\Delta_y}$$

where Δ_{i1} is the cofactor of the first row and the column corresponding to x_i, Δ_{1i} is the cofactor of the first row and the column corresponding to y_i, Δ_x is the system determinant for (5-1-1), and Δ_y is the system determinant for (5-1-2).

Suppose graph G has t edges in a forest. Then the determinant Δ_x (or Δ_y) can be formed by the following algorithm.

Algorithm 3

If Δ_x is wanted, consider the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ R_1 & 0 & E_{11} & E_{12} & 0 & E_{21} & E_{22} \end{bmatrix}, \qquad (5\text{-}1\text{-}5)$$

obtained from (5-1-2).

If Δ_y is wanted, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ R_1 & 0 & E_{11} & E_{12} & 0 & E_{21} & E_{22} \end{bmatrix}, \qquad (5\text{-}1\text{-}6)$$

obtained from (5-1-3). Let

$$E_{1x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & E_{11} & E_{12} \end{bmatrix} \quad \text{and} \quad E_{2x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{21} & E_{22} \end{bmatrix}$$

so (5-1-5) is equal to

$$\begin{bmatrix} O & E_{1x} & E_{2x} \\ R_1 & & \end{bmatrix}.$$

Similarly let

$$E_{1y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{11} & E_{12} \end{bmatrix} \quad \text{and} \quad E_{2y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & E_{21} & E_{22} \end{bmatrix}$$

so (5-1-5) is equal to

$$\begin{bmatrix} O & E_{1y} & E_{2y} \\ R_1 & & \end{bmatrix}.$$

Let E_{1x}, E_{2x}, E_{1y}, E_{2y} each have n columns, and number them from 1 to n in their natural order. If Δ_x is wanted, determine the values of all $\binom{n}{t}^2$ determinants of the maximum order submatrices, L_t^x, of (5-1-5) that contain all columns of R_1 and t columns of E_{1x}. If Δ_y is wanted, determine the values of all $\binom{n}{t}^2$ determinants of the maximum order submatrices, L_t^y, of (5-1-5) that contain all columns of R_1 and t columns of E_{1y}.

For each matrix, B, of L_t, let (k) represent the sequence of columns of B from E_{1x} (or E_{1y}, if Δ_y is desired), and (m) the sequence of columns of E_{2x} (or E_{2y}) not in B.

Then proceed as follows:

1. Form the square matrix J_t^x (J_t^y) made up of the determinants of each element of L_t^x (L_t^y), where the rows of

J_t^x (J_t^y) correspond to the sequences (k), the columns of J_t^x (J_t^y) correspond to the sequences (m), and if if $(k) = (m)$ the corresponding entry is on the diagonal of J_t^x (J_t^y). Therefore, the (k,m) entry of J_t^x (J_t^y) corresponds to the sequence (k) and to the sequence (m).

2. For each column of J_t^x (J_t^y), evaluate the sum

$$\Sigma(m) = \sum_{h=1}^{t} m_h, \text{ where } (m) = (m_1, \ldots, m_t).$$

If the sum is odd, change the sign of all entries in this column of J_t^x (J_t^y). Let the resulting matrix be called J_{ts}^x (J_{ts}^y). (By definition (3-1-5), this corresponds to evaluating the function sgn. $[(m)]$.)

3. Form the outer product, P, of (5-1-1) for graph G.
4. From P, obtain x as the row vector of coefficients of the terms of P in its canonical form.
5. Then

$$\Delta_x = (\pm 1)\ [X\ J_{ts}^x\ X^T] \tag{5-1-7}$$

and

$$\Delta_y = (\pm 1)\ [X\ J_{ts}^y\ X^T] \tag{5-1-8}$$

where it is understood that entries corresponding to the same sequences are <u>multiplied together</u>.

The proof of this Algorithm, as that for Algorithm-1, follows from Theorem 3-1-4. The (± 1) sign is in (5-1-7) and

(5-1-8) since the cutset and circuit unimodular matrices of (5-1-2) and (5-1-3) can be chosen and their rows ordered many different ways, thereby changing the sign of the determinants, Δ_x and Δ_y. Certainly the cutset and circuit matrices can be chosen to make both (5-1-7) and (5-1-8) a (+1).

The cofactors Δ_{i1} and Δ_{1i} of (5-1-4), can be written as follows:

$$\Delta_{i1} = (-1)^{n+r} \text{ det. } \begin{bmatrix} R_1 & O & E_{11} & O & E_{21} & E_{22} \\ O & B_1 & B & O & O & O \\ O & O & O & A_1 & A_2 & A_3 \end{bmatrix} \tag{5-1-9}$$

$$\Delta_{1i} = (-1)^{2n+r} \text{ det. } \begin{bmatrix} R_1 & O & E_{11} & E_{12} & O & E_{21} \\ O & B_1 & B_2 & B_3 & O & O \\ O & O & O & O & A_1 & A_2 \end{bmatrix} \tag{5-1-10}$$

where r is the number of columns of R_1.

By a procedure analogous to the proofs of Theorems 3-1-4 and 3-1-5, the following algorithm can be devised to evaluate (5-1-9) and (5-1-10).

<u>Algorithm 4</u>

If Δ_{i1} is wanted, consider the matrix

$$M_o = [R_1 \quad O \quad E_{11} \; E_{21} \; E_{22}] \tag{5-1-11}$$

obtained from (5-1-2), or (5-1-3), where the O matrix is a 1 column matrix.

If Δ_{1i} is wanted, consider the matrix

$$M_1 = [R_1 \quad E_{11}\ E_{12}\ O \quad E_{21}] \tag{5-1-12}$$

obtained from (5-1-2), or (5-1-3), where the O matrix is a 1 column matrix.

Let $E_{i1} = [O \quad E_{11}]$ and $E_{i2} = [E_{21} \quad E_{22}]$ so (5-1-11) is equal to $[R_1 \quad E_{i1}\ E_{i2}]$. Similarly let $E_{1i} = [E_{11}\ E_{12}]$ and $E_{2i} = [O \quad E_{21}]$ so (5-1-12) is equal to $[R_1 \quad E_{1i}\ E_{2i}]$. Let E_{i1}, E_{1i}, E_{i2}, E_{21} each have (n-1) columns and give them the same numbers as their corresponding columns in (5-1-5) and (5-1-6).

If Δ_{i1} is wanted, determine the values of all $\binom{n-1}{t-1}^2$ determinants of the maximum order submatrices, L^{i1}_{t-1} of (5-1-11) that contain all columns of R_1 and t-1 columns of E_{i1}.

If Δ_{1i} is wanted, determine the values of all $\binom{n-1}{t}^2$ determinants of the maximum order submatrices, L^{1i}_{t}, of (5-1-12) that contain all columns of R_1 and t columns of E_{1i}.

For each matrix, B, of L^{i1}_{t-1} (or L^{1i}_{t}), let (k) represent the sequence of columns of B from E_{i1} (E_{1i}), and (m) the sequence of columns of E_{i2} (E_{2i}) not in B. Then proceed as follows:

1. Form the square matrix J^{i1}_{t-1} (J^{1i}_{t}) made up of the determinants of each element of L^{i1}_{t-1} (L^{1i}_{t}), where the rows of J^{i1}_{t-1} (J^{1i}_{t}) correspond to the sequences (k), and the columns of J^{i1}_{t-1} (J^{1i}_{t}) correspond to the sequences (m), and if $(k) = (m)$ the corresponding entry is on the diagonal of J^{i1}_{t-1} (J^{1i}_{t}). There the (k,m) entry of J^{i1}_{t-1} (J^{1i}_{t}) corresponds to the sequence (k) of E_{i1} (E_{1i}) and to the sequence (m) of E_{i2} (E_{2i}).

2. For each column of J^{i1}_{t-1} (J^{1i}_{t}) evaluate the sum $\Sigma(m) = \sum_{h=1}^{t-1} m_h$ $(\Sigma(m) = \sum_{h=1}^{t} m_h)$ where $(m) = (m_1,\ldots,m_{t-1})$ or $(m) = (m_1,\ldots,m_t)$. If the sum is odd, change the sign of all entries in this column of J^{i1}_{t-1} (J^{1i}_{t}). Let the resulting matrix be called $J^{i1}_{(t-1)s}$ (J^{1i}_{ts}). (By definition (3-1-5), this corresponds to evaluating the function sgn. [(m)].)

3. If Δ_{i1} is to be evaluated, from X obtained in steps 3 and 4 of Algorithm-3, obtain X_{+i} by deleting all entries from X that correspond to a sequence not containing edge i. Remove edge i from each corresponding sequence. Similarly obtain X_{+j}.

   If Δ_{1i} is to be evaluated, from X obtained in steps 3 and 4 of Algorithm-3, obtain X_{-i} by deleting all entries from X containing edge i. Similarly obtain X_{-j}.

4. Then

$$\Delta_{i1} = (\pm 1) \;[X_{+i}\, J^{i1}_{(t-1)s}\, X^T_{+j}] \tag{5-1-11}$$

$$\Delta_{1i} = (\pm 1) \;[X_{-j}\, J^{1i}_{ts}\;\; X^T_{-i}] \tag{5-1-12}$$

   where in (5-1-11) and (5-1-12) it is understood that entries corresponding to the same sequences are multiplied together.

By keeping track of the sign of the determinants, the following equations can be obtained:

$$\frac{x_i}{x_j} = (-1)^{i+n-t} \frac{X_i \; J^{i1}_{(t-1)s} \; X_j^T}{X \; J^x_{ts} \; X^T}$$

$$\frac{y_i}{x_j} = (-1)^{i+1} \frac{X_{-j} \; J^{1i}_{ts} \; X_{-i}^T}{X \; J^x_{ts} \; X^T}$$

$$\frac{x_i}{y_j} = (-1)^{i+n-t} \frac{X_i \; J^{i1}_{(t-1)s} \; X_j^T}{X \; J^y_{ts} \; X^T} \qquad (5\text{-}1\text{-}13)$$

$$\frac{y_i}{y_j} = (-1)^{i+1} \frac{X_{-j} \; J^{1i}_{ts} \; X_{-i}^T}{X \; J^y_{ts} \; X^T}$$

The Classical Methods

As opposed to the above two algorithms the classical methods are based on the results obtained by Maxwell and Kirchoff [MA-1]. For two terminal networks they obtain the matrix J_{ts} or something equivalent. However, to find the vector X, the techniques given in the literature are sadly lacking. Various "rules" have been devised for stepping from one tree to another with a corresponding determination of sign [MA-2], [MA-3], [MA-4]. However, this technique has the disadvantage that after a certain number of steps it is possible to obtain the same tree (or forest) twice and it is not easy to determine when all trees have been found. The major defect here is that there is no clear and simple rule to find all trees. The Grassman outer product, on the other hand, provides a clear and simple evaluation of X, which is easily programmed on a computer. In fact, since for two terminal networks, the sign of the entries of X are of no consequence, the Grassman product can be formed in (mod. 2) algebra.

For the general network where the impedance (or admittance) matrix exists, Mason, Coates, and Mayeda have all given methods based upon the formulation of two graphs and whose common tree must be found to accomplish the analysis. Again there is no clear and simple method to find all common trees and their relative signs. Using (5-1-1) and Algorithms 3 and 4, Grassman algebra handles all such numerical work simply and effectively, not necessitating the formation of two graphs and modified equations. To use this algorithm, neither the impedance nor admittance matrix need exist.

Besides the above advantages, (5-1-1) handles tree sign and forest determination of the general linear system program of Coates and Mayeda by a simple formation of the Grassman outer product of incidence sets.

The advantages and the generality thus obtained alone show that the techniques of Grassman algebra as used in this thesis are an important mathematical tool for system analysis.

Part II. Examples

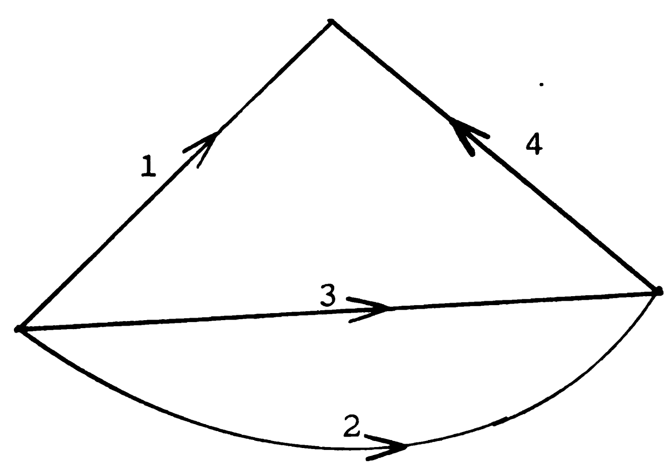
Example 1

Determine $\frac{y_2}{x_1}$ for the following system.

1. The component equations are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & r_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & r_3 & g_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & g_4 & r_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2. The graph is:



Solution:

$$J_{2s}^{x} = \begin{array}{c} \begin{matrix} (12) & (13) & (14) & (23) & (24) & (34) \end{matrix} \\ \begin{bmatrix} -(r_3r_4-g_3g_4) & 0 & 0 & 0 & 0 & 0 \\ 0 & -r_2r_4 & +r_2g_4 & 0 & 0 & 0 \\ 0 & r_2g_3 & -r_2r_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & r_3 & g_3 \\ 0 & 0 & 1 & 0 & g_4 & r_4 \end{bmatrix}$$

$$J_{2s}^{1i} = \begin{array}{c} \\ (23) \\ (24) \\ (34) \end{array} \begin{array}{c} \begin{matrix} (13) & (14) & (34) \end{matrix} \\ \begin{bmatrix} +r_4 & -g_4 & 0 \\ -g_3 & +r_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\rho_1 = e_1 + e_4, \quad \rho_2 = e_1 + e_2 + e_3$$

$$[\rho_1 \;\; \rho_2] = e_1e_2 + e_1e_3 - e_1e_4 - e_2e_4 - e_3e_4.$$

Therefore

$$X = \begin{array}{c} \begin{matrix} (12) & (13) & (14) & (23) & (24) & (34) \end{matrix} \\ [\begin{matrix} 1 & 1 & -1 & 0 & -1 & -1 \end{matrix}] \end{array}$$

$$X_{-1} = \begin{array}{ccc} (23) & (24) & (34) \\ [\;0 & -1 & -1\;] \end{array}$$

$$X_{-2} = \begin{array}{ccc} (13) & (14) & (34) \\ [\;1 & -1 & -1\;] \end{array} .$$

By (5-1-13)

$$\frac{y_2}{x_1} = \frac{X_{-1}\; J_{2s}^{1i}\; X_{-2}^T}{X\;\; J_{2s}^{x}\;\; X^T} = \frac{-g_3\; -r_3}{-r_3r_4+g_3g_4\; -r_2r_4\; -r_2r_3\; -r_2g_3\; -r_2g_4}$$

Example 2

Does the following system have a unique solution?

1. The component equations are:

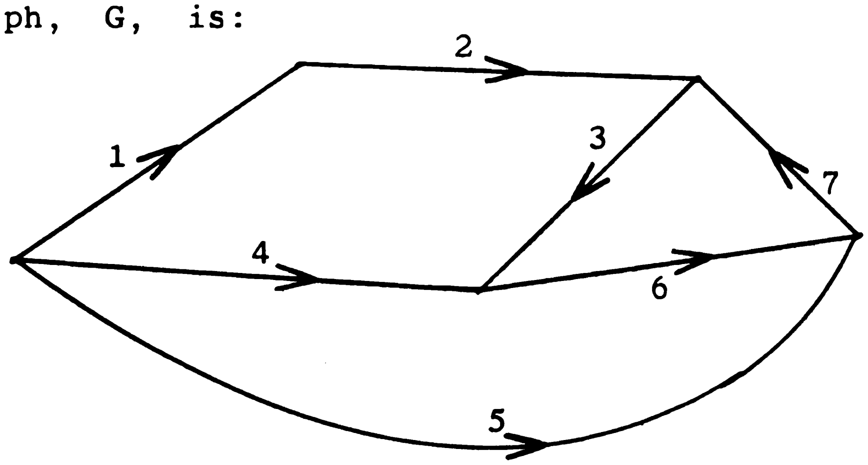
$$\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} f_3(t) \\ f_4(t) \end{bmatrix}$$

$$x_5 - (s^3 + s + 1)\; y_5 = f_5(t)$$

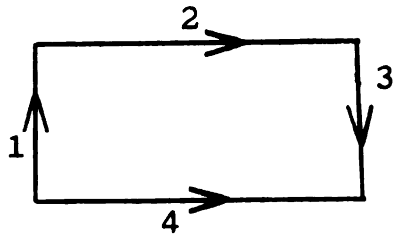
$$\begin{bmatrix} s+5 & 1 \\ 2 & s-3 \end{bmatrix} \begin{bmatrix} x_6 \\ x_7 \end{bmatrix} - \begin{bmatrix} s^2 + 5s & s + 1 \\ s - 1 & s^2 + 5 \end{bmatrix} \begin{bmatrix} y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. The graph, G, is:

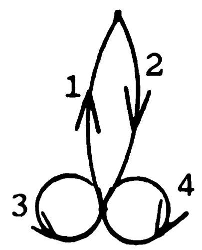


Solution: Let $S = \{e_1, e_2, e_3, e_4\}$, where $e_i \in E(G)$, $i = 1, 2, 3, 4$.

Then $G \cdot S =$



and $G \times S =$



By Corollary 3-1-5 , the system has a unique solution for $a \neq 1$ and no unique solution for $a = 1$.

Example 3

Does the following system have a unique solution on $L_{2,7}$ (5,15)?

1. The component equations are:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \int_5^{15} ts\,[y_1(s)y_2(s)+2y_1(s)+\sin y_1(s)]\,ds \\ ts\,[-y_1(s)y_2(s)+\sin y_2(s)+4y_2(s)]\,ds \end{bmatrix} +$$

$$\begin{bmatrix} 2y_1(t) + \cos.\ y_1(t) \\ y_2(t) + 6\,t\,y_2(t) \end{bmatrix}$$

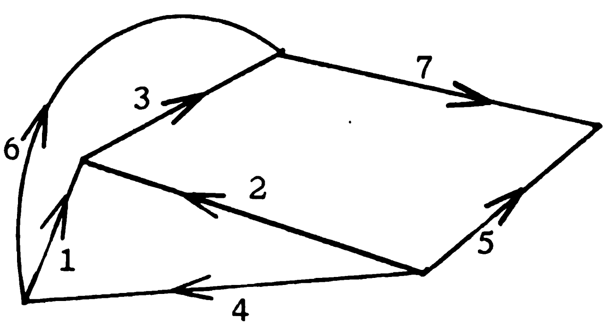
$$\begin{bmatrix} x_3(t) \\ y_4(t) \end{bmatrix} = \begin{bmatrix} 2e^t x_4(t) \\ -2e^t y_3(t) \end{bmatrix}$$

$$x_5(t) = \sin y_5(t)$$

$$y_6(t) = 4x_6(t)$$

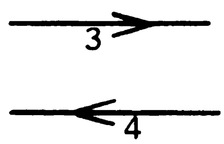
$$y_7(t) = f_7(t)$$

2. The system graph, G, is:

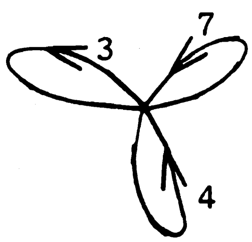


Solution: By Lemmas B8, B9, and B10, the components corresponding to edges 1, 2, and 6 are strongly monotonic. By Lemma B9 the component corresponding to edges 3 and 4 is monotonic. All components also satisfy a Lipschitz condition.

The graph $G \cdot (S_{p3} \cup S_{p4} \cup S^{x}_{p6})$ is



The graph $G \times (S_{p3} \cup S_{p4} \cup S^{y}_{p6})$ is



Therefore, by Corollary 3-2-2, the system has a unique solution.

Example 4

Is the following system stable?

1. The component equations of Type 1 are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \end{bmatrix} = \begin{bmatrix} 0.5t^2+1 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\dot{x}_6 = 2y_6$$

$$\begin{bmatrix} x_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} y_7 \\ x_8 \end{bmatrix}, \qquad a \neq 0$$

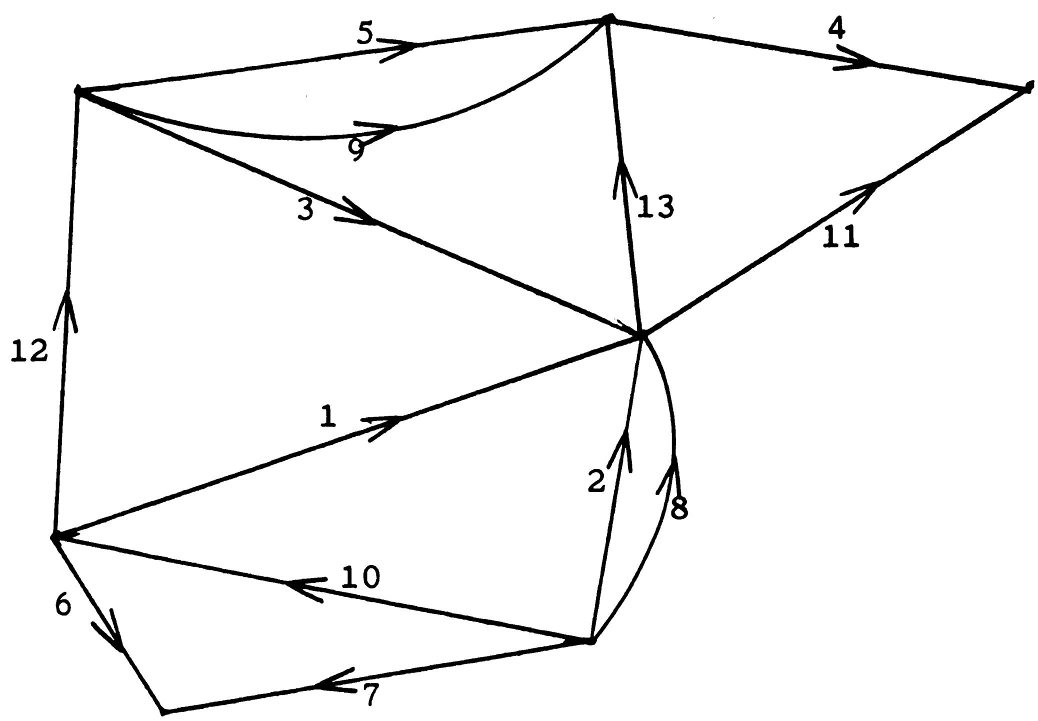
$$\begin{bmatrix} x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} y_9 \\ y_{10} \end{bmatrix}$$

$$x_{11} = 5y_{11}$$

$$y_{12} = f_{12}(t)$$

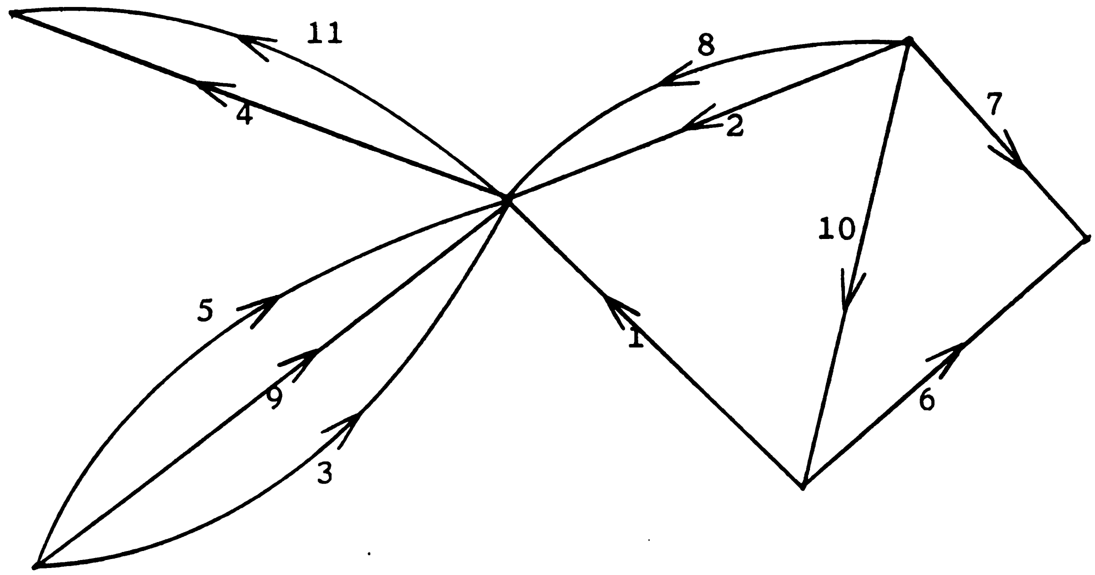
$$x_{13} = f_{13}(t)$$

2. The graph, G, is:



Solution: Take forest $T_o = \{1, 2, 6, 9, 11, 13$.

The graph $\left[[\, G \; X \; (E(G) - S^{x}_{p5})] \cdot (E(G) - S_{p5}) \right]$ is:



By Corollary 4-1-1, the system is not stable for $a < 0$, and stable for $a > 0$.

CHAPTER VI

SUMMARY

The principal results of this thesis are as follows:

1. Grassman algebra is shown (in Algorithms 1, 2, 3, and 4) to be the abstract mathematical discipline that enables one to calculate more simply and concisely the determinant functions that describe the behavior of linear time-stationary systems.
2. In Theorems 2-1-3 and 2-1-6 and Corollary 2-1-2, two new results are given in the theory of Grassman algebra which are used in the algorithms.
3. The chains of integers of Tutte's matroid theory are extended to real vector spaces.
4. A foundation is given for the relationship between graph theory and the Grassman outer product of a vector subspace. This knowledge is extremely useful for the methods of topological analysis and synthesis.
5. For the first time this thesis has shown the intermediate role which matroid theory plays in the establishment of the connection between the graph and its two related vector subspaces.

6. An important new theorem (Theorem 3-1-3), is derived which gives the necessary and sufficient condition for a unique solution for linear systems with semidefinite components.
7. Mappings in Hilbert spaces are analyzed in the light of present system theory and two new theorems given (Theorems 3-2-1 and 3-2-2).
8. Two very general theorems for systems of algebraic and differential equations are given. These theorems encompass practically all previously published results (Theorems 3-2-3 and 3-2-4).
9. The graph theoretical Lemmas of Chapters III and IV provide some important direct connections between graphs and the representative matrices for their subspaces.
10. The stability theorems provide direct relations between the graph equations, the component equations, and the system solution characteristic of stability.

BIBLIOGRAPHY

[BAR-1] Barrett, J. F. "The Use of Functionals in the Analysis of Nonlinear Physical Systems," Statistical Advisory Report, Ministry of Supply, Great Britain, Jan. 1957.

[BE-1] Bellman, R. Stability Theory of Differential Equations, McGraw-Hill, 1953.

[BE-2] Bellman, R. Introduction to Matrix Analysis, McGraw-Hill, 1960.

[BEN-1] Bendixson, M. I. "Sur Les Racines D'une Equation Fondamentale," Acta Mathematica, Vol. 24, 1902, pp. 367-370.

[BI-1] Birkhoff, G., MacLane, S. A Survey of Modern Algebra, MacMillan Company, 1953.

[BI-2] Birkhoff, G., Diaz, J. B. "Nonlinear Network Problems," Quart. of Applied Math., Vol 13, Jan. 1956, pp. 431-443.

[BOS-1] Bose, A. G. "A Theory of Nonlinear Systems," M.I.T. Research Laboratory of Electronics Report No. 309, Cambridge, Massachusetts, 1956.

[BR-1] Browder, F. E. "The Solvability of Nonlinear Functional Equations," Duke Math. Journal, 1963, p. 557.

[CI-1] Civita, T. Levi. The Absolute Differential Calculus, Blackie & Son, Ltd., 1929.

[CR-1] Chrystal, G. "A Fundamental Theorem Regarding the Equivalence of Systems of Ordinary Linear Differential Equations, and Its Application to the Determination of the Order and the Systematic Solution of a Determinate System of Such Equations," Trans. of the Royal Soc. of Edin., Vol. 38, Pt. I, 1895, pp. 163-178.

[DE-1] Desoer, C. A., Katzenelson, J. "Nonlinear R.L.C. Networks," Bell System Technical Journal, Jan. 1965, p. 161.

[DU-1] Duffin, R. J. "Nonlinear Networks I," Bull. Am. Math. Soc., Vol. 52, 1946, pp. 833-838.

[DU-2] Duffin, R. J. "Nonlinear Networks II," Bull. Am. Math. Soc., Vol. 53, 1947.

[DU-3] Duffin, R. J. "Nonlinear Networks III," Bull. Am. Math. Soc., Vol. 55, 1949, p. 12.

[DU-4] Duffin, R. J. "Nonlinear Networks IV," Proc. Am. Math. Soc., Vol. 4, 1953, p. 233.

[DU-5] Duffin, R. J. "Analysis of the Wang Algebra of Networks," Trans. Am. Math. Soc., 1959.

[FO-1] Forder, H. G. The Calculus of Extension, Cambridge University Press, 1941.

[FR-1] Frame, J. S., Koenig, H. E. "Applications of Matrices to Systems Analysis," IEEE Spectrum, May 1964.

[GA-1] Gantmacher, F. R. The Theory of Matrices, Vol. 1 & 2, Chelsea Publishing Co., 1959.

[GR-1] Graves, R. L., Wolf, P. Recent Advances in Mathematical Programming, McGraw-Hill, 1963.

[HA-1] Hahn, Wolfgang. Theory and Applications of Lyapunov's Direct Method, Prentice-Hall, 1963.

[HO-1] Hohn, F. E. Elementary Matrix Algebra, MacMillan Company, 1958.

[IN-1] Ince, E. L. Ordinary Differential Equations, Dover Publications, 1926.

[KA-1] Kaplan, W. Advanced Calculus, Wiley & Sons, 1953.

[KI-1] Kim, W. H., Chien, R. T. Topological Analysis and Synthesis of Communication Networks, Columbia Univ. Press, 1962.

[KO-1] Koenig, H. E., Tokad, Y. A., Kesavan, H. K. Analysis of Discrete Physical Systems, McGraw-Hill, to be published in 1965.

[LE-1] Lefschetz, S. Differential Equations: Geometric Theory, Interscience Publishers, Inc., 1957.

[MA-1] Mayeda, W., Seshu, S. "Topological Formulas for Network Functions," Bull. No. 446, Univ. of Ill. Engr. Exper. Station, 1957.

[MA-2] Mayeda, W., Van Valkenburg, M. E. "Analysis of Non-Reciprocal Networks by Digital Computers," Inst. Radio Engrs. Nat'l. Convention Record, Vol. 6, Pt. 2, 1958, pp. 70-75.

[MA-3] Mayeda, W. "Topological Formulas for Active Networks," Proc. of Nat'l. Elec. Conference, Chicago, Vol. 15, 1958, pp. 1-13.

[MA-4] Mayeda, W. "Reducing Computation Time in the Analysis of Networks by Digital Computers," IRE-Trans. PGCT 6, 1959, pp. 136-137.

[MAC-1] MacLane, Saunders. "Grassman Algebra and Determinants," Lecture Notes, Univ. of Chicago.

[MC-1] Frank, P., McFee, R. "Determining Input-Output Relationships of Nonlinear Systems by Inversions," IEEE Trans. on Circuit Theory, CT-10, June, 1963.

[MI-1] Minty, G. J. "Monotone Networks," Proc. Roy'l Soc. London, 257A, 1960, pp. 194-212.

[MI-2] Minty, G. J. "Monotone (Nonlinear) Operators in Hilbert Space," Duke Math. Journal, Vol. 29, 1962, pp. 341-346.

[NE-1] Nemytskii, and Stepanov. Qualitative Theory of Differential Equations, Princeton Univ. Press, 1960.

[OR-1] Ore, Oystein. "Theory of Graphs," Am. Math. Soc., Providence, R. I., 1962.

[PA-1] Paul, John C. "An Application of Modern Algebra to Engineering Graph Theory," Unpublished Thesis, Oklahoma State Univ., Dept. of Elec. Engr., May 1963.

[RI-1] Riesz, F., Nagy, S. Functional Analysis, Frederick Ungar Publishing Company, 1955.

[RO-1] Royden, H. L. Real Analysis, MacMillan Company, 1963.

[SE-1] Seshu, S., Reed, M. B. Linear Graphs and Electrical Networks, Addison-Wesley Publishing Company, 1961.

[TU-1] Tutte, W. T. "A Class of Abelian Groups," Canadian J. of Math., Vol. 8, 1956, pp. 13-28.

[TU-2] Tutte, W. T. "Homotopy Theorem for Matroids I," Trans. Am. Math. Soc., Vol. 88, 1958, pp. 144-160.

[TU-3] Tutte, W. T. "Homotopy Theorem for Matroids II," Trans. Am. Math. Soc., Vol. 88, 1958, pp. 160-174.

[TU-4] Tutte, W. T. "Matroids and Graphs," Trans. Am. Math. Soc., Vol. 90, 1959, pp. 527-552.

[TU-5] Tutte, W. T. "An Algorithm for Determining Whether a Given Binary Matroid is Graphic," Proc. Am. Math. Soc., Vol. 11, 1960, pp. 905-917.

[WI-1] Wirth, J. L. "Time Domain Models of Physical Systems and Existence of Solutions," An Unpublished Thesis, Elec. Engr. Dept., Michigan State University, 1962.

[WIE-1] Wiener, N. Nonlinear Problems in Random Theory, Wiley and Sons, 1958.

[ZAH-1] Zadeh, L. T. Linear System Analysis, McGraw-Hill, 1963.

[ZAH-2] Zadeh, L. T. "A Contribution to the Theory of Nonlinear Systems," Journal of the Franklin Institute, Vol. 255, 1953, pp. 387-408.

[ZAM-1] Zames, G. "Functional Analysis Applied to Nonlinear Feedback Systems," IEEE Trans. on Circuit Theory, CT-10, 1963, pp. 392-404.

APPENDIX A

ON POSITIVE DEFINITE AND SEMIDEFINITE MATRICES

Lemma A1: Any positive definite matrix is nonsingular.

Proof: Since A is positive definite it follows that $A + A^T$ is positive definite and has positive eigenvalues [BE-2].

By Bendixson's theorem [BEN-1], the real parts of the eigenvalues of A are between the minimum and maximum eigenvalues of $A + A^T$. Therefore, A has eigenvalues with positive real parts, and A is nonsingular.

Lemma A2: The inverse of a positive definite matrix is positive definite.

Proof: Let A be positive definite and $Y \neq 0$. By Lemma A1, A is nonsingular. If $Y = AX$ then $X \neq 0$ and it follows that $Y^T A^{-1} Y = X^T A^T X = X^T A X > 0$.

Lemma A3: If A is positive semidefinite, $B^T AB$ is positive semidefinite for arbitrary B.

Proof: Let Y be an arbitrary vector and let $X = BY$, then

$$Y^T B^T A B Y = X^T AX \geq 0$$

<u>Lemma A4</u>: Let

$$C_5 = \begin{bmatrix} C_1 & 0 \\ C_3 & -U \end{bmatrix}$$

Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

be positive semidefinite, Q_{11} and Q_{22} skew and U the unit matrix and assume

$$\left[\begin{bmatrix} U & O \\ -Q_{12}^T & Q_{22} \end{bmatrix} - \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ O & U \end{bmatrix}\right] = D$$

is nonsingular. Then $D^{-1}\ C_5$ is positive semidefinite.

<u>Proof</u>: Let $Y \neq O$. $Y = D^TX$ for some X, since D is nonsingular, and $X \neq O$.

$$Y^T\ D^{-1}\ C_5Y = X^T\ C_5D^TX \geq O$$

since

$$C_5D^T = \begin{bmatrix} C_1 & O \\ C_3 & -U \end{bmatrix} \left[\begin{bmatrix} U & -Q_{12} \\ O & -Q_{22}^T \end{bmatrix} - \begin{bmatrix} Q_{11}^T & O \\ Q_{12}^T & U \end{bmatrix} \begin{bmatrix} C_1^T & C_3^T \\ C_2^T & C_4^T \end{bmatrix}\right] =$$

$$= \begin{bmatrix} C_1 & 0 \\ C_3+C_2^T & C_4^T \end{bmatrix} + \begin{bmatrix} -C_1Q_{11}^TC_1^T & -C_1Q_{12}+C_1Q_{11}C_3^T \\ -C_3Q_{11}^TC_1^T+Q_{12}^TC_1^T & -C_3Q_{12}-C_3Q_{11}^TC_3^T+Q_{12}^TC_3^T+Q_{22}^T \end{bmatrix}$$

so C_5D^T has the same symmetric part as C, and is therefore positive semidefinite.

Corollary: If in addition to the above, C is positive definite, D is positive definite.

Lemma A5: If A is positive definite and B has maximum column rank, then B^TAB is positive definite.

Proof: Let $Y \neq 0$, $X = BY$. Since B has maximum column rank $X \neq 0$ and it follows that

$$Y^T B^T ABY = X^T AX > 0$$

Lemma A6: The matrix

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

is positive definite, if and only if the matrix

$$\begin{bmatrix} C_1 - C_2C_4^{-1}C_3 & C_2C_4^{-1} \\ -C_4^{-1}C_3 & C_4^{-1} \end{bmatrix} = K_0$$

is positive definite.

Proof: Let $A = (C_1 - C_2C_4^{-1}C_3)^{-1}$. If either K_0 or C is positive definite A^{-1} exists and is nonsingular. The matrix

$$\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}^{-1} = \begin{bmatrix} A & -A\ C_2C_4^{-1} \\ -C_4^{-1}C_3A & C_4^{-1} + C_4^{-1}C_3A\ C_2C_4^{-1} \end{bmatrix}$$

is positive definite if and only if C is positive definite by Lemma A2. Therefore the matrix

$$K_1 = \begin{bmatrix} A^{-1} & 0 \\ C_4^{-1}C_3 & U \end{bmatrix} \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}^{-1} \begin{bmatrix} (A^{-1})^T & C_3^T(C_4^{-1})^T \\ 0 & U \end{bmatrix}$$

$$= \begin{bmatrix} (A^{-1})^T & (C_3^T(C_4^{-1})^T - C_2C_4^{-1}) \\ 0 & C_4^{-1} \end{bmatrix}$$

is positive definite if C is positive definite. Let $K_2 = K_1 + K_1^T$, so K_2 is positive definite if and only if K_1 is positive definite, but K_2 is twice the symmetric part of K_0. Therefore K_0 is positive definite if and only if C is positive definite.

Lemma A7: If the matrix

$$H = \begin{bmatrix} Q_{111} & Q_{112} & Q_{211} & Q_{212} \\ 0 & U & 0 & 0 \\ -Q_{211}^T & -Q_{213}^T & Q_{311} & Q_{312} \\ 0 & 0 & 0 & U \end{bmatrix} - \begin{bmatrix} C_{11} & C_{12} & & \\ C_{21} & C_{22} & & \\ & & C_{33} & C_{34} \\ & & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} U & 0 & 0 & 0 \\ -Q_{112}^T & Q_{113} & Q_{213} & Q_{214} \\ 0 & 0 & U & 0 \\ -Q_{212}^T & -Q_{214}^T & -Q_{312}^T & Q_{313} \end{bmatrix}$$

where $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ is positive definite and $\begin{bmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{bmatrix}$ is positive semidefinite, Q_{111}, Q_{113}, Q_{311}, and Q_{313} are skew and

$\begin{bmatrix} Q_{211}^T & Q_{213}^T \\ Q_{212}^T & Q_{214}^T \end{bmatrix}$ has maximum row rank, then H is nonsingular.

Proof: Premultiply H by the nonsingular matrix

$$\begin{bmatrix} U & -Q_{112} & 0 & -Q_{212} \\ 0 & U & 0 & 0 \\ 0 & Q_{213}^T & U & -Q_{312} \\ 0 & 0 & 0 & U \end{bmatrix}$$

to obtain:

$$
H_1 = \begin{bmatrix} Q_{111} & 0 & Q_{211} & 0 \\ 0 & U & 0 & 0 \\ -Q_{211}^T & 0 & Q_{311} & 0 \\ 0 & 0 & 0 & U \end{bmatrix}
$$

$$
- \begin{bmatrix} U & -Q_{112} & 0 & -Q_{212} \\ 0 & U & 0 & 0 \\ 0 & Q_{213}^T & U & -Q_{312} \\ 0 & 0 & 0 & U \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & & \\ C_{21} & C_{22} & & \\ & & C_{33} & C_{34} \\ & & C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} U & 0 & 0 & 0 \\ -Q_{112} & U & Q_{213} & 0 \\ 0 & 0 & U & 0 \\ -Q_{212}^T & 0 & -Q_{312}^T & U \end{bmatrix} \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & Q_{113} & 0 & Q_{214} \\ 0 & 0 & U & 0 \\ 0 & -Q_{214}^T & 0 & Q_{313} \end{bmatrix}
$$

$$
= \begin{bmatrix} Q_{111} & 0 & Q_{211} & 0 \\ 0 & U & 0 & 0 \\ -Q_{211}^T & 0 & Q_{311} & 0 \\ 0 & 0 & 0 & U \end{bmatrix}
$$

$$
- \begin{bmatrix} C_{55}+Q_{212}C_{44}Q_{212}^T & C_{25} & C_{25}Q_{213} - Q_{212}C_{64} & -Q_{212}C_{44} \\ C_{52} & C_{22} & C_{22}\,Q_{213} & 0 \\ Q_{213}^T C_{52} - C_{46}Q_{212}^T & Q_{213}^T C_{22} & C_{66}+Q_{213}^T\,C_{22}Q_{213} & C_{46} \\ -C_{44}Q_{212}^T & 0 & C_{64} & C_{44} \end{bmatrix} \times
$$

$$\times\begin{bmatrix} U & O & O & O \\ O & Q_{113} & O & Q_{214} \\ O & O & U & O \\ O & Q_{214}^T & O & Q_{313} \end{bmatrix}$$

where

$$C_{55} = C_{11} - Q_{112}\, C_{21} - C_{12}\, Q_{112}^T + Q_{112}\, C_{22}\, Q_{112}^T$$

$$C_{52} = C_{21} - C_{22}\, Q_{112}^T$$

$$C_{25} = C_{12} - Q_{112}\, C_{22}$$

$$C_{66} = C_{33} - Q_{312}\, C_{43} - C_{34}\, Q_{312}^T + Q_{312}\, C_{44}\, Q_{312}^T$$

$$C_{46} = C_{34} - Q_{312}\, C_{44}$$

$$C_{64} = C_{43} - C_{44}\, Q_{312}^T$$

The matrix

$$\begin{bmatrix} C_{55} & C_{25} \\ C_{52} & C_{22} \end{bmatrix} = \begin{bmatrix} U & -Q_{112} \\ O & U \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} U & O \\ -Q_{112}^T & U \end{bmatrix}$$

so is positive definite.

The matrix

$$\begin{bmatrix} C_{66} & C_{46} \\ C_{64} & C_{44} \end{bmatrix} = \begin{bmatrix} U & -Q_{312} \\ O & U \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{bmatrix} \begin{bmatrix} U & O \\ -Q_{312}^T & U \end{bmatrix}$$

so is positive semidefinite.

The matrix C_{22} is positive definite so has a positive definite inverse by Lemmas A1 and A2. Premultiply H_1 by the nonsingular matrix

$$\begin{bmatrix} -U & C_{25}C_{22}^{-1} & 0 & -Q_{212} \\ 0 & C_{22}^{-1} & 0 & 0 \\ 0 & Q_{213}^T & -U & 0 \\ 0 & 0 & 0 & U \end{bmatrix}$$

to obtain

$$H_2 = \begin{bmatrix} C_{77} & C_{78} & -Q_{211} & -Q_{212} \\ C_{87} & C_{88} & -Q_{213} & -Q_{214} \\ Q_{211}^T - C_{46}Q_{212}^T & Q_{213}^T - C_{46}Q_{214}^T & -Q_{311} + C_{66} & C_{46}Q_{313} \\ C_{44}\,Q_{212}^T & +C_{44}^T Q_{214}^T & -C_{64} & U - C_{44}Q_{313} \end{bmatrix}$$

where

$$C_{77} = -Q_{111} + C_{55} - C_{25}\,C_{22}^{-1}\,C_{52}$$

$$C_{78} = C_{25}\,C_{22}^{-1}$$

$$C_{87} = -C_{22}^{-1}\,C_{52}$$

$$C_{88} = C_{22}^{-1} - Q_{113}$$

By Lemma A6, and the fact that Q_{111} and Q_{113} are skew

$$\begin{bmatrix} C_{77} & C_{78} \\ C_{87} & C_{88} \end{bmatrix} \quad \text{is positive definite.}$$

Let

$$\begin{bmatrix} Q_{211}^T - C_{46}\, Q_{212}^T & Q_{213}^T - C_{46}\, Q_{214}^T \\ C_{44}\, Q_{212}^T & C_{44}\, Q_{214}^T \end{bmatrix} \begin{bmatrix} C_{77} & C_{78} \\ C_{87} & C_{88} \end{bmatrix}^{-1} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

Premultiply H_2 by the nonsingular matrix

$$\begin{bmatrix} U & O & O & O \\ O & U & O & O \\ -K_1 & -K_2 & U & O \\ -K_3 & -K_4 & O & U \end{bmatrix}$$

to obtain

$$H_3 = \begin{bmatrix} C_{77} & C_{78} & -Q_{211} & -Q_{212} \\ C_{87} & C_{88} & -Q_{213} & -Q_{214} \\ O & O & K_5 & K_6 \\ O & O & K_7 & K_8 \end{bmatrix}$$

where

$$H_4 = \begin{bmatrix} K_5 & K_6 \\ K_7 & K_8 \end{bmatrix} = \begin{bmatrix} -Q_{311} + C_{66} & C_{46}\, Q_{313} \\ -C_{64} & U - C_{44}\, Q_{313} \end{bmatrix}$$

$$+ \begin{bmatrix} Q^T_{211} - C_{46}\, Q^T_{212} & Q^T_{213} - C_{46}\, Q^T_{214} \\ C_{44}\, Q^T_{212} & C_{44}\, Q^T_{214} \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \begin{bmatrix} Q_{211} & Q_{212} \\ Q_{213} & Q_{214} \end{bmatrix}$$

where $\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} = \begin{bmatrix} C_{77} & C_{78} \\ C_{87} & C_{88} \end{bmatrix}^{-1}$ so is positive definite.

Now H_3, and hence H is nonsingular if and only if the matrix H_4 is nonsingular. Expand H_4 as

$$H_4 = \begin{bmatrix} C_{66} - Q_{311} & C_{46}\, Q_{313} \\ -C_{64} & U - C_{44}\, Q_{313} \end{bmatrix}$$

$$+ \begin{bmatrix} U & -C_{46} \\ O & C_{44} \end{bmatrix} \begin{bmatrix} D_5 & D_6 \\ D_7 & D_8 \end{bmatrix}$$

where

$$\begin{bmatrix} D_5 & D_6 \\ D_7 & D_8 \end{bmatrix} = \begin{bmatrix} Q^T_{211} & Q^T_{213} \\ Q^T_{212} & Q^T_{214} \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \begin{bmatrix} Q_{211} & Q_{212} \\ Q_{213} & Q_{214} \end{bmatrix}$$

Since

$$\begin{bmatrix} Q^T_{211} & Q^T_{213} \\ Q^T_{212} & Q^T_{214} \end{bmatrix}$$

is maximum row rank, by Lemma A5 the matrix

$$\begin{bmatrix} D_5 & D_6 \\ D_7 & D_8 \end{bmatrix}$$

is positive definite.

Now combine all factors of H_4 so

$$H_4 = \begin{bmatrix} C_{66} - Q_{311} + D_5 - C_{46}\ D_7 & C_{46}\ Q_{313} + D_6 - C_{46}\ D_8 \\ -C_{64} + C_{44}\ D_7 & U - C_{44}\ Q_{313} + C_{44}\ D_8 \end{bmatrix}$$

$$= \begin{bmatrix} C_{66} + E_1 - C_{46}\ E_3 & -C_{46}\ E_4 + E_2 \\ -C_{64} + C_{44}\ E_3 & U + C_{44}\ E_4 \end{bmatrix}$$

where

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} = \begin{bmatrix} D_5 - Q_{311} & D_6 \\ D_7 & D_8 - Q_{313} \end{bmatrix}$$

and is positive definite. Postmultiply H_4 by the non-singular matrix

$$\begin{bmatrix} U & O \\ -E_4^{-1} E_3 & E_4^{-1} \end{bmatrix}$$

to obtain

$$H_5 = \begin{bmatrix} C_{66} + E_1 - E_2 E_4^{-1} E_3 & -C_{46} + E_2 E_4^{-1} \\ -C_{64} - E_4^{-1} E_3 & E_4^{-1} + C_{44} \end{bmatrix}$$

$$= \begin{bmatrix} C_{66} & -C_{46} \\ -C_{64} & C_{44} \end{bmatrix} + \begin{bmatrix} E_1 - E_2 E_4^{-1} E_3 & E_2 E_4^{-1} \\ -E_4^{-1} E_3 & E_4^{-1} \end{bmatrix}$$

The matrix

$$\begin{bmatrix} C_{66} & -C_{46} \\ -C_{64} & C_{44} \end{bmatrix} = \begin{bmatrix} -U & O \\ O & U \end{bmatrix} \begin{bmatrix} C_{66} & C_{46} \\ C_{64} & C_{44} \end{bmatrix} \begin{bmatrix} -U & O \\ O & U \end{bmatrix}$$

so is positive semidefinite.

The matrix

$$\begin{bmatrix} E_1 - E_2 E_4^{-1} E_3 & E_2 E_4^{-1} \\ - E_4^{-1} E_3 & E_4^{-1} \end{bmatrix}$$

is positive definite by Lemma A6. Therefore H_5 is positive definite, so is nonsingular and therefore H is nonsingular.

Lemma A8: If M is a rectangular matrix with maximum row rank, then $M^T AM$ is positive semidefinite if, and only if, A is positive semidefinite.

Proof: If A is positive semidefinite, let X be any vector and let $Y = MX$. Therefore,

$$X^T M^T AMX = Y^T AY \geq 0$$

If $M^T AM$ is positive semidefinite, let Y be any vector, since M is maximum row rank there exists at least one vector X, such that $Y = MX$. Therefore,

$$Y^T AY = X^T M^T AMX \geq 0$$

Lemma A9: Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

be positive definite, Q_{11} and Q_{22} skew and U the unit matrix. Then

$$\left[\begin{bmatrix} U & O \\ -Q_{12}^T & Q_{22} \end{bmatrix} - \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ O & U \end{bmatrix} \right] = D$$

is nonsingular.

Proof: Premultiply D by the nonsingular matrix

$$\begin{bmatrix} U & O \\ -Q_{12}^T & -U \end{bmatrix}$$

to obtain

$$D_1 = \begin{bmatrix} U - C_5 Q_{11} & C_6 \\ -C_7 Q_{11} & C_8 \end{bmatrix}$$

where

$$\begin{bmatrix} C_5 & C_6 \\ C_7 & C_8 \end{bmatrix} = \begin{bmatrix} U & O \\ -Q_{12}^T & -U \end{bmatrix} \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} U & -Q_{12} \\ O & -U \end{bmatrix} + \begin{bmatrix} O & O \\ O & -Q_{22} \end{bmatrix}$$

so is positive definite. The matrix C_8 is positive definite, and its inverse is positive definite by Lemma A2. Postmultiply D_1 by

$$\begin{bmatrix} U & O \\ C_8^{-1} C_7 Q_{11} & U \end{bmatrix}$$

to obtain

$$D_2 = \begin{bmatrix} U - C_5 Q_{11} + C_6 C_8^{-1} C_7 Q_{11} & C_6 \\ O & C_8 \end{bmatrix}$$

The matrix

$$C_9 = C_5 - C_6\, C_8^{-1}\, C_7 = \begin{bmatrix} U & -C_6\, C_8^{-1} \end{bmatrix} \begin{bmatrix} C_5 & C_6 \\ C_7 & C_8 \end{bmatrix} \begin{bmatrix} U \\ -(C_8^{-1})^T\, C_6^T \end{bmatrix}$$

so is positive definite. Therefore premultiply D_2 by

$$\begin{bmatrix} C_9^{-1} & O \\ O & U \end{bmatrix}$$

to obtain

$$D_3 = \begin{bmatrix} C_9^{-1} - Q_{11} & C_9^{-1}\, C_6 \\ O & C_8 \end{bmatrix}$$

and since C_9^{-1} is positive definite, D_3 is nonsingular.

<u>Lemma A10</u>: If $\dot{P}(t)^{-1} = \frac{d}{dt}\; P^{-1}(t)$ exists and is negative semidefinite, and if $K(t)$ is as defined in (4-1-6), then $\dot{K}(t)^{-1} = \frac{d}{dt}\; K^{-1}(t)$ is negative semidefinite.

<u>Proof</u>: From (4-1-7),

$$\frac{d}{dt} \begin{bmatrix} K^{-1} & \xi_{12} + Q_{14}\, \xi_{22} \\ \xi_{21} + \xi_{22}\, Q_{14}^T & \xi_{22} \end{bmatrix} = \begin{bmatrix} \dot{K}^{-1} & \xi_{12} + Q_{14}\, \xi_{22} \\ \xi_{21} + \xi_{22}\, Q_{14}^T & \xi_{22} \end{bmatrix}$$

$$= \begin{bmatrix} U & O \\ Q_{14}^T & U \end{bmatrix}^T \left[\frac{d}{dt}\; P^{-1}(t)\right] \begin{bmatrix} U & O \\ Q_{14}^T & U \end{bmatrix}$$ which is negative

semidefinite so $\dot{K}^{-1}$ is negative semidefinite.

APPENDIX B

NONLINEAR MAPPINGS IN HILBERT SPACE

In Part 2 of Chapter III, some simple concepts from Hilbert and Banach space were introduced. The following sequence of results utilize the definitions given there.

Let $\langle\ ,\ \rangle$ denote the inner product in a Hilbert space.

<u>Theorem B1</u>: Let G be a continuous mapping of the Hilbert space H into itself such that for all x_1 and x_2 in H,

$$\langle G(x_1)-G(x_2),\ x_1-x_2\rangle \geq c\left(\max\left\{\|x_1\|\ ,\ \|x_2\|\right\}\right)\|x_1-x_2\|^2$$

where $c(r)$ is a positive non-increasing function of r such that

$$\int_1^\infty c(r)\,dr = +\infty$$

Then G is 1-1, onto, and its inverse is continuous.

<u>Proof</u>: [BR-1].

<u>Condition (C1)</u>: A mapping $G: H \rightarrow H$ satisfies condition (C1) if it is continuous and for all x_1 and x_2 in H,

$$\langle G(x_1) - G(x_2),\ x_1-x_2\rangle \geq c\left(\max\left\{\|x_1\|\ ,\ \|x_2\|\right\}\right)\|x_1-x_2\|^2$$

where $c(r)$ is a positive non-increasing function of r such that

$$\int_1^\infty c(r)\, dr = \infty .$$

Condition (L1): A mapping $G: H \to H$ satisfies condition (L1) if, for all x_1 and x_2 in H,

$$\langle x_1 - x_2,\ G(x_1) - G(x_2) \rangle \geq c\left(\max\left\{\|x_1\|\ ,\ \|x_2\|\right\}\right) \|x_1 - x_2\|^2$$

and

$$\langle x_1 - x_2,\ G(x_1) - G(x_2) \rangle \geq c\left(\max\left\{\|G(x_1)\|,\ \|G(x_2)\|\right\}\right) \|G(x_1) - G(x_2)\|^2$$

where $c(r)$ satisfies the requirements of condition (C1).

Remark 1: If G is strongly monotonic and satisfies a Lipschitz condition. G satisfies condition (L1).

Lemma B2: Suppose each mapping of a set of mappings on E_m (the m dimensional Euclidean space), or on $L_{2,m}$, satisfies condition (C1). Then their direct sum satisfies condition (C1).

Proof: Let G be the direct sum of the mappings G_i, and $c_i(r)$ be the $c(r)$ in condition (C1) corresponding to G_i. If $h(r) = \min_i \left[\left\{c_i(r)\right\}\right]$, then

$$\int_1^\infty h(r)\, dr = \infty$$

and, if $X_i = \begin{bmatrix} x_1^{\,i} \\ \vdots \\ x_m^{\,i} \end{bmatrix}$, i=1, 2,

$$\left\langle G(X_1) - G(X_2),\ X_1 - X_2 \right\rangle \geq$$

$$h\left(\max\left\{ \|x_1^{\,1}\|, \ldots, \|x_m^{\,1}\|, \|x_1^{\,2}\|, \ldots, \|x_m^{\,2}\| \right\}\right) \|X_1 - X_2\|^2.$$

But $\max\left\{ \|x_1^{\,i}\|, \ldots, \|x_m^{\,i}\| \right\}$ is a norm equivalent to $\|X_i\|$ on E_m or $L_{2,m}$.

Therefore there exists some function $h_1(r)$ such that

$$\int_1^{\infty} h_1(r)\, dr = \infty \quad \text{and}$$

$$\left\langle G(X_1) - G(X_2),\ X_1 - X_2 \right\rangle \geq h_1\left(\max\left\{ \|X_1\|, \|X_2\| \right\}\right) \|X_1 - X_2\|^2$$

<u>Lemma B3</u>: If $X = F_o(Z, Y)$ is a continuous function of Z and Y from one Banach space to another, and there exists a continuous inverse in X for each Y, such that $Z = F_1(X, Y)$, then F_1 is a continuous function of X and Y.

<u>Proof</u>: Fix X so that

$$F_o(Z^1, Y^1) - F_o(Z^2, Y^2) = 0.$$

Then $\| F_o(Z^1, Y^1) - F_o(Z^2, Y^1) \| = \| F_o(Z^2, Y^1) - F_o(Z^2, Y^2) \|$

By the continuity of F_o as a function of Y, for every $\epsilon > 0$, there exists $\delta > 0$, such that for

$$\|Y^1 - Y^2\| < \delta, \|F_o(Z^2, Y^1) - F_o(Z^2, Y^2)\| < \epsilon_1$$

and $\|F_o(Z^1, Y^1) - F_o(Z^2, Y^1)\| < \epsilon_1$.

Now let

$$X^1 = F_o(Z^1, Y^1) \quad \text{and} \quad X^2 = F_o(Z^2, Y^1).$$

Then

$$Z^1 = F_1(X^1, Y^1), \quad Z^2 = F_1(X^2, Y^1).$$

Therefore, for

$$\|Y^1 - Y^2\| < \delta,$$

$$\|X^1 - X^2\| < \epsilon_1 \rightarrow \|Z^1 - Z^2\| < \epsilon_2$$

since F_1 is continuous at Y^1.

For each ϵ_2, there exists ϵ_1 and a δ such that

$$\|Y^1 - Y^2\| < \delta \text{ implies } \|Z^1 - Z^2\| < \epsilon_2.$$

It follows that for each fixed X, F_1 is a continuous function of Y.

Also,

$$\|F_1(X^3, Y^3) - F_1(X^4, Y^4)\| \leq$$

$$\|F_1(X^3, Y^3) - F_1(X^4, Y^3)\| + \|F_1(X^4, Y^3) - F_1(X^4, Y^4)\|$$

so F_1 is a continuous function of X and Y.

Corollary: If $X = F_o(Z, Y)$ is a function from one Banach space to another that satisfies a Lipschitz condition for all Z and Y, and there exists an inverse that satisfies a Lipschitz condition in X for all fixed Y (the Lipschitz constant is independent of Y), such that $Z = F_1(X, Y)$, then F_1 satisfies a Lipschitz condition in X and Y.

Proof: By the technique of Lemma B3, for each fixed X, F_1 satisfies a Lipschitz condition in Y and the Lipschitz constant is independent of X.

Lemma B4: Suppose a set of equations are given in the form

$$\begin{bmatrix} Z_o \\ Z_1 \end{bmatrix} = \begin{bmatrix} F_o\ (Z_2, Z_3) \\ F_1\ (Z_2, Z_3) \end{bmatrix} \tag{1}$$

where F_o and F_1 are continuous. Let $Z_h = \begin{bmatrix} z_{h_1} \\ \vdots \\ z_{h_i} \end{bmatrix}$,

$h = (0, 1, 2, 3)$, Z_o, F_o, and Z_2 be m-tuples, Z_1, F_1, and Z_3 be p-tuples, and $\langle Z_2, Z_o\rangle + \langle Z_3, Z_1\rangle$ be the inner product that is strongly monotonic and satisfies a Lipschitz condition, (abbreviated - STRMLC). Then (1) can be solved explicitly for Z_o and Z_3 to give

$$\begin{bmatrix} Z_o \\ Z_3 \end{bmatrix} = \begin{bmatrix} F_2\ (Z_2, Z_1) \\ F_3\ (Z_2, Z_1) \end{bmatrix} \tag{2}$$

where F_2 and F_3 are continuous, and (2) is (STRMLC).

Proof: If Z_2 is kept fixed, $F_1(Z_2, Z_3)$ satisfies the requirements of Theorem B1 so has a continuous inverse defined everywhere for each fixed Z_2.

$$Z_3 = F_3 (Z_2, Z_1).$$

Now F_3 is continuous by Lemma B3. Let

$$F_2 (Z_2, Z_1) = F_o (Z_2, F_3 (Z_2, Z_1)) \quad \text{and} \quad F_2$$

is continuous since F_o and F_3 are continuous. Since (1) is (STRMLC), it follows that (2) is (STRMLC).

Lemma B5: If Q is a finite matrix transformation with maximum column rank and a mapping F satisfies condition (C1), then $Q^T F Q$ (the composite function) satisfies condition (C1).

Proof:

$$\langle x_1 - x_2, Q^T F (Q x_1) - Q^T F (Q x_2) \rangle =$$

$$\langle Q x_1 - Q x_2, F (Q x_1) - F (Q x_2) \rangle \geq$$

$$c_1(Q x_1, Q x_2) \| Q x_1 - Q x_2 \|^2 =$$

$$c_1(Q x_1, Q x_2) \langle Q^T Q(x_1 - x_2), x_1 - x_2 \rangle \geq$$

$$a \cdot c_1(Q x_1, Q x_2) \| x_1 - x_2 \|^2 = c_2(x_1, x_2) \| x_1 - x_2 \|^2$$

where $c_2(x_1, x_2)$ satisfies the condition (C1). The constant a is positive and $\|Q\,x\| \geq a\,\|x\|$, since Q^TQ is positive definite by Lemma A5, and strongly monotonic by Theorem A1 of [WI-1]. Also since Q is a matrix, it represents a bounded linear transformation so $\|Q\,x\| \leq a_1 \|x\|$. Therefore $\|x\|$ and $\|Q\,x\|$ are equivalent, [RO-1], so there exists a $c_2(x_1, x_2)$ which satisfies the requirements of condition (C1).

<u>Lemma B6</u>: If F is monotonic,

$$\langle x_1 - x_2,\ F\,(x_1 + y) - F\,(x_2 + y)\rangle \geq 0$$

for all x_1 and x_2.

<u>Proof</u>: $\langle x_1 - x_2,\ F\,(x_1 + y) - F\,(x_2 + y)\rangle =$

$$\langle (x_1 + y) - (x_2 + y),\ F\,(x_1 + y) - F\,(x_2 + y)\rangle \geq 0.$$

<u>Corollary</u>: If F satisfies condition (C1), then $F(x + y)$ satisfies condition (C1) for each y when considered as a function of x.

<u>Lemma B7</u>: If X is a Banach space, and F is defined on a convex set $D \subset X$, a necessary and sufficient condition that F be monotone over D is that each point $x \in D$ has a spherical neighborhood $B(x)$ such that F is monotone over $D \cap B(x)$.

Proof: [MI-2].

<u>Corollary</u>: If X is a Banach space and F is defined on a convex set $D \subset X$, a necessary and sufficient condition that F be strongly monotonic (or satisfy condition C1) is that each point $x \in D$ has a spherical neighborhood $B(x)$ such that F is strongly monotone (or satisfies condition C1) over $D \cap B(x)$.

<u>Proof</u>: (After MI-2.) For any distinct $x_1, x_2 \in D$, the straight line segment $t\, x_1 + (1-t)\, x_2$ $(0 \leq t \leq 1)$ is totally bounded so is compact and contained in D; therefore there is a finite covering by neighborhoods of the hypothesis. Choose $\epsilon > 0$ smaller than the smallest radius such that $\dfrac{\|x_1 - x_2\|}{\epsilon}$ is an integer n. Let

$$t_m = \frac{\epsilon\, m}{\|x_1 - x_2\|} \qquad (m = 0, \ldots, n),$$

and

$$y_m = t_m\, x_1 + (1-t_m)\, x_2$$

so that

$$y_m - y_{m-1} = \Delta y$$

for all m. For each m, y_m and y_{m-1} lie in one of the neighborhoods of the hypothesis, so

$$\left\langle y_m - y_{m-1},\ F(y_m) - F(y_{m-1}) \right\rangle = \left\langle \Delta y,\ F(y_m) - F(y_{m-1}) \right\rangle \geq$$

$$c\ (\max \left\{ \|y_m\|\ ,\ \|y_{m-1}\| \right\})\ \|\Delta y\|^2$$

Sum over m, to obtain

$$\left\langle \Delta y,\ F(x_1) - F(x_o) \right\rangle \geq c\ (\max \left\{ \|y_o\ ,\ldots,\ y_n\| \right\})\ n\ \|\Delta y\|^2$$

Also

$$n(\Delta y) = x_1 - x_o .$$

Therefore

$$\left\langle x_1 - x_o,\ F(x_1) - F(x_o) \right\rangle \geq c(\max \left\{ \|y_o\|,\ldots, \|y_n\| \right\})\ n^2 \|\Delta y\|^2$$

$$c(\max \left\{ \|y_o\|,\ldots, \|y_n\| \right\})\ \|x_1 - x_o\|^2$$

But $c(\max \left\{ \|y_o\|,\ldots, \|y_n\| \right\}) = c(\max \left\{ \|x_1\|\ ,\ \|x_o\| \right\})$

since all y_m are on a straight line and maximum norm is an end point.

<u>Definition</u>: A mapping, F, from a Banach space, X, into another Banach space Y, is said to have a directional derivative $\frac{d}{dt} F(x_o + t\,x)$ (or Gateau differential) at x_o if

$$\lim_{t \to 0} \left\| \frac{F(x_o + tx) - F(x_o)}{t} - \frac{d}{dt}\ (F(x_o + t\ x) \right\| = 0$$

and $\frac{d}{dt} F(x_o + t\ x)$ is linear in x. Let $K = \left\{ x \in X : \|x\| = 1 \right\}$. If the convergence relationship above is satisfied uniformly with respect to $x \in K$, F is said to be Frechet differentiable at x_o.

<u>Theorem B8</u>: (This theorem is modeled after a theorem in [MI-2].)

Let $F: D \to H$ where D is a convex subset of a Hilbert space H. If the directional derivative exists, then

$$\left\langle h, \frac{d}{dt} F(x + th) \right\rangle$$

is equal to the value of

$$\left\langle x_1 - x_2, F(x_1) - F(x_2) \right\rangle, \quad \text{for} \quad h = x_1 - x_2;\ x_1, x_2 \in D,$$

and $\quad x = \xi x_1 + (1 - \xi) x_2 \quad (0 < \xi < 1)$

<u>Proof</u>: Consider any $x_1, x_2 \in D$.

Let

$$f(s) = \left\langle x_1 - x_2, F(s\, x_1 + (1-s)\, x_2) \right\rangle ,$$

and f is differentiable for $0 \leq s \leq 1$ since the directional derivative of F exists. Now

$$\left\langle x_1 - x_2, F(x_1) - F(x_2) \right\rangle = f(1) - f(0).$$

By the mean value theorem,

$$f(1) - f(0) = \left[\frac{d}{ds} \left\langle x_1 - x_2, F(s\, x_1 + (1-s)\, x_2) \right\rangle \right]_{s = \xi}$$

$$= \lim_{\Delta s \to 0} \frac{\left\langle x_1 - x_2, F(x + \Delta s\, (x_1 - x_2)) - F(x) \right\rangle}{\Delta s}$$

where $0 < \xi < 1$ and $x = \xi\ x_1 + (1-\xi)\ x_2$

$$\lim_{\Delta s \to 0} = \left\langle x_1 - x_2,\ \frac{F(x + \Delta s(x_1 - x_2)) - F(x)}{\Delta s} \right\rangle$$

$$\left\langle x - x,\ \lim_{\Delta s \to 0} \frac{F(x + \Delta s(x_1 - x_2)) - F(x)}{\Delta s} \right\rangle$$

$$\left\langle h\ ,\ \frac{d}{dt}\ (F(x + t\ h) \right\rangle$$

Lemma B9: If F is a mapping from a Euclidean space into itself and the Jacobian matrix $\frac{\delta F(s)}{\delta x}$ exists and is continuous, then its quadratic form $h^T \frac{\delta F(x)}{\delta x}\ h$ is equal to the value of $\langle x_1 - x_2,\ F(x_1) - F(x_2) \rangle$, where $(h = x_1 - x_2)$ and $(x = \xi\ x_1 + (1-\xi)\ x_2)$, for some $0 < \xi < 1$.

Proof: The directional derivative of $\langle h,\ F(x + t\ h) \rangle$ is

$$\lim_{\Delta s \to 0} \left\langle \frac{h,\ F(x + (\Delta s)(h)) - F(x)}{\Delta s} \right\rangle = \left\langle h,\ \lim_{\Delta s \to 0} \frac{F(x + (\Delta s)(h)) - F(x)}{\Delta s} \right\rangle$$

$= h^T \frac{\delta F(x)}{\delta x}\ h$ (see [KA-1]).

This last equation follows since F is continuously differentiable.

Lemma B10: Suppose that the function $K(s,\ t,\ u)$, maps $E_m \longrightarrow E_m$, where s and t are parameters, $K(s,\ t,\ u)$ is Lebesgue measurable with respect to s and t and twice continuously differentiable with respect to u for $a \leq t \leq b$, $a \leq s \leq b$, $-\infty < u < \infty$. Suppose also for these values of u, s, t,

(a) $\|K''_{u^2}(s, t, u)\|_{E_m \times E_m \times E_m} \leq \|J(s, t)\|_{E_m \times E_m}$

for all u

(b) $\|K(s, t, 0)\|_{E_m} \in L_2 \times L_2$

(c) $\|K'_u(s, t, 0)\|_{E_m \times E_m} \in L_2 \times L_2$

where $\| \ \|_{E_m \times E_m \times E_m}$, $\| \ \|_{E_m \times E_m}$, and $\| \ \|_{E_m}$

denote the norms of $E_m \times E_m \times E_m$, $E_m \times E_m$, and E_m respectively, and $J(s, t)$ is an element of $L_{2,m} \times L_{2,m}$. Consider the operation

$$y(s) = P(x(s)) = \int_a^b K(s, t, x(t))\, dt \tag{1}$$

Then the operation (1) maps the space $L_{2,m}$ into itself, has a Frechet differential $P'(x_o, x)$ at every point $x_o \in L_{2,m}$, and

$$P'(x_o, x) = \int_a^b K'_{x_o}(s, t, x_o(t))\, x(t)\, dt \tag{2}$$

<u>Proof</u>: Since $K(s, t, u)$ is continuous in u and measurable in s and t, and since u is an element in $L_{2,m}$, u is measurable; so K is measurable. By Taylor's theorem:

$$K(s, t, x(t)) = K(s, t, 0) + K_u'(s, t, 0)\, x(t)$$

$$+ 1/2 \quad x^T(t) \left[K_{u^2}''(s, t, \theta\, x(t))\right] x(t)$$

$$(0 < \theta < 1)$$

Therefore:

$$\int_a^b \int_a^b \left\|K(s,t,x(t))\right\|_{E_m}^2 \, dt\, ds \leq \int_a^b \int_a^b \left\|K(s,t,0)\right\|_{E_m}^2 \, dt\, ds$$

$$+ \int_a^b \int_a^b \left\|K_u'(s,t,0)x(t)\right\|_{E_m}^2 dt\, ds + 1/2 \int_a^b \int_a^b \left\|x^T\, K_{u^2}''(s,t,\theta\, x)x\right\|_{E_m}^2 dt\, ds$$

by Minkowski's inequality,

$$\leq \int_a^b \int_a^b \left\|K(s,t,0)\right\|_{E_m}^2 dt\, ds + \left[\int_a^b \int_a^b \left\|K_u'(s,t,0)\right\|_{E_m X\, E_m}^2 dt\, ds\right] X$$

$$\left[\int_a^b \int_a^b \left\|x(t)\right\|_{E_m}^2 dt\right] + 1/2 \left[\int_a^b \int_a^b \left\|J(s,t)\right\|_{E_m\, X\, E_m}^2 dt\, ds\right] X$$

$$\left[\int_a^b \int_a^b \left\|x(t)\right\|_{E_m}^2 dt\right]^2 < \infty$$

by (a), (b), and (c) and by Holders inequality. Therefore, P maps $L_{2,m}$ into $L_{2,m}$. To show P is Frechet differentiable and P' is given by (2):

$$\lim_{\tau\to 0}\left\|\int_a^b \left[\frac{K(s,t,x_o+\tau x(t))-K(s,t,x_o(t))}{\tau} - K'_{x_o}(s,t,x_o(t))x(t)\right]dt\right\|_{L_{2,m}}$$

$$= \lim_{\tau\to 0}\left\|\int_a^b \frac{(\tau)}{2}\, x^T(t)\left[K''_{u^2}(s,t,x_o(t)+\theta\tau x(t))\right]x(t)\,dt\right\|_{L_{2,m}}$$

$$\leq \lim_{\tau\to 0}\left[\int_a^b\int_a^b \frac{(\tau)}{2}\left\|J(s,t)\right\|^2_{E_m \times E_m} dt\,ds\right]\left[\int_a^b\int_a^b \left\|x(t)\right\|^2_{E_m} dt\right]^2$$

$$\leq \lim_{\tau\to 0} A\,\tau\left\|x(t)\right\|_{L_{2,m}} = 0$$

by Taylor's theorem, (a), Holder's inequality, and A is a positive constant. The limit is uniform for $\left\|x(t)\right\|_{L_{2,m}} = 1$ with respect to $x(t)$. Therefore,

$$P'(x_o,\ x) = \int_a^b K'_{x_o}(s,\ t,\ x_o(t))\ x(t)\ dt.$$

Lemma B11: Suppose that the function $K(s,t,u)$ is continuously differentiable with respect to u for $a \leq t \leq b$, $a \leq s \leq b$, $-\infty < u < \infty$, and that for these values of s, t, $\left\|K(s,t,0)\right\|_{E_m} \in L_2 \times L_2$ and $\left\|K'_x(s,t,x)\right\|_{E_m \times E_m} \leq \left\|J(s,t)\right\|_{E_m \times E_m}$ for all $x \in E_m$ where $\left\|\ \right\|_{E_m}$ denotes Euclidean norm on the space of dimension m, and J is an element of $L_{2,m} \times L_{2,m}$ and $\| \ \|_{E_m \times E_m}$ the direct product norm. Consider the operation:

$$y = P(x), \ y(s) = \int_a^b K(s,t,x(t)) \, dt. \tag{1}$$

Then the operation (1) maps the space $L_{2,m}$ into itself, has a Gateau differential $P'(x_o, x)$ at every point $x_o \in L_{2,m}$, and

$$P'(x_o, x) = \int_a^b K'_{x_o}(s,t,x_o(t)) \, x(t) \, dt. \tag{2}$$

<u>Proof</u>: Observe that since $K(s,t,u)$ is continuous, $K(s,t,x(t))$ is measurable. By Taylor's theorem:

$$K(s,t,x(t)) = K(s,t,0) + K'_u(s,t,\theta\, x(t)) \, x(t) \quad (0 < \theta < 1)$$

Therefore:

$$\int_a^b \int_a^b \|K(s,t,x(t))\|^2_{E_m} \, ds \, dt \leq \int_a^b \int_a^b \|K(s,t,0)\|^2_{E_m} \, ds \, dt$$

$$+ \left[\int_a^b \int_a^b \|J(s,t)\|^2_{E_m \times E_m} \, ds \, dt \right] \left[\int_a^b \|x(t)\|^2_{E_m} \, dt \right] < \infty$$

since $\|K(s,t,0)\|_{E_m} \in L_2 \times L_2$, and by Holder's inequality.

Therefore P maps $L_{2,m}$ into $L_{2,m}$.

To show P is differentiable and P' is given by (2): By Taylor's theorem, and Holder's inequality,

$$\left\| \frac{K(s,t,x_o(t)+\tau x(t))-K(s,t,x_o(t))}{\tau} - K'_{x_o}(s,t,x_o(t))\,x(t) \right\|^2_{E_m}$$

$$\leq \left\| K'_{x_o}(s,t,x_o(t)+\theta\tau x(t)) - K'_{x_o}(s,t,x_o(t)) \right\|^2_{E_m \times E_m} \left\| x(t) \right\|^2_{E_m}$$

$$\leq 4 \left\| J(s,t) \right\|^2_{E_m \times E_m} \left\| x(t) \right\|^2_{E_m} \qquad (0 < \theta < 1)$$

Since this latter function is integrable, by using the Lebesgue convergence theorem, [RO-1], it is permissible to pass to the limit under the integral sign.

Therefore:

$$\lim_{\tau \to 0} \left\| \int_a^b \left[\frac{K(s,t,x_o+\tau x(t))-K(s,t,x_o(t))}{\tau} - K'_{x_o}(x,t,x_o(t))x(t) \right] dt \right\|^2_{L_{2m}}$$

$$= \lim_{\tau \to 0} \left\| \int_a^b \left[\left[K'_{x_o}(s,t,x_o(t)+\theta\tau x(t))-K'_{x_o}(s,t,x_o(t)) \right] x(t) \right] dt \right\|^2_{L_{2m}}$$

$$\leq \int_a^b \int_a^b \lim_{\tau \to 0} \left\| K'_{x_o}(s,t,x_o(t)+\theta\tau x(t))x(t)-K'_{x_o}(s,t,x_o(t))x(t) \right\|^2_{E_m} dt\, ds$$

$= 0$, in view of the continuity of K'_{x_o}.

Therefore:

$$P'(x_o(t),\, x(t)) = \int_a^b K'_{x_o}(s,t,x_o(t))\; x\,(t)\, dt$$