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OF THE REGRESSION PARAMETER

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WEIGHTED EMPIRICAL-TYPE ESTIMATION
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ABSTRACT

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We consider three estimators for the slope parameter β in the simple linear regression model, each of which is based on the minimization of a statistic for testing H_0: $\beta = 0$ versus the alternatives H_1: $\beta \neq 0$. The statistics include a Cramér-von Mises statistic and its rank analogue, and the Kolmogorov-Smirnov statistic.

Invariance and symmetry properties of the estimators are studied for finite samples, and their asymptotic distributions are derived. The Cramér-von Mises-type estimators are shown to be asymptotically normal, while the asymptotic distribution of the Kolmogorov-Smirnov-type estimator is expressed in terms of functionals of a Brownian bridge.

The Cramér-von Mises-type estimators are compared with some common estimators of β by an examination of asymptotic variances at various underlying distributions. Comparisons for the Kolmogorov-Smirnov-type estimator are made via a Monte Carlo study and by comparing asymptotic upper bounds for the lengths of associated confidence intervals.

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INTRODUCTION AND SUMMARY

1. The Model. Consider the simple linear regression model $X_{ni} = \beta_0 + \beta c_{ni} + \varepsilon_{ni}$, $1 \leq i \leq n$, where $c_{n1} \leq c_{n2} \leq \cdots \leq c_{nn}$ are known constants, not all equal, β_0 and β are unknown parameters, and the ε_{ni} are iid F for F an absolutely continuous distribution. We regard β_0 as a nuisance parameter and consider three methods for estimating β.

Throughout this paper we will, for the sake of convenience, suppress the dependence of $\{X_{ni}\}$, $\{\varepsilon_{ni}\}$, and $\{c_{ni}\}$ on n. The vectors $(c_1, c_2, \ldots, c_n)'$ and $(X_1, X_2, \ldots, X_n)'$ will be denoted by $\underset{\sim}{c}$ and $\underset{\sim}{X}$, respectively, d_i, $1 \leq i \leq n$, will denote the centered c_i's, and we will take $\sigma_c^2 = \sum_{i=1}^n d_i^2$. Furthermore, many of the statements made are true w.p. 1, even though it may not be stated explicitly.

2. Cramér-von Mises Type Estimation of β; the Rank Analogue. As given by Hájek and Šidák (1967, p. 103), the rank analogue of the Cramér-von Mises test for H_0: $\beta = \Delta$, where Δ is a given constant, is based on the statistic

$$M_1(\Delta) = \int_0^1 \Big[\sum_{i=1}^n d_i I(R_{ni\Delta} \leq nt)\Big]^2 dt$$

where $R_{ni\Delta}$ is the rank of $X_i - \Delta c_i$ among $\{X_j - \Delta c_j, 1 \leq j \leq n\}$. Since H_0 is rejected only for large values of $M_1(\Delta)$, it seems reasonable to attempt to define an estimator $\hat{\beta}_1$ for β based on the minimization of M_1 in Δ.

In section 1.1 we propose a unique definition for such an estimator and give a numerical example to illustrate its computation. When so defined $\hat{\beta}_1$ is translation invariant and has a distribution which is symmetric about the true parameter provided the underlying distribution is symmetric or the centered regression constants are skew-symmetric (section 1.2). Section 1.3 contains intermediate results which are used to derive the asymptotic distribution of the normalized estimator in section 1.4. Finally, in section 1.5, we consider the asymptotic efficiency of $\hat{\beta}_1$. With asymptotic variance as the basis for comparison, $\hat{\beta}_1$ performs remarkably well against some common estimators for β, particularly when the underlying distribution has heavy tails. Comparisons are made with the Wilcoxon, median, normal scores, and least squares estimates at the normal, double exponential, and logistic distributions. At the double exponential, $\hat{\beta}_1$ out-performs all of the above but the optimal median-type estimator. Similarly, at the logistic, only the optimal Wilcoxon-type estimator is more efficient. At the normal, $\hat{\beta}_1$ beats only the median-type estimator, but shows only a slight loss of efficiency against the other estimators.

3. <u>Cramér-von Mises Type Estimation of β</u>. We base our second estimator on a statistic which is similar to M_1 of the previous section, but which uses the observations themselves rather than their ranks. Here we consider the process

$$M_2(\Delta) = \int_{-\infty}^{\infty}[\sum_{i=1}^{n} d_i I(X_i - \Delta c_i \leq x)]^2 dx, \ \Delta \in R,$$

and seek to define an estimator $\hat{\beta}_2$ for β based on the minimization of M_2 in Δ.

In section 2.1 we propose a unique definition for such an estimator and give a numerical example to illustrate its computation. So defined, $\hat{\beta}_2$ possesses invariance and symmetry properties analogous to those of $\hat{\beta}_1$ (section 2.2).

When specialized to the two sample location problem (i.e., $c_1 = c_2 = \ldots = c_m = 0$; $c_{m+1} = c_{m+2} = \ldots = c_n = 1$; $1 \leq m < n$),

$$M_2(\Delta) = \sigma_c^{-2} \int_{-\infty}^{\infty}[\frac{1}{n-m} \sum_{i=m+1}^{n} I(X_i \leq x + \Delta) - \frac{1}{m} \sum_{i=1}^{m} I(X_i \leq x)]^2 \ dx.$$

Thus $\sigma_c^2 M_2(\Delta)$ represents the squared L_2-distance between the empirical distribution of one sample and a shifted empirical distribution for the other sample. Fine (1966) showed that in this situation the Wilcoxon-type estimator $\hat{\beta}_W$ for β satisfies $M_2(\hat{\beta}_W) = \inf_{-\infty<\Delta<\infty} M_2(\Delta)$. This raises the question as to whether $\hat{\beta}_2 = \hat{\beta}_W$ in general. In section 2.1 we give an example which shows that this is not the case. It is true, however, that the two estimators are asymptotically equivalent, as shown in section 2.4.

It should also be noted that $\hat{\beta}_2$ is related to the weighted median estimators for β considered by Scholz (1978). In section 2.1 we show that

$$M_2(\Delta) = -\sum_{1 \leq i < j \leq n} d_i d_j |X_j - X_i - \Delta(d_j - d_i)|.$$

The (a.e.) derivative of M_2 is

$$\eta + \sum_{i<j} d_i d_j [1 - 2I(X_j - X_i - \Delta(d_j - d_i)],$$

where η is a constant which is independent of Δ. This statistic is analogous to that upon which the Scholz estimates are based. We remark that the weights $-d_i d_j (d_j - d_i)$, $1 \leq i < j \leq n$, satisfy the Scholz optimality condition and that $\sigma_c(\hat{\beta}_2 - \beta)$ achieves the minimal asymptotic variance for his class of estimators. Our results do not follow from Scholz's work, however, since the weights here need not be nonnegative.

4. Kolmogorov-Smirnov Type Estimation of β. Since the Kolmogorov-Smirnov test for H_0: $\beta = \Delta$, Δ fixed, is based on the statistic

$$D_c(\Delta) = \sup_{-\infty<x<\infty} \left| \sum_{i=1}^{n} d_i I(X_i \leq x + \Delta c_i) \right|,$$

with small values of $D_c(\Delta)$ favoring H_0, we seek to define an estimator $\hat{\beta}_3$ for β which satisfies

$$D_c(\hat{\beta}_3) = \inf_{-\infty<\Delta<\infty} D_c(\Delta) .$$

We note that when specialized to the two sample location problem

$$D_c(\Delta) = \sup_{-\infty<x<\infty} \frac{m(n-m)}{n} \left| \frac{1}{n-m} \sum_{i=m+1}^{n} I(X_i \leq x + \Delta) - \frac{1}{m} \sum_{i=1}^{m} I(X_i \leq x) \right|,$$

which is a constant multiple of the sup-norm distance between the empirical distribution of one sample and a shifted empirical distribution for the other sample.

In section 3.1 we propose a unique definition for $\hat{\beta}_3$ and illustrate its computation with a numerical example. So defined, $\hat{\beta}_3$ agrees, in the case of the two sample location problem, with the estimator of location proposed by Rao et al. (1975).

Section 3.2 establishes that the invariance and symmetry properties enjoyed by $\hat{\beta}_1$ and $\hat{\beta}_2$ are valid for $\hat{\beta}_3$ as well.

The asymptotic properties of $\hat{\beta}_3$ are discussed in section 3.3 where it is shown that the asymptotic distribution of the normalized estimator can be expressed in terms of functionals of a Brownian bridge.

In section 3.4 we consider $100(1-\alpha)\%$ confidence sets for β of the form

$$\{\Delta;\ D_c(\Delta) \leq \gamma_{c,\alpha}\}$$

where $\gamma_{c,\alpha}$ is the critical value for which one rejects H_0: $\beta = \Delta$ whenever $D_c(\Delta) > \gamma_{c,\alpha}$. We derive asymptotic upper bounds for the lengths of such intervals, both in probability and w.p. 1.

Finally, in sections 3.5 and 3.6, we consider the efficiency of $\hat{\beta}_3$ and the associated confidence intervals. Comparisons are made for $\hat{\beta}_3$ versus $\hat{\beta}_1$ and $\hat{\beta}_W$ via a Monte Carlo study, while the confidence intervals are compared to normal scores and Wilcoxon-type intervals using the upper bounds of section 3.4.

CHAPTER 1

CRAMÉR-VON MISES TYPE ESTIMATION OF β; THE RANK ANALOGUE

1. Notation and Preliminaries. To the assumptions of the model introduced in section 1 of the introduction we add the following:

(1.1) F has a continuous bounded density f satisfying

$f(x) > 0$ a.e. on $\{x;\ 0 < F(x) < 1\}$.

(1.2) $$\lim_{n\to\infty} \sigma_c^{-1} \max_{1\le i\le n} |d_i| = 0.$$

In what follows let the vectors $\underset{\sim}{c}$ and $\underset{\sim}{X}$ be given and define the quantities $B = \int_{-\infty}^{\infty} f^2(x)dx$ and $K = \int_{-\infty}^{\infty} f^3(x)dx$. For each real Δ and for each $t \in [0,1]$ define

$$S(t,\Delta) := \sum_{i=1}^{n} d_i I(R_{ni\Delta} \le nt)$$

where

$$R_{ni\Delta} := \sum_{j=1}^{n} I(X_j - \Delta c_j \le X_i - \Delta c_i) .$$

We also define, for each real Δ,

$$W_1(\Delta) := \int_0^1 S(t,\Delta)dt$$

and note that

$$W_1(0) = -n^{-1} \sum_{i=1}^{n} d_i R_{ni0} \,.$$

Next consider the process

$$\{M_1(\Delta),\ -\infty < \Delta < \infty\}$$

where

$$M_1(\Delta) := \int_0^1 S^2(t,\Delta)dt \,.$$

If we let $(D_{n1\Delta}, D_{n2\Delta}, \ldots, D_{nn\Delta})$ denote the vector of anti-ranks for $(\underset{\sim}{X} - \Delta \underset{\sim}{c})'$, it is interesting to note that

$$\sigma_c^{-2} M_1(0) = n^{-1} \sigma_c^{-2} \sum_{j=1}^{n-1} [\sum_{i=1}^{j} d_{D_{ni0}}]^2 \,.$$

This is the Cramér-von Mises statistic (Hájek-Šidák, 1967) for testing the regression slope parameter $\beta = 0$ against the alternatives $\beta \neq 0$.

For a fixed sample, $M_1(\Delta)$ is a step function (in Δ) whose points of discontinuity are contained in the set

$$\Gamma_1 = \{(X_j - X_i)/(c_j - c_i);\ i < j \text{ and } c_i < c_j\}.$$

Set $\Delta_0 = \min\{\Delta;\ \Delta \in \Gamma_1\}$ and $\Delta_1 = \max\{\Delta;\ \Delta \in \Gamma_1\}$. Then for $c_i < c_j$, $\Delta < \Delta_0$ implies $\Delta < (X_j - X_i)/(c_j - c_i)$ and hence $R_{ni\Delta} < R_{nj\Delta}$. Thus the residuals $\{X_i - \Delta c_i,\ 1 \leq i \leq n\}$ are naturally ordered and therefore (w.p.1)

$$S(t,\Delta) = \begin{cases} \sum_{i=1}^{j} d_i & \text{for} \quad t \in [\frac{j}{n}, \frac{j+1}{n}),\ 1 \le j \le n-1 \\ \\ 0 & \text{for} \quad t \in [0, \frac{1}{n}) \cup \{1\}. \end{cases}$$

Hence,

$$M_1(\Delta) = \int_0^1 S^2(t,\Delta)dt = n^{-1} \sum_{j=1}^{n-1} [\sum_{i=1}^{j} d_i]^2$$

and thus

$$M_1(\Delta_0^-) = n^{-1} \sum_{j=1}^{n-1} [\sum_{i=1}^{j} d_i]^2 .$$

Similarly, for $\Delta > \Delta_1$, the residuals are in a reversed natural ordering. Using $\sum_{i-1}^{n} d_i = 0$ one obtains $M_1(\Delta) = n^{-1}\sum_{j=1}^{n-1}[\sum_{i=1}^{j} d_i]^2 = M_1(\Delta_1^+)$.

As Δ crosses Δ_0 only one pair of adjacent residuals cross. Let $c_k < c_{k+1}$ denote their respective regression constants. Then

$$\begin{aligned} M_1(\Delta_0^-) - M_1(\Delta_0^+) &= n^{-1} \sum_{j=1}^{n-1} [\sum_{i=1}^{j} d_i]^2 \\ &\quad - n^{-1}\{\sum_{\substack{j=1 \\ j\neq k}}^{n-1} [\sum_{i=1}^{j} d_i]^2 + [d_{k+1} + \sum_{j=1}^{k-1} d_i]^2\} \\ &= n^{-1}\{[\sum_{i=1}^{k} d_i]^2 - [d_{k+1} + \sum_{i=1}^{k-1} d_i]^2\}. \end{aligned}$$

Now $c_1 \le c_2 \le \cdots \le c_n$ and $c_k < c_{k+1}$ imply

$$\sum_{i=1}^{k} d_i < d_{k+1} + \sum_{i=1}^{k-1} d_i \le 0 .$$

Thus $M_1(\Delta_0^-) > M_1(\Delta_0^+)$. Similarly it follows that $M_1(\Delta_1^+) > M_1(\Delta_1^-)$.

As a result, the following quantities are finite:

$$\beta_1^* = \min\{s \in \Gamma_1;\ M_1(s^+) = \inf_{\Delta \in \Gamma_1^c} M_1(\Delta)\}$$

$$\beta_1^{**} = \max\{s \in \Gamma_1;\ M_1(s^-) = \inf_{\Delta \in \Gamma_1^c} M_1(\Delta)\} .$$

We now define our estimator $\hat{\beta}_1$ for β by

$$\hat{\beta}_1 = \tfrac{1}{2}(\beta_1^* + \beta_1^{**}) .$$

Numerical example

By the preceding remarks we may determine the value of $\hat{\beta}_1$ by identifying the set of slopes Γ_1 and computing $M_1(\Delta^-)$ for each $\Delta \in \Gamma_1$. Computation of $M_1(\Delta)$ is facilitated by using the formula

$$M_1(\Delta) = -n^{-1} \sum_{1 \le i < j \le n} d_i d_j |R_{ni\Delta} - R_{nj\Delta}| . \tag{1.2}$$

We consider here a two-sample problem; that is, we take $c_1 = c_2 = 0$ and $c_3 = c_4 = c_5 = c_6 = 1$ for the data

160, -26, 17, -150, -30, 12

TABLE I

Values of $M_1(\Delta^-)$ for $\Delta \in \Gamma_1$

$\Delta \in \Gamma$	-310	-190	-148	-143	-124	-4	38	43
$M_1(\Delta^-)$	.6296	.3519	.1852	.1296	.1852	.1296	.1852	.3519

From the above table it is clear that

$$\beta_1^* = -148;\ \beta_1^{**} = -4$$

so that $\hat{\beta}_1 = -76$.

2. Finite Sample Properties

(a) Invariance

A useful property of the estimator $\hat{\beta}_1$ is its translation invariance; that is, for all real γ,

$$\hat{\beta}_1(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \hat{\beta}_1(\underset{\sim}{X}) + \gamma. \tag{2.1}$$

To verify (2.1) we note that

$$\begin{aligned}
&M_1(\Delta - \gamma)(\underset{\sim}{X}) \\
&= \int_0^1 \{\sum_{i=1}^n d_i I[\sum_{j=1}^n I(X_j - (\Delta - \gamma)c_j \leq X_i - (\Delta - \gamma)c_i) \leq nt]\}^2 dt \\
&= \int_0^1 \{\sum_{i=1}^n d_i I[\sum_{j=1}^n I(X_j + \gamma c_j - \Delta c_j \leq X_i + \gamma c_i - \Delta c_i) \leq nt]\}^2 dt \\
&= M_1(\Delta)(\underset{\sim}{X} + \gamma\underset{\sim}{c}).
\end{aligned}$$

Thus for all real γ

$$\begin{aligned}
\beta^*(\underset{\sim}{X} + \gamma\underset{\sim}{c}) &= \min\{s \in \Gamma + \gamma;\ M(s^+)(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \inf_{\Delta \in (\Gamma+\gamma)^c} M(\Delta)(\underset{\sim}{X} + \gamma\underset{\sim}{c})\} \\
&= \min\{s \in \Gamma + \gamma;\ M((s - \gamma)^+)(\underset{\sim}{X}) = \inf_{\Delta \in (\Gamma+\gamma)^c} M(\Delta - \gamma)(\underset{\sim}{X})\} \\
&= \min\{s + \gamma;\ s \in \Gamma,\ M(s^+)(\underset{\sim}{X}) = \inf_{\Delta \in \Gamma^c} M(\Delta)(\underset{\sim}{X})\} \\
&= \beta^*(\underset{\sim}{X}) + \gamma.
\end{aligned}$$

Similarly, $\beta^{**}(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \beta^{**}(\underset{\sim}{X}) + \gamma$ for all real γ, and (2.1) follows.

From (2.1) we conclude that

$$(2.2) \qquad P_\beta(\hat{\beta} - \beta \leq z) = P_0(\hat{\beta} \leq z) ,$$

for all real z, where P_β and P_0 indicate that the true parameter is assumed to be β and 0, respectively. Because of (2.2) we may assume throughout the rest of the paper that $\beta = 0$ without loss of generality.

(b) Symmetry

Theorem 2.1. $\hat{\beta}$ is symmetric about β if one of the following conditions hold:

(i) F is symmetric

(ii) $d_i = -d_{n-i+1}$, $1 \leq i \leq n$.

Proof. Assume, without loss of generality, that $\beta = 0$.

Proof of (i). Since for any real number a, $M_1(\Delta)(\underset{\sim}{X}) = M_1(\Delta)(\underset{\sim}{X} + a\underset{\sim}{1})$, we may assume that F is symmetric about 0 without loss of generality. Now $\underset{\sim}{X} \sim -\underset{\sim}{X}$ so that $\hat{\beta}_1(\underset{\sim}{X}) \sim \hat{\beta}_1(-\underset{\sim}{X})$ and hence it suffices to show that $\hat{\beta}_1(-\underset{\sim}{X}) \sim -\hat{\beta}_1(\underset{\sim}{X})$. Let $\Theta = \{\Delta;\ X_j - X_i = \Delta(c_j - c_i)$, some $1 \leq i < j \leq n\}$. Then for $\Delta \in \Theta^c$, (2.3) $$R_{ni\Delta}(-\underset{\sim}{X}) = \sum_{j=1}^{n} I(-X_j - \Delta c_j \leq -X_i - \Delta c_i)$$

$$= \sum_{j=1}^{n} I(X_j + \Delta c_j \geq X_i + \Delta c_i)$$

$$= \sum_{j=1}^{n} \{1 - I(X_j + \Delta c_j \leq X_i + \Delta c_i)\}$$

$$+ \sum_{j=1}^{n} I(X_j + \Delta c_j = X_i + \Delta c_i)$$

$$= n + 1 - R_{ni(-\Delta)}(\underset{\sim}{X}) .$$

Using (1.2), (2.3) and the fact that $\Theta = \Gamma_1$ w.p.1, we have (w.p.1)

$$M_1(-\Delta)(\underset{\sim}{X}) = -n^{-1} \sum_{i<j} d_i d_j |R_{ni(-\Delta)}(\underset{\sim}{X}) - R_{nj(-\Delta)}(\underset{\sim}{X})|$$

$$= -n^{-1} \sum_{i<j} d_i d_j |R_{ni\Delta}(-\underset{\sim}{X}) - R_{nj\Delta}(-\underset{\sim}{X})|$$

$$= M_1(\Delta)(-\underset{\sim}{X})$$

for $\Delta \in \Gamma_1^c$. But this implies that $\hat{\beta}_1(-\underset{\sim}{X}) = -\hat{\beta}_1(\underset{\sim}{X})$, completing the proof of (i). □

<u>Proof of (ii)</u>. Let $\underset{\sim}{X}^*$ denote $(X_n, X_{n-1}, \ldots, X_1)'$. Since $\beta = 0$ we have $\underset{\sim}{X} \sim \underset{\sim}{X}^*$ and hence $\hat{\beta}_1(\underset{\sim}{X}) \sim \hat{\beta}_1(\underset{\sim}{X}^*)$. We show that $\hat{\beta}_1(\underset{\sim}{X}^*) \sim -\hat{\beta}_1(\underset{\sim}{X})$. Now

$$R_{ni\Delta}(\underset{\sim}{X}^*) = \sum_{j=1}^{n} I(X_j^* - \Delta c_j \leq X_i^* - \Delta c_i)$$

$$= \sum_{j=1}^{n} I(X_{n-j+1} + \Delta c_{n-j+1} \leq X_{n-i+1} + \Delta c_{n-i+1})$$

$$= R_{n,n-i+1,-\Delta}(\underset{\sim}{X}).$$

Thus,

$$S(t,\Delta)(\underset{\sim}{X}^*) = \sum_{i=1}^{n} d_i I(R_{n,n-i+1,-\Delta}(\underset{\sim}{X}) \le nt)$$

$$= -\sum_{i=1}^{n} d_{n-i+1} I(R_{n,n-i+1,-\Delta}(\underset{\sim}{X}) \le nt)$$

$$= -S(t,-\Delta)(\underset{\sim}{X}).$$

It is now clear that $M_1(\Delta)(\underset{\sim}{X}^*) = M_1(-\Delta)(\underset{\sim}{X})$ so that $\hat{\beta}_1(\underset{\sim}{X}^*) = -\hat{\beta}_1(\underset{\sim}{X})$. Since $\underset{\sim}{X}^* \sim \underset{\sim}{X}$ we have $\hat{\beta}_1(\underset{\sim}{X}^*) \sim -\hat{\beta}_1(\underset{\sim}{X})$ and the proof of (ii) is completed. □

3. Asymptotic Behavior of $M_1(\Delta)$. Throughout this section we retain the notation of section 1 and assume that (1.1) holds. The following theorem is proved as in Koul (1977).

Theorem 3.1. Let $0 < a < \infty$. Then

$$\sup_{\substack{0<t<1 \\ |\Delta|\le a}} \sigma_c^{-1} |S(t,\Delta\sigma_c^{-1}) - S(t,0) - \Delta\sigma_c f(F^{-1}(t))| \overset{P_0}{\to} 0 . \tag{3.1}$$

A consequence of the above theorem is

Lemma 3.1. Let $0 < a < \infty$ and $T_1(\Delta) = \int_0^1 [S(t,0) + \Delta\sigma_c f(F^{-1}(t))]^2 dt$. Then

$$\sup_{|\Delta|\le a} \sigma_c^{-2} |M_1(\Delta\sigma_c^{-1}) - T_1(\Delta)| \overset{P_0}{\to} 0 , \tag{3.2}$$

$$\sup_{|\Delta|\le a} \sigma_c^{-2} |W_1^2(\Delta\sigma_c^{-1}) - (\int [S(t,0) + \Delta\sigma_c f(F^{-1}(t))] dt)^2| \overset{P_0}{\to} 0 . \tag{3.3}$$

Proof. To establish (3.2) we note that

$$(3.4)\qquad \sup_{\substack{|\Delta|\le a\\ 0\le t\le 1}} \sigma_c^{-1}|S(t,0) + \Delta\sigma_c f(F^{-1}(t))| \le \sigma_c^{-1} \sup_{0\le t\le 1} |S(t,0)| + a\|f\|_\infty .$$

But $\sigma_c^{-1} \sup_{0\le t\le 1} |S(t,0)|$ is the Kolmogorov-Smirnov statistic which has a limiting distribution (Hájek and Šidák, 1967). Thus the LHS of (3.4) is bounded in probability so that in view of (3.1)

$$\sup_{\substack{|\Delta|\le a\\ 0\le t\le 1}} \sigma_c^{-2}|S^2(t,\Delta\sigma_c^{-1}) - [S(t,0) + \Delta\sigma_c f(F^{-1}(t))]^2| \xrightarrow{P_0} 0$$

and (3.2) follows.

To establish (3.3) we note that

$$(3.5)\qquad \sup_{|\Delta|\le a} \sigma_c^{-1}|W_1(\Delta\sigma_c^{-1}) - \int_0^1[S(t,0) + \Delta\sigma_c f(F^{-1}(t))]dt| \xrightarrow{P_0} 0$$

follows from (3.1). We also have (recall $B = \int_{-\infty}^{\infty} f^2(x)dx$)

$$(3.6)\qquad \sup_{|\Delta|\le a} \sigma_c^{-1}|\int_0^1[S(t,0) + \Delta\sigma_c f(F^{-1}(t))]dt| \le \sigma_c^{-1}|W_1(0)| + aB.$$

But $-\sigma_c^{-1}W_1(0)$ is the Wilcoxon statistic and hence has a limiting distribution (Hájek and Šidák, 1967). Thus the LHS of (3.6) is bounded in probability so that (3.3) follows from (3.5). □

For fixed $0 < a < \infty$ and $0 < d < \infty$, define the event

$$E_{n1}(a,d) = \{\sigma_c^{-2}W_1^2(0) < d,\ \inf_{|\Delta|=a} \sigma_c^{-2}W_1^2(\Delta\sigma_c^{-1}) \ge d\} .$$

Lemma 3.2. For every $\varepsilon > 0$ there exist positive real numbers N, a and d such that whenever $n \ge N$,

$$P_0(E_{n1}(a,d)) \geq 1 - \varepsilon.$$

Proof. Since $\sigma_c^{-1}W_1(0)$ has a limiting distribution, there exists a positive real number b such that

$$P_0[\sigma_c^{-1}|W_1(0)| \leq b] \geq 1 - \varepsilon \;\forall\; n .$$

If we also take $d > 2b^2$ and choose a so that $a > b + (3d/2)^{\frac{1}{2}} B^{-1}$ we have

$$\sigma_c^{-2}W_1^2(0) \leq b^2 < d/2$$

and

$$\inf_{|\Delta|=a} \sigma_c^{-2}(\int_0^1[S(t,0) + \Delta\sigma_c f(F^{-1}(t))]dt)^2$$

$$= \min\{(\sigma_c^{-1}W_1(0) - aB)^2, (\sigma_c^{-1}W_1(0) + aB)^2\}$$

$$\geq (aB - \sigma_c^{-1}|W_1(0)|)^2 \geq (aB - b)^2 \geq 3d/2$$

on $\{\sigma_c^{-1}|W_1(0)| \leq b\}$. Choosing N according to (3.3) completes the proof. □

Lemma 3.3. For every $\varepsilon > 0$ and $d' > 0$ there exist positive real numbers a and N such that $n \geq N$ implies

$$P_0(\inf_{|\Delta|\geq a} \sigma_c^{-2}W_1^2(\Delta\sigma_c^{-1}) \geq d') \geq 1 - \varepsilon .$$

Proof. In the proof of lemma 3.2, take $d > \max\{d',2b^2\}$. The proof is completed by using the fact that W_1 is nonincreasing in Δ (Hájek, 1969, p. 35). □

4. Asymptotic Distribution of $\sigma_c\hat{\beta}_1$. Throughout this section we retain the notation of sections 1-3 and assume that (1.1) and (1.2) hold. In addition we assume, without loss of generality, that $\beta = 0$.

Lemma 4.1. For $0 \le t \le 1$ and $n \ge 1$ define

$$\begin{aligned} a_n(i,t) &= 0 && i \le tn \\ &= i - tn && tn \le i \le tn + 1 \\ &= 1 && tn + 1 \le i \quad . \end{aligned}$$

Then the process

$$\{Z_n(t) := \sigma_c^{-1} \sum_{i=1}^{n} d_i a_n(R_i,t),\ 0 \le t \le 1\}$$

converges in distribution in $(B, C[0,1])$ to the Brownian Bridge $\{B(t),\ 0 \le t \le 1\}$.

Proof. Hájek and Sidák (1967), Theorem V.3.5. □

Remark. $\{Z_n(t),\ 0 \le t \le 1\}$ is a process with continuous sample paths which is related to $\{S(t,0),\ 0 \le t \le 1\}$ in the following manner:

$$\sup_{0 \le t \le 1} |Z_n(t) + \sigma_c^{-1} S(t,0)| \le \sigma_c^{-1} \max_{1 \le i \le n} |d_i| .$$

For y a bounded integrable function on $C[0,1]$ define

$$h(y) = K^{-1} \int_0^1 y(t) f(F^{-1}(t)) dt$$

where $K = \int_{-\infty}^{\infty} f^3(x) dx$.

Note that since h is a continuous functional on $C[0,1]$ we have

Lemma 4.2. $L_0(h(Z_n)) \Rightarrow N(0,\sigma_1^2)$ where

$$\sigma_1^2 = K^{-2}[\int_0^1 G^2(t)dt - (\int_0^1 G(t)dt)^2]$$

and

$$G(t) = \int_0^t f(F^{-1}(s))ds, \quad 0 \le t \le 1 .$$

We now define

$$\sigma_c\hat{\hat{\beta}}_1 := -h(\sigma_c^{-1}S(\cdot,0)) .$$

For $0 < b < a$ and $\rho > 0$ define

$$G_{n1}(a,b) := \{|\sigma_c\hat{\hat{\beta}}_1| \le b, \inf_{|\Delta|\ge a\sigma_c^{-1}} M_1(\Delta) > \inf_{\substack{|\Delta|<a\sigma_c^{-1}\\ \Delta\notin\Gamma}} M_1(\Delta)\}$$

$$H_{n1}(a,\rho) := \{\sup_{|\Delta|\le a\sigma_c^{-1}} \sigma_c^{-2}|M_1(\Delta) - T_1(\Delta\sigma_c)| \ge K\rho^2/2\}$$

$$\Delta_{n1}(a) := \sup\{\sigma_c|\beta_1 - \hat{\hat{\beta}}_1| : \beta_1 \in \Gamma_1^c, M_1(\beta_1) = \inf_{\substack{|\Delta|\le a\sigma_c^{-1}\\ \Delta\in\Gamma_1}} M_1(\Delta)\} .$$

Lemma 4.3. Let $\varepsilon > 0$ and $0 < b < a$. Then

$$P_0(\{\Delta_{n1}(a) > \varepsilon\} \cap G_{n1}(a,b)) \to 0.$$

Proof. Suppose there exist γ and ρ positive such that $P_0(\{\Delta_{n1}(a) > 2\rho\} \cap G_{n1}(a,b)) > \gamma$ for infinitely many n. By (3.2) there exists an $N > 0$ such that $n \ge N$ implies

$$P_0(H_{n1}(a,\rho)) < \gamma/2$$

and hence

$$P_0[\{\Delta_{n1}(a) > 2\rho\} \cap G_{n1}(a,b) \cap H^c_{n1}(a,\rho)] > \gamma/2 .$$

Since the above event is contained in $\{\Delta_{n1}(a) > 2\rho\} \cap G_{n1}(a,b)$, we can find for any given $\underset{\sim}{X} \in \{\Delta_{n1}(a) > 2\rho\} \cap G_{n1}(a,b) \cap H^c_{n1}(a,\rho)$, a $\Delta^* \in \Gamma^c_1 \cap (-a\sigma_c^{-1}, a\sigma_c^{-1})$ satisfying $|\hat{\hat{\beta}}_1 - \Delta^*_1| > \rho$ and

$$M_1(\Delta^*_1) = \inf_{\substack{|\Delta|<a\sigma_c^{-1} \\ \Delta\notin\Gamma_1}} M_1(\Delta) . \tag{4.1}$$

Because we also have $\underset{\sim}{X} \in H^c_{n1}(a,\rho)$ it follows that

$$\sigma_c^{-2}|M_1(\Delta^*) - T_1(\sigma_c\Delta^*)| < K\rho^2/2 . \tag{4.2}$$

Noting that $T_1(\sigma_c\Delta)$ is quadratic in Δ with leading coefficient $K\sigma_c^2$ and minimum occurring at $\Delta = \hat{\hat{\beta}}_1$, we conclude that

$$\sigma_c^{-2}[T_1(\sigma_c\Delta^*) - T_1(\sigma_c\hat{\hat{\beta}}_1)] > K\rho^2 \tag{4.3}$$

from $|\hat{\hat{\beta}}_1 - \Delta^*| > \rho$. Since for any $\Delta \in \Gamma^c_1 \cap (-a\sigma_c^{-1}, a\sigma_c^{-1})$,

$$\sigma_c^{-2}|M_1(\Delta) - T_1(\hat{\hat{\beta}}_1\sigma_c)| \leq \sigma_c^{-2}|M_1(\Delta) - T_1(\Delta\sigma_c)|$$
$$+ \sigma_c^{-2}|T_1(\Delta\sigma_c) - T_1(\hat{\hat{\beta}}_1\sigma_c)| < \tfrac{1}{2}K\rho^2 + |T_1(\Delta\sigma_c) - T_1(\hat{\hat{\beta}}_1\sigma_c)|,$$

by the continuity of T_1 in Δ there is a $\Delta^{**} \in \Gamma^c_1 \cap (-a\sigma_c^{-1}, a\sigma_c^{-1})$ for which

$$\sigma_c^{-2}|M_1(\Delta^{**}) - T_1(\hat{\hat{\beta}}_1\sigma_c)| < K\rho^2/2 . \tag{4.4}$$

But combining (4.2) - (4.4) we see that $M_1(\Delta^*) > T_1(\sigma_c\Delta^*) - K\rho^2/2 > T_1(\sigma_c\hat{\hat{\beta}}_1) + K\rho^2/2 > M_1(\Delta^{**})$, contradicting (4.1). □

<u>Lemma 4.4</u>. $L_0(\sigma_c\beta_1) \Rightarrow N(0,\sigma_1^2)$.

Proof.

$$|\sigma_c\hat{\hat{\beta}}_1 - h(Z_n)| = |h(\sigma_c^{-1}S(\cdot,0)) + h(Z_n)|$$

$$\leq K^{-1}\|f\|_\infty \int_0^1 |\sigma_c^{-1}S(t,0) + Z_n(t)|dt \leq (K\sigma_c)^{-1}\|f\|_\infty \max_{1\leq i\leq n} |d_i| \to 0.$$

Thus $L_0(\sigma_c\hat{\hat{\beta}}_1) \Rightarrow N(0,\sigma_1^2)$. □

Lemma 4.5. Given $\varepsilon > 0$ there exist positive real numbers a,b and N with $a > b$ such that $n \geq N$ implies

$$P_0[G_{n1}(a,b)] \geq 1 - \varepsilon .$$

Proof. Since $\sigma_c\hat{\hat{\beta}}_1$ and $\sigma_c^{-2}M_1(0)$ have limiting distributions there exists a positive real number b such that

$$P_0[\sigma_c|\hat{\hat{\beta}}_1| \leq b, \ \sigma_c^{-2}M_1(0) \leq b] \geq 1 - \varepsilon/2$$

for all n. Taking $d > b$ and noting that $M_1(\Delta) \geq W_1^2(\Delta)$ by the Cauchy-Schwarz inequality, it follows from lemma 3.3 that there exist $a > b$ and $0 < N < \infty$ such that

$$P_0[\inf_{|\Delta|\geq a\sigma_c^{-1}} \sigma_c^{-2}M_1(\Delta) \geq d] \geq 1 - \varepsilon/2$$

for $n \geq N$. But then

$$\sigma_c^{-2}M_1(0) \geq \inf_{\substack{|\Delta|<a\sigma_c^{-1} \\ \Delta\notin\Gamma_1}} \sigma_c^{-2}M_1(\Delta)$$

implies

$$P_0[G_{n1}(a,b)] \geq P_0[\sigma_c|\hat{\hat{\beta}}_1| \leq b,\ \sigma_c^{-2}M_1(0) \leq b,$$

$$\inf_{|\Delta|\geq a\sigma_c^{-1}} \sigma_c^{-2}M_1(\Delta) \geq d] \geq 1 - \varepsilon. \qquad \square$$

Theorem 4.1. $L_0(\sigma_c\hat{\hat{\beta}}_1) \to N(0,\sigma_1^2)$.

Proof. We prove that $\sigma_c|\hat{\beta}_1 - \hat{\hat{\beta}}_1| \xrightarrow{P_0} 0$. The theorem them follows from lemma 4.4. Let $\varepsilon > 0$ and $\delta > 0$ be given. By lemma 4.5 there exist positive real numbers a,b and N_1 with $a > b$ such that

$$P_0[G_{n1}(a,b)] \geq 1 - \delta/2 \quad \forall\ n \geq N_1 \quad .$$

Now use lemma 4.3 to choose $N > N_1$ such that

$$P_0(\{\Delta_{n1}(a) > \varepsilon\} \cap G_{n1}(a,b)) < \delta/2 \quad \forall \quad n \geq N\ .$$

Then for $n \geq N$ we have

$$\varepsilon \geq \sup\{\sigma_c|\tilde{\beta}_1 - \hat{\hat{\beta}}_1| : M_1(\tilde{\beta}_1) = \inf_{\substack{|\Delta|<a\sigma_c^{-1} \\ \Delta\notin\Gamma_1}} M_1(\Delta)\}$$

$$= \sup\{\sigma_c|\tilde{\beta}_1 - \hat{\hat{\beta}}_1| : M_1(\tilde{\beta}_1) = \inf_{\Delta\notin\Gamma} M_1(\Delta)\} \geq \tfrac{1}{2}\sigma_c[|\beta_1^* - \hat{\hat{\beta}}_1|$$

$$+ |\beta_1^{**} - \hat{\hat{\beta}}_1|] \geq \sigma_c|\hat{\beta}_1 - \hat{\hat{\beta}}_1|$$

on the event $\{\Delta_{n1}(a) \leq \varepsilon\} \cap G_{n1}(a,b)$. Since $P_0(\{\Delta_{n1}(a) \leq \varepsilon\} \cap G_{n1}(a,b)) \geq 1 - \delta$, we have established $\sigma_c|\hat{\beta}_1 - \hat{\hat{\beta}}_1| \xrightarrow{P_0} 0$. $\square$

5. Asymptotic Efficiency of $\hat{\beta}_1$. We define the asymptotic efficiency of $\hat{\beta}_1$ relative to any other estimator $\hat{\beta}_.$ of β as the ratio $\sigma_1^2/\sigma_.^2$ of the asymptotic variances of $\sigma_c\hat{\beta}_.$ and $\sigma_c\hat{\beta}$.

The following tables indicate that $\hat{\beta}_1$ does quite well relative to some common estimators for β when the underlying distribution has heavy tails.

TABLE II

Asymptotic Variances of Various Estimators for β

	σ_W^2	σ_1^2	σ_M^2	$\sigma_{\Phi^{-1}}^2$	σ_{LS}^2
D. Exp.	1.333	1.2	1	$\Pi/2 = 1.5707$	2
Logistic	3	3.0357	4	$\Pi = 3.1416$	$\Pi^2/3 = 3.2899$
Normal	$\Pi/3 = 1.0472$	1.0946	$\Pi/2 = 1.5707$	1	1

TABLE III

$\sigma_1^2/\sigma_{\cdot}^2$ for the Variances in Table II

	σ_W^2	σ_M^2	$\sigma_{\Phi^{-1}}^2$	σ_{LS}^2
D. Exp.	.90	1.20	.7639	.6
Logistic	1.0119	.7589	.966	.9227
Normal	1.0453	.6969	1.0946	1.0946

<u>Computation of</u> σ_1^2. σ_1^2 can be computed from either of the formulas

$$\sigma_1^2 = K^{-2}[\int_0^1 G^2(t)dt - (\int_0^1 G(t)dt)^2] \tag{5.1}$$

where $G(t) = \int_0^t f(F^{-1}(s))ds$, $0 \le t \le 1$ or

$$\sigma_1^2 = K^{-2}\int\int[F(x) \wedge F(y) - F(x)F(y)]f^2(x)f^2(y)dxdy . \tag{5.2}$$

For F double exponential or logistic, (5.1) was used since $f(F^{-1}(s))$ is easily obtained.

For $F = \Phi$, (5.2) implies that

$$\sigma_1^2 = (4\pi)^{-1}K^{-2}E[\Phi(X \wedge Y) - \Phi(X)\Phi(Y)]$$

For X and Y independent $N(0,.5)$. One then uses the fact that

$$E\Phi(X \wedge Y) = P(Z_1 \leq 0,\ Z_2 \leq 0)$$

where (Z_1,Z_2) has a bivariate normal distribution with $E(Z_1) = E(Z_2) = 0$, $\sigma^2(Z_1) = \sigma^2(Z_2) = 3$, and $\sigma(Z_1,Z_2) = 2$.

CHAPTER 2.

CRAMÉR-VON MISES TYPE ESTIMATION OF β

1. Notation and Preliminaries. To the assumptions of the model introduced in section 1 of the introduction we add the following:

(1.1) F has a finite mean and a continuous density f satisfying $\int f^2(x)dx < \infty$.

(1.2) $H(w) := \int F(w + x)F(dx)$ has a positive derivative $H'(0)$ at $w = 0$.

(1.3) $\lim_{n\to\infty} \sigma_c^{-1} \max_{1\le i\le n} |d_i| = 0.$

(1.4) $\sigma_c^{-4} \sum_{i<j} |d_i d_j|(d_j - d_i)^2 = O(1)$ as $n \to \infty$.

(1.5) $n\sigma_c^{-6} \sum_{i=1}^{n} \sum_{j=1}^{n} [d_i d_j(d_j - d_i)]^2 = O(1)$ as $n \to \infty$.

We retain the notational conventions of preceding sections and define the additional quantities

$$d_{ij} = -d_i d_j(d_j - d_i), \quad 1 \le i, j \le n .$$

$$d_{ij}^{+} = \max\{0, d_{ij}\} , \quad 1 \le i, j \le n$$

$$d_{ij}^{-} = \max\{0, -d_{ij}\}, \quad 1 \le i, j \le n$$

$$K_n^+ = \sigma_c^{-4} \sum_{i<j} d_{ij}^+(d_j - d_i)$$

$$K_n^- = \sigma_c^{-4} \sum_{i<j} d_{ij}^-(d_j - d_i) .$$

Remark. Note that (1.4) implies that $K_n^+ + K_n^-$ remains bounded. Furthermore, both (1.4) and (1.5) follow from the more common assumption

$$(1.6) \qquad \sqrt{n}\, \sigma_c^{-1} \max_{1\le i\le n} |d_i| = O(1) \quad \text{as} \quad n \to \infty$$

since

$$\sigma_c^{-4} \sum_{i<j} |d_i d_j| (d_j - d_i)^2 \le 4\sigma_c^{-4} \max_{1\le i\le n} d_i^2 \sum_{i<j} |d_i d_j|$$

$$\le 4\sigma_c^{-2} \max_{1\le i\le n} d_i^2 \sqrt{\sum_{i<j} d_i^2 \sum_{i<j} d_j^2}$$

$$\le 4n\sigma_c^{-2} \max_{1\le i\le n} d_i^2 = O(1)$$

and

$$n\sigma_c^{-6} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i^2 d_j^2 (d_j - d_i)^2 \le 2n\sigma_c^{-6} \max_{1\le i\le n} d_i^2 \sum_{i<j} |d_i d_j| (d_j - d_i)^2$$

$$\le 8[n\sigma_c^{-2} \max d_i^2]^2 = O(1) .$$

In what follows let the vectors $\underset{\sim}{c}$ and $\underset{\sim}{X}$ be given. For each real Δ and each real x define

$$U(x,\Delta) = \sum_{i=1}^{n} d_i I(X_i \le x + \Delta c_i)$$

and consider the process

$$\{M_2(\Delta),\ -\infty < \Delta < \infty\}$$

where

$$M_2(\Delta) = \int_{-\infty}^{\infty} U^2(x,\Delta)dx \ .$$

Recall that in the case of the two sample location problem, $M_2(\Delta)$ is a constant multiple of the squared L_2-distance between two empirical distribution functions. Thus we are led to estimate β in the general regression problem by attempting to minimize the quantity $M_2(\Delta)$ in Δ. To this end we investigate the properties of M_2 as a function of Δ.

Taking $X_i(\Delta) = X_i - \Delta c_i$ and using the identity $2\max(a,b) = a + b + |a - b|$ we see that

$$(1.7) \quad M_2(\Delta) = \int_{X_{(1)}(\Delta)}^{X_{(n)}(\Delta)} [\sum_{i=1}^{m} d_i I(X_i(\Delta) \leq x)]^2 dx$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} d_i d_j \int_{\max(X_i(\Delta),X_j(\Delta))}^{X_n(\Delta)} 1 \, dx$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} d_i d_j \max(X_i(\Delta),\ X_j(\Delta))$$

$$= -\sum_{1\leq i<j\leq n}\sum d_i d_j |X_j - X_i - \Delta(d_j - d_i)|.$$

It is immediate from (1.7) that, for fixed $\underset{\sim}{X}$ and $\underset{\sim}{c}$, $M_2(\Delta)$ is piecewise linear in Δ and that any changes in slope occur at points contained in the set

$$\Gamma_2 = \{(X_j - X_i)/(c_j - c_i);\ 1 \leq i < j \leq n \ \text{ and } \ c_i < c_j\}.$$

Now consider the set

$$A = \{\Delta;\ M_2(\Delta) = \inf_s M_2(s)\}.$$

If we establish that

$$(1.8) \qquad \lim_{|\Delta|\to\infty} M_2(\Delta) = +\infty$$

it will follow from the piecewise linear nature of M_2 that A is a nonempty subset of Γ_2. It can be seen from (1.7) that the slope of $M_2(\Delta)$ is $\sum_{1\le i<j\le n} d_i d_j (d_j - d_i)$ for $\Delta < \min \Gamma_2$ and $-\sum_{1\le i<j\le n} d_i d_j (d_j - d_i)$ for $\Delta > \max \Gamma_2$. Computations similar to those of (1.7) yield

$$-\sum_{1\le i<j\le n} d_i d_j (d_j - d_i) = \int_{-\infty}^{\infty} [\Sigma\, d_i I(x \ge c_i)]^2\, dx\ .$$

Since $c_1 \le c_2 \le \ldots \le c_n$ and the c_i's are not all equal, $d_1 \ne 0$ and there is a $K \le n$ such that $d_K \ne d_1$. Let K^* denote the first such K. For x between c_1 and c_{K^*} we have

$$[\Sigma\, d_i I(x \ge c_i)]^2 = (K^* - 1)^2 d_1^2 > 0\ .$$

Hence

$$-\sum_{1\le i<j\le n} d_i d_j (d_j - d_i) > 0\ .$$

This establishes (1.8).

We now define

$$\hat{\beta}_2 = \text{ave}(A).$$

<u>Remark</u>. In the two sample location problem

$$\hat{\beta}_2 = \text{med}\{X_j - X_i;\ 1 \le i,\ j \le n,\ c_i = 0,\ c_j = 1\}.$$

A regression example

That $\hat{\beta}_2$ need not agree with the Wilcoxon estimate $\hat{\beta}_W$ in the general regression problem is illustrated by the following example. Here we consider the sample

$$-1,\ -2,\ 10,\ -3,\ 15,\ -28$$

with the weights $c_i = i$, $1 \le i \le 6$. As was indicated earlier in this section, we can determine $\hat{\beta}_2$ once we have computed $n^{-1}M_2(\Delta)$ for each $\Delta \in \Gamma_2$. To compute $\hat{\beta}_W$, it suffices to calculate $nW_1(\Delta^-)$ for each $\Delta \in \Gamma_2$ (Adichie, 1967).

TABLE I

Values of $n^{-1}M_2(\Delta)$ and $nW_1(\Delta^-)$ for $\Delta \in \Gamma_2$

$\Delta \in \Gamma_2$	$n^{-1}M_2(\Delta)$	$nW_1(\Delta^-)$
-43	454.3333	17.5
-13	93.0833	16.5
-12.6667	89.0972	15.5
-12.5	87.3125	12.5
- 6.5	18.0625	10.5
- 5.4	10.8667	6.5
- 1.0	27.9167	1.5
- .6667	28.7917	.5
- .5	29.4375	-2.5
2.5	42.5625	-4.5
4.0	49.8750	-6.5
5.5	64.6875	-10.5
5.6667	66.1944	-12.5
12	137.7083	-15.5
18	203.9583	-16.5

Here we see that

$$\hat{\beta}_2 = -5.4; \quad \hat{\beta}_W = -.6667 .$$

Although $\hat{\beta}_2$ and $\hat{\beta}_W$ may differ for any given finite sample, as the above example illustrates, we prove in section 2.4 that $\sigma_c|\hat{\beta}_2 - \hat{\beta}_W|$ converges in probability to 0.

2. Finite Sample Properties

(a) Invariance. A useful property of the estimator $\hat{\beta}_2$ is its translation invariance; that is,

$$(2.1) \quad \hat{\beta}_2(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \hat{\beta}_2(\underset{\sim}{X}) + \gamma \quad \text{for all real } \gamma.$$

To verify (2.1) we note that

$$\begin{aligned} M_2(\Delta - \gamma)(X) &= -\sum_{i<j} d_i d_j |X_j - X_i - (\Delta - \gamma)(d_j - d_i)| \\ &= -\sum_{i<j} d_i d_j |(X_j + \gamma c_j) - (X_i + \gamma c_i) - \Delta(d_j - d_i)| \\ &= M_2(\Delta)(\underset{\sim}{X} + \gamma\underset{\sim}{c}). \end{aligned}$$

Thus $A(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \{a + \gamma;\ a \in A(\underset{\sim}{X})\}$ from which (2.1) follows. As a result of (2.1) we have

$$(2.2) \quad P_\beta(\hat{\beta}_2 - \beta \le z) = P_0(\hat{\beta}_2 \le z) \quad \text{for all real } z.$$

(b) Symmetry. We assume, without loss of generality, that $\beta = 0$.

Theorem 2.1. $\hat{\beta}_2$ is symmetric about β if one of the following conditions hold:

(i) F is symmetric

(ii) $d_i = -d_{n-i+1}$, $1 \leq i \leq n$.

Proof of (i). $\hat{\beta}_2(\underset{\sim}{X}) \sim \hat{\beta}_2(-\underset{\sim}{X})$ follows from a proof similar to that of theorem 1.2.1. Using (1.7) one obtains $M_2(-\Delta)(\underset{\sim}{X}) = M_2(\Delta)(-\underset{\sim}{X})$ and hence $A(-\underset{\sim}{X}) = -A(\underset{\sim}{X})$. Thus $\hat{\beta}_2(\underset{\sim}{X}) \sim \hat{\beta}_2(-\underset{\sim}{X}) = -\hat{\beta}_2(\underset{\sim}{X})$, completing the proof of (i).

Proof of (ii). As in the proof of theorem 1.2.1, $\hat{\beta}_2(\underset{\sim}{X}) \sim \hat{\beta}_2(\underset{\sim}{X}^*)$ where $\underset{\sim}{X}^* = (X_n, X_{n-1}, \ldots, X_1)'$. From (1.7), $M_2(\Delta)(\underset{\sim}{X}^*) = M_2(-\Delta)(\underset{\sim}{X})$ and hence $A(\underset{\sim}{X}^*) = -A(\underset{\sim}{X})$. Thus $\hat{\beta}_2(\underset{\sim}{X}^*) = -\hat{\beta}_2(\underset{\sim}{X})$ and hence $\hat{\beta}_2(\underset{\sim}{X}) \sim \hat{\beta}_2(\underset{\sim}{X}^*) = -\hat{\beta}_2(\underset{\sim}{X})$, completing the proof of (ii).

3. Asymptotic Behavior of $M_2(\Delta)$. Throughout this section we retain the notation of section 1 and assume that (1.1) through (1.4) hold. Assume, without loss of generality, that $\beta = 0$. For convenience we introduce the following additional notation; for $-\infty < \Delta < \infty$ define

$$T_2^+(\Delta) = \sum_{i<j} d_{ij}^+ I(X_j - X_i \leq \Delta_{ij})$$

$$T_2^-(\Delta) = \sum_{i<j} d_{ij}^- I(X_j - X_i \leq \Delta_{ij})$$

$$T_2(\Delta) = T_2^+(\Delta) - T_2^-(\Delta)$$

$$V^+(\Delta) = \sum_{i<j} (d_i d_j)^- I(X_j - X_i \leq \Delta_{ij})(X_j - X_i)$$

$$V^-(\Delta) = \sum_{i<j} (d_i d_j)^+ I(X_j - X_i \leq \Delta_{ij})(X_j - X_i)$$

$$V(\Delta) = V^+(\Delta) - V^-(\Delta)$$

where

$$\Delta_{ij} = \Delta(d_j - d_i) \ .$$

Remark 3.1. In terms of the notation introduced above, we show that for $\Delta \in \Gamma_2^c$

$$M_2(\Delta) = M_2(0) - 2[V(\Delta) - V(0)] + 2\Delta[T_2(\Delta) - T_2(0)]$$
$$+ 2\Delta[T_2(0) - E_0[T_2(0)]] \ .$$

Proof.

$$M_2(\Delta) = -\sum_{i<j} d_i d_j |X_j - X_i - \Delta_{ij}|$$
$$= -\sum_{i<j} d_i d_j [1 - 2I(X_j - X_i \le \Delta_{ij})](X_j - X_i - \Delta_{ij})$$
$$= -\sum_{i<j} d_i d_j (X_j - X_i) + 2\sum_{i<j} d_i d_j I(X_j - X_i \le \Delta_{ij})(X_j - X_i)$$
$$- 2\sum_{i<j} d_i d_j \Delta_{ij} I(X_j - X_i \le \Delta_{ij}) + \sum_{i<j} d_i d_j \Delta_{ij}$$
$$= -\sum_{i<j} d_i d_j [1 - 2I(X_j - X_i \le 0)](X_j - X_i)$$
$$+ 2\sum_{i<j} d_i d_j I(0 < X_j - X_i \le \Delta_{ij})(X_j - X_i)$$
$$+ 2\Delta\sum_{i<j} d_{ij} I(0 < X_j - X_i \le \Delta_{ij})$$
$$+ 2\Delta\sum_{i<j} d_{ij}[I(X_j - X_i \le 0) - .5]$$
$$= M_2(0) - 2[V(\Delta) - V(0)] + 2[T_2(\Delta) - T_2(0)]$$
$$+ 2\Delta[T_2(0) - E(T_2(0))]. \qquad \square$$

In lemmas 3.1-3.3 we investigate the asymptotic behavior of T_2 and V.

<u>Lemma 3.1</u>. Let $0 < b < \infty$. Then under the assumptions of section 1,

$$\sup_{|\Delta|\le b} |\sigma_c^{-3}[T_2(\Delta/\sigma_c) - T_2(0)] - \Delta H'(0)| \xrightarrow{P_0} 0 .$$

<u>Proof</u>. The lemma is proved by combining analogous results for T_2^+ and T_2^-. We consider only T_2^+; the proof for T_2^- is similar.

Fix $\Delta \in R$. Using (1.4) one proves as in Scholz (1978) that

$$E_0\{\sigma_c^{-3}[T_2^+(\Delta/\sigma_c) - T_2^+(0)] - \Delta K_n^+ H'(0)\} \to 0 . \tag{3.1}$$

We next show that

$$\mathrm{Var}_0\{\sigma_c^{-3}[T_2^+(\Delta/\sigma_c) - T_2^+(0)]\} \to 0 . \tag{3.2}$$

Set

$$Z_{ij} = I(0 < X_j - X_i \le \Delta_{ij}/\sigma_c) , \qquad 1 \le i, j \le n,$$

$$S_n^+ = \sigma_c^{-3} \sum_{i<j} d_{ij}^+ Z_{ij} ,$$

$$\hat{S}_n^+ = \sigma_c^{-3} \sum_{k=1}^{n} E_0(S_n^+ | X_k) .$$

One shows as in Scholz (1978) that

$$\begin{aligned} &\mathrm{Var}_0(S_n^+) - \mathrm{Var}_0(\hat{S}_n^+) \\ &\le \sigma_c^{-6} \sum_{i<j} (d_{ij}^+)^2 [E_0\mathrm{Var}_0(Z_{ij}|X_i) + E_0\mathrm{Var}_0(Z_{ij}|X_j)]. \end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned}
\text{LHS } (3.3) &\le \frac{1}{2}\,\sigma_c^{-6} \sum_{i<j} (d_{ij}^{+})^2 \\
&\le \sigma_c^{-6} \max_{1\le k\le n} d_k^2 \sum_{i<j} |d_i d_j| (d_j - d_i)^2 \\
&= \sigma_c^{-2} \max_{1\le k\le n} d_k^2 \,[K_n^{+} + K_n^{-}] \\
&= o(1) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

For each $1 \le k \le n$ we have

$$\begin{aligned}
E_0(S_n^{+}|X_k) &= \sigma_c^{-3} \sum_{i<j} d_{ij}^{+}\, E_0(Z_{ij}|X_k) \\
&= \sigma_c^{-3}[\sum_{i<k} d_{ik}^{+}\, E_0(Z_{ik}|X_k) + \sum_{k<i} d_{ki}^{+}\, E_0(Z_{ki}|X_k)] \\
&= \sigma_c^{-3} \sum_{i<k} d_{ik}^{+}\, [F(X_k) - F(X_k + \Delta_{ki}/\sigma_c)] \\
&\quad + \sigma_c^{-3} \sum_{k<i} d_{ki}^{+}[F(X_k + \Delta_{ik}/\sigma_c) - F(X_k)].
\end{aligned}$$

Using $\int_{-\infty}^{\infty} f^2(x)dx < \infty$ and the Cauchy-Schwarz inequality it follows that F is uniformly continuous and hence that

$$\begin{aligned}
\mathrm{Var}_0[E_0(S_n^{+}|X_k)] &\le E_0[E_0(S_n^{+}|X_k)]^2 \\
&\le \sigma_c^{-6}[\sum_{i=1}^{n} |d_{ik}| \sup_{i,j,x} |F(x + \Delta_{ij}/\sigma_c) - F(x)|]^2 \\
&\le \sigma_c^{-6} \sup_{i,j,x} |F(x + \Delta_{ij}/\sigma_c) - F(x)|^2 n \sum_{i=1}^{n} d_{ik}^2 \\
&= n\sigma_c^{-6} \sum_{i=1}^{n} d_{ik}^2\, o(1) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Applying (1.5) we obtain

$$\mathrm{Var}_0(\hat{S}_n^+) \leq [n\sigma_c^{-6} \sum_{k=1}^{n} \sum_{i=1}^{n} d_{ik}^2] o(1) = o(1) \quad \text{as} \quad n \to \infty.$$

Combining this result with (3.3) yields (3.2). From (3.1) and (3.2) we conclude

$$(3.4) \qquad |\sigma_c^{-3}[T^+(\Delta/\sigma_c) - T^+(0)] - \Delta K_n^+ H'(0)| \xrightarrow{P_0} 0 .$$

We next verify that

$$(3.5) \qquad \sup_{|\Delta| \leq b} |\sigma_c^{-3}[T^+(\Delta/\sigma_c) - T^+(0)] - K_n^+ \Delta H'(0)| \xrightarrow{P_0} 0 .$$

Let $\varepsilon, \delta > 0$ be given and let

$$-b = \Delta_1 < \Delta_2 < \ldots < \Delta_{k(\varepsilon)} = b$$

be a partition of $[-b,b]$ such that

$$\max_{1 \leq i \leq k(\varepsilon)-1} (\Delta_{i+1} - \Delta_i) < \frac{\varepsilon}{2}[\sup_n K_n^+ H'(0)]^{-1} .$$

By the above we have

$$|\sigma_c^{-3}[T^+(\Delta_j/\sigma_c) - T^+(0)] - K_n^+ \Delta_j H'(0)| \xrightarrow{P_0} 0$$

for each $1 \leq j \leq k(\varepsilon)$. Thus we may choose $0 < N < \infty$ such that

$$1-\delta < P[\max_{1 \leq j \leq k(\varepsilon)} |\sigma_c^{-3}[T^+(\Delta_j/\sigma_c) - T^+(0)] - K_n^+ \Delta_j H'(0)| < \varepsilon]$$

whenever $n \geq N$.

Now suppose that $\Delta_0 \in (\Delta_j, \Delta_{j+1})$ for some $1 \leq j \leq k(\varepsilon) - 1$.

Then since $\sigma_c^{-3}[T^+(\Delta/\sigma_c) - T^+(0)]$ and $K_n^+ \Delta H'(0)$ are nondecreasing in Δ we have

$$\begin{aligned}
&\sigma_c^{-3}[T^+(\Delta_0/\sigma_c) - T^+(0)] - K_n^+ \Delta_0 H'(0) \\
&\leq \sigma_c^{-3}[T^+(\Delta_{j+1}/\sigma_c) - T^+(0)] - K_n^+ \Delta_j H'(0) \\
&\leq \sigma_c^{-3}[T^+(\Delta_{j+1}/\sigma_c) - T^+(0)] - K_n^+\Delta_{j+1} H'(0) \\
&\qquad + K_n^+ H'(0)[\Delta_{j+1} - \Delta_j] \\
&\leq \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_c^{-3}[T^+(\Delta_0/\sigma_c) - T^+(0)] - K_n^+\Delta_0 H'(0) \\
&\geq \sigma_c^{-3}[T^+(\Delta_j/\sigma_c) - T^+(0)] - K_n^+\Delta_j H'(0) + K_n^+ H'(0)[\Delta_j - \Delta_{j+1}] \\
&\geq -\varepsilon \quad .
\end{aligned}$$

Hence

$$|\sigma_c^{-3}[T^+(\Delta_0/\sigma_c) - T^+(0)] - K_n^+\Delta_0 H'(0)| < \varepsilon$$

and it follows that

$$P_0[\sup_{|\Delta|\leq b} |\sigma_c^{-3}[T^+(\Delta/\sigma_c) - T^+(0)] - K_n^+ \Delta H'(0)| < \varepsilon] \geq 1 - \delta,$$

completing the proof of (3.5).

In a similar fashion one can show that

$$(3.6) \quad \sup_{|\Delta|\leq b} |\sigma_c^{-3}[T^-(\Delta/\sigma_c) - T^-(0)] - K_n^- \Delta H'(0)| \xrightarrow{P_0} 0 \ .$$

Combining (3.5) and (3.6) completes the proof of the lemma. □

Lemma 3.2. Suppose $X \sim H$. Then

$$\lim_{t \to 0} t^{-2} E_H\{X[I(X \leq t) - I(X \leq 0)]\} = H'(0)/2.$$

Proof. We consider only the right limit; the proof for the left limit is similar. Let $\varepsilon > 0$ be given. Since $H'(0)$ exists there is a $t_0 > 0$ such that $|H(y) - H(0) - yH'(0)| < 2y\varepsilon$ whenever $0 \leq y \leq t_0$. Thus $0 \leq t \leq t_0$ implies that

$$|t^{-2} \int_0^t [H(y) - H(0) - yH'(0)]dy|$$

$$\leq t^{-2} \int_0^t 2y\, \varepsilon\, dy \leq \varepsilon$$

and hence

$$\lim_{t \downarrow 0} t^{-2} \int_0^t [H(y) - H(0) - yH'(0)]dy = 0 .$$

Defining $X_t = XI(0 < X \leq t)$ we see that

$$\lim_{t \downarrow 0} t^{-2} E_H\{X[I(X \leq t) - I(X \leq 0)]\}$$

$$= \lim_{t \downarrow 0} t^{-2} \int_0^t P_H(X_t > y)dy$$

$$= \lim_{t \downarrow 0} t^{-2} \int_0^t [H(t) - H(y)]dy$$

$$= \lim_{t \downarrow 0} t^{-2}\{\int_0^t H(t)dy - \int_0^t [H(0) + yH'(0)]dy$$

$$- \int_0^t [H(y) - H(0) - yH'(0)]dy\}$$

$$= \lim_{t \downarrow 0} \{t^{-1}[H(t) - H(0)] - H'(0)/2\}$$

$$= H'(0)/2 . \qquad \square$$

<u>Lemma 3.3</u>. Let $0 < b < \infty$. Then under the assumptions of section 1

$$\sup_{|\Delta|\le b} |\sigma_c^{-2}[V(\Delta/\sigma_c) - V(0)] - \frac{1}{2}\Delta^2 H'(0)| \xrightarrow{P_0} 0 .$$

<u>Proof</u>. Fix $0 < \Delta \le b$. The proof for $-b \le \Delta < 0$ is similar. Then

$$E_0\{\sigma_c^{-2}[V^+(\Delta/\sigma_c) - V^+(0)]\}$$

$$= E_0\{\sigma_c^{-2} \sum_{i<j} (d_i d_j)^- I(0 < X_j - X_i \le \Delta_{ij}/\sigma_c)(X_j - X_i)\}$$

$$= \Delta^2\sigma_c^{-4} \sum_{\substack{i<j \\ d_{ij}>0}} d_{ij}(d_j - d_i)E_0\{(\Delta_{ij}/\sigma_c)^{-2} I(0 < X_j - X_i \le \Delta_{ij}/\sigma_c)(X_j - X_i) - H'(0)/2\}$$

$$+ \frac{1}{2}\Delta^2\sigma_c^{-4} \sum_{\substack{i<j \\ d_{ij}>0}} d_{ij}(d_j - d_i)H'(0) .$$

Now

$$\frac{1}{2}\Delta^2\sigma_c^{-4} \sum_{\substack{i<j \\ d_{ij}>0}} d_{ij}(d_j - d_i)H'(0) = \frac{1}{2}\Delta^2 K_n^+ H'(0)$$

and by lemma 3.2 and (1.3),

$$|\Delta^2\sigma_c^{-4} \sum_{\substack{i<j \\ d_{ij}>0}} d_{ij}(d_j - d_i)E_0\{(\Delta_{ij}/\sigma_c)^{-2} I(0 < X_j - X_i \le \Delta_{ij}/\sigma_c)(X_j - X_i) - H'(0)/2\}|$$

$$\le \Delta^2 K_n^+ \max_{i<j} |E_0\{(\Delta_{ij}/\sigma_c)^{-2} I(0 < X_j - X_i \le \Delta_{ij}/\sigma_c)(X_j - X_i) - H'(0)/2\}|$$

$$\le \Delta^2 K_n^+ \, o(1) = o(1) \quad \text{as} \quad n \to \infty.$$

Thus

(3.7) $\quad E_0\{\sigma_c^{-2}[V^+(\Delta/\sigma_c) - V^+(0)] - \frac{1}{2} K_n^+\Delta^2 H'(0)\} \to 0$.

To establish the lemma for V^+ we show that

(3.8) $$\mathrm{Var}_0(\sigma_c^{-2}[V^+(\Delta/\sigma_c) - V^+(0)]) \to 0 .$$

Set

$$Z_{ij} = I(0 < X_j - X_i \le \Delta_{ij}/\sigma_c)(X_j - X_i)$$

$$S_n^+ = \sigma_c^{-2} \sum_{i<j} (d_i d_j)^- Z_{ij}$$

and

$$\hat{S}_n^+ = \sigma_c^{-2} \sum_{k=1}^{n} E_0(S_n^+ | X_k) .$$

It follows as in theorem 1 of Scholz (1978) that

$$\mathrm{Var}_0(S_n^+) - \mathrm{Var}_0(\hat{S}_n^+)$$

$$\le \sigma_c^{-4} \sum_{i<j} [(d_i d_j)^-]^2 [E_0 \mathrm{Var}_0(Z_{ij}|X_i) + E_0 \mathrm{Var}_0(Z_{ij}|X_j)] .$$

Hence, by (1.2) and (1.5),

(3.9) $$\begin{aligned} &\mathrm{Var}_0(S_n^+) - \mathrm{Var}_0(\hat{S}_n^+) \\ &\le 2\sigma_c^{-4} \sum_{i<j} [(d_i d_j)^-]^2 E_0 Z_{ij}^2 \\ &\le 2\sigma_c^{-4} \sum_{i<j} [(d_i d_j)^-]^2 (\Delta_{ij}/\sigma_c)^2 [H(\Delta_{ij}/\sigma_c) - H(0)] \\ &\le 2\Delta^2\sigma_c^{-6} \sum_{i<j} (d_{ij}^+)^2 [H(\Delta_{ij}/\sigma_c) - H(0)] \\ &\le 2\Delta^2\sigma_c^{-6} \sum_{i<j} (d_{ij}^+)^2 \max_{i<j} [H(\Delta_{ij}/\sigma_c) - H(0)]. \end{aligned}$$

For each $1 \le k \le n$ we have

$$E_0(S_n^+|X_k) = \sigma_c^{-2} \sum_{i<k} (d_i d_k)^- E_0(Z_{ik}|X_k)$$

$$+ \sigma_c^{-2} \sum_{k<i} (d_i d_k)^- E_0(Z_{ki}|X_k)$$

where

$$E_0(Z_{ik}|X_k) = \int I(0 \leq X_k - y \leq \Delta_{ik}/\sigma_c)(X_k - y)F(dy)$$

$$= \int_{X_k-\Delta_{ik}/\sigma_c}^{X_k} (X_k - y)F(dy)$$

$$\leq (\Delta_{ik}/\sigma_c)[F(X_k) - F(X_k - \Delta_{ik}/\sigma_c)], \ i < k,$$

and

$$E_0(Z_{ki}|X_k) \leq (\Delta_{ik}/\sigma_c)[F(X_k + \Delta_{ik}/\sigma_c) - F(X_k)], \ i > k.$$

Therefore

$$\mathrm{Var}_0(\hat{S}_n^+) = \sum_{k=1}^{n} \mathrm{Var}_0 E_0(S_n^+|X_k)$$

$$\leq \sigma_c^{-4}[\sup_{x,i<k} |F(x + \Delta_{ik}/\sigma_c) - F(x)|]^2 \sum_k [\sum_i |d_i d_k|(\Delta_{ik}/\sigma_c)]^2$$

$$\leq 2\Delta^2\sigma_c^{-6} \sum_{i<k} d_{ik}^2 \, o(1) = o(1) \quad \text{as} \quad n \to \infty.$$

Combining this result with (3.9) yields (3.8). Using (3.7), (3.8) and the fact that $\sigma_c^{-2}[V^+(\Delta/\sigma_c) - V^+(0)]$ and $\frac{1}{2} K^+\Delta^2 H'(0)$ are both nondecreasing (nonincreasing) on $0 \leq \Delta \leq b$ $(-b \leq \Delta \leq 0)$ it follows that

$$\text{(3.10)} \quad \sup_{|\Delta|\leq b} |\sigma_c^{-2}[V^+(\Delta/\sigma_c) - V^+(0)] - \frac{1}{2} K^+\Delta^2 H'(0)| \xrightarrow{P_0} 0 .$$

In a similar fashion one shows that

$$(3.11)\quad \sup_{|\Delta|\le b} |\sigma_c^{-2}[V^-(\Delta/\sigma_c) - V^-(0)] - \tfrac{1}{2} K^-\Delta^2 H'(0)| \xrightarrow{P_0} 0 .$$

Combining (3.10) and (3.11) completes the proof of the lemma. □

<u>Theorem 3.1</u>. Let $0 < b < \infty$. Then

$$\sup_{|\Delta|\le b} |\sigma_c^{-2}[M_2(\Delta/\sigma_c) - M_2(0)] - \Delta^2 H'(0) + 2\Delta\sigma_c^{-3}T_2^*(0)| \xrightarrow{P_0} 0$$

where $T_2^*(0) = -[T_2(0) - E_0(T_2(0))]$.

<u>Proof</u>. The theorem is a consequence of remark (3.1) and lemmas (3.1) and (3.3). □

We conclude the section by proving three lemmas which will be useful in showing that the sequence $\{\sigma_c\hat{\beta}_2\}$ is bounded in probability.

<u>Lemma 3.4</u>. The sequence $\{\sigma_c^{-2}M_2(0)\}$ is bounded in probability.

<u>Proof</u>. $$E_0[\sigma_c^{-2}M_2(0)] = E_0[-\sigma_c^{-2} \sum_{i<j} d_i d_j |X_j - X_i|]$$

$$= -\sigma_c^{-2}E_0|X_1 - X_2| \sum_{i<j} d_i d_j$$

$$= \frac{1}{2} E_0|X_1 - X_2|.$$

Since $\sigma_c^{-2}M_2(0)$ is nonnegative for each n, application of the Markov inequality completes the proof. □

<u>Lemma 3.5</u>. Let

$$W_2(\Delta) = \sum_{i=1}^{n} d_i \int_{X_i'(\Delta)}^{X_{(n)}'(\Delta)} \sqrt{f(x)}\, dx, \qquad -\infty < \Delta < \infty,$$

where $X_i'(\Delta) = X_i - \Delta d_i$. Then

(i) W_2 is a nondecreasing function of Δ.

(ii) $|\sigma_c^{-2}[W_2(\Delta/\sigma_c) - W_2(0)] - \Delta\, E_0 \sqrt{f(X_1)}| \overset{P_0}{\to} 0 \quad \forall\, \Delta \in R.$

(iii) $W_2^2(\Delta) \le M_2(\Delta) \quad \forall \quad \Delta \in R.$

<u>Proof of (i)</u>. Let $\Delta < \Delta'$ and set $A = \max(X'_{(n)}(\Delta), X'_{(n)}(\Delta'))$. Then

$$W_2(\Delta') - W_2(\Delta) = \sum_{i=1}^{n} d_i[\int_{X_i'(\Delta')}^{A} \sqrt{f(x)}\, dx - \int_{X_i'(\Delta)}^{A} \sqrt{f(x)}\, dx]$$

$$= \sum_{i=1}^{n} d_i^+ \int_{X_i'(\Delta')}^{X_i'(\Delta)} \sqrt{f(x)}\, dx + \sum_{i=1}^{n} d_i^- \int_{X_i'(\Delta)}^{X_i'(\Delta')} \sqrt{f(x)}\, dx$$

$$\ge 0 .$$

<u>Proof of (ii)</u>. We consider the case when $\Delta \ge 0$; the proof for $\Delta < 0$ is similar.

$$\sigma_c^{-1}[W_2(\Delta/\sigma_c) - W_2(0)]$$

$$= \sigma_c^{-1}[\sum_{i=1}^{n} d_i^+ \int_{X_i'(\Delta/\sigma_c)}^{X_i} \sqrt{f(x)}\, dx + \sum_{i=1}^{n} d_i^- \int_{X_i}^{X_i'(\Delta/\sigma_c)} \sqrt{f(x)}\, dx]$$

$$= \sigma_c^{-1} \sum_{i=1}^{n} d_i^+[(\Delta d_i^+/\sigma_c)^{-1} \int_{X_i'(\Delta/\sigma_c)}^{X_i} \sqrt{f(x)}\, dx - \sqrt{f(X_i)}]\Delta d_i^+/\sigma_c$$

$$+ \sigma_c^{-1} \sum_{i=1}^{n} d_i^-[(\Delta d_i^-/\sigma_c)^{-1} \int_{X_i}^{X_i'(\Delta/\sigma_c)} \sqrt{f(x)}\, dx - \sqrt{f(X_i)}]\Delta d_i^-/\sigma_c$$

$$+ \Delta\sigma_c^{-2} \sum_{i=1}^{n} d_i^2 \sqrt{f(X_i)} .$$

Therefore, by assumption (1.1) and the above,

$$|\sigma_c^{-1}[W_2(\Delta/\sigma_c) - W_2(0)] - \Delta\sigma_c^{-2} \sum_{i=1}^{n} d_i^2 \sqrt{f(X_i)}|$$

$$\leq \Delta \max_{1\leq i\leq n} |(\Delta d_i/\sigma_c)^{-1} \int_{X_i'(\Delta/\sigma_c)}^{X_i} \sqrt{f(x)}\, dx - \sqrt{f(X_i)}| = o(1) \quad \text{as} \quad n \to \infty.$$

We complete the proof of (ii) by showing that

$$|\Delta\sigma_c^{-2} \sum_{i=1}^{n} d_i^2 \sqrt{f(X_i)} - \Delta\, E_0 \sqrt{f(X_1)}| \overset{P_0}{\to} 0 \ .$$

But this follows from the WLLN since

$$\mathrm{Var}_0(\sqrt{f(X_1)}) \leq \int f^2(x)dx < \infty.$$

<u>Proof of (iii)</u>. By the Cauchy-Schwarz inequality

$$M_2(\Delta) = \int_{\{f>0\}}\{[\sum_{i=1}^{n} d_i I(X_i - \Delta d_i \leq x)]^2/f(x)\}f(x)dx$$

$$\geq (\int_{-\infty}^{\infty} \sum_{i=1}^{n} d_i I(X_i - \Delta d_i \leq x)\sqrt{f(x)}\, dx)^2$$

$$= (\int_{X'_{(1)}(\Delta)}^{X'_{(n)}(\Delta)} \sum_{i=1}^{n} d_i I(X_i - \Delta d_i \leq x)\sqrt{f(x)}\, dx)^2$$

$$= (\sum_{i=1}^{n} d_i \int_{X_i'(\Delta)}^{X'_{(n)}(\Delta)} \sqrt{f(x)}\, dx)^2$$

$$= W_2^2(\Delta). \qquad \square$$

Before stating our final lemma we define, for each $0 < a < \infty$, $0 < d < \infty$ and $n \geq 1$, the event

$$E_{n2}(a,d) = \{\sigma_c^{-2}M_2(0) < d, \inf_{|\Delta|=a} \sigma_c^{-2}W_2^2(\Delta/\sigma_c) \geq d\}.$$

<u>Lemma 3.6</u>. For every $\varepsilon > 0$ there exist positive real numbers N, a and d such that $n \geq N$ implies

$$P_0(E_{n2}(a,d)) \geq 1 - \varepsilon.$$

Proof. Since $0 \leq W_2^2(0) \leq M_2(0)$ for all n, the sequence $\{\sigma_c^{-1} W_2(0)\}$ is bounded in probability by lemma 3.4. Hence for fixed $|\Delta| < \infty$,

$$(3.12) \quad |\sigma_c^{-2} W_2^2(\Delta/\sigma_c) - [\sigma_c^{-1} W_2'(0) + \Delta\, E_0 \sqrt{f(X_1)}]^2| \overset{P_0}{\to} 0$$

by lemma 3.5. Now let b be such that

$$P_0[\sigma_c^{-2} M_2(0) \leq b] \geq 1 - \varepsilon \quad \forall \quad n \geq 1 .$$

If we take $d > 2b$ and choose a so that

$$a > (\sqrt{b} + \sqrt{3d/2})(E_0 \sqrt{f(X_1)})^{-1}$$

we have

$$\sigma_c^{-2} W_2^2(0) \leq \sigma_c^{-2} M_2(0) \leq b < d/2$$

and hence

$$\inf_{|\Delta|=a} [\sigma_c^{-1} W_2(0) + \Delta\, E_0 \sqrt{f(X_1)}]^2 \geq [a\, E_0 \sqrt{f(X_1)} - \sigma_c^{-1} |W_2(0)|]^2 \geq 3d/2$$

on $\{\sigma_c^{-2} M_2(0) \leq b\}$. The proof is completed by applying (3.12). □

4. Asymptotic Distribution of $\sigma_c \hat{\beta}_2$. Throughout this section we retain the notation of sections 1-3 and assume that (1.1) - (1.4) hold. In addition we assume, without loss of generality, that $\beta = 0$.

Lemma 4.1. $\{\sigma_c \hat{\beta}_2\}$ is bounded in probability.

Proof. Let $\varepsilon > 0$ be given. By lemma 3.6 there exist positive real numbers a,d and N such that

$$P_0[E_{n2}(a,d)] \geq 1 - \varepsilon \quad \forall \ n \geq N.$$

By parts (i) and (iii) of lemma 3.5

$$\inf_{|\Delta|\geq a} \sigma_c^{-2} M_2(\Delta/\sigma_c) \geq \inf_{|\Delta|\geq a} \sigma_c^{-2} W_2^2(\Delta/\sigma_c) \geq d$$

and

$$\sigma_c^{-2} W_2^2(0) \leq \sigma_c^{-2} M_2(0) < d$$

on $E_{n2}(a,b)$ whenever $n \geq N$. Thus

$$\{|\sigma_c\hat{\beta}_2| \leq a\} \supseteq \{\sigma_c^{-2} M_2(0) < d, \inf_{|\Delta|\geq a} M_2(\Delta/\sigma_c) \geq d\} \supseteq E_{n2}(a,d) \quad \forall \ n \geq N$$

implies

$$P[|\sigma_c\hat{\beta}_2| \leq a] \geq P[E_{n2}(a,d)] \geq 1 - \varepsilon \quad \forall \ n \geq N . \qquad \square$$

The result of theorem 3.1 suggests that an approximating statistic for $\sigma_c\hat{\beta}_2$ is $\sigma_c^{-3} T_2^*(0)/H'(0)$. The next lemma gives the asymptotic distribution of that statistic.

Lemma 4.2. Under assumptions (1.1) and (1.3) $\mathcal{L}_0(\sqrt{12}\ \sigma_c^{-3} T_2^*(0) \Rightarrow N(0,1)$.

Proof. Since

$$T_2^*(0) = -\sum_{i<j} d_{ij}[I(X_j \leq X_i) - .5] = -\sum_{i<j} d_{ij}[I(R_{ni0} \leq R_{nj0}) - .5],$$

the projection of $T_2^*(0)$ into the family of linear rank statistics is

$$W_2^* := n^{-1}\sigma_c^2 \sum_{i=1}^{n} d_i R_{ni0}$$

(Hájek and Šidák (1967), p. 61). Since

$$\sqrt{12}\ \sigma_c^{-3}|T_2^*(0) - W_2^*| \xrightarrow{P_0} 0$$

(Sievers, 1976), the proof is completed by noting that

$$L_0(\sqrt{12}\ \sigma_c^{-3}\ W_2^*) \Rightarrow N(0,1)$$

under assumptions (1.1) and (1.3) (Hájek and Sidák (1967), p. 163). □

For $0 < a < b < \infty$ define

$$G_{n2}(a,b) = \{|\sigma_c^{-3}\ T_2^*(0)/H'(0)| \leq b,\ \inf_{|\Delta|>a} M_2(\Delta/\sigma_c) > \inf_{|\Delta|\leq a} M_2(\Delta/\sigma_c)\}$$

and

$$\Delta_{n2}(a) = \sup\{|\Delta^* - \sigma_c^{-3}\ T_2^*(0)/H'(0)|;\ |\Delta^*| \leq a,\ M_2(\Delta^*/\sigma_c)$$

$$= \inf_{|\Delta|\leq a} M_2(\Delta/\sigma_c)\}.$$

<u>Lemma 4.3</u>. Let $\varepsilon > 0$ and $0 < b < a$ be given. Then

$$P_0(\{\Delta_{n2}(a) > \varepsilon\} \cap G_{n2}(a,b)) \to 0$$

as $n \to \infty$.

<u>Proof</u>. Suppose there exist ε_1 and δ positive such that

$$P_0(\{\Delta_{n2}(a) > \varepsilon\} \cap G_{n2}(a,b)) \geq \delta$$

for infinitely many n. For each such n there exists a $|\Delta_0| \leq a$ such that

$$(4.1)\qquad |\Delta_0 - \sigma_c^{-3}\, T_2^*(0)/H'(0)| \geq \varepsilon_1$$

and

$$(4.2)\qquad M_2(\Delta_0/\sigma_c) = \inf_{|\Delta|\leq a} M_2(\Delta/\sigma_c)$$

on $G_{n2}(a,b)$. Since

$$Q(\Delta) := \sigma_c^{-2}M_2(0) - 2\Delta\sigma_c^{-3}\, T_2^*(0) + \Delta^2 H'(0)$$

is quadratic in Δ and achieves its minimum at $\Delta = \sigma_c^{-3}T_2^*(0)/H'(0)$, (4.1) implies that

$$Q(\Delta_0) - Q(\sigma_c^{-3}\, T_2^*(0)/H'(0)) \geq H'(0)\varepsilon_1^2 \; .$$

By (4.2) we also have

$$M_2(\sigma_c^{-4}\, T_2^*(0)/H'(0)) \geq M_2(\Delta_0/\sigma_c) \; .$$

But

$$\sup_{|\Delta|\leq a} |\sigma_c^{-2}M_2(\Delta/\sigma_c) - Q(\Delta)|$$

$$\geq \max\{|\sigma_c^{-2}M_2(\Delta_0/\sigma_c) - Q(\Delta_0)|,\ |\sigma_c^{-2}M_2(\sigma_c^{-4}T_2^*(0)/H'(0))$$

$$- Q(\sigma_c^{-3}\, T_2^*(0)/H'(0))|\}$$

$$\geq H'(0)\varepsilon_1^2/2$$

on $G_{n2}(a,b)$ since

$$|\sigma_c^{-2}M_2(\Delta_0/\sigma_c) - Q(\Delta_0)| < H'(0)\varepsilon_1^2/2$$

implies

$$\sigma_c^{-2} M_2(\sigma_c^{-4}\, T_2^*(0)/H'(0)) - Q(\sigma_c^{-3}\, T_2^*(0)/H'(0))$$

$$\geq \sigma_c^{-2}\, M_2(\Delta_0/\sigma_c) - Q(\sigma_c^{-3}\, T_2^*(0)/H'(0))$$

$$> Q(\Delta_0) - H'(0)\varepsilon_1^2/2 - Q(\sigma_c^{-3}\, T_2^*(0)/H'(0))$$

$$\geq H'(0)\varepsilon_1^2/2\ .$$

Thus

$$\limsup_{n\to\infty} \sup_{|\Delta|\leq a} |\sigma_c^{-2} M_2(\Delta/\sigma_c) - Q(\Delta)| \geq H'(0)\varepsilon_1^2/2\ ,$$

contradicting theorem 3.1. □

We are now ready to give the asymptotic distribution of $\sigma_c\hat{\beta}_2$.

<u>Theorem 4.1</u>. $L_0(\sigma_c\hat{\beta}_2) \to N(0,\ (12H'(0)^2)^{-1})$.

<u>Proof</u>. We prove that $|\sigma_c\hat{\beta}_2 - \sigma_c^{-3}\, T_2^*(0)/H'(0)| \xrightarrow{P_0} 0$. The theorem then follows from lemma 4.2. Let $\varepsilon > 0$ and $\delta > 0$ be given. By lemma 4.2 and the proof of lemma 4.1 we may choose $0 < b < a < \infty$ and $N_1 > 0$ such that

$$P_0[G_{n2}(a,b)] \geq 1 - \tfrac{1}{2}\delta \quad \text{for} \quad n \geq N_1\ .$$

Now use lemma 4.3 to choose $N > N_1$ such that

$$P_0(\{\Delta_{n2}(a) > \varepsilon\} \cap G_{n2}(a,b)) < \tfrac{1}{2}\delta$$

whenever $n \geq N$. Then for $n \geq N$ we have

$$\varepsilon \le \Delta_{n2}(a) = \sup\{|\Delta^* - \sigma_c^{-3}T_2^*(0)/H'(0)| \ : M_2(\Delta^*/\sigma_c) = \inf_{|\Delta|\le a} M_2(\Delta/\sigma_c)\}$$

$$= \sup\{|\Delta^* - \sigma_c^{-3}T_2^*(0)/H'(0)| \ : M_2(\Delta^*/\sigma_c) = \inf_{\Delta} M_2(\Delta/\sigma_c)\}$$

$$\ge |\sigma_c\hat{\beta}_2 - \sigma_c^{-3}T_2^*(0)/H'(0)|$$

on $\{\Delta_{n2}(a) \le \varepsilon\} \cap G_{n2}(a,b)$. Since

$$P(\{\Delta_{n2}(a) \le \varepsilon\} \cap G_{n2}(a,b)) \ge 1 - \delta$$

for n large,

$$|\sigma_c\hat{\beta}_2 - \sigma_c^{-3}T_2^*(0)/H'(0)| \overset{P_0}{\to} 0 . \qquad \square$$

The asymptotic relationship between $\hat{\beta}_2$ and the Wilcoxon-type estimator $\hat{\beta}_W$ is established in the following

Corollary 4.1. Under assumptions (1.1) - (1.5) $\sigma_c|\hat{\beta}_2 - \hat{\beta}_W| \overset{P_0}{\to} 0.$

Proof. An immediate consequence of the asymptotic uniform linearity (in Δ) of W^* is

$$|\sqrt{12}\ \sigma_c^{-3}\ W_2^* - \sigma_c\hat{\beta}_W| \overset{P_0}{\to} 0 .$$

Since lemma 4.2 and theorem 4.1 yield

$$\sqrt{12}\ \sigma_c^{-3}|T_2^*(0) - W_2^*| \overset{P_0}{\to} 0$$

and

$$|\sigma_c\hat{\beta}_2 - \sigma_c^{-3}\ T_2^*(0)/H'(0)| \overset{P_0}{\to} 0,$$

we have

$$\sigma_c|\hat{\beta}_2 - \hat{\beta}_W| \overset{P_0}{\to} 0 . \qquad \square$$

CHAPTER 3

KOLMOGOROV-SMIRNOV TYPE ESTIMATION OF β

1. Notation and Preliminaries. In chapter 3 we retain the notation of previous sections. To the assumptions of the model introduced in section 1 of the introduction we add the following:

(1.1) F has a continuous bounded density f satisfying

$f(x) > 0$ a.e. on $\{x: 0 < F(x) < 1\}$.

(1.2) $$\lim_{n\to\infty} \sigma_c^{-1} \max_{1\le i\le n} |d_i| = 0 .$$

In what follows let the vectors $\underset{\sim}{c}$ and $\underset{\sim}{X}$ be given. For Δ real define

(1.3) $$D_c(\Delta) = \sup_{-\infty<x<\infty} |U(x,\Delta)|.$$

Remark. We now verify that

(1.4) $$D_c(\Delta) = \sup_{0<t<1} |S(t,\Delta)| \qquad \forall \ \Delta \in R,$$

a result which will be useful in establishing the asymptotic distribution of our estimator.

For each $1 \le j \le n-1$ we have

$$U(x,\Delta) = \sum_{i=1}^{j} d_{D_{ni\Delta}} , \quad X_{(j)}(\Delta) \le x < X_{(j+1)}(\Delta)$$

and

$$S(t,\Delta) = \sum_{i=1}^{j} d_{D_{ni}} \quad , \quad j/n \leq t < (j+1)/n.$$

Since we also have

$$U(x,\Delta) = 0 \ , \ x \notin [X_{(1)}(\Delta), \ X_{(n)}(\Delta))$$

and

$$S(t,\Delta) = 0 \ , \ x \in (0, \tfrac{1}{n}) \ ,$$

(1.3) is established.

To aid in investigating the properties of D_c as a function of Δ we define, for $\Delta \in R$,

$$D_c^+(\Delta) = \sup_{-\infty<x<\infty} [U(x,\Delta)]$$

and

$$D_c^-(\Delta) = - \inf_{-\infty<x<\infty} [U(x,\Delta)].$$

<u>Lemma 1.1</u>. D_c^+ (D_c^-) is a left-continuous non-decreasing (right-continuous non-increasing) step function in Δ whose points of discontinuity are a subset of

$$\Gamma_3 := \{(X_j - X_i)/(d_j - d_i);\ d_j \geq 0,\ d_i < 0,\ 1 \leq i < j \leq n\}.$$

<u>Proof</u>. We consider only D_c^+; the proof for D_c^- is similar. That D_c^+ is a step function follows from the fact that $D_c^+(\Delta)$ is a function of the ranks of $\{X_i(\Delta),\ 1 \leq i \leq n\}$. To establish its non-decreasing nature we make use of (1.3). Let $\Delta_1 \leq \Delta_2 \leq \cdots \leq \Delta_m$ denote the ordered members of $\{(X_j - X_i)/(d_j - d_i);\ 1 \leq i < j \leq n,\ d_i \neq d_j\}$ and set $\Delta_0 = -\infty$, $\Delta_{m+1} = +\infty$. For $1 \leq j \leq m$ choose any

Δ', Δ'' such that

$$\Delta_{j-1} < \Delta' < \Delta_j < \Delta'' < \Delta_{j+1} .$$

Now $\Delta_j = (X_\ell - X_k)/(d_\ell - d_k)$ for some $k < \ell$ such that $d_k < d_\ell$. Hence

$$X_k(\Delta') < X_\ell(\Delta')$$

$$X_\ell(\Delta'') < X_k(\Delta'')$$

$$X_\ell(\Delta_j) = X_k(\Delta_j) .$$

For Δ real let $x_0(\Delta) = (X_k(\Delta) + X_\ell(\Delta))/2$. Then

$$(1.5) \quad D_c^+(\Delta') = \max\{ \sup_{x<X_k(\Delta')} U(x,\Delta'), U(x_0(\Delta'),\Delta'), \sup_{x>X_\ell(\Delta'')} U(x,\Delta'')\}$$

and

$$(1.6) \quad D_c^+(\Delta'') = \max\{ \sup_{x<X_\ell(\Delta'')} U(x,\Delta''), U(x_0(\Delta''),\Delta''), \sup_{x>X_k(\Delta'')} U(x,\Delta'')\} .$$

Note that as $\Delta \in (\Delta_{j-1}, \Delta_{j+1})$ crosses Δ_j, only the residuals $X_k(\Delta)$ and $X_\ell(\Delta)$ cross. The other residuals remain distinct and in their same relative order with probability one. Hence the following are valid:

$$(1.7) \quad \sup_{x<X_k(\Delta')} U(x,\Delta') = \sup_{x<X_\ell(\Delta'')} U(x,\Delta'') = \sup_{x<X_\ell(\Delta_j)=X_k(\Delta_j)} U(x,\Delta_j)$$

$$(1.8) \quad \sup_{x>X_\ell(\Delta')} U(x,\Delta') = \sup_{x>X_k(\Delta'')} U(x,\Delta'') = \sup_{x>X_k(\Delta_j)=X_\ell(\Delta_j)} U(x,\Delta_j)$$

$$(1.9) \quad U(x_0(\Delta'),\Delta') = U(x_0(\Delta_j),\Delta_j) - d_\ell$$

$$(1.10) \qquad U(x_0(\Delta''),\Delta'') = U(x_0(\Delta_j),\Delta_j) - d_k .$$

We complete the proof by considering three cases:

Case I. If $d_k < d_\ell \leq 0$ then by (1.9) and (1.10)

$$\begin{aligned}\sup_{x<X_k(\Delta')} U(x,\Delta') &\geq U(X_k(\Delta')^-,\Delta') \\ &= U(x_0(\Delta_j),\Delta_j) - d_k - d_\ell \\ &\geq \max\{U(x_0(\Delta_j),\Delta_j) - d_\ell , U(x_0(\Delta_j),\Delta_j) - d_k\} \\ &= \max\{U(x_0(\Delta'),\Delta'), U(x_0(\Delta''),\Delta'')\}.\end{aligned}$$

Thus

$$D_c^+(\Delta') = \max\{\sup_{x<X_k(\Delta')} U(x,\Delta'), \sup_{x>X_\ell(\Delta')} U(x,\Delta')\} = D_c^+(\Delta'')$$

follows from (1.5) - (1.8).

Case II. If $0 \leq d_k < d_\ell$ then

$$\begin{aligned}\sup_{x>X_\ell(\Delta')} U(x,\Delta') &\geq U(X_\ell(\Delta')^+,\Delta') = U(x_0(\Delta_j),\Delta_j) \\ &\geq \max\{U(x_0(\Delta_j),\Delta_j) - d_k, U(x_0(\Delta_j),\Delta_j) - d_\ell\} \\ &= \max\{U(x_0(\Delta'),\Delta'), U(x_0(\Delta''),\Delta'')\}\end{aligned}$$

by (1.9) and (1.10). Thus

$$D_c^+(\Delta') = \max\{\sup_{x<X_k(\Delta')} U(x,\Delta'), \sup_{x>X_\ell(\Delta')} U(x,\Delta')\} = D_c^+(\Delta'')$$

follows from (1.5) - (1.8).

Case III. If $d_k \leq 0 \leq d_\ell$, $d_k < d_\ell$, then by (1.9)

$$\sup_{x>X_\ell(\Delta')} U(x,\Delta') \geq U(X_\ell(\Delta')^+,\Delta') = U(x_0(\Delta_j),\Delta_j)$$
$$\geq U(x_0(\Delta_j),\Delta_j) - d_\ell$$
$$= U(x_0(\Delta'),\Delta').$$

Thus

$$D_c^+(\Delta') = \max\{ \sup_{x<X_k(\Delta')} U(x,\Delta'), \sup_{x>X_\ell(\Delta')} U(x,\Delta')$$
$$\leq \max\{ \sup_{x<X_k(\Delta')} U(x,\Delta'), \sup_{x>X_\ell(\Delta')} U(x,\Delta'), U(x_0(\Delta''),\Delta'')\}$$
$$= D_c^+(\Delta'')$$

follows from (1.5) - (1.8).

To establish left continuity, first note that

$$D_c^+(\Delta_j) = \max\{ \sup_{x<X_\ell(\Delta_j)} U(x,\Delta_j), U(x_0(\Delta_j),\Delta_j), \sup_{x>X_k(\Delta_j)} U(x,\Delta_j)\}.$$

Applying (1.7), (1.8) and

$$\sup_{x>X_k(\Delta_j)} U(x,\Delta_j) \geq U(X_k(\Delta_j)^+,\Delta_j) = U(x_0(\Delta_j),\Delta_j),$$

we have

$$D_c^+(\Delta_j) = \max\{ \sup_{x<X_\ell(\Delta')} U(x,\Delta'), \sup_{x>X_k(\Delta')} U(x,\Delta')\} = D_c^+(\Delta).$$

Finally, since $D_c^+(\Delta') = D_c^+(\Delta'')$ in cases I and II, the points of discontinuity of D_c^+ are seen to be a subset of Γ_3. □

Before defining $\hat{\beta}_3$ we need one additional

Lemma 1.2. $D_C^+(\Delta_1^-) = 0 = D_c^-(\Delta_m^+)$ and $D_c^-(\Delta_1^-) = \sum_{i=1}^n d_i^+ = D_c^+(\Delta_m^+)$.

<u>Proof</u>. Note that for $\Delta < \Delta_1$, $1 \le i < j \le n$ and $d_i < d_j$ imply $X_i(\Delta) < X_j(\Delta)$, so that the $\{X_k(\Delta), 1 \le k \le n\}$ are naturally ordered with respect to $d_1 \le d_2 \le \dots \le d_n$. Using the monotonicity of the d_i's and $\sum_{i=1}^n d_i = 0$ we have, for $1 \le k \le n-1$,

$$U(x,\Delta) = \sum_{i=1}^{k} d_i \le 0, \quad x \in [X_k(\Delta), X_{k+1}(\Delta)).$$

Since

$$U(x,\Delta) = 0 \ , \ x \notin [X_{(1)}(\Delta), X_{(n)}(\Delta))$$

we have

$$\sup_{-\infty<x<\infty} [U(x,\Delta)] = 0$$

and

$$-\inf_{-\infty<x<\infty} [U(x,\Delta)] = \sum_{i=1}^{n} d_i^- = \sum_{i=1}^{n} d_i^+ .$$

Because D_c^+ and D_c^- are constant for $\Delta < \Delta_1$ it follows that

$$D_c^+(\Delta_1^-) = D_c^+(\Delta) = 0$$

and

$$D_c^-(\Delta_1^-) = D_c^-(\Delta) = \sum_{i=1}^{n} d_i^+ .$$

The proof that $D_c^+(\Delta_m^+) = \sum_{i=1}^{n} d_i^+$ and $D_c^-(\Delta_m^+) = 0$ is similar and uses the fact that the residuals $\{X_k(\Delta), 1 \le k \le n\}$ are in a reversed natural ordering with respect to the d_i's when $\Delta > \Delta_m$. □

Lemmas 1.1 and 1.2 guarantee that the following exist and are finite:

$$\beta_3^{*} := \inf\{\Delta \in R;\ D_c^{+}(\Delta) \geq D_c^{-}(\Delta)\},$$

$$\beta_3^{**} := \sup\{\Delta \in R;\ D_c^{-}(\Delta) \geq D_c^{+}(\Delta)\}.$$

Note that by the monotonic nature of $D_c^{+}(\Delta)$ and $D_c^{-}(\Delta)$, $\beta_3^{**} \geq \beta_3^{*}$ w.p. 1.

We are now ready to define the estimator

$$\hat{\beta}_3 = \frac{1}{2}(\beta_3^{*} + \beta_3^{**}) .$$

<u>Lemma 1.3</u>. D_c is nondecreasing for $\Delta \geq \hat{\beta}_3$ and nonincreasing for $\Delta \leq \hat{\beta}_3$.

<u>Proof</u>. Note that

$$(1.11) \qquad D_c(\Delta) = \max(D_c^{+}(\Delta),\ D_c^{-}(\Delta)) \quad \forall \quad \Delta \in R .$$

By the definition of β_3^{*}, $\Delta > \beta_3^{*}$ implies $D_c^{+}(\Delta) \geq D_c^{-}(\Delta)$ and hence $D_c(\Delta) = D_c^{+}(\Delta)$. By the definition of β_3^{**}, $\Delta < \beta_3^{**}$ implies $D_c^{-}(\Delta) \geq D_c^{+}(\Delta)$ and hence $D_c(\Delta) = D_c^{-}(\Delta)$. Thus, since $\beta_3^{*} \leq \hat{\beta}_3 \leq \beta_3^{**}$, $D_c(\Delta) = D_c^{+}(\Delta)$ for $\Delta > \hat{\beta}_3$ and $D_c(\Delta) = D_c^{-}(\Delta)$ for $\Delta < \hat{\beta}_3$ and it remains to show that

$$(1.12) \qquad D_c(\hat{\beta}_3) \leq \min(D_c(\hat{\beta}_3^{-}),\ D_c(\hat{\beta}_3^{+})) .$$

But by lemma 1.1,

$$D_c^{+}(\hat{\beta}_3) = D_c^{+}(\hat{\beta}_3^{-}) \leq D_c^{+}(\hat{\beta}_3^{+})$$

and

$$D_c^{-}(\hat{\beta}_3) = D_c^{-}(\hat{\beta}_3^{+}) \leq D_c^{-}(\hat{\beta}_3^{-}) .$$

Therefore,

$$(1.13)\qquad \begin{aligned} D_c(\hat{\beta}_3) &= \max(D_c^+(\hat{\beta}_3),\ D_c^-(\hat{\beta}_3)) \\ &\le \max(D_c^+(\hat{\beta}_3^-),\ D_c^-(\hat{\beta}_3^-)) \\ &= D_c(\hat{\beta}_3^-) \end{aligned}$$

and

$$(1.14)\qquad \begin{aligned} D_c(\hat{\beta}_3) &= \max(D_c^+(\hat{\beta}_3),\ D_c^-(\hat{\beta}_3)) \\ &\le \max(D_c^+(\hat{\beta}_3^+),\ D_c^-(\hat{\beta}_3^+)) \\ &= D_c(\hat{\beta}_3^+)\ . \end{aligned}$$

Combining (1.13) and (1.14) establishes (1.12), completing the proof of the lemma. □

Remark. Lemma 1.3 shows that

$$D_c(\hat{\beta}_3) = \inf_{-\infty<\Delta<\infty} D_c(\Delta)\ .$$

Numerical Example

According to its definition, $\hat{\beta}_3$ can be determined by identifying the set Γ_3 and computing $D_c^+(\Delta^-)$ and $D_c^-(\Delta^-)$ for each $\Delta \in \Gamma_3$. As an illustration we take $c_i = i,\ 1 \le i \le 4$ and consider the sample

$$1,\ 5,\ 3,\ 6\ .$$

Simple computations yield:

TABLE I

Values of $D_c(\Delta^-)$ and $D_c^+(\Delta^-)$ for $\Delta \in \Gamma_3$

$\Delta \in \Gamma_3$	$D_c^-(\Delta^-)$	$D_c^+(\Delta^-)$
-2	2	0
.5	1.5	0
1	1.5	.5
5/3	1	.5

Here $\beta^* = 5/3 = \beta^{**}$ so that $\hat{\beta}_3 = 5/3$.

2. <u>Finite Sample Properties</u>

(a) <u>Invariance</u>. A useful property of $\hat{\beta}_3$ is its translation invariance; that is,

$$(2.1) \qquad \hat{\beta}_3(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \gamma + \hat{\beta}_3(\underset{\sim}{X}) \qquad \forall \ \gamma \in R \ .$$

To verify (2.1) we note that

$$\begin{aligned} D_c^{\pm}(\Delta)(\underset{\sim}{X} + \gamma\underset{\sim}{c}) &= \sup_{0<t<1} [\pm S(t,\Delta)(\underset{\sim}{X} + \gamma\underset{\sim}{c})] \\ &= \sup_{0<t<1} [\pm S(t,\Delta-\gamma)(\underset{\sim}{X})] \\ &= D_c^{\pm}(\Delta-\gamma)(\underset{\sim}{X}) \ . \end{aligned}$$

Thus,

$$\begin{aligned} \beta_3^*(\underset{\sim}{X} + \gamma\underset{\sim}{c}) &= \inf\{\Delta;\ D_c^+(\Delta-\gamma)(\underset{\sim}{X}) \geq D_c^-(\Delta-\gamma)(\underset{\sim}{X})\} \\ &= \gamma + \inf\{\Delta;\ D_c^+(\Delta)(\underset{\sim}{X}) \geq D_c^-(\Delta)(\underset{\sim}{X})\} \\ &= \gamma + \beta_3^*(\underset{\sim}{X}) \qquad \forall \ \gamma \in R. \end{aligned}$$

Similarly,

$$\beta_3^{**}(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \gamma + \beta_3^{**}(\underset{\sim}{X}) \qquad \forall\ \gamma \in R$$

so that

$$\hat{\beta}_3(\underset{\sim}{X} + \gamma\underset{\sim}{c}) = \gamma + \hat{\beta}_3(\underset{\sim}{X}) \qquad \forall\ \gamma \in R\ .$$

(b) Symmetry. We assume, without loss of generality, that $\beta = 0$.

Theorem 2.1. $\hat{\beta}_3$ is symmetric about β if one of the following conditions holds:

(i) F is symmetric.

(ii) $d_i = -d_{n-i+1}$, $1 \leq i \leq n$.

Proof of (i). $\hat{\beta}_3(\underset{\sim}{X}) \sim \hat{\beta}_3(-\underset{\sim}{X})$ follows from a proof similar to that of theorem 1.2.1. Using the definitions of D_c^+ and D_c^- one obtains

$$D_c^+(-\Delta)(\underset{\sim}{X}) = D_c^-(\Delta)(-\underset{\sim}{X})$$

and

$$D_c^-(-\Delta)(\underset{\sim}{X}) = D_c^+(\Delta)(-\underset{\sim}{X})\ .$$

Thus the definitions of β_3^* and β_3^{**} yield

$$\beta_3^*(-\underset{\sim}{X}) = -\beta_3^*(\underset{\sim}{X})$$

and

$$\beta_3^{**}(-\underset{\sim}{X}) = -\beta_3^{**}(\underset{\sim}{X})\ .$$

Therefore

$$\hat{\beta}_3(\underset{\sim}{X}) \sim \hat{\beta}_3(-\underset{\sim}{X}) = -\hat{\beta}_3(\underset{\sim}{X}),$$

completing the proof of (i).

Proof of (ii). As in the proof of theorem 1.2.1, $\hat{\beta}_2(\underset{\sim}{X}) \sim \hat{\beta}_2(\underset{\sim}{X}^*)$ where $\underset{\sim}{X}^* = (X_n, X_{n-1}, \ldots, X_1)$. Using the definitions of D_c^+ and D_c^- and the proof of (i) one obtains

$$D_c^+(\Delta)(\underset{\sim}{X}^*) = D_c^-(-\Delta)(\underset{\sim}{X}) = D_c^+(\Delta)(-\underset{\sim}{X})$$

and

$$D_c^-(\Delta)(\underset{\sim}{X}^*) = D_c^+(-\Delta)(\underset{\sim}{X}) = D_c^-(\Delta)(-\underset{\sim}{X}) \ .$$

Thus

$$\hat{\beta}_3(\underset{\sim}{X}) \sim \hat{\beta}_3(\underset{\sim}{X}^*) = \hat{\beta}_3(-\underset{\sim}{X}) \sim -\hat{\beta}_3(\underset{\sim}{X})$$

as in the proof of (i). $\square$

3. Asymptotic Distribution of $\sigma_c \hat{\beta}_3$. In this section we assume, without loss of generality, that $\beta = 0$.

To aid in the proof of theorem 3.1 we first define a class of functionals $\{T_z, z \in R\}$ on the set Ψ of bounded functions on $[0,1]$ by

$$T_z(h) = \sup_{0<t<1} \{[h(t) + zf(F^{-1}(t))] \vee 0\}$$

$$+ \inf_{0<t<1} \{[h(t) + zf(F^{-1}(t))] \wedge 0\} \quad \forall \ h \in \Psi, z \in R \ .$$

For $h, g \in \Psi$ and $z \in R$ we have

$$|[h(t) + zf(F^{-1}(t))] \vee 0 - [g(t) + zf(F^{-1}(t))] \vee 0|$$

$$\leq |h(t) - g(t)| \leq \|h - g\|_\infty \quad \forall \ t \in [0,1]$$

and

$$|[h(t) + zf(F^{-1}(t))] \wedge 0 - [g(t) + zf(F^{-1}(t))] \wedge 0|$$

$$\leq |h(t) - g(t)| \leq \|h - g\|_\infty \quad \forall \quad t \in [0,1].$$

Thus

$$|T_z(h) - T_z(g)| \leq 2\|h - g\|_\infty,$$

establishing that T_z is a continuous functional on $C[0,1]$.

Remark. Let $\{B(t), 0 \leq t \leq 1\}$ denote a Brownian bridge. Then since $B(0+) = S(0+,0) = f(F^{-1}(0+)) = 0$ w.p. 1, we have (w.p.1)

$$T_z(B) = \sup_{0<t<1} \{B(t) + zf(F^{-1}(t))\}$$

$$+ \inf_{0<t<1} \{B(t) + zf(F^{-1}(t))\}$$

and

$$T_z(\sigma_c^{-1}S(\cdot,0)) = \sup_{0<t<1} \{\sigma_c^{-1}S(t,0) + zf(F^{-1}(t))\}$$

$$+ \inf_{0<t<1} \{\sigma_c^{-1}S(t,0) + zf(F^{-1}(t))\} .$$

We are now ready to state

Theorem 3.1. Under conditions (1.1) and (1.2),

$$P_0[\sigma_c\hat{\beta}_3 \leq z] \to P_0[T_z(B) \geq 0] \quad \forall \quad z \in R .$$

Proof. We assume that $z > 0$; the proof for $z \leq 0$ is similar. From the definitions of β_3^* and β_3^{**},

$$\beta_3^{**} < z/\sigma_c \Rightarrow D_c^+(z/\sigma_c) > D_c^-(z/\sigma_c)$$

and

$$D_c^+(z/\sigma_c) \geq D_c^-(z/\sigma_c) \Rightarrow \beta_3^* \leq z/\sigma_c .$$

Thus $\beta_3^* \leq \hat{\beta}_3 \leq \beta_3^{**}$ implies that

$$P_0[D_c^+(z/\sigma_c) > D_c^-(z/\sigma_c)] \leq P_0[\hat{\beta}_3 \leq z/\sigma_c] \leq P_0[D_c^+(z/\sigma_c) \geq D_c^-(z/\sigma_c)].$$

Applying theorem 1.3.1 and using the inequalities

$$\sigma_c^{-1} D_c^+(z/\sigma_c) \geq 0,$$

$$\sup_{0<t<1} \{\sigma_c^{-1} S(t,0) + zf(F^{-1}(t))\} \geq 0,$$

we obtain

$$\begin{aligned} &|\sigma_c^{-1} D_c^+(z/\sigma_c) - \sup_{0<t<1} \{\sigma_c^{-1} S(t,0) + zf(F^{-1}(t))\}| \\ &\quad \leq \sup_{0<t<1} |\sigma_c^{-1}[S(t,z/\sigma_c) - S(t,0)] - zf(F^{-1}(t))| \xrightarrow{P_0} 0 . \end{aligned}$$

Similarly,

$$|\sigma_c^{-1} D_c^-(z/\sigma_c) + \inf_{0<t<1} \{\sigma_c^{-1} S(t,0) + zf(F^{-1}(t))\}| \xrightarrow{P_0} 0 .$$

Hence

$$(3.1) \qquad |D_c^+(z/\sigma_c) - D_c^-(z/\sigma_c) - T_z(\sigma_c^{-1} S(t,0))| \xrightarrow{P_0} 0 .$$

We now show that

$$(3.2) \qquad L_0(T_z(\sigma_c^{-1} S(\cdot,0))) \Rightarrow L_0(T_z(B)) .$$

Recall from lemma 1.4.1 the process $\{Z_n(t),\ 0 \leq t \leq 1\}$ with continuous sample paths which converges in distribution in $(B, C[0,1])$ to $\{B(t),\ 0 \leq t \leq 1\}$. Since

$$\|Z_n(\cdot) + \sigma_c^{-1}S(\cdot,0)\|_\infty \leq \sigma_c^{-1} \max_{1\leq i\leq n} |d_i| ,$$

the continuity of T_z yields

$$|T_z(\sigma_c^{-1}S(\cdot,0)) - T_z(-Z_n(\cdot))| \xrightarrow{P_0} 0 .$$

Since

$$L_0(T(-Z_n(\cdot))) \Rightarrow L_0(T(-B)) \sim L_0(T(B)),$$

(3.2) is proved. Combining (3.1) and (3.2) gives

$$P_0[D_c^+(z/\sigma_c) \geq D_c^-(z/\sigma_c)] \to P_0[T_z(B) \geq 0]$$

and

$$P_0[D_c^+(z/\sigma_c) > D_c^-(z/\sigma_c)] \to P_0[T_z(B) > 0].$$

But $P_0\ T_z(B) = 0 = 0$ by lemma 1 of Rao et al. (1975) and the proof is completed. □

4. Interval Estimation of β. Throughout this section we assume, without loss of generality, that $\beta = 0$ and that (1.1) and (1.2) hold.

Let $\gamma_{c,\alpha}$ denote the critical value for which one rejects H_0: $\beta = 0$ at level α whenever $D_c(0) > \gamma_{c,\alpha}$. Then a $100(1-\alpha)\%$ confidence set for β is given by

$$I_{c,\alpha} := \{\Delta;\ D_c(\Delta) \leq \gamma_{c,\alpha}\} .$$

Lemma 1.1 and (1.1) imply that $I_{c,\alpha}$, when nonempty, must be an interval. Note that

$$I_{c,\alpha} = \phi \Rightarrow D_c(0) > \gamma_{c,\alpha};$$

hence empty intervals are obtained only on an event where one rejects the true null hypothesis.

It has been shown (Hájek and Šidák, 1967, p. 189) that

$$L_0(\sigma_c^{-1} D_c(0)) = L_0(\sup_{0 \le t \le 1} |B(t)|).$$

Defining K_α to be the $1-\alpha$ percentile of $L_0(\sup_{0 \le t \le 1} |B(t)|)$ we then have

$$\lim_{n\to\infty} \sigma_c^{-1} \gamma_{c,\alpha} = K_\alpha .$$

__Lemma 4.1__. Given $0 < B < \infty$ and $\varepsilon > 0$ there exist b and N positive such that $n \geq N$ implies

$$P_0[\inf_{|z| \ge b} \sigma_c^{-1} D_c(z/\sigma_c) > B] > 1 - \varepsilon.$$

__Proof__. Using lemma 1.3, the proof is similar to that of lemma 1.3.3. $\square$

Define $K_\alpha^* = \sup_{n \ge 1} \gamma_{c,\alpha}$ and let μ denote Lebesgue measure on R. We are now ready to prove

__Theorem 4.1__. Suppose that (1.1) and (1.2) hold. Then for each $\eta > 0$, $\limsup_n \sigma_c\, \mu(I_{c,\alpha})$ is bounded in probability by $2K_\alpha \|f\|_\infty^{-1} + \eta$.

__Proof__. Let $\varepsilon > 0$, $\|f\|_\infty > \delta > 0$. By lemma 4.1 there exist b and N positive such that $n \geq N$ implies

$$P_0[\inf_{|\Delta| \ge b} D(z/\sigma_c) > K_\alpha^*] > 1 - \varepsilon/2 .$$

Choose $t_0 \in (0,1)$ such that

$$f(F^{-1}(t_0)) \geq \|f\|_\infty - \delta$$

and $N_1 \geq N$ such that

$$P_0[\sup_{|\Delta|\geq b} \sigma_c^{-1}|S(t_0,\Delta/\sigma_c) - S(t_0,0) - \Delta\sigma_c f(F^{-1}(t_0))| \leq \delta] > 1 - \varepsilon/2 \quad \forall\, n \geq N_1.$$

Then $n \geq N_1$ implies

$$\begin{aligned}
\sigma_c I_{c,\alpha} &\subseteq \{\Delta;\ |S(t_0,\Delta/\sigma_c)| \geq \gamma_{c,\alpha}\} \cap \{\Delta;\ |\Delta| \geq b\} \\
&\subseteq \{\Delta;\ |S(t_0,0) - \Delta\sigma_c f(F^{-1}(t_0))| \leq \gamma_{c,\alpha} + \sigma_c\delta\} \cap \{\Delta;\ |\Delta| \leq b\} \\
&\subseteq \{\Delta;\ |S(t_0,0) - \Delta\sigma_c f(F^{-1}(t_0))| \leq \gamma_{c,\alpha} + \sigma_c\delta\} \\
&= \{\Delta;\ -(\gamma_{c,\alpha} + \sigma_c\delta + S(t_0,0))/\sigma_c f(F^{-1}(t_0)) \leq \Delta \\
&\qquad \leq (\gamma_{c,\alpha} + \sigma_c\delta - S(t_0,0))/\sigma_c f(F^{-1}(t_0))\}
\end{aligned}$$

with probability greater than $1 - \varepsilon$. Hence

$$P_0[\limsup_n \sigma_c\mu(I_{c,\alpha}) \leq 2(\sigma_c^{-1}\gamma_{c,\alpha} + \delta)/(\|f\|_\infty - \delta)] > 1 - \varepsilon.$$

Since δ and ε were arbitrary and $\lim_{n\to\infty} \sigma_c^{-1}\, \gamma_{c,\alpha} = K_\alpha$, the proof is completed. $\square$

It is possible, under more restrictive conditions, to show that the bounds of theorem 4.1 hold w.p.1. Such a result is given in the following

Theorem 4.2. Assume, in addition to (1.1), that $f'(x)$ exists and is bounded for a.a. x. Regarding $\underset{\sim}{c}$, assume that

(4.1) $$n^{\frac{1}{2}}\sigma_c^{-1} \max_{1\le i\le n} |d_i| = O(1) \quad \text{as} \quad n \to \infty,$$

(4.2) $$\liminf_n n^{-1}\sigma_c^2 > 0 \ .$$

Then

$$\limsup_n \sigma_c \mu(I_{c,\alpha}) \le 2K_\alpha \|f\|_\infty^{-1} \quad \text{w.p. 1.}$$

Proof. Let $\varepsilon > 0$ and $0 < b < \infty$. Define, for $t \in (0,1)$ and $x, \Delta \in R$,

$$U^*(t,\Delta) = \sum_{i=1}^n d_i I[F(X_i) \le F(F^{-1}(t) + \Delta d_i)],$$

$$U'(t,\Delta) = \sum_{i=1}^n d_i I(X_i \le x + \Delta d_i).$$

From theorem 3.1 of Ghosh and Sen (1972),

$$\sup_{\substack{0<t<1 \\ |\Delta|\le b}} |U^*(t,\Delta/\sigma_c) - U^*(t,0) - \Delta\sigma_c f(F^{-1}(t))| \to 0 \quad \text{w.p. 1} \quad (P_0) \ .$$

The theorem is proved by modifying the proofs of lemma 4.1 and theorem 4.1 once we show that for $|\Delta| \le b$ and n sufficiently large,

(4.3) $$\sup_{0<t<1} |U^*(t,\Delta/\sigma_c)| = \sup_{x\in R} |U'(x,\Delta/\sigma_c)| = D(\Delta/\sigma_c) \ .$$

Let $S(F)$ denote the support of F. By (1.1) $S(F)$ is a real interval. In case $S(F) = R$, F^{-1} is a bijection and hence

$$\sup_{0<t<1} |U^*(t,\Delta/\sigma_c)| = \sup_{0<t<1} |U'(F^{-1}(t),\Delta/\sigma_c)|$$

$$= \sup_{x\in R} |U'(x,\Delta/\sigma_c)|$$

$$= D(\Delta/\sigma_c),\ |\Delta| \le b,$$

establishing (4.3). In case $S(F) \neq R$,

$$\sup_{0<t<1} |U^*(t,\Delta/\sigma_c)| = \sup_{0<F(x)<1} |U'(x,\Delta/\sigma_c)|$$

and we must prove that

$$\sup_{0<F(x)<1} |U'(x,\Delta/\sigma_c)| = \sup_{x\in R} |U'(x,\Delta/\sigma_c)| \tag{4.4}$$

for n sufficiently large and $|\Delta| \le b$. To this end choose $0 < N_1 < \infty$ such that

$$b\sigma_c^{-1} \max_{1\le i\le n} |d_i| < \mu(S(F)) \quad \forall \quad n \ge N_1 .$$

Then for $n \ge N_1$ and $x \le x_0 = \inf S(F)$,

$$0 \le U'(x,\Delta/\sigma_c) \le \sum_{i=1}^{n} d_i^+ = U'(x_0^+,\Delta/\sigma_c) \ , \quad \Delta > 0,$$

$$U'(x,\Delta/\sigma_c) = 0 = U'(x_0^+,\Delta/\sigma_c) \qquad , \quad \Delta = 0,$$

and

$$|U'(x,\Delta/\sigma_c)| \le \sum_{i=1}^{n} d_i^- = |U'(x_0^+,\Delta/\sigma_c)| \ , \quad \Delta < 0$$

imply

$$|U'(x,\Delta/\sigma_c)| \le |U'(x_0^+,\Delta/\sigma_c)|. \tag{4.5}$$

Similarly, for $n \ge N_1$ and $x \ge x_1 = \sup S(F)$,

$$(4.6) \qquad |U'(x,\Delta/\sigma_c)| \leq |U'(x_1^-,\Delta/\sigma_c)| \ .$$

Combining (4.5) and (4.6) yields (4.4) and hence (4.3). □

5. Asymptotic Efficiency of the $I_{c,\alpha}$. Using the bounds of section 4 one can compare the $I_{c,\alpha}$ to other common confidence intervals for which asymptotic lower bounds can be computed (Rao et al., 1975). Koul (1971) has computed the asymptotic lengths of the normalized confidence intervals based on a wide class of linear rank statistics. Although his bounds were in probability bounds, they can be strengthened to w.p.1 bounds by applying the results of Ghosh and Sen (1972). As an example of the type of results which can be obtained and to demonstrate the efficiency of the Kolmogorov-Smirnov type intervals, we compute bounds for the asymptotic efficiency of the $I_{c,\alpha}$ with respect to confidence intervals based on the Normal scores and Wilcoxon type rank statistics.

In what follows let Φ denote the standard normal c.d.f., let z_α be defined by $\Phi(z_\alpha) = 1 - \alpha$ and define

$$\varphi(t,f) = f'(F^{-1}(t))/f(F^{-1}(t)), \quad 0 < t < 1.$$

Assume, without loss of generality, that $\beta = 0$.

(a) Comparison with Wilcoxon-type intervals. Let

$$K_{c,\alpha} = \{\Delta; \ |n^{-1} \sum_{i=1}^{n} d_i R_{ni\Delta}| \leq \delta_{c,\alpha}\}$$

where $\delta_{c,\alpha}$ is such that one rejects H_0: $\beta = 0$ at level α whenever $|n^{-1} \sum_{i=1}^{n} d_i R_{ni\Delta}| > \delta_{c,\alpha}$ and accepts H_0 otherwise. Under the assumptions of theorem 4.2

$$\lim_{n\to\infty} \sigma_c \mu(K_{c,\alpha}) = z_{\alpha/2}/\sqrt{3} \int f^2(x)dx \qquad \text{w.p.1.}$$

Thus,

$$\text{(5.1)} \quad \limsup_{n\to\infty} \frac{\mu(I_{c,\alpha})}{\mu(K_{c,\alpha})} \leq \Psi_\alpha(F) := \sqrt{12}\, K_\alpha \int f^2(x)dx/z_{\alpha/2}\|f\|_\infty \quad \text{w.p.1.}$$

To obtain an upper bound on

$$\limsup_{\alpha\to 0} \Psi_\alpha(F)$$

for fixed F, we investigate the behavior of

$$\limsup_{\alpha\to 0} K_{2\alpha}/z_\alpha .$$

From Hájek and Šidák (1967, p. 182) we obtain

$$1 - \alpha = P[\sup_{0<t<1} |B(t)| \leq K_\alpha] \geq 1 - 2\exp(-2K_\alpha^2)$$

and hence

$$-\ln(\alpha/2) \geq K_\alpha^2 .$$

Thus

$$\limsup_{\alpha\to 0} K_{2\alpha}/z_\alpha \leq \limsup_{\alpha\to 0} -[2\Phi^{-1}(\alpha)]^{-2}\ln(1-\alpha)$$

$$= \limsup_{x\to\infty} -2x^{-2}\ln(1-\Phi(x)).$$

Using

$$(x^{-1} - x^{-3})\varphi(x) \leq 1 - \Phi(x) \leq x^{-1}\varphi(x),$$

where φ denotes the standard normal density, one obtains

$$\limsup_{x\to\infty} -2x^{-2}\ln(1-\Phi(x)) = .25 \ .$$

Therefore,

$$\limsup_{\alpha\to 0} \Psi_\alpha(F) \le \sqrt{3} \int f^2(x)dx/2\|f\|_\infty \ . \tag{5.2}$$

The following table gives the upper bound $\Psi_\alpha(F)$ for various choices of F and α. Here $\Psi_0(F) :=$ RHS (5.2).

TABLE I

Values of $\Psi_\alpha(F)$ for Comparison to Wilcoxon Intervals

$\alpha\backslash F$	Std. Normal	Logistic	Dbl. Exp.	Cauchy
.5	3.005	2.834	2.125	2.125
.1	1.823	1.718	1.289	1.289
.05	1.697	1.600	1.200	1.200
.025	1.618	1.525	1.144	1.144
.01	1.548	1.459	1.094	1.094
.005	1.510	1.424	1.068	1.068
0	.612	.577	.433	.433

(b) <u>Comparison with Normal Scores-type Confidence Intervals</u>. Let

$$J_{c,\alpha} = \{\Delta;\ |\sum_{i=1}^{n} d_i\Phi^{-1}(R_{ni\Delta}/n)| \le \delta_{c,\alpha}\}$$

where $\delta_{c,\alpha}$ is such that one rejects H_0: $\beta = 0$ at level whenever $|\sum_{i=1}^{n} d_i\Phi^{-1}(R_{ni\Delta}/n)| > \delta_{c,\alpha}$ and accepts H_0 otherwise. Under the assumptions of theorem 4.2,

$$\lim_{n\to\infty} \sigma_c\mu(J_{c,\alpha}) = 2z_{\alpha/2}/\int_0^1 \Phi^{-1}(u)\varphi(u,f)du \qquad \text{w.p. } 1 \ .$$

Thus,

$$\limsup_{n\to\infty} \frac{\mu(I_{c,\alpha})}{\mu(J_{c,\alpha})} \leq \Psi_\alpha(F) = K_\alpha \int_0^1 \Phi^{-1}(u)\varphi(u,f)du/z_{\alpha/2}\|f\|_\infty \quad \text{w.p.1.}$$

The following table gives the upper bound $\Psi_\alpha(F)$ for various choices of F and α. For $\alpha = 0$, $\Psi_\alpha(F) := \int_0^1 \Phi^{-1}(u)\varphi(u,f)du/4\|f\|_\infty$.

TABLE II

Values of $\Psi_\alpha(F)$ for Comparison to
Normal Scores Intervals

$\alpha\backslash F$	Std. Normal	Logistic	Dbl. Exp.	Cauchy
.5	3.076	2.769	1.958	1.8
.10	1.865	1.679	1.187	1.1
.05	1.737	1.564	1.106	1.0
.025	1.655	1.490	1.054	.96
.01	1.584	1.426	1.008	.92
.005	1.546	1.392	.984	.90
0	.627	.564	.399	.36

6. Monte Carlo Study. In order to compare $\hat{\beta}_3$ with other point estimators of β, 5000 samples of size 40 were generated from each of the standard normal, double exponential, logistic, and Cauchy (median 0) distributions. Taking $c_i = i$, $1 \leq i \leq 40$, we then computed $\hat{\beta}_1$, $\hat{\beta}_3$, and the Wilcoxon estimate, $\hat{\beta}_W$, for each sample. The following table gives $s^2(\sigma_c\hat{\beta}_.)$ for each set of 5000 samples.

TABLE III

Values of $s^2(\sigma_c\hat{\beta}_.)$

s^2 F	Std. Normal	Logistic	Dbl. Exp.	Cauchy
$s^2(\sigma_c\hat{\beta}_w)$	1.0668	2.9883	1.4541	.3153
$s^2(\sigma_c\hat{\beta}_1)$	1.1755	3.1985	1.4532	.3666
$s^2(\sigma_c\hat{\beta}_3)$	1.1181	3.1138	1.5497	.3386

Each set of observations was based on a corresponding sample of uniform (0,1) variates generated by the Fortran subroutine RANF on the Michigan State University CDC 6500. The logistic and double exponential variates were generated by computing $F^{-1}(U)$ for each uniform variate U, the Cauchy variates were generated by computing $\tan[(U - .5)/\pi]$, and each normal variate was generated by computing $(-2 \ln U_1)^{\frac{1}{2}} \cos(2\pi\ U_2)$ for independent uniform (0,1) variates U_1 and U_2.

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