THE REALIZATION OF ORLICZ SEQUENCE SPACES AND HARMONIC ANALYSIS

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ABSTRACT

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By

James Milford Boyett

In this thesis we consider the inter-relation between the realization problem of L. Schwartz and harmonic analysis for Orlicz sequence spaces. A solution to the realization problem generalizing the work of Mustari on ℓ_p-spaces is presented in Chapter II. Harmonic analysis for such Orlicz sequence spaces is then carried out in Chapter III. The latter work generalizes some work of Kuelbs and Mandrekar. Finally, in Chapter IV an explicit form of the Fourier transform for Gaussian measures on an interesting subclass of these Orlicz sequence spaces is obtained and is exploited to study a Central Limit Theorem for this class. The results of the final chapter include and extend some work of M.N. Vakhania.

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

We consider in this thesis two related problems. The first is the problem due to L. Schwartz of realizations for Orlicz sequence spaces, which in the case of classical ℓ_p-spaces was first studied by Mustari [22]. In a subsequent paper [23], Mustari also studied the question for separable Banach spaces and obtained necessary conditions for realization. His work does not provide sufficient conditions in the case of separable Banach spaces, nor does it apply if the space is an ℓ_p-space $(0 < p < 1)$.

Once the realization problem is settled we show that it provides the extension of methods in [13] to the Orlicz sequence spaces. The Bochner theorem and Lévy continuity theorem proved in [13] can then be generalized rather simply to the case of realizable Orlicz sequence spaces. In essence, we essentially give our proofs as to adapt methods in [13]. Recently, using different methods other than those in [13], J. Kuelbs [12] has studied the generalization of the results of [13]. We note that although his method is general, it is also more complicated than the method of [13], and in case of Orlicz sequence spaces, which is his main application, his results are included in ours. As a matter of fact our realization result shows that the most general Orlicz sequence spaces for which methods in [12] are applicable are precisely the ones studied in this thesis. Our results bring out both the potential as as well as the limitations of the methods in [13] while providing very simple proofs of the extensions of the work in [13].

As an application of methods involved in the Lévy Continuity Theorem we establish a form of the central limit theorem for realizable Orlicz sequence spaces generated by a convex function. This is done in the last chapter after obtaining the form of the Fourier transform of Gaussian random variables (vectors) taking values in these spaces. This last result generalizes some of the work of M.N. Vakhania [28].

Chapter II begins with some basic properties of Orlicz functions and spaces and other well-known results that are used throughout this thesis. Our main result -- the realization theorem -- is given in section 2.3 and from it we obtain, as a corollary, Mustari's results of [22]. Using this theorem we also obtain a partial solution to a problem formulated by Lindenstrauss and Tzafriri [18].

In chapter III we give a Lévy continuity theorem and a Bochner theorem for realizable Orlicz sequence spaces and show that these theorems for certain Orlicz sequence spaces considered in [11] are contained in [14].

CHAPTER II

REALIZATIONS

§2.1 INTRODUCTION

Let $\mathcal{A}$ be a σ-algebra of subsets of a set Ω, and let P be a complete probability measure on $(\Omega,\mathcal{A})$. We shall denote by $\mathcal{M}(\Omega,\mathcal{A},P)$ the vector space of real valued $\mathcal{A}$-measurable functions where equality of functions is understood to be almost everywhere (a.e.). On $\mathcal{M}(\Omega,\mathcal{A},P)$ we define the distance $d(f,g) = \int_\Omega \frac{|f(\omega) - g(\omega)|}{1 + |f(\omega) - g(\omega)|} dP(\omega)$. Then $\mathcal{M}(\Omega,\mathcal{A},P)$ with the topology induced by this distance (topology of convergence in probability) is a topological vector space (t.v.s.). We shall consider structural conditions on a t.v.s. E in order that there should exist a probability space $(\Omega,\mathcal{A},P)$ and a vector space isomorphism T mapping E into $\mathcal{M}(\Omega,\mathcal{A},P)$ $(T : E \to \mathcal{M}(\Omega,\mathcal{A},P))$ such that both T and T^{-1} are continuous. Topological vector spaces E for which this happens are said to be realizable with the linear homeomorphism T being called the realization. We consider a special case of this problem when E is a real F-space of real sequences with a Schauder basis, and the realization T is assumed to satisfy additional conditions (cf. §2.3).

Such realizations have proved useful in the study of harmonic analysis on certain vector spaces (e.g. [13] and [11]). In particular, realizations for $E = \ell_p$ and $E = L_p[0,1]$ $(0 < p \le 2)$ with the usual topologies have been established in [13] and [25]. D.H. Mushtari [22] examined this problem for the sequence spaces ℓ_p and showed that only in the case $0 < p \leq 2$ does such a linear homeomorphism exist.

In this chapter we examine the problem for the Orlicz spaces of sequences. We obtain necessary and sufficient conditions on the associated Orlicz function in order that a realization exists. This is achieved through careful analysis of the works of [3] and [26] where the sufficiency of these conditions in the context of function spaces was studied. After giving necessary notation and terminology in the next section, we present our main result in the last section.

§2.2 BASIC FACTS AND PROPERTIES OF ORLICZ SPACES

2.2.1 DEFINITION. An Orlicz function φ is a continuous, even, non-negative function non-decreasing for positive x such that $\varphi(0) = 0$, $\varphi(x) > 0$ for $x \neq 0$.

For a sequence of real scalars $a = \{a_n\}$ we write $\rho_\varphi(a) = \Sigma_{n=1}^{\infty}\varphi(a_n)$ and let $\ell_\varphi = \{a = \{a_n\} : \exists\, \lambda \in \mathcal{R}^+ = [0,\infty) \ni \rho_\varphi(\lambda^{-1}a) < \infty\}$. We also write $\rho(a)$ when no misunderstanding is likely from the omission of the subscript. If for a sequence $a^k = \{a_n^k\} \subset \ell_\varphi$ and $a = \{a_n\} \in \ell_\varphi$ $\rho(a - a^k) = \Sigma_{n=1}^{\infty}\varphi(a_n - a_n^k)$ converges to zero as k tends to infinity, then we say "$\{a^k\}$ converges in the mean to a". If φ is a convex Orlicz function, then ℓ_φ with the norm $\|a\|_\varphi = \inf\{\lambda > 0 : \rho(\lambda^{-1}a) \leq 1\}$ (Luxemburg norm) is a Banach space. In the Banach space $(\ell_\varphi, \|\cdot\|_\varphi)$ convergence to zero in the norm is equivalent to convergence to zero in the mean. The space ℓ_φ with the norm $\|\cdot\|_\varphi$ is called an Orlicz sequence space. For the most part properties relating to convex Orlicz functions are taken from [10], [17] and [29].

2.2.2 EXAMPLES. 1) Let $\varphi(x) = |x|^p$ $(1 \leq p < \infty)$. Then ℓ_φ is the classical ℓ_p space with the usual topology. In fact the Luxemburg norm is the ℓ_p-norm. This example shows that in a very natural way Orlicz sequence spaces are a generalization of the ℓ_p-spaces.
2) Let $\varphi(x) = (1 + |x|)\ln(1 + |x|) - |x|$. Then it is well known ([10], p. 20) that φ generates an Orlicz sequence space distinct from any ℓ_p space $(1 \leq p < \infty)$.

Convex Orlicz functions have the representation $\varphi(x) = \int_0^{|x|}p(t)dt$ ([10], p. 5) where $p(t)$, the right derivative of φ, is a non-decreasing, right-continuous, non-negative function defined for $t \geq 0$. The following proposition shows that all convex Orlicz functions with $p(0) > 0$ generate the same Orlicz sequence space.

2.2.3 PROPOSITION ([17], p. 127). Let φ be a convex Orlicz function with $\varphi(x) = \int_0^{|x|} p(t)dt$. Then $p(0) > 0$ if and only if (iff) ℓ_φ is isomorphic to ℓ_1.

As our problem has been studied for the above case, we consider Orlicz functions which include convex functions with $p(0) = 0$. The function $g(s) = \sup_{p(t) \leq s} t$ is then a right-continuous, non-decreasing function defined on the non-negative reals such that $q(0) = 0$ and $q(s) > 0$ for $s > 0$. The function $\psi(x) = \int_0^{|x|} q(s)ds$ is a convex Orlicz function, and following [10] we call it the complementary function of φ. It is easy to see that the relation of being complementary is symmetric.

2.2.4 REMARK. If $\varphi(x) = |x|^p$ $(1 < p < \infty)$, then $\psi(x) = |x|^q$ where $1/p + 1/q = 1$. The terminology "complementary functions" originated with this example. It should be noted that in many cases it is impossible to find an explicit formula for the complementary function, i.e., $\varphi(x) = e^{x^2} - 1$ ([10], p. 14).

We can now define a sequence space $\bar{\ell}_\varphi$ as the space of all real sequences $a = \{a_n\}$ such that $|||a|||_\varphi = \sup_{\rho_\psi(b) \leq 1} \Sigma a_n b_n < \infty$. $\bar{\ell}_\varphi$ with this norm is a Banach space and is related to ℓ_φ as follows.

2.2.5 PROPOSITION. Let φ be a convex Orlicz function such that $p(0) = 0$. Then $a \in \ell_\varphi$ iff $a \in \bar{\ell}_\varphi$ and $\|a\|_\varphi \leq |||a|||_\varphi \leq 2\|a\|_\varphi$ (i.e., $(\ell_\varphi, \|\cdot\|_\varphi)$ and $(\bar{\ell}_\varphi, |||\cdot|||_\varphi)$ are isomorphic as Banach spaces).

As stated in remark 2.2.4, for $\varphi(x) = |x|^p$ $(p > 1)$, $\psi(x) = |x|^q$ where $1/p + 1/q = 1$. The following proposition is an analogue of the classical result on inequalities involving complementary functions.

2.2.6 PROPOSITION. Suppose that φ is a convex Orlicz function with the complementary Orlicz function ψ. Then,

i) For all $x,y \geq 0$, $xy \leq \varphi(x) + \psi(y)$ (Young's inequality)

ii) For all $x \in \ell_\varphi$,

$$\Sigma\, x_n y_n \leq |||x|||_\varphi \rho_\psi(y) \quad \text{if} \quad \rho_\psi(y) \leq 1 .$$

For a convex Orlicz function φ we can define another vector space of real sequences by $h_\varphi = \{a = \{a_n\} : \forall\, \lambda > 0,\ \rho(\lambda^{-1}a) < \infty\}$, and since h_φ is a subset of ℓ_φ we can consider it with the norm $\|\cdot\|_\varphi$. This new space h_φ with norm $\|\cdot\|_\varphi$ has played a significant role in the study of the topological duals of Orlicz sequence spaces, and was introduced by Gribanov [7] who established the following result.

2.2.7 PROPOSITION. Let φ be a convex Orlicz function. Then h_φ is a closed subspace of ℓ_φ.

The dual space of a t.v.s. E is the vector space E' whose elements are the continuous linear functionals on E. E' will always be considered as having the weak-star topology ([24], p. 66). That is the topology induced by pointwise convergence. For $y \in E'$ and $x \in E$ we denote as (x,y) ($\langle x,y\rangle$) the evaluation of y at x (i.e., $y(x) = (x,y)$).

2.2.8 PROPOSITION. If φ is a convex Orlicz function having complementary Orlicz function ψ, then h'_ψ is isomorphic to ℓ_φ.

Given a convex Orlicz function the sequence space ℓ_φ is linear. In general, however, the space ℓ_φ associated with an Orlicz function φ need not be linear. An important class of Orlicz functions for which ℓ_φ is a vector space are those which satisfy the so called Δ_2-condition in a neighborhood of the origin. In addition the Δ_2-condition ensures us that the unit vectors ($e^k = (0,\ldots,0,1,0,\ldots)$, 1 in k^{th} coordinate) form a Schauder basis for the space ([17], [18]).

2.2.9 DEFINITION. An Orlicz function φ is said to satisfy the Δ_2-condition for small x if there exists $x_0 > 0$, $h > 0$ such that

$$\varphi(2x) \le h\varphi(x) \quad \text{for} \quad 0 \le x \le x_0.$$

Since φ is assumed to be non-decreasing it is obvious that $h \ge 1$; furthermore, for every $\lambda > 0$ there exists $h(\lambda) > 1$ such that

$$\varphi(\lambda x) \le h(\lambda)\varphi(x) \quad \text{for} \quad 0 \le x \le x_0$$

(cf. section 3.3).

2.2.10 PROPOSITION. Let φ be a convex Orlicz function with $p(0) = 0$. Then the following are equivalent:

i) φ satisfies the Δ_2-condition for small x.

ii) $\ell_\varphi = h_\varphi$

iii) ℓ_φ is separable.

2.2.11 EXAMPLES. 1) $\varphi(x) = e^{|x|} - |x| - 1$ is an example of a convex Orlicz function which does not satisfy the Δ_2-condition. Incidently, the complementary function to φ does satisfy the Δ_2-condition ([10], p. 27). 2) $\varphi(x) = |x|^p$ $(0 < p < 1)$ is an example of a non-convex Orlicz function which satisfies the Δ_2-condition for all x.

For the Orlicz functions satisfying the Δ_2-condition, the associated space ℓ_φ need not be a Banach space. For example, consider the case $\varphi(x) = |x|^p$ $(0 < p < 1)$. However it is a t.v.s. as the following proposition [21] shows.

2.2.12 PROPOSITION. Let φ be an Orlicz function satisfying the Δ_2-condition for small x. Then ℓ_φ with the quasi-norm $\|a\|_\varphi = \inf\{\lambda > 0 : \rho(\lambda^{-1}a) \le \lambda\}$ is an F-space in which convergence to zero in this quasi-norm is equivalent to convergence to zero in the mean.

When the Orlicz function φ satisfies the Δ_2-condition for small x, ℓ_φ with the quasi-norm given above will also be called an Orlicz sequence space.

EXAMPLE: Let $\varphi(x) = |x|^p$ $(0 < p < 1)$. Then $(\ell_\varphi, \|\cdot\|_\varphi)$ is the classical ℓ_p-space with the usual topology. Note however that $\|\cdot\|_\varphi \neq \|\cdot\|_p$ $(\|a\|_p = \Sigma_{n=1}^{\infty} |a_n|^p$: the usual quasi-norm on $\ell_p)$.

Since we are interested in the Orlicz sequence spaces as topological vector spaces, we need to know when if ever, distinct Orlicz functions generate the same topological vector space of sequences. This depends on the behavior of the function in a neighborhood of the origin as the next definition and subsequence proposition show.

2.2.13 DEFINITION. Two Orlicz functions φ and ψ are said to be equivalent for small x (for large x) denoted $\varphi \overset{s}{\sim} \psi$ $(\varphi \overset{\ell}{\sim} \psi)$ if there exist an $x_0 > 0$ and strictly positive constants b_1, b_2, k_1, k_2 such that

$$b_1\varphi(k_1x) \leq \psi(x) \leq b_2\varphi(k_2x) \quad \text{for} \quad 0 < x \leq x_0 \quad (\text{for} \quad x \geq x_0).$$

If $\varphi \overset{\ell}{\sim} \psi$ and x_0 in the definition can be taken to be zero, then we say φ is equivalent to ψ for all x and write $\varphi \overset{a}{\sim} \psi$.

It is easily seen that these are equivalence relations on the collection of Orlicz functions. These relations are simpler for Orlicz functions satisfying the Δ_2-condition: $\varphi \overset{s}{\sim} \psi$ $(\varphi \overset{\ell}{\sim} \psi)$ if there exist k_1, k_2 strictly positive constants and $x_0 > 0$ such that $k_1\varphi(x) \leq \psi(x) \leq k_2\varphi(x)$ for $0 < x \leq x_0$ $(x \geq x_0)$. The next proposition is basic to the theory of Orlicz spaces and is taken from [21].

2.2.14 PROPOSITION. Let φ and ψ be Orlicz functions. Then,

i) $\ell_\varphi = \ell_\psi$ iff $\varphi \overset{s}{\sim} \psi$

ii) If $\varphi \overset{s}{\sim} \psi$, then $\rho_\varphi(a^k) \to 0$ as $k \to \infty$ iff $\rho_\psi(a^k) \to 0$ as $k \to \infty$.

iii) If φ satisfies the Δ_2-condition for small x and $\varphi \overset{s}{\sim} \psi$, then ψ satisfies the Δ_2-condition for small x.

2.2.15 REMARK. As an example of the above, observe that, in view of propositions 2.2.12 and 2.2.14, if φ and ψ are two Orlicz functions such that $\varphi \overset{s}{\sim} \psi$ and φ satisfies the Δ_2-condition for small x, then ℓ_φ and ℓ_ψ are isomorphic as topological vector spaces. Since the topology and vector space structure of an Orlicz sequence space is dependent only on the behavior of the Orlicz function in a neighborhood of the origin, any given Orlicz function satisfying the Δ_2-condition for small x can be replaced by an equivalent Orlicz function satisfying the Δ_2-condition for all x ([10], p. 24). Similarly, an Orlicz function which satisfies the Δ_2-condition for small x can be replaced by an equivalent Orlicz function which is strictly increasing ([21], p. 104). In particular Orlicz functions in chapter III will be assumed to be strictly increasing as well as satisfying the Δ_2-condition for all x.

Two real valued functions f and g are said to be asymptotically equivalent at $x = 0$ if $\lim\limits_{x\to 0} f(x)/g(x) = c > 0$, denoted by $f \sim g$ as $x \to 0$. It is easy to see that for two Orlicz functions φ and ψ, $\varphi \sim \psi$ as $x \to 0$ is sufficient for $\varphi \overset{s}{\sim} \psi$, but the converse is not true.

Facts concerning Orlicz function spaces are very similar to the above for Orlicz sequence spaces. Aside from obvious differences some distinction is made necessary because Orlicz function spaces over an interval of finite measure are determined by the behavior of the Orlicz function in a neighborhood of infinity and not the origin. We also need to assume that an Orlicz function is increasing with $\lim_{x\to\infty} \varphi(x) = \infty$.

For $\mathcal{M} = \mathcal{M}([0,1], \mathcal{B}[0,1]$, Lebesgue measure) let $\rho_\varphi(f) = \int_0^1 \varphi(f(x))dx$ where dx indicates the integration with respect to Lebesgue measure on $[0,1]$. We let $L_\varphi = \{f \in \mathcal{M} : \exists\ \lambda \in \mathcal{R}^+ \ni \rho_\varphi(\lambda^{-1}f) < \infty\}$. Analogous to the situation existing in the sequence space case a further assumption is needed to ensure that L_φ will be a linear space.

2.2.16 <u>DEFINITION</u>. An Orlicz function φ is said to satisfy the Δ_2-condition for large x with Δ_2-constant $h \geq 0$ if there exists an $x_0 > 0$ such that

$$\varphi(2x) \leq h\varphi(x) \quad \text{for} \quad x \geq x_0 \ .$$

Suppose that φ is an Orlicz function satisfying the Δ_2-condition for large x. Then L_φ with the usual vector addition and multiplication by a scalar is a linear space and becomes an F-space under the quasi-norm $\|f\|_\varphi = \inf\{\lambda > 0 : \rho(\lambda^{-1}f) \leq \lambda\}$. L_φ with this quasi-norm will be called an Orlicz function space. A sequence $\{f_k\} \subset L_\varphi$ is said to "converge to zero in the mean" if $\lim_{k\to\infty} \rho_\varphi(f_k) = 0$. It is well known ([10], p. 76) that convergence in the quasi-norm $\|\cdot\|_\varphi$ is equivalent to convergence in the mean.

Since our study deals with realizations of Orlicz function spaces as t.v. spaces we need to know when distinct Orlicz functions generate the same function spaces.

2.2.17 <u>PROPOSITION</u>. Let φ and ψ be Orlicz functions. Then,

i) $L_\varphi = L_\psi$ iff $\varphi \overset{\ell}{\sim} \psi$

ii) If $\varphi \overset{\ell}{\sim} \psi$, then $\rho_\varphi(f_k) \to 0$ as $k \to \infty$ iff $\rho_\psi(f_k) \to 0$ as $k \to \infty$.

iii) If φ satisfies the Δ_2-condition for large x and $\varphi \overset{\ell}{\sim} \psi$, then ψ satisfies the Δ_2-condition for large x.

Two real valued functions f and g are said to be asymptotically equivalent at infinity if $\lim_{x\to\infty} f(x)/g(x) = c > 0$ denoted $f \sim g$ as $x \to \infty$. Then for two Orlicz functions φ and ψ, $\varphi \sim \psi$ as $x \to \infty$ implies $\varphi \overset{\ell}{\sim} \psi$, but the converse need not be true.

The study of Orlicz spaces seems to naturally divide into two classes depending on how the rate of growth of the associated Orlicz function compares to the function $f(x) = x^2$, and with this motivation we make the following definition.

2.2.18 <u>DEFINITION</u>. Let $K(2,0)$ $(K(2,\infty))$ be the collection of Orlicz functions φ such that there exists Orlicz function ψ with $\varphi \overset{a}{\sim} \psi$ and $\psi(x)/x^2$ is a non-increasing (non-decreasing) function of x.

The usefulness of this definition evolves from the next proposition which can be found in [20].

2.2.19 <u>PROPOSITION</u>. Let φ be an Orlicz function such that $\varphi \in K(2,0)$ $(\varphi \in K(2,\infty))$. Then there exists Orlicz function ψ such that $\varphi \overset{a}{\sim} \psi$ and $\psi(\sqrt{x})$ is a concave (convex) function of x.

2.2.20 <u>REMARK</u>. Due to this proposition and the fact that equivalent Orlicz functions generate the same Orlicz spaces, when $\varphi \in K(2,0)$ $(\varphi \in K(2,\infty))$ we will assume without any loss of generality that $\varphi \circ \sqrt{\ }$ is a concave (convex) function.

We conclude this section by giving notation and well known definitions that will be used throughout this thesis.

For two real valued functions f and g we will write $f(x) \wedge g(x)$ for $\min_x\{f(x), g(x)\}$ and $f(x) \vee g(x)$ for $\max_x\{f(x), g(x)\}$.

The indicator function of a set A will be denoted by $[A]$, i.e. $[A] = [A](\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$.

If X is a symmetric (about zero) infinitely divisible random variable, then the characteristic function of X is given by $\chi_X(t) = \exp\{-\sigma^2 t^2/2 - \int_0^\infty (1 - \cos ut)\, dM(u)\}$ where $\sigma \geq 0$ and M is a Lévy measure on $\mathcal{R}^+$. That is, M is a measure on $\mathcal{R}^+$ such that $\int_1^\infty dM(u) < \infty$ and $\int_0^1 u^2 dM(u) < \infty$. A complete discussion of this representation for infinitely divisible distributions appears in ([6], p. 70).

§2.3 REALIZATIONS OF ORLICZ SEQUENCE SPACES

Let E be a vector space on which an invariant metric d is defined. Furthermore, suppose that the vector space operations are continuous with respect to the topology on E induced by d, and that the metric space (E,d) is complete. On E define the quasi-norm $\|\cdot\|$ by $\|x\| = d(x,0)$. Then $(E,\|\cdot\|)$ is an F-space ([24], p. 8). Now suppose that $(E,\|\cdot\|)$ is a real F-space of sequences of real numbers with a Schauder basis $\{e^k\}$. We say that the sequence space $(E,\|\cdot\|)$ is realizable if there exist a probability space $(\Omega,\mathcal{A},P)$ and a linear homeomorphism $T : E \to \mathcal{M}(\Omega,\mathcal{A},P)$ such that the random variables in $T(E)$ are symmetric about the origin and $\{T(e^k)\}$ is a sequence of independent identically distributed (iid) random variables. ($\mathcal{M}(\Omega,\mathcal{A},P)$ is assumed to have the topology of convergence in probability.)

In order to study the realizations of sequence spaces a different type of space is needed. Suppose that X is a random variable symmetric about the origin, and let $\{X_n\}$ be a sequence of independent random variables distributed as X. Define by ℓ_X the space of all real sequences such that $\{\Sigma_{n=1}^{k} a_n X_n\}$ is Cauchy in probability. We say that the space ℓ_X is generated by the random variable X. Note that since we are dealing with independent random variables, $\ell_X = \{\{a_n\} : \Sigma_{n=1}^{\infty} a_n X_n \text{ converges a.s.}\}$.

Our main effort is to establish that $\varphi \in K(2,0)$ is necessary and sufficient for the sequence space ℓ_φ (section 2.2) to be realizable. In particular we exhibit a symmetric random variable X such that $\ell_\varphi = \ell_X$.

The key to the development is the following lemma which can be found in [3] and [26].

2.3.1 LEMMA. Suppose that φ is an Orlicz function satisfying the Δ_2-condition for small x and $\varphi \in K(2,0)$. Then there exists $\sigma \geq 0$ and a Lévy measure M such that

$$\varphi(x) \overset{s}{\sim} \sigma^2 x^2 + \int_0^\infty (x^2 u^2 \wedge 1)\, dM(u) .$$

2.3.2 EXAMPLE. 1) If $\varphi(x) = |x|^p$ $(0 < p < 2)$ then $\sigma = 0$ and $dM(u) = u^{-1-p}du$. 2) If $\varphi(x) = x^2$, then $\sigma = 1$ and M is the zero measure on $\mathbb{R}^+$.

The next theorem characterizes those Orlicz sequence spaces which can be generated by a symmetric random variable. In fact it shows that the space ℓ_X for X a symmetric random variable is always an Orlicz sequence space.

2.3.3 THEOREM. If X is a random variable symmetric about zero, then there exists an Orlicz function φ such that $\ell_X = \ell_\varphi$ and $\varphi \in K(2,0)$. Conversely, if φ is an Orlicz function from the class $K(2,0)$, then there exists a random variable symmetric about zero such that $\ell_\varphi = \ell_X$.

The equalities that appear above are equalities between vector spaces. In fact the mapping $I(x) = x$ is a vector space isomorphism.

PROOF. First suppose that X is a random variable symmetric about zero and consider $E = \ell_X$. Then $\{a_n\} \in E$ iff $\Sigma_{n=1}^\infty a_n X_n$ converges a.s. The Kolmogorov three series theorem ([19], p. 237) implies the existence of $A > 0$ such that

i) $\Sigma_{n=1}^\infty P\{|a_n X_n| > A\} < \infty$

ii) $\Sigma_{n=1}^\infty E(a_n X_n [|a_n X_n| \leq A]) < \infty$

iii) $\Sigma_{n=1}^{\infty} \operatorname{Var}(a_n X_n [|a_n X_n| \le A]) < \infty$.

Since the random variables $\{X_n\}$ are symmetric, the series ii) is zero.

Notice that using the symmetry of X i) and iii) can be written as

$$2\Sigma_{n=1}^{\infty}\int_{A/|a_n|}^{\infty} dF(x) < \infty, \text{ and } 2\Sigma_{n=1}^{\infty} a_n^2\int_0^{A/|a_n|} x^2 dF(x) < \infty$$

where $F(x)$ is the cumulative distribution function of X. Now define the function q_A by $q_A(\lambda) = \int_0^{\infty}(\lambda^2 x^2 \wedge A^2)dF(x)$. Then the above can be stated as $\{a_n\} \in E$ iff $\Sigma_{n=1}^{\infty} q_A(a_n) < \infty$.

Clearly we can see the function q_A is an even function of λ non-decreasing for $\lambda \ge 0$ with $q_A(0) = 0$ and $q(\lambda) > 0$ for $\lambda \ne 0$, and the Lebesgue dominated convergence theorem shows that q_A is continuous. Hence q_A is an Orlicz function and since for $x \ge 0$, $(4\lambda^2 x^2 \wedge A^2) \le 4(\lambda^2 x^2 \wedge A^2)$ for all $\lambda > 0$, q_A also satisfies the Δ_2-condition for all $\lambda > 0$. Now let $q(\lambda) = q_1(\lambda)$ and observe that $q_A(\lambda) = A^2 q(\lambda/A)$; thus, $q_A \overset{s}{\sim} q$ and we conclude that $\{a_n\} \in E$ iff $\Sigma_{n=1}^{\infty} q(a_n) < \infty$, by proposition 2.2.14.

The above shows that $E = \ell_q$ and so there remains only to show that $q \in K(2,0)$, but this follows from $q(\lambda)/\lambda^2 = \int_0^{\infty}(x^2 \wedge 1/\lambda^2)dF(x)$.

Conversely suppose that φ is an Orlicz function satisfying the Δ_2-condition for small x and $\varphi \in K(2,0)$. Then we exhibit a symmetric random variable X so that $\ell_\varphi = \ell_X$. Lemma 2.3.1 implies there exist $\sigma \ge 0$ and a Lévy measure M such that $\varphi(x) \overset{s}{\sim} \sigma^2 x^2 + \int_0^{\infty}(x^2 u^2 \wedge 1)dM(u)$.

To simplify the details we consider three cases; the first case is to assume that the Lévy measure M is identically zero. Then $\ell_\varphi = \ell_2$ and Chebyshev's inequality implies that if $\{a_n\} \in \ell_2$ then

$\{a_n\} \in \ell_{X_1}$ where X_1 is a Gaussian distributed random variable with mean zero and variance 1. Using ([2], proposition 8.37, p. 177) it is easy to see that $\{a_n\} \in \ell_{X_1}$ implies $\{a_n\} \in \ell_2$; thus, we have $\ell_\varphi = \ell_{X_1}$.

For the second case we will assume that $\sigma = 0$. Let X_M be a random variable with the characteristic function, $\chi_{X_M}(t) = \exp\{-\int_0^\infty (1 - \cos xt)dM(x)\}$ and so X_M is a symmetric infinitely divisible random variable. Recall that $\ell_{X_M} = \{\{a_n\} : \Sigma_{n=1}^\infty a_n X_n$ converges a.s.$\}$ where $\{X_n\}$ is a sequence of iid random variables distributed as X_M. We show that $\ell_\varphi = \ell_{X_M}$. Suppose that $\{a_n\} \in \ell_\varphi$. Then $\Sigma_{n=1}^\infty \int_0^\infty (a_n^2 u^2 \wedge 1)dM(u) < \infty$, and in particular $\Sigma_{n=1}^\infty \int_{1/|a_n|}^\infty dM(u) < \infty$; hence

$$(2.3.4) \qquad \Sigma_{n=1}^\infty \int_{1/|a_n|}^\infty (1 - \cos u\, a_n t)dM(u) < \infty \quad \text{for all } t.$$

It can be easily shown that $(1 - \cos x/2) \sim x^2/2$ as $x \to 0$ and so there exists $\epsilon > 0$ such that for $|x| < \epsilon$, $(1 - \cos x) \le x^2/2$. Thus, for $0 \le u \le 1/|a_n|$, $|t| < \epsilon$ we note

$$\int_0^{1/|a_n|} (1 - \cos a_n u\ t)dM(u) \le \int_0^{1/|a_n|} \frac{t^2 a_n^2 u^2}{2} M(du)$$

$$\le \frac{\epsilon^2}{2} \int_0^{1/|a_n|} a_n^2 u^2 M(du)$$

and because $\Sigma_{n=1}^\infty \int_0^{1/|a_n|} a_n^2 u^2 F(du) < \infty$ we must have for $|t| < \epsilon$, $\Sigma_{n=1}^\infty \int_0^{1/|a_n|} (1 - \cos a_n u\ t)dM(u) < \infty$. Together with (2.3.4) this implies that

$$\Sigma_{n=1}^\infty \int_0^\infty (1 - \cos a_n u\ t)dM(u) < \infty \quad \text{for } |t| < \epsilon .$$

Hence ([2], theorem 8.38, p. 177) implies that $\Sigma_{n=1}^{\infty} a_n X_n$ converges a.s. That is $\ell_\varphi \subset \ell_{X_M}$.

Now suppose $\{a_n\} \in \ell_{X_M}$. Then $\Sigma_{n=1}^{\infty} a_n X_n$ converges a.s. and $\Sigma_{n=k}^{\infty} a_n X_n \to 0$ in probability as $k \to \infty$. Since $\chi_{\Sigma_{n=k}^{\infty} a_n X_n}(t) = \exp\{-\Sigma_{n=k}^{\infty} \int_0^\infty (1 - \cos u\, a_n t) dM(u)\}$, we note $\Sigma_{n=k}^{\infty} \int_0^\infty (1 - \cos u\, a_n t) dM(u) \to 0$ uniformly on compact subsets of $\mathcal{R}$ as $n \to \infty$. In particular this implies $\lim_{k\to\infty} \int_0^1 \Sigma_{n=k}^{\infty} \int_0^\infty (1 - \cos u\, a_n t) dM(u) dt = 0$. Using Tonelli's theorem and integrating out the variable t we get $\lim_{k\to\infty} \Sigma_{n=k}^{\infty} \int_0^\infty (1 - \frac{\sin a_n u}{a_n u}) dM(u) = 0$, but since there exists a constant $c > 0$ such that $1 - \frac{\sin x}{x} \geq c\,(1 \wedge x^2)$ for all $x > 0$, $\lim_{k\to\infty} \Sigma_{n=k}^{\infty} \int_0^\infty (a_n^2 u^2 \wedge 1) dM(u) = 0$. Thus, $\Sigma_{n=1}^{\infty} \int_0^\infty (a_n^2 u^2 \wedge 1) dM(u) < \infty$ implying that $\{a_n\} \in \ell_\varphi$. This concludes the proof of this case since $\{a_n\}$ was an arbitrary vector from ℓ_{X_M}.

For the final case, suppose that $\sigma > 0$ and M is not the zero measure on $\mathcal{R}^+$. Then let $\varphi_1(x) = \int_0^\infty (x^2 u^2 \wedge 1) dM(u)$ where $\varphi(x) \overset{s}{\sim} \sigma^2 x^2 + \int_0^\infty (x^2 u^2 \wedge 1) dM(u)$. Observe that $\varphi \in K(2,0)$ implies that $\varphi_1 \in K(2,0)$ and furthermore that $\ell_{\varphi_1} \subset \ell_2$. Thus, $\ell_\varphi = \ell_{\varphi_1}$ and the question reduces to the second case.

As a corollary to the above proof we have the following.

2.3.5 <u>COROLLARY</u>. Suppose that φ is an Orlicz function satisfying the Δ_2-condition for small x and $\varphi \in K(2,0)$. Then $\ell_\varphi = \ell_X$ where the random variable X is as given in the above theorem for each case, and for $\{a^k\} \subset \ell_\varphi$, $a^k \to 0$ as $k \to \infty$ in ℓ_φ iff $\{\Sigma_{n=1}^{\infty} a_n^k X_n\} \to 0$ as $k \to \infty$ in probability.

<u>PROOF</u>. For $|t| < \varepsilon$ with ε as chosen in the above proof, $\Sigma_{n=1}^{\infty} \varphi(a_n^k) \geq -\log \chi_{\Sigma_{n=1}^{\infty} a_n^k X_n}(t)$; hence, if $a^k \to 0$ in ℓ_φ as

as $k \to \infty$ then $\sum_{n=1}^{\infty} a_n^k X_n \to 0$ in probability. Conversely, we saw that if $-\log \chi_{\sum_{n=1}^{\infty} a_n^k X_n}(t) \to 0$ as $k \to \infty$ uniformly on compact subsets of $\mathcal{R}$, then $\sigma^2 \sum_{n=1}^{\infty} (a_n^k)^2 + \sum_{n=1}^{\infty} \int_0^1 ((a_n^k)^2 u^2 \wedge 1) dM(u) \to 0$ as $k \to \infty$; therefore, $a^k \to 0$ in ℓ_φ as $k \to \infty$.

Now we prove our main theorem.

2.3.6 THEOREM. Suppose that φ is an Orlicz function satisfy- the Δ_2-condition for small x. The orlicz sequence space ℓ_φ is realizable as a space of random variables iff $\varphi \in K(2,0)$.

PROOF. If $\varphi \in K(2,0)$, then by theorem 2.3.3 there exists a symmetric infinitely divisible random variable X such that $\ell_\varphi = \ell_X$. Let $E = \{\sum_{n=1}^{\infty} a_n X_n : \{a_n\} \in \ell_X\}$ with $\{X_n\}$ iid as X.
Then E with the topology of convergence in probability is a t.v.s. Define T mapping ℓ_φ into E ($T : \ell_\varphi \to E$) by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n X_n$. This linear mapping is well-defined and injective since $\{a_n\} \in \ell_\varphi$ iff $\{a_n\} \in \ell_X$, and corollary 2.3.5 shows that T is bi-continuous. Thus ℓ_φ has the realization given by T.

Now suppose that ℓ_φ is realizable, i.e. there exist $\mathcal{M}(\Omega,\mathcal{A},P)$ and $T : \ell_\varphi \to \mathcal{M}(\Omega,\mathcal{A},P)$ such that T is a linear homeomorphism and $\{T(e^k)\}$ is a sequence of iid symmetric random variables where e^k is the k^{th} unit vector of ℓ_φ. Then for $\{a_n\} \in \ell_\varphi$, $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n T(e^n)$. Let $X = T(e^1)$. Then $\ell_\varphi = \ell_X$ and since T is a linear isomorphism this is equality between vector spaces. Theorem 2.3.3 shows that $\ell_X = \ell_\psi$ where ψ is an Orlicz function from the class $K(2,0)$. Then using the bi-continuity of T and corollary 2.3.5 we conclude $\ell_\varphi = \ell_\psi$ (equality as t.v. spaces) and hence, $\varphi \in K(2,0)$.

2.3.7 EXAMPLE. 1) Let $\varphi(x) = |x|^p$ $(0 < p < 1)$. Then from example 2.3.2 we get $\sigma = 0$ and $dM(u) = u^{-1-p} du$ and recognize that

$\exp\{-\int_0^\infty(1 - \cos xu)u^{-1-p}du\}$ is the characteristic function of a symmetric stable random variable with index p. Thus $\{a_n\} \in \ell_\varphi$ iff $\Sigma_{n=1}^\infty a_n X_n$ converges a.s. with $\{X_n\}$ iid symmetric stable of index p.

As a corollary to theorem 2.3.6 we obtain the results of Mushtari [22].

2.3.8 COROLLARY. The sequence space ℓ_p for $p > 2$ is not realizable as a space of random variables.

This result can also give some information on a problem suggested by J. Lindenstrauss and L. Tzafriri [18]. There they prove the following:

2.3.9 PROPOSITION. Every Orlicz sequence space generated by a convex Orlicz function φ contains a subspace isomorphic to ℓ_p for some $p \geq 1$.

The question raised is can more be said about the possible values of p. Our results show the following:

2.3.10 THEOREM. If φ is a convex Orlicz function and $\varphi \in K(2,0)$, then ℓ_φ contains a subspace isomorphic to ℓ_p for some p, $1 \leq p \leq 2$.

PROOF. The proof is immediate from theorem 2.3.6, corollary 2.3.8, and proposition 2.3.9.

CHAPTER III

LÉVY CONTINUITY THEOREM FOR ORLICZ SEQUENCE SPACES

§3.1 INTRODUCTION

In this chapter we generalize the Lévy continuity theorem to the case of Orlicz sequence spaces when the associated Orlicz function is in the class $K(2,0)$. This extends the results of [13] which handles the situation for the classical ℓ_p $(0 < p < \infty)$ spaces. For completeness and ease of reference a statement of the Bochner theorem proved in [11] is included. We also show that the results of Kuelbs and Mandrekar can be applied to ℓ_φ when the Orlicz function φ is in the class $K(2,\infty)$; thus, [14] includes the results given in [11] for this class of Orlicz spaces.

Our proof of the Lévy continuity theorem is based on the realization theorem and adaptation of techniques from [13] to this case. Recently, by using an extension of characteristic functions due to L. LeCam [16], J. Kuelbs has obtained some results which constitute generalization of the work in [13]. However, his proofs are complicated due to the generality of his approach and the precise sequence spaces to which such results are applicable are not known. Extensions of techniques of [8], [13] are of interest due to the simplicity of the approach as brought out in the recent work of [11]. Our approach shows that the problem of Lévy continuity theorem is intimately connected with the existence of a realization of the space. In turn the realization problem is intimately related to the structural problems of Banach spaces as our theorem 2.3.10 indicates.

First we establish some terminology and preliminaries in section 3.2 and give our results in the last section.

§3.2 PRELIMINARIES

Let ℓ denote the vector space of all sequences of real numbers with the topology of coordinate-wise convergence (Tychonoff topology). We will frequently think of the ℓ_φ spaces as being subsets of ℓ (cf. section 4.2) and if $x \in \ell$ we define

$$P_N x = (x_1, \ldots, x_N, 0, \ldots)$$

$$Q_N x = (0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots) .$$

We denote the coordinate functionals on ℓ by β_j, $j = 1,2,\ldots$. That is, $\beta_j : \ell \to \mathcal{R}$ is given by $\beta_j(x) = x_j$ for all $x \in \ell$ and for all j.

For $\varphi \in K(2,0)$ we denote by λ_φ the probability measure on the Borel subsets, $\mathcal{B}$, of ℓ by taking the product measure on ℓ such that the coordinate functionals have independent infinitely divisible laws with Fourier transforms $\exp\{-\sigma^2 t^2/2 - \int_0^\infty (1 - \cos ut)dM(u)\}$ where σ^2 and the Lévy measure M are given by lemma 2.3.1.

3.2.1 DEFINITION. The Fourier transform (or characteristic functional) of a probability measure μ on the Borel subsets of a topological vector space E is the function χ defined on E' (the topological dual of E) by

$$\chi(x) = \int_E \exp\{i(y,x)\}d\mu(y) \quad \text{for } x \in E' .$$

3.2.2 REMARK. If E' contains enough linear functionals to separate points of E and if μ is a tight Borel probability measure (cf. definition 3.2.7), then χ determines μ uniquely on the Borel subsets of E. In particular we study the case where $E = \ell_\varphi$ and $\varphi \in K(2,0)$. While ℓ_φ may not be locally convex, the coordinate functionals are contained in ℓ_φ' and so ℓ_φ' does separate points of

ℓ_φ. More will be said later concerning the tightness of Borel probabilities on ℓ_φ.

3.2.3 LEMMA. If μ is a probability measure on the Borel subsets $\mathcal{C}$ of ℓ_φ, $\varphi \in K(2,0)$, then the function

$$(x,y)^{\sim} = \lim_N \Sigma_{k=1}^N \beta_k(x)\beta_k(y)$$

is a $\mathcal{B} \times \mathcal{C}$-measurable function on $\ell \times \ell_\varphi$, and

$$(\lambda_\varphi \times \mu)(\{|(x,y)^{\sim}| < \infty\}) = 1.$$

The proof of this lemma is very similar to the proof for the ℓ_p $(0 < p \leq 2)$ case as given in ([13], lemma 3.1, p. 222) and will be omitted.

3.2.4 DEFINITION. If μ is a probability measure on the Borel subsets of ℓ_φ, $\varphi \in K(2,0)$, then we define the extended Fourier transform of μ on ℓ by

$$\tilde{\chi}(x) = \int_{\ell_\varphi} \exp\{i(y,x)^{\sim}\}d\mu(y) \quad \text{for} \quad x \in \ell .$$

3.2.5 REMARK. The extended Fourier transform of a probability μ on the Borel subsets of ℓ_φ is a Borel measurable function on ℓ which is finite almost everywhere with respect to the measure λ_φ and which is equal to $\chi(x) = \int_{\ell_\varphi} \exp\{i(y,x)\}d\mu(y)$ for all $x \in \ell_\varphi'$. Thus $\tilde{\chi}$ is truly an extension of χ from ℓ_φ' to ℓ.

Let S be a metric space with $\mathcal{J}$ denoting the Borel sets in S. We need the following concepts from the theory of weak convergence of measures [1].

3.2.6 DEFINITION. A sequence of probability measures $\{P_n\}$ on $\mathcal{J}$ is said to converge weakly to the probability measure P on

$\mathcal{T}$ if $\int_S f dP_n \to \int_S f dP$ for every bounded continuous real valued function defined on S.

If $\{P_n\}$ converges weakly to P, we write $P_n \Rightarrow P$.

3.2.7 DEFINITION. A probability measure P on $(S, \mathcal{T})$ is said to be tight if for every positive ε there exists a compact set $K \subset S$ such that $P(K) > 1 - \varepsilon$.

We shall deal only with Orlicz spaces for which φ satisfies the Δ_2-condition, and by proposition 2.2.10 these ℓ_φ spaces are separable. The ℓ_φ spaces are a priori complete being F-spaces; hence, every probability measure on ℓ_φ is tight ([1], p. 10).

3.2.8 DEFINITION. A family $\{\mu_\alpha : \alpha \in A\}$ of probability measures on S are said to be tight if for every positive ε there exists a compact set $K \subset S$ such that $\mu_\alpha(K) > 1 - \varepsilon$ for all $\alpha \in A$.

3.2.9 DEFINITION. A family $\{\mu_\alpha : \alpha \in A\}$ of probability measures on $(S, \mathcal{T})$ is said to be relatively (conditionally) compact if every sequence of elements $\{\mu_{\alpha_k}\}$ $(\alpha_k \in A,\ k = 1,2,\ldots)$ contains a weakly convergent subsequence.

The next proposition is due to Prohorov and can be found in [1].

3.2.10 PROPOSITION. Let S be a separable and complete metric space. Then a family $\{\mu_\alpha : \alpha \in A\}$ of probability measures on $(S, \mathcal{T})$ is tight iff the family is relatively compact.

We shall be dealing with tight sequences of measures on ℓ_φ for $\varphi \in K(2,0)$ and satisfying the Δ_2-condition. In order to do this we need a description of the compact subsets of such spaces. For the ℓ_p case such a description is readily available [4]. We give the following generalizations which do not seem to be available in the literature.

Suppose that φ is an Orlicz function satisfying the Δ_2-condition for $0 \le x \le x_0$ with constant $h \ge 1$ (i.e. $\varphi(2x) \le h\varphi(x)$ for $0 \le x \le x_0$). We now assume without any loss of generality, that φ is strictly increasing (remark 2.2.15). If $\max\{|t|, |s|\} \le x_0$ we have the following

$$\varphi(t-s) = \varphi(|t-s|) \le \varphi(|t| + |s|) \le \varphi(2\max\{|t|,|s|\})$$

$$\le h\varphi(\max\{|t|,|s|\}) \le h\varphi(|t|) + h\varphi(|s|)$$

(3.2.11) $\quad \varphi(t-s) \le h\varphi(t) + h\varphi(s) \quad$ if $\max\{|t|,|s|\} \le x_0$.

Thus if φ satisfies the Δ_2-condition for all $x \ge 0$,

(3.2.12) $$\rho_\varphi(x+y) \le h\rho_\varphi(x) + h\rho_\varphi(y) .$$

Note that the characterization we give for the compact subsets is in terms of the mean function and not in terms of the quasi-norm on the space.

3.2.13 <u>THEOREM</u>. Suppose φ is an Orlicz function satisfying the Δ_2-condition with constant h for $0 \le x \le x_0$. Then a set $K \subset \ell_\varphi$ is compact iff the following conditions are satisfied:

i) $\sup_{f \in K} \rho(f) < \infty$

ii) $\lim_{N\to\infty} \sup_{f\in K} \rho^N(f) = 0$ where $\rho^N(f) = \Sigma_{n=N+1}^{\infty}\varphi(\beta_n(f))$.

<u>PROOF</u>. Suppose $K \subset \ell_\varphi$ is such that i) and ii) are satisfied. Let $\{f_k\}$ be a sequence from K. Then by i), $\varphi(\beta_n(f_k)) \le \rho(f_k) \le \sup_{f\in K} \rho(f) < \infty$ for $n = 1,2,\ldots; k = 1,2,\ldots$. So by the usual diagonal process we can extract a subsequence $\{f_{k_j}\}$ from $\{f_k\}$ such that $\lim_{j\to\infty} \varphi(\beta_n(f_{k_j}))$ exists for each n, $n = 1,2,\ldots$. Since φ is strictly increasing with $\lim_{x\to\infty} \varphi(x) = \infty$, we must have $\lim_{j\to\infty} \beta_n(f_{k_j})$ existing for

for each n. Now to show $\{f_{k_j}\}$ converges in ℓ_φ. Since φ satisfies the Δ_2-condition, it will suffice to show for $\varepsilon > 0$, $\exists\, J \ni$ for all $i,j \geq J$, $\rho(f_{k_i} - f_{k_j}) < \varepsilon$. By assumption ii) $\exists\, N \ni \sup_{f \in K} \rho^N(f) < \min\{\varphi(x_0), \varepsilon/3h\}$. Then,

$$\rho(f_{k_i} - f_{k_j}) = \Sigma_{n=1}^{N} \varphi(\beta_n(f_{k_i}) - \beta_n(f_{k_j})) + \rho^N(f_{k_i} - f_{k_j}) .$$

Since $\lim_{i,j\to\infty} |\beta_n(f_{k_i}) - \beta_n(f_{k_j})| = 0$ and φ is continuous at the origin, $\exists\, J \ni \forall\, i,j \geq J$, $\Sigma_{n=1}^{N} \varphi(\beta_n(f_{k_i}) - \beta_n(f_{k_j})) < \varepsilon/3$. Then by the assumption $\max\{|\beta_n(f_{k_i})|, |\beta_n(f_{k_j})|\} < x_0$ for all i,j and $n \geq N + 1$, using the fact that φ is increasing, and equation 3.2.11, we find

$$\rho^N(f_{k_i} - f_{k_j}) \leq h\rho^N(f_{k_i}) + h\rho^N(f_{k_j}) \leq 2\varepsilon/3 .$$

Hence, $\rho(f_{k_i} - f_{k_j}) < \varepsilon$ for all $i,j \geq J$.

Conversely, suppose that $K \subset \ell_\varphi$ is compact. Then clearly i) holds since ρ is a continuous function on ℓ_φ. Now take $1 > \varepsilon > 0$. Then $\exists\, f_1,\ldots,f_\ell \in K$ such that for $f \in K\ \exists\, i$, $1 \leq i \leq \ell \ni \|f - f_i\|_\varphi < \alpha$ where $\alpha = \min\{\varphi(x_0), \varepsilon/2h\}$, but if $\|f\|_\varphi \leq 1$ then $\rho(f) \leq \|f\|_\varphi$, so $\rho(f - f_i) < \alpha$. Now choose N so that $\rho^N(f_i) < \alpha$ for all i, $1 \leq i \leq \ell$. Take $f \in K$ and choose i so that $\rho(f - f_i) \leq \alpha$. Then, $\rho^N(f) = \rho^N(f - f_i + f_i) \leq h\rho^N(f - f_i) + h\rho^N(f_i)$ implying that $\rho^N(f) < \varepsilon$. Since $1 > \varepsilon > 0$ was arbitrary and f was an arbitrary vector from K, condition ii) is verified.

3.2.14 <u>THEOREM</u>. Suppose φ is an Orlicz function satisfying the Δ_2-condition for all x with Δ_2-constant h. A set $K \subset \ell_\varphi$ is compact if for every $\delta > 0$ $\exists\, x_1,\ldots,x_r \in \ell_\varphi$ such that $K \subset \cup_{j=1}^{r} S(x_j,\delta)$ where $S(x_j,\delta) = \{y \in \ell_\varphi : \rho(y - x_j) \leq \delta\}$.

PROOF. For the proof it will suffice to show that under the stated hypothesis, conditions i) and ii) of theorem 3.2.13 are satisfied. Take $\varepsilon > 0$. Then by assumption $\exists\ x_1, \ldots, x_r \in \ell_\varphi$ such that $K \subset \bigcup_{j=1}^{r} S(x_j, \varepsilon/2h)$. For $f \in K$ choose i, $1 \le i \le r$, such that $f \in S(x_i, \varepsilon/2h)$.

$$\rho(f) = \rho(f - x_i + x_i) \le h\rho(f - x_i) + h\rho(x_i)$$

$$\le \varepsilon/2 + h \sup_{1 \le i \le r} \rho(x_i) < \infty .$$

Hence $\sup_{f \in K} \rho(f) < \infty$ and the first condition is shown. Now choose N_0 such that $N \ge N_0$ implies $\rho^N(x_i) < \varepsilon/2h$ for all i, $1 \le i \le r$. Then again we see $\rho^N(f) \le h\rho^N(f - x_i) + h\rho^N(x_i)$ yielding $\rho^N(f) \le \varepsilon/2 + h\rho^N(x_i) < \varepsilon$. Therefore, $\lim_{N \to \infty} \sup_{f \in K} \rho^N(f) = 0$, and the theorem is proved.

§3.3 LÉVY CONTINUITY THEOREM AND BOCHNER'S THEOREM

We now prove Lévy Continuity Theorem for Orlicz sequence spaces with $\varphi \in K(2,0)$. The section also includes a statement of Bochner's Theorem and concludes by showing that the case $\varphi \in K(2,\infty)$ is derivable from the work in [14]. This is done to show that the work in [11] which uses techniques similar to [13] is included in [14].

In this section until further notice, φ will denote an Orlicz function satisfying the Δ_2-condition for all x with Δ_2-constant h, and we define $\psi(t)$ by

$$(3.3.1) \qquad \psi(t) = \sigma^2 t^2/2 + \int_0^\infty (1 - \cos xt)\,dM(x)$$

where $\sigma > 0$ is the constant and M is the Lévy measure of lemma 2.3.1. Recall that corollary 2.3.5 implies $\varphi \overset{s}{\sim} \psi$; hence, ψ will generate the same topology on ℓ_φ as does φ. In particular for $\{x^k\} \subset \ell_\varphi$, $\rho_\varphi(x^k) \to 0$ as $k \to \infty$ iff $\rho_\psi(x^k) \to 0$ as $k \to 0$; hence, for every $\epsilon > 0$ there exists $\delta(\epsilon) = \delta > 0$ such that

$$(3.3.2) \qquad \text{if } \rho_\varphi(x) > \epsilon \text{ then } \rho_\psi(x) > \delta .$$

We begin the proof of Lévy's Continuity Theorem with the following lemmas.

3.3.3 LEMMA. If $\{\mu_\alpha : \alpha \in A\}$ is a family of probability measures on ℓ_φ such that

$$\lim_{N\to\infty;\ \gamma\downarrow 0}\ \sup_{\alpha\in A} J_{N,\gamma}(\mu_\alpha) = 0$$

where

$$J_{N,\gamma}(\mu_\alpha) = \int_{\ell_\varphi} 1 - \exp\{-[\Sigma_{n=1}^N \psi(\gamma x_n) + \rho_\psi^N(x)]\}\,d\mu_\alpha(x) .$$

Then $\{\mu_\alpha : \alpha \in A\}$ is conditionally compact.

PROOF. Take $\varepsilon > 0$, $0 < \beta < 1$ and define $E = \{y \in \ell_\varphi : \Sigma_1^N \varphi(\gamma y_n) + \rho_\varphi^N(y) \geq \beta/2h\}$, where h is the Δ_2-constant for φ. For $0 \leq t < 1$, $t/2 \leq 1 - e^{-t}$, so we find $\mu_\alpha(E) = \frac{1}{\delta}\int_E \delta \, d\mu_\alpha(x)$ where δ corresponds to $\beta/2h$ as in 3.3.2, and clearly δ can be chosen less than 1. Thus

$$\mu_\alpha(E) \leq \frac{2}{\delta}\int_E 1 - e^{-\delta} d\mu_\alpha(x) \leq \frac{2}{\delta}\int_E 1 - \exp\{-[\Sigma_1^N \psi(\gamma x_n) + \rho_\psi^N(x)]\} d\mu_\alpha(x),$$

$$\mu_\alpha(E) \leq \frac{1}{\delta} J_{N,\gamma}(\mu_\alpha) .$$

By our assumption there exists $N_0(\varepsilon,\delta)$ and $\gamma_0(\varepsilon,\delta)$ such that for $N \geq N_0$, $\gamma \leq \gamma_0$, $J_{N,\gamma}(\mu_\alpha) \leq \varepsilon\delta/2$ for all $\alpha \in A$. Hence, $\mu_\alpha\{x \in \ell_\varphi : \Sigma_1^N \varphi(\gamma x_n) + \rho^N(x) < \beta/2h\} > 1 - \varepsilon$ for $\gamma \leq \gamma_0$, $N \geq N_0$ and all $\alpha \in A$. With no loss of generality we take $\gamma_0 < 1$. For $x \in E^c$, $\Sigma_1^N \varphi(\gamma x_n) < \beta/2h < 1 < \frac{1}{\gamma}$. therefore, $\|P_N x\|_\varphi < \frac{1}{\gamma}$. Since $\{x \in P_N\ell_\varphi : \|P_N x\|_\varphi < 1/\gamma\}$ is a bounded subset of a finite dimensional space, we can find $x_1,\ldots,x_r$ in $P_N\ell_\varphi$ such that $\|x_i\|_\varphi < 1/\gamma$ and for all $x \in \ell_\varphi$ with $\|P_N x\|_\varphi < 1/\gamma$ we get $\min_{1\leq j\leq r} \|P_N x - x_j\|_\varphi < \beta/2h < 1$, hence, $\min_{1\leq j\leq r} \rho(P_N x - x_j) < \beta/2h$.

Then using 3.2.12 we can write

$$S(x_j,\beta) = \{y \in \ell_\varphi : \rho_\varphi(y - x_j) \leq \beta\}$$

$$\supseteq \{y \in \ell_\varphi : \rho_\varphi(P_N y - x_j) \leq \beta/2h,\ \rho_\varphi^N(y) \leq \beta/2h\} .$$

So for $y \in E^c$, $\rho_\varphi^N(y) \leq \beta/2h$ and $\exists\ j : 1 \leq j \leq r$ such that $\rho(P_N y - x_j) \leq \beta/2h$; thus, $y \in S(x_j,\beta)$. This shows $\overline{E^c} \subset \cup_{j=1}^r S(x_j,\beta)$ and theorem 3.2.14 yields that $\overline{E^c}$ is compact. Then $\mu_\alpha(\overline{E^c}) \geq 1 - \varepsilon$ for all $\alpha \in A$, and $\overline{E^c}$ compact implies using proposition 3.2.10 that the family $\{\mu_\alpha : \alpha \in A\}$ is conditionally compact.

3.3.4 LEMMA. Let $\{\mu_k\}$ be a sequence of probability measures on ℓ_φ such that $\{\tilde{\chi}_k\}$ converges in λ_φ-measure to $\tilde{\chi}$ where $\tilde{\chi}$ is the extended Fourier transform of a measure μ on ℓ_φ. Then

$$\lim_{N\to\infty;\ \gamma\downarrow 0}\ \sup_k J_{N,\gamma}(\mu_k) = 0.$$

The proof of this lemma is similar to the proof of lemma 3.3 in [13] with the λ_φ-measure replacing the symmetric stable measure used there, and hence, is omitted.

3.3.5 LÉVY CONTINUITY THEOREM. Let $\{\mu_k\}$ be a sequence of probability measures on ℓ_φ with Fourier transforms $\{\chi_k\}$ defined on ℓ'_φ. Then $\{\mu_k\}$ converges weakly to a measure μ with Fourier transform χ iff $\{\tilde{\chi}_k\}$ converges in λ_φ-probability to $\tilde{\chi}$ and $\{\chi_k\}$ converges to χ on ℓ'_φ.

PROOF. First assume that $\{\mu_k\}$ converges weakly to μ on ℓ_φ. Then it is obvious that $\lim_{k\to\infty} \chi_k(x) = \chi(x)$ on ℓ'_φ.

$$\int_\ell |\tilde{\chi}_k - \tilde{\chi}|^2 d\lambda_\varphi = \int_\ell |\tilde{\chi}_k|^2 d\lambda_\varphi - \int_\ell \tilde{\chi}_k \bar{\tilde{\chi}}\, d\lambda_\varphi$$
$$- \int_\ell \bar{\tilde{\chi}}_k \tilde{\chi}\, d\lambda_\varphi + \int_\ell |\tilde{\chi}|^2 d\lambda_\varphi .$$

Furthermore, $\{\mu_k\}$ converging weakly to μ implies $\{\mu_k \times \mu_k\}$ converges weakly to $\{\mu \times \mu\}$ and hence,

$$\begin{aligned}
\lim_k \int_\ell |\tilde{\chi}_k|^2 d\lambda_\varphi &= \lim_k \int_\ell \int_{\ell_\varphi} \exp\{i(x,y)\} d\mu_k(y) \int_{\ell_\varphi} \exp\{-(x,z)\} d\mu_k(z)\, d\lambda_\varphi \\
&= \lim_k \int_{\ell_\varphi} \int_{\ell_\varphi} \int_\ell \exp\{i(x,y-z)\} d\lambda_\varphi(x)\, d\mu_k(y)\, d\mu_k(z) \\
&= \lim_k \int_{\ell_\varphi} \int_{\ell_\varphi} \exp\{-\rho_\psi(y-z)\} d\mu_k(y)\, d\mu_k(z) \\
&= \int_{\ell_\varphi} \int_{\ell_\varphi} \exp\{-\rho_\psi(y-z)\} d\mu(y)\, d\mu(z) = A .
\end{aligned}$$

Similarly $\int_\ell \tilde{\chi}_k \bar{\tilde{\chi}}\, d\lambda_\varphi$ converges to A which is $\int_\ell |\tilde{\chi}|^2 d\lambda_\varphi$. Thus $\{\tilde{\chi}_k\}$ converges in mean-square to $\tilde{\chi}$ and so also in λ_φ-measure.

Conversely, using the preceding two lemmas we find $\{\mu_k\}$ is a conditionally compact sequence. Thus there exists a subsequence $\{\mu_{k_j}\}$ converging weakly to a probability measure ν with Fourier transform ξ. Then $\xi = \lim_j \chi_{k_j} = \chi$, and the uniqueness of the Fourier transforms for measures implies $\{\mu_{k_j}\}$ must converge weakly to μ (i.e. $\mu = \nu$ on ℓ_φ). Furthermore this shows that any convergent subsequence of $\{\mu_k\}$ must converge to μ. Hence, every subsequence of $\{\mu_k\}$ in turn has a subsequence which converges to μ and so $\{\mu_k\}$ converges weakly to μ, since weak convergence in this case is metric convergence.

The Bochner theorem for these spaces was proved in [11] using techniques developed in [13]. In the following statement of the theorem let C denote the complex numbers and a positive definite function is a function satisfying $\Sigma_{i,j=1}^N z_i \bar{z}_j f(r_i - r_j) \geq 0$ for any finite collection of real numbers $r_1, \ldots, r_N$ and complex scalars $z_1, \ldots, z_N$.

3.3.6 BOCHNER THEOREM. Suppose that φ is an Orlicz function satisfying the Δ_2-condition for small x and $\varphi \in K(2,0)$. Let $f : \ell_\varphi' \to C$. Then

$$f(y) = \int_{\ell_\varphi} \exp\{i(x,y)\} d\mu(x) \quad (y \in \ell_\varphi')$$

for some Borel probability measure on ℓ_φ iff f is positive definite, sequentially weak-star continuous and

$$1 = f(0) = \lim_N \int_{\ell_\varphi'} f(y) d\lambda_\varphi(\epsilon_N, y)$$

where

$$\int_\ell \exp\{i(x,y)\} d\lambda_\varphi(\epsilon_N, y) = \Pi_{j=1}^N \exp(-\epsilon_{N,j} \psi(\beta_j(x))),$$

for any sequence of $\epsilon_N = (\epsilon_{N,1}, \ldots, \epsilon_{N,N})$ such that $\epsilon_{N,j} \geq 0$ (for all N and j) and $\lim_{N\to\infty} \max_{1\leq j\leq N} \epsilon_{N,j} = 0$, and when ψ is given by equation 3.3.1.

Now we consider the case where the Orlicz function is from the class $K(2,\infty)$: still satisfying the Δ_2-condition for all $x \geq 0$. We derive the Lévy continuity theorem and Bochner theorem for the sequence space ℓ_φ by showing that ℓ_φ satisfies conditions given in the work of Kuelbs and Mandrekar. First we give necessary notation and terminology from [14], state the theorems, and then show that their hypotheses are satisfied by this class of Orlicz sequence spaces.

Let ℓ_∞ denote the Banach space of all bounded sequences of real numbers with the usual supremum norm, and ℓ_∞^+ the positive cone of ℓ_∞. E will be a real F-space with a basis $\{b_n\}$.

3.3.7 DEFINITION. If $\lambda \in \ell_\infty^+$ and $\{\mu_\alpha : \alpha \in A\}$ is a family of probability measures on E such that

$$\mu_\alpha\{x \in E : \Sigma_{k=1}^\infty \lambda_k x_k^2 < \infty\} = 1$$

for each $\alpha \in A$, we say λ is sufficient for the family $\{\mu_\alpha : \alpha \in A\}$.

3.3.8 DEFINITION. A family of probability measures $\{\mu_\alpha : \alpha \in A\}$ on E is a λ-family for some $\lambda \in \ell_\infty^+$ if λ is sufficient for $\{\mu_\alpha : \alpha \in A\}$ and for every $\epsilon, \delta > 0$ there is a sequence $\{\epsilon_N\}$ such that

$$\mu_\alpha\{x \in E : \Sigma_{k=N+1}^\infty \lambda_k x_k^2 < \delta\} > 1 - \epsilon$$

implies

$$\mu_\alpha\{x \in E : \|\Sigma_{k=N+1}^\infty x_k b_k\| < b(\delta)\} > 1 - (\epsilon + \epsilon_N)$$

where $\lim_N \epsilon_N = 0$ and b is a strictly increasing function on $[0,\infty)$ with $b(0) = 0$.

Now let $\alpha(\cdot)$ be a convex function on $[0,\infty)$ such that $\alpha(0) = 0$ and $\alpha(s) > 0$ if $s > 0$. Further, assume that for every compact K of E there exists an $r > 0$ such that $y \in K$ implies $A(y) = \Sigma_{i=1}^{\infty} \alpha(y_i^2) < r$, and for every $r > 0$ there exists $M > 0$ such that $A(y) < r$ implies $\Sigma_{i=1}^{\infty}\alpha(y_i^2) \leq M\gamma(\|y'\|)$ where $\gamma(\cdot)$ is another continuous function on $[0,\infty)$ such that $\gamma(0) = 0$.

3.3.9 DEFINITION. If the quasi-norm, $\|\cdot\|$, on E admits the existence of functions $\alpha(\cdot)$ and $\gamma(\cdot)$ having the above properties we will say that it is accessible.

If the quasi-norm on E is accessible then by the τ_α-topology we will mean the topology on E' generated by taking as a subbase all translates of sets of the form $\{x \in E' : T(x,x) < 1\}$ as $T(\cdot,\cdot)$ varies over the symmetric, positive definite, bilinear forms on E' which are jointly weak-star sequentially continuous on E' and satisfying $\Sigma_{k=1}^{\infty}\alpha(t_{kk}) < \infty$ where $t_{kk} = T(\beta_k,\beta_k)$ with β_k the k^{th} coordinate functional on E (i.e. $\beta_k(x) = x_k$).

3.3.10 BOCHNER THEOREM [14]. If E has an accessible quasi-norm then a function χ on E' is the Fourier transform of a probability measure iff

i) $\chi(0) = 1$, χ is positive definite,

ii) χ is continuous in the τ_α-topology,

iii) the family of measures $\{\mu_N\}$ corresponding to $\chi(P_N(\cdot))$ has a subsequence which is a λ-family for some $\lambda \in \ell_\infty^+$ satisfying $\lim_k \Sigma_{i=k}^{\infty}\lambda_i t_{ii} = 0$ whenever $\Sigma_{i=1}^{\infty}\alpha(t_{ii}) < \infty$. Here $\alpha(\cdot)$, of course, is the function associated with the accessibility of the quasi-norm.

For $\varphi \in K(2,\infty)$ we now let λ_φ denote the product probability on ℓ such that the i^{th} coordinate is Gaussian with mean zero and variance $\lambda_i > 0$, and $\lambda \in \ell_\infty^+$ is chosen so that it is sufficient for the probability measure μ on E.

3.3.11 <u>LÉVY CONTINUITY THEOREM</u> [14]. Let $\{\mu_k\}$ be a sequence of probability measures on E with Fourier transforms $\{\chi_k\}$. Then $\{\mu_k\}$ converges weakly to a measure μ with Fourier transform χ iff $\{\mu_k\}$ is a λ-family for some $\lambda \in \ell_\infty^+$ which is also sufficient for μ, $\{\chi_k\}$ converges to χ on a subset of E' which is dense in E' with respect to weak-star sequential convergence, and $\{\tilde{\chi}_k\}$ converges in λ_φ measure to $\tilde{\chi}$.

To show these results apply to ℓ_φ when $\varphi \in K(2,\infty)$ we need only show $\|\cdot\|_\varphi$ is accessible since we already know ℓ_φ is an F-space with a basis (φ satisfies Δ_2-condition).

Let $\alpha(s) = \varphi(\sqrt{s})$. Then by proposition 2.2.19 we know α is a convex function on $[0,\infty)$ such that $\alpha(0) = 0$ and $\alpha(s) > 0$ for $s > 0$. For K a compact subset of ℓ_φ, theorem 3.2.13 implies $\sup_{y \in K} \rho_\varphi(y) < \infty$ and $\rho_\varphi(y) = \Sigma_{i=1}^\infty \varphi(y_i) = \Sigma_{i=1}^\infty \alpha(y_i^2)$; thus the first conditions for accessibility is met. There remains only to be exhibited for $r > 0$ a constant M and continuous function $\gamma(\cdot)$ such that $\gamma(0) = 0$ such that $A(y) < r$ implies $A(y) \le M\gamma(\|y\|_\varphi)$.

Since α is convex so is φ (composition of convex functions) and then using the Δ_2-condition we note $\varphi(\lambda x) \le \lambda\varphi(x)$ for $0 < \lambda \le 1$ and $\varphi(\lambda x) \le h^n \varphi(x)$ for $2^{n-1} < \lambda \le 2^n$, $n = 1,2,\dots$. Now define

$$\gamma(\lambda) = \begin{cases} h\lambda & 0 \le \lambda \le 1 \\ \frac{1}{2^{n-1}} (h^{n+1} - h^n)\lambda + 2h^n - h^{n+1} & 2^{n-1} \le \lambda \le 2^n,\ n = 1,2,\dots \end{cases}$$

Then clearly γ is a continuous piece-wise linear function and $\gamma(0) = 0$. Recall that since φ is convex $\|y\|_\varphi = \inf\{\varepsilon > 0 : \rho_\varphi(y/\varepsilon) \leq 1\}$. Take $y \in \ell_\varphi$ and choose the smallest n such that $\|y\|_\varphi/2^n \leq 1$.

$$\rho_\varphi(y) = \rho_\varphi(\|y\|_\varphi \frac{y}{\|y\|_\varphi}) = \rho_\varphi(\frac{\|y\|_\varphi}{2^n} 2^n \frac{y}{\|y\|_\varphi})$$

$$\leq \rho_\varphi(2^n \frac{y}{\|y\|_\varphi}) \leq h^n \rho_\varphi(\frac{y}{\|y\|_\varphi}) \leq h^n .$$

But $2^{n-1} \leq \|y\|_\varphi \leq 2^n$, so $\gamma(\|y\|_\varphi) \geq h^n$. Therefore $\rho_\varphi(y) = A(y) \leq \gamma(\|y\|_\varphi)$.

Thus the results of [14] apply to Orlicz sequence spaces when $\varphi \in K(2,\infty)$.

CHAPTER IV

GAUSSIAN MEASURES ON ORLICZ SEQUENCE SPACES

§4.1 INTRODUCTION AND PRELIMINARIES

In this chapter the problem of characterizing the Fourier transforms of Gaussian measures on Orlicz sequence spaces is considered with the main theorem being for the case $\varphi \in K(2,0)$ and convex. This extends the results of ([28], p. 1561) in which Gaussian measures on the classical ℓ_p spaces $(1 \leq p < \infty)$ are treated. In the last section our result is used to derive a central limit theorem for these spaces.

What follows are some known results and notation that we use in this chapter.

4.1.1 DEFINITION. A probability measure μ on a topological vector space E is said to be Gaussian with mean $a \in E$ if the distribution of $<\cdot,y>$ is Gaussian with mean $<a,y>$ for each linear functional $y \in E'$ (the topological dual of E).

If ℓ (the space of all real sequences $x = \{x_n\}$) is given the Tychonoff topology, then it becomes a separable reflexive Fréchet ([24], p. 8) space. Let ℓ_0 denote the linear subspace of ℓ which consists of those elements of ℓ containing only a finite number of nonzero coordinates. Then the topological dual of ℓ is ℓ_0, i.e. $\ell' = \ell_0$.

4.1.2 PROPOSITION. Let φ be an Orlicz function, then ℓ_φ is a Borel subset of ℓ.

PROOF. The coordinate functionals defined on ℓ are Borel measurable maps (i.e. $\beta_j : \ell \to \mathbb{R}$ defined by $\beta_j(x) = x_j$ for all $x \in \ell$ is a Borel measurable map for all j). Since φ is a continuous function, $\varphi \circ \beta_j$ is Borel measurable for all j; thus, $\Sigma_{j=1}^N \varphi \circ \beta_j$ defines a Borel measurable map on ℓ for all N. That ℓ_φ is a Borel subset of ℓ now follows from these remarks and that

$$\ell_\varphi = \cup_{n=1}^\infty \cap_{m=1}^\infty \{x \in \ell : \Sigma_{j=1}^m \varphi(\beta_j(x)) \le n\} .$$

The above proposition implies that the Borel subsets of ℓ_φ are Borel subsets of ℓ; hence, a measure μ defined on the Borel subsets of ℓ_φ can be considered as a Borel measure on ℓ.

4.1.3 DEFINITION. A matrix $S = (s_{ij})$ is said to be positive definite if $eSe' \ge 0$ for all $e \in \ell$ (e' denotes transpose of the vector e).

If $S = (s_{ij})$ is a symmetric positive definite matrix then

(4.1.4) $$s_{ij} \le \sqrt{s_{ii}}\ \sqrt{s_{jj}} \quad \text{for all } i \text{ and } j .$$

This is easy to see by considering in definition 4.1.3, the vector e having $-\sqrt{s_{jj}}$ in the i^{th} coordinate, $\sqrt{s_{ii}}$ in the j^{th} coordinate and zero's elsewhere.

The next proposition is well known and can be found in [28].

4.1.5 PROPOSITION. If μ is a Gaussian measure on ℓ then the Fourier transform of μ is given by

(4.1.6) $$\chi(f) = \exp\{i <f,m> - 1/2 <Sf,f>\} \quad f \in \ell_0 ,$$

where $<f,m> = \Sigma_{n=1}^{\infty} f_n m_n$, $m = \{m_n\} \in \ell$ and $<Sf,f> = \Sigma_{j,k=1}^{\infty} s_{jk} f_j f_k$ with the matrix (s_{jk}) being symmetric and positive definite. Conversely, all functionals of the form 4.1.6 defined on ℓ_0 are the Fourier transforms of Gaussian measures on ℓ.

A relation between moments of a Gaussian random variable and its standard deviation is given by the next proposition [27]; the proof of which is straightforward and can be found through a change of variables and the definition of the gamma function, Γ.

4.1.7 PROPOSITION. Let X be a Guassian random variable with mean zero and variance σ^2. Then,

$$E|X|^p = c(p,2)(EX^2)^{p/2} , \tag{4.1.8}$$

where $c(p,2) = \Gamma(p + 1/2)/(2\pi)^{1/2}$.

We also have need for the following special case of a result of Fernique [5] pertaining to the integrability of Gaussian vectors.

4.1.9 PROPOSITION. If μ is a Gaussian measure on an F-space $(E,\|\cdot\|)$, then

$$\int_E \|x\| d\mu(x) < \infty .$$

§4.2 REPRESENTATION OF GAUSSIAN MEASURES ON ORLICZ SEQUENCE SPACES

In this section we will characterize Gaussian measures on Orlicz sequence spaces which are generated by a convex Orlicz function φ where $\varphi \in K(2,0)$ by giving a representation for their characteristic functionals.

For convex Orlicz functions φ with a right derivative $p(t)$ that is strictly positive at the origin ($p(0) > 0$) proposition 2.2.3 says that ℓ_φ is isomorphic (as a topological vector space) to ℓ_1. Since the form of Fourier transforms of Gaussian measures on ℓ_1 are well known (e.g. [28]), we assume that $p(0) = 0$.

4.2.1 THEOREM. Let φ be a convex Orlicz function satisfying the Δ_2-condition for all x with $\varphi(x) = \int_0^{|x|} p(t)dt$. Furthermore, suppose that $\varphi \in K(2,0)$ with $p(0) = 0$. Then μ is a Gaussian measure on ℓ_φ with mean $a \in \ell_\varphi$ iff the Fourier transform χ of μ defined on ℓ_φ' has the form

(4.2.2) $\chi(f) = \exp\{i \langle a,f\rangle - 1/2 \langle Sf,f\rangle\}$ for all $f \in \ell_\varphi'$,

where $\langle Sf,f\rangle = \Sigma_{j,k=1}^{\infty} s_{jk} f_j f_k$ with (s_{jk}) being a symmetric positive definite matrix such that $\Sigma_{k=1}^{\infty} \varphi(s_{kk}^{1/2}) < \infty$. In addition, S can be taken as $s_{jk} = \int_{\ell_\varphi} \beta_j(x-a)\beta_k(x-a)d\mu(x)$ for $j = 1,2,\ldots$; $k = 1,2,\ldots$.

PROOF. Let φ be a convex Orlicz function satisfying the hypothesis of the theorem, and let ψ denote the Orlicz function complementary to φ (2.2.4).

Now suppose that μ is a Gaussian measure on ℓ_φ with mean $a \in \ell_\varphi$. The Bochner theorem 3.3.6 for ℓ_φ implies that the Fourier transform χ of μ on ℓ_φ' is weak-star sequentially continuous; thus, $\chi(f) = \lim_N \chi(P_N f)$ for $f \in \ell_\varphi'$. Now $\ell_\varphi' \subset \ell$ and $P_N f \in \ell_0$

for all N, so thinking of μ as a Gaussian measure on ℓ and applying proposition 4.1.5 we get the existence of a symmetric positive definite matrix $S = (s_{jk})$ with $s_{jk} = \int_{\ell}\beta_j(x-a)\beta_k(x-a)d\mu(x) = \int_{\ell_\varphi}\beta_j(x-a)\beta_k(x-a)d\mu(x)$ for $j = 1,2,\ldots$; $k = 1,2,\ldots$; and such that

$$\chi(f) = \lim_N\chi(P_Nf) = \lim_N\exp\{i <a,P_Nf> - 1/2\ \Sigma_{j,k=1}^N s_{jk}f_jf_k\}$$

for $f \in \ell_\varphi'$. Note that $\lim_N<a,P_Nf> = <a,f>$ for $f \in \ell_\varphi'$. To complete this part of the proof we need to show $\lim_N\Sigma_{j,k=1}^N s_{jk}f_jf_k$ exists for all $f \in \ell_\varphi'$ and that $\{s_{kk}^{\frac{1}{2}}\} \in \ell_\varphi$. We do this by first proving the following lemma.

4.2.3 LEMMA. With the notation as given above $\{s_{kk}^{\frac{1}{2}}\} \in \ell_\varphi$.

PROOF. Since μ is Gaussian with mean vector $a \in \ell_\varphi$, $\beta_j(\cdot)$ is Gaussian with mean $<a,\beta_j> = \beta_j(a)$ for $j = 1,2,\ldots$. By proposition 4.1.7,

$$\int_{\ell_\varphi}|\beta_j(x-a)|d\mu(x) = E|\beta_j(X-a)| = c(1,2)[E(\beta_j(X-a))^2]^{\frac{1}{2}},$$

i.e.,

$$s_{kk}^{\frac{1}{2}} = (1/c(1,2))\int_{\ell_\varphi}|\beta_j(x-a)|d\mu(x)\ . \tag{4.2.4}$$

Proposition 2.2.8 implies that h_ψ' is isomorphic to ℓ_φ, and so to complete the proof of the lemma it will suffice to show $\Sigma_{k=1}^\infty s_{kk}^{\frac{1}{2}}\beta_k(y) < \infty$ for all $y \in h_\psi$. Take $y \in h_\psi$. Then there exists $r > 0$ such that $\rho_\psi(\beta_k(y)/r) \le 1$. Now using 4.2.4,

$$|\Sigma_{k=1}^N s_{kk}^{\frac{1}{2}}\beta_k(y)| \le (1/c(1,2))\int_{\ell_\varphi}\Sigma_{k=1}^N|\beta_k(x-a)||\beta_k(y)|d\mu(x).$$

Multiplying the right hand side by 1 and applying proposition 2.2.6,

$$\Sigma_{k=1}^{N} s_{kk}^{\frac{1}{2}} \beta_k(y) \le (r/c(1,2))\rho_\psi(y/r)\int_{\ell_\varphi} |||x-a|||_\varphi d\mu(x),$$

but the right hand side is finite since $\rho_\psi(y/r) \le 1$ and proposition 4.1.9 yields $\int_{\ell_\varphi} |||x-a|||_\varphi d\mu(x) < \infty$. Now define $S_N : h_\psi \to \mathcal{R}$ by $S_N(y) = \Sigma_{k=1}^{N} s_{kk}^{\frac{1}{2}} \beta_k(y)$. The above shows that the limit on N of $S_N(y)$ exists for every $y \in h_\psi$, and applying the uniform boundness principle [24], $S(y) = \lim_N S_N(y)$ is a bounded linear functional on h_ψ. By the earlier remark that h_ψ' can be identified with ℓ_φ, there must exist a unique $x \in \ell_\varphi$ such that $S(y) = \langle y,x\rangle = \Sigma_{k=1}^{\infty} \beta_k(x)\beta_k(y)$ for all $y \in h_\psi$, but $S(y) = \Sigma_{k=1}^{\infty} s_{kk}^{\frac{1}{2}} \beta_k(y)$ implies that $x = \{s_{kk}^{\frac{1}{2}}\}$ and so $\Sigma_{k=1}^{\infty} \varphi(s_{kk}^{\frac{1}{2}}) < \infty$ completing the proof of the lemma.

Upon proving the lemma we can finish the "if" portion of the proof of the theorem by showing that $\lim_N \Sigma_{j,k=1}^{N} s_{jk} f_j f_k$ exists for all $f \in \ell_\varphi'$. Since (s_{jk}) is a symmetric positive definite matrix, inequality 4.1.4 can be used to derive,

$$|\Sigma_{j,k=1}^{N} s_{jk} f_j f_k| \le \Sigma_{j,k=1}^{N} s_{jk} |f_j||f_k| \le (\Sigma_{j=1}^{N} s_{jj}^{\frac{1}{2}} |f_j|)\Sigma_{k=1}^{N} s_{kk}^{\frac{1}{2}} |f_k| .$$

Then by the above lemma $\{s_{kk}^{\frac{1}{2}}\} \in \ell_\varphi$ implying that we have a bound independent of N by proposition 2.2.6(i) completing the proof.

Conversely suppose that $\chi(f) = \exp\{i \langle a,f\rangle - 1/2 \langle Sf,f\rangle\}$ where (s_{jk}) is a symmetric positive definite matrix such that $\Sigma_{k=1}^{\infty} \varphi(s_{kk}^{\frac{1}{2}}) < \infty$ and $\langle Sf,f\rangle = \Sigma_{j,k=1}^{\infty} s_{jk} f_j f_k$. Let $\{\mu_N\}$ denote the sequence of Gaussian measures on ℓ_φ with Fourier transform $\chi_N(f) = \chi(P_N f)$ for $N = 1,2,\ldots$. Since for each N we are dealing with a measure on a finite dimensional space, the form of $\chi_N(f)$ implies that μ_N is a Gaussian measure. Now to show that the sequence of measures $\{\mu_N\}$ is conditional compact. Take $\varepsilon,\delta > 0$.

$$\mu_N\{y \in \ell_\varphi : \rho_\varphi^L(y) > \delta\} \le 1/\delta \int_{\ell_\varphi} \Sigma_{i=L+1}^{\infty} \varphi(\beta_i(y))d\mu_N(y)$$

$$= 1/\delta \Sigma_{i=L+1}^{\infty} \int_{\ell_\varphi} \varphi(\beta_i(y))d\mu_N(y) \ .$$

Since $\varphi \in K(2,0)$ proposition 2.2.19 implies that $\varphi \circ \surd$ is concave; thus,

$$\mu_N\{y \in \ell_\varphi : \rho_\varphi^L(y) > \delta\} \le 1/\delta \Sigma_{i=L+1}^{\infty} \varphi([\int_{\ell_\varphi} \beta_i^2(y)d\mu(y)]^{\frac{1}{2}}).$$

However,

$$\int_{\ell_\varphi} \beta_i^2(y)d\mu(y) = \begin{cases} s_{ii} & \text{for} \quad 1 \le i \le N \\ 0 & \text{for} \quad N < i \end{cases}$$

and since $\{s_{kk}^{\frac{1}{2}}\} \in \ell_\varphi$ we can choose L sufficiently large independent of N so that $\rho_\varphi^L(\{s_{kk}^{\frac{1}{2}}\}) < \delta/\varepsilon$; therefore, $\mu_N\{y \in \ell_\varphi : \rho^L(y) > \delta\} < \varepsilon$ for $N = 1,2,\ldots$. From this fact and utilizing the techniques in the proof of lemma 3.3.3, we see that the sequence of measures $\{\mu_N\}$ is conditionally compact and must converge weakly to some measure μ on ℓ_φ. Since $\chi(f) = \lim_N \chi(P_N f)$ for each $f \in \ell_\varphi'$, χ is the Fourier transform of the measure μ. We are done if μ can be concluded to be Gaussian. Choose $y \in \ell_\varphi'$. Then the sequence of Gaussian random variables $\{\langle P_N \cdot, y\rangle\}$ converges to the random variable $\langle \cdot, y\rangle$; hence, the random variable $\langle \cdot, y\rangle$ must be Gaussian for each $y \in \ell_\varphi'$. Therefore μ is a Gaussian measure on ℓ_φ.

As a corollary to this theorem we get the result of M. Nicolas Vakhania [28] (see also [15]).

4.2.5 <u>COROLLARY</u>. $\chi(f) = \exp\{i \langle a, f\rangle - 1/2 \langle Sf, f\rangle\}$ for $f \in \ell_p'$ is the Fourier transform of a Gaussian measure μ on ℓ_p

with mean $a \in \ell_p$ $(1 < p \leq 2)$ iff (s_{jk}) is a symmetric positive definite matrix such that $\langle Sf,f\rangle = \Sigma_{j,k=1}^{\infty} s_{jk} f_j f_k$ and $\Sigma_{k=1}^{\infty} s_{kk}^{p/2} < \infty$.

§4.3 A CENTRAL LIMIT THEOREM FOR ORLICZ SEQUENCE SPACES

We now study a central limit theorem for sequences of independent identically distributed random variables (vectors) taking values in an Orlicz sequence space ℓ_φ.

4.3.1 THEOREM. Suppose that φ is a convex Orlicz function satisfying the Δ_2-condition for all x with $\varphi \in K(2,0)$. Let $Z_1, Z_2, \ldots$ be independent identically distributed random vectors with values in ℓ_φ and having zero expected values. Let μ_N denote the measure induced on ℓ_φ by $Y_N = N^{-\frac{1}{2}}(Z_1 + \ldots + Z_N)$ and μ denote a measure equivalent to μ_1. Then if

$$(4.3.2) \qquad \Sigma_{k=1}^{\infty}\varphi([\int_{\ell_\varphi} \beta_k^2(y)d\mu(u)]^{\frac{1}{2}}) < \infty$$

and if $\langle Sf,f\rangle = \Sigma_{j,k=1}^{\infty} s_{jk} f_j f_k$ $(f \in \ell_\varphi')$ is a norm continuous function on ℓ_φ' where $s_{jk} = \int_{\ell_\varphi} \beta_j(y)\beta_k(y)d\mu(y)$ (for all j and k), then $\{\mu_N\}$ converges weakly to the Gaussian measure ν on ℓ_φ with Fourier transform $\chi(f) = \exp\{-1/2 \langle Sf,f\rangle\}$ for all $f \in \ell_\varphi'$.

4.3.3 REMARK. Noting that $S = (s_{jk})$ is a symmetric positive definite matrix and that $\{s_{kk}^{\frac{1}{2}}\} \in \ell_\varphi$, we have the hypothesis of theorem 4.2.1 satisfied; thus, we conclude that χ is the Fourier transform of a Gaussian measure on ℓ_φ. Now we prove the theorem.

PROOF. For $f \in \ell_\varphi'$ define

$$\chi_N(f) = E\{\exp\{i\langle Y_N, f\rangle\}\} = \int_{\ell_\varphi} \exp\{i \langle y,f\rangle\}d\mu_N(y)$$

and

$$\chi(f) = E\{\exp\{i\langle Z_1, f\rangle\}\} = \int_{\ell_\varphi} \exp\{i \langle y,f\rangle\}d\mu(y).$$

Then for all $f \in \ell_\varphi'$, $\chi_N(f) = [\chi(f/\sqrt{N})]^N$. Using the series expansion for the exponential function we write,

$$\chi(f/\sqrt{N}) = \int_{\ell_\varphi} [1 + i\langle y, f/\sqrt{N}\rangle - 1/2 \langle y, f/\sqrt{N}\rangle^2$$
$$+ o(1/N)]d\mu(y),$$

but Z_1 having zero mean implies

$$\chi(f/\sqrt{N}) = 1 - (1/2)\int_{\ell_\varphi} \langle y, f/\sqrt{N}\rangle^2 d\mu(y) + o(1/N)$$
$$= 1 - (1/2N)\Sigma_{j,k=1}^{\infty} s_{jk} f_j f_k + o(1/N);$$

thus,

$$\lim_N \chi_N(f) = \lim_N [\chi(f/\sqrt{N})]^N$$
$$= \exp\{-1/2 \langle Sf, f\rangle\} = \chi(f).$$

Hence χ must be a positive definite function on ℓ_φ' with $\chi(0) = 1$. Now we show that the sequence of measures $\{\mu_N\}$ is conditionally compact and must converge weakly to a measure ν.

Take $\varepsilon, \delta > 0$. Since $\varphi \in K(2,0)$, by proposition 2.2.19 $\varphi \circ \sqrt{\ }$ is concave, implying,

$$\mu_N\{y \in \ell_\varphi : \rho^L(y) > \delta\} \le (1/\delta)\Sigma_{j=L+1}^{\infty}\int_{\ell_\varphi} \varphi(\beta_j(y))d\mu_N(y)$$
$$\le (1/\delta)\Sigma_{j=L+1}^{\infty}\varphi(s_{jj}^{\frac{1}{2}}).$$

Since $\int_{\ell_\varphi} \beta_j^2(y)d\mu_N(y) = \int_{\ell_\varphi} \beta_j^2(y)d\mu(y)$ for $j = 1,2,\ldots$; and by assumption 4.3.2 as in the proof of theorem 4.2.1, we conclude that the sequence of measures $\{\mu_N\}$ is conditionally compact. Therefore $\{\mu_N\}$ converges weakly to a measure ν on ℓ_φ, and since $\lim_N \chi_N(f) = \chi(f)$ for all $f \in \ell_\varphi'$, the Fourier transform of ν is χ. Then by theorem 4.2.1 (remark 4.3.3) ν is a Gaussian measure on ℓ_φ. Thus we know $\mu_N \Rightarrow \nu$, and ν is the Gaussian measure with the required Fourier transform.

§4.4 CONCLUDING REMARKS

The question of the realization of Orlicz function spaces can be discussed following these definitions taken from [3] and [26]. Let E denote a topological vector space of Borel measurable functions defined on the interval $[0,1]$.

4.4.1 DEFINITION. A random linear form (r.l.f.) X, defined on E is a linear mapping of E into a space of measurable functions $\mathcal{M}(\Omega,\mathcal{A},P)$.

4.4.2 DEFINITION. A r.l.f. X defined on E is said to have independent increments if for all finite collections of elements $\{f_1,\ldots,f_n\} \subset E$ having disjoint support, the random variables $X(f_1),\ldots,X(f_n)$ are independent; furthermore, $f_n \to 0$ a.e. implies $X(f_n) \to 0$ in probability.

4.4.3 DEFINITION. A r.l.f. X defined on E is said to be homogeneous if for congruent Borel subsets E_1,E_2 of $[0,1]$, $\mathcal{L}(X([E_1])) = \mathcal{L}(X([E_2]))$.

4.4.4 DEFINITION. A r.l.f. X defined on E is said to be symmetric if the random variable $X(f)$ is symmetric about zero for all $f \in E$.

A t.v.s. E is said to be realizable as a space of random variables if there exists a probability space $(\Omega,\mathcal{A},P)$ and a linear homeomorphism T mapping E into $\mathcal{M}(\Omega,\mathcal{A},P)$ such that the map T when considered as a r.l.f., is symmetric, homogeneous with independent increments.

The problem as formulated above is considered by Urbanik and Woyczynski [26], and Bretagnolle and Dacunha-Castelle [3] when the topological vector space E is an Orlicz function space $L_\varphi[0,1]$. Their results can be used to show that as in the sequence space case $\varphi \in K(2,0)$

is necessary and sufficient for the function space L_φ to be realizable.

The following problems are presently under study: 1) Harmonic analysis on realizable Orlicz function spaces. 2) The domain of attraction problem for Gaussian measures on realizable Orlicz sequence spaces. 3) Representation of Fourier transforms of infinitely divisible distributions on realizable Orlicz sequence spaces. The results of this study will be presented elsewhere.

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