A METHOD TO IMPROVE THE ANALOG COMPUTER SOLUTION OF LINEAR SYSTEMS

Thesis for the Degree of Ph. D.

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WILLIAM THOMAS MARQUITZ

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This is to certify that the

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By

William Thomas Marquitz

AN ABSTRACT

Submitted to
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In the analysis of linear time-invariant systems, the solution of a system of first-order differential equations of the form,

$$\frac{d}{dt} X(t) = -A\, X(t),\ X(0)$$

where

X(t) is an n-vector and A is an nxn constant matrix, is required. The solution may be obtained, for example by analytical methods, digital computer, or analog computer. In this thesis, the solution by analog computer is considered.

In the analog computer solution of the system of first order linear differential equations with constant coefficients, certain errors and stability problems result. Some work appearing in the literature for reducing the error concerns itself with improving the internal construction of the operational amplifiers. Other work deals actually with the calculation of the errors.

In this thesis, a new approach, called the "correction procedure" is followed. From the given models of the active computing elements, it is determined what changes must be made in the potentiometers of the computer set up for the given state equations such that a vector, which is the output of n active

elements, is of the same order as that of $X(t)$ and has the solution to

$$\frac{d}{dt} X(t) = - A X(t), \quad X(0) .$$

The correction procedure is easy to apply and only the potentiometer settings are affected. Also the number of active computing elements is not increased.

It is shown that the time and amplitude scaled form of any system of first-order linear differential equations with constant coefficients may be solved using the correction procedure, and an "improved" analog computer solution is obtained.

In applying the correction procedure, it is also shown that no internal instabilities are introduced within the corrected form of the set up.

An algorithm is given to illustrate the application of the correction procedure, in a step by step manner.

Several numerical examples are included for demonstrating the use of the correction procedure. Finally, some of these examples are actually run on an analog computer with the operational amplifiers deliberately modified to have poor frequency response characteristics.

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CHAPTER I

INTRODUCTION

1.1. Previous Work

In the analysis of linear time-invariant physical systems one has to deal with the solution of a set of first order linear differential equations with constant coefficients of the form,

$$\frac{d}{dt}\overline{X}(t) = -\overline{A}\,\overline{X}(t) + \overline{B}\,E(t) \;;\; \overline{X}(0) \qquad (1.1.1)$$

where $\overline{X}(t)$ is an n-vector, called the state, and $E(t)$ is an m-vector, called the excitation or driver. There are several methods for the solution of (1.1.1). It may be solved analytically, by digital computer, or by analog computer. The solution by analytical and digital computer methods has been described by several authors, in particular Koenig, Tokad, Kesavan [1]. The solution of the state model by analog computer is also described by several authors ([1], [2] and others).

The primary interest of this work is the analog computer solution of the state model in (1.1.1) with a specific interest which will be described shortly. The work is concerned with a solution technique when non-ideal or practical computing elements are used.

In practice, the analog computer does not yield the exact or analytical solution to (1.1.1). Several papers ([3], [4], [5], [6] and others) and many books ([2], [7], [8], [9], [10], [11], and others) have considered various sources of errors. It has been shown by these authors that bandwidth limitations inherent in the active devices produce shifts in the characteristic roots of the system being solved. Macnee [3] has applied the error analysis to the analog computer solution of the equation of harmonic motion and has found an expression for the bound on the error as a function of time. The primary reason for the error analysis in previous work by Macnee, Marsocci, Cederbaum [12] and others is to insure stability of the analog computer solution without regard to accuracy in the solution.

Gilbert and McHugh [13] have described a method whereby a set of algebraic equations of the form

$$AX = Y$$

may be solved and the instability problems are eliminated. Although the method is involved, the procedure is an attempt to change the set up of the problem in order to yield a stable solution.

Turner [14] has given a gradient technique which modifies the set up in order to satisfy the equality constraints that may exist for the problem. This type of approach may be useful for some types of control problems.

Walter [15] has considered the instability problems that may exist due to regenerative or algebraic loops.

Otterman [16] develops the idea of using loop analysis or node analysis in order to use the fewest possible number of integrators and is related to the work in [15]. This is the first step toward the use of state equations in the analysis of electrical networks. In a related paper, Anderson and Newcomb [17] consider the same problem as Otterman looked into except that the scattering matrix rather than the imittance matrix is used. The problem of a minimum number of integrators is, of course, automatically eliminated by the use of the state model formulation procedure.

It will be shown that (1.1.1) can be placed into the form,

$$\frac{d}{dt} X(t) = - A\, X(t), \; X(0) \; . \tag{1.1.2}$$

The analog computer set up is based on the modification of (1.1.2) in the following form,

$$\frac{d}{dt} X(t) = A_p[-X(t)] + (-A_m)X(t) \; ; \; X(0) \tag{1.1.3}$$

where A_p is a matrix of the same order as A but containing all non-negative entries while A_m is a matrix of the same order as A but containing all non-positive entries. It is clear that

$$A = A_p + A_m \tag{1.1.4}$$

and this decomposition is, of course, not unique. This, as will be seen later, will provide some flexibilities in the correction procedure. The analog computer diagram of (1.1.3) is shown in Figure 1.1.1.

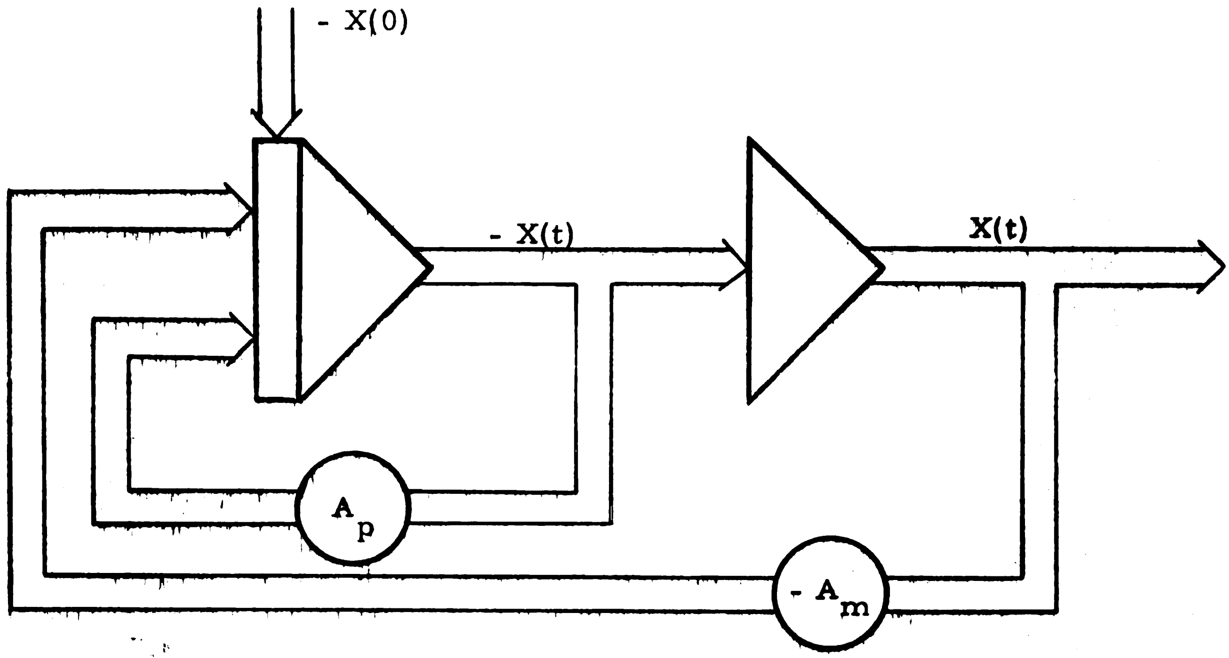


Figure 1.1.1. Computer Set Up of State Model.

The symbols used for the integrators and summing amplifiers (in heavy lines) refers to a vector of active components of the same order as X(t).

In the classical approach to system analysis, usually an n-th order differential equation for each of the functions $x_i(t)$ is developed. It is known [18] that any n-th order linear differential equation with constant coefficients may be replaced by an equivalent set of linear first order differential equations with a coefficient matrix in the companion matrix form. Although this procedure yields a state model similar to (1.1.2) it actually solves only one function and hence involves more work than the direct state model formulation described in [1]. Note that the companion matrix form of the n-th order differential equation is a special case of the class described by (1.1.2).

1.2. Outline of Thesis

In the practical solution of (1.1.2) on an analog computer, the computing elements are assumed to have a given model. In this thesis, for the given models, a formulation technique is derived in order to get exact solutions for the class of differential equations described by (1.1.2).

A correction procedure is developed for the cases in which different types of component models are considered. This correction procedure does not increase the number of active computing elements as would be used in a normal set up but changes the coefficient potentiometer settings.

The analysis is carried out in the time domain and relies on the properties of differential equations rather than those equations obtained by transform methods.

The idea involved in the correction procedure is somewhat similar to that of the predistortion technique used in filter synthesis [19]. However, except for a very special case, the correction procedure cannot be considered as a transformation of a function of a complex variable, neither can it be considered as a method of linear transformation.

The technique used in this thesis involves the construction of the state model of the following form,

$$\frac{d}{dt} Y(t) = - C\, Y(t)\ , \quad Y(0) \qquad (1.2.1)$$

where the order of $Y(t)$ is greater than the order of $X(t)$ in (1.1.2).

In general, if X(t) is the exact solution of (1.1.2) the correction procedure is applied to (1.2.1) so that a subvector of Y(t), say Z(t), becomes identical to X(t) for $t \geq 0$. Thus the correction procedure is based on the selection of the coefficient matrix in (1.2.1). This, as will be seen later, implies that only the coefficient potentiometer settings need be adjusted.

In regard to the stability problem considered by previous authors [3] and [12], if the system corresponding to the state model (1.1.2) is stable then the solution Z(t) = X(t) will automatically be stable. Further, by the method of analysis of the analog computer solution of (1.2.1), the state model actually solved by the analog computer, all state variables are available and whether used or not they can be observed. Thus hidden modes at the output of any active component do not exist.

The remaining chapters will now be briefly described.

Chapter Two presents in detail the frequency and time domain models for the active components used in the analog computer set up.

Chapter Three deals with the theoretical foundation for the correction procedure for several different types of models used in describing the active components.

In Chapter Four the explicit formulation of the correction procedure is given and it is expressed in the form of an algorithm. This chapter also includes a discussion on the fact that the correction procedure sometimes requires time and amplitude scaling of the original equations. This, however, does not introduce any new problems since there usually will be a need for such scaling of the original equations as they would be solved on the analog computer without the use of the correction procedure.

Chapter Five contains examples and actual analog computer solutions. The harmonic generator equation is analyzed in detail and plots of the corrected outputs are given.

The final chapter summarizes the findings of the thesis and suggests some ideas for further studies.

CHAPTER II

CHARACTERIZATION OF THE ACTIVE ELEMENTS USED IN THE ANALOG COMPUTER

2.1. Introduction

This chapter deals with the characterizations of the active elements used in the analog computer. A complete set of characteristics for each element is given as well as the equations of operation. Initially, the ideal computing elements are considered followed by various models for practical computing elements.

2.2. Ideal Computing Elements

One of the basic active devices used on the analog computer is the high-gain D. C. amplifier. The symbol used to represent this device is in Figure 2.2.1,

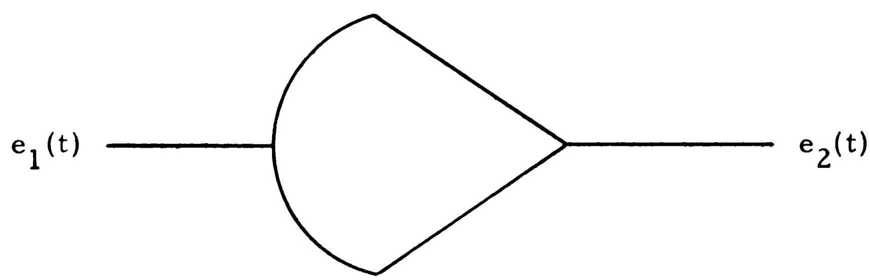


Figure 2.2.1. Symbol for D. C. Amplifier.

The characteristics of this device are:

1) Infinite voltage gain, $\frac{e_2}{e_1} = -A = -\infty$.
2) Voltage gain constant with frequency.
3) Infinite input impedance (implies zero input current).

4) Zero output impedance.

5) Very low input D. C. offset and drift with time and temperature.

Another computing element used for sign changing and multiplication by a constant is the summing amplifier. The symbol used for the summing amplifier is shown in Figure 2. 2. 2,

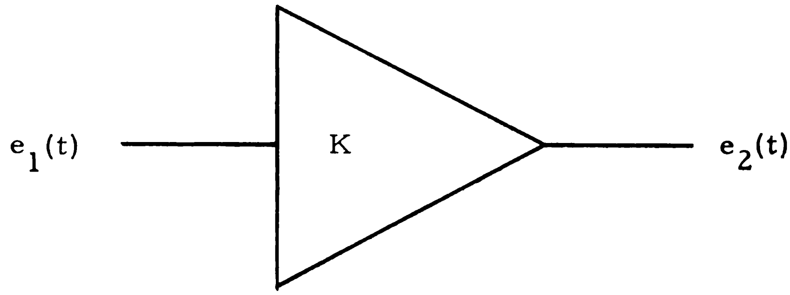


Figure 2. 2. 2. Summing Amplifier

The equation of operation for the ideal summing amplifier is:

$$e_2(t) = - K e_1(t) \quad . \qquad (2.2.1)$$

The interconnection scheme for the summing amplifier is given in Figure 2. 2. 3.

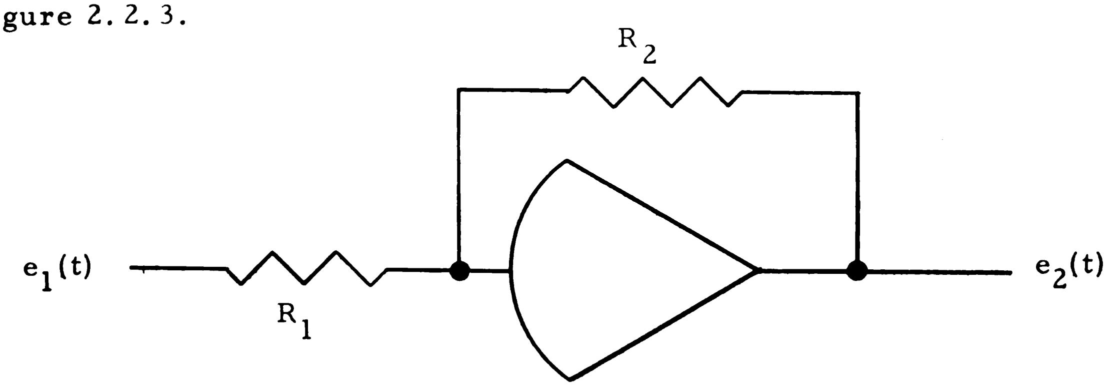


Figure 2. 2. 3. Ideal Summing Amplifier.

The frequency response of the ideal summing amplifier is constant.

The computing element used to obtain the integral of a scalar multiple of a time varying signal, $e_1(t)$, is the integrator. The symbol for the integrator is shown in Figure 2.2.4.

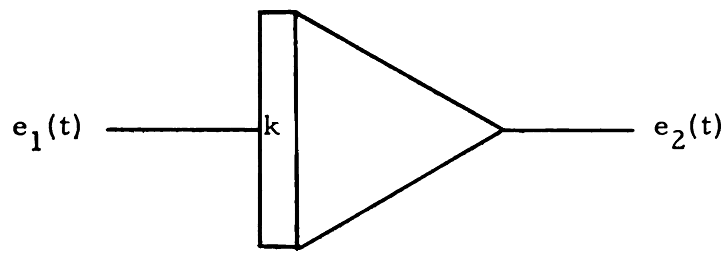


Figure 2.2.4. Integrator.

The equation of operation of this device can be written in two different forms,

1) $$e_2(t) = -\int_0^t k\, e_1(t)\, dt + e_2(0) \qquad (2.2.2a)$$

2) $$\frac{d}{dt} e_2(t) = -k\, e_1(t) \qquad (2.2.2b)$$

The interconnection scheme for the ideal integrator is given in Figure 2.2.5.

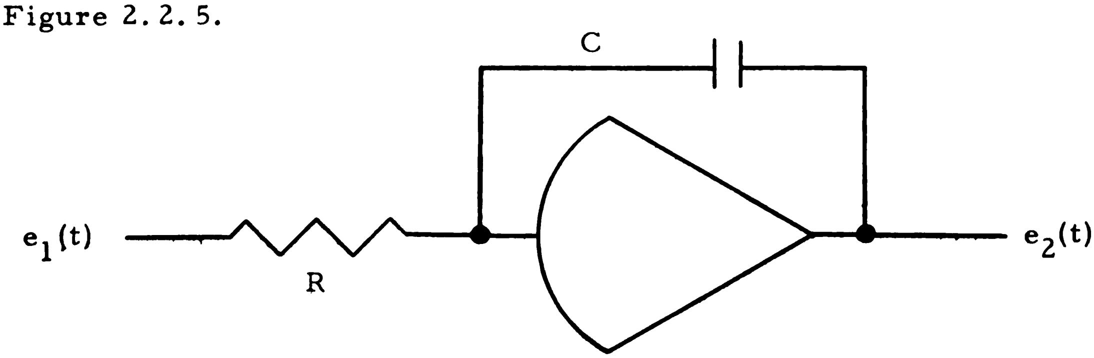


Figure 2.2.5. Ideal Integrator.

The transfer function for the ideal integrator is

$$\frac{e_2(s)}{e_1(s)} = \frac{-K}{s} \tag{2.2.3}$$

where s is the conventional differential or Laplace transform operator. In general more than one input may be connected to the integrator.

2.3. Practical Computing Elements

In practice the five characteristics for the high-gain amplifier given above are not physically realizable. The practical high-gain amplifier has:

1) Very high but finite voltage gain

$$\frac{e_2(s)}{e_1(s)} = -A(s) \quad (|A| \neq \infty) .$$

Note that the voltage gain is frequency dependent so that the bandwidth of the high-gain amplifier is finite. In practice the gain remains constant up to 100-150 K.C., then it rolls off at 20 db/decade.

2) Input impedance large but finite.

3) Output impedance which is small but nonzero.

4) D.C. offset and drift changes with temperature and time.

In this work the primary interest lies in the errors in computer solutions due to the variation from the ideal characteristics to the practical characteristic of 1). Characteristics 2), 3), and 4) are mainly electronics problems which depend on the internal structure of the amplifier. These are investigated by several authors ([2], [7], [8], [9], [10], [11] and others). With the advent of solid state operational amplifiers the problem of errors due to the low input impedance inherent with these components was considered by Hellerman [20] and others. These problems will not be considered in this thesis.

First-order Effects

Several authors ([3], [21] and others) have shown that the transfer function for a practical integrator is of the form,

$$\frac{e_2(s)}{e_1(s)} = -\frac{T_o}{T_o s + 1} \qquad (2.3.1)$$

where $T_o = A_o$, the low frequency open-loop gain of the D.C. amplifier. In the time domain the equation of operation is:

$$\frac{d}{dt} e_2(t) = -\left(\frac{1}{T_o}\right) e_2(t) - e_1(t) \qquad (2.3.2)$$

Similarly, for the practical summing amplifier, the limited bandwidth of the amplifier yields a transfer function of the form:

$$\frac{e_2(s)}{e_1(s)} = -\frac{1}{(T_1 s + 1)} \qquad (2.3.3)$$

where T_1 is the time constant of the summing amplifier. In the time domain, (2.3.3) can be written as,

$$\frac{d}{dt} e_2(t) = -\left(\frac{1}{T_1}\right)[e_2(t) + e_1(t)] \qquad (2.3.4)$$

It can be seen that when $T_o = \infty$ and $T_1 = 0$, equations (2.3.1) and (2.3.3) yield the transfer function for the ideal elements.

Second-order Effects

It has been shown ([3], [4], [5], and others) that a more accurate model for the high-gain D.C. amplifier is,

$$\frac{e_2(s)}{e_1(s)} = - \frac{A_o}{(T_2 s + 1)} \qquad (2.3.5)$$

where A_o is the low frequency open-loop gain and T_2 is the time constant of the high-gain amplifier. In practice, the high-gain amplifier must have a roll-off of 12 db. or less in order to insure stability [22] so that designers strive to design the high-gain amplifier to have the transfer function given in (2.3.5).

Using the equation (2.3.5), the time domain equation of operation for the integrator is

$$\frac{d^2}{dt^2} [e_2(t)] = - a_1 \frac{d}{dt} e_2(t) - a_2 e_2(t) - a_3 e_1(t) \qquad (2.3.6)$$

where:

$a_1 = [T_2 + C(1 + A_o)] / T_2$

$a_2 = 1/(C T_2)$

$a_3 = A_o/(RCT_2)$

R, input resistor

C, feedback capacitor

Similarly the summing amplifiers have an equation of operation of the form

$$\frac{d}{dt} e_2(t) = - b_1 e_2(t) - b_2 e_1(t) \qquad (2.3.7)$$

where:

$$b_1 = (2 + A_o)/(2 T_2)$$

$$b_2 = A_o/(2 T_2) .$$

<u>Higher-order Effects</u>

In general, the transfer function for the high-gain D. C. amplifier may be given by,

$$\frac{e_2(s)}{e_1(s)} = - \frac{P(s)}{Q(s)} \qquad (2.3.8)$$

where P(s) and Q(s) are polynomials with real coefficients. The form (2.3.8) is considered in Chapter Six in a general discussion on the extension of the material presented in the first part of Chapter Three.

CHAPTER III

THEORETICAL CONSIDERATIONS FOR THE ANALOG COMPUTER SOLUTION OF STATE-MODEL EQUATIONS

3.1. Introduction

In this chapter a new formulation for the analog computer solutions of state model equations is introduced. The correction procedure presented in the next chapter requires such preliminary formulation. Since the formulation depends on the type of model used for characterizing the active computing elements, corresponding to each case described in Chapter 2, a different formulation is given. The case where only one type of computing element is used shows promise in the area of active network synthesis.

3.2. Augmented Form of State Model [1]

The general form of state model to be solved is as follows:

$$\frac{d}{dt}\,\overline{X}(t) = -\,\overline{A}\,\overline{X}(t) + \overline{B}\,E(t)\,, \quad \overline{X}(0) \tag{3.2.1}$$

where

$\overline{X}(t)$ is an n-vector

$\overline{A}$ is an nxn constant real matrix

$E(t)$ is an m-vector

$\overline{B}$ is an nxm constant real matrix.

In this thesis we shall assume that $E(t)$, the driving function, satisfies the relations,

$$E(t) = -K\,F(t) \tag{3.2.2}$$

$$\frac{d}{dt}F(t) = -T\,F(t)\,, \quad F(0) \tag{3.2.3}$$

where T and K are constant matrices and K is not necessarily square. The class of functions that satisfy (3.2.2) and (3.2.3) has been shown [1, pp. 159-161] to be quite large. Any function that is expressible by a convergent power series can be placed in the form of (3.2.3).

If we let

$$X(t) = \begin{bmatrix} \overline{X}(t) \\ F(t) \end{bmatrix}$$

then the following homogeneous equation is obtained

$$\frac{d}{dt}\begin{bmatrix} \overline{X}(t) \\ F(t) \end{bmatrix} = -\begin{bmatrix} \overline{A} & \overline{B}\,K \\ 0 & T \end{bmatrix}\begin{bmatrix} \overline{X}(t) \\ F(t) \end{bmatrix}; \quad \begin{bmatrix} \overline{X}(0) \\ F(0) \end{bmatrix} \tag{3.2.4}$$

where

F(t) is a p-vector so that X(t) is a (n+p)-vector.

Further, let

$$\begin{bmatrix} \overline{A} & (\overline{B}\,K) \\ 0 & T \end{bmatrix} = A$$

then equation (3.2.4) can be written as

$$\dot{X}(t) = \frac{d}{dt}X(t) = -AX(t)\,, \quad X(0) \tag{3.2.5}$$

Since it is assumed that all drivers satisfy equations (3.2.2) and (3.2.3), the homogeneous form (3.2.5) is the only form to be considered throughout the thesis.

3.3. Modified Form of State Model

The form of the state model may be expanded into a higher order vector differential equation. Such an expansion is required in the later development considered in section 3.4.

Consider the state model in (3.2.5) which is rewritten for convenience,

$$\dot{X}(t) = \frac{d}{dt} X(t) = - A X(t) , \quad X(0) \tag{3.2.5}$$

To expand this first order vector differential equation into a second order vector differential equation, consider the relations,

$$\dot{X}(t) = \frac{d}{dt} X(t) = - A X(t) \tag{3.3.1a}$$

$$\ddot{X}(t) = \frac{d^2}{dt^2} X(t) = - A \frac{d}{dt} X(t) \tag{3.3.1b}$$

Now let $A = N_1 + N_2$, where N_1 and N_2 are to be chosen to satisfy certain constraints to be discussed later. Then

$$\frac{d^2}{dt^2} X(t) = - N_1 \frac{d}{dt} X(t) + N_2 A X(t) . \tag{3.3.1c}$$

Considering (3.3.1a), now (3.3.1c) can be put into an augmented form as follows

$$\frac{d}{dt}\begin{bmatrix}\dot{X}(t)\\ X(t)\end{bmatrix} = -\begin{bmatrix}N_1 & -N_2A\\ -U & 0\end{bmatrix}\begin{bmatrix}\dot{X}(t)\\ X(t)\end{bmatrix}; \begin{bmatrix}\dot{X}(0)\\ X(0)\end{bmatrix} \qquad (3.3.2)$$

where $\dot{X}(0) \equiv -A\,X(0)$ and U is a unit matrix.

Observe that the solution of (3.2.5) is

$$X(t) = e^{-At}\,X(0) \qquad (3.3.3)$$

This solution also satisfies (3.3.1c). Indeed, after substituting (3.3.3) into (3.3.1c) with

$$\dot{X}(0) = -A\,X(0)$$

one has

$$(A^2 - N_1A - N_2A)\,e^{-At}X(0) = 0 \qquad (3.3.4)$$

since

$$A = N_1 + N_2 \quad . \qquad (3.3.5)$$

On the other hand, (3.3.4) is satisfied because the coefficient matrix is a zero matrix, i.e.,

$$A^2 - (N_1 + N_2)A = (A^2 - A^2) = 0$$

for all values of t.

Similarly to expand the state model (3.2.5) from a first order vector differential equation to a third order vector differential equation, consider the following relations,

$$\dot{X}(t) = -A\,X(t) \qquad (3.3.6a)$$

$$\ddot{X}(t) = -A\,\dot{X}(t) \tag{3.3.6b}$$

$$\dddot{X}(t) = -A\,\ddot{X}(t)\ . \tag{3.3.6c}$$

Let $A = n_1 U + N_1 + N_2$ (3.3.6d)

then

$$\begin{aligned} \dddot{X}(t) &= -n_1\,\ddot{X}(t) - N_1\,\ddot{X}(t) - N_2\,\ddot{X}(t) \\ &= -n_1\,\ddot{X}(t) + N_1 A\,\dot{X}(t) + N_2 A\,\dot{X}(t) \\ &= -n_1\,\ddot{X}(t) + N_1 A\,\dot{X}(t) - N_2 A^2 X(t) \end{aligned} \tag{3.3.7}$$

where n_1 is a positive constant.

Equation (3.3.7) has the following augmented first order differential equation form,

$$\frac{d}{dt}\begin{bmatrix} \ddot{X}(t) \\ \dot{X}(t) \\ X(t) \end{bmatrix} = -\begin{bmatrix} n_1 U & -N_1 A & N_2 A^2 \\ -U & 0 & 0 \\ 0 & -U & 0 \end{bmatrix}\begin{bmatrix} \ddot{X}(t) \\ \dot{X}(t) \\ X(t) \end{bmatrix}$$

with

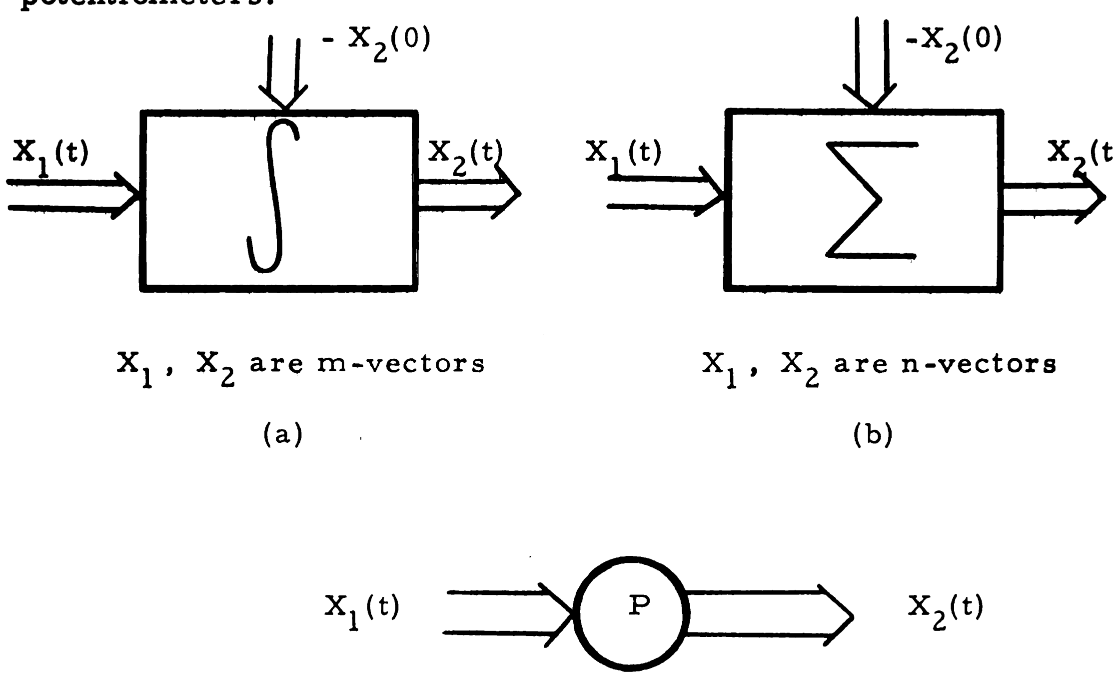
$$\begin{bmatrix} \ddot{X}(0) \\ \dot{X}(0) \\ X(0) \end{bmatrix} = \begin{bmatrix} A^2 \\ -A \\ U \end{bmatrix} X(0) \tag{3.3.8}$$

The reasons for decomposing A in the form of (3.3.6d) will become apparent in a later section of this chapter.

3.4. Computer Diagram of the State Model

In this section, the symbolism used for the computing elements is considered. Using the defined symbols, first the form for the analog computer diagram with ideal computing elements is given, then the correction procedure for analog computer set ups with practical computing elements is considered.

The symbols for non-ideal or practical computer elements are listed in Figure 3.4.1. These are, vector of integrators, vector of summing amplifiers and the matrix of coefficient potentiometers.



X_1 is an m-vector, X_2 is an n-vector

(c)

Figure 3.4.1. (a) Vector of Integrators
(b) Vector of Summing Amplifiers
(c) Matrix of Coefficient Potentiometers

The broad connecting lines are assumed to be n channels where n is the order of the vector represented on the particular line.

Computer Diagram with Ideal Components

Consider the state model in (3.2.5),

$$\frac{d}{dt} X(t) = - A X(t); \quad X(0)$$

the constraint that must be satisfied for the analog computer solution concerns the realizability of the entries of A. More specifically if the entries of A do not satisfy the condition

$$0 \leq |a_{ij}| \leq 1 \qquad (3.4.1)$$

then the process of time and amplitude scaling must be used.

Definition 3.4.1. A real matrix A_p is said to be a "positive matrix" if all the entries of A_p are non-negative.

Definition 3.4.2. A real matrix A_m, whose entries are non-positive is called a "negative matrix."

The following theorem is evident.

Theorem 3.4.1.

Any real matrix of order mxn may be decomposed into the sum of a positive and a negative matrix of the same order.

Note that the decomposition stated in the above theorem is, of course, not unique. This non-uniqueness, as will be seen later, will provide some flexibilities in the solution of matrix equations discussed in Chapter Four.

Consider the state model in (3.2.5) and decompose A into a positive matrix A_p and a negative matrix A_m. The state model takes on the form

$$\dot{X}(t) = -(A_p + A_m)X(t), \quad X(0) \quad . \qquad (3.4.2)$$

For an entry in a matrix to be realizable as a coefficient potentiometer setting, that entry must be non-negative.

As a result, the realizable matrices in (3.4.2) are,

1) A_p and
2) $-A_m$.

Thus, to afford only realizable entries in the form of (3.4.2), let the state model (3.2.5) have the form,

$$\dot{X}(t) = A_p(-X(t)) + (-A_m)X(t), \quad X(0) \quad . \qquad (3.4.3)$$

Using the symbols for the ideal computing elements given in Figure 3.4.2, the analog computer diagram for (3.4.3) will be as in Figure 3.4.3. In this diagram since ideal computing elements are used, the output $X(t)$ is the exact solution of (3.2.5).

In Figure 3.4.3, it should be noted that, there are exactly n integrators and n summing amplifiers, with $X(t)$ an n-vector. In the correction procedure to follow, the number of integrators and summing amplifiers will remain the same but the form of A_p and A_m will be changed.

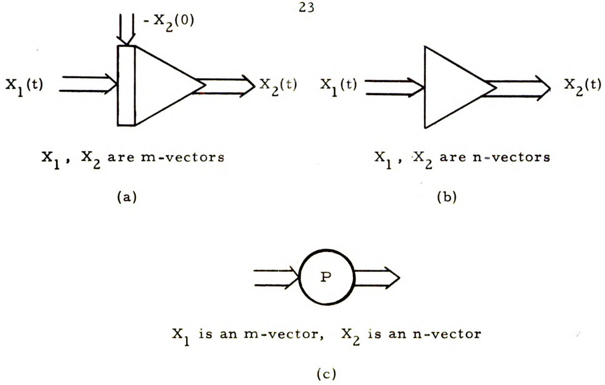


Figure 3.4.2. (a) Ideal Vector of Integrators.
(b) Ideal Vector of Summing Amplifiers.
(c) Matrix of Potentiometers.

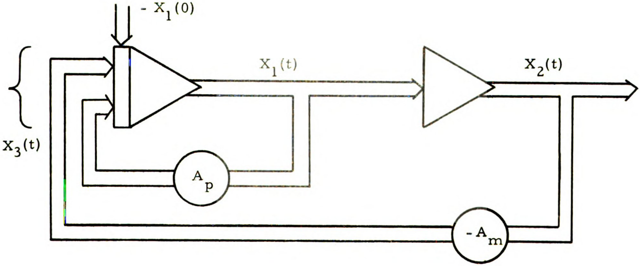


Figure 3.4.3. Computer Diagram.

Computer Solutions Using Practical Computing Elements

In the computer diagram given in Figure 3.4.3 if the ideal components are replaced by practical computing elements new vectors are generated. For this reason the labeling on Figure 3.4.3 will be modified to yield the diagram given in Figure 3.4.4.

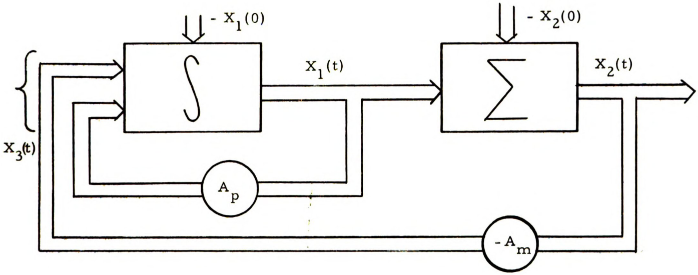


Figure 3.4.4. Practical Model of the Computer Diagram.

In general, $X_1(t) \neq -X(t)$ and $X_2(t) \neq X(t)$, where $X(t)$ is the solution of (3.2.5). Corresponding to the models used for the computing elements, there exists a new set of matrices $\overline{A}_p$ and $\overline{A}_m$ such that $X_1(t) \equiv X(t)$ or $X_2(t) \equiv X(t)$ for $t \geq 0$. The remainder of the chapter is devoted to finding these matrices, $\overline{A}_p$ and $\overline{A}_m$, as functions of A and the models used for the computing elements.

A. First-Order Models for the Computing Elements and the First-Order Correction Procedure

Consider the first-order models for the active computing elements given in (2.3.2) and (2.3.4) for the integrator:

$$\dot{X}_1(t) = -(\frac{1}{T_o}) X_1(t) - X_3(t) \qquad (3.4.4)$$

where $T_o = A_o$ = low frequency open-loop gain of the D.C. amplifier and for the summing amplifier we have

$$\dot{X}_2(t) = -\frac{1}{T_1} [X_2(t) + X_1(t)] \qquad (3.4.5)$$

where T_1 is the time constant of the summing amplifier.

Consider the diagram of an arbitrary state model as shown in Figure 3.4.5.

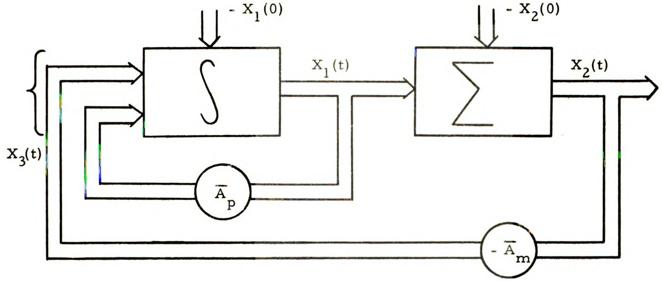


Figure 3.4.5. Computer Diagram for an Arbitrary State Model.

At this point, it is important to note that $\bar{A}_p$ and $\bar{A}_m$ are not known. However through analysis of the circuit in Figure 3.4.3 and by requiring $X_2(t) \equiv X(t)$, for $t \geq 0$, the relationship among the matrices $\bar{A}_p$, $\bar{A}_m$ and A will be established.

From Figure 3.4.5 one has

$$X_3 = \bar{A}_p X_1 - \bar{A}_m X_2 \tag{3.4.6a}$$

Taking $T_o = \frac{1}{k_1}$, where k_1 is a known positive constant, from (3.4.4) one has

$$\dot{X}_1(t) = - k_1 X_1(t) - \bar{A}_p X_1(t) + \bar{A}_m X_2(t) \tag{3.4.6b}$$

Similarly, taking $T_1 = \frac{1}{k_2}$, where k_2 is a known positive constant, one obtains

$$\dot{X}_2(t) = - k_2[X_2(t) + X_1(t)] \tag{3.4.6c}$$

These equations can be written together in the form of a first-order vector differential equation of the following form.

$$\frac{d}{dt}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = - \left[\begin{array}{c|c} k_1 U + \bar{A}_p & - A_m \\ \hline k_2 U & k_2 U \end{array}\right] \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \tag{3.4.7}$$

In essence, for a given $\bar{A}_p$ and $\bar{A}_m$, the circuit in Figure 3.4.5 actually solves equation (3.4.7).

We shall now obtain from (3.4.7), a differential equation involving only the vector $X_2(t)$. Since

$$\dot{X}_2(t) = - k_2 [X_1(t) + X_2(t)]$$

or

$$X_1(t) = - T_1 \dot{X}_2(t) - X_2(t) \tag{3.4.8}$$

Differentiating (3.4.8) one has

$$\dot{X}_1(t) = - T_1 \ddot{X}_2(t) - \dot{X}_2(t)$$

From (3.4.7),

$$\begin{aligned} \dot{X}_1(t) &= - (k_1 U + \overline{A}_p) X_1(t) + \overline{A}_m X_2(t) \\ &= (k_1 U + \overline{A}_p)[T_1 \dot{X}_2(t) + X_2(t)] + \overline{A}_m X_2(t) \end{aligned}$$

therefore

$$T_1 \ddot{X}_2(t) + \dot{X}_2(t) = - (k_1 U + \overline{A}_p)[T_1 \dot{X}_2(t) + X_2(t)] - \overline{A}_m X_2(t) . \tag{3.4.9}$$

In first-order vector differential equation form (3.4.9) takes on the following form

$$\frac{d}{dt} \begin{bmatrix} \dot{X}_2(t) \\ X_2(t) \end{bmatrix} = - \left[\begin{array}{c|c} (k_1 + k_2)U + \overline{A}_p & k_2 (k_1 U + \overline{A}_p + \overline{A}_m) \\ - U & O \end{array}\right] \begin{bmatrix} \dot{X}_2(t) \\ X_2(t) \end{bmatrix} \begin{bmatrix} \dot{X}_2(0) \\ X_2(0) \end{bmatrix} \tag{3.4.10}$$

From the work of section 3.3 it is known that equation (3.2.5) could be put into an expanded first-order vector differential equation (3.3.2), i.e.,

$$\frac{d}{dt} \begin{bmatrix} \dot{X}(t) \\ X(t) \end{bmatrix} = - \begin{bmatrix} N_1 & - N_2 A \\ - U & 0 \end{bmatrix} \begin{bmatrix} \dot{X}(t) \\ X(t) \end{bmatrix} ; \begin{bmatrix} \dot{X}(0) \\ X(0) \end{bmatrix}$$

and if $\dot{X}(0) \equiv - A X(0)$ then the solution for $X(t)$ is $e^{-At} X(0)$.

Note that $\overline{A}_p$ and $\overline{A}_m$ are still arbitrary positive and negative matrices, respectively, however there is now an explicit relation among the matrices $\overline{A}_p$, $\overline{A}_m$ and A. Note also that

$$A = N_1 + N_2 \tag{3.4.11}$$

In order to have $X_2(t) \equiv X(t)$ for $t \geq 0$, several conditions must be satisfied. These conditions come from the fact that for (3.4.10) and (3.3.2) to have the same solution, the initial conditions as well as the coefficient matrices must be equal. More specifically, in order for $X_2(t) \equiv X(t)$ for $t \geq 0$, one must have

$$\text{1)} \quad \begin{bmatrix} \dot{X}_2(0) \\ X_2(0) \end{bmatrix} \equiv \begin{bmatrix} -A \\ U \end{bmatrix} \quad X(0)$$

$$\text{2)} \quad (k_1 + k_2)U + \overline{A}_p = N_1 \tag{3.4.12}$$

$$\text{3)} \quad k_2(k_1 U + \overline{A}_p + \overline{A}_m) = -N_2 A$$

$$\text{4)} \quad A = (N_1 + N_2)$$

The following observations can be made directly from (3.4.12). First, $(k_1 + k_2)U + \overline{A}_p$ is a positive matrix as is N_1. Further

$$\overline{A}_p = [N_1 - (k_1 + k_2)U] \tag{3.4.13}$$

must be a positive matrix.

Second, using relations 2), 3), 4) of (3.4.12), an explicit form for $\overline{A}_m$ in terms of A and N_1 is given as:

$$k_2 \overline{A}_m = N_1(A - k_2 U) - A^2 + k_2^2 U \tag{3.4.14}$$

Since $k_2 > 0$ and $\overline{A}_m$ is a negative matrix then N_1 must be found from (3.4.14) such that

$$N_1(A - k_2U) - A^2 + k_2^2 U \tag{3.4.15}$$

is a negative matrix.

The algorithm used for the determination of N_1 satisfying (3.4.13) and (3.4.15) will be discussed in Chapter Four.

Assuming that N_1 is found, then $\overline{A}_p$ and $\overline{A}_m$ are obtained from (3.4.12) and (3.4.14), respectively, and the improved computer solution $X_2(t) = X(t)$ in Figure 3.4.5 is insured except that $X_1(0)$ must be found.

To find $X_1(0)$, consider (3.4.6c) which is rewritten as

$$\dot{X}_2(t) = - k_2 X_2(t) - k_2 X_1(t) \tag{3.4.6c}$$

At $t = 0$,

$$\dot{X}_2(0) \equiv \dot{X}(0) \equiv - A X(0)$$

and

$$X_2(0) \equiv X(0) \quad .$$

Therefore,

$$- T_1[\dot{X}_2(0) + k_2 X_2(0)] = X_1(0)$$

or

$$T_1 A X(0) - X(0) = X_1(0)$$

hence

$$(T_1 A - U) X(0) = X_1(0) \tag{3.4.16}$$

Another important property which may be obtained from equation (3.4.6c) is considered next. Since

$$X(t) = X_2(t)$$

and

$$\dot{X}_2(t) = -A X_2(t)$$

then from (3.4.7)

$$-T_1 [\dot{X}_2(t) + k_2 X_2(t)] = X_1(t)$$

or

$$X_1(t) = (T_1 A - U) X_2(t) \tag{3.4.17}$$

This relation yields an interesting interpretation of the over-all analog computer solution to the state model in (3.4.7). Indeed the part of the state variables, $X_1(t)$, are linear combinations of the remaining state variables, $X_2(t)$, hence in the system (3.4.7) only those modes appearing in the expression of $X_2(t)$ will exist. Note that $X_2(t) \equiv X(t)$ and hence $X_2(t)$ contains the modes corresponding to the eigenvalues of the A matrix in (3.2.5). On the other hand some of the eigenvalues of the coefficient matrix in (3.4.7), which we shall denote momentarily as $\tilde{A}$, are usually different than those of A. By the selection of matrix N_1, as is seen from (3.4.17), all the eigenvalues of A are now contained in those of $\tilde{A}$. Further note that the order of $\tilde{A}$ is greater than the order of A, therefore $\tilde{A}$ has additional eigenvalues. However these eigenvalues will never appear in the solution of (3.4.7) due to the choice of initial conditions [23]. In summary, the selection of N_1 and $X_1(0)$ from (3.4.16) will cause the state model in (3.4.7) to have only a part of the modes in the system, which are the modes contained in the original state model. In Chapter Five, through an example the implications of this discussion will be further demonstrated.

B. Second-Order Models for the Computing Elements and the Second-Order Correction Procedure

In some applications, it may be desirable to give a more accurate model for the integrators and summing amplifiers. In Chapter 2 equations (2.3.6) and (2.3.7) are the second-order models for the integrator and summing amplifier. Applying these equations to the variables in Figure 3.4.6 for the integrator, we have

$$\ddot{X}_1(t) = -a_1 \dot{X}_1(t) - a_2 X_1(t) - a_3 X_3(t) \qquad (3.4.18)$$

where

$$a_1 = [T_2 + C(1 + A_o)]/T_2$$

$$a_2 = 1/(C T_2)$$

$$a_3 = A_o/(RCT_2) \cong A_o/T_2$$

and for the summing amplifier,

$$\dot{X}_2(t) = -b_1 X_2(t) - b_2 X_1(t) \qquad (3.4.19)$$

where

$$b_1 = (2 + A_o)/(2 T_2)$$

$$b_2 = A_o/(2 T_2)$$

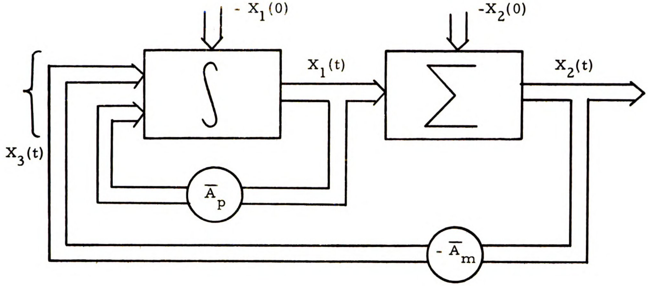


Figure 3.4.6. Computer Diagram for an Arbitrary State Model.

At this point it is important to recall that $\overline{A}_p$ and $\overline{A}_m$ are not known. However through the analysis of the circuit in Figure 3.4.6 and requiring that $X_2(t) \equiv X(t)$, $X(t)$ being the solution of (3.2.5) for $t \geq 0$, the relationships among the matrices $\overline{A}_p$, $\overline{A}_m$ and A can now be established. From Figure 3.4.6,

$$X_3(t) = \overline{A}_p X_1(t) - \overline{A}_m X_2(t) . \tag{3.4.20a}$$

Combining (3.4.18) and (3.4.20a), one can get an expression in terms of $X_1(t)$ and $X_2(t)$ alone.

$$\ddot{X}_1(t) = -a_1 \dot{X}_1(t) - a_2 X_1(t) - a_3[\overline{A}_p X_1(t) - \overline{A}_m X_2(t)] \tag{3.4.20b}$$

Expressing (3.4.20b) and (3.4.19) in first order vector differential equation form we obtain,

$$\frac{d}{dt}\begin{bmatrix}\dot{X}_1(t)\\ X_1(t)\\ X_2(t)\end{bmatrix} = -\begin{bmatrix}a_1 U & (a_2 U + a_3 \overline{A}_p) & -a_3 \overline{A}_m\\ -U & 0 & 0\\ 0 & b_2 U & b_1 U\end{bmatrix}\begin{bmatrix}\dot{X}_1(t)\\ X_1(t)\\ X_2(t)\end{bmatrix} \quad (3.4.20c)$$

In reality, for some given $\overline{A}_p$ and $\overline{A}_m$, the circuit in Figure 3.4.6 actually solves equation (3.4.20c).

At this point, it is desired to find the first order vector differential equation in terms of $X_2(t)$ and its required derivatives. From (3.4.20c) we have

$$b_2 X_1(t) = -[\dot{X}_2(t) + b_1 X_2(t)] \quad \text{and}$$

$$b_2 \dot{X}_1(t) = -[\ddot{X}_2(t) + b_1 \dot{X}_2(t)] \quad \text{and}$$

$$b_2 \ddot{X}_1(t) = -[\dddot{X}_1(t) + b_1 \ddot{X}_2(t)]$$

Therefore,

$$\begin{aligned}\ddot{X}_1(t) &= -\frac{1}{b_2}[\dddot{X}_2(t) + b_1 \ddot{X}_2(t)]\\ &= \Bigg[\frac{a_1}{b_2}[\ddot{X}_2(t) + b_1 \dot{X}_2(t)]\\ &\quad + \frac{(a_2 U + a_3 \overline{A}_p)}{b_2}[\dot{X}_2(t) + b_1 X_2(t)]\\ &\quad + a_3 \overline{A}_m X_2(t)\Bigg] \end{aligned} \quad (3.4.20d)$$

Arranging (3.4.20d) into first-order vector differential equation form we obtain,

$$\frac{d}{dt}\begin{bmatrix}\ddot{X}_2(t)\\ \dot{X}_2(t)\\ X_2(t)\end{bmatrix} = -\left[\begin{array}{c|c|c}(a_1+b_1)U & (a_2+a_1b_1)U+a_3\bar{A}_p & a_2U+a_3(\bar{A}_p+b_2\bar{A}_m)\\ \hline -U & 0 & 0\\ \hline 0 & -U & 0\end{array}\right]\begin{bmatrix}\ddot{X}_2(t)\\ \dot{X}_2(t)\\ X_2(t)\end{bmatrix}$$

$$\text{and}\quad \begin{bmatrix}\ddot{X}_2(0)\\ \dot{X}_2(0)\\ X_2(0)\end{bmatrix} \qquad (3.4.21)$$

From the work of section 3.3, (3.2.5) could be put into an expanded first-order vector differential equation of the following form

$$\frac{d}{dt}\begin{bmatrix}\ddot{X}(t)\\ \dot{X}(t)\\ X(t)\end{bmatrix} = -\begin{bmatrix}n_1U & -N_1A & N_2A^2\\ -U & 0 & 0\\ 0 & -U & 0\end{bmatrix}\begin{bmatrix}\ddot{X}(t)\\ \dot{X}(t)\\ X(t)\end{bmatrix};$$

$$\begin{bmatrix}\ddot{X}(0)\\ \dot{X}(0)\\ X(0)\end{bmatrix} \equiv \begin{bmatrix}A^2\\ -A\\ U\end{bmatrix} X(0) \qquad (3.3.8)$$

Note that,

$$A = n_1U + N_1 + N_2 . \qquad (3.4.22)$$

Since $\ddot{X}(0) \equiv A^2 X(0)$ and $\dot{X}(0) \equiv -A X(0)$, then the solution for $X(t)$ is $e^{-At} X(0)$. Again, it must be pointed out that, up to this point, $\overline{A}_p$ and $\overline{A}_m$ are arbitrary positive and negative matrices, respectively. However, now there is an explicit relation among the matrices $\overline{A}_p$, $\overline{A}_m$ and A.

In order to have $X_2(t) \equiv X(t)$ for $t \geq 0$, several conditions must be satisfied. These conditions come from the equivalance of (3.4.21) and (3.3.8) which implies the identity of the initial conditions as well as the coefficient matrices. More specifically in order for $X_2(t) = X(t)$ for $t \geq 0$ one has

$$1)\quad \begin{bmatrix} \ddot{X}_2(0) \\ \dot{X}_2(0) \\ X_2(0) \end{bmatrix} \equiv \begin{bmatrix} A^2 \\ -A \\ U \end{bmatrix} X(0)$$

$$2)\quad (a_1 + b_1)\, U = n_1 U$$

$$3)\quad (a_2 + a_1 b_1)\, U + a_3 \overline{A}_p = -N_1 A \qquad (3.4.23)$$

$$4)\quad a_2 U + a_3(\overline{A}_p + b_2 \overline{A}_m) = N_2 A^2$$

$$5)\quad A = n_1 U + N_1 + N_2 \qquad (3.4.22)$$

Several points can be made from (3.4.23). First, $(a_2 + a_1 b_1)U + a_3 \overline{A}_p$ is a positive matrix so that $-N_1 A = \overline{N}_1$ must be a positive matrix and, in fact,

$$a_3 \overline{A}_p = (\overline{N}_1 - (a_2 + a_1 b_1)U) \qquad (3.4.24)$$

must be a positive matrix.

Second, using relations 3), 4), 5) of (3.4.23), an explicit form for $\overline{A}_m$ in terms of $\overline{N}_1$ is given by

$$a_3 b_2 \overline{A}_m = \overline{N}_1 (A - U) + (A - n_1 U) A + a_1 b_1 U \qquad (3.4.25)$$

Since a_3 and b_2 are positive scalar constant, then

$$[\overline{N}_1 (A - U) + (A - n_1 U) A + a_1 b_1 U] \qquad (3.4.26)$$

must be a negative matrix. The algorithm to find $\overline{N}_1$ to satisfy equations (3.4.24) and (3.4.26) will be discussed in Chapter Four.

Assuming that $\overline{N}_1$ is found then $\overline{A}_p$ and $\overline{A}_m$ can be determined from (3.4.23) and (3.4.25), also the improved computer solution $X_2(t) = X(t)$ in Figure 3.4.6 is guaranteed except that the initial conditions $X_1(0)$ must be found.

To find $X_1(0)$, consider (3.4.19) for $t = 0$.

$$\dot{X}_2(0) = - b_1 X_2(0) - b_2 X_1(0) \qquad (3.4.27)$$

since

$$X_2(0) = X(0)$$

$$\dot{X}_2(0) = \dot{X}(0) = - A X(0)$$

one has

$$- A X(0) + b_1 X(0) = - b_2 X_1(0)$$

$$X_1(0) = \frac{(A - b_1 U)}{b_2} X(0) \qquad (3.4.28)$$

Clearly knowing b_1 and b_2 from (3.4.19), the initial conditions may be formed for the integrators in Figure 3.4.6.

In regard to the problem of extraneous modes existing at the output of the integrators, consider (3.4.19) for any time t and note that

$$X_1(t) = \frac{(A - b_1 U)}{b_2} X_2(t) \qquad (3.4.29)$$

The discussion at the end of section 3.4A applies to this section, as well, in regard to the modes that can be measured on this computer set up. This implies that all the outputs of the active elements in Figure 3.4.6 are bounded if (3.2.5) has a stable solution.

C. Analog Computer Solution of the State Model Using One Type of Active Computing Element

The formulation of the correction procedure for first-order models for the computing elements has been given. Since the equations of operation for the integrator and summing amplifier given in (3.4.4) and (3.4.5) respectively, are similar in form, it seems advisable to consider an active computing element which replaces both the integrator and summing amplifier.

Such an approach to analog computer solutions has several advantages. First, since there is only one type of active device, more of the interconnections can be made inside the machine. Second, since the operational amplifier must perform only one operation, the device may be designed to perform this operation with a savings in cost. Third, this approach suggests a method of active network synthesis of the state model.

Let there exist a general active device D, as shown in Figure 3.4.7.

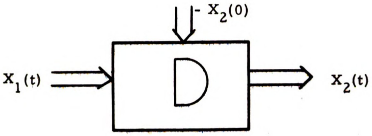


Figure 3.4.7. An Active Component Described by (3.4.30).

Assume that the relation between $X_1(t)$ and $X_2(t)$ is given by

$$\dot{X}_2(t) = -\frac{1}{T_1}\left[X_2(t) + X_1(t)\right] \tag{3.4.30}$$

where T_1 represents the time constant of the actice device D.

The computer set up for the state model in (3.2.5), using the general active element D, is shown in Figure 3.4.8.

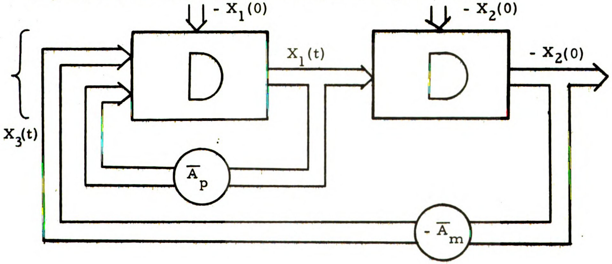


Figure 3.4.8. Computer Diagram for (3.2.5) Using the D Element in Figure 3.4.7.

At this point of the development, it is important to note that $\overline{A}_p$ and $\overline{A}_m$ are not known. However, through the analysis of the circuit in Figure 3.4.8 and requiring that $X_2(t) \equiv X(t)$, where $X(t)$ is the solution of (3.2.5), for $t \geq 0$, a relation among matrices $\overline{A}_p$, $\overline{A}_m$ and A can be established. From Figure 3.4.8,

$$X_3(t) = \overline{A}_p X_1(t) - \overline{A}_m X_2(t) \quad . \tag{3.4.31}$$

Then since $T_1 = 1/k_2$ and from (3.4.30) one has

$$\dot{X}_1(t) = -k_2 [X_1(t) + X_3(t)]$$

and

$$\dot{X}_2(t) = -k_2[X_2(t) + X_1(t)] \tag{3.4.32}$$

Solving these equations for $X_1(t)$ and $X_2(t)$, also considering the required derivatives,

$$\dot{X}_1(t) = -k_2 X_1(t) - k_2 [\overline{A}_p X_1(t) - \overline{A}_m X_2(t)]$$

$$\dot{X}_2(t) = -k_2 X_2(t) - k_2 X_2(t)$$

one finally obtains,

$$\frac{d}{dt} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = -k_2 \begin{bmatrix} (U+\overline{A}_p) & -\overline{A}_m \\ U & U \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \tag{3.4.33}$$

For given $\overline{A}_p$ and $\overline{A}_m$, the circuit in Figure 3.4.8 actually solves (3.4.33) with some initial vector.

It is now desired to find from (3.4.33) a differential equation which involves only $X_2(t)$. This is indeed possible, since

$$\dot{X}_2(t) = -k_2 X_1(t) - k_2 X_2(t)$$

then

$$k_2 X_1(t) = -[\dot{X}_2(t) + k_2 X_2(t)] \qquad (3.4.34)$$

and differentiating this last equation one has

$$k_2 \dot{X}_1(t) = -[\ddot{X}_2(t) + k_2 \dot{X}_2(t)] \quad .$$

Therefore, the first equation of (3.4.33) can be rewritten as

$$\begin{aligned} \dot{X}_1(t) &= -T_1 [\ddot{X}_2(t) + k_2 \dot{X}_2(t)] \\ &= k_2 (U + \overline{A}_p) T_1 [\dot{X}_2(t) + k_2 X_2(t)] \\ &\quad + k_2 \overline{A}_m X_2(t) \end{aligned}$$

Writing this result in first-order vector differential form, one finally has

$$\frac{d}{dt}\begin{bmatrix} \dot{X}_2(t) \\ X_2(t) \end{bmatrix} = -\left[\begin{array}{c|c} k_2(2U+\overline{A}_p) & k_2^2 (U+\overline{A}_p+\overline{A}_m) \\ \hline -U & 0 \end{array}\right]\begin{bmatrix} \dot{X}_2(t) \\ X_2(t) \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} \dot{X}_2(0) \\ X_2(0) \end{bmatrix} \qquad (3.4.35)$$

where $k_2 = 1/T_1$.

As is established in section (3.3) that, equation (3.2.5) can be put into an expanded first-order vector differential equation form as follows.

$$\frac{d}{dt}\begin{bmatrix}\dot{X}(t)\\ X(t)\end{bmatrix} = -\begin{bmatrix}N_1 & -N_2 A\\ -U & 0\end{bmatrix}\begin{bmatrix}\dot{X}(t)\\ X(t)\end{bmatrix}, \quad \begin{bmatrix}\dot{X}(0)\\ X(0)\end{bmatrix} \tag{3.3.2}$$

If $\dot{X}(0) \equiv -A\,X(0)$ then the solution of (3.3.2) for $X(t)$ is $e^{-At}X(0)$. From the work in section 3.3 one has

$$A = N_1 + N_2 \tag{3.4.11}$$

In order to have $X_2(t) \equiv X(t)$ for $t \geq 0$, several conditions must be satisfied. These conditions follow from the equivalence of (3.4.35) and (3.3.2), i.e., the initial conditions as well as the respective coefficient matrices must be identical. More specifically $X_2(t) \equiv X(t)$ for $t \geq 0$ is obtained only if

1) $$\begin{bmatrix}\dot{X}_2(0)\\ X_2(0)\end{bmatrix} \equiv \begin{bmatrix}-A\\ U\end{bmatrix} X(0)$$

2) $k_2(2\,U + \overline{A}_p) = N_1$ (3.4.36)

3) $k_2^2\,(U + \overline{A}_p + \overline{A}_m) = -N_2 A$

4) $A = (N_1 + N_2)$

Several conclusions follow from the relations in (3.4.36).

First, $k_2(2\,U + \overline{A}_p)$ is a positive matrix so that N_1 must be a positive matrix and, in fact,

$$(N_1 - 2 k_2 U) = k_2 \overline{A}_p \tag{3.4.37}$$

must be a positive matrix.

Second, from the relations in (3.4.36), an explicit form for $\overline{A}_m$ in terms of A and N_1 is given as:

$$k_2^2 \overline{A}_m = N_1 (A - k_2 U) - A^2 + k_2^2 U \tag{3.4.38}$$

Since $k_2 > 0$ and $\overline{A}_m$ is a negative matrix then N_1 must be found to satisfy (3.4.38).

The algorithm to find N_1 to satisfy equations (3.4.37) and (3.4.38) will be found in Chapter 4.

Assuming now N_1 is found $\overline{A}_p$ and $\overline{A}_m$ can be found from (3.4.37) and (3.4.38), respectively. Hence, the improved solution $X_2(t) = X(t)$ in Figure 3.4.8 is guaranteed. The only problem now, is to find $X_1(0)$ and $X_2(0)$. To determine $X_1(0)$ consider (3.4.32),

$$\dot{X}_2(t) = - k_2 [X_1(t) + X_2(t)] \tag{3.4.32}$$

Since

$$X_2(t) = X(t)$$

then

$$\dot{X}_2(0) = \dot{X}(0) = - A X(0) .$$

Hence from (3.4.32) one has

$$- T_1 [\dot{X}_2(t) + k_2 X_2(t)] = X_1(t) \tag{3.4.39}$$

and

$$- T_1 \dot{X}_2(0) - X_2(0) = X_1(0)$$

or

$$(T_1 A - U)' X(0) = X_1(0) \qquad (3.4.40)$$

Finally, in terms of the problems of extraneous modes existing at the output of the operational amplifiers, consider (3.4.39). At any time $t \geq 0$

$$X_1(t) = (T_1 A - U) X_2(t) \qquad (3.4.41)$$

and the discussion at the end of section 3.4A applies as well for this section.

3.5. Conclusion

In this chapter a new formulation for the analog computer solutions of state model equations is developed. The correction procedure which is presented in the next chapter requires this preliminary formulation. Since the formulation depends on the type of model used for the active components described in chapter 2, three distinct developments are given.

Although the output of the amplifiers forming $X_2(t)$, the amplifier outputs that form $X_1(t)$ are also investigated. As was seen, only the modes in $X_2(t)$ can exist in the vector $X_1(t)$.

The case where only one type of computing element is used shows promise in the area of active network synthesis of the state model.

CHAPTER IV

CORRECTION ALGORITHM FOR THE VARIOUS COMPUTER DIAGRAMS

4.1. Introduction

In this chapter the problem involved in finding the matrix N_1 or $\overline{N}_1$, introduced in the previous chapter, will be considered.

It will be shown that, after a suitable time and amplitude scaling, any state model can be made to fit the class of state models for which the solution can be improved by the use of the correction procedure.

4.2. Correction Algorithm for the First-Order Models for the Computing Elements

From the theoretical derivation for first-order models for the computer solution, it was found that for $X_2(t) = X(t)$, the exact solution of (3.2.5), the matrix N_1 has to satisfy a set of constraints.

From equations (3.4.12) and (3.4.14) one has

$$\overline{A}_p = [N_1 - (k_1 + k_2) U] \qquad (4.2.1)$$

and

$$\overline{A}_m = T_1 [N_1 (A - k_2 U) - A^2 + k_2^2 U] \qquad (4.2.2)$$

The matrix N_1 must be chosen such that $\overline{A}_p$ is a positive matrix and $\overline{A}_m$ is a negative matrix.

In general, to satisfy (4.2.1) the entries of N_1 must be all positive and the diagonal entries must not be less than the sum of known positive constants, k_1 and k_2. On the other hand, from

(4.2.2) the matrix

$$[\, N_1 (A - k_2 U) - A^2 + k_2^2 U\,] \qquad (4.2.3)$$

must be a negative matrix. In the above relation the only unknown matrix is N_1. Since N_1 is a square matrix of the same order as A (order of A is n) then there are n^2 entries in N_1 where

1) $n_{ii} \geq (k_1 + k_2) \geq 0$

2) $n_{ij} \geq 0$; $i \neq j$.

By performing the matrix operations on the matrices in (4.2.3), a set of n^2 inequalities can be obtained and from these inequalities the choice for each entry in N_1 can be made.

Consider the following definition.

Definition 4.2.1. If a real square matrix, A has no eigenvalues in the right half plane and its minimal polynomial has simple zeros on the imaginary axis, then A is said to be a stable matrix.

However, not all systems described by

$$\dot{X}(t) = -A\, X(t) \;;\; X(0)$$

where $-A$ is a stable matrix, can be corrected in order to get the analog computer solution to have $X_2(t) = X(t)$.

To show this consider the system

$$\dot{x}(t) = -x(t) \;,\; x(0) = 1 \qquad (4.2.4)$$

In this case the solution of (4.2.4) is

$$x(t) = e^{-t}$$

and $x(t)$ describes a stable system. However, in considering equations (4.2.1) and (4.2.3),

$$n_1 - (k_1 + k_2) \geq 0$$
$$[\, n_1 (1 - k_2) - (1 + k_2^2)] \leq 0 \tag{4.2.5}$$

Since k_1 and k_2 are known from the models of the amplifiers used in the analog computer, one may not be able to select n_1 to satisfy both of the above inequalities. To see this, let $k_2 = 1/2$ and $k_1 = 3/2$. From (4.2.5)

$$n_1 \geq 2$$

and

$$n_1 \leq 3/2 \ .$$

Clearly, there is no n_1 which satisfies these inequalities. In the above example, if $A < k_2$ (A is a scalar) then $(A - k_2)$ would have been a scalar with negative entries and n_1 could be made large enough to allow both inequalities in (4.2.5) to be satisfied.

If the procedure of amplitude and time scaling discussed by several authors ([21] and others) was applied to the example until $A < k_2$ then the scaled problem would exist in the class of problems in which the correction procedure may be applied and a positive N_1 (scalar in this case) would exist.

Since the example is a scalar system, only time scaling may be applied to get $A < k_2$.

Let $bT = t$ then, by the chain rule for derivatives,

$$\frac{d}{dt} = \frac{d}{d(bT)} = \left(\frac{1}{b}\right)\frac{d}{dt} \qquad (4.2.6)$$

so that for the example

$$\frac{d}{dt}\, x(T) = -\, b\, x(T) \; ; \; x(0) \; .$$

Choose $b = 0.4$ then $bA = 0.4$ and in (4.2.5) with $k_1 = 3/2$ and $k_2 = 1/2$

$$n_1 \geq 3/2$$

and (4.2.7)

$$-\, n_1\,(0.1) - 0.16 + 0.25 \leq 0$$

Clearly $n_1 \geq 3/2$ satisfies both inequalities in (4.2.7).

In the case where A is a matrix, the off diagonal entries of A may be reduced in magnitude by amplitude scaling. To show this, for the 3x3 case,

$$\frac{d}{dt}\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} = - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} \qquad (4.2.8)$$

let

$$y_1(t) = m_1\, X_1(t)$$
$$y_2(t) = m_2\, X_2(t)$$
$$y_3(t) = m_3\, X_3(t)$$

and let $bT = t$ and employing (4.2.6), the scaled equations become

$$\frac{d}{dt}\begin{bmatrix} y_1(T) \\ y_2(T) \\ y_3(T) \end{bmatrix} = -b\begin{bmatrix} a_{11} & (m_1/m_2)\,a_{12} & (m_1/m_3)\,a_{13} \\ (m_2/m_1)\,a_{21} & a_{22} & (m_2/m_3)\,a_{23} \\ (m_3/m_1)\,a_{31} & (m_3/m_2)\,a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} y_1(T) \\ y_2(T) \\ y_3(T) \end{bmatrix} \tag{4.2.9}$$

The scalars m_1, m_2, m_3 will allow the programmer to force the entries in the coefficient matrix in (4.2.9) closer in magnitude to one another. This idea is not essential to the procedure but may be of some help in programming the problems. The scale factor for time then may be used to satisfy the matrix inequality relation in (4.2.1).

From the foregoing discussion of this section it is clear that an algorithm may be generated such that, through the use of time and amplitude scaling, any state model may be solved using the correction procedure to yield exact solutions. This fact will now be proved more rigorously.

Definition 4.2.2. Scaled coefficient matrix, A'. Let $A = [a_{ij}]_n$ be the coefficient matrix for the state model in (3.2.5) and consider $M = \text{diag}(m_1, m_2, \ldots, m_n)$; $m_i > 0$ for all i. If $Y(t) = M\,X(t)$ then

$$\frac{d}{dt}\,Y(t) = -A'\,Y(t) \tag{4.2.10}$$

is called the scaled state model, and the coefficient matrix

$$A' = MAM^{-1} = [(m_i/m_j)\,a_{ij}]_n \tag{4.2.11}$$

is called the amplitude scaled coefficient matrix.

The reason to use M in (4.2.11) is evident; it will modify the entries of A so that the entries may be made closer in magnitude to one another.

Assume that A is scaled by M to obtain A' as in (4.2.11). The new form of constraint equations on N_1 are,

$$[N_1 - (k_1 + k_2 U] \qquad (4.2.1)$$

and

$$T_1 [N_1 (A' - k_2 U) - (A')^2 + k_2^2 U] \qquad (4.2.12)$$

At this point applying the time scale factor

$$t = bT$$

equation (4.2.10) becomes

$$\frac{d}{dt} Y(T) = - b A' Y(T), \quad Y(0) = M X(0) . \qquad (4.2.13)$$

Now a set of inequalities may be set up so that,

$$T_1 [N_1 (bA' - k_2 U) - (bA')^2 + k_2^2 U]$$

becomes a negative matrix and N_1 is a positive matrix.

Theorem 4.2.1.

Consider an analog computer whose computing elements are characterized by first-order models. The time and amplitude scaled form obtained for any state model of the form,

$$\dot{X}(t) = - A X(t) ; X(0)$$

may be solved exactly on the analog computer through the use of the correction procedure.

Proof:

Consider a special form for the N_1 matrix defined as follows.

$$N_1 = \bar{n} V \qquad (4.2.14)$$

where $V = [1]_n$ i.e., all n^2 entries of V are 1. The constraints on N_1, as established earlier are

1) $\bar{n} \geq (k_1 + k_2)$; k_1 and k_2 are known positive constants and
2) $[N_1 (bA' - k_2U) - (bA')^2 + k_2^2 U]$

is a negative matrix.

The time scale factor b will be chosen so that all the entries of

$$P = V(bA' - k_2U)$$

are strictly negative. In order words, if A' has typical entries a'_{ij} then the typical entries of P are

$$p_{ij} = [b(\sum_{i=1}^{n} a'_{ij}) - k_2] ,$$

for all i and j and can be made negative by adjusting b. As a result of the above, b can be selected such that all p_{ij} are strictly negative.

The choice of $\bar{n}$ will be made in order to satisfy constraints 1) and 2) above. Since P is a matrix with all entries negative, by increasing $\bar{n}$ to the point where constraints 1) and 2) are both satisfied then the choice of $N_1 = \bar{n}\, V$ is established. This completes the proof.

In the proof of the above theorem the matrix N_1 is taken in a very special form. Although this form of N_1 is sufficient for the proof, it by no means implies that N_1 always must be taken in this special form. It is conceivable that, in practice, the choice of N_1 in (4.2.14) may not yield reasonable $\bar{A}_p$ and $\bar{A}_m$ matrices from an analog computer programming standpoint. A better choice of N_1 may be with arbitrary entries,

$$N_1 = [n_{ij}] \tag{4.2.15}$$

so that each entry in (4.2.15) may be adjusted to satisfy the constraints in Theorem 4.2.1 while giving more reasonable $\bar{A}_p$ and $\bar{A}_m$ matrices. Since the process of choosing N_1 in (4.2.15) will be an optimization scheme with a digital computer the choice of N_1 in (4.2.14) would be a good initial guess.

To observe the affects of scaling on the eigenvalues of A consider the following theorem.

<u>Theorem 4.2.2.</u>

Let the system

$$\dot{X}(t) = -A\,X(t)\ ;\ X(0)$$

have the characteristic equation

$$D(\lambda) = |\lambda U + A| = 0\ . \tag{4.2.16}$$

The characteristic equation for the scaled state model (4.2.13) is

$\overline{D}(\lambda)$ where

$$\overline{D}(\lambda) = b\, D(\lambda/b) = 0 \tag{4.2.17}$$

Proof:

The characteristic equation for the scaled state model in (4.2.13) is,

$$\begin{aligned}\overline{D}(\lambda) &= |\lambda U + b\, M A M^{-1}| = 0 \\ &= b|M|\ \left|\frac{\lambda M^{-1}}{b} + AM^{-1}\right| \\ &= b\ \left|\frac{\lambda U}{b} + A\right| = 0\end{aligned}$$

Therefore $\overline{D}(\lambda) = b\, D(\lambda/b)$. Q. E. D.

The above theorem, of course, simply states that, since $b > 0$, the locations of eigenvalues of the matrices A and A' on the complex plane are the same, i.e., if A corresponds to a stable matrix so does A'.

4.3. Correction Algorithm for the Second-Order Models for the Computing Elements

In this section, the results obtained in the previous section for the first-order model of the computing elements will now be extended to the second-order model of these elements. The formulation for this case is already given in Chapter 3 where, for the exact solution of (3.2.5), the matrix $\overline{N}_1 = -N_1 A$ need be selected to satisfy equations (3.4.24) and (3.4.25) which are repeated here for ready reference.

$$a_3 \overline{A}_p = [\overline{N}_1 - (a_2 + a_1 b_1)U] \tag{4.3.1}$$

and

$$a_3 b_2 \overline{A}_m = [\, \overline{N}_1 \,(A-U) + (A-n_1 U)A + a_1 b_1 U \,] \qquad (4.3.2)$$

In the above equations a_3, b_2, n_1, a_1 and b_1 are known positive constants while $\overline{A}_p$ is a positive matrix and $\overline{A}_m$ is a negative matrix.

In general, to satisfy (4.3.1) and (4.3.2), an $\overline{N}_1$ matrix with n^2 entries must be entered into these inequality relations. From (4.3.1) it is evident that

$$\overline{n}_{ii} \geq (a_2 + a_1 b_1) \,, \quad \text{for all } i \,,$$

and from (4.3.2) there are n^2 inequality relations involving the n^2 entries of $\overline{N}_1$.

As is shown in section 4.2, not all state models can fit directly into the class of state models which may be solved for the exact solution through the use of the correction procedure. However by time and amplitude scaling, as will be shown in the following, the state model can be modified to fit in this class. Using the matrix M where

$$M = \text{diag}\,(m_1, m_2, \ldots, m_n); \quad m_i > 0 \quad \text{for all } i \,,$$

and letting

$$Y(t) = M\, X(t)$$

the scaled state model for that in (3.2.5) is,

$$\dot{Y}(t) = -A'\, Y(t) \,, \quad Y(0) \qquad (4.3.3)$$

where

$$A' = M A M^{-1} \text{ and } Y(0) = M X(0).$$

Using the transformed state model, the constraint equations on $\overline{N}_1$ take on the form,

$$a_3 \overline{A}_p = [\overline{N}_1 - (a_2 + a_1 b_1) U]$$

which is a positive matrix and,

$$a_3 b_2 \overline{A}_m = [\overline{N}_1 (A'-U) + (A'-n_1 U) A' + a_1 b_{1U}]$$

which is a negative matrix. If the time scaling is used,

$$t = bT$$

then (4.3.3) becomes

$$\dot{Y}(T) = - bA' Y(T), \quad Y(0) \tag{4.3.4}$$

and the set of inequality constraints for the solution of (4.3.4) are of the form,

$$a_3 \overline{A}_p = [\overline{N}_1 - (a_3 + a_1 b_1) U] \tag{4.3.5a}$$

and

$$a_3 b_2 \overline{A}_m = [\overline{N}_1 (bA'-U) + (bA'-n_1 U) bA' + a_1 b_1 U] \tag{4.3.5b}$$

By properly choosing b and $\overline{N}_1$, the inequality constraints implied in (4.3.5) may be satisfied. This is stated and proved in the following.

<u>Theorem 4.3.1.</u>

The time and amplitude scaled form of any state model of the form

$$\dot{X}(t) = - A\ X(t)\ ;\quad X(0)$$

may be solved exactly on the analog computer, whose computing elements have second-order models, through the use of the correction procedure.

<u>Proof:</u>

Consider the following special form form for the $\overline{N}_1$ matrix,

$$\overline{N}_1 = \overline{n}\, V \tag{4.3.6}$$

where

$$V = [1]_n,\ \text{i.e., all } n^2 \text{ entries of } V \text{ are } 1\,.$$

The constraints on $\overline{N}_1$ are

1) $\overline{n} \geq (a_2 + a_1 b_1)$

and

2) $a_3 b_2 \overline{A}_m = [\,\overline{N}_1\ (bA'-U) + (bA'-n_1 U)\, bA' + a_1 b_1 U\,]$

where $\overline{A}_m$ is a negative matrix and a_1, b_1, n_1, b_2, and a_3 are known positive constants.

The time scale factor b will be chosen such that

$$P = V(bA' - U)$$

is a matrix with all entries strictly negative. If A' has typical entries a'_{ij} then P has typical entries

$$p_{ij} = [b (\sum_{i=1}^{n} a'_{ij}) - 1] \quad \text{for all } i \text{ and } j .$$

Clearly, b may be chosen so that all p_{ij} are negative.

The choice of $\overline{n}$ in (4.3.6) will be made in order to satisfy constraints 1) and 2) above. Since P is a matrix whose entries are all negative by increasing $\overline{n}$ to the point where constraints 1) and 2) are both satisfied and the choice of $\overline{N}_1$ is now complete.

As was discussed in section 4.2, the choice of $\overline{N}_1$ in (4.3.6) may result in matrices $\overline{A}_p$ and $\overline{A}_m$ which are difficult to program on the analog computer. However, $\overline{N}_1$ as defined in (4.3.6) may be a good initial choice for $\overline{N}_1$ where

$$\overline{N}_1 = [\overline{n}_{ij}]$$

which may be chosen to improve the $\overline{A}_p$ and $\overline{A}_m$ from an analog computer programming standpoint.

4.4. Justification of the Correction Algorithm for the Analog Computer Solution Using One Type of Active Computing Element

In the last part of section 3.4, the formulation of the correction procedure for the analog computer set up, using one type of active element, is given. For the exact solution of (3.2.5) with $X_2(t) = X(t)$, several constraint relations were obtained for the $\overline{A}_p$ and $\overline{A}_m$ matrices. From (3.4.37) and (3.4.38),

$$k_2 \overline{A}_p = (N_1 - 2 k_2 U) \qquad (3.4.37)$$

must be a positive matrix and,

$$k_2^2 \overline{A}_m = [N_1 (A - k_2 U) - A^2 + k_2^2 U] \tag{3.4.38}$$

is a negative matrix. In these equations, k_2 is a known positive constant.

In general, to satisfy (3.4.37) and (3.4.38) an N_1 matrix with n^2 (n is the order of A) entries, n_{ij}, must be entered into the above relations.

From (3.4.37) it is evident that

$$n_{ii} \geq 2 k_2 \quad \text{for all } i$$

and from (3.4.38) there will be n^2 inequality relations, involving the n^2 entries of N_1, which must be solved simultaneously.

As shown in section 4.2, only a limited class of state models can be solved directly on the analog computer using the correction procedure. However by time and amplitude scaling the state model in scaled form may be obtained for which the correction procedure yields an improved solution on the analog computer.

Using the M matrix and scale factor b, the scaled state model will have the form,

$$\frac{d}{dt} Y(T) = - bA' Y(T), \quad Y(0) \tag{4.4.1}$$

where

1) $A' = M A M^{-1}$

2) $Y(0) = M X(0)$

3) $t = bT$.

By properly choosing b and N_1, it is possible to make

$$k_2 \overline{A}_p = (N_1 - 2 k_2 U) \qquad (4.4.2)$$

a positive matrix, while

$$k_2^2 \overline{A}_m = [N_1 (bA' - k_2 U) - (bA')^2 + k_2^2 U] \qquad (4.4.3)$$

is a negative matrix. To show this is always possible the following theorem is stated.

Theorem 4.4.1.

The time and amplitude scaled form of any state model of the form,

$$\dot{X}(t) = - A X(t) ; \quad X(0)$$

may be solved exactly on the analog computer through the use of the correction procedure for analog computer set ups using only one type of active computing element.

Proof:

Consider the following special form for the N_1 matrix.

$$N_1 = \overline{n} V \qquad (4.4.4)$$

where $V = [1]_n$, i.e., all entries of V are 1.

The constraints on N_1 are

1) $\overline{n} \geq 2 k_2$

and

2) $k_2^2 \overline{A}_m = [N_1 (bA' - k_2 U) - (bA')^2 + k_2^2 U]$

is a negative matrix. In these constraints, k_2 is a known positive constant.

The time scale factor is chosen as before by forcing

$$P = V(bA' - k_2 U)$$

to be a matrix with all entries negative.

The choice of $\bar{n}$ in (4.4.4) will be made in order to satisfy the above constraints. Since P is a matrix with all entries negative, by increasing $\bar{n}$ to the point where constraints 1) and 2) are both satisfied, the choice of N_1 in (4.4.1) is complete. Therefore, by forcing P to be a matrix with all negative entries, $\bar{n}$ may always be found to satisfy the above constraints. Q. E. D.

It should be noted again that the choice of N_1 in (4.4.4) may result in $\bar{A}_p$ and $\bar{A}_m$ matrices which are difficult to program on the analog computer. However, N_1 as defined above may be a good initial choice for N_1 where

$$N_1 = [n_{ij}]$$

which may be chosen to improve the $\bar{A}_p$ and $\bar{A}_m$ from an analog computer programming standpoint.

So far, in the above discussion modification on the initial conditions due to the time and amplitude scaling is not considered. Therefore, consider the relations in (3.4.7) and (3.4.33) for the development using first-order models for the active computing elements. Clearly

$$X_1(t) = -T_1 [\dot{X}_2(t) + k_2 X_2(t)]$$

and in the scaled form one has

$$Y_1(T) = -T_1[\dot{Y}_2(T) + k_2 Y_2(T)] \quad . \qquad (4.4.5)$$

However, from the other established relations,

$$Y_2(T) = Y(T)$$

$$\dot{Y}_2(T) = -bA' Y_2(T)$$

and at $T = 0$, these relations yield,

$$Y_1(0) = (T_1 bA' - U) Y(0)$$

or

$$Y_1(0) = (T_1 b MAM^{-1} - U) M X(0)$$

so that

$$Y_1(0) = (T_1 b M A - M) X(0) \qquad (4.4.6)$$

and

$$Y_2(0) = M X(0) \quad .$$

Similarly, for the case where second-order models for the active elements are considered, the initial conditions are given by

$$Y_1(0) = \frac{1}{b_2} (b M A - b_1 M) X(0) \qquad (4.4.7)$$

and

$$Y_2(0) = M X(0) \quad .$$

This completes the development of the algorithm for each of the three cases considered. A brief summary of the procedure is presented in the form of a flow chart in section 4.5.

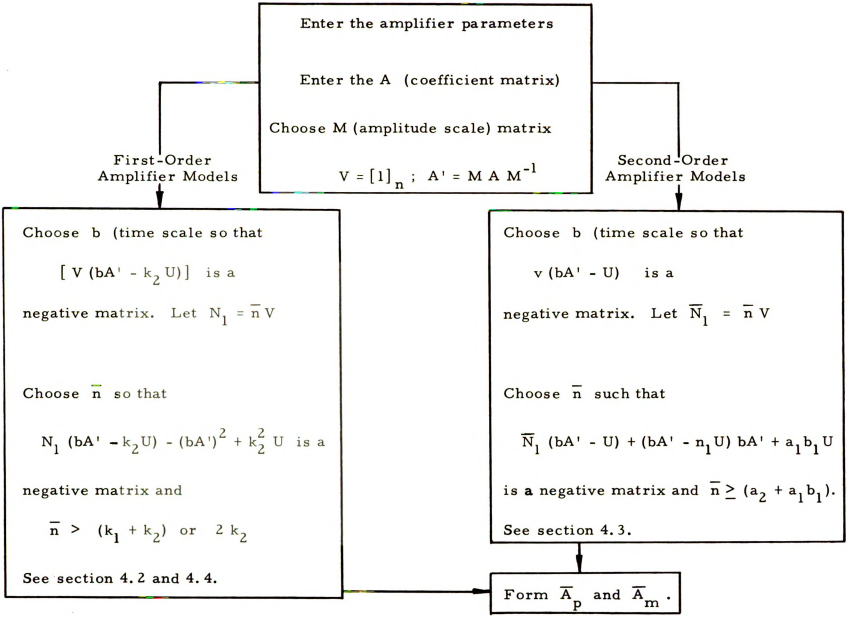
4.5. Flow Diagram for the Correction Procedure

The flow diagram given at the end of this section summarizes the steps used in the application of the correction procedure. As is seen from the flow diagram, the form of the matrix N_1 or $\overline{N}_1$ is taken in a special form: $\bar{n}V$, where V is a square matrix all of whose entries are 1. From a practical standpoint, in finding the parameter b and the matrix N_1 or $\overline{N}_1$, the resultant $\overline{A}_p$ and $\overline{A}_m$ matrices may have entries unsuitable for programming on the analog computer. In this case a more general form for N_1 or $\overline{N}_1$ may be sought. The most general form is, of course, the one in which all the n^2 entries of N_1 or $\overline{N}_1$ are considered as parameters (where n is the order of N_1 or $\overline{N}_1$). However for the determination of these parameters, in general, an analytical method cannot be used. One must use a more sophisticated search technique, on the digital computer, for an optimal solution of N_1 or $\overline{N}_1$ subject to the matrix inequalities and the satisfaction of some desirable forms for the $\overline{A}_p$ and $\overline{A}_m$ matrices. In any event, when such a search technique is employed a good initial guess is required for b and N_1 or $\overline{N}_1$. However, in the algorithm, taking the special form $\bar{n}V$ for N_1 or $\overline{N}_1$, for a chosen b, one actually has a solution to the problem as far as the inequalities are concerned. Therefore, the search technique merely has to improve this solution in order to obtain more desirable forms for N_1 or $\overline{N}_1$.

In order to give an indication as to how to obtain a formulation for this general case, consider the situation where first-order models used for the computing elements. It will be seen that, in reality, the formulation yields a set of linear equations to be solved simultaneously subject to some restrictions on the entries of $\overline{A}_p$ and $\overline{A}_m$. It will also be seen that, in this formulation, the matrix N_1 does not play any essential role, in fact it does not appear in this formulation at all. Indeed, from (4.2.1) and (4.2.12) corresponding to the case under consideration, eliminating N_1 and after simple algebraic manipulation, we have

$$\overline{A}_p [(b/k_2 A' - U] - \overline{A}_m = - (k_1 + k_2) [(b/k_2) A' - U] + (b^2/k_2)(A')^2 - k_2 U \qquad (4.5.1)$$

The above matrix equation is actually equivalent to n^2 linear equations in $2n^2$ unknowns. These unknowns are the entries of the matrices $\overline{A}_p$ and $\overline{A}_m$. The magnitude of the non-zero entries of $\overline{A}_p$ and $\overline{A}_m$ are restricted to lie in the interval $[0.1, 10.0]$. This restriction is placed so the entries of $\overline{A}_p$ and $\overline{A}_m$ may be easily programmed on the analog computer. Since some n^2 of the $2n^2$ variables may be selected arbitrarily, it is possible that the requirements imposed on the entries of $\overline{A}_p$ and $\overline{A}_m$ may be satisfied. Note that the set of n^2 equations considered above are consistent for values of b taken in some intervals. The Theorem 4.2.1 justifies this statement at least for a special form of the matrix N_1.


Enter the amplifier parameters
Enter the A (coefficient matrix)
Choose M (amplitude scale) matrix
$V = [1]_n$; $A' = M A M^{-1}$
First-Order Amplifier Models
Second-Order Amplifier Models
Choose b (time scale so that
$[V(bA' - k_2 U)]$ is a
negative matrix. Let $N_1 = \bar{n} V$
Choose $\bar{n}$ so that
$N_1 (bA' - k_2 U) - (bA')^2 + k_2^2 U$ is a
negative matrix and
$\bar{n} > (k_1 + k_2)$ or $2 k_2$
See section 4.2 and 4.4.
Choose b (time scale so that
$v(bA' - U)$ is a
negative matrix. Let $\bar{N}_1 = \bar{n} V$
Choose $\bar{n}$ such that
$\bar{N}_1 (bA' - U) + (bA' - n_1 U) bA' + a_1 b_1 U$
is a negative matrix and $\bar{n} \geq (a_2 + a_1 b_1)$.
See section 4.3.
Form $\bar{A}_p$ and $\bar{A}_m$.

4.6. Conclusions

This chapter outlines a method of solution for the determination of the corrected matrices $\overline{A}_p$ and $\overline{A}_m$. The algorithm uses the formulation of the previous chapter. Since this formulation depends on the type of models used for the active elements, the correction algorithm is applied separately for each of the three forms considered for the active element models.

In all cases the algorithm yields a method to find the time scale factor b and matrices M and N_1 or $\overline{N}_1$.

In the next chapter, the correction algorithm will be applied to several problems where the analog computer has active elements with first-order models.

CHAPTER V

EXAMPLES AND EXPERIMENTAL RESULTS

5.1. Introduction

This chapter contains examples of the use of the correction algorithm for a computer set up using one type of active computing element. A detailed analysis of the proposed computer set up is given followed by the actual solution obtained on the computer.

A second example using the first-order models for the active computing elements is given for the equation of the harmonic generator.

The uncorrected form of the computer set up is given first in order that the instability of the set up in this form can be seen. The corrected form follows the uncorrected form and the desired results are obtained.

Note that the models used for the active computing elements have extremely poor frequency response characteristics compared to the actual computing elements normally found in a good quality analog computer. The elements with extremely poor response characteristics are purposefully constructed in the laboratory which also provide simplifications in the calculation of the corrected matrices $\overline{A}_p$ and $\overline{A}_m$. Also, by choosing the models with poor frequency response characteristics the bandwidth and open-loop gain limitations of the actual computing elements do not affect these models.

5.2. Example - A Single Differential Equation

We shall now apply the theory developed in section 3.4 for computer set ups involving only one type of active computing element to the following equation

$$\dot{x}(t) = -a\,x(t); \quad x(0) \qquad (5.2.1)$$

Assume that the time constant used for the degraded amplifier (equation 2.3.3) is

$$T_1 = 0.1 \quad (k_2 = 10) \qquad (5.2.2)$$

For the calculation of the matrices $\bar{A}_p$ and $\bar{A}_m$ (from (4.4.2) and (4.4.3)) with $a = 1.0$, we have the relations,

$$\bar{A}_p = \frac{\bar{n}}{10} - 2 \qquad (4.4.2)$$

$$\bar{A}_m = \frac{\bar{n}(b - 10) - b^2 + 100}{100} \qquad (4.4.3)$$

Choosing $b = 1$ we note that

$$\bar{A}_m = \frac{-9\bar{n} + 99}{100}$$

From (4.4.2)

$$\bar{n} \geq 20$$

and if $\bar{n} = 20$ then

$$\bar{A}_p = 0.0$$

$$\bar{A}_m = -0.81 \quad . \qquad (5.2.3)$$

Since $M = 1$ and $b = 1$, then by (4.4.6) with $x(0) = 1$,

$$x_1(0) = (0.1 \ -1)\ x(0) = -0.9 \qquad (5.2.4)$$

The computer set up for the problem is shown in Figure 5.2.1.

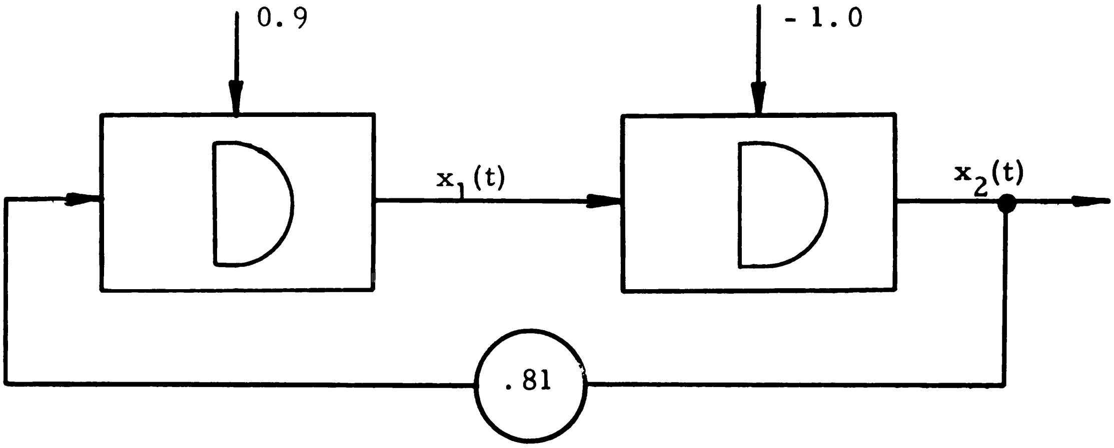


Figure 5.2.1. Computer Diagram Using One-Type of Active Component(Corrected).

To check the accuracy of the solution obtained for $x_2(t)$ in Figure 5.2.1, consider the solution of (3.4.33) for a given

$$A_p = 0$$

$$A_m = -0.81 ,$$

which is,

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = - \begin{bmatrix} 10 & 8.1 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \begin{bmatrix} -0.9 \\ 1.0 \end{bmatrix} . \qquad (5.2.5)$$

By methods described in [1, pp. 191-250] the analytical solution of (5.2.5) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \begin{bmatrix} -0.9 \\ 1.0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} e^{-19t} .$$

In theory $x_2(t) = x(t) = e^{-t}$ but the zeros in the matrix multiplied by e^{-19t} are the differences of two numbers which, in practice may not be exactly equal.

A scheme to reduce the affects of errors in potentiometer settings would further improve the solution.

In order to show that the choice of N_1 may not be unique, choose $\overline{n} = 30$. This value of $\overline{n}$ satisfies (4.4.2) and (4.4.3) and

$$\begin{aligned} \overline{A}_p &= 1.0 \\ \overline{A}_m &= -1.71 \end{aligned} \tag{5.2.7}$$

The computer diagram using $\overline{A}_p$ and $\overline{A}_m$ from (5.2.7) is given in Figure 5.2.2.

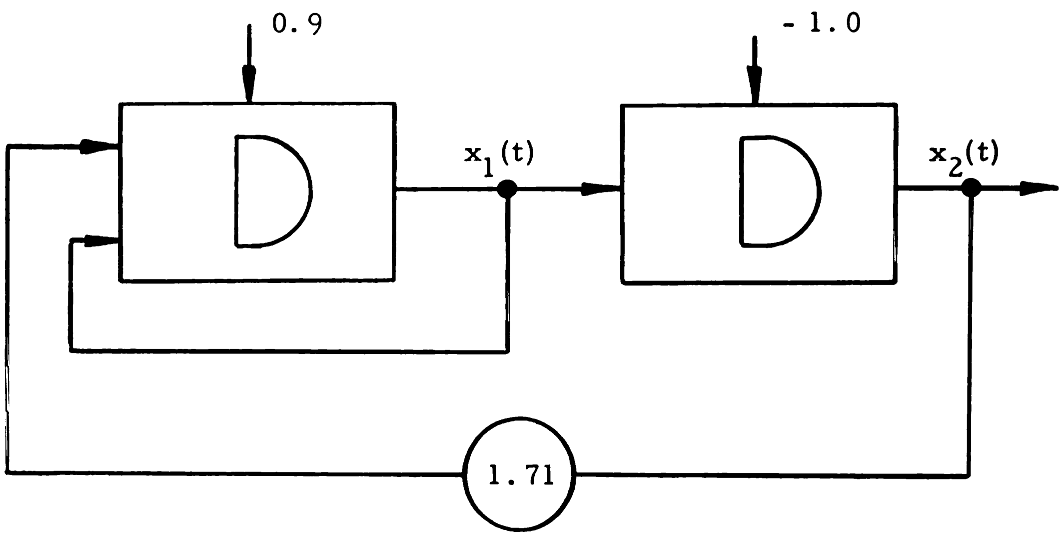


Figure 5.2.2. Computer Diagram Using One-Type of Active Component (Corrected).

To check the accuracy of the solution of the circuit in (3.4.33) for $A_p = 1.0$ and $\overline{A}_m = -1.71$, consider (3.4.33),

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -\begin{bmatrix} 20 & 17.1 \\ 10 & 10 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ and } \begin{bmatrix} -0.9 \\ 1.0 \end{bmatrix} \qquad (5.2.8)$$

From equation (3.4.40), $x_1(0)$ and $x_2(0)$ are not functions of b or $\overline{n}$ so that the initial conditions for (5.2.8) are the same as those in (5.2.5). The solution of (5.2.7) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -0.9 \\ 1.0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} e^{-29t} \qquad (5.2.9)$$

The solution

$$x_2(t) = e^{-t}$$

is the exact solution of equation (5.2.1) with $a = 1.0$ and $x(0) = 1.0$.

An important point may be seen from the solutions (5.2.6) and (5.2.9). In theory (5.2.6) and (5.2.9) have the same solution but in practice the exact initial conditions given in (5.2.5) and (5.2.8) will not be entered on the analog computer. Because of errors in the setting of potentiometers the computer set up in Figure (5.2.2) would be more desirable. The undesired mode in (5.2.9) will approach zero faster than the undesired mode in (5.2.6).

This idea may be used in order to make a good choice of N_1.

Consider the equation (4.2.13) and the corresponding characteristic polynomial.

$$D(s) = |sU + bA'|$$

The eigenvalues of (3.4.33) and (3.4.35) are related, due to the fact that for $X_2(t) = X(t)$, then (3.4.41) implies that the modes that appear in $X_1(t)$ also appear in $X_2(t)$. In other words, the solution of (3.4.33) and (3.4.35) are related if $X_2(t) = X(t)$.

Using the form of (3.4.35) given in (3.3.2), the characteristic equation of (3.4.35) is,

$$\overline{D}(s) = \begin{vmatrix} sU + N_1 & -N_2 bA' \\ -U & sU \end{vmatrix} \qquad (5.2.10)$$

By elementary operations (row and column) on the above equation, note that

$$\overline{D}(s) = \begin{vmatrix} sU - N_2 & O \\ -U & sU + bA' \end{vmatrix}$$

which has a solution

$$\overline{D}(s) = |Us + N_1 - bA'| \; |Us + bA'| \; . \qquad (5.2.11)$$

Note that if N_1 is chosen such that

$$|Us + N_1 - bA'| \qquad (5.2.12)$$

has eigenvalues with real parts more negative than each eigenvalue in $D(s)$, then the effects of set up errors in potentiometer settings will be reduced.

The only practical limit on the size of $\bar{n}$ will probably be the magnitude of the entries of the resultant $\bar{A}_p$ and $\bar{A}_m$ matrices.

A type of constraint limiting the magnitude of the entries in $\bar{A}_p$ and $\bar{A}_m$ could be easily incorporated into a digital program to choose $\bar{n}$ in order to make

$$|U_s + N_1 - bA'|$$

have eigenvalues with the real parts more negative than the real part of each eigenvalue of $\bar{D}(s)$ where

$$\bar{D}(s) = |sU + bA'| \quad .$$

To consider the region of possible choices of $\bar{n}$ and b to see if a third choice may be more desirable than the previous two, return to the above example. From (4. 4. 2) and (4. 4. 3) the graph in Figure 5. 2. 3 yields the acceptable regions of $\bar{n}$ and b.

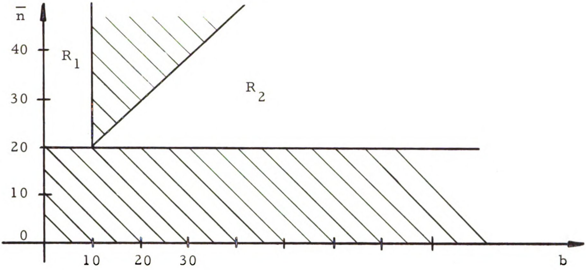


Figure 5.2.3. Regions of Acceptable $\bar{n}$ and b are Unshaded.

In region R_1, as $\bar{n}$ is increased, the undesired mode approaches zero more quickly but the magnitude of the entries in $\bar{A}_p$ and $\bar{A}_m$ also increase.

In region R_2 holding $\bar{n}$ fixed will not change $\bar{A}_p$. If b = 10 (regardless of $\bar{n}$) $\bar{A}_m = 0$. If b = 10 and $\bar{n} = 20$ the characteristic equation of the system is

$$D(s) = (s + 10)^2$$

which implies the unwanted mode would be te^{-10t}. For large t, the unwanted mode may give some error if there are errors in the potentiometer settings. If b = 20 and $\bar{n} = 20$ then $\bar{A}_p = 0$ and $\bar{A}_m = -1.0$ and the characteristic equation is

$$D(s) = s(s + 20)$$

The unwanted mode would be a constant which just shifts the solution by a DC value.

For $\bar{n} = 20$ and $b = 30$, $\bar{A}_p = 0$ and $\bar{A}_m = -4.0$ and the characteristic polynomial is

$$D(s) = (s + 30)(s - 10) .$$

In this case the unwanted mode is e^{10t} which is highly undesirable.

As can be seen, the criterion stated in relation (5.2.12) can be quite important when considering practial problems in computer set ups.

The results of the computer set up of Figure 5.2.1 is shown in Figure 5.2.4.

The value of the function (Figure 5.2.3) at $t = 1$ is 0.370 where the analytical answer is 0.368.

5.3. Example Two - Harmonic Generator

Consider the sine function

$$x(t) = a \sin (wt + \phi)$$

Since

$$\dot{x}(t) = a w \cos (wt + \phi)$$

$$\ddot{x}(t) = - w^2 x(t)$$

$x(t)$ may be rewritten in the following first order form,

$$\frac{d}{dt} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} = - \begin{bmatrix} 0 & w^2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x}(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} a w \cos \phi \\ a \sin \phi \end{bmatrix} \qquad (5.3.1)$$

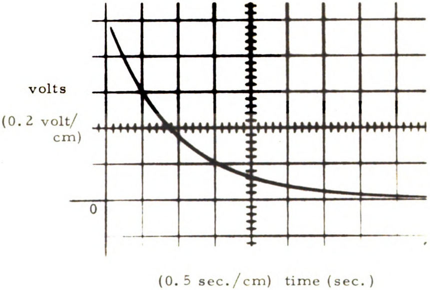


Figure 5. 2. 4. **Output** of the Computer Diagram in Figure 5. 2. 1.

The characteristic equation is

$$D(s) = \begin{vmatrix} s & w^2 \\ -1 & s \end{vmatrix} = (s^2 + w^2) = 0$$

We shall use first-order models for the integrator (3.4.4) and summing amplifier (3.4.5), and the constants will be chosen as,

$$T_o = 10 \quad (k_1 = 0.1)$$

$$T_1 = 0.1 \quad (k_2 = 10)$$

These constants are chosen to degrade the existing components so that calculations are simplified.

If the correction procedure of section (4.2) is not used, the equation actually solved by the computer (Figure 5.3.1) will be equation (3.4.7).

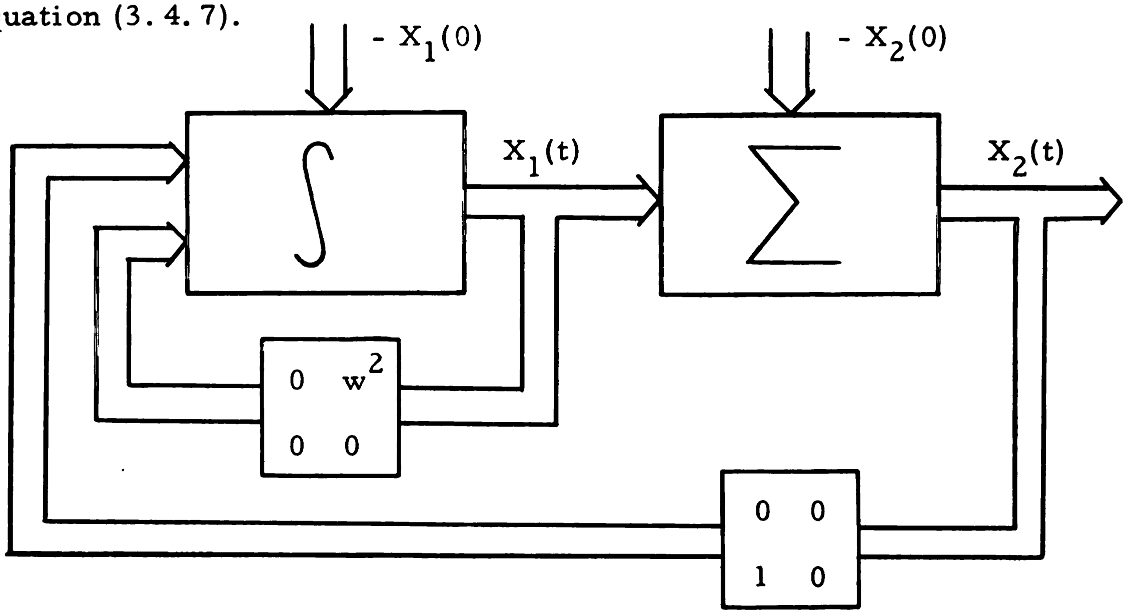


Figure 5.3.1. Uncorrected Diagram for Example 5.3.

$$\frac{d}{dt}\begin{bmatrix} X_{11}(t) \\ X_{12}(t) \\ X_{21}(t) \\ X_{22}(t) \end{bmatrix} = -\begin{bmatrix} 0.1 & w^2 & 0 & 0 \\ 0 & 0.1 & 1 & 0 \\ 10 & 0 & 10 & 0 \\ 0 & 10 & 0 & 10 \end{bmatrix}\begin{bmatrix} X_{11}(t) \\ X_{12}(t) \\ X_{21}(t) \\ X_{22}(t) \end{bmatrix} \qquad (5.3.3)$$

The characteristic equation for (3.4.7) is

$$\overline{D}(s) = (s + 10)\,(s^3 + 10.2s^3 + 2.01s + 10(w^2 + 0.01))$$

For the system described by (5.3.3), to be unstable

$$(2.01 - 0.98\,(w^2 + 0.01)) < 0 \qquad (5.3.4)$$

For example for $w = 1.50$, the system is unstable. The coefficient matrix with $w = 1.5$ radian is

$$A = \begin{bmatrix} 0 & 2.25 \\ -1 & 0 \end{bmatrix}$$

Since the entries of A are within the range 1 to 10, no amplitude scaling is needed.

For the calculation of $\overline{A}_p$ and $\overline{A}_m$, consider equations (4.2.1) and (4.2.12)

$$\overline{A}_p = N_1 - (k_1 + k_2)\,U \qquad (4.2.1)$$

$$= n\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 10.1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\overline{A}_m = \left(\frac{\overline{n}}{10}\begin{bmatrix} -(10+b) & 2.25b-10 \\ -(10+b) & 2.25b-10 \end{bmatrix} + (0.225\,b^2 + 10)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \qquad (4.2.12)$$

Choose $b = 1.0$ so that in (4.2.12)

$$\bar{A}_m = 0.1 \begin{bmatrix} -11 & -7.75 \\ -11 & -7.75 \end{bmatrix} + 10.225 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The constraints on $\bar{n}$ are

$$\bar{n} \geq 10.1$$

and

$$\bar{n} \geq 13.2$$

so that $\bar{A}_p$ and $\bar{A}_m$ are positive and negative matrices, respectively.

For $\bar{n} = 13.2$,

$$\bar{A}_p = \begin{bmatrix} 3.1 & 13.2 \\ 13.2 & 3.1 \end{bmatrix}$$

and (5.3.5)

$$\bar{A}_m = - \begin{bmatrix} 4.3 & 10.225 \\ 14.525 & 0 \end{bmatrix}$$

The initial conditions needed to implement this set up are

$$X_2(0) = X(0)$$

$$X_1(0) = (T_1 A - U) \; X(0) \qquad (3.4.16)$$

If $a = 1$ and $\phi = \pi/2$ then in (5.3.1)

$$X(0) = \begin{bmatrix} \dot{x}(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As a result,

$$X_1(0) = \begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 0.2467 \\ -1.00 \end{bmatrix}$$

In (5.3.5) the entries of $\overline{A}_p$ and $\overline{A}_m$ are all greater than one. Although these values may be programmed on the analog computer, the set up would be difficult. If N_1 is chosen as

$$N_1 = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$$

then for (4.2.1) and (4.2.12), with $b = 1$

$$\overline{A}_p = \begin{bmatrix} n_{11} - 10.1 & n_{12} \\ n_{21} & n_{22} - 10.1 \end{bmatrix}$$

and

$$\overline{A}_m = -0.1 \begin{bmatrix} 10\, n_{11} + n_{12} & 10\, n_{12} - 2.25\, n_{11} \\ 10\, n_{21} + n_{22} & 10\, n_{22} - 2.25\, n_{21} \end{bmatrix} + 10.225 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If

$$N_1 = \begin{bmatrix} 10.1 & 2.2725 \\ 0 & 10.225 \end{bmatrix}$$

then

$$\overline{A}_p = \begin{bmatrix} 0 & 2.2725 \\ 0 & 0.125 \end{bmatrix} \qquad (5.3.6a)$$

and

$$\bar{A}_m = - \begin{bmatrix} 0.10225 & 0 \\ 1.0225 & 0 \end{bmatrix} \qquad (5.3.6b)$$

Clearly the set up of the program is greatly simplified with $\bar{A}_p$ and $\bar{A}_m$ given in (5.3.6) rather then the matrices $\bar{A}_p$ and $\bar{A}_m$ given in (5.3.5)

The relation in (4.5.1) for this particular case, with $k_1 = 0.1$, $k_2 = 10.0$, and $b = 1$ takes the form

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0.225 \\ -0.1 & -1 \end{bmatrix} - \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} -0.125 & -2.2725 \\ 1.01 & -0.125 \end{bmatrix} \qquad (5.3.7)$$

where $\bar{A}_p = [p_{ij}]$ and $\bar{A}_m = [m_{ij}]$.

The above selection for $\bar{A}_p$ and $\bar{A}_m$ in (5.3.6) satisfies (5.3.7). This can be seen by simple substitution. However, if other more satisfactory solutions are sought, one can conveniently use (5.3.7) for this purpose.

The computer set up for this problem using (5.3.6) is given in Figure 5.3.2.

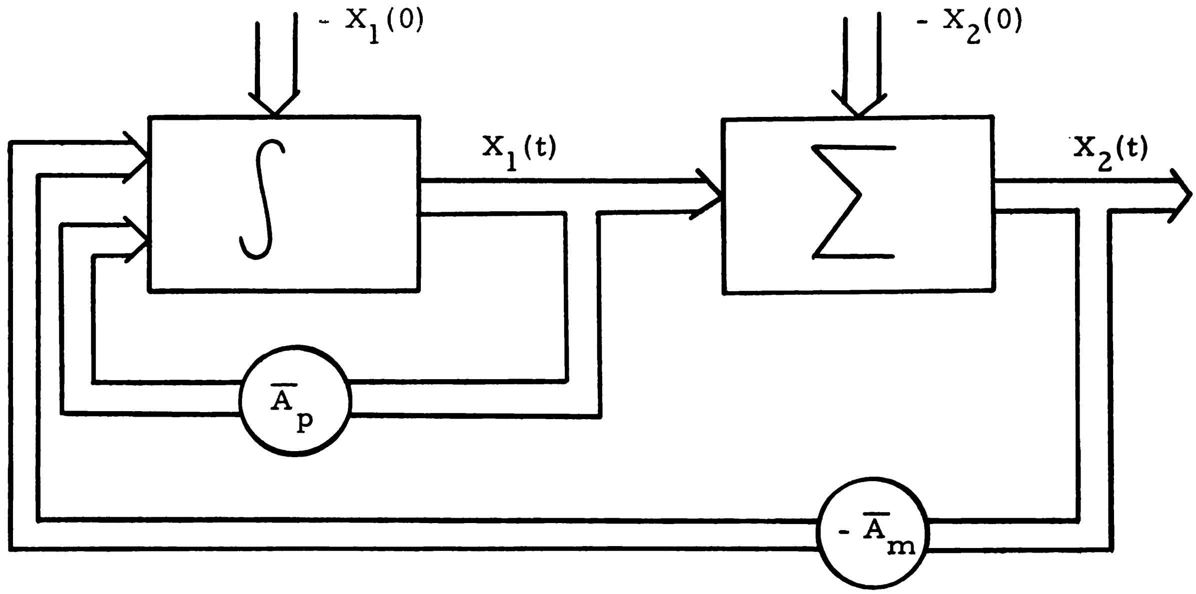


Figure 5.3.2. Corrected Diagram for the Sine Wave.

This example points out the fact that even though a solution for N_1 in (4.2.1) and (4.2.12) is always possible, a better choice can probably be found for N_1. The experimental results are shown for the uncorrected (Figure 5.3.1) and corrected (Figure 5.3.2) forms.

The output voltage shown in the following figures is $x_{22}(t)$ where

$$X_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} .$$

The output $x_{22}(t)$ for the uncorrected form (Figure 5.3.1) is shown in Figure 5.3.3.

Clearly, the system is unstable as the theory has shown.

After the correction procedure has been applied, the output $x_{22}(t)$ is shown in Figure 5.3.4.

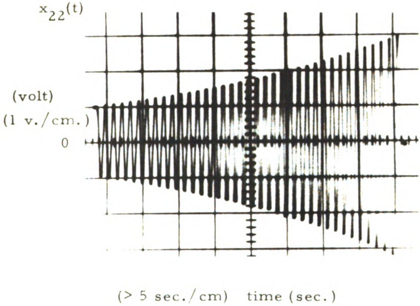


Figure 5.3.3. Output of the Uncorrected Form of the Computer Diagram in Figure 5.3.1.

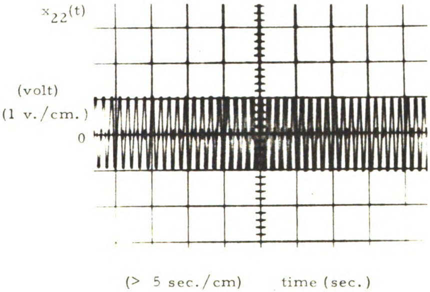


Figure 5.3.4. Output of the Corrected Form of the Computer Diagram in Figure 5.3.2.

Figure 5.3.4 shows the output over a period of several minutes and the output $x_{22}(t)$ varies from 1.0 to - 1.0.

After running the problem for ten minutes a measurement of $x_{22}(t)$ was taken at a faster sweep speed. The result of this measurement is shown in Figure 5.3.5.

Evidently the output $x_{22}(t)$ is a sine wave with a radian frequency of 1.55 radians per second.

5.4. Example Three - Higher Order State Models

The purpose of this example is to show that the application of the correction algorithms of sections (4.2) and (4.4) is not difficult or time consuming.

Consider the following third order system,

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 1/2 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} \tag{5.4.1}$$

Since the element values in the coefficient matrix are within a decade of one another, no amplitude scaling is needed. We shall assume that for the integrators and summing amplifiers only first-order models are used (section 4.2). Let a choice be made for the amplifier time constants as,

$$k_1 = \left(\frac{1}{T_o}\right) = 0.1$$

$$k_2 = \left(\frac{1}{T_1}\right) = 10.0$$

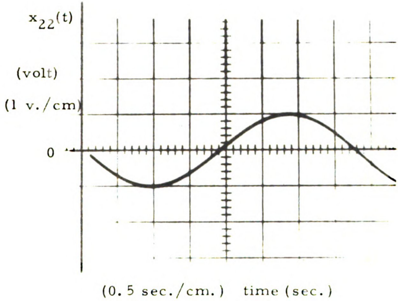


Figure 5.3.5. Output of the Corrected Form of the Computer Diagram in Figure 5.3.2 After Ten Minutes of Operation.

Using the algorithm in section (4.2),

$$\overline{A}_p = \overline{n}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} - (10.1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (5.4.2)$$

and

$$\overline{A}_m = \overline{n}\begin{bmatrix}-1 & (0.1b-1) & (0.05b-1)\\ -1 & (0.1b-1) & (0.05\,b-1)\\ -1 & (0.1b-1) & (0.05b-1)\end{bmatrix}$$

$$+\ \frac{b^2}{40}\begin{bmatrix}0 & 4 & 6\\ 4 & -4 & -2\\ -6 & 2 & 7\end{bmatrix} + 10\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (5.4.3)$$

If, $b = 1$ and $\overline{n} = 11$ then

$$\overline{A}_p = \begin{bmatrix}0.9 & 11 & 11\\ 11 & 0.9 & 11\\ 11 & 11 & 0.9\end{bmatrix}$$

and (5.4.4)

$$\overline{A}_m = -\begin{bmatrix}1 & 9.8 & 10.3\\ 9.9 & 0 & 10.5\\ 10.15 & 9.85 & 0.625\end{bmatrix}.$$

Again note that if a general N_1 could have been chosen, smaller entry values in $\overline{A}_p$ and $\overline{A}_m$ would have been obtained.

For the situation where one type of active computing element is to be used, assume that k_2 is chosen as,

$$k_2 = \frac{1}{T_1} = 10.0 \ .$$

Using the relations for $\overline{A}_p$ and $\overline{A}_m$ in section 4. 4,

$$\overline{A}_p = 0.1 \begin{bmatrix} \overline{n} - 20 & \overline{n} & \overline{n} \\ \overline{n} & \overline{n} - 20 & \overline{n} \\ \overline{n} & \overline{n} & \overline{n} - 20 \end{bmatrix} \tag{5.4.5}$$

and

$$100\,\overline{A}_m = \overline{n} \begin{bmatrix} -10 & (b - 10) & (\frac{b}{2} - 10) \\ -10 & (b - 10) & (\frac{b}{2} - 10) \\ -10 & (b - 10) & (\frac{b}{2} - 10) \end{bmatrix}$$

$$- \frac{b^2}{40} \begin{bmatrix} 0 & 4 & 6 \\ 4 & -4 & -2 \\ -6 & 2 & -7 \end{bmatrix} + 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.4.6}$$

Clearly, for $b = 1$ and $\overline{n} = 20$

$$\overline{A}_p = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

and (5. 4. 7)

$$\overline{A}_m = - \begin{bmatrix} 1 & 1.801 & 1.8985 \\ 2.001 & .799 & 1.8995 \\ 1.9985 & 1.8005 & 0.8983 \end{bmatrix}$$

The calculations for this example were not difficult and could be completed in a short amount of time using only hand calculations.

The use of a general N_1 matrix would enable one to reduce the magnitude of the entries of $\overline{A}_p$ and $\overline{A}_m$.

5.5. Conclusions

From the examples of this chapter several interesting observations are made. Example one was performed to show the validity of the procedure by the corrected system by analytical and analog computer methods. A criterion that may be used in the choice of N_1 is given for the general problem where the effects of the unwanted modes in the computer solution are reduced. In practice small errors may result in initial conditions or potentiometer settings. These deviations may allow the unwanted modes to appear in the analog computer solution. If these modes correspond to an unstable solution, the errors in the analog computer solution will be quite short.

Example Two shows the use of the correction procedure for a system which is affected by shifts in the eigenvalues.

Example Three shows that the calculation of a general matrix, N_1 for a third order system is not difficult and in the case where the models are chosen properly, the calculations may be carried out by hand.

CHAPTER VI

CONCLUSIONS

Review of Thesis

In the solution of a state model for a linear time-invariant system, several methods may be used. The method investigated in the thesis is the solution by analog computer.

Chapter One reviews the previous work in the area of error analysis and methods to reduce error by construction techniques within the operational amplifier. In this thesis the emphasis is placed on a correction technique for given models of the computing elements. In this technique one need not improve the operating characteristics of the computing elements.

Chapter Two deals with the models to be used for the active components in various sections of Chapter Three. The assumptions made for ideal computing elements are given. For the practical models, the assumptions are stated and for the given set of assumptions, the models are given.

Chapter Three includes the formulation technique which serves as the basis for the correction procedure. It is shown that for the class of driving functions which can be generated as the solution of linear first order differential equations with constant coefficients, the state model in the homogeneous form is general enough for the study of solutions on the analog computer.

A section dealing with a method of expanding the first-order linear differential equation with constant coefficients into a higher-order linear differential equation with constant coefficients is discussed. These higher-order forms of the state model are used to match coefficient matrices of the differential equations actually solved by the analog computer in order that a vector which has the same order as the state vector also has the solution of the given state model.

A separate development for the set up is given for each of the following three cases:

1) First-order models for the integrator and summing amplifier.

2) Second-order model for the integrator and first-order model for the summing amplifier.

3) One type of active computing element.

The type of computer considered in the third case has application in the area of synthesis of the state model using active components.

Chapter Four continues with the development of the previous chapter and in each of the three cases considered, an algorithm is generated which gives a method (but not the only means) to obtain the corrected matrices $\overline{A}_p$ and $\overline{A}_m$.

As an extension of the algorithm, it is proved that the time and amplitude scaled form of any state model may be solved exactly (in the sense that the model for the computing elements is assumed to be exact) on the analog computer.

Chapter Five covers several numerical examples. The actual solution on an analog computer is given for several of these examples. The active elements have been degraded quite drastically in terms of gain and bandwidth to be able to show the procedure works for extremely poor computing elements. A brief development is presented with the first example which examines a criterion of choosing the N_1 matrix in such a way as to reduce the effects of small errors in potentiometer settings.

In the development of the thesis several desirable features have been incorporated; these are listed in the following:

1) In order to apply the correction algorithm, the analytic solution of the state model is not required.

2) The correction procedure does not require any more active components than the uncorrected form. The only changes that are required are in the coefficient potentiometer settings.

3) The output of every active component in the computer set up is available. This means that all the modes, generated within the computer may be observed.

4) The solution on the analog computer of the state model has the same stability properties as the state model itself. In other words, from a theoretical standpoint, if the state model has a stable solution so will the analog computer. From a practical standpoint, errors in initial conditions and potentiometer settings may result in some of the unwanted modes appearing in the analog computer solution.

Further Suggested Work

This thesis presents a new approach for analog computer solutions of the time invariant state model. Although the problem is solved theoretically, further work is needed in several areas listed in the following:

1) Correction procedure for computer set ups using more complicated models for the active elements. Although the method used in the thesis would apply, the mathematics involved is time consuming.

2) The analog computer solution of nonlinear differential equations (including time-varying linear systems) is of major importance. The accurate modeling of the multiplier needed for the analog computer solution of nonlinear differential equations seems to be the biggest problem in this area.

3) Minimization of the number of summing amplifiers is of practical importance and since there is some control over $\overline{A}_m$, a correction procedure for $X_1(t) = X(t)$ and minimizing the entries in $\overline{A}_m$ may be one method to solve this problem.

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