OPTIMAL SAMPLED-DATA CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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ABSTRACT

OPTIMAL SAMPLED-DATA CONTROL OF
DISTRIBUTED PARAMETER SYSTEMS

By

Kwang Yun Lee

This thesis is concerned with the sampled-data control problem for distributed parameter systems with quadratic cost criteria, where the system operators are the infinitesimal generators of semigroups of operators.

An equivalent discrete-time problem is formulated in the variational framework. The existence and uniqueness of an optimal control is proved and a necessary condition for optimality is derived. The optimal control is given by a linear feedback law of sampled states. The feedback operator is shown to be the bounded, positive semi-definite and self-adjoint solution of a nonlinear operator difference equation of Riccati type. This operator is represented by an integral operator whose kernel satisfies an integro-difference equation. These results are shown to hold for the control problem on the infinite time interval with an additional assumption.

The results obtained above for general distributed controls are then specialized to the case of pointwise control. The optimal discrete-time pointwise control is given by a

Kwang Yun Lee

simplified linear feedback law which depends on the control point location. A finite dimensional eigenfunction approximation is obtained by a suitable choice of cost functional. The structure of feedback controls for this approximation is composed of an observer which is independent of control point location and a gain matrix which depends on control point location. These are illustrated by an example of the scalar heat equation.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

One of the many problems arising in optimal control is the problem of controlling distributed parameter systems. In general this problem is concerned with determining the system inputs which minimize some given performance criteria when the systems are described by partial differential equations. Most of the research in this problem has been oriented to the use of continuous, rather than discrete, controls in the time evolution. In many cases of practical interest, however, it is strongly desired to control distributed parameter systems by means of discrete-time controls. Furthermore, since the nature of continuous-time evolution is desirable and should be retained, sampled-data control, that is, sampling the continuous-time data, is considered to be the most practical scheme. Examples of such cases are the control of traffic flow in an urban freeway using digital speed metering and the control of temperature distribution in a rolling plate employing scanning-type thermocouple measurements. In both of these examples the controls are piecewise-constant controls, that is, controls which are constant for each time interval. Moreover, controls are determined on the basis of sampling continuous data. Other examples for which sampled-data control might be applied are systems in which measurement data for feedback control is not available continuously in time, systems having a long range of time evolution which implies that periodic sampling is inherent in the process, or systems employing a digital computer in the feedback loop as a part of the controller. Biological systems such as the growth of cereal leaf beetles, algal production in lakes, detritus processing in streams, or plankton distribution in nutrient pools require experimental measurements at discrete instants in time and thus typify processes of the first type. Population models for ecological systems of great diversity would require dynamic equations having a long range of time evolution.

The purpose of this dissertation is to formulate the sampled-data control problem for distributed parameter systems and to present a general approach to obtaining the solution of this problem. The system to be considered is described by an evolution equation in function space, where the system operator is an infinitesimal generator of a semigroup of bounded linear operators. The cost functional is quadratic in the deviation of the state distribution from a desired distribution and in the control energy.

Many approaches have been used to the solution of general continuous-time distributed parameter systems. Basically these might be classified into four types of approaches. The first one is the approach of Butkovskii's [B-8], [B-9]. The distributed parameter systems he considers are those described by nonlinear integral equations.

For these systems he has employed the calculus of variations to develop a maximum principle, that is, a necessary condition for the optimal control. Sakawa's [S-3], Yeh and Tou's [Y-2], and Yavin and Sivan's [Y-3] works fall into this category.

The second approach is that of Wang [W-1] and Wang and Tung [W-2]. They derive a maximum principle for distributed parameter systems described by partial differential equations by suing a dynamic programming procedure, which has been extended to function spaces by Bellman and Kalaba [B-11]. Using the same approach Kim and Erzberger [K-4] have obtained a set of functional equations analogous to the matrix Riccati equation for lumped systems. The works of Katz [K-5], Egorov [E-1], [E-2], Sirazetdinov [S-4] and Brogan [B-10] are in the spirit of this approach.

The third type is the well-known system theoretic concepts initiated by Balakrishnan [B-4], who developed a general theory of optimal control problems in Banach spaces using the theory of semigroups of linear operators and has applied some of the results to a class of control problems for distributed parameter systems. Specifically, he considers both the time-optimal and final value control problem. Fattorini [F-3], Axelband [A-3] and Freedman [F-2] have worked along these same lines. Russell [R-3], Lukes and Russell [L-4] and Datko [D-2] have considered the quadratic cost functional for the systems in Hilbert spaces and derived feedback control laws.

The last type of approach is the variational approach of Lions [L-1]. He characterizes the control problems of systems described by partial differential equations as variational problems, and generates a new maximum principle, giving necessary and sufficient conditions to the solution of the variational problems associated with distributed parameter systems. Greenberg [G-1] extended the results of Lions to the systems whose spatial differential operators are infinitesimal generators of semigroups of operators.

Most of the works that have been performed for a decade are concerned with continuous-time systems. Recently Matsumoto and Ito [M-1] formulated a discrete-time pointwise control problem from a second order parabolic system which has a green's function associated with it. They obtained a feedback control law using dynamic programming.

The lack of works in the discrete-time optimal control of distributed parameter systems has motivated this research. We shall consider general distributed parameter systems whose spatial differential operators are infinitesimal generators of semigroups of operators. We shall formulate the discrete-time control problem in the variational framework which was adopted by Lions for continuous-time problems, and will generate new results for the control problem associated with discrete-time distributed parameter systems. We shall also show that distributed systems driven by finite dimensional controls (the pointwise control problem) fall within the framework of this formulation and the results obtained for a general class of controls are specialized to the case of pointwise control in a straightforward manner.

The outline of the thesis is as follows. Chapter II is devoted to the mathematical background which provides the basic structure of distributed systems and their transformations into ordinary differential equations in infinite dimensional function spaces. The characterization of solutions to these equations is provided using the theory of semigroups of operators.

In Chapter III we formulate a sampled-data control problem, and thus an equivalent discrete-time control problem. The necessary condition for optimal control is derived, and the control is given by a feedback law in which the feedback operator is the solution of a Riccati operator difference equation. The remainder of the chapter contains a discussion of the behavior of optimal solutions when the terminal time approaches infinity and the integral representation of the feedback operators.

The results obtained above for general distributed controls are then specialized to the case of pointwise control in Chapter IV and a pointwise feedback control is obtained which is of simpler form than the distributed feedback control from a computational point of view. By a suitable choice of cost functional, the finite dimensional approximation by an eigenfunction expansion is obtained. The approximation is illustrated by an example of the scalar heat equation.

Chapter V contains a summary of the results in the thesis and recommendations for further research.

CHAPTER II

MATHEMATICAL BACKGROUND

This chapter is devoted to the mathematical foundation for the developments which will be presented in the sequel. Partial differential equations are formulated into ordinary differential equations in the function spaces in order to obtain the analogy of distributed parameter systems with the lumped parameter theory, that is, state and state space, matrix operators, state equations, transition matrices, and the variation of constants formulae. Section 2.1 is concerned with the concept of state in distributed parameter systems and the discussion of particular spaces of generalized functions which serve as state spaces. The spatial differential operators are defined on the state spaces in Section 2.2. Parabolic partial differential equations are converted into ordinary differential equations in function spaces in Section 2.3. In Section 2.4 the concept of a semigroup of operators is introduced. In Section 2.5 the characterization of solutions for the distributed parameter systems with the aid of semigroup theory is considered.

2.1 DISTRIBUTION AND THE SOBOLEV SPACES

In finite dimensional systems the state is a point in a finite dimensional Euclidean vector space. But the <u>state</u> of a distributed parameter system at each instant of time is a function defined on the given spatial region, or, in other words, the state is a point in an infinite dimensional (function) space. Since quadratic cost criteria are of interest, this space is chosen to be the Hilbert space of square integrable functions on the spatial region of definition. Again, as will be shown, this space is not quite suitable for distributed parameter systems, but certain subspaces, namely the Sobolev spaces, are suitable.

Let D be an open set in R^n with boundary ∂D. Throughout it is assumed that D is a bounded, open set with boundary ∂D which is a C^∞-manifold of dimension $(n-1)$. The symbol $z = (z_1, \cdots, z_n)$ denotes the spatial variable in D. Further let $C_0^\infty(D)$ be the space of infinitely differentiable functions of compact support on D, i.e., $\varphi \in C_0^\infty(D)$ vanishes on the outside of the compact support of D (cf. [Y-1],p.62). The space of bounded linear functionals on $C_0^\infty(D)$, i.e., the dual of $C_0^\infty(D)$, is called the space of <u>distributions</u>, or generalized functions, on D, and is denoted by $\mathcal{D}(D)$. A element $F \in \mathcal{D}(D)$ has the form

$$F(\varphi) = \int_D f(z)\ \varphi(z)dz \qquad \forall\ \varphi \in C_0^\infty(D)$$

where $f(\cdot)$ is some Lebesgue integrable function on D.

We present two of the properties on the space of distributions. First, the space of square integrable functions on D, $L^2(D)$, is a subset of $\mathcal{D}(D)$. This is easily seen by noting the fact that $C_0^\infty(D) \subset L^2(D)$ (i.e., any infinitely differentiable function with compact support on D is square integrable on D) and, therefore, the dual space of $L^2(D)$ must be contained in the dual space of $C_0^\infty(D)$, namely $\mathcal{D}(D)$. Since $L^2(D)$ is its own dual the following inclusion relation holds

$$(2.1.1) \qquad C_0^\infty(D) \subset L^2(D) \subset \mathcal{D}(D) .$$

The second property is the differentiation of distribution. If $F \in \mathcal{D}(D)$, the distributional derivative or the generalized derivative of F (with respect to z_i, $i = 1,2,\cdots, n$) is defined by (cf. [Y-1], p. 49)

$$(2.1.2) \qquad \frac{\partial}{\partial z_i} F(\varphi) = - F\left(\frac{\partial \varphi}{\partial z_i}\right) \quad \forall \varphi \in C_0^\infty(D) .$$

Remark 2.1.1: The above notion is an extension of the usual notion of the derivative. For, if the function f is continuously differentiable with respect to z_1, then we have

$$\frac{\partial}{\partial z_1} F_f(\varphi) = - F_f\left(\frac{\partial \varphi}{\partial z_1}\right) = - \int \cdots \int_D f(z) \frac{\partial \varphi}{\partial z_1} dz_1 \cdots dz_n$$

$$(2.1.3) \qquad = \int \cdots \int_D \frac{\partial}{\partial z_1} f(z) \cdot \varphi(z) dz_1 \cdots dz_n = F_{\partial f/\partial z_1}(\varphi) ,$$

as may be seen by integration by parts observing that $\varphi(z)$ vanishes identically outside a compact subset of D. Thus, in view of (2.1.1) we may define the differentiation (in the sense of distribution) for all elements of $L^2(D)$.

This generalized approach to differentiation can be extended to any order of differential operators:

Corollary 2.1.2 [Y-1,p.50]: A distribution $F \in \mathcal{D}(D)$ is infinitely differentiable in the sense of distributions defined above and

(2.1.4) $$D^q F(\varphi) = (-1)^{|q|} F(D^q\varphi) \quad \forall\ \varphi \in C_0^\infty(D) ,$$

where

(2.1.5) $$q = (q_1, q_2, \cdots, q_n), \quad |q| = \sum_{i=1}^{n} q_i ,$$

and

(2.1.6) $$D^q = D_1^{q_1} D_2^{q_2} \cdots D_n^{q_n} , \quad D_i = \frac{\partial}{\partial z_i} .$$

We now have the following definition.

<u>Definition 2.1.3</u>: The <u>Sobolev space</u> of order m, denoted by $H^m(D)$, is defined by

$$H^m(D) = \{F\colon F \in L^2(D),\ D^q F \in L^2(D) \quad \forall\ q,\ |q| \le m\} .$$

The space $H^m(D)$ can be shown to be complete in the topology induced by the inner product

(2.1.7) $$\langle F,G\rangle_{H^m(D)} = \sum_{|q|\le m} \langle D^q F ,\ D^q G\rangle_{L^2(D)} ,$$

and, hence, it is a Hilbert space (cf.[Y-1], p.55). Thus, with the completeness of $H^m(D)$, the Sobolev spaces can be considered as <u>the state spaces</u> of distributed parameter systems.

Next we describe the subspaces of $H^m(D)$ which incorporate with certain boundary conditions. For each $x \in H^m(D)$ we may associate the trace of x on ∂D as well as that of its normal derivative $\frac{\partial^k}{\partial n^k} x$, for $1 \le k \le m-1$, and in this way characterize the image of $H^m(D)$ by the map

$$(2.1.8) \qquad x \to x|_{\partial D}, \ \frac{\partial x}{\partial n}\Big|_{\partial D}, \ldots, \frac{\partial^{m-1} x}{\partial n^{m-1}}\Big|_{\partial D},$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on ∂D. This characterization requires Sobolev spaces of non-integral order and it is therefore essential to introduce such spaces.

The Sobolev spaces of non-integral order are defined by means of Fourier transformation. We first consider the space $H^m(D)$ with $D = R^n$. The Fourier transform of x, $\mathcal{J}x(\zeta)$ is defined by

$$(2.1.9) \qquad \mathcal{J}x(\zeta) = \int_{R^n} \exp(-2\pi j(\zeta \cdot z))x(z)\,dz,$$

where $(\zeta \cdot z) = \zeta_1 z_1 + \cdots + \zeta_n z_n$, the usual inner product on R^n. It can be shown that

$$(2.1.10) \quad \mathcal{J}D^q x(\zeta) = (2\pi j)^{|q|}\, \zeta^q\, \mathcal{J}x(\zeta) \quad \forall\ x \in L^2(R^n).$$

where $\zeta^q = \zeta_1^{q_1} \cdots \zeta_n^{q_n}$. This results in an alternative definition of the Sobolev space, namely

$$H^m(R^n) = \{x:\ \zeta^q \mathcal{J}x(\zeta) \in L^2(R^n) \quad \forall\ |q| \le m\}$$

or, equivalently,

$$(2.1.11) \quad H^m(R^n) = \{x:\ (1 + |\zeta|^2)^{m/2}\, \mathcal{J}x(\zeta) \in L^2(R^n)\}.$$

Note that there is no restriction in allowing m to be an integer or positive in (2.1.11), rather than requiring it to be any number in R. Thus, the Sobolev spaces can be defined for any number $m \in R$. In general $H^m(D)$ is defined by

$$(2.1.12) \quad H^m(D) = \{x:\ (1 + |\zeta|^2)^{m/2}\, \mathcal{J}x(\zeta) \in L^2(D)\},$$

which is a Hilbert space with inner product

$$(x,y)_{H^m(D)} = \langle (1+|\zeta|^2)^{m/2} \mathcal{J}x(\zeta), (1+|\zeta|^2)^{m/2} \mathcal{J}y(\zeta)\rangle_{L^2(D)} .$$

It can be shown that the dual space of $H^m(D)$ is $H^{-m}(D)$, i.e., $(H^m(D))' = H^{-m}(D)$ (cf.[Y-1,p.99, p.155], and [L-3]).

We are now in a position to state the trace theorem:

<u>Theorem 2.1.4</u> [Lions-Magenes, L-2]: For any $x \in H^m(D)$, we may define in a unique manner its traces

$$x\big|_{\partial D}, \frac{\partial x}{\partial n}\Big|_{\partial D}, \ldots, \frac{\partial^{m-1}x}{\partial n^{n-1}}\Big|_{\partial D} .$$

Moreover, we have

$$\frac{\partial^k x}{\partial n^k}\Big|_{\partial D} \in H^{m-k-\frac{1}{2}}(\partial D), \ 0 \le k \le m-1 ,$$

and the map $x \to \{\frac{\partial^k x}{\partial n^k}, \ 0 \le k \le m-1\}$ is a linear, continuous mapping of $H^m(D)$ onto $\prod_{k=0}^{m-1} H^{m-k-\frac{1}{2}}(\partial D)$.

Lions and Magenes [L-2] also showed that the kernel of the map (2.1.8) (i.e., the space of $x \in H^m(D)$ such that $\frac{\partial^k x}{\partial n^k}\Big|_{\partial D} = 0, \ 0 \le k \le m-1)$ is the closure of $C_0^\infty(D)$ in the norm of $H^m(D)$. We denote this subspace by $H_0^m(D)$. Thus

$$(2.1.13) \quad H_0^m(D) = \{x \in H^m(D): \frac{\partial^k x}{\partial n^k}\Big|_{\partial D} = 0, \ 0 \le k \le m-1\} .$$

Since $H_0^m(D)$ is a closed subspace of $H^m(D)$, and therefore a Hilbert space (with the inner product of $H^m(D)$), it may just as easily be considered as a candidate for a state space as $H^m(D)$. Here, the condition imposed on the boundary ∂D

in the definition of $H_0^m(D)$, i.e., $\left.\frac{\partial^k x}{\partial n^k}\right|_{\partial D} = 0,\ 0 \le k \le m-1$, is called the Dirichlet boundary condition.

2.2 DIFFERENTIAL OPERATORS

In this section we will discuss the properties of spatial differential operators, which play the role in distributed parameter systems which matrices play in lumped parameter systems. It has been shown in the preceding section that a differential operator of order m is everywhere defined and closed on $H^m(D)$. However, the operator is not bounded, which gives rise a major distinction to matrices in finite dimensional systems.

Let $a_q(z)$ be a real valued function, where q is the n-tuple defined by (2.1.5). Define the formal differential operator A of order m:

$$A = \sum_{|q| \le m} a_q(z)\, D^q , \tag{2.2.1}$$

where D^q is the differential operator defined in (2.1.6) and

$$\sum_{|q| \le m} = \sum_{\substack{q \\ |q|=0}} + \sum_{\substack{q \\ |q|=1}} + \cdots + \sum_{\substack{q \\ |q|=m}} .$$

Similarly we define the formal adjoint differential operator of A (cf.[C-1], p.59), denoted by A', as

$$A' = \sum_{|q| \le m} (-1)^{|q|}\, D^q\, a_q(z) . \tag{2.2.2}$$

In general, the formal adjoint operator A' is not equal to the adjoint operator A^*, where A^* is defined by

$$\langle x, Ay \rangle_{H^m(D)} = \langle A^* x, y \rangle_{H^m(D)} .$$

It can be shown, by means of Green's formula (cf.[C-1,p.63], and [Y-1,p.50]), that

$$\langle x, Ay \rangle_{H^m(D)} = \langle A' x, y \rangle + C ,$$

where the constant C depends on conditions at the boundary ∂D. In the case of Dirichlet boundary conditions, discussed in Section 2.1, $C = 0$ and thus $A' = A^*$.

If $a_q(z) \in L^\infty(D)$, i.e., if it is essentially bounded (cf.[R-1], p.112), then the m^{th} order differential operator (2.2.1) is said to be <u>elliptic</u> (cf.[D-1], p.1704) if

$$\sum_{|q|=m} a_q(z) \zeta^q \neq 0 \quad \forall\ \zeta \in R^n,\ z \in D,\ \zeta \neq 0 .$$

Note that this is a condition on the highest order term, i.e., the terms containing partial derivatives of order m. If we restrict our attention to elliptic differential operators which contain only even order partial derivatives, we define the concept of coercivity in the following manner:

<u>Definition 2.2.1</u>: If A is an elliptic differential operator of the form

$$A = \sum_{|q| \le 2p} a_q(z)\ D^q$$

where $a_q(z) = 0$ if $|q| \neq 2k$, for $k = 0,1,\cdots, p$, then A is said to be <u>coercive</u> if

$$(2.2.3) \qquad (-1)^k \sum_{|q|=2k} a_q(z)\zeta^q \leq -a \sum_{|q|=2k} \zeta^q$$

for some $a > 0$, for $k = 0,1,\cdots, p$, and for all $\zeta \in R^n$ and $z \in D$.

This concept of coercivity describes the property of operators more commonly referred to as negative definiteness, namely the condition

$$\langle Ax,x\rangle_{H^m(D)} \leq -a \|x\|^2_{H^m(D)}$$

for some $a > 0$ and for all $x \in H^m(D)$. It might be noted that just as negative definiteness of a matrix implies that the eigenvalues of the matrix lie on the negative real axis, the spectrum of a coercive operator is a subset of the left half-plane. We also define the strong ellipticity which has milder condition.

Definition 2.2.2: If A is an elliptic differential equation of even order $2p$, then A is said to be strongly elliptic if

$$(2.2.4) \qquad (-1)^p \sum_{|q|=2p} a_q(z)\zeta^q \leq -a \sum_{|q|=2p} \zeta^q$$

for some $a > 0$, and for all $\zeta \in R^n$ and $z \in D$. Note that (2.2.4) is a special case of (2.2.3), which applies only for the highest order terms. Thus, coercivity implies strong ellipticity, but the converse is not true.

A more general operator than those considered above which will be introduced later is the differential operator which plays the role of an infinitesimal generator of a semigroup of operators. This will be studied in Section 2.4 and 2.5.

2.3 STATE EQUATIONS

Utilizing the concepts of state, state space, and system differential operators discussed in the preceding sections, we describe the state equations in the form of partial differential equations with an additional time variable.

Let $x(t)$ be a function defined on $t \in [0,T]$ with values in the Sobolev space $H^m(D)$, i.e., $x(t) \in H^m(D) \quad \forall\ t \in [0,T]$. For each $t \in [0,T]$, $x(t)$ may be considered as a point in the function space $H^m(D)$. We define the space $L^2(0,T;\ H^m(D))$:

Definition 2.3.1: The space of square integrable Sobolev space-valued functions is

$$L^2(0,T;H^m(D)) = \{x : x(t) \in H^m(D) \quad \forall\ t \in [0,T],\ \int_0^T \|x(t)\|^2_{H^m(D)}\, dt < \infty\}.$$

Note that this is a Hilbert space with inner product

$$\langle x,y \rangle_{L^2(0,T;H^m(D))} = \int_0^T \langle x(t),y(t) \rangle_{H^m(D)}\, dt\ .$$

In order to describe the dynamics (i.e., evolution in time) of distributed parameter systems, we may again introduce the notion of distributions on $[0,T]$ as we did on D in Section 1. If we consider the space of infinitely differentiable Sobolev space-valued functions with compact support in

$[0,T]$ and its corresponding dual space of distributions, which may be denoted by $\mathcal{D}[0,T]$, then $L^2(0,T;H^m(D)) \subset \mathcal{D}[0,T]$ and the following Sobolev space of Sobolev space-valued functions may be defined (cf.[L-1], p. 102).

Definition 2.3.2: The Sobolev space of Sobolev space-valued functions on $[0,T]$, denoted by $W(0,T)$, is defined by

$$W(0,T) = \{x\colon\ x \in L^2(0,T;H^m(D)),\ \frac{d}{dt}\,x \in L^2(0,T;H^m(D))\}\ .$$

This is a Hilbert space with inner product

$$\langle x,y\rangle_{W(0,T)} = \langle x,y\rangle_{L^2(0,T;H^m(D))} + \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle_{L^2(0,T;H^m(D))}\ .$$

We are now in a position to describe partial differential equations by ordinary differential equations in the space of Sobolev space-valued functions. The parabolic equations are of the form:

$$\frac{\partial x(t,z)}{\partial t} = A\ x(t,z) + f(t,z) \tag{2.3.1}$$

where A is an elliptic partial differential operator in the spatial variable z. If $x(t,z)$, $t \in [0,T]$, $z \in D$ is assumed to be the element $x(t) \in W(0,T)$, then (2.3.1) has the equivalent formulation as the ordinary differential equation in $L^2(0,T;\ H^m(D))$

$$\frac{d}{dt}\,x(t) = A\ x(t) + f(t) \tag{2.3.2}$$

where $f \in L^2(0,T;\ L^2(D))$. If the initial condition is given by $x(0,z) = x_0(z) \in H^m(D)$, then we write the initial condition of (2.3.2) as $x(0) = x_0$.

2.4 SEMIGROUPS

In this section we shall introduce the notion of semigroups which play the similar role of the transition matrices in finite dimensional systems. Let $\mathfrak{X}$ be a Banach space and let $\mathcal{B}(\mathfrak{X})$ be the Banach algebra of endomorphisms of $\mathfrak{X}$, i.e., the space of bounded linear transformations on $\mathfrak{X}$ to itself (cf.[H-1], p.51).

<u>Definition 2.4.1</u> ([B-1], p.7): If a mapping $\Phi(t): [0,\infty) \to \mathcal{B}(\mathfrak{X})$ satisfies the following conditions:

(2.4.1) (i) $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2), \quad t_1, t_2 \geq 0,$

(2.4.2) (ii) $\Phi(0) = I, \quad I = \text{identity operator},$

then $\{\Phi(t), t \geq 0\}$ is called a one-parameter semigroup of operators in $\mathcal{B}(\mathfrak{X})$. The semigroup $\{\Phi(t), t \geq 0\}$ is said to be of class (C_0) if it satisfies the further property

(2.4.3) (iii) $\operatorname*{s-lim}_{t \to 0+} \Phi(t)x = x, \quad x \in \mathfrak{X},$

refered to as the strong continuity of $\Phi(t)$ at the origin.

In the sequel we shall generally assume, unless otherwise stated, that the family of bounded linear operators $\{\Phi(t), t \geq 0\}$ mapping $\mathfrak{X}$ into itself is a semigroup of class (C_0), thus that all three conditions of the above definition are satisfied. We further state some of the properties of the semigroup in the following:

Lemma 2.4.2 ([B-1], Proposition 1.1.2): (a) $\|\Phi(t)\|$ is bounded on every finite subinterval of $[0,\infty)$.

(b) For each $x \in \mathfrak{X}$, the vector-valued function $\Phi(t)x$ on $[0,\infty)$ is strongly continuous.

(c) One has

$$(2.4.4) \qquad \omega_0 \equiv \inf_{t>0} \frac{1}{t} \log \|\Phi(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t)\| < \infty .$$

(d) For each $\omega > \omega_0$, there exists a constant M_ω such that for all $t \geq 0$

$$(2.4.5) \qquad \|\Phi(t)\| \leq M_\omega e^{\omega t} .$$

In part (b) of the lemma we have seen that the operator function $\Phi(t)$ is continuous on $[0,\infty)$ in the strong operator topology, i.e., $\lim_{t\to t_0} \|\Phi(t)x - \Phi(t_0)x\| = 0$ for any $t_0 \geq 0$ and for all $x \in \mathfrak{X}$ (cf.[B-1],p.290). Thus the family $\{\Phi(t),\ t \geq 0\}$ is often called a strongly continuous semigroup in $\mathcal{B}(\mathfrak{X})$. If, in addition, the map $t \to \Phi(t)$ is continuous on $[0,\infty)$ in the uniform operator topology, i.e., $\lim_{t\to t_0} \|\Phi(t) - \Phi(t_0)\| = 0$ for any $t_0 \geq 0$ (cf.[B-1], p.290), where the norm is the usual induced operator norm on $\mathfrak{X}$ (cf.[T-1], p.86), then $\{\Phi(t),\ t \geq 0\}$ is said to be a uniformly continuous semigroup in $\mathcal{B}(\mathfrak{X})$. In case the norms of the semigroup operators are bounded uniformly with respect to t, i.e., $\|\Phi(t)\| \leq M$ (M a constant larger than or equal to one) for all $t \geq 0$, then $\{\Phi(t),\ t \geq 0\}$ is called an equi-bounded semigroup of class (C_0) in $\mathcal{B}(\mathfrak{X})$, and if the constant M is equal to or less than one a contraction semigroup of class (C_0) in $\mathcal{B}(\mathfrak{X})$.

Definition 2.4.3 ([B-1], p.9): The infinitesimal generator A of the semigroup $\{\Phi(t),\ t \geq 0\}$ is defined by

$$(2.4.6) \qquad A\,x \equiv \operatorname*{s-lim}_{\eta\to 0+} A_\eta\, x\,, \qquad A_\eta = \frac{1}{\eta}\,[\,\Phi(\eta) - I\,]$$

whenever the limit exists; the domain of A, in symbol $Do(A)$, being the set of elements x for which this limit exists.

Lemma 2.4.4 ([B-1], Proposition 1.1.4): (a) $Do(A)$ is a linear manifold in $\mathfrak{X}$ and A is a linear operator.

(b) If $x \in Do(A)$, then $\Phi(t)\,x \in Do(A)$ for each $t \geq 0$ and

$$(2.4.7) \qquad \frac{d}{dt}\,\Phi(t)\,x = A\,\Phi(t)x = \Phi(t)A\,x\,, \qquad t \geq 0\,;$$

furthermore,

$$(2.4.8) \qquad \Phi(t)x - x = \int_0^t \Phi(\tau)A\,x\,d\tau\,, \qquad t \geq 0\,.$$

(c) $Do(A)$ is dense in $\mathfrak{X}$, i.e., $\overline{Do(A)} = \mathfrak{X}$, and A is a closed operator.

Remark 2.4.5: (a) [B-1, p.13] If B is any operator in $\mathcal{B}(\mathfrak{X})$, then the operator function

$$(2.4.9) \qquad \Phi(t) = \exp(tB) \equiv I + \sum_{k=1}^{\infty} \frac{(tB)^k}{k!}\,, \qquad 0 \leq t < \infty$$

defines a semigroup in $\mathcal{B}(\mathfrak{X})$. Indeed, by the definition and the properties of the ordinary exponential function one obtains that $\|\Phi(t)\| \leq \exp(t\ \|B\|)$ for all $t \geq 0$ and that $\{\Phi(t),\ t \geq 0\}$ satisfies the semigroup properties (2.4.1) and (2.4.2). Moreover, it can be shown that $\Phi(t)$ is continuous, even continuously differentiable in $[0,\infty)$ in the uniform operator topology. Thus $\{\Phi(t),\ t \geq 0\}$ forms a uniformly continuous semigroup in $\mathcal{B}(\mathfrak{X})$, its infinitesimal generator is given by B and

$$\frac{d}{dt}\exp(tB) = B\exp(tB) = \exp(tB)\ B\ , \quad 0 \leq t < \infty .$$

(b) [H-1, Theorem 9.4.2] On the other hand, it may be shown that every uniformly continuous semigroup in $\mathcal{B}(\mathfrak{X})$ is of the form (2.4.9).

As noted in the above remark every uniformly continuous semigroup has a bounded infinitesimal generator (cf.[D-1], p.621), and therefore has an exponential representation (2.4.9). But in the case the infinitesimal generator A of a strongly continuous semigroup is in general not bounded, i.e., is not defined over the whole space, so that the exponential expression (2.4.9) involving A would be meaningless. To avoid this difficulty it might be valuable to use a limiting argument employing A_η which is a bounded operator as defined in (2.4.6). In fact it has been shown (cf.[B-1, p.19], [R-2,p.401]) that a limiting exponential solution does exist, which is referred to as Hille's first exponential formula:

$$(2.4.10) \qquad \Phi(t)x = \lim_{\eta\to 0}\exp(t\ A_\eta)x \qquad \forall\ x \in \mathfrak{X}\ ,\ t \geq 0\ ,$$

where the convergence is uniform with respect to t in any finite interval $[0,b]$. So every strongly convergent semigroup has the representation (2.4.10).

An alternate representation for the semigroup is in terms of the resolvent $R(\lambda;A)$ of the infinitesimal generator A. The resolvent is the operator $R(\lambda;A) = (\lambda I-A)^{-1}$ defined for all values of λ for which the inverse exists (cf.[H-1], p.123). It can be shown (cf.[H-1], p.342) that the resolvent operator is the Laplace transform of the semigroup:

$$R(\lambda;A)x = \int_0^\infty e^{-\lambda t}\,\Phi(t)x\,dt\,, \quad x \in \mathfrak{X}$$

for all $\mathrm{Re}(\lambda) > \omega_0$, where ω_0 is defined in (2.4.4), moreover, we have the Laplace inversion formula

$$\Phi(t)x = \operatorname*{s-lim}_{\gamma\to\infty} \frac{1}{2\pi j}\int_{\omega-j\gamma}^{\omega+j\gamma} e^{\lambda t}\,R(\lambda;A)x\,d\lambda$$

for each $x \in Do(A)$ and $t > 0$ with $\omega > \max(0,\omega_0)$.

One of the important questions is under what conditions will a closed linear operator A be the infinitesimal generator of a semigroup of class (C_0). The Hille-Yosida theorem ([H-1], p.364) tells us that a necessary and sufficient condition for a closed linear operator A to generate a semigroup $\{\Phi(t), t \geq 0\}$ of class (C_0) is that there exist real numbers M and ω such that for every real $\lambda > \omega$, and

(2.4.11) $$\|R(\lambda;A)^n\| \leq M\,(\lambda-\omega)^{-n}, \quad n = 1,2,\cdots .$$

We now can determine whether the spatial differential operators of Section 2 are infinitesimal generators of semigroups. Dunford and Schwartz [D-1, p.1767] showed that the necessary condition for the elliptic partial differential operator A defined in (2.2.1) to be an infinitesimal generator is

$$(2.4.12) \qquad (-1)^{m/2} \sum_{|q|=m} a_q(z)\,\zeta^q \leq 0\,, \quad z \in D\,, \quad \zeta \in R^n\,.$$

Note that the condition for strong ellipticity (2.2.4) clearly satisfies the necessary condition (2.4.12). In this respect a series of extensive works has been devoted to the strongly elliptic partial differential operators (cf.[A-1], [A-2], [B-2] and [B-3]). An important result is that if $\{\Phi(t), t \geq 0\}$ is the semigroup of operators generated by a strongly elliptic operator, than the bounded operator $\Phi(t)$ has the exponential bound

$$(2.4.13) \qquad \|\Phi(t)\| \leq M\ e^{-\lambda t}$$

where M and λ are positive constants (cf.[F-1], p.72 and p.158).

It remains to characterize the solutions of partial differential equations with the aid of semigroup theory.

2.5 REPRESENTATION OF SOLUTIONS FOR DISTRIBUTED PARAMETER SYSTEMS

In this section we will characterize solutions of partial differential equations. First let us consider the homogeneous equation of (2.3.2)

$$\dot{x}(t) = A\, x(t). \tag{2.5.1}$$

Then as we noticed in Lemma 2.4.4 we may invoke the theory of semigroups of operators: Indeed, Phillips [P-1] has shown that a necessary and sufficient condition for (2.5.1) to have a unique solution in $[0,\infty)$ for each initial value $x(0) \in Do(A)$ such that

$$\underset{t\to 0^+}{\text{s-lim}}\; x(t) = x(0) \tag{2.5.2}$$

is that A be the infinitesimal generator of a strongly continuous semigroup $\{\Phi(t), t \geq 0\}$ of class (C_0) (cf. Definition 2.4.1). The solution itself is given by

$$x(t) = \Phi(t)\; x(0)\;, \tag{2.5.3}$$

where of course, from Definition 2.4.1 and Lemma 2.4.4,

$$\Phi(t_1+t_2) = \Phi(t_1)\; \Phi(t_2), \qquad t_1,\ t_2 \geq 0$$
$$\Phi(0) = I$$

and

$$\frac{d}{dt}\, \Phi(t)x = A\Phi(t)x = \Phi(t)A\,x\,, \quad x \in Do(A).$$

Thus the operator $\Phi(t)$ is the obvious analog of the transition matrix in finite dimensional systems. This result can be extended to characterize solutions of forced equations in a form analogous to the variation of constants formula in finite dimensional systems.

We introduce the concept of measurability and Bochner integrability for our own purpose.

<u>Definition 2.5.1</u> ([H-1], p.72): A function on $[0,\infty)$ to $\mathfrak{X}$ is strongly measurable if there exists a sequence of countably-valued functions converging almost everywhere in $[0,\infty)$ to $f(t)$.

<u>Definition 2.5.2</u> ([H-1], p.78): (a) A countably-valued function $f(t)$ on $[0,\infty)$ to $\mathfrak{X}$ is integrable (Bochner) iff $\|f(t)\|$ is integrable (Lebesgue). We define

$$(B) \int_I f(t)\, d\mu = \sum_{k=1}^{\infty} f_k\, \mu(I_k \cap I)$$

where $f(t) = f_k$ on I_k, $I \subset [0,\infty)$ and μ is a Lebesgue measure.

(b) A function $f(t)$ on $[0,\infty)$ to $\mathfrak{X}$ is integrable (Bochner) if there exists a sequence of countably-valued function $\{f_n(t)\}$ converging almost everywhere to $f(t)$ and such that

$$\lim_{n\to\infty} \int_{[0,\infty)} \|f(t) - f_n(t)\|\, d\mu = 0 ,$$

and we define

$$(B) \int_I f(t)\, d\mu = \lim_{n\to\infty} (B) \int_I f_n(t)\, d\mu .$$

Now we are in a position to discuss the solution of a forced system

$$\frac{d}{dt} x(t) = A\, x(t) + f(t), \quad x(0) = x_0 . \tag{2.5.4}$$

The construction of a solution for this system can be found elsewhere (cf.[T-2], [K-1], [Y-1], [P-2], [F-1], and [B-4]). We state here one of the results due to Balakrishnan.

<u>Theorem 2.5.3</u> (Balakrishnan, 1965): Let A be the infinitesimal generator of a strongly continuous semigroup $\{\Phi(t), t \geq 0\}$. Let $f(t)$ be strongly measurable and Bochner integrable in every finite interval in $[0,\infty)$. Further let $f(t) \in Do(A)$ for almost every t. Then, (2.5.4) has a unique solution given by

$$x(t) = \Phi(t)\ x(0) + \int_0^t \Phi(t-\tau)\ f(\tau)\ d\tau \,. \tag{2.5.5}$$

CHAPTER III

OPTIMAL SAMPLED-DATA CONTROL

The purpose of this chapter is to formulate the sampled-data control problem for a distributed parameter system and to solve the equivalent discrete-time problem (DTP). The continuous-time system constrained by piecewise constant controls is transformed into DTP, and the DTP is treated in the framework of a variational problem, i.e., that of characterizing extremals to a given functional, constrained by an infinite dimensional difference equation.

The sampled-data problem and it's equivalent DTP are formulated in Sections 3.1 and 3.2. In Section 3.3 the existence and uniqueness of solutions for DTP is proved and the necessary condition for optimality is derived. In Section 3.4 the optimal control is given by a feedback operator which satisfies an operator difference equation of Riccati type. The control on the infinite time interval is investigated in Section 3.5. In Section 3.6 it is shown that the Riccati operator equation is equivalent to an integro-difference equation.

3.1 CONTROL PROBLEM

In this section we will discuss the space of controls, and then define the quadratic cost criteria for general distributed parameter systems. In Chapter II we have seen that the Sobolev spaces and the elliptic partial differential operators are specific examples of an abstract Banach space $\mathfrak{X}$ and the infinitesimal generator A of strongly continuous semigroup on $\mathfrak{X}$, respectively. Since we already have an expression for the solution of evolution equation in an abstract space by (2.5.5), we may begin with the general distributed parameter systems rather than the specific cases.

Let H and U be Hilbert spaces, and $x(t) \in H$ and $u(t) \in U$ be the state and control of the system at time $t \in [0,T]$, respectively. We denote $\mathcal{L}(X;Y)$ to be the space of bounded linear transformations from X into Y, and let $B(t) \in \mathcal{L}(U;H)$ for all $t \in [0,T]$. We assume that A is a closed linear operator defined on a dense domain $Do(A) \subseteq H$ and generates a strongly continuous semigroup $\Phi(t)$ for $t \geq 0$.

We now consider a control system

$$(3.1.1) \quad \frac{d}{dt} x(t) = A\, x(t) + B(t)\, u(t), \quad x(0) = x_0 \in H .$$

We further assume that $B(t)\, u(t)$ is strongly measurable and integrable in the sense of Bochner (cf. Section 2.5). Then by Theorem 2.5.3 the existence and the uniqueness of solution to (3.1.1) is guaranteed and its solution is given by

$$(3.1.2) \quad x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)B(\tau)u(\tau)\, d\tau .$$

Next, the quadratic cost criteria weighting the state and the control will be introduced. The notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ will be used for the inner product and the norm on H, respectively (or on U, which can be distinguished in the context).

<u>Definition 3.1.1</u>: Let $R \in \mathcal{L}(H;H)$. (a) [P-3,p.203] The adjoint operator R^* of R is defined by

$$< Rx,y > = < x,R^*y > \quad \forall\, x \in H .$$

(b) R is called self-adjoint if $R = R^*$.

(c) R is called positive definite if $\langle x,Rx\rangle \geq a\|x\|$ for some $a > 0$, $\forall\, x \in H$.

(d) R is called positive semi-definite if $\langle x,Rx\rangle \geq 0$, $\forall\, x \in H$.

If we denote the desired state distribution as $x_d(t) \in H$, $t \in [0,T]$, we may state the cost criterion for the system (3.1.1) as:

$$(3.1.3)\quad J = \int_0^T [\langle x(t)-x_d(t),Q(t)(x(t)-x_d(t))\rangle+\langle u(t),R(t)ut)\rangle]dt$$
$$+ \langle x(T) - x_d(T),\ F(x(T) - x_d(T))\rangle ,$$

where, for each $t \in [0,T]$, $Q(t)$, $F \in \mathcal{L}(H,H)$ are bounded self-adjoint positive semi-definite operators and $R(t) \in \mathcal{L}(U,U)$ is a bounded self-adjoint positive definite operator.

The minimization of cost functional (3.1.3) over all control $u(t) \in U$ has appeared elsewhere (cf.[L-1], [L-4], [D-2], [G-1], [G-2], and [F-2]). In many cases of practical interest, however, it is actually desired to control distributed system by means of discrete-time controls. Thus sampled-data control is desired and is formulated in the next section.

3.2 SAMPLED-DATA FORMULATION

The term sampled-data is used to describe systems in which the sampling operation occurs between the plant and the controller, such as systems that have a telemeter link in the feedback loop or use a single instrument to monitor several variables in a sequential manner. An indirect way of introducing sampling is to define an index set $\sigma = \{0,1,2,\cdots, N\}$ and the corresponding time set on σ, i.e., $\{t_i\} = \{t_i = i\delta : i \in \sigma\}$, where δ is a sampling period. If the terminal time is finite, then $t_N = T = N\delta$, where it has been assumed that T is an integral multiple of the sampling period. This assumption is not essential, but is made for convenience (cf. [L-5]). The control discretization on σ requires that the inputs be piecewise-constant functions of time and that changes of values of $u(t)$ occur only at the sampling instants t_i, that is

$$u(t) = u(t_i) \equiv u_i \quad \text{for} \quad t \in [t_i, t_{i+1}). \tag{3.2.1}$$

Note that $u(t)$ is strongly measurable and integrable in the sense of Bochner (cf. Section 2.5), hence by Theorem 2.5.3 we have a unique solution in the form of (3.1.2). Now we define a basic sampled-data control problem (BP):

<u>Basic Problem (BP)</u>: Given a system (3.1.1) with a cost functional J by (3.1.3), find a sequence of controls $u^* = \{u_i^* \in U, i \in \sigma\}$ such that for all $u = \{u_i \in U, i \in \sigma\}$

$$J(u^*) = \inf_u J(u).$$

It should be noted that the purpose of using the cost functional in continuous form (3.1.3) is to penalize the system for error or excessive control inputs continuously in time rather than at the sampling instants, thus to achieve a better performance (cf.[L-6]).

This BP cannot be solved directly because the admissible controls are constrained to be piecewise-constant. As was done in the finite-dimensional case (cf.[L-5]), we transform this problem from a constrained one to an unconstrained one by integrating the differential equation and the cost functional and thus going from a continuous-time problem to a discrete-time one. The transformation is accomplished through the use of solution (3.1.2) evaluated for $t \in [t_i, t_{i+1})$. We therefore obtain

$$x(t) = \Phi(t-t_i)\, x(t_i) + D(t,t_i)\, u_i \,, \tag{3.2.2}$$

where for each $t \in [t_i, t_{i+1})$ $D(t,t_i) \in \mathcal{L}(U;H)$ such that

$$D(t,t_i)\, u_i = \int_{t_i}^{t} \Phi(t-\tau)\, B(\tau)\, u_i \, d\tau \,. \tag{3.2.3}$$

Letting $t = t_{i+1}$, we obtain a state difference equation

$$x_{i+1} = \Phi\, x_i + D_i\, u_i \,, \quad x_0 \in H \tag{3.2.4}$$

where $x_i = x(t_i)$, $\Phi = \Phi(\delta) = \Phi(t_{i+1}-t_i)$, and $D_i = D(t_{i+1}, t_i)$. If $u(t)$ is constant over the sampling period, using (3.2.2) we obtain the following expression for the cost criterion (3.1.3):

$$(3.2.5) \quad J = \sum_{i=0}^{N-1} [\langle x_i, Q_i x_i\rangle + 2\langle x_i, M_i u_i\rangle + \langle u_i, R_i u_i\rangle$$
$$- 2\langle x_i, S_i x_{d_i}\rangle - 2\langle u_i, P_i x_{d_i}\rangle + \langle x_{d_i}, E_i x_{d_i}\rangle]$$
$$+ \langle x_N, F\, x_N\rangle - 2\langle x_N, F\, x_{d_N}\rangle + \langle x_{d_N}, F\, x_{d_N}\rangle ,$$

where for each $i \in \sigma$, $Q_i, S_i, E_i \in \mathcal{L}(H;H)$, $R_i \in \mathcal{L}(U;U)$, $M_i \in \mathcal{L}(U;H)$, and $P_i \in \mathcal{L}(H;U)$ such that

$$(3.2.6) \qquad Q_i x_i = \int_{t_i}^{t_{i+1}} \Phi^*(t-t_i) Q(t)\, \Phi(t-t_i)\, x_i \, dt ,$$

$$(3.2.7) \qquad M_i u_i = \int_{t_i}^{t_{i+1}} \Phi^*(t-t_i) Q(t) D(t,t_i)\, u_i \, dt ,$$

$$(3.2.8) \qquad R_i u_i = \int_{t_i}^{t_{i+1}} [R(t) + D^*(t,t_i) Q(t) D(t,t_i)] u_i \, dt ,$$

$$(3.2.9) \qquad S_i x_{d_i} = \int_{t_i}^{t_{i+1}} \Phi^*(t-t_i) Q(t)\, x_{d_i} \, dt ,$$

$$(3.2.10) \qquad P_i x_{d_i} = \int_{t_i}^{t_{i+1}} D^*(t,t_i) Q(t)\, x_{d_i} \, dt ,$$

$$(3.2.11) \qquad E_i x_{d_i} = \int_{t_i}^{t_{i+1}} Q(t)\, x_{d_i} \, dt .$$

Note that for each $i \in \sigma$ Q_i and E_i are self-adjoint positive semi-definite operators, and R_i is a self-adjoint positive definite operator. Thus the continuous-time system (3.1.1), with the cost functional (3.1.3) and the control constraint (3.2.1), has been transformed into the discrete-time system (3.2.4) with the cost functional (3.2.5).

We therefore define a discrete-time problem (DTP) which is equivalent to BP.

Discrete-Time Problem (DTP): Given a system (3.2.4) with a cost functional J by (3.2.5), find a sequence of controls $u^* = \{u_i^* \in U, i \in \sigma\}$ such that for all $u = \{u_i \in U, \ i \in \sigma\}$

$$J(u^*) = \inf_u J(u) .$$

Throughout this dissertation we will be concerned with solving DTP.

3.3 DISCRETE-TIME PROBLEM (DTP)

In order to solve DTP we will introduce the function spaces on which DTP can be handled easily. We denote x and u to be sequences of states and controls on σ, respectively, such that $x = \{x_0, x_1, \cdots, x_{N-1}\}$ and $u = \{u_0, u_1, \cdots, u_{N-1}\}$ with $x_i \in H$ and $u_i \in U$. Let $\ell^2(0,N;H)$ be the family of all functions x on σ with values in H.

Remark 3.3.1 [D-1,p.257]: $\ell^2(0,N;H)$ is a Hilbert space with usual addition and scalar multiplication, and with an inner product defined by, for $x, y \in \ell^2(0,N;H)$,

$$(3.3.1) \qquad \langle x,y \rangle_{\ell^2(0,N;H)} = \sum_{i=0}^{N-1} \langle x_i, y_i \rangle_H .$$

Similarly, we may define a Hilbert space $\ell^2(0,N;U)$ with inner product analogous to (3.3.1).

To prove the existence and the uniqueness of solution for DTP we require that the solution of difference equation

(3.2.4) depends continuously on the control. Hence we prove the following lemmas.

<u>Lemma 3.3.2</u>: The mapping $u \to x$ of $\ell^2(0,N;U)$ into $\ell^2(0,N;H)$ defined by the difference equation (3.2.4) is continuous.

<u>Proof</u>: The solution x of (3.2.4) can be expressed as

$$x_i = \Phi(i)x_0 + \sum_{k=0}^{i-1} \Phi(i-1-k)\, D_k\, u_k, \quad x_0 \in H, \tag{3.3.2}$$

where $\Phi(i) \equiv \Phi(i\delta)$. Let $x^1, x^2 \in \ell^2(0,N;H)$ be the solutions (3.3.2) corresponding to controls $u^1, u^2 \in \ell^2(0,N;U)$, respectively. Then we deduce that

$$\begin{aligned}
\|x^1 - x^2\|^2_{\ell^2(0,N;H)} &= \sum_{i=0}^{N-1} \|x_i^1 - x_i^2\|_H^2 \\
&= \sum_{i=0}^{N-1} \Big\| \sum_{k=0}^{i-1} \Phi(i-1-k) D_k (u_k^1 - u_k^2) \Big\|^2 \\
&\leq \sum_{i=0}^{N-1} \left[\sum_{k=0}^{i-1} \|\Phi(i-1-k) D_k (u_k^1 - u_k^2)\| \right]^2 .
\end{aligned}$$

Since $\Phi(i)$ and D_i are bounded we have

$$\|\Phi(i-1-k) D_k (u_k^1 - u_k^2)\| \leq \|\Phi(i-1-k) D_k\| \; \|u_k^1 - u_k^2\| .$$

Note that $\|\Phi(i)\|$ is bounded by Lemma 2.4.2, and $\|D_k\|$ is bounded by the uniform boundedness theorem (cf.[Y-1], p.69), i.e., $\|\Phi(i)\| \leq M$ and $\|D_k\| \leq d$. Thus we have

$$\sum_{i=0}^{i-1} \|\Phi(i-1-k) D_k\|^2 \leq i\, M^2 d^2 < \infty$$

so that by Schwartz' inequality

$$\left[\sum_{k=0}^{i-1} \|\Phi(i-1-k) D_k (u_k^1-u_k^2)\|\right]^2$$

(3.3.5) $$\leq \sum_{i=0}^{i-1} \|\Phi(i-1-k) D_k\|^2 \sum_{i=0}^{i-1} \|u_k^1-u_k^2\|^2$$

$$\leq i\ M^2 d^2\ \|u^1-u^2\|^2_{\ell^2(0,N;U)} .$$

Substituting (3.3.5) into (3.3.4), we have

$$\|x^1-x^2\|^2_{\ell^2(0,N;H)} \leq \sum_{i=0}^{N-1} i\ M^2 d^2 \|u^1-u^2\|^2_{\ell^2(0,N;U)}$$

$$\leq N^2 M^2 d^2\ \|u^1-u^2\|^2_{\ell^2(0,N;U)} ,$$

or equivalently,

$$\|x^1-x^2\|_{\ell^2(0,N;H)} \leq C\ \|u^1-u^2\|_{\ell^2(0,N;U)}$$

which implies that the mapping $u \to x$ of $\ell^2(0,N;U)$ into $\ell^2(0,N;H)$ is continuous.

Q.E.D.

<u>Lemma 3.3.3</u>: The mapping $u \to x_N$ of $\ell^2(0,N;U)$ into H defined by the difference equation (3.2.4) is continuous.

<u>Proof</u>: Let $x_N^1, x_N^2 \in H$ be the terminal state due to controls $u^1, u^2 \in \ell^2(0,N;U)$, respectively. Then by (3.3.2)

$$\|x_N^1-x_N^2\|_H = \|\sum_{k=0}^{N-1} \Phi(N-1-k) D_k (u_k^1-u_k^2)\|^2$$

$$\leq \left[\sum_{k=0}^{N-1} \|\Phi(N-1-k) D_k (u_k^1-u_k^2)\|\right]^2$$

$$\leq \left[\sum_{k=0}^{N-1} \|\Phi(N-1-k) D_k\|\ \|u_k^1-u_k^2\|\right]^2 .$$

Using the same argument as in the proof of Lemma 3.3.2 we have

$$\|x_N^1 - x_N^2\|_H \leq N^{\frac{1}{2}} \, Md \, \|u^1 - u^2\|_{\ell^2(0,N;U)} . \qquad \text{Q.E.D.}$$

Lions [L-1, pp. 6-10] proved a general existence and uniqueness theorem for controls minimizing a certain cost functional. He also showed that this theorem covers the existence and uniqueness of optimal controls for the continuous-time control problem. Now, the discrete-time problem (DTP), defined in Section 3.2, will be shown to fall into Lion's framework in the function spaces $\ell^2(0,N;H)$ and $\ell^2(0,N;U)$.

<u>Theorem 3.3.4</u>: The discrete-time problem (DTP), defined in Section 3.2, has a unique solution $u^* \in \ell^2(0,N;U)$.

To prove this we need the following definition and lemmas due to Lions. Let V be a Hilbert space.

<u>Definition 3.3.5</u>: A continuous symmetric coercive bilinear form $\pi(u,v)$ is a continuous function in both arguments which maps $V \times V$ into the reals for which there exists a $C > 0$ such that

$$\pi(u,u) \geq C \, \|u\|^2 \qquad \forall \; u \in V$$

$$\pi(u,v) = \pi(v,u) \qquad \forall \; u, \; v \in V$$

If we consider a functional

$$C(u) = \pi(u,u) - 2L(u), \qquad u \in V, \tag{3.3.6}$$

where L is a bounded linear functional defined on V, then we have the following:

<u>Lemma 3.3.6</u> (Lions, [L-1]): If $\pi(u,v)$ is a continuous symmetric coercive bilinear form, then there exists a unique $u^* \in V$ such that

$$C(u^*) = \inf_{u \in V} C(u) .$$

<u>Lemma 3.3.7</u> (Lions, [L-1]): If the hypotheses of Lemma 3.3.6 are satisfied, then the minimizing element $u^* \in V$ is characterized by

$$\pi(u^*,v) = L(v) \quad \forall\, v \in V . \tag{3.3.7}$$

The proofs of above lemmas will be given in Appendix for reference. Now we return to the proof of the theorem.

<u>Proof of Theorem 3.3.4</u>: We may write the cost functional (3.2.5) with inner products in the function spaces $\ell^2(0,N;H)$ and $\ell^2(0,N;U)$, i.e.,

$$\begin{aligned} J = {} & \langle x,Qx\rangle_{\ell^2(0,N;H)} + 2\langle x,Mu\rangle_{\ell^2(0,N;H)} + \langle u,Ru\rangle_{\ell^2(0,N;U)} \\ & -2\langle x,Sx_d\rangle_{\ell^2(0,N;H)} - 2\langle u,Px_d\rangle_{\ell^2(0,N;U)} + \langle x_d,Ex_d\rangle_{\ell^2(0,N;H)} \\ & + \langle x_N,Fx_N\rangle_H - 2\langle x_N,Fx_{d_N}\rangle_H + \langle x_{d_N},Fx_{d_N}\rangle_H , \end{aligned} \tag{3.3.8}$$

where $Q,S,E \in \mathcal{L}(\ell^2(0,N;H);\ell^2(0,N;H))$, $R \in \mathcal{L}(\ell^2(0,N;U);\ell^2(0,N;U))$, $M \in \mathcal{L}(\ell^2(0,N;U);\ell^2(0,N;H))$, and $P \in \mathcal{L}(\ell^2(0,N;H);\ell^2(0,N;U))$ such that for $i \in \{0,1,\cdots, N-1\}$

$$Qx = \{Q_i x_i\}, \quad Mu = \{M_i u_i\}, \quad Ru = \{R_i u_i\} ,$$

$$Sx_d = \{S_i x_{d_i}\}, \; Px_d = \{P_i x_{d_i}\}, \; Ex_d = \{E_i x_{d_i}\} .$$

Note that Q and E are self-adjoint positive semi-definite operators, and R is a self-adjoint positive definite operator. We will simplify the notations by deleting subscripts in the expression of norms in the spaces $\ell^2(0,N;H)$ and $\ell^2(0,N;U)$ unless they are necessary.

Let x^u denote the response of the system (3.2.4) due to control $u \in \ell^2(0,N;U)$. We define the bilinear form $\pi(u,v)$ on $\ell^2(0,N;U) \times \ell^2(0,N;U)$ to be

$$\pi(u,v) \equiv \langle x^v - x^0, Q(x^u - x^0)\rangle + \langle x^v - x^0, Mu\rangle + \langle x^u - x^0, Mv\rangle$$

$$+ \langle x_N^v - x_N^0,\ F(x_N^u - x_N^0)\rangle + \langle v, Ru\rangle, \tag{3.3.9}$$

and a linear functional $L(v)$ on $\ell^2(0,N;U)$ to be

$$L(v) \equiv -\langle x^v - x^0, Qx^0\rangle - \langle x^0, Mv\rangle - \langle x_N^v - x_N^0,\ Fx_N^0\rangle \tag{3.3.10}$$
$$+ \langle x^v - x^0, Sx_d\rangle + \langle x_N^v - x_N^0, Fx_{d_N}\rangle + \langle v, Px_d\rangle .$$

Then the cost functional (3.3.8) becomes

$$J(u) = \pi(u,u) - 2L(u)$$

$$+ \langle x^0, Qx^0\rangle + \langle x_N^0, Fx_N^0\rangle - 2\langle x^0, Sx_d\rangle \tag{3.3.11}$$
$$- 2\langle x_N^0, Fx_{d_N}\rangle + \langle x_d, Ex_d\rangle + \langle x_{d_N}, Fx_{d_N}\rangle .$$

Since the last six terms are independent of u, the problem is equivalent to minimizing $J_1(u) = \pi(u,u) - 2L(u)$. The continuities of $\pi(u,v)$ and $L(v)$ follow from Lemma 3.3.2 and Lemma 3.3.3. Clearly $\pi(u,v)$ is symmetric. The coercivity of $\pi(u,u)$ follows from (3.3.9) and the definition of

$R(t)$, i.e.,

$$\pi(u,u) = \int_0^T [\langle x^u(t)-x^0(t), Q(t)(x^u(t)-x^0(t))\rangle + \langle u(t), R(t)u(t)\rangle]dt$$

$$+ \langle x^u(T) - x^0(T), F(x^u(T) - x^0(T)\rangle$$

$$\geq \int_0^T \langle u(t), R(t)u(t)\rangle dt \geq c \int_0^T \|u(t)\|^2 dt$$

$$= c \sum_{i=0}^{N-1} \|u_i\|^2 = c \|u\|_{\ell^2(0,N;U)} .$$

Since $\pi(u,v)$ satisfies the hypotheses of Lemma 3.3.6, there exists a unique $u^* \in \ell^2(0,N;U)$ such that

$$J_1(u^*) = \inf_{u\in\ell^2(0,N;U)} J_1(u) . \qquad \text{Q.E.D.}$$

Next we derive the necessary condition for optimality which is analogous to the result for finite-dimensional systems.

<u>Theorem 3.3.8</u>: If $u^* \in \ell^2(0,N;U)$ is the optimal control for the discrete-time problem (DTP) with optimal response $x^* \in \ell^2(0,N;H)$, then there necessarily exists a unique adjoint state $p^* \in \ell^2(0,N;H)$ such that

$$u_i^* = R_i^{-1}P_i x_{d_i} - R_i^{-1}M_i^* x_i^* - R_i^{-1}D_i^* p_{i+1}^* , \tag{3.3.12}$$

$$p_i^* = \Phi^* p_{i+1} + Q_i x_i^* - S_i x_{d_i} + M_i u_i^* ,$$

$$p_N^* = F(x_N^* - x_{d_N}) , \tag{3.3.13}$$

where R^{-1} is the inverse of R, and M_i^*, D_i^* and Φ^* are the adjoints of M_i, D_i and Φ, respectively.

<u>Proof</u>: We note x^* satisfies

$$x^*_{i+1} = \Phi x^*_i + D_i u^*_i , \quad x^*_0 = x_0 . \tag{3.3.14}$$

From the proof of Theorem 3.3.4, $\pi(u,v)$, defined by (3.3.9), is a continuous symmetric coercive bilinear form. Hence, by Lemma 3.3.7, the optimal control must satisfy

$$\pi(u^*,v) = L(v) \quad \forall\, v \in \ell^2(0,N;U) ,$$

or equivalently,

$$\pi(u^*,v-u^*) = L(v-u^*) \quad \forall\, v \in \ell^2(0,N;U) . \tag{3.3.15}$$

Further, let us introduce the adjoint equation

$$\begin{aligned} p_i &= \Phi^* p_{i+1} + Q_i x_i - S_i x_{d_i} + M_i u_i , \\ p_N &= F(x_N - x_{d_N}) . \end{aligned} \tag{3.3.16}$$

Here p_i is called the adjoint state and it should be noted that a unique solution $p \in \ell^2(0,N;H)$ exists for (3.3.16). In fact, by changing i to $N-i$ and realizing Φ^* is a semigroup (cf. [B-1], p.47), we can have an explicit solution for p_i for all $i \in \sigma$, which is a similar form to (3.3.2). Let us denote (x^u,p^u) as the solution pair of a system (3.3.14) and (3.3.16) due to a control $u \in \ell^2(0,N;U)$. Forming the inner product on $\ell^2(0,N;H)$ between p^u and $x^v - x^u$ we obtain

$$\langle p^u, x^v - x^u \rangle = \sum_{i=0}^{N-1} \langle p_i^u, x_i^v - x_i^u \rangle_H$$

$$= \sum_{i=0}^{N-1} \langle \Phi^* p_{i+1}^u,\ x_i^v - x_i^u \rangle_H + \langle M u, x^v - x^u \rangle$$

$$(3.3.17) \qquad + \langle Qx^u - Sx_d,\ x^v - x^u \rangle$$

$$= \sum_{i=0}^{N-1} \langle p_{i+1}^u,\ \Phi(x_i^v - x_i^u) \rangle + \langle M u, x^v - x^u \rangle$$

$$+ \langle Qx^u - Sx_d,\ x^v - x^u \rangle .$$

Note that the left-hand side of (3.3.17) can be expressed as:

$$\langle p^u, x^v - x^u \rangle = \sum_{i=0}^{N-1} \langle p_{i+1}^u, x_{i+1}^v - x_{i+1}^u \rangle_H - \langle p_N^u, x_N^v - x_N^u \rangle_H .$$

$$(3.3.18)$$

$$= \sum_{i=0}^{N-1} \langle p_{i+1}^u, x_{i+1}^v - x_{i+1}^u \rangle_H - \langle F(x_N^u - x_{d_N}), x_N^v - x_N^u \rangle$$

Equating (3.3.17) and (3.3.18), using (3.3.14), and letting $u = u^*$ we obtain

$$\sum_{i=0}^{N-1} \langle p_{i+1}^{u^*}, D_i(v_i - u_i^*) \rangle_H = \langle Qx^{u^*} - Sx_d,\ x^v - x^{u^*} \rangle$$

$$(33.19)$$

$$+ \langle F(x_N^{u^*} - x_{d_N}),\ x_N^v - x_N^{u^*} \rangle + \langle M u^*,\ x^v - x^{u^*} \rangle .$$

Now from (3.3.15),

$$\pi(u^*, v - u^*) - L(v - u^*)$$

$$= \langle Qx^{u^*} - Sx_d,\ x^v - x^{u^*} \rangle + \langle F(x_N^{u^*} - x_{d_N}), x_N^v - x_N^{u^*} \rangle$$

$$(3.3.20)$$

$$+ \langle M u^*, x^v - x^{u^*} \rangle + \langle x^{u^*},\ M(v - u^*) \rangle$$

$$+ \langle R u^*, v - u^* \rangle - \langle Px_d,\ v - u^* \rangle = 0 .$$

Combining (3.3.19) and (3.3.20), we obtain

$$\sum_{i=0}^{N-1} \langle p^{u^*}_{i+1}, D_i(v_i - u^*_i)\rangle_H + \langle x^{u^*}, M(v-u^*)\rangle$$

$$+ \langle Ru^*, v-u^*\rangle - \langle Px_d, v-u^*\rangle = 0 ,$$

or equivalently,

$$\sum_{i=0}^{N-1} [\langle D^*_i p^{u^*}_{i+1}, v_i - u^*_i\rangle_H + \langle M^*_i x^{u^*}, v_i - u^*_i\rangle_H$$

(3.3.21)

$$+ \langle R_i u^*_i, v_i - u^*_i\rangle_H - \langle P_i x_{d_i}, v_i - u^*_i\rangle] = 0 .$$

Since (3.3.21) hold for all $v \in \ell^2(0,N:U)$, we obtain

(3.3.22) $$R_i u^*_i = P_i x_{d_i} - M^*_i x^{u^*} - D^*_i p^{u^*}_{i+1} .$$

Moreover, since R is positive definite (cf.Definition 3.1.1) it has an inverse (cf.[Y-1],p.43) and so (3.3.22) reduces to (3.3.12). Q.E.D.

3.4 DECOUPLING AND THE RICCATI OPERATOR DIFFERENCE EQUATION

In this section we derive a feedback form of the optimal control given by (3.3.12). The feedback operator is shown to be bounded, self-adjoint and positive semi-definite, and the cost functional is expressed in terms of the feedback operator.

We define bounded operators on H:

(3.4.1) $$\Theta_i = \Phi - D_i R_i^{-1} M^*_i ,$$

(3.4.2) $$\Gamma_i = Q_i - M_i R_i^{-1} M^*_i .$$

Lemma 3.4.1: The operator $\Gamma_i \in \mathcal{L}(H;H)$ is bounded self-adjoint and positive semi-definite for all $i \in \sigma$.

Proof: The self-adjointness and boundedness comes directly from those of Q_i and R_i. It is clear that for all $i \in \sigma$

$$L_i = \int_{t_i}^{t_{i+1}} [\langle x(t), Q(t)x(t)\rangle + \langle u(t), R(t)u(t)\rangle] dt \geq 0$$

or equivalently,

$$L_i = \langle x_i, Q_i x_i\rangle + 2\langle x_i, M_i u_i\rangle + \langle u_i, R_i u_i\rangle \geq 0$$

for all $u_i \in U$. Now let u_i be given by $u_i = -R_i^{-1}M_i^* x_i$. Then we have

$$\begin{aligned} L_i &= \langle x_i, (Q_i - M_i R_i^{-1} M_i^*) x_i\rangle \geq 0 \\ &= \langle x_i, \Gamma_i x_i \rangle \geq 0 , \end{aligned}$$

which implies that Γ_i is positive semi-definite. Q.E.D.

Using (3.4.1) and (3.4.2), the system of equations (3.3.14) and (3.3.13) can be simplified into the form:

$$x_{i+1} = \Theta_i x_i - D_i R_i^{-1} D_i^* p_{i+1} + D_i R_i^{-1} P_i x_{d_i}$$

(3.4.3)

$$p_i = \Theta_i^* p_{i+1} + \Gamma_i x_i - S_i x_{d_i} + M_i R_i^{-1} P_i x_{d_i}$$

$$x_s = h \in H, \; p_N = F(x_N - x_{d_N}); \; i \in \{s, s+1, \cdots, N\}, \; s \in \sigma .$$

This system admits a unique solution pair $(x,p) \in \ell^2(s,N;H) \times \ell^2(s,N;H)$. This fact is easily seen if the cost functional J in (3.2.5) is defined on the interval $[s,N]$ instead of on $[0,N]$. The system (3.4.3) has the following properties.

<u>Lemma 3.4.2</u>: The mapping $h \to (x,p)$, solution of (3.4.3), is continuous from H into $\ell^2(s,N;H) \times \ell^2(s,N;H)$.

<u>Proof</u>: Without loss of generality we let $x_d = 0$. Let us denote by $x^n(v)$ the state of the system given by

(3.4.4) $$x_{i+1} = \Phi x_i + D_i v_i , \quad x_s = h_n, \quad \text{on } [s,N].$$

For a fixed v, if $h_n \to h$,

(3.4.5) $$x^n(v) \to x(v) \quad \text{in} \quad \ell^2(s,N;H).$$

Let $J_s^h(v)$ denote the cost with control v and initial condition h at time s. Let u^n and u be the optimal control for $J_s^{h_n}(v)$ and J_s^h, respectively. Then $J_s^{h_n}(u^n) = \inf J_s^{h_n}(v) \leq J_s^{h_n}(u)$ and $J_s^{h_n}(u) \to J_s^h(u)$ from (3.4.5). Hence

(3.4.6) $$\overline{\lim}\, J_s^{h_n}(u^n) \leq J_s^h(u) = \inf J_s^h(v) .$$

But

$$J_s^{h_n}(u^n) \geq c \sum_{i=s}^{N-1} \|u_i^n\|^2 ,$$

which when combined with (3.4.6) shows that u^n belongs to a bounded subset of $\ell^2(s,N;U)$ as $h_n \to h$. Then we can choose (cf.[Y-1], p.126) a subsequence u^k such that

(3.4.7) $$u^k \to w \quad \text{weakly in} \quad \ell^2(s,N;U).$$

Therefore, $x^k(u^k) \to x(w)$ weakly in $\ell^2(s,N;H)$ and hence

$$\underline{\lim}\ J_s^{h_k}(u^k) \geq J_s^h(w)\ ,$$

which when combined with (3.4.6) shows $J_s^h(w) \leq J_s^h(u)$ and hence necessarily $w = u$. Therefore,

$$u^n \to u \quad \text{weakly in} \quad \ell^2(s,N;U),$$

$$J_s^{h_n}(u^n) \to J_s^h(u)\ , \tag{3.4.8}$$

and

$$x^n(u^n) \to x(u) \quad \text{weakly in} \quad \ell^2(s,N;H),$$

$$p^n(u^n) \to p(u) \quad \text{weakly in} \quad \ell^2(s,N;H).$$

This proves the continuity of the linear mapping $h \to (x,p)$ from H into $\ell^2(s,N;H) \times \ell^2(s,N;H)$. Furthermore, (3.4.8) implies that $u^n \to u$ strongly in $\ell^2(s,N;U)$ and hence the mapping $h \to (x,p)$ is in fact continuous in strong topologies. Q.E.D.

<u>Corollary 3.4.3</u>: For $h \in H$, let (x,p) be the solution of (3.4.3). Then the mapping

$$h \to p_s \tag{3.4.9}$$

is continuous from H into H.

<u>Proof</u>: The proof follows from the fact that the mapping (4.7) is the composition of the mapping $h \to (x,p)$ and the mapping $(x,p) \to p_s$. But $(x,p) \in \ell^2(s,N;H) \times \ell^2(s,N;H)$ implies that for every $i \in [s,N]$ $\|x_i\|$ and $\|p_i\|$ are bounded. Hence we may take

subsequences x_i^n, p_i^n such that

$$x_i^n \to \bar{x}_i \quad \text{weakly in } H \quad \forall\, i \in [s,N],$$

$$p_i^n \to \bar{p}_i \quad \text{weakly in } H \quad \forall\, i \in [s,N].$$

The second equation of (3.4.3) becomes in the limit,

$$\bar{p}_i = \Theta_i^* \, \bar{p}_{i+1} + \Gamma_i \, \bar{x}_i \quad \text{on} \quad [s,N],$$

hence, we may take $\bar{p}_i = p$ for all $i \in [s,N]$, and in particular $\bar{p}_s = p_s$. Thus the mapping $(x,p) \to p_s$ is continuous from $\ell^2(s,N;H) \times \ell^2(s,N;H)$ into H.

Q.E.D.

Now we have the feedback representation of the optimal control.

<u>Theorem 3.4.4</u>: The optimal control $u^* \in \ell^2(O,N;U)$ for the discrete-time problem (DTP), defined in Section 3.2, is given by the feedback form

$$\begin{aligned} u_i = & -[R_i^{-1}M_i^* + R_i^{-1}D_i^*K_{i+1}(I + D_iR_i^{-1}D_i^*K_{i+1})^{-1}\Theta_i]x_i \\ & -[R_i^{-1}D_i^* - R_i^{-1}D_i^*K_{i+1}(I + D_iR_i^{-1}D_i^*K_{i+1})^{-1}D_iR_i^{-1}D_i^*]g_{i+1} \qquad (3.4.10) \\ & +[R_i^{-1}P_i - R_i^{-1}D_i^*K_{i+1}(I + D_iR_i^{-1}D_i^*K_{i+1})^{-1}D_iR_i^{-1}P_i]x_{d_i}, \end{aligned}$$

where for $i \in \sigma$, K_i is the solution of the Riccati-type operator difference equation

$$(3.4.11) \qquad K_i = \Theta_i^* \, K_{i+1}(I + D_iR_i^{-1}D_i^*K_{i+1})^{-1}\Theta_i + \Gamma_i,$$

$$K_N = F,$$

and g_i is the solution of the linear difference equation

$$g_i = [\Theta_i^* - \Theta_i^* K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} D_i R_i^{-1} D_i^*] g_{i+1}$$

$$(3.4.12)\quad + [\Theta_i^* K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} D_i R_i^{-1} P_i + M_i R_i^{-1} P_i - S_i] x_{d_i} ,$$

$$g_N = - F x_{d_N} .$$

<u>**Proof**</u>: From the continuity of $h \to p_s$, p_s can be written uniquely (cf.[L-1], p.135) in the form

$$(3.4.13)\qquad p_s = K_s h + g_s ,$$

where $K_s \in \mathcal{L}(H;H)$ and $g_s \in H$. Since s is arbitrary in σ and h is the evaluation of x_s, (3.4.13) implies that

$$(3.4.14)\qquad p_i = K_i x_i + g_i \qquad \forall\ i \in \sigma ,$$

where (x,p) is the solution pair of the system (3.4.3). Using (3.4.14) we can rewrite the system (3.4.3) as

$$x_{i+1} = \Theta_i x_i + D_i R_i^{-1} P_i x_{d_i} - D_i R_i^{-1} D_i^* (K_{i+1} x_{i+1} + g_{i+1}) ,$$

$$(3.4.15)\quad p_i = \Theta_i^* (K_{i+1} x_{i+1} + g_{i+1}) + \Gamma_i x_i - S_i x_{d_i} + M_i R_i^{-1} P_i x_{d_i}$$

$$= K_i x_i + g_i ,$$

$$p_N = K_N x_N + g_N = F x_N - F x_{d_N} ;\ x_0 \in H .$$

Rearranging the first equation in (3.4.15), we obtain

$$(3.4.16)\quad x_{i+1} = (I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} [\Theta_i x_i + D_i R_i^{-1} P_i x_{d_i} - D_i R_i^{-1} D_i^* g_{i+1}] ,$$

where the inverse is well-defined since K_{i+1} is positive semi-definite by the next theorem (Theorem 3.4.5). Substituting (3.4.16) into the second equation in (3.4.15) and rearranging terms, we have

$$[K_i - \Theta_i^* K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1}\Theta_i - \Gamma_i]x_i$$

$$(3.4.17) \qquad = - g_i + \Theta_i^* g_{i+1} - \Theta_i^* K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} D_i R_i^{-1} D_i^* g_{i+1}$$

$$+ \Theta_i^* K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} D_i R_i^{-1} P_i x_{d_i} - S_i x_{d_i} + M_i R_i^{-1} P_i x_{d_i}$$

Since x_i is arbitrary in the sense that it depends on an arbitrary choice of x_0, it is necessary that K_i and g_i satisfies (3.4.11) and (3.4.12), respectively. The feedback control (3.4.10) follows from (3.3.12), (3.4.14), (3.4.11) and (3.4.12). Q.E.D.

Next we examine the properties of the feedback operator K_s and express the optimal cost in terms of K_s.

<u>Theorem 3.4.5</u>: The feedback operator K_s on H is self-adjoint, positive semi-definite and bounded.

<u>Proof</u>: (Self-adjointness). Let (x^1, p^1) and (x^2, p^2) be solution pairs corresponding to initial conditions h^1 and h^2, respectively, for system (3.4.3) with $x_d = 0$. Then

$$0 = \sum_{i=s}^{N-1} \langle p_i^1 - \Theta_i^* p_{i+1}^1 - \Gamma_i\, x_i^1 \,,\, x_i^2 \rangle$$

$$= \sum_{i=s}^{N-1} \langle p_i^1, x_i^2\rangle - \sum_{i=s}^{N-1} \langle p_{i+1}^1, \Theta_i x_i^2 \rangle - \sum_{i=s}^{N-1} \langle \Gamma_i^1 x_i^1, x_i^2\rangle \,,$$

$$= \langle p_s^1, x_s^2\rangle - \langle p_N^1, x_N^2\rangle - \sum_{i=s}^{N-1} \langle p_{i+1}^1, D_i R_i^{-1} D_i^* p_{i+1}^2 \rangle - \sum_{i=s}^{N-1} \langle \Gamma_i x_i^1, x_i^2 \rangle \,,$$

Hence by (3.4.13),

$$(3.4.18)\quad \langle K_s h^1, h^2\rangle = \langle F x_N^1, x_N^2\rangle + \sum_{i=s}^{N-1} \langle p_{i+1}^1, D_i R_i^{-1} D_i^* p_{i+1}^2\rangle + \sum_{i=s}^{N-1} \langle \Gamma_i x_i^1, x_i^2\rangle .$$

Thus self-adjointness of K_s comes from those of F, R_i, and Γ_i.

(Positivity). Let $u \in \ell^2(s,N;U)$ be the optimal control with cost $J_s^h(u)$ of the system (3.4.3) with $x_d = 0$. Optimality implies that u satisfies the necessary condition

$$(3.4.19)\qquad R_i u_i + M_i^* x_i + D_i^* p_{i+1} = 0 .$$

If $h^1 = h^2 = h$, the equality (3.4.18) becomes

$$(3.4.20)\qquad \langle K_s h, h\rangle = \langle F x_N, x_N\rangle + \sum_{i=s}^{N-1} \langle p_{i+1}, D_i R_i^{-1} D_i^* p_{i+1}\rangle + \sum_{i=s}^{N-1} \langle Q_i x_i, x_i\rangle - \sum_{i=s}^{N-1} \langle M_i R_i^{-1} M_i^* x_i, x_i\rangle .$$

But, by virtue of (3.4.19), we deduce

$$(3.4.21)\qquad \begin{aligned} &2\langle Mu, x\rangle_{\ell^2(s,N;H)} + \langle Ru, u\rangle_{\ell^2(s,N;U)} \\ &= - \sum_{i=s}^{N-1} \langle M_i R_i^{-1} M_i^* x_i, x_i\rangle + \sum_{i=s}^{N-1} \langle p_{i+1}, D_i R_i^{-1} D_i^* p_{i+1}\rangle . \end{aligned}$$

Combining (3.4.20) and (3.4.21), we obtain

$$(3.4.22)\qquad \begin{aligned} \langle K_s h, h\rangle &= \langle Qx, x\rangle_{\ell^2(s,N;H)} + \langle 2Mu, x\rangle_{\ell^2(s,N;H)} \\ &\quad + \langle Ru, u\rangle_{\ell^2(s,N;U)} + \langle F x_N, x_N\rangle_H \\ &= J_s^h(u) \geq 0 , \end{aligned}$$

which proves the positive semi-definite property of K_s.

(Boundedness). Since, by Lemma 3.4.2, the transformations $h \to x$, $h \to p$, and $h \to x_N$ are continuous in the strong topology, we have $\|x\|_{\ell^2(s,N;H)} \le C_1 \|h\|_H$, $\|p\|_{\ell^2(s,N;H)} \le C_2\|h\|_H$, and $\|x_N\|_H \le C_3 \|h\|_H$. Since Γ, R, and F are bounded, we have $\langle \Gamma x,x\rangle \le M_1 C_1^2 \|h\|^2$, $\langle p, DR^{-1}D^* p\rangle \le M_2 C_2^2 \|h\|^2$, and $\langle F x_N, x_N\rangle \le M_3 C_3^2 \|h\|^2$. This implies that, by (3.4.20)

$$\langle K_s h,h\rangle \le (M_1C_1^2 + M_2C_2^2 + M_3C_3^2)\ \|h\|^2 .$$

Thus K_s is bounded. Q.E.D.

<u>Theorem 3.4.6</u>: The optimal cost of system (3.4.3) with initial state h at time s is given by

$$J_s^h(u^*) = \langle K_s h,h\rangle + 2\langle g_s , h\rangle + \varphi_s ,$$

where K_s and g_s are solutions of (3.4.11) and (3.4.12), respectively, and φ_s is the solution of

$$\begin{aligned}
\varphi_i &= \varphi_{i+1} - \langle p_i x_{d_i}, R_i^{-1} P_i x_{d_i}\rangle + \langle D_i R_i^{-1} P_i x_{d_i}, H_i D_i R_i^{-1} P_i x_{d_i}\rangle \\
&\quad - 2\langle g_{i+1}, D_i R_i^{-1} D_i^* H_i D_i R_i^{-1} P_i x_{d_i}\rangle + 2\langle g_{i+1}, D_i R_i^{-1} D_i^* x_{d_i}\rangle \\
&\quad + \langle g_{i+1} D_i R_i^{-1} D_i^* H_i D_i R_i^{-1} D_i^* g_{i+1}\rangle - \langle g_{i+1}, D_i R_i^{-1} D_i^* g_{i+1}\rangle , \qquad (3.4.23)\\
\varphi_N &= \langle F x_{d_N}, x_{d_N}\rangle ,
\end{aligned}$$

where $H_i \in \mathcal{L}(H,H)$ is a self-adjoint positive semi-definite operator such that $H_i = K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1}$.

<u>Proof</u>: From the second equation of (3.4.3) we have

$$\langle \Gamma x - Sx_d + MR^{-1}Px_d, x - x_d \rangle_{\ell^2(s,N;H)}$$

(3.4.24)
$$= \sum_{i=s}^{N-1} \langle p_i - \Theta_i^* p_{i+1}, x_i - x_{d_i} \rangle_H$$

The left hand side of (3.4.24) can be expanded to

$$\langle Qx,x\rangle - \langle MR^{-1}M^* x,x\rangle - \langle Sx_d,x\rangle + \langle MR^{-1}Px_d,x\rangle$$
$$- \langle Qx,x_d\rangle + \langle MR^{-1}M^* x,x_d\rangle + \langle Sx_d,x_d\rangle - \langle MR^{-1}Px_d,x_d\rangle .$$

Using the first equation of (3.4.3), and using the similar way for the derivation of (3.4.18), the right hand side of (3.4.24) can be shown to be

$$\langle K_s h + g_s,h\rangle - \langle F(x_N - x_{d_N}),x_N - x_{d_N}\rangle - \langle F(x_N - x_{d_N}),x_{d_N}\rangle$$
$$- \langle Q\ x,x_d\rangle + \langle MR^{-1}M^* x,x_d\rangle - \langle MR^{-1}Px_d,x_d\rangle + \langle Sx_d,x_d\rangle$$
$$+ \sum_{i=s}^{N-1} \langle p_{i+1},D_i R_i^{-1} P_i x_{d_i}\rangle - \sum_{i=s}^{N-1} \langle p_{i+1},D_i R_i^{-1} D^* p_{i+1}\rangle .$$

Thus (3.4.24) yields

$$\langle Qx,x\rangle - 2\langle Sx_d,x\rangle + \langle F(x_N - x_{d_N}),x_N - x_{d_N}\rangle$$
$$= - \langle Sx_d,x\rangle + \langle MR^{-1}M^* x,x\rangle - \langle MR^{-1}Px_d,x\rangle$$

(3.4.25)
$$- \langle F(x_N - x_{d_N}),x_{d_N}\rangle + \langle K_s h + g_s,h\rangle$$
$$+ \sum_{i=s}^{N-1} \langle p_{i+1},D_i R_i^{-1} P_i x_{d_i}\rangle - \sum_{i=s}^{N-1} \langle p_{i+1},D_i R_i^{-1} D_i^* p_{i+1}\rangle .$$

From the necessary condition (3.3.12) it can be shown that

$$2\langle x,Mu\rangle + \langle u,Ru\rangle - 2\langle u,Px_d\rangle$$

(3.4.26)
$$= - \langle MR^{-1}M^* x,x\rangle - \langle Px_d,R^{-1}Px_d\rangle$$
$$+ 2\langle Px_d,R^{-1}M^* x\rangle + \sum_{i=s}^{N-1} \langle p_{i+1},D_i R_i^{-1} D_i^* p_{i+1}\rangle .$$

Thus, the cost functional (3.3.8) becomes, by (3.4.25) and (3.4.26),

$$J_s^h(u^*) = \langle K_s h,h\rangle + \langle g_s,h\rangle - \langle F(x_N - x_{d_N}),x_{d_N}\rangle$$

(3.4.27)

$$- \langle Px_d,R^{-1}Px_d\rangle + \langle MR^{-1}Px_d,x\rangle$$

$$- \langle Sx_d,x\rangle + \sum_{i=s}^{N-1} \langle p_{i+1},D_iR_i^{-1}P_ix_{d_i}\rangle .$$

When we use (3.4.12), (3.4.14), (3.4.15), and (3.4.16), then (3.4.27) can be expressed as

$$\begin{aligned}J_s^h(u^*) &= \langle K_s h,h\rangle + 2\langle g_s,h\rangle + \langle Fx_{d_N}, x_{d_N}\rangle \\ &+ \sum_{i=s}^{N-1}[-\langle P_ix_{d_i},R_i^{-1}P_ix_{d_i}\rangle + \langle D_iR_i^{-1}P_ix_{d_i},H_iD_iR_i^{-1}P_ix_{d_i}\rangle \\ &- 2\langle g_{i+1}, D_iR_i^{-1}D_i^*H_iD_iR_i^{-1}P_ix_{d_i}\rangle \\ &+ 2\langle g_{i+1}, D_iR_i^{-1}D_i^*x_{d_i}\rangle+\langle g_{i+1},D_iR_i^{-1}D_i^*H_iD_iR_i^{-1}D_i^*g_{i+1}\rangle \\ &- \langle g_{i+1}, D_iR_i^{-1}D_i^*g_{i+1}\rangle] ,\end{aligned} \tag{3.4.28}$$

where $H_i = K_{i+1}(I + D_iR_i^{-1}D_i^*K_{i+1})^{-1}$, hence we prove (3.4.23)

Q.E.D.

Note that H_i is bounded, self-adjoint, and positive semi-definite because of K_i, F_i and the Riccati equation (3.4.11), i.e.,

$$\begin{aligned}\langle(K_i-\Gamma_i)x_i,y_i\rangle &= \langle\Theta_i^*H_i\Theta_ix_i,y_i\rangle \\ &= \langle H_i\Theta_ix_i,\Theta_iy_i\rangle ,\end{aligned}$$

and

$$\langle x_i, (K_i - \Gamma_i) y_i \rangle = \langle x_i, \Theta_i^* H_i \Theta_i y_i \rangle$$
$$= \langle \Theta_i x_i, H_i \Theta_i y_i \rangle ,$$

which implies H_i is bounded, self-adjoint and positive semi-definite by those properties of K_i and Γ_i.

Remark 3.4.7: (a) The equation (3.4.13) is reduced to $p_s = K_s h$ when $x_d = 0$, and to $p_s = g_s$ when $h = 0$.

(b) The existences and uniquenesses of the solutions of the Riccati equation (3.4.11) and the linear equation (3.4.12) are the consequences of the strong continuity of the transformation $h \to p_s$.

With the Riccati operator equation on hand, we will consider one way of solving the operator equation. If we assume H is a separable Hilbert space, then there exists a basis $\{\varphi_i\}_{i=1}^{\infty}$, $\varphi_i \in H$, such that any element $x \in H$ has a unique representation

$$x = \sum_{j=1}^{\infty} x_j \varphi_j ,$$

where $x_j = \langle x, \varphi_j \rangle$ (cf.[R-1], p.212). Thus, we may consider an element $x \in H$ to be alternatively represented by a infinite dimensional vector $\underline{x}$ with j^{th} component x_j. If L is a any linear operator on H, we have for $x \in H$,

$$Lx = L \sum_{j=1}^{\infty} x_j \varphi_j = \sum_{j=1}^{\infty} x_j L\varphi_j .$$

Now, $L\varphi_j$ is an element of H so that

$$L\varphi_j = \sum_{i=0}^{\infty} L_{ij} \varphi_j ,$$

where $L_{ij} = \langle L\varphi_j, \varphi_i \rangle$. Thus Lx may be represented by $\underline{L}\,\underline{x}$, where $\underline{L}$ is the infinite matrix with ij^{th} element L_{ij}. Similarly if U is a separable Hilbert space with basis $\{\psi_i\}_{i=1}^{\infty}$, each element $u \in U$ may be considered to be an infinite dimensional vector $\underline{u}$, and the control operator D may be considered to be the infinite matrix $\underline{D}$ with ij^{th} element $D_{ij} = \langle D\psi_j, \psi_i \rangle$.

Let us, for the purpose of illustration assume that DTP is time invariant and $R = I$. Then we may rewrite the Riccati operator equation (3.4.11) as the <u>infinite dimensional</u> matrix Riccati equation

$$\underline{K}_i = \underline{\Theta}^* \underline{K}_{i+1} (\underline{I} + \underline{D}\, \underline{D}^* \underline{K}_{i+1})^{-1} \underline{\Theta} + \underline{\Gamma} \,,$$

$$\underline{K}_N = \underline{F} \,,$$

where all of the matrices are uniquely determined in the fashion prescribed above.

It is possible to truncate above matrices and solve the resulting finite dimensional matrix equation for an approximate value of the $\underline{K}_i$ matrix. An alternative way of solving the Riccati equation will be considered in Section 3.6.

3.5 CONTROL ON THE INFINITE TIME INTERVAL

In this section we will develop a treatment of the discrete-time problem (DTP), defined in Section 3.2, on the infinite time interval, i.e., on $\sigma = \sigma_\infty = \{0,1,2,\cdots\}$. We assume that the operators D_i, Q_i, M_i, R_i, S_i, P_i, and E_i are uniformly bounded on σ, and that $F = 0$.

Let $\ell^2(0,\infty;H)$ be the family of all sequences $x = \{x_i\}_{i\in\sigma}$ with $x_i \in H$ such that $\sum_\sigma \|x_i\|^2 < \infty$.

<u>Remark 3.5.1</u> [D-1,p.257]: $\ell^2(0,\infty;H)$ is a Hilbert space with the usual addition and scalar multiplication, and with an inner product defined by, for $x, y \in \ell^2(0,\infty;H)$,

$$(3.5.1) \qquad < x,y >_{\ell^2(0,\infty;H)} = \sum_\sigma < x_i,y_i >_H .$$

Similarly $\ell^2(0,\infty;U)$ is a Hilbert space with inner product analogous to (3.5.1).

We make the following definition.

<u>Definition 3.5.2</u>: (a) A control u defined on σ is said to be admissible if $u \in \ell^2(0,\infty;U)$.

(b) A state x is said to be a solution of system (3.2.4) if it satisfies (3.2.4) with an admissible control u and initial condition $x_0 \in H$, and if $x(u) \in \ell^2(0,\infty;H)$, where $x(u) = \{x(u)_i\}_{i\in\sigma}$.

Notations: The state of the system will be denoted by x^N (to emphasize the dependence on N), that is, x^N is the solution of (3.2.4) on $\sigma_N = \{0,1,2,\cdots, N\}$. The cost functional in (3.2.5) is denoted by $J_N(u)$. Let (x^N,p^N) be the unique solution pair of the system (3.4.3) on σ_N and let K_i^N and g_i^N be the corresponding operator and function K_i and g_i in (3.4.11) and (3.4.12) respectively.

On the infinite time interval we make the following hypothesis, which was trivially guaranteed on the finite time interval.

<u>Hypothesis I</u>: For every $u \in \ell^2(0,\infty;U)$ and $x_o \in H$, there exists a unique solution of system (3.2.4) on σ.

It should be pointed out that stronger hypotheses have been used in the continuous-time problem (cf.[L-4] and [D-2]).

<u>Lemma 3.5.3</u>: Hypothesis I implies that the mapping $u \to x$ defined by the difference equation (3.2.4) is continuous from $\ell^2(0,\infty;U)$ into $\ell^2(0,\infty;H)$.

<u>Proof</u>: Let T be the mapping $u \to x(u)$. Clearly T is linear. It remains to show that T is bounded. Define $T_n(u) = \{x_1(u), x_2(u), \cdots, x_n(u)\} \in \ell^2(0,\infty;H)$. Clearly T_n is linear. Assuming $x_o = 0$ and by a simple calculation we can show that T_n is bounded for all n, i.e., $\|T_n(u)\| \leq C_n\|u\|$ for all $u \in \ell^2(0,\infty;U)$ and for all n, where C_n is a constant which depends on n (cf. Lemma 3.3.2). Since for each $u \in \ell^2(0,\infty;U)$ $\|T_n(u)\|$ is bounded by a constant $\|x(u)\|$ for all n, by the uniform boundedness theorem (cf.[R-1],p.196) there is a constant C such that $\|T_n\| \leq C$ for all n. Again by Yosida [Y-1, p.69, Corollary 2] T is the strong limit of the sequence $\{T_n\}$ and T is a bounded linear operator. Q.E.D.

<u>Remark 3.5.4</u>: Lemma 3.5.3 implies that $x \in \ell^2(0,\infty;H)$ and hence $J_\infty(u) < \infty$ for all $u \in \ell^2(0,\infty;U)$. The requirement $J_\infty(u) < \infty$ is the hypothesis adopted by Lukes and Russell [L-4] and Datko [D-2].

Now we have the existence and uniqueness of DTP.

Theorem 3.5.5: Assume that Hypothesis I holds. Then the discrete-time problem (DTP), defined in Section 3.2, on the infinite time interval has a unique optimal control $u^\infty \in \ell^2(0,\infty;U)$.

Proof: Because of Lemma 3.5.3, Lemma 3.3.6 can be used and the proof is identical to that of Theorem 3.3.4.

We now give a sufficient condition for Hypothesis I to hold.

Theorem 3.5.5: If $\|\Phi\| < 1$, then Hypothesis I will hold true.

Proof: We are concerned with showing the existence and uniqueness of solution $x(u) \in \ell^2(0,\infty;H)$ satisfying

(3.5.2) $$x_{i+1} = \Phi x_i + D_i u_i\,; \quad x_0 \in H\,.$$

(Existence). Let $\|\Phi\| = a < 1$. Let $x(0)^N \in \ell^2(0,N;H)$ be the solution of (3.2.4) with zero control and $\hat{x}(u)^N \in \ell^2(0,N;H)$ be the solution of (3.2.4) corresponding to a control $u \in \ell^2(0,\infty;U)$ and $x_0 = 0$. Then the solution of (3.2.4) can be written in the form

(3.5.3) $$x(u)^N = x(0)^N + \hat{x}(u)^N\,.$$

Now for $x(0)^N$, we derive

$$\begin{aligned} 0 &= \sum_{i=0}^{N-1} \langle x(0)^N_{i+1}, x(0)^N_{i+1}\rangle - \sum_{i=0}^{N-1} \langle \Phi x(0)^N_i\,, x(0)^N_{i+1}\rangle \\ &\geq \sum_{i=0}^{N-1} \|x(0)^N_{i+1}\|^2 - \sum_{i=0}^{N-1} \|\Phi x(0)^N_i\|\ \|x(0)^N_{i+1}\| \end{aligned}$$

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$$\geq \sum_{i=0}^{N-1} \|x(0)_i^N\|^2 - \|x_o\|^2 - a\left(\sum_{i=0}^{N-1} \|x(0)_i^N\|^2 \sum_{i=0}^{N-1} \|x(0)_{i+1}^N\|^2\right)^{1/2}$$

$$\geq \sum_{i=0}^{N-1} \|x(0)_i^N\|^2 - \|x_o\|^2 - a \sum_{i=0}^{N-1} \|x(0)_i^N\|^2$$

$$= (1-a)\sum_{i=0}^{N-1} \|x(0)_i^N\|^2 - \|x_o\|^2 ,$$

hence,

$$\|x(0)^N\|_{\ell^2(0,N;H)} \leq C , \quad \text{(independent of N)}. \tag{3.5.4}$$

Similarly for $\hat{x}(u)^N$ with $x_o = 0$ we derive

$$\sum_{i=0}^{N-1} \langle \hat{x}(u)_{i+1}^N, \hat{x}(u)_{i+1}^N \rangle - \sum_{i=0}^{N-1} \langle \Phi\hat{x}(u)_i^N, \hat{x}(u)_{i+1}^N \rangle = \sum_{i=0}^{N-1} \langle D_i u_i, \hat{x}(u)_{i+1}^N \rangle ,$$

or

$$\sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2 - \sum_{i=0}^{N-1} \|\Phi\hat{x}(u)_i^N\| \|\hat{x}(u)_{i+1}^N\| \leq \sum_{i=0}^{N-1} \|D_i u_i\| \|\hat{x}(u)_{i+1}^N\| ,$$

or

$$\sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2 - a\left(\sum_{i=0}^{N-1} \|\hat{x}(u)_i^N\|^2 \sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2\right)^{1/2}$$
$$\leq \left(\sum_{i=0}^{N-1} \|D_i u_i\|^2 \sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2\right)^{1/2} ,$$

or

$$\sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2 - a \sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2 \leq \left(\sum_{i=0}^{N-1} \|D_i u_i\|^2\right)^{1/2} \times$$
$$\left(\sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2\right)^{1/2} ,$$

or

$$(1 - a)^2 \sum_{i=0}^{N-1} \|\hat{x}(u)_{i+1}^N\|^2 \le \sum_{i=0}^{N-1} \|D_i u_i\|^2 \le \sum_{i=0}^{\infty} \|D_i u_i\|^2 .$$

Hence

$$\sum_{i=0}^{N-1} \|\hat{x}(u)_i^N\|^2 \le C_1 \sum_{i=0}^{\infty} \|u_i\|^2 \le C . \tag{3.5.5}$$

Therefore, from (3.5.3), (3.5.4) and (3.5.5), we have

$$\|x(u)^N\|_{\ell^2(0,N;H)} \le C_2 \|u\|^2_{\ell^2(0,\infty;U)} \le C . \tag{3.5.6}$$

Now if

$$\tilde{x}_i^N = \text{extension of } \; x(u)_i^N \; \text{ by } \; 0 \; \text{ for } \; i > N ,$$

$$\tilde{u}_i^N = \begin{cases} u_i & \text{in } \; [0,N), \\ 0 & \text{for } \; i \ge n , \end{cases}$$

we have

$$\tilde{x}_{i+1}^N = \Phi \tilde{x}_i^N + D_i \tilde{u}_i^N - \Phi x_N^N \delta(i - N) \quad \text{on} \quad [0,\infty) , \tag{3.5.7}$$

where $\delta(0) = 1$, and $\delta(r) = 0$ for $r \ne 0$, and from (3.5.6)

$$\|\tilde{x}^N\|^2_{\ell^2(0,\infty;H)} \le C . \tag{3.5.8}$$

We may then take a subsequence $N_n \to \infty$ such that

$$\tilde{x}^{N_n} \to x \quad \text{weakly in} \quad \ell^2(0,\infty;H) .$$

We then pass to the limit in (3.5.7) and hence

$$x_{i+1} = \Phi x_i + D_i u_i \quad \text{on} \quad [0,\infty) .$$

(Uniqueness). Suppose x^1 and x^2 are solutions of (3.2.4) on σ with control $u \in \ell^2(0,\infty;U)$. Then

$$\|x^1_{i+1} - x^2_{i+1}\| \le a\|x^1_i - x^2_i\| . \tag{3.5.9}$$

Since $x^1_0 = x^2_0$, by iterating (3.5.9) we conclude that $x^1_i = x^2_i$ for all $i \in \sigma$.

Thus we have shown that Hypothesis I holds.

Q.E.D.

<u>Remark 3.5.6</u>: The assumption that $\|\Phi\| < 1$ can be well-satisfied if $\Phi(t)$ is the semigroup of operators generated by a strongly elliptic operator. Because the operator $\Phi(t)$ has an exponential bound (2.4.13):

$$\|\Phi(t)\| \le M\, e^{-\lambda t} , \quad M, \lambda > 0 ,$$

and hence by a suitable choice of sampling interval $t = \delta$, we can always have $\|\Phi\| = \|\Phi(\delta)\| < 1$.

<u>Theorem 3.5.7</u>: Suppose $\|\Phi\| < 1$, then the adjoint state p is defined in a unique manner by

$$p_i = \Phi^* p_{i+1} + Q_i x_i - S_i x_{d_i} + M_i u_i , \tag{3.5.10}$$

and

$$p \in \ell^2(0,\infty;H) . \tag{3.5.11}$$

<u>Proof</u>: Let us simplify the notation by defining $f_i = Q_i x_i - S_i x_{d_i} + M_i u_i \in \ell^2(0,\infty;H)$. Then we are concerned with showing the existence and uniqueness of $p \in \ell^2(0,\infty;H)$ satisfying

$$p_i = \Phi^* p_{i+1} + f_i \tag{3.5.12}$$

We note that $p \in \ell^2(0,\infty;H)$ implies that $p_\infty = 0$.

(Uniqueness). Let $\|\Phi\| = a < 1$. Suppose (3.5.12) holds with $f = 0$. Then

$$\begin{aligned}
0 &= \sum_{i=0}^{\infty} \langle p_i, p_i\rangle - \sum_{i=0}^{\infty} \langle \Phi^* p_{i+1}, p_i\rangle \\
&\geq \sum_{i=0}^{\infty} \|p_i\|^2 - \sum_{i=0}^{\infty} \|\Phi^* p_{i+1}\| \; \|p_i\| \\
&\geq \sum_{i=0}^{\infty} \|p_i\|^2 - \Big(\sum_{i=0}^{\infty} \|\Phi^* p_{i+1}\|^2 \sum_{i=0}^{\infty} \|p_i\|^2\Big)^{1/2} \\
&\geq \sum_{i=0}^{\infty} \|p_i\|^2 - a\Big(\sum_{i=0}^{\infty} \|p_{i+1}\|^2 \sum_{i=0}^{\infty} \|p_i\|^2\Big)^{1/2} \\
&\geq \sum_{i=0}^{\infty} \|p_i\|^2 - a \sum_{i=0}^{\infty} \|p_i\|^2 \\
&= (1 - a) \sum_{i=0}^{\infty} \|p_i\|^2 , \qquad (3.5.13)
\end{aligned}$$

hence $p = 0$.

(Existence). Let q^N be the solution in $\ell^2(0,N;H)$ of

$$q_i^N = \Phi^* q_{i+1}^N + f_i \quad \text{in} \quad [0,N) , \tag{3.5.14}$$

where $q_N^N = 0$. Then

$$\sum_{i=0}^{N-1} \langle q_i^N, q_i^N\rangle - \sum_{i=0}^{N-1} \langle \Phi^* q_{i+1}^N, q_i^N\rangle = \sum_{i=0}^{N-1} \langle f_i, q_i^N\rangle . \tag{3.5.15}$$

With similar arguments for (3.5.13), we deduce, from (3.5.15),

$$(1 - a) \sum_{i=0}^{N-1} \|q_i^N\|^2 \leq \Big(\sum_{i=0}^{N-1} \|f_i\|^2 \sum_{i=0}^{N-1} \|q_i^N\|^2\Big)^{1/2} ,$$

or,

$$(1-a)^2 \sum_{i=0}^{N-1} \|q_i^N\|^2 \le \sum_{i=0}^{N-1} \|f_i\|^2 \le \sum_{i=0}^{\infty} \|f_i\|^2 .$$

Hence

$$(3.5.16) \qquad \sum_{i=0}^{N-1} \|q_i^N\|^2 \le c_1 \sum_{i=0}^{\infty} \|f_i\|^2 \le c_2 , \quad \text{(independent of } N\text{)}.$$

Now if

$$\tilde{q}_i^N = \text{extension of } q_i^N \text{ by } 0 \text{ for } i > N ,$$

$$\tilde{f}_i^N = \begin{cases} f_i & \text{in } [0,N), \\ 0 & \text{for } i \ge N , \end{cases}$$

we have

$$(3.5.17) \qquad \tilde{q}_i^N = \Phi^* \tilde{q}_{i+1}^N + \tilde{f}_i^N \quad \text{on } [0,\infty) ,$$

and hence from (3.5.16),

$$(3.5.18) \qquad \sum_{i=0}^{\infty} \|\tilde{q}_i^N\|^2 \le c_2 .$$

We may then take a subsequence $N_n \to \infty$ such that

$$\tilde{q}^{N_n} \to p \quad \text{weakly in } \ell^2(0,\infty;H).$$

We then pass to the limit in (3.5.17), and hence,

$$p_i = \Phi^* p_{i+1} + f_i \quad \text{on } [0,\infty). \qquad \text{Q.E.D.}$$

Now we state the necessary condition for the optimal control on the infinite interval.

<u>Theorem 3.5.8</u>: Let $\|\Phi\| < 1$. If $u^\infty \in \ell^2(0,\infty;U)$ is the optimal control for the discrete-time problem (DTP),

defined in Section 3.2, with optimal response $x^{\infty} \in \ell^2(0,\infty;H)$, then there necessarily exists a unique adjoint state $p^{\infty} \in \ell^2(0,\infty;H)$ such that

$$u_i^{\infty} = R_i^{-1}P_i x_{d_i} - R_i^{-1}M_i^* x_i^{\infty} - R_i^{-1}D_i^* p_{i+1}^{\infty}, \tag{3.5.19}$$

$$p_i^{\infty} = \Phi^* p_{i+1}^{\infty} + Q_i x_i^{\infty} - S_i x_{d_i} + M_i u_i^{\infty} \tag{3.5.20}$$

where the optimal response $x^{\infty} \in \ell^2(0,\infty;H)$ satisfies

$$x_{i+1}^{\infty} = \Phi x_i^{\infty} + D_i u_i^{\infty}; \quad x_0^{\infty} = x_0 . \tag{3.5.21}$$

<u>Proof</u>: Proof is identical to that of Theorem 3.3.8 hence omitted here.

<u>Theorem 3.5.9</u>: Let $\|\Phi\| < 1$. The optimal control $u^{\infty} \in \ell^2(0,\infty;U)$ for the discrete-time problem (DTP) on the infinite interval is given by

$$\begin{aligned} u_i^{\infty} = &-[R_i^{-1}M_i^* + R_i^{-1}D_i^*K_{i+1}^{\infty}(I + D_iR_i^{-1}D_i^*K_{i+1}^{\infty})^{-1}\Theta_i]x_i^{\infty} \\ &-[R_i^{-1}D_i^* - R_i^{-1}D_i^*K_{i+1}^{\infty}(I + D_iR_i^{-1}D_i^*K_{i+1}^{\infty})^{-1}D_iR_i^{-1}D_i^*]g_{i+1} \\ &+[R_i^{-1}P_i - R_i^{-1}D_i^*K_{i+1}^{\infty}(I + D_iR_i^{-1}D_i^*K_{i+1}^{\infty})^{-1}D_iR_i^{-1}P_i]x_{d_i}, \end{aligned} \tag{3.5.22}$$

where for all $i \in \sigma$, K_i^{∞} is the bounded, positive semi-definite, self-adjoint operator, satisfying

$$K_i^{\infty} = \Theta_i^* K_{i+1}^{\infty}(I + D_iR_i^{-1}D_i^*K_{i+1}^{\infty})^{-1}\Theta_i + \Gamma_i, \tag{3.5.23}$$

and $g_i^{\infty} \in \ell^2(0,\infty;H)$ is the solution of

$$g_i^\infty = [\Theta_i^* - \Theta_i^* K_{i+1}^\infty (I + D_i R_i^{-1} D_i^* K_{i+1}^\infty)^{-1} D_i R_i^{-1} D_i^*] g_{i+1}$$

(3.5.24)

$$+[\Theta_i^* K_{i+1}(I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} D_i R_i^{-1} P_i + M_i R_i^{-1} P_i - S_i] x_{d_i} .$$

Moreover, the optimal cost on the interval $[s,\infty)$ is given by

(3.5.25) $$J = \langle K_s^\infty x_s^\infty, x_s^\infty \rangle + 2\langle g_s^\infty, x_s^\infty \rangle + \varphi_s^\infty ,$$

where $\varphi_i^\infty \in \ell^2(0,\infty;H)$ is the solution of

$$\varphi_i^\infty = \varphi_{i+1}^\infty - \langle P_i x_{d_i}, R_i^{-1} P_i x_{d_i} \rangle + \langle D_i R_i^{-1} P_i x_{d_i}, H_i^\infty D_i R_i^{-1} P_i x_{d_i} \rangle$$

$$- 2\langle g_{i+1}, D_i R_i^{-1} D_i^* H_i^\infty D_i R_i^{-1} P_i x_{d_i} \rangle + 2\langle g_{i+1}, D_i R_i^{-1} D_i^* x_{d_i} \rangle$$

$$+ \langle g_{i+1}, D_i R_i^{-1} D_i^* H_i^\infty D_i R_i^{-1} D_i^* g_{i+1} \rangle - \langle g_{i+1}, D_i R_i^{-1} D_i^* g_{i+1} \rangle ,$$

with $H_i^\infty = K_{i+1}^\infty (I + D_i R_i^{-1} D_i^* K_{i+1}^\infty)^{-1}$.

<u>Proof</u>: If s is any fixed integer in $[0,\infty)$ and $x_s = h \in H$, then, using the same arguments in the proofs of Lemma 3.4.2 and Corollary 3.4.3, it can be shown that the transformation $h \to p_s$ is continuous from H into H so that we have

(3.5.23) $$p_s^\infty = K_s^\infty h + g_s^\infty ,$$

or, since $s \in [0,\infty)$ is arbitrary, we have

(3.5.24) $$p_i^\infty = K_i^\infty x_i^\infty + g_i^\infty \quad \forall \ i \in [0,\infty) ,$$

where (x_i^∞, p_i^∞) is a solution pair of (3.5.20) and (3.5.21).

Thus, the same arguments in the proofs of Theorem 3.4.4, Theorem 3.4.5 and Theorem 3.4.6 follows to complete the theorem. Q.E.D.

Next we consider the time invariant problem, i.e., we assume that $D_i = D$, $M_i = M$, $R_i = R$, $S_i = S$, $P_i = P$, and $E_i = E$.

Theorem 3.5.10: If the system (3.2.4) and the cost functional (3.2.5) are time invariant, then the feedback operator is also time invariant, i.e., $K_i^\infty = K^\infty$ for all $i \in \sigma$, and K^∞ is the solution of the algebraic operator equation

$$K^\infty = \Theta^* K^\infty (I + DR^{-1}D^* K^\infty)^{-1}\Theta + \Gamma . \tag{3.5.25}$$

Proof: Let $x_d = 0$. Since the system (3.5.20) and (3.5.21) become autonomous, it is independent of initial time s in the sense that if we translate the origin such that $i \to i-s$ then $p_s^\infty = p_o^\infty$. Hence from (3.5.24) with $g_s^\infty = 0$ we conclude that $K_s^\infty = K_o^\infty$. Q.E.D.

Thus we have completed the study of DTP on the infinite time interval. The important results remaining is the convergence properties as $N \to \infty$.

We denote u^N and u^∞ be the optimal controls on the intervals $[0,N)$ and $[0,\infty)$ respectively.

Theorem 3.5.11: Let $\|\Phi\| < 1$. Let $\tilde{u}^N (\tilde{x}^N, \tilde{p}^N$ resp.) be the extension of $u^N (x^N, p^N$ resp.) on $[0,\infty)$ by 0 outside $[0,N)$. Then as $N \to \infty$,

(3.5.26) $\quad \tilde{u}^N \rightarrow u^\infty \quad$ weakly in $\ell^2(0,\infty;U)$,

(3.5.27) $\quad \tilde{x}^N \rightarrow x^\infty \quad$ weakly in $\ell^2(0,\infty;H)$,

(3.5.28) $\quad \tilde{p}^N \rightarrow p^\infty \quad$ weakly in $\ell^2(0,\infty;H)$,

(3.5.29) $\quad K_s^N h \rightarrow K_s^\infty h \quad$ weakly in H, $\forall\, s \in \sigma$, $\forall\, h \in H$.

__Proof:__ Let $j_N = \inf J_N(v)$, $j_\infty = \inf J_\infty(v)$. For $v \in \ell^2(0,\infty;U)$ we have, $J_N(v) \leq J_\infty(v)$ and hence $j_N \leq j_\infty$. Thus $j_N = J_N(u^N) \geq C \sum_{i=0}^{N-1} \|u_i^N\|^2$ and if $\tilde{u}^N$ is defined as in the statement of the theorem, we have

(3.5.30) $$\|\tilde{u}^N\|_{\ell^2(0,\infty;U)} \leq C .^*$$

But then due to Remark 3.5.4 and (3.5.18),

(3.5.31) $$\|\tilde{x}^N\|_{\ell^2(0,\infty;H)} \leq C ,$$

(3.5.32) $$\|\tilde{p}^N\|_{\ell^2(0,\infty;H)} \leq C ,$$

and again by virtue of (3.5.31),

(3.5.33) $$\|x_N^N\| \leq C .$$

Hence we have from (3.4.3)

(3.5.34) $$\tilde{p}_i^N = \Theta_i^* \tilde{p}_{i+1}^N + \Gamma_i \tilde{x}_i^N + \tilde{e}_i^N - (\Gamma_N x_N^N + e_N)\,\delta(i-N) ,$$

(3.5.35) $$\tilde{x}_{i+1}^N = \Theta_i \tilde{x}_i^N - D_i R_i^{-1} D_i^* \tilde{p}_{i+1}^N + \tilde{f}_i^N - (\Theta_N x_N^N + f_N)\,\delta(i-N) ,$$

where $\delta(0) = 1$, and $\delta(r) = 0$ for $r \neq 0$, and $\tilde{e}_i^N$ and

---

*The C's denoting constants independent of N.

$\tilde{f}_i^N$ are extensions of $e_i = -S_i x_{d_i} + M_i R_i^{-1} P_i x_{d_i}$ and $f_i = D_i R_i^{-1} P_i x_{d_i}$ on $[0,N]$ respectively.

We may then find a sequence $N_n \to \infty$ such that

$$\tilde{u}^{N_n} \to \bar{u} \qquad \text{weakly in } \ell^2(0,\infty;U)\ ,$$

$$\tilde{x}^{N_n} \to \bar{x}\ ,\ \tilde{p}^{N_n} \to \bar{p} \quad \text{weakly in } \ell^2(0,\infty;H)\ .$$

Thus (3.5.34) and (3.5.35) become

$$\bar{p}_i = \Theta_i^* \bar{p}_{i+1} + \Gamma_i \bar{x}_i + e_i \tag{3.5.36}$$

$$\bar{x}_{i+1} = \Theta_i \bar{x}_i - D_i R_i^{-1} D_i^* \bar{p}_{i+1} + f_i \tag{3.5.37}$$

By comparing the above with (3.5.19), (3.5.20) and (3.5.21), we deduce that $\bar{p} = p^\infty$ and $\bar{x} = x^\infty$. The relation

$$u_i^N = -R_i^{-1} D_i^* p_{i+1}^N - R_i^{-1} M_i^* x_i^N + R_i P_i x_{d_i} \tag{3.5.38}$$

gives us in the limit (if necessary, take a subsequence),

$$\bar{u}_i = -R_i^{-1} D_i \bar{p}_{i+1} - R_i^{-1} M_i^* \bar{x}_i + R_i P_i x_{d_i} \tag{3.5.39}$$

hence we have, by comparing with (3.5.19), that $\bar{u} = u^\infty$. Thus we have proved (3.5.26), (3.5.27) and (3.5.28).

To prove (3.5.29), we observe that $K_s^N h$ is defined by

$$\begin{aligned} x_{i+1}^N &= \Theta_i x_i^N - D_i R_i^{-1} D_i^* p_{i+1}^N && \text{in } [s,N)\ , \\ p_i^N &= \Theta_i^* p_{i+1}^N + \Gamma_i x_i^N && \text{in } [s,N)\ , \\ x_s^N &= h\ , \quad p_N^N = 0\ , \end{aligned} \tag{3.5.40}$$

and then

$$K_s^N h = p_s^N .$$

But (x^N, p^N) corresponds to the optimal control of a system whose state is given by

$$x_{i+1} = \Phi x_i + D_i u_i \quad \text{in} \quad [s,N), \quad x_s = h ,$$

and whose cost is given by

$$J^h_{s,N}(u) = \sum_{i=s}^{N-1} [\langle x_i, Q_i x_i \rangle + 2\langle x_i, M_i u_i \rangle + \langle u_i, R_i u_i \rangle]$$

By (3.4.22), we have

$$\inf J^h_{s,N}(u) = \langle K_s^N h, h \rangle \le J^h_{s,N}(0) \le C\|h\|^2 ,$$

hence

(3.5.41) $\quad \|K_s^N h\| \le C\|h\| , \quad C =$ constant independent of s and N.

Now if w^N is optimal control of this problem, we obtain from (3.5.30)

$$\sum_{i=s}^{N-1} \|w_i^N\|^2 \le C ,$$

and extending w^N by 0 for $i \ge N$, we deduce that $\tilde{w}^N$ (resp. $\tilde{x}^N$, $\tilde{p}^N$) ranges in a bounded set of $\ell^2(0,\infty;U)$ (resp. $\ell^2(0,\infty;H)$). We may then find a sequence $N_n \to \infty$ such that $\tilde{w}^{N_n} \to w$, $\tilde{x}^{N_n} \to \bar{x}$, $\tilde{p}^{N_n} \to \bar{p}$ in the corresponding weak topologies and hence satisfies (3.5.19), (3.5.20) and (3.5.21). Hence $\bar{x} = x^\infty$, $\bar{p} = p^\infty$ and $\bar{p} = p^\infty$ and $p_s^{N_n} \to p_s^\infty$ weakly in H, completing the proof.

Q.E.D.

3.6 THE RICCATI INTEGRO-DIFFERENCE EQUATION

The optimal feedback operator K_i is found to be the solution of the Riccati operator difference equation. But there are no straightforward procedures for solving operator equations directly. In this section we derive an equation from the Riccati operator equation which can be solved analytically or numerically. This will be done by showing that K_i can be represented by an integral operator, and thus an integro-difference equation will then be derived for the kernel of this integral operator. Throughout this section we choose H to be $L^2(D)$.

<u>Theorem 3.6.1</u>: The optimal feedback operator $K_i \in \mathcal{L}(L^2(D);L^2(D))$ has a unique kernel $K_i(z,\zeta)$ such that

(3.6.1) $$K_i x = \int_D K_i(z,\zeta)x(\zeta)d\zeta \quad \forall\, x \in L^2(D) .$$

To prove this we need the following theorem, so-called Schwartz Kernel Theorem.

<u>Theorem 3.6.2</u> (Schwartz Kernel Theorem, [S-1]): If H_1 and H_2 are locally convex spaces and T is a continuous linear operators from H_1 into H_2, and if the following are true:

(i) $C_o^\infty(D) \subset H_i \subset H_i' \subset \mathcal{D}(D)$, $i = 1,2$,

(ii) $C_o^\infty(D)$ is dense in $H_1 \cap H_2$,

then L can be represented by a unique integral operator whose kernel $L(z,\zeta)$ is a distribution on $D \times D$.

<u>Proof of Theorem 3.6.1</u>: For any $i \in \sigma$, K_i is bounded linear operator from $L^2(D)$ into itself, implying

that K_i is continuous. By (2.1.1), $C_o^\infty(D) \subset L^2(D) \subset \mathcal{D}(D)$, and $C_o^\infty(D)$ is dense in $L^2(D)$. Thus by Theorem 3.6.2 there exists a unique kernel $K_i(z,\zeta)$ satisfying (3.6.1) for all $x \in C_o^\infty(D)$, and since $C_o^\infty(D)$ is dense in $L^2(D)$, (3.6.1) holds for all $x \in L^2(D)$ with the limiting arguments.

Q.E.D.

Before we derive the Riccati integro-difference equation we define operators $L_i, H_i \in \mathcal{L}(H;H)$:

(3.6.2) $$L_i = D_i R_i^{-1} D_i^* ,$$

(3.6.3) $$H_i = K_{i+1}(I + L_i K_{i+1})^{-1} .$$

<u>Theorem 3.6.3</u>: The kernel $K_i(z,\zeta)$ corresponding to the optimal feedback operator $K_i \in \mathcal{L}(L^2(D), L^2(D))$ characterized in Theorem 3.4.4, by (3.4.11), satisfies the integro-difference equation

(3.6.4) $$K_i(z,\zeta) = \Theta_{i,z}^* \Theta_{i,\zeta}^* H_i(z,\zeta) + \Gamma_i(z,\zeta) ,$$

(3.6.5) $$K_{i+1}(z,\zeta) = H_i(z,\zeta) + \int_D \int_D H_i(z,\rho) L_i(\rho,\sigma) K_{i+1}(\sigma,\zeta)\, d\sigma\, d\rho ,$$

(3.6.6) $$K_N(z,\zeta) = F(z,\zeta) ,$$

where $L_i(z,\sigma)$, $H_i(z,\rho)$, $\Gamma_i(z,\zeta)$, $F(z,\zeta)$ and $K_i(z,\zeta)$ are the symmetric kernels corresponding to operators L_i, H_i, Γ_i, F and K_i respectively, and $\Theta_{i,z}^*$ is the operation of Θ_i^* on the argument z.

<u>Proof</u>: Since all the operators L_i, H_i, Γ_i, F and K are bounded on $L^2(D)$, by Theorem 3.6.1, they have

corresponding integral representation whose kernels are distributions on $D \times D$. The Riccati equation for $x \in L^2(D)$ is from (3.4.11)

$$K_i x = \Theta_i^* K_{i+1} (I + D_i R_i^{-1} D_i^* K_{i+1})^{-1} \Theta_i x + \Gamma_i x \; ; \; K_N x = F x , \tag{3.6.7}$$

or equivalently, combining with (3.6.2) and (3.6.3), is a system of equations

$$\begin{aligned} K_i x &= \Theta_i^* H_i \Theta_i x + \Gamma_i x , \\ K_{i+1} x &= H_i x + H_i L_i K_{i+1} x , \\ K_N x &= F x . \end{aligned} \tag{3.6.8}$$

Note that, since the kernel $H_i(z,\zeta)$ is a distribution,

$$\begin{aligned} H_i \Theta_i x &= \int_D H_i(z,\zeta)\, \Theta_{i,\zeta}\, x(\zeta)\, d\zeta = \langle H_i(z,\zeta), \Theta_{i,\zeta}\, x(\zeta)\rangle_{L^2(D)} \\ &= \langle \Theta_{i,\zeta}^* H_i(z,\zeta), x(\zeta)\rangle_{L^2(D)} = \int_D \Theta_{i,\zeta}^* H_i(z,\zeta) x(\zeta)\, d\zeta . \end{aligned}$$

Thus, applying the integral representations to (3.6.8), we obtain

$$\begin{aligned} \int_D K_i(z,\zeta) x(\zeta)\, d\zeta &= \int_D \Theta_{i,z}^* \Theta_{i,\zeta}^* H_i(z,\zeta) x(\zeta)\, d\zeta + \int_D \Gamma_i(z,\zeta) x(\zeta)\, d\zeta , \\ \int_D K_{i+1}(z,\zeta) x(\zeta)\, d\zeta &= \int_D \int_D \int_D H_i(z,\rho) L_i(\rho,\sigma) K_{i+1}(\sigma,\zeta) x(\zeta)\, d\zeta\, d\sigma\, d\rho \\ &\quad + \int_D H_i(z,\zeta)\, x(\zeta)\, d\zeta , \\ \int_D K_N(z,\zeta) x(\zeta)\, d\zeta &= \int_D F(z,\zeta) x(\zeta)\, d\zeta . \end{aligned} \tag{3.6.9}$$

Since (3.6.9) holds for all $x \in L^2(D)$, we deduce (3.6.4), (3.6.5) and (3.6.6). The symmetry of kernels comes from the self-adjointness of operators. Q.E.D.

Boundary conditions can be specified for the Riccati integro-difference equation (3.6.4) - (3.6.6). Let y be the value of the transformation (3.6.1), then the evaluation of y at $z \in \partial D$, the boundary, is

$$y(z)\Big|_{z\in\partial D} = \int_D K_i(z,\zeta)\Big|_{z\in\partial D} x(\zeta)\,d\zeta .$$

Thus, the Dirichlet boundary condition, i.e.,

$\left.\frac{\partial^k y}{\partial n^k}\right|_{\partial D} = 0$, $0 \le k \le m-1$, gives us a condition for $K_i(z,\zeta)$:

$$K_i(z,\zeta)\Big|_{z\in\partial D} = \frac{\partial}{\partial n} K_i(z,\zeta)\Big|_{z\in\partial D} = \cdots = \frac{\partial^{m-1}}{\partial n^{m-1}} K_i(z,\zeta)\Big|_{z\in\partial D} = 0,$$

where n is the outward normal at the boundary ∂D. Moreover, by the symmetry of $K_i(z,\zeta)$, the above boundary condition also holds for $\zeta \in \partial D$, i.e.,

$$K_i(z,\zeta)\Big|_{\zeta\in\partial D} = \frac{\partial}{\partial n} K_i(z,\zeta)\Big|_{\zeta\in\partial D} = \cdots = \frac{\partial^{m-1}}{\partial n^{m-1}} K_i(z,\zeta)\Big|_{\zeta\in\partial D} = 0.$$

With the similar arguments in the derivation of the Riccati integro-difference equation, we can also derive an integro-difference equation for g_i characterized in Theorem 3.4.4, by (3.4.12), i.e.,

$$g_i(z) = \Theta^*_{i,z} g_{i+1}(z) - \int_D\int_D \Theta^*_{i,z} H_i(z,\zeta) L_i(\zeta,\sigma) g_i(\sigma)\,d\sigma\,d\zeta$$
$$+ \int_D\int_D \Theta^*_{i,z} H_i(z,\zeta) G_i(\zeta,\sigma) x_{d_i}(\sigma)\,d\sigma\,d\zeta$$

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(3.6.10)

$$+ \int_D W_i(z,\zeta)x_{d_i}(\zeta)d\zeta\,,$$

$$g_N(z) = -\int_D F(z,\zeta)x_{d_N}(\zeta)d\zeta\,,$$

where $G_i(\zeta,\sigma)$ and $W_i(z,\zeta)$ are the corresponding kernels of bounded operators $G_i = D_i R_i^{-1} P_i$, and $W_i = M_i R_i^{-1} P_i - S_i$, respectively.

In sequel we may also represent the optimal cost in terms of the feedback kernel. Recall that the optimal cost is given by

$$J = \langle K_o x_o, x_o \rangle_{L^2(D)} + 2\langle g_o, x_o \rangle_{L^2(D)} + \varphi_o\,, \tag{3.6.11}$$

where φ_i is the solution of (3.4.23). Using the integral operator representation for K_i, and evaluating the inner products in $L^2(D)$, we obtain

$$J = \int_D \int_D K_o(z,\zeta)x_o(\zeta)x_o(z)d\zeta\,dz$$

(3.6.12)

$$+ 2\int_D g_o(z)x_o(z)dz + \varphi_o\,,$$

where φ_i satisfies the difference equation

$$\varphi_i(z) = \varphi_{i+1}(z) - \int_D \int_D Y_i(z,\zeta)x_{d_i}(\zeta)x_{d_i}(z)d\zeta\,dz$$

$$+ \int_D \int_D \int_D \int_D G_i^*(z,\rho)H_i(\rho,\sigma)G_i(\sigma,\zeta)x_{d_i}(\zeta)x_{d_i}(z)d\zeta\,d\sigma\,d\rho\,dz$$

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$$-2 \int_D \int_D \int_D \int_D L_i(z,\rho) H_i(\rho,\sigma) G_i(\sigma,\zeta) x_{d_i}(\zeta) g_{i+1}(z)\, d\zeta\, d\sigma\, d\rho dz$$

$$+2 \int_D \int_D L_i(z,\zeta) x_{d_i}(\zeta) g_{i+1}(z)\, d\zeta\, dz$$

$$+ \int_D \int_D \int_D \int_D L_i(z,\zeta) H_i(\rho,\sigma) L_i(\sigma,\zeta) g_{i+1}(\zeta) g_{i+1}(z)\, d\zeta\, d\sigma\, d\rho\, dz$$

$$- \int_D \int_D L_i(z,\zeta) g_{i+1}(\zeta) g_{i+1}(z)\, d\zeta\, dz\,,$$

$$\varphi_N(z) = \int_D \int_D F(z,\zeta) x_{d_N}(\zeta) x_{d_N}(z)\, d\zeta\, dz\,,$$

with $Y_i(z,\zeta)$ to be the corresponding kernel to the bounded operator $Y_i = P_i^* R_i^{-1} P_i$.

The results obtained in this section can also be true for the problem on the infinite time interval. Moreover, if the problem (i.e., the state equation and the cost functional) is time invariant, then we have the following set of equations

$$K(z,\zeta) = \Theta_z^* \Theta_\zeta^* H(z,\zeta) + \Gamma(z,\zeta)\,, \tag{3.6.13}$$

$$K(z,\zeta) = H(z,\rho) + \int_D \int_D H(z,\rho) L(\rho,\sigma) K(\sigma,\zeta)\, d\sigma\, d\rho\,. \tag{3.6.14}$$

CHAPTER IV

SAMPLED-DATA POINTWISE CONTROL

In this chapter the results obtained in Chapter III are specialized to the pointwise control problem, where the control is applied only at a finite number of points in the spatial domain. The pointwise control problem (PCP) is defined in Section 4.1. In Section 4.2 the optimal pointwise feedback control is derived, and it is shown to be computationally simpler than the general feedback form of Chapter III. Section 4.3 deals with the approximation of PCP, where a suitable choice of state weighting operator results in the finite dimensional approximation by an eigenfunction expansion. An example for PCP is considered in Section 4.4.

4.1 POINTWISE CONTROL PROBLEM (PCP)

In this section we shall formulate the pointwise control problem, where controls are applied only at a fixed number of points within the spatial domain of the system.

We suppose that control is concentrated on some finite number of spatial domains in D, say, $E_1, \cdots, E_M$, and the control is respectively constant with respect to z in each domain. Here, we introduce the characteristic function $\chi^k(z)$ to each E_k defined as

$$(4.1.1) \qquad \chi^k(z) = \begin{cases} 1, & z \in E_k , \\ 0, & z \notin E_k , \end{cases} \quad k = 1, 2, \cdots, M .$$

When the measure of E_k approaches to zero (in the sense of Lebesgue), we obtain the ideal case such that

$$(4.1.2) \qquad \chi^k(z) = \delta(z-z^k), \quad z^k \in D, \quad k = 1,2,\cdots, M .$$

where $\delta(z)$ is the Dirac's delta function in R^n. Note that the control space U to be considered is M-dimensional Euclidean space R^M, and the control is M-vector $\underline{u}$.

<u>Definition 4.1.1</u>: The pointwise control operator D_i^o, defined for all $i \in \sigma$ on R^M, is as following:

$$(4.1.3) \qquad D_i^o \, \underline{u}_i = \sum_{k=1}^{M} \chi^k(z) d_i^k \, u_i^k \quad \forall \; \underline{u}_i \in R^M ,$$

where $\chi^k(z)$ is given by (4.1.1), and for all $i \in \sigma$, $d_i^k \in R$.

Strictly speaking the operator defined by (4.1.3) should be called as a spacially concentrated control operator, but since later on we will approximate the results obtained to the pointwise case, we prefer to call it as pointwise control operator.

In order to apply the results of Section 3.6 we require that D_i^o to be a bounded linear operator from R^M into $L^2(D)$.

<u>Lemma 4.1.2</u>: For each $i \in \sigma$, D_i^o is a bounded linear operator from R^M into $L^2(D)$.

<u>Proof</u>:

$$\int_D (D_i^o \, \underline{u}_i)^2 dz = \int_D \left(\sum_{k=1}^{M} \chi^k(z) d_i^k \, u_i^k\right)^2 dz$$

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$$= \sum_{k=1}^{M} \int_D \chi^k(z)(d_i^k\, u_i^k)^2 dz = \sum_{k=1}^{M} (d_i^k\, u_i^k)^2 \int_D \chi^k(z)\,dz$$

Since $\int_D \chi^k(z)\,dz = \mu(E_k)$, the Lebesgue measure of E_k, and since this is less than $\mu(D)$, we have

$$\int_D (D_i^o\, \underline{u}_i)^2 dz \le \mu(D) \sum_{k=1}^{M} (b_i^k\, u_i^k)^2 = \mu(D)\ \|\underline{D}_i \underline{u}_i\|^2_{R^M} ,$$

where $\underline{D}_i$ is the $M \times M$ diagonal matrix with entries $D_i^{kk} = d_i^k$, $k = 1,2,\cdots, M$. If $\|\underline{D}_i\|_{R^M}$ is the induced matrix norm of $\underline{D}_i$, it follows that D_i^o is a bounded linear operator from R^M into $L^2(D)$. Q.E.D.

<u>Remark 4.1.3</u>: If the pointwise control operator D_i^o is defined in terms of delta function (4.1.2) instead of the characteristic function (4.1.1), then $D_i^o\, \underline{u}_i \notin L^2(D)$ since the delta function is not square integrable, and thus the theory of Section 3.6 is not applicable.

Since the control space is R^M, it remains to revise the cost functional in this framework. Furthermore, for the attraction of simplified results we formulate a simpler cost functional than the original one, (3.2.5), in DTP, i.e., the cost functional for the pointwise control is defined by

$$(4.1.4) \qquad J = \sum_{i=0}^{N-1} [\langle x_i, Q_i x_i\rangle + \langle \underline{u}_i, \underline{R}_i \underline{u}_i\rangle] + \langle x_N, F\, x_N\rangle ,$$

where $\underline{R}_i$ is a symmetric, positive definite $M \times M$ matrix for all $i \in \sigma$, and, Q_i and F are as defined in Section 3.2.

Now we define the pointwise control problem (PCP):

Pointwise Control Problem (PCP): Given the discrete-time system (3.2.4) with $D_i = D_i^o$, defined in (4.1.3), and the cost functional (4.1.4). Find a sequence of controls $\underline{u}^* = \{\underline{u}_i^* \in R^M, i \in \sigma\}$ such that for all $\underline{u} = \{\underline{u}_i \in R^M, i \in \sigma\}$

$$J(\underline{u}^*) = \inf_{\underline{u}} J(\underline{u}) .$$

4.2 THE SOLUTION OF PCP

In this section the pointwise control problem (PCP) is solved by applying the results of Section 3.6, that is the feedback integral operator and the Riccati integro-difference equation. If we apply the results of Chapter III to the pointwise control problem (PCP), which has the control space $U = R^M$ and the pointwise control operator D_i^o defined in (4.1.3), then the optimal control is given by

$$\underline{u}_i^* = -\underline{R}_i^{-1} D_i^{o*} K_{i+1} (I + D_i^o \underline{R}_i^{-1} D_i^{o*} K_{i+1})^{-1} \Phi\, x_i , \tag{4.2.1}$$

where K_i is the feedback integral operator whose kernel $K_i(z,\zeta)$ satisfies the Riccati integro-difference equation

$$\begin{aligned} K_i(z,\zeta) &= \Phi_z^* \Phi_\zeta^* H_i(z,\zeta) + Q_i(z,\zeta) , \\ K_{i+1}(z,\zeta) &= H_i(z,\zeta) + \int_D \int_D H_i(z,\rho) L_i^o(\rho,\sigma) K_{i+1}(\sigma,\zeta)\, d\sigma d\rho , \\ K_N(z,\zeta) &= F(z,\zeta) , \end{aligned} \tag{4.2.2}$$

where $L_i^o(\rho,\sigma)$ is the kernel of the operator $L_i^o = D_i^o \underline{R}_i^{-1} D_i^{o*}$.

We will now simplify the above equation and hence the optimal control (4.2.1). The adjoint pointwise control operator $D_i^{o^*} \in \mathcal{L}(L^2(D); R^M)$ is obtained as following: If $y \in L^2(D)$ and $\underline{u}_i \in R^M$, then

$$
\begin{aligned}
\langle D_i^{o^*} y, u_i\rangle_{R^M} &= \langle y, D_i^o \underline{u}_i\rangle_{L^2(D)} \\
&= \int_D y(z) \sum_{k=1}^{M} \chi^k(z)\, d_i^k u_i^k \, dz \\
&= \sum_{k=1}^{M} \Big[\int_D y(z)\ \chi^k(z)\, dz\Big] d_i^k\ u_i^k ,
\end{aligned}
$$

thus $D_i^{o^*} y$ is considered to be a vector in R^M such that

$$
D_i^{o^*} y = \left[\ d_i^k \int_D \chi^k(z)\, y(z)\, dz\ \right] , \tag{4.2.3}
$$

where the bracket denotes a M-dimensional column vector. The kernel $L_i^o(\rho,\sigma)$ of the operator $L_i^o = D_i^o \underline{R}_i^{-1} D_i^{o^*}$ is then obtained as following: Using (4.2.3) we deduce

$$
\begin{aligned}
(L_i^o y)(\rho) &= D_i^o \underline{R}_i^{-1} \left[\ d_i^k \int_D \chi^k(\sigma)\, y(\sigma)\, d\sigma\ \right] \\
&= D_i^o \left[\ \sum_{k=1}^{M} R_i^{-1}(j,k)\, d_i^k \int_D \chi^k(\sigma)\, y(\sigma)\, d\sigma\ \right] \\
&= \sum_{j=1}^{M} \chi^j(\rho)\, d_i^j \sum_{k=1}^{M} R_i^{-1}(j,k)\, d_i^k \int_D \chi^k(\sigma)\, y(\sigma)\, d\sigma \\
&= \int_D \Big[\sum_{j=1}^{M} \sum_{k=1}^{M} \chi^j(\rho)\, d_i^j R_i^{-1}(j,k)\, d_i^k\ \chi^k(\sigma) \Big] y(\sigma)\, d\sigma ,
\end{aligned}
$$

where $R_i^{-1}(j,k)$ is the jk^{th} entry of matrix $\underline{R}_i^{-1}$. Thus the kernel of $L_i^o y$ is given by

$$(4.2.4) \qquad L_i^o(\rho,\sigma) = \sum_{j=1}^{M} \sum_{k=1}^{M} \chi^j(\rho) d_i^j R_i^{-1}(j,k) d_i^k \chi^k(\sigma) .$$

The double integration term in (4.2.2) may now be written as

$$\int_D \int_D H_i(z,\rho) L_i^o(\rho,\sigma) K_{i+1}(\sigma,\zeta) d\sigma \, d\rho$$

(4.2.5)

$$= \int_D \int_D H_i(z,\rho) \sum_{j=1}^{M} \sum_{k=1}^{M} \chi^j(\rho) d_i^j R_i^{-1}(j,k) d_i^k \chi^k(\sigma) K_{i+1}(\sigma,\zeta) d\sigma \, d\rho$$

$$= \sum_{j=1}^{M} \sum_{k=1}^{M} [d_i^j \int_D \chi^j(\rho) H_i(z,\rho) d\rho] R_i^{-1}(j,k) [d_i^k \int_D \chi^k(\sigma) K_{i+1}(\sigma,\zeta) d\sigma] ,$$

and if we define vector functions $\underline{h}_i(z)$ and $\underline{k}_{i+1}(z)$ to be

(4.2.6)

$$\underline{h}_i(z) = \begin{bmatrix} \uparrow \\ d_i^k \int_D \chi^k(\rho) H_i(z,\rho) d\rho \\ \downarrow \end{bmatrix} , \underline{k}_{i+1}(z) = \begin{bmatrix} \uparrow \\ d_i^k \int_D \chi^k(\rho) K_{i+1}(z,\rho) d\rho \\ \downarrow \end{bmatrix} ,$$

then (4.2.5) becomes $\underline{h}_i^T(z) \underline{R}_i^{-1} \underline{k}_{i+1}(\zeta)$, where T denotes the transpose. Thus the Riccati integro-difference (4.2.2) becomes

$$K_i(z,\zeta) = \Phi_z^* \Phi_\zeta^* H_i(z,\zeta) + Q_i(z,\zeta) ,$$

$$(4.2.7) \qquad K_{i+1}(z,\zeta) = H_i(z,\zeta) + \underline{h}_i^T(z) \underline{R}_i^{-1} \underline{k}_{i+1}(\zeta) ,$$

$$K_N(z,\zeta) = F(z,\zeta) ,$$

and the optimal pointwise control (4.2.1) becomes

$$
\begin{aligned}
\underline{u}_i^* &= -\underline{R}_i^{-1} D_i^{o^*} H_i \; \Phi \; x_i \\
&= -\underline{R}_i^{-1} D_i^{o^*} \int_D H_i(z,\zeta)\, \Phi_\zeta x_i(\zeta)\, d\zeta \\
&= -\underline{R}_i^{-1} D_i^{o^*} \int_D \Phi_\zeta^* H_i(z,\zeta)\, x_i(\zeta)\, d\zeta \\
&= -\underline{R}_i^{-1} \left[d_i^k \int_D \chi^k(z) \int_D \Phi_\zeta^* \updownarrow H_i(z,\zeta)\, x_i(\zeta)\, d\zeta\, dz \right] \\
(4.2.8) \qquad &= -\underline{R}_i^{-1} \left[\int_D \Phi_\zeta^* (d_i^k \int_D \chi^k(z) \updownarrow H_i(z,\zeta)\, dz)\, x_i(\zeta)\, d\zeta \right] \\
&= -\underline{R}_i^{-1} \int_D \Phi_\zeta^* \underline{h}_i(\zeta)\, x_i(\zeta)\, d\zeta \,.
\end{aligned}
$$

So far, because of the difficulty stated in Remark 4.1.3, i.e., $D_i^o\, \underline{u}_i \not\in L^2(D)$, we have been forced to use the characteristic function (4.1.1) rather than the delta function (4.1.2) in order to apply the theory of Chapter III. Thus, as a result, we have obtained the equation (4.2.7) for which we know a solution exists. Since we are interested in the pointwise control (even though it is impossible to apply ideal point source), we may replace the characteristic function (4.1.1) with the delta function (4.1.2) in order to solve the equation (4.2.7) approximately (cf.[P-4], [M-1]). Thus, if we substitute (4.1.2) into (4.2.6) we obtain

$$
(4.2.9) \qquad \underline{h}_i(z) = \left[d_i^k \; \updownarrow H_i(z,z^k) \right], \; \underline{k}_{i+1}(z) = \left[d_i^k \; \updownarrow K_{i+1}(z,z^k) \right] ,
$$

and the optimal control (4.2.8) becomes

$$
\begin{aligned}
\underline{u}_i^* &= -\underline{R}_i^{-1} \int_D \Phi_\zeta^* \, \underline{h}_i(\zeta) x_i(\zeta) d\zeta \\
(4.2.10) \qquad &= -\underline{R}_i^{-1} \begin{bmatrix} \int_D \Phi_\zeta^* \, d_i^k \, H_i(\zeta, z^k) x_i(\zeta) d\zeta \end{bmatrix} \\
&= -\underline{R}_i^{-1} \underline{D}_i \int_D \hat{\underline{h}}_i(\zeta) x_i(\zeta) d\zeta \\
&= -\underline{R}_i^{-1} \underline{D}_i \, \underline{y}_i \, ,
\end{aligned}
$$

where $\underline{D}_i$ is $M \times M$ diagonal matrix with entries $D_i^{kk} = d_i^k$, the k^{th} control coefficient in (4.1.3), and $\hat{\underline{h}}_i$ and $\underline{y}_i$ may be called the <u>weighting</u> and <u>output</u> M-vectors of the averaging device respectively such that

$$
(4.2.11) \qquad \underline{y}_i = \int_D \hat{\underline{h}}_i(\zeta) x_i(\zeta) d\zeta \, , \quad \hat{\underline{h}}_i(\zeta) = \begin{bmatrix} \Phi_\zeta^* \, H_i(\zeta, z^k) \end{bmatrix} .
$$

Note that it is necessary to determine the M functions $H_i(z, z^k)$, $k = 1, 2, \cdots, M$, in order to completely specify the optimal feedback control. In the computational point of view this is certainly simpler than the computation of the entire kernel, i.e., the computation of $H_i(z, \zeta)$ for all $(z, \zeta) \in D \times D$.

We illustrate the feedback structure thus obtained in Figure 4.2.1 where "---->" and " $\Longrightarrow$ " indicate the flow of distributed quantity and M-vector respectively.

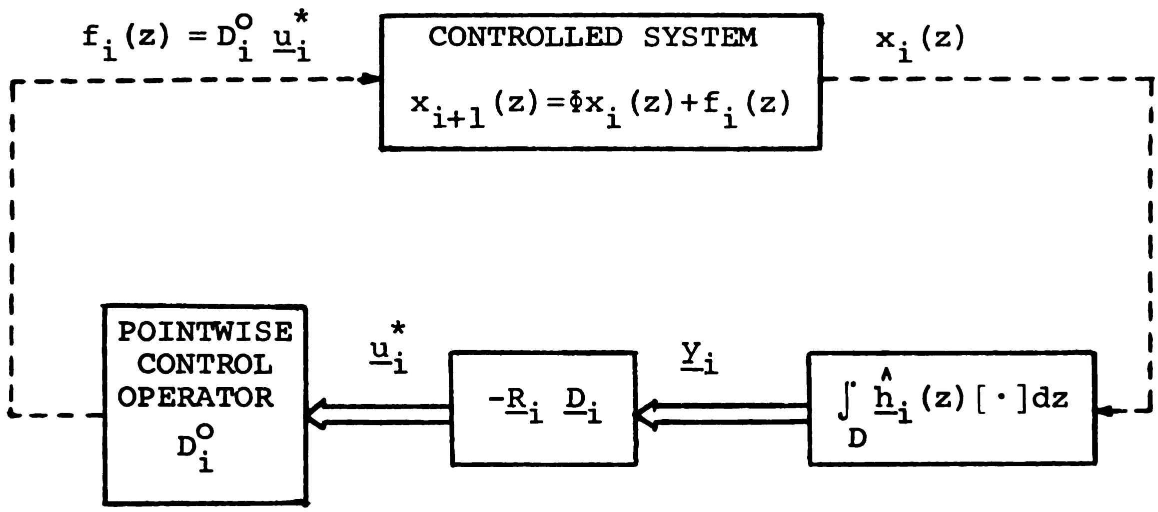

Figure 4.2.1 Optimal feedback pointwise control system.

It is interesting to note that this feedback structure is analogous to that of continuous-time problem (cf.[G-1]). We may conclude here that obtaining a solution for the set of functions $\{H_i(z,z^k)\}_{k=1}^{M}$ enables one to design appropriate instruments (not necessarily physical devices; could be computers, managements, etc. ...) with weighting functions equal to $\Phi_z^* H_i(z,z^k)$.

Let us consider the pointwise control problem (PCP) on the infinite time interval. If Hypothesis I, given in Section 3.5, is satisfied, then an optimal control exists for PCP according to the results in Section 3.5. Moreover, if the system is time-invariant, the time-invariant feedback and weighting kernels are $K(z,\zeta)$ and $H(z,\zeta)$ respectively, and corresponding $k(z)$ and $h(z)$ are obtained from (4.2.9). Thus (4.2.7) becomes the time-invariant algebraic Riccati equation

$$K(z,\zeta) = \Phi_z^* \, \Phi_\zeta^* \, H(z,\zeta) + Q(z,\zeta) \, ,$$

(4.2.12)

$$K(z,\zeta) = H(z,\zeta) + \underline{h}^T(z)\underline{R}^{-1}\underline{k}(\zeta) \, ,$$

and the optimal control (4.2.10) becomes

$$\underline{u}_i^* = - \underline{R}^{-1}\underline{D} \left[\int_D \Phi_\zeta^* \, H(\zeta, z^k) x_i(\zeta) d\zeta \right]$$

(4.2.13)

$$= - \underline{R}^{-1}\underline{D} \int_D \hat{\underline{h}}(\zeta) x_i(\zeta) d\zeta \; .$$

The optimal cost for PCP is given by

$$J = \langle K x_o , x_o \rangle$$

(4.2.14)

$$= \int_D \int_D K(z,\zeta) x_o(z) x_o(\zeta) d\zeta \, dz \; .$$

We shall consider a special class of solutions of (4.2.12) in the next section.

4.3 APPROXIMATION

The pointwise control problem (PCP) on the infinite time interval will be considered in this section. The focus of the development is finding a method for obtaining a solution for the Riccati equation (4.2.12). The kernel $Q(z,\zeta)$ is approximated by the eigenfunctions of system operator, and as a result an algebraic matrix equation is obtained.

We rewrite the Riccati equation (4.2.12) here:

$$K(z,\zeta) = \Phi_z^* \Phi_\zeta^* H(z,\zeta) + Q(z,\zeta)$$

(4.3.1)

$$K(z,\zeta) = H(z,\zeta) + \underline{h}^T(z)\underline{R}^{-1}\underline{k}(\zeta)$$

where $\underline{h}(z)$ and $\underline{k}(z)$ are M-vectors with i^{th} components $d^i H(z,z^i)$ and $d^i K(z,z^i)$ respectively, and the points z^i, $i = 1,2,\cdots, M$ are control locations in D.

It is assumed that the system operator Φ^* has a countable number of eigenvalues λ^i, $i = 1,2,\cdots,$ and that the sequence of eigenfunctions $\{w^i\}$ forms a complete orthonormal basis for the space $L^2(D)$. Note that this assumption is well satisfied when Φ^* is a compact and normal or self-adjoint operator (cf.[C-2,p.359],[P-3,p.411],[P-5]). We choose the kernel of the state weighting operator to be

$$Q(z,\zeta) \equiv \underline{w}^T(z)\underline{Q}\underline{w}(\zeta) , \tag{4.3.2}$$

where $\underline{Q}$ is an $n \times n$ positive definite constant matrix and $\underline{w}(z)$ is the n-vector whose i^{th} component is the eigenfunction $w^i(z)$. It may be shown that the state weighting operator Q with the kernel (4.3.2) is positive semi-definite, i.e.,

$$\begin{aligned}\langle Qx,x\rangle_{L^2(D)} &= \int_D \int_D x(z)\underline{w}^T(z)\underline{Q}\ \underline{w}(\zeta)\ x(\zeta)d\zeta\ dz \\ &= [\int_D x(z)\underline{w}^T(z)dz]\underline{Q}[\int_D \underline{w}(\zeta)x(\zeta)d\zeta] \\ &= \underline{x}^T\ \underline{Q}\ \underline{x} \geq 0 ,\end{aligned}$$

where $\underline{x}$ is the n-vector whose i^{th} component is $\langle x,w^i\rangle_{L^2(D)}$. Note that the operator Q is only semi-definite even though the matrix $\underline{Q}$ is definite, because there exist nonzero vectors $x \in L^2(D)$ which are orthogonal to the subspace generated by the first n eigenfunctions, resulting in $\langle Qx,x\rangle_{L^2(D)} = 0$. Note that if $n \to \infty$, the kernel $Q_\infty(z,\zeta)$ of a positive definite operator is obtained.

Now we derive an equivalent matrix Riccati equation from the kernel equation (4.3.1).

We assume that the optimal feedback and weighting kernels are of the forms

$$K(z,\zeta) = \underline{w}^T(z)\underline{K}\,\underline{w}(\zeta), \tag{4.3.3}$$

$$H(z,\zeta) = \underline{w}^T(z)\underline{H}\,\underline{w}(\zeta)\ , \tag{4.3.4}$$

where $\underline{K}$ and $\underline{H}$ are unknown $n \times n$ symmetric positive semi-definite matrices. Substituting (4.3.2), (4.3.3) and (4.3.4) into (4.3.1) we obtain

$$\begin{aligned} \underline{w}^T(z)\underline{K}\,\underline{w}(\zeta) &= \Phi_z^*\,\underline{w}^T(z)\underline{H}\,\Phi_\zeta^*\,\underline{w}(\zeta) + \underline{w}^T(z)\underline{Q}\,\underline{w}(\zeta)\ , \\ \underline{w}^T(z)\underline{K}\,\underline{w}(\zeta) &= \underline{w}^T(z)\underline{H}\,\underline{w}(\zeta) + \underline{a}^T(z)\ \underline{R}^{-1}\underline{b}(\zeta)\ , \end{aligned} \tag{4.3.5}$$

where $\underline{a}(z)$ and $\underline{b}(z)$ are the M-vectors with i^{th} components

$$a^i(z) = d^i\underline{w}^T(z)\underline{H}\,\underline{w}(z^i),\ b^i(z) = d^i\underline{w}^T(z)\underline{K}\,\underline{w}(z^i)\ .$$

The vectors $a(z)$ and $b(z)$ may be written in the form

(4.3.6) $$\underline{a}(z) = \underline{D}\ \underline{W}\ \underline{H}\ \underline{w}(z),\ \underline{b}(z) = \underline{D}\ \underline{W}\ \underline{K}\ \underline{w}(z)\ ,$$

where $\underline{D}$ is the diagonal $M \times M$ matrix with i^{th} entry $D^{ii} = D^i$, the i^{th} control coefficient (cf.(4.1.3)), and $\underline{W}$ is the $M \times n$ matrix with ij^{th} element $W^{ij} = w^j(z^i)$. Since $\underline{w}(z)$ is the vector of eigenfunctions of Φ_z^*, we have

(4.3.7) $$\Phi_z^*\ \underline{w}(z) = \underline{\Lambda}\ \underline{w}(z)\ ,$$

where $\underline{\Lambda}$ is the diagonal $n \times n$ matrix with i^{th} entry $\Lambda^{ii} = \lambda^i$, the i^{th} eigenvalue.

Substituting (4.3.6) and (4.3.7) into (4.3.5) we obtain the equivalent matrix equation:

(4.3.8) $$\begin{aligned} \underline{K} &= \underline{\Lambda}\ \underline{H}\ \Lambda + Q\ , \\ \underline{K} &= \underline{H} + \underline{H}\ \underline{W}^T\underline{D}\ \underline{R}^{-1}\underline{D}\ \underline{W}\ \underline{K}\ . \end{aligned}$$

Solving the second equation of (4.3.8) in terms of $\underline{H}$, and then substituting into the first one, we obtain the matrix Riccati equation:

(4.3.9) $$\underline{K} = \underline{\Lambda}\ \underline{K}\ (\underline{I} + \underline{W}^T\underline{D}\ \underline{R}^{-1}\underline{D}\ \underline{W}\ \underline{K})^{-1}\underline{\Lambda} + \underline{Q}\ .$$

We note that this Riccati equation is associated with the following finite-dimensional control problem:

Finite-dimensional Control Problem (FCP): Given the n-dimensional system

(4.3.10) $$\underline{x}_{i+1} = \underline{\Lambda}\ \underline{x}_i + \underline{W}^T\ \underline{D}\ \underline{u}_i\ ;\ \ \underline{x}_o \in R^n\ ,$$

and the cost functional

$$(4.3.11) \qquad J = \sum_{i=0}^{\infty} [\underline{x}_i^T \, \underline{Q} \, \underline{x}_i + \underline{u}_i^T \, \underline{R} \, \underline{u}_i] \, ,$$

find a sequence of control $\underline{u}^* = \{\underline{u}_i^*, i \in \sigma, \underline{u}_i^* \in R^M\}$ such that for all $\underline{u} = \{\underline{u}_i, i \in \sigma, \underline{u}_i \in R^M\} \in \ell^2(0,\infty;R^M)$,

$$J(\underline{u}^*) = \inf_{\underline{u}} J(\underline{u}) \, .$$

It is well known that (4.3.9) has a positive semi-definite solution $\underline{K}$ if the system (4.3.10) is completely controllable, and furthermore if the system is also observable, then $\underline{K}$ is positive definite (cf.[K-1], [L-6]). The necessary and sufficient condition for the system to be completely controllable is found elsewhere (cf.[K-1], [S-2]), that is, the $n \times nM$ matrix $\underline{G}$ defined by

$$(4.3.12) \qquad \underline{G} \equiv [\underline{W}^T \underline{D} \mid \underline{\Lambda} \, \underline{W}^T \underline{D} \mid \text{-----} \mid \underline{\Lambda}^{n-1} \underline{W}^T \underline{D}]$$

is of rank n if and if only the system (4.3.10) is completely controllable.

Note that the system (4.3.10) is precisely the n^{th} order eigenfunction approximation of the distributed parameter system (3.2.4) with $D = D^O$ by the method of Galerkin (cf.[P-6], p.15), for example, the n-vector coefficient of the forcing term $D^O \underline{u}_i$ by the eigenfunction expansion is

$$\langle D^O \underline{u}_i, \underline{w} \rangle_{L^2(D)} = \int_D \sum_{j=1}^{M} \chi^j(z) d^j u_i^j \, \underline{w}(z) dz$$

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$$= \sum_{j=1}^{M} d^j u_i^j \int_D \chi^j(z)\underline{w}(z)\,dz = \sum_{j=1}^{M} \underline{w}(z^j)\, d^j u_i^j$$

$$= \underline{W}^T \underline{D}\ \underline{u}_i ,$$

which is equal to the forcing term in (4.3.10).

The weighting matrix $\underline{H}$ can also be computed from (4.3.8) using the obtained solution $\underline{K}$ of the Riccati equation (4.3.9). Thus, utilizing (4.3.4) and (4.3.7), the optimal pointwise control (4.2.13) becomes

$$\underline{u}_i^* = -\underline{R}^{-1}\underline{D} \left[\int_D \Phi_\zeta^* \ \underline{w}^T(z^k)\underline{H}\ \underline{w}(\zeta)\, x_i(\zeta)\, d\zeta \right]$$

$$= -\underline{R}^{-1}\underline{D} \left[\underline{w}^T(z^k)\underline{H}\ \underline{\Lambda} \int_D \underline{w}(\zeta)\, x_i(\zeta)\, d\zeta \right]$$

$$= -\underline{R}^{-1}\underline{D}\ \underline{W}\ \underline{H}\ \underline{\Lambda} \int_D \underline{w}(\zeta)\, x_i(\zeta)\, d\zeta = -\underline{R}^{-1}\underline{D}\ \underline{W}\ \underline{H}\ \underline{\Lambda}\ \underline{x}_i , \qquad (4.3.13)$$

where $\underline{x}_i$ is the n-vector coefficient of $x_i(z)$ in the eigenfunction expansion. Employing (4.3.3), the optimal cost (4.2.14) with initial state $x_o \in L^2(D)$ becomes

$$J = \int_D \int_D x_o(z) K(z,\zeta)\, x_o(\zeta)\, d\zeta dz = \int_D x_o(z)\underline{w}^T(z)\,dz\ \underline{K} \int_D \underline{w}(\zeta)\, x_o(\zeta)\, d\zeta$$

$$= \underline{x}_o^T\ \underline{K}\ \underline{x}_o , \qquad (4.3.14)$$

where $\underline{x}_o$ is the n-vector coefficient of $x_o(z)$. Thus we have shown that by choosing $Q(z,\zeta)$ to be of the form (4.3.2), both the optimal control and optimal cost depend only on the first n coefficients of the state variable in the eigenfunction expansion.

4.4 AN EXAMPLE FOR PCP

We illustrate the approximation scheme by an example.

Consider the one-dimensional heat equation with pointwise control operator given by

$$\text{(4.4.1)} \qquad \frac{\partial x(t,z)}{\partial t} = \frac{\partial^2 x(t,z)}{\partial z^2} + B_o\underline{u}(t) \ , \quad 0 \leq z \leq 1 \ ,$$

$$x(t,0) = x(t,1) = 0 \ ,$$

where B_o is a pointwise control operator. The system operator $A = \frac{\partial^2}{\partial z^2}$ is self-adjoint and the eigenvalues are $\mu_i = -i^2\pi^2$, $i = 1,2,\cdots$, with orthonormal eigenvectors $w^i(z) = \sqrt{2}\sin i\pi z$. The semigroup generated by A is given by

$$\text{(4.4.2)} \qquad \Phi(t)y = \sum_{i=1}^{\infty} e^{-i^2\pi^2 t}\, y^i w^i(z) \ ,$$

where y^i is the i^{th} coefficient of y in the eigenvalue expansion (cf.[P-5]). The equivalent discrete system is

$$\text{(4.4.3)} \qquad x_{i+1} = \Phi\, x_i + D^o\underline{u}_i \ ,$$

where the eigenvalues corresponding to $\Phi = \Phi(\delta)$ are

$\lambda_i = e^{-i^2\pi^2\delta}$, $i = 1,2,\cdots,$ and D^o is the pointwise control operator corresponding to B_o in (4.4.1).

We suppose that we are applying two pointwise controls at z^1 and z^2, and choose the state weighting kernel $Q(z,\zeta)$ to be

$$Q(z,\zeta) = [w^1(z)w^2(z)]\ \underline{Q}\begin{bmatrix} w^1(\zeta) \\ w^2(\zeta) \end{bmatrix},$$

where $\underline{Q}$ is a symmetric positive definite 2×2 matrix. The matrix $\underline{W}$ (cf. (4.3.6)) is

$$\underline{W} = \begin{bmatrix} \sqrt{2}\sin \pi z^1 & \sqrt{2}\sin 2\pi z^1 \\ \sqrt{2}\sin \pi z^2 & \sqrt{2}\sin 2\pi z^2 \end{bmatrix}.$$

The controllability matrix $\underline{G}$ is the 2×4 matrix

$$\underline{G} = \sqrt{2}\begin{bmatrix} d_1\sin \pi z^1 & d_2\sin \pi z^2 & \lambda_1 d_1\sin \pi z^1 & \lambda_1 d_2\sin \pi z^2 \\ d_1\sin 2\pi z^1 & d_2\sin 2\pi z^2 & \lambda_2 d_1\sin 2\pi z^1 & \lambda_2 d_2\sin 2\pi z^2 \end{bmatrix}.$$

It is easy to show that the rank of $\underline{G}$ is 2, thus the solution of Riccati equation (4.3.9) is guaranteed. If there is only one control at z^1, we have

$$\underline{G} = \sqrt{2}\begin{bmatrix} d_1\sin \pi z^1 & \lambda_1 d_1\sin \pi z^1 \\ d_1\sin 2\pi z^1 & \lambda_2 d_1\sin 2\pi z^1 \end{bmatrix}$$

which is of rank 2 since $\lambda_1 \neq \lambda_2$. Note that in the two point control case λ_1 and λ_2 need not be distinct.

To compute analytically we choose that $\underline{D} = \underline{R} = \underline{Q} = \underline{I}$, the 2×2 identity matrix. If we denote the matrix $\underline{K}$ to be

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix},$$

and choose the control points z^1 and z^2 to lie symmetrically about the midpoint $z = \frac{1}{2}$, i.e., $z^2 = 1 - z^1$, then the Riccati equation (4.3.9) yields the following

$$2A\, k_{11}^2 + (1 - 2A - e^{-2\pi^2\delta})\, k_{11} - 1 = 0$$

$$2B\, k_{22}^2 + (1 - 2B - e^{-8\pi^2\delta})\, k_{22} - 1 = 0$$

$$k_{12} = 0,$$

where $A = 2\sin^2\pi z^1$ and $B = 2\sin^2 2\pi z^1$. Thus we obtain

$$k_{12} = 0,$$

$$k_{11} = \frac{-1+e^{-2\pi^2\delta}+4\sin^2\pi z^1+\sqrt{1+2e^{-2\pi^2\delta}+e^{-4\pi^2\delta}+16\sin^4\pi z^1+8\sin^2\pi z^1}}{8\sin^2\pi z^1}$$

(4.4.4)

$$k_{22} = \frac{-1+e^{-8\pi^2\delta}+4\sin^2 2\pi z^1+\sqrt{1+2e^{-8\pi^2\delta}+e^{-16\pi^2\delta}+16\sin^4 2\pi z^1+8\sin^2 2\pi z^1}}{8\sin^2 2\pi z^1}$$

The optimal cost (4.3.14) becomes

(4.4.5)

$$J(x_o) = \int_D \int_D x_o(z)[k_{11}\sin\pi z\sin\pi\zeta + k_{22}\sin 2\pi z\sin 2\pi\zeta]x_o(\zeta)\,d\zeta\,dz$$

The weighting matrix $\underline{H}$ is obtained from (4.3.8), i.e.,

(4.4.6) $$\underline{H} = \underline{\Lambda}^{-1}(\underline{K} - \underline{Q})\underline{\Lambda}^{-1} ,$$

which yields the entries $h_{11} = e^{2\pi^2\delta}(k_{11}-1)$, $h_{22} = e^{8\pi^2\delta}(k_{22}-1)$, and $h_{12} = 0$. Thus, from (4.3.13), the optimal control is

(4.4.7) $$u_i^* = \begin{bmatrix} g_1 & g_2 \\ g_1 & -g_2 \end{bmatrix} \begin{bmatrix} \int_D \sin \pi\zeta \ \ x_i(\zeta)d\zeta \\ \int_D \sin 2\pi\zeta \ x_i(\zeta)d\zeta \end{bmatrix} ,$$

where

$$g_1 = - \frac{-1+e^{-2\pi^2\delta}-4\sin^2\pi z^1+\sqrt{1+2e^{-2\pi^2\delta}+e^{-4\pi^2\delta}+16\sin^4\pi z^1+8\sin^2\pi z^1}}{4e^{-\pi^2\delta}\sin \pi z^1}$$

$$g_2 = - \frac{-1+e^{-8\pi^2\delta}-4\sin^2 2\pi z^1+\sqrt{1+2e^{-8\pi^2\delta}+e^{-16\pi^2\delta}+16\sin^4 2\pi z^1+8\sin^2 2\pi z^1}}{4e^{-4\pi^2\delta}\sin 2\pi z^1}$$

Note that if we have the measuring devices which yield the output

$$\underline{y}_i = \begin{bmatrix} \int_D \sin \pi\zeta \ \ x_i(\zeta)d\zeta \\ \int_D \sin 2\pi\zeta \ x_i(\zeta)d\zeta \end{bmatrix} ,$$

then $\underline{y}_i$ can be fed directly through the gain matrix

$$\underline{G} = \begin{bmatrix} g_1 & g_2 \\ g_1 & -g_2 \end{bmatrix}$$

to obtain the optimal control

$$\underline{u}_i^* = \underline{G}\ \underline{y}_i\ .$$

It should be noted that the measurement does not depend on the control point locations, and only the gain matrix $\underline{G}$ does. Thus the measurement and control problems are decoupled and the changing control point location does not alter the structure of measuring devices. This provides us the choice of control point location which minimizes the average cost in some sense. The system in the example is illustrated in Figure 4.4.1.

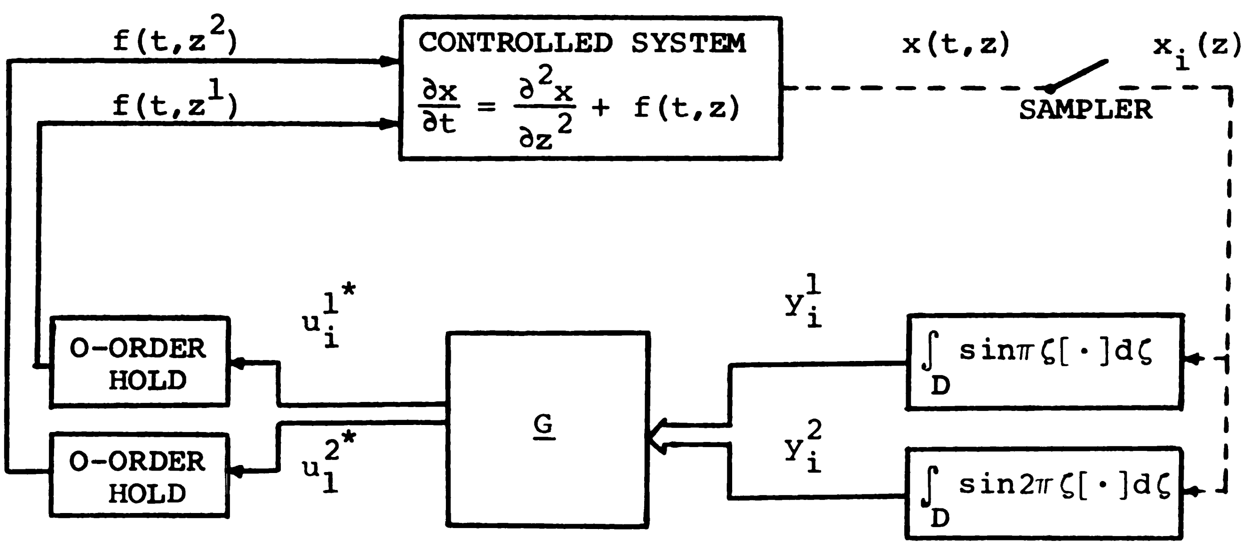

Figure 4.4.1 Sampled-data control system for Example 4.4.

The optimal control point location can be found analytically as a parameter optimization problem. The optimal control is given by (4.3.14), i.e.,

$$J(\underline{x}_o) = \underline{x}_o^T\ \underline{K}\ \underline{x}_o\ ,$$

where $\underline{x}_o$ is the n-vector coefficient of initial state $x_o(z)$.

If we assume that $\underline{x}_o$ is uniformly distributed over the unit sphere $|\underline{x}_o| = 1$, then it can be shown that (cf.[K-3]) the average cost of $J(\underline{x}_o)$ over the unit sphere is

$$J_{avg} = \frac{1}{2}\,\text{trace}\,(\underline{K}) = \frac{1}{2}(k_{11} + k_{22})\ ,$$

where k_{11} and k_{22} is given by (4.4.4). Thus the optimal point location of z^1 (and z^2) can be found simply by differentiating J_{avg} and setting it equal to zero.

We have completed the example in this section. Now we summarize the results of approximations for PCP, i.e., a suitable choice of the state weighting kernel $Q(z,\zeta)$ yields the finite dimensional approximation by eigenfunction expansion, from which we conclude that the optimal control law which feeds back only the first n modes under consideration is optimal over the class of all feedback control laws. In the example it was shown that the feedback structure of the PCP can be separated into a measurement part, which is independent of control point location, and a gain part, which depends on control point location.

CHAPTER V

SUMMARY AND CONCLUSIONS

This dissertation is concerned with the sampled-data control of distributed parameter systems with quadratic cost criteria, where the system operator is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators. It is shown that the optimal control exists and is given by a bounded linear transformation of the sampled states of the system. The resulting optimal feedback operator is shown to be the solution of an operator difference equation of the Riccati type. The feedback operator is represented by an integral operator whose kernel satisfies an integro-difference equation.

These results for general infinite dimensional control problem are specialized to the problem of pointwise control, where control is applied only at a finite number of points in the spatial domain. The pointwise feedback control is shown to be of simpler form than the distributed feedback control. It is also shown that a particular choice of the state weighting operator yields a finite dimensional approximation by the method of eigenfunction expansion, and a decoupling of the measurement and control problems.

In summary, this dissertation creates a new problem of infinite dimensional discrete-time optimal control, gives a simple approach to solving control problems of this type and generates new results for infinite dimensional systems. This approach enables us to make direct extensions of the results for finite dimensional systems to infinite dimensional discrete-time systems.

There are a number of topics for further research based on this work, for example:

(1) The feedback control law obtained requires the distributed state $x(z)$ over the entire spatial domain. However, in many cases, the state is observed only at a finite number of points. Thus the pointwise output feedback problem analogous to that of finite dimensional systems (cf.[L-7]) should be investigated.

(2) In Section 4.4 the optimal control point location was briefly studied. This problem can also be generalized to a problem with an arbitrary set of control points by minimizing the averaged cost in some sense over the set of possible control points.

(3) The computation of solutions to Riccati operator equations might be developed applying the Galerkin technique, successive approximations (cf.[P-6]), or steepest descent methods (cf.[B-5]).

(4) The stochastic optimal control of discrete-time distributed parameter systems should be investigated. This problem is analogous to that of continuous-time distributed parameter systems studied by Bensoussan [B-7].

(5) The variational approach might be used in the identification of discrete-time distributed systems in a manner analogous to that of continuous-time systems (cf.[P-6], [B-6]).

The problems (4) and (5) are under investigation by the author and will be reported in a subsequent work.

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APPENDIX

APPENDIX

PROOFS OF LEMMAS 3.3.6 AND 3.3.7

<u>Lemma 3.3.6</u> (Lions, [L-1]): If $\pi(u,v)$ is a continuous symmetric coercive bilinear form, then there exists a unique $u^* \in V$ such that

$$C(u^*) = \inf_{u \in V} C(u) .$$

<u>Proof</u>: (Existence). Let $u_n \in V$ be a minimizing sequence, i.e.,

$$C(u_n) \to \inf_{u \in V} C(u) . \tag{A.1}$$

Note from (3.3.6) that $C(u) \geq C\|u\|^2 - C_1\|u\|$, thus with (A.1) u_n is bounded. Therefore we can choose a subsequence u_k of u_n which converges to a weak limit, say, $w \in V$. Since $\pi(u,u)$ is lower semicontinuous and $L(u)$ is continuous in the weak topology of V, $C(u)$ in (3.3.6) is lower semicontinuous and $\underline{\inf}\, C(u_k) \geq C(w)$. Applying (A.1) we have $\inf_{u \in V} C(u) \geq C(w)$, $w \in V$. Hence it is necessary that $C(w) = \inf_{u \in V} C(u)$ and $u^* = w$.

(Uniqueness). The function $\pi(u,u)$ is strictly convex, because for $\lambda \in (0,1)$, $u_1, u_2 \in V$,

$$\pi((1-\lambda)u_1 + \lambda\, u_2,\ (1-\lambda)u_1 + \lambda\, u_2)$$

$$= \pi(u_1+\lambda(u_2-u_1),\ u_1 + \lambda(u_2-u_1))$$

$$= \pi(u_1,u_1) + 2\lambda\pi(u_1,u_2-u_1) + \lambda^2\ \pi(u_2-u_1,u_2-u_1)$$

$$< \pi(u_1,u_1) + 2\lambda\pi(u_1,u_2-u_1) + \lambda\ \ \pi(u_2-u_1,u_2-u_1)$$

$$=(1-\lambda)\ \pi(u_1,u_1) + \lambda\ \pi(u_2,u_2)\ .$$

Hence (3.3.6) implies $C(u)$ is also strictly convex. Suppose both u_1 and u_2 assume the infimum of $C(u)$ on V, then V is convex (since it is a whole space) implies $\frac{1}{2}(u_1+u_2) \in V$ and

$$C(\tfrac{1}{2}\ (u_1+u_2)) < \tfrac{1}{2}\ C(u_1) + \tfrac{1}{2}\ C(u_2) = \inf_{u \in V} C(u)\ ,$$

which contradicts the assumptions on u_1 and u_2 unless $u_1 = u_2$. Q.E.D.

<u>Lemma 3.3.7</u> (Lions, [L-1]): If the hypotheses of Lemma 3.3.6 are satisfied, then the minimizing element $u^* \in V$ is characterized by

$$\text{(A.2)} \qquad \pi(u^*,v) = L(v) \quad \forall\ v \in V\ .$$

<u>Proof</u>: Suppose u^* is the minimizing element in V, then

$$C(u^*) \leq C((1-\lambda)u^* + \lambda w) \quad \forall\ w \in V \text{ and } \lambda \in [0,1]$$

or

$$C(u^* + \lambda(w-u^*)) - C(u^*) \geq 0\ .$$

Using (3.3.6),

$$\pi(u^* + \lambda(w-u^*), u^* + \lambda(w-u^*)) - 2L(u^*+\lambda(w-u^*))$$
$$- [\pi(u^*,u^*) - 2L(u^*)] \geq 0$$

or

$$2[\pi(u^*,w-u^*) - L(w-u^*)] + \lambda\, \pi(w-u^*,w-u^*) \geq 0$$

As $\lambda \to 0$ we have

$$\pi(u^*,w-u^*) \geq L(w-u^*) .$$

Since w is any vector in V, choosing $w = u^* \pm v$, $v \in V$ implies (A.2).

The necessity is proved by using the convexity of $C(u)$, i.e., for $\lambda \in [0,1]$,

$$C(v) - C(u^*) \geq \frac{1}{\lambda} [C((1-\lambda)u^* + \lambda v) - C(u^*)] \quad \forall\, v \in V$$

Using (3.3.6) we obtain

$$C(v) - C(u^*) \geq 2[\pi(u^*,v-u^*) - L(v-u^*)] + \lambda\, \pi(v-u^*,v-u^*)$$

As $\lambda \to 0$ we have

$$C(v) - C(u^*) \geq 2[\pi(u^*,v-u^*) - L(v-u^*)] ,$$

and thus (A.2) implies $C(u^*) \leq C(v) \quad \forall\ v \in V$.

Q.E.D.