

THE HARDY SPACES AND OTHER
RELATED FUNCTION SPACES

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ABSTRACT

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by

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The Hardy spaces H^p are closely related to certain other spaces of analytic functions. For $0 < p < 1$, let B^p denote the class of all functions f analytic in the unit disk satisfying

$$\|f\|_p = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} (1-r)^{(1/p)-2} |f(re^{i\theta})| d\theta dr < \infty.$$

B^p with the above norm is a Banach space. For $0 < p, q < \infty$, let $H^{p,q}$ denote the space of all functions f analytic in the unit disk for which

$$\|f\|_{p,q} = \left\{ \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{q/p} dr \right\}^{1/q} < \infty$$

and define $H^{p,\infty}$ to be the Hardy space H^p . If $0 < p, q \leq 1$ or $0 < p < 1$, $q = \infty$ and $\sigma = \left(\frac{1}{p} + \frac{1}{q}\right)^{-1}$, then B^σ is the "containing Banach space" for $H^{p,q}$ in the sense that $H^{p,q}$ is a dense subset of B^σ and $H^{p,q}$ and B^σ have the same continuous linear functionals.

The relationships between the spaces H^σ , $H^{p,q}$, and B^σ ($\sigma = \left(\frac{1}{p} + \frac{1}{q}\right)^{-1}$) are studied for all p and q . In particular, if $0 < p, q \leq 1$, then $H^{p,q}$ is an intermediate space between H^σ and B^σ . This relationship, $H^\sigma \subset H^{p,q} \subset B^\sigma$, may be used

to determine certain coefficient properties of $H^{p,q}$ functions.

The general properties of the corresponding spaces h^p , b^p , and $h^{p,q}$ of harmonic functions are also studied. If $0 < p < 1$, it is shown that h^p is a non-locally convex F-space with enough continuous linear functionals to separate points. Next, the properties of b^p are discussed and its dual space is determined. Finally, the spaces $h^{p,q}$ are studied and the relationships between h^σ , $h^{p,q}$, and b^σ ($\sigma = (\frac{1}{p} + \frac{1}{q})^{-1}$) are examined. In particular, it is shown that $b^{p/2}$ is the containing Banach space for $h^{p,p}$, $0 < p \leq 1$.

The last topic to be considered is composition operators on B^p . If ϕ is an analytic function mapping the unit disk into itself and f is in B^p , the composition operator C_ϕ is defined by $C_\phi(f) = f \circ \phi$. It is shown that C_ϕ is a bounded linear operator on B^p . Conditions are also given on ϕ in order that C_ϕ be a bounded operator from B^p into H^q , $0 < q \leq \infty$. Isometric and invertible composition operators are characterized, and finally compact operators and their spectra are discussed.

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CHAPTER I
INTRODUCTION

1. Fundamentals.

The unit disk in the complex plane will be denoted by D . We will assume throughout that $z \in D$ has the form $re^{i\theta}$ and will often write r in place of $|z|$.

For a function u harmonic in D , the integral means of order p , $0 < p < \infty$ are given by

$$M_p(r, u) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right\}^{1/p}.$$

The infinity means of u are given by

$$M_\infty(r, u) = \max_{0 \leq \theta < 2\pi} |u(re^{i\theta})|.$$

We denote by h^p , $0 < p \leq \infty$, the class of functions u harmonic in D such that

$$\sup_{0 \leq r < 1} M_p(r, u) < \infty.$$

Similarly H^p denotes the class of all functions f analytic in D such that

$$\sup_{0 \leq r < 1} M_p(r, f) < \infty.$$

It is well known that if $f \in H^p$, $0 < p \leq \infty$, then f is of bounded characteristic and consequently has radial

limits a.e. Furthermore, the boundary function, $f(e^{i\theta})$, defined by the radial limits is in $L^p[0, 2\pi]$. (See, for example, [2] or [11].)

The integral means M_p were first studied by G. H. Hardy [8] in 1914. He showed that if f is analytic in $\mathbb{D}$ and $0 < p \leq \infty$, then $M_p(r, f)$ is a nondecreasing function of r and $\log M_p(r, f)$ is a convex function of $\log r$. In further studies, G. H. Hardy and J. E. Littlewood proved the following theorems.

Theorem 1.1 (G. H. Hardy and J. E. Littlewood [10], p. 406).

If $p > 0$, $a \geq 0$, $p < q \leq \infty$ and

$$M_p(r, f) \leq C(1 - r)^{-a},$$

then

$$M_q(r, f) \leq K(p, a)C(1 - r)^{-a - (1/p) + (1/q)},$$

where C is an absolute constant and $K(p, a)$ depends only on p and a .

Notation: $C(a_1, a_2, \dots, a_n)$ will denote a constant depending only on the numbers $a_1, a_2, \dots, a_n$.

Theorem 1.2 (G. H. Hardy and J. E. Littlewood [9], p. 413).

If $p > 0$, $a > 0$ and

$$M_p(r, f') \leq C(1 - r)^{-a-1}$$

then

$$M_p(r, f) \leq K(p, a)(1 - r)^{-a}.$$

Notation: $f(r) \asymp g(r)$ means $f(r)/g(r) \rightarrow 1$ as $r \rightarrow 1$.

$f(r) \sim g(r)$ means $f(r)/g(r)$ and $g(r)/f(r)$ are both bounded for r sufficiently close to 1.

The next theorem may be proved by simple computations (see [2], p. 65).

Theorem 1.3. If $\alpha > 1$, then

$$\int_0^{2\pi} |1 - z|^{-\alpha} d\theta \sim (1 - r)^{-\alpha+1}.$$

There are various definitions of fractional derivatives and integrals. We shall use the one given by P. Duren, B. W. Romberg, and A. Shields in [3]. This definition differs only by a factor of z^β from the one given by G. H. Hardy and J. E. Littlewood in [10].

Definition 1.1. If $f(z) = \sum a_n z^n$, the fractional derivative of order β of f is defined as

$$f^{[\beta]}(z) = \sum \frac{\Gamma(n+1+\beta)}{n!} a_n z^n$$

and the fractional integral of order β is defined as

$$f_{[\beta]}(z) = \sum \frac{n!}{\Gamma(n+1+\beta)} a_n z^n.$$

A function u harmonic in D can be written in the form

$$u(z) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta}.$$

We define

$$u^{[\beta]}(z) = \sum_{-\infty}^{\infty} \frac{\Gamma(|n|+1+\beta)}{|n|!} c_n r^{|n|} e^{in\theta}$$

and

$$u_{[\beta]}(z) = \sum_{-\infty}^{\infty} \frac{|n|!}{\Gamma(|n|+1+\beta)} c_n r^{|n|} e^{in\theta}.$$

We will assume many of the elementary properties of harmonic functions. For example, each u harmonic in D can be completed analytically and any two analytic completions of u differ by a constant. If $f = u + iv$ is an analytic completion of u and $v(0) = 0$, then we say that f is the normalized analytic completion of u or simply "the" analytic completion of u .

2. Background.

If $f \in H^p$, then we define

$$(1.1) \quad \|f\|_{H^p} = \sup_{r < 1} M_p(r, f).$$

If $p \geq 1$, then (1.1) defines a norm on H^p . If $p < 1$, then (1.1) does not satisfy the triangle inequality and hence fails to be a norm. However, $\|\cdot\|_{H^p}^p$ satisfies the triangle inequality and induces a metric on H^p . The spaces H^p , $0 < p \leq \infty$, are complete. Indeed, H^p is a closed subspace of $L^p(D)$, the class of complex-valued functions f satisfying:

$$\|f\|_{L^p} = \sup_{r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.$$

Thus, if $p \geq 1$, H^p is a Banach space.

Definition 1.2. An F space is a linear space with a complete translation invariant metric under which scalar multiplication is a continuous operation ([6], p. 51).

If $p < 1$, then H^p is an F space. F spaces have many of the important properties of Banach spaces. In particular, the Open Mapping Theorem, the Closed Graph Theorem, and the Principle of Uniform Boundedness all hold for F spaces. Furthermore, the Hahn Banach Theorem holds for locally convex F spaces. For precise statement of these theorems, we refer the reader to [6].

Definition 1.3. We say that an F space has the Hahn Banach Extension Property, H.B.E.P., if every continuous linear functional on each closed subspace has a continuous linear extension to the whole space.

In particular, a locally convex F space has the H.B.E.P. J. H. Shapiro [17] has shown that for an F space with a basis, the H.B.E.P. is equivalent to local convexity.

3. The spaces H^p and B^p , $0 < p < 1$.

The properties of H^p , $0 < p < 1$, as a linear space, were first studied by S. S. Walters in [18] and [19]. He showed in [18] that H^p has enough continuous linear functionals to separate points, in contrast to the L^p spaces ($0 < p < 1$) which have no continuous linear functionals other than the zero functional. In [19], he conjectured that the H^p spaces were not locally convex. This was later proved by P. Duren, B. W. Romberg, and A. Shields in [3].

If A is a linear space, we will denote its dual by A^* . We will also need the following definitions.

Definition 1.3. If f and g are harmonic in D , and $g \in h^1$, then define

$$\langle f, g \rangle = \lim_{r \rightarrow 1} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta}) d\theta$$

provided the limit exists.

Definition 1.4. Two Banach spaces A and B are equivalent if there is a one-to-one linear mapping T of A onto B such that both T and T^{-1} are continuous, i.e., A and B are equivalent if they are linearly homeomorphic.

S. S. Walters showed in [19] that corresponding to each $\varphi \in (H^p)^*$, there is an analytic function g such that

$$(1.2) \quad \varphi(f) = \langle f, g \rangle$$

for each $f \in H^p$, and conversely if g is a function such that $\langle f, g \rangle$ exists for every f in H^p , then (1.2) defines a continuous linear functional on H^p . B. W. Romberg [14] continued this study improving upon Walters' results.

In the case p is not the reciprocal of an integer, Romberg gave a condition on g which is necessary and sufficient in order that (1.2) define a continuous linear functional on H^p . Romberg also gave a partial characterization of $(H^p)^*$ in the case that p is the reciprocal of an integer. Finally in [3], P. Duren, B. W. Romberg, and A. Shields gave a characterization of $(H^p)^*$ up to equivalence for

all $p < 1$. To present these results, we will need to define certain Lipschitz classes of functions.

Definition 1.5. If f is a complex-valued function defined on $|z| = 1$, the modulus of continuity of f is given by

$$w(h;f) = \sup_{|\alpha-\beta| \leq h} |f(e^{i\alpha}) - f(e^{i\beta})|.$$

f is said to belong to the Lipschitz class Λ_α ($0 < \alpha \leq 1$) if

$$w(h;f) = O(h^\alpha) \text{ as } h \rightarrow 0.$$

Furthermore, f is said to belong to class Λ_* if

$$|f(e^{i(t+h)}) - 2f(e^{it}) + f(e^{i(t-h)})| = O(h).$$

The classes λ_α and λ_* are defined in a similar manner with "O" replaced by "o". For a function f analytic in D , we say $f \in \Lambda_\alpha(\lambda_\alpha, \lambda_*, \lambda_*)$ if f is continuous in $\bar{D}$ and $f(e^{i\theta}) \in \Lambda_\alpha(\lambda_\alpha, \lambda_*, \lambda_*)$.

Definition 1.6. Let Λ_α^n , ($n = 0, 1, \dots$, $0 < \alpha \leq 1$), be the space of functions $f(z)$ analytic in D and continuous in $\bar{D}$ such that $f^{(n)} \in \Lambda_\alpha$. Λ_α^n is a Banach space under the norm

$$\|f\| = \|f\|_{H^\infty} + \sup_{\substack{t, \theta \\ t > 0}} t^{-\alpha} |f^{(n)}(e^{i(\theta+t)}) - f^{(n)}(e^{i\theta})|.$$

Similarly, Λ_*^n denotes the Banach space of functions f analytic in D and continuous on $\bar{D}$ such that $f^{(n)} \in \Lambda_*$ and

$$\|f\| = \|f\|_{H^\infty} + \sup_{\substack{t, \theta \\ t > 0}} t^{-1} |f^{(n)}(e^{i(\theta+t)}) - 2f^{(n)}(e^{i\theta}) + f^{(n)}(e^{i(\theta-t)})|$$

The spaces λ_{α}^n , λ_{*}^n are defined in the same manner.

Theorem 1.4. (P. Duren, B. W. Romberg, and A. Shields [3], p. 35).

If $\frac{1}{n+1} < p < \frac{1}{n}$, $\alpha = \frac{1}{p} - n$ and $\varphi \in (H^p)^*$, then there is a unique $g \in \Lambda_{\alpha}^{n-1}$ such that

$$\varphi(f) = \langle f, g \rangle$$

for each $f \in H^p$. Conversely, if $g \in \Lambda_{\alpha}^{n-1}$, then $\langle f, g \rangle$ exists for each $f \in H^p$ and defines a continuous linear functional on H^p .

If $p = \frac{1}{n+1}$, then $g \in \Lambda_{*}^{n-1}$ and conversely any function $g \in \Lambda_{*}^{n-1}$ defines a continuous linear functional on H^p .

Theorem 1.5. (Duren, Romberg, Shields [3], p. 39).

If $\frac{1}{n+1} < p < \frac{1}{n}$, $\alpha = \frac{1}{p} - n$, then $(H^p)^*$ is equivalent to Λ_{α}^{n-1} . If $p = \frac{1}{n+1}$, $(H^p)^*$ is equivalent to Λ_{*}^{n-1} .

We may also talk about the spaces Λ_{α} , Λ_{*} , λ_{α} , λ_{*} , Λ_{α}^n , etc., of harmonic functions. The following theorem allows us to characterize these spaces in terms of growth conditions.

Theorem 1.6. (See Zygmund [20], vol. 1, p. 263).

(i) A necessary and sufficient condition for a harmonic function u to be in Λ_{α} , $0 < \alpha \leq 1$ is that

$$\frac{\partial u}{\partial \theta} = o((1 - r)^{\alpha-1}).$$

(ii) A necessary and sufficient condition for u to be in Λ_* is that

$$\frac{\partial^2 u}{\partial \theta^2} = O((1 - r)^{-1}).$$

Theorem 1.7. (Duren, Romberg, Shields [3], p. 44).

If f is analytic in D and $f(z) = O((1 - r)^{-a})$, $a > 0$, then

$$(i) \quad f^{[\beta]}(z) = O((1 - r)^{-(a+\beta)}), \quad \beta > 0$$

and

$$(ii) \quad f_{[\beta]}(z) = O((1 - r)^{\beta-a}), \quad 0 < \beta < a.$$

Theorems 1.6 and 1.7 may be stated for either analytic or harmonic functions since in general a harmonic function and its analytic completion have the same order of growth, [20] vol. I, p. 253.

Definition 1.7. Let ψ^p , $0 < p < 1$, denote the space of all functions g analytic (harmonic) in D such that

$$g^{[1/p]} = O((1 - r)^{-1}),$$

endowed with the norm

$$\|g\|_{\psi^p} = \sup_{r < 1} \{M_\infty(r, g^{[1/p]})(1 - r)\}.$$

The spaces ψ^p are defined similarly replacing "O" by "o".

It has been noted (J. H. Shapiro [17], p. 27) that Ψ^p is equivalent to Λ_α^{n-1} , $\alpha = \frac{1}{p} - n$ for $\frac{1}{n+1} < p < \frac{1}{n}$ and equivalent to Λ_*^{n-1} if $p = \frac{1}{n+1}$. This may be proved easily using Theorems 1.6 and 1.7.

It was remarked earlier that H^p , $0 < p < 1$, is not locally convex. Actually the following stronger theorem was proved.

Theorem 1.8. (Duren, Romberg, Shields [3], p. 51). There exists a proper closed subspace $H^p(E)$ of H^p and a continuous linear functional φ on $H^p(E)$ which cannot be extended to all of H^p . Thus, H^p does not have the H.B.E.P. and hence is not locally convex.

In [3], Duren, Romberg, and Shields found the "containing Banach space" of H^p , that is, they found a Banach space B^p which has the same continuous linear functionals as H^p and which contains H^p as a dense subspace.

Definition 1.8. For $0 < p < 1$, let B^p denote the space of all functions $f(z)$ analytic in D such that

$$(1.3) \quad \|f\|_{B^p} = \int_0^1 (1-r)^{(1/p)-2} M_1(r, f) dr < \infty.$$

The spaces b^p of harmonic functions are defined similarly.

Theorem 1.9. (Duren, Romberg, Shields [3], p. 40). The space B^p , $0 < p < 1$, with norm (1.3) is a Banach space. Furthermore,

$$(i) \quad |f(z)| \leq C(p) \|f\|_{B^p} (1-r)^{-1/p}$$

for each $f \in B^p$, and

$$f(z) = o((1-r)^{-1/p});$$

(ii) for each $f \in B^p$,

$$\lim_{\rho \rightarrow 1} \|f_\rho - f\|_{B^p} = 0,$$

where $f_\rho(z) = f(\rho z)$;

(iii) H^p is dense in B^p ;

$$(iv) \quad \|f\|_{B^p} \leq C(p) \|f\|_{H^p}$$

for each $f \in H^p$.

Coefficients of B^p functions satisfy the same growth condition as coefficients of H^p functions.

Theorem 1.10. (Duren, Romberg, Shields [3], p. 41). If $f(z) = \sum a_n z^n \in B^p$, then

$$|a_n| \leq C(p) \|f\|_{B^p} n^{(1/p)-1}$$

and $a_n = o(n^{(1/p)-1})$. Conversely, if $0 < p < 1$ and $a_n = O(n^\alpha)$, $\alpha < (1/p) - 3/2$, then $f \in B^p$. The $(1/p) - 3/2$ is best possible, in that, there exists $g(z) = \sum b_n z^n$ such that $b_n = O(n^{(1/p)-3/2})$ and g is not in B^p .

The next theorem shows that for $0 < p < q < 1$, the spaces B^q and B^p are equivalent under the correspondence $f \rightarrow f[(1/p)-(1/q)]$ for each $f \in B^q$.

Theorem 1.11. (Duren, Romberg, Shields [3], p. 43). If $0 < p < q < 1$ and $\beta = (1/p) - (1/q)$, then

(i) $f \in B^p$ implies $f_{[\beta]} \in B^q$ and

$$\|f_{[\beta]}\|_{B^q} \leq C(p,q)\|f\|_{B^p};$$

(ii) $f \in B^q$ implies $f^{[\beta]} \in B^p$ and

$$\|f^{[\beta]}\|_{B^p} \leq C(p,q)\|f\|_{B^q}.$$

Theorem 1.12. (Duren, Romberg, Shields [3], p. 46).

Theorems 1.4 and 1.5 remain true with H^p replaced by B^p .

Theorems 1.9 and 1.12 imply B^p is the containing Banach space of H^p . The next theorem relates B^p to the closure of H^p in $(H^p)^{**}$.

Theorem 1.13. (Duren, Romberg, Shields [3], p. 46). For each $f \in H^p$ ($0 < p < 1$),

$$C(p)\|f\|_{B^p} \leq \|f\|_{(H^p)^{**}} \leq K(p)\|f\|_{B^p}.$$

Hence B^p is equivalent to the closure of H^p in $(H^p)^{**}$.

Finally, Duren, Romberg, and Shields [3] showed that B^p is itself a conjugate space.

Theorem 1.14. (Duren, Romberg, Shields [3], p. 49). If

$\frac{1}{n+1} < p < \frac{1}{n}$, $\alpha = (1/p) - n$, and $\varphi \in (\lambda_\alpha^{n-1})^*$, then there is

a unique function $f \in B^p$ such that

$$\varphi(g) = \langle f, g \rangle$$

for each $g \in \lambda_{\alpha}^{n-1}$. Conversely for each $f \in B^p$, $\langle f, g \rangle$ determines a bounded linear functional on λ_{α}^{n-1} . Furthermore, $(\lambda_{\alpha}^{n-1})^*$ and B^p are equivalent. If $p = \frac{1}{n+1}$, the above statements remain true with λ_{α}^{n-1} replaced by λ_{α}^{n-1} .

The coefficient properties of H^p and B^p functions have been studied by P. Duren and A. Shields, [4] and [5]. These properties are generally stated in terms of coefficient multipliers.

Definition 1.9. Let A and B be two complex sequence spaces. We say that a sequence $\{\lambda_n\}$ multiplies A into B if $\{\lambda_n a_n\} \in B$ whenever $\{a_n\} \in A$.

Each analytic function can be associated with its sequence of Taylor coefficients and hence H^p and B^p can be treated as sequence spaces.

Theorem 1.15. (P. Duren and A. Shields [5], p. 70). If $0 < p \leq 1$, then $\{\lambda_n\}$ multiplies H^p into ℓ^{∞} if and only if

$$(1.4) \quad \lambda_n = o(n^{1-(1/p)}).$$

If $p < 1$, then $\{\lambda_n\}$ multiplies B^p into ℓ^{∞} if and only if (1.4) holds.

Theorem 1.16. (P. Duren and A. Shields [5], p. 70). Let $0 < p < 1$, then

(i) $\{\lambda_n\}$ multiplies H^p into ℓ^q ($p \leq q < \infty$) if and only if

$$(1.5) \quad \sum_{n=1}^N n^{q/p} |\lambda_n|^q = o(N^q);$$

(ii) if $1 \leq q < \infty$, $\{\lambda_n\}$ multiplies B^p into ℓ^q if and only if (1.5) holds.

CHAPTER II

MIXED NORM SPACES

In [1], Benedick and Panzone studied the spaces $L^{p,q}$ of functions $f(x_1, x_2)$ such that

$$\|f\|_{p,q} = \left\{ \int_{X_2} \left[\int_{X_1} |f(x_1, x_2)|^p d\mu_1 \right]^{q/p} d\mu_2 \right\}^{1/q} < \infty,$$

where $x_i \in X_i$ and μ_i is a measure on X_i ($i = 1, 2$). A complex function $f(z) = f(re^{i\theta})$ may be considered as a function of r and θ . In particular, we will be concerned with the $L^{p,q}$ functions which are analytic in D . These classes of functions were introduced by J. H. Shapiro in [17] and are denoted $H^{p,q}$. Shapiro showed that if $0 < p, q \leq 1$, and $\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q}$, then B^σ is the containing Banach space of $H^{p,q}$. He also considered the relationships between these spaces in the case $0 < p \leq 1$, $q > 1$. In section 2, we continue this study. In particular, we consider the relationships between $H^{p,q}$ and B^σ when $p > 1$, and the relationships between H^σ and $H^{p,q}$ for all p and q .

In section 3, we make use of the relationships given in section 2 to study coefficients of $H^{p,q}$ functions.

1. Preliminaries.

We begin by defining the $H^{p,q}$ spaces and stating some of their basic properties.

Definition 2.1. Let $H^{p,q}$ denote the class of functions f analytic in D such that

$$(2.1) \quad \|f\|_{p,q} = \left\{ \int_0^1 (M_p(r,f))^q dr \right\}^{1/q} < \infty.$$

If $q = \infty$, define $H^{p,\infty}$ to be H^p . The corresponding classes $h^{p,q}$ of harmonic functions are defined similarly.

If $1 \leq p, q \leq \infty$, then (2.1) defines a norm and $H^{p,q}$, $h^{p,q}$ are Banach spaces. If $m = \min(p, q) < 1$, then $\|\cdot\|_{p,q}$ fails to be a norm, however, $\|\cdot\|_{p,q}^m$ induces a complete metric. The next two theorems summarize some of the general properties of $H^{p,q}$ determined by J. H. Shapiro in [17].

Theorem 2.1. (J. H. Shapiro [17], pp. 28-30). The spaces $H^{p,q}$ ($0 < p, q \leq \infty$) are complete with respect to the metric mentioned above. Furthermore,

$$(i) \quad |f(z)| \leq C(p,q) \|f\|_{p,q} (1-r)^{-((1/p)+(1/q))}$$

for each $f \in H^{p,q}$;

$$(ii) \quad \|f_\rho - f\|_{p,q} \rightarrow 0 \text{ as } \rho \rightarrow 1 \text{ for each } f \in H^{p,q}$$

($f_\rho(z) = f(\rho z)$).

Theorem 2.2. (J. H. Shapiro [17], pp. 40 and 52). If $m = \min(p, q) < 1$, then $H^{p,q}$ is not locally convex. If $0 < p < 1$, $0 < q \leq \infty$, then $H^{p,q}$ does not have the H.B.E.P.

To avoid repetition, we introduce the following notation.

Notation: If p and q are positive numbers such that

$\frac{1}{p} + \frac{1}{q} > 1$, then σ will denote the number defined by the equation:

$$\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q}.$$

2. Relationships between $H^{p,q}$, H^σ , B^σ .

We consider first the relationships between $H^{p,q}$ and B^σ . The case $0 < p, q \leq 1$ has been studied by J. H. Shapiro [17]. We present his results in the following two theorems.

Theorem 2.3. (J. H. Shapiro [17], p. 30). If $f \in H^{p,q}$, $0 < p, q \leq 1$, then $f \in B^\sigma$ and

$$\|f\|_{B^\sigma} \leq C(p,q) \|f\|_{p,q}.$$

Theorem 2.4. (J. H. Shapiro [17], pp. 35 and 37). Let $0 < p, q \leq 1$. If φ is a continuous linear functional on $H^{p,q}$, then there exists a unique $g \in \Psi^\sigma$ such that

$$(2.2) \quad \varphi(f) = \langle f, g \rangle$$

for each $f \in H^{p,q}$. Conversely, if $g \in \Psi^\sigma$, then (2.2) defines a continuous linear functional on $H^{p,q}$. Moreover, the spaces Ψ^σ and $(H^{p,q})^*$ are equivalent.

It follows then that $(B^\sigma)^*$ is equivalent to $(H^{p,q})^*$, $0 < p, q \leq 1$, and B^σ is the containing Banach space of $H^{p,q}$. Furthermore, if we consider $H^{p,q}$ and B^σ as subspaces of their second dual, then B^σ can be associated with the closure of $H^{p,q}$ in $(H^{p,q})^{**}$.

Theorem 2.5. For each $f \in H^{p,q}$, $0 < p, q \leq 1$,

$$C(p,q) \|f\|_{B^\sigma} \leq \|f\|_{(H^{p,q})^{**}} \leq K(p,q) \|f\|_{B^\sigma}.$$

Hence, B^σ is equivalent to the closure of $H^{p,q}$ in $(H^{p,q})^{**}$.

Proof: By Theorem 1.12, $(B^\sigma)^*$ and ψ^σ are equivalent under the correspondence $g \leftrightarrow \varphi$, where $\varphi(f) = \langle f, g \rangle$ for each $f \in B^\sigma$. Let $\tilde{\varphi}$ be φ restricted to $H^{p,q}$. Then $\tilde{\varphi}(f) = \langle f, g \rangle$ for each $f \in H^{p,q}$ and, by Theorem 2.4, $(H^{p,q})^*$ and ψ^σ are equivalent under the correspondence $g \leftrightarrow \tilde{\varphi}$. It follows that $(B^\sigma)^*$ and $(H^{p,q})^*$ are equivalent under the correspondence $\varphi \leftrightarrow \tilde{\varphi}$ and

$$C(p,q)\|\tilde{\varphi}\| \leq \|\varphi\| \leq K(p,q)\|\tilde{\varphi}\|.$$

If $f \in H^{p,q}$, then $f \in B^\sigma$ and since B^σ is a Banach space, we have

$$\begin{aligned} \|f\|_{B^\sigma} &= \|f\|_{(B^\sigma)^{**}} \\ &= \sup_{\|\varphi\| \neq 0} \frac{|\varphi(f)|}{\|\varphi\|} \\ &\leq \frac{1}{C(p,q)} \sup_{\|\tilde{\varphi}\| \neq 0} \frac{|\tilde{\varphi}(f)|}{\|\tilde{\varphi}\|} \\ &= \frac{1}{C(p,q)} \|f\|_{(H^{p,q})^{**}}. \end{aligned}$$

Similarly,

$$\|f\|_{(H^{p,q})^{**}} \leq K(p,q)\|f\|_{(B^\sigma)^{**}} = K(p,q)\|f\|_{B^\sigma}.$$

Let $\{f_n\}$ be a sequence in $H^{p,q}$, then $\{f_n\}$ is Cauchy in the $(H^{p,q})^{**}$ norm if and only if it is Cauchy in the B^σ norm. Thus, each element in the closure of $H^{p,q}$ in

$(H^{p,q})^{**}$ can be associated with an equivalence class of Cauchy sequences in $(H^{p,q})^{**}$ norm and hence with a unique B^σ function.

Let $0 < p \leq 1$, then B^σ is the containing Banach space for $H^{p,q}$ if either $0 < q \leq 1$ or $q = \infty$. It is natural to ask whether this is still true if $1 < q < \infty$.

Theorem 2.6. If $1 < q < \infty$, $0 < p < \frac{q}{q-1}$, then $H^{p,q}$ is not contained in B^σ .

Proof: The functions

$$(2.3) \quad f_{\alpha,\beta}(z) = (1 - z)^{-1/\alpha} \left\{ \frac{1}{z} \log \frac{1}{1-z} \right\}^{-1/\beta}, \quad \alpha > 0$$

were examined by J. E. Littlewood ([12], p. 93). He showed that if $\lambda > \alpha$, then

$$(2.4) \quad M_\lambda(r, f_{\alpha,\beta}) \simeq A(\alpha, \beta, \lambda) (1 - r)^{-(1/\alpha) + (1/\lambda)} (\log \frac{1}{1-r})^{-1/\beta}.$$

Thus, if we set $\alpha = \sigma$, then (i) $f_{\alpha,\beta} = f_{\sigma,\beta}$ is in $H^{p,q}$ if and only if $\beta < q$; (ii) $f_{\sigma,\beta}$ is in B^σ if and only if $\beta < 1$. In particular, if $q > 1$, we may choose $\beta = \frac{1}{2}(q+1)$. Then $1 < \beta < q$ and hence $f_{\sigma,\beta} \in H^{p,q}$ but $f_{\sigma,\beta} \notin B^\sigma$.

Although $H^{p,q}$ is not contained in B^σ if $q > 1$, we do have the following result.

Theorem 2.7. (J. H. Shapiro [17], p. 33). If $f \in H^{p,q}$, $0 < p \leq 1$, $1 < q < \infty$ then $f \in B^t$ for each $t < \sigma$ and

$$\|f\|_{B^t} \leq K(p, q, t) \|f\|_{p,q}.$$

We next turn our attention to the case $p > 1$.

Theorem 2.8. If $p > 1$, $0 < q < \frac{p}{p-1}$, $t < p$, then B^σ is contained in $H^{t,q}$ and

$$\|f\|_{t,q} \leq K(p,q,t) \|f\|_{B^\sigma}$$

for each $f \in B^\sigma$.

Proof: We may assume $t \geq 1$. If $f \in B^\sigma$, then

$$\begin{aligned} \|f\|_{B^\sigma} &\geq \int_r^1 M_1(\rho, f) (1 - \rho)^{(1/\sigma)-2} d\rho \\ &\geq M_1(r, f) (1 - r)^{(1/\sigma)-1} ((1/\sigma)-1)^{-1}. \end{aligned}$$

Thus,

$$(2.5) \quad M_1(r, f) \leq C(p, q) \|f\|_{B^\sigma} (1 - r)^{1-(1/\sigma)}.$$

If $t = 1$,

$$\begin{aligned} (2.6) \quad M_t^q(r, f) &= M_1^q(r, f) \\ &\leq C(p, q) \|f\|_{B^\sigma}^q (1 - r)^{q(\frac{p-1}{p})-1}. \end{aligned}$$

Hence

$$(2.7) \quad M_t^q(r, f) \leq K(p, q)^q \|f\|_{B^\sigma}^q (1 - r)^{q((1/t)-(1/p))-1}.$$

The conclusion follows by integrating (2.6) and (2.7) and then taking $1/q$ powers.

Theorem 2.9. If $p \geq 1$, $0 < q < \frac{p}{p-1}$, and $s < q$, then $B^\sigma \subset H^{p,s}$ and

$$\|f\|_{p,s} \leq K(p,q,s) \|f\|_{B^\sigma}$$

for each $f \in B^\sigma$.

Proof: If $f \in B^\sigma$, then we have by (2.5) that

$$M_1(r,f) \leq C(p,q) \|f\|_{B^\sigma} (1-r)^{1-(1/\sigma)}.$$

It follows from Theorem 1.1 that

$$M_p(r,f) \leq K(p,q) \|f\|_{B^\sigma} (1-r)^{-1/q}$$

and hence

$$M_p^s(r,f) \leq K_{p,q}^s \|f\|_{B^\sigma}^s (1-r)^{-s/q}.$$

Corollary 2.10. If $s < q \leq 1$, then

$$H^{1,q} \subset B^{q/(q+1)} \subset H^{1,s}.$$

If $q > 1$ and $s < t < q$, then

$$H^{1,q} \subset B^{t/(t+1)} \subset H^{1,s}.$$

Proof: The first statement follows from Theorems 2.4 and 2.9. The second statement follows from Theorems 2.7 and 2.9.

We next consider the relationships between $H^{p,q}$ and H^σ . These relationships can be determined using a theorem of Hardy and Littlewood.

Theorem 2.11. (G. H. Hardy and J. E. Littlewood [10], p. 411).

If $0 < \lambda < p \leq \infty$, $\alpha = \frac{1}{\lambda} - \frac{1}{p}$, $\ell \geq \lambda$ and $f \in H^\lambda$, then

$$\int_0^1 (M_p(r,f))^\ell (1-r)^{\ell\alpha-1} dr \leq K(\lambda,p,\ell) \|f\|_{H^\lambda}^\ell.$$

In particular, if we set $\lambda = \sigma$, $\ell = q = 1/\alpha$, then we obtain the following corollary.

Corollary 2.12. If $0 < p, q \leq \infty$ and $f \in H^\sigma$, $\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q}$, then $f \in H^{p,q}$ and

$$\|f\|_{p,q} \leq K(p,q) \|f\|_{H^\sigma}.$$

It is easy to see that the above containment is strict, since $H^{p,q}$ contains all functions f satisfying

$$f(z) = O((1 - r)^{-a}), \quad a < 1/q.$$

Thus, $H^{p,q}$ contains functions which are not of bounded characteristic. In fact, there exist functions having radial limits a.e. which are in $H^{p,q}$, but are not in H^σ .

The functions $f_{\alpha,\beta}$ defined by (2.3) have radial limits a.e. and J. E. Littlewood ([12], p. 96) showed that for $\lambda = \alpha$, $\lambda < \beta$

$$(2.8) \quad M_\lambda(r, f_{\alpha,\beta}) \approx A(\alpha,\beta) \left(\log \frac{1}{1-r} \right)^{\beta+(1/\lambda)}.$$

Thus, if we set $\alpha = \sigma$ and $\beta = \frac{1}{2}(q + \sigma)$, then $\sigma < \beta < q$ and it follows from (2.4) that $f_{\sigma,\beta} \in H^{p,q}$; however, (2.8) implies $f_{\sigma,\beta}$ is not in H^σ .

3. Coefficients.

It follows from Theorem 2.3 and Corollary 2.12 that if $0 < p, q \leq 1$, then $H^{p,q}$ is an intermediate space between H^σ and B^σ , i.e.,

$$H^\sigma \subset H^{p,q} \subset B^\sigma.$$

Thus, $H^{p,q}$ functions possess properties common to H^σ and B^σ functions. In particular, coefficient results for $H^{p,q}$ functions are obtained as immediate consequences of the H^σ and B^σ properties given in Chapter I.

Theorem 2.13. Let $0 < p, q \leq 1$.

- (i) If $f(z) = \sum a_n z^n \in H^{p,q}$, then $a_n = o(n^{(1/\sigma)-1})$, the exponent $(1/\sigma)-1$ is best possible.
- (ii) If $1 \leq s < \infty$, then $\{\lambda_n\}$ multiplies $H^{p,q}$ into ℓ^s if and only if

$$(2.9) \quad \sum_{n=1}^N n^{s/\sigma} |\lambda_n| = O(N^s);$$
- (iii) If $\sigma \leq s < 1$ and $\{\lambda_n\}$ multiplies $H^{p,q}$ into ℓ^s then (2.9) holds.
- (iv) $\{\lambda_n\}$ multiplies $H^{p,q}$ into ℓ^∞ if and only if $\lambda_n = o(n^{1-(1/q)})$.

Proof: The "o" condition in (i) holds for $H^{p,q}$ since it holds for B^σ . The exponent is best possible for H^σ (see [2], p. 98) and hence for $H^{p,q}$. If $\{\lambda_n\}$ satisfies (2.9), then by Theorem 1.16, $\{\lambda_n\}$ multiplies B^σ into ℓ^s and hence multiplies $H^{p,q}$ into ℓ^s . Conversely, if $\{\lambda_n\}$

multiplies $H^{p,q}$ into $\mathcal{L}^s$, then it multiplies H^σ into $\mathcal{L}^s$ and hence satisfies (2.9). This proves (ii) and (iii). (iv) is proved similarly using Theorem 1.15.

We remark that the "o" estimate in (i) is actually best possible in a much stronger sense. In [5], p. 70, P. Duren and A. Shields showed that if $\{\delta_n\}$ is any sequence of positive numbers such that $a_n = o(\delta_n)$ for every $f(z) = \sum a_n z^n$ in H^σ , then there exists an $\epsilon > 0$ such that

$$\delta_n n^{1-(1/\sigma)} \geq \epsilon > 0, \quad n = 1, 2, \dots, \quad .$$

In [4], p. 259, P. Duren and A. Shields showed that if $f(z) = \sum a_n z^n$ is in H^σ , then

$$(2.10) \quad \sum n^{-\delta} |a_n|^s < \infty$$

for $s > \sigma$, $\delta = 1 + s((1/\sigma)-1)$, $0 < \sigma \leq 1$. They then showed that B^σ and H^σ functions differ in allowable moduli of coefficients by giving an example of a B^σ function whose coefficients do not satisfy (2.10). A similar example may be provided for $H^{p,q}$. Indeed, the function $f_{\sigma,\beta}$ defined by (2.3) is in $H^{p,q}$ for $\beta = \frac{1}{2}(q + \sigma)$. However, J. E. Littlewood ([12], p. 93) has shown

$$a_n \simeq C(p,q) n^{(1/\sigma)-1} (\log n)^{-1/\beta}$$

so that

$$\sum n^{-\delta} |a_n|^s \geq K(p,q,s) \sum \frac{1}{n \log n} = \infty.$$

CHAPTER III

HARMONIC FUNCTIONS

In this chapter, the spaces h^p , b^p , $h^{p,q}$ of harmonic functions are studied. These spaces are defined in the same manner as the corresponding spaces of analytic functions.

Section 1 deals with the question of whether the harmonic conjugate of a function in one of the above mentioned classes is in the same class. The spaces h^p , b^p , and $h^{p,q}$ are treated in sections 2, 3 and 4 respectively. The general properties of each of the spaces as well as the relationships between the three spaces are discussed in these sections.

1. Conjugate functions.

If $p > 1$, the spaces h^p and H^p are very much alike. In fact, if $f = u + iv$ is analytic in D , then $f \in H^p$ if and only if $u \in h^p$. This is a consequence of the following well-known theorem of M. Riesz (see, for example, [2], p. 54).

Theorem 3.1. (M. Riesz). If $f = u + iv$ is analytic in D and $p > 1$, then

$$M_p(r, v) \leq C(p)M_p(r, u), \quad 0 \leq r < 1.$$

Thus, if $u \in h^p$, then $v \in h^p$.

In the case $p = 1$, $u \in h^1$ does not imply its harmonic conjugate v is in h^1 . However, we do have the following theorem (see [2], p. 57).

Theorem 3.2. (A. Kolmogorov). If $u \in h^1$, then its harmonic conjugate v is in h^p for all $p < 1$ and

$$M_p(r, v) \leq C(p) M_1(r, u), \quad 0 \leq r < 1.$$

If $p < 1$, the situation is much worse. G. H. Hardy and J. E. Littlewood ([9], p. 419) have shown that the function

$$\eta(z) = \operatorname{Re} \tau(z) = \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{z^n}{1+z^n} \right)$$

is in h^p for all $p < 1$, however, $\eta(z)$ has radial limits existing on a set of at most measure zero. If $\tau(z)$ is in H^p for some p then $\tau(z)$ must have radial limits a.e. But this would imply $\eta(z)$ has radial limits a.e. Thus, $\tau(z)$ is not in H^p for any p .

We next investigate whether theorems similar to Theorem 3.1 hold for $h^{p,q}$ and b^p . As an immediate consequence of Theorem 3.1, we have that if $u \in h^{p,q}$, $p > 1$, $0 < q \leq \infty$, then its harmonic conjugate $v \in h^{p,q}$ and

$$\|v\|_{p,q} \leq C(p) \|u\|_{p,q}.$$

The question of whether $u \in h^{p,q}$ implies $v \in h^{p,q}$ for $0 < p \leq 1$, $0 < q < \infty$ is still open, although it has been answered affirmatively in the case $q = p$.

Theorem 3.3. (G. H. Hardy and J. E. Littlewood [9], p. 413).

If $u \in h^{p,p}$, $0 < p < \infty$, then its conjugate $v \in h^{p,p}$ and

$$\|v\|_{p,p} \leq C(p)\|u\|_{p,p}.$$

The situation for b^p is much nicer.

Theorem 3.4. (P. Duren and A. Shields [4], p. 256). If $u \in b^p$, then its harmonic conjugate $v \in b^p$ and

$$\|v\|_{b^p} \leq C(p)\|u\|_{b^p}.$$

We may use Theorem 3.4 to show that most of the theorems concerning B^p given in Chapter I hold also for b^p .

2. The spaces h^p , $0 < p < 1$.

Recall that $u \in h^p$ if and only if

$$\|u\|_{h^p} = \sup_{r < 1} M_p(r, u) < \infty.$$

As was the case for H^p , $p < 1$, $\|\cdot\|_{h^p}$ does not satisfy the triangle inequality and hence is not a norm. However, $\|\cdot\|_{h^p}^p$ does obey the triangle inequality and defines a metric on h^p .

The properties of $M_p(r, u)$, u harmonic, have been studied by Hardy and Littlewood [9]. We sum up a number of their results in the following theorem.

Theorem 3.5. (Hardy and Littlewood [9], pp. 410-415). Let u be harmonic in D and $f(z) = \sum \gamma_n z^n$ be its analytic completion. If $0 < p \leq 1$, $a \geq 0$ and

$$M_p(r, u) \leq C(1 - r)^{-a}$$

then

- (i) $|\gamma_n| \leq B(p,a)C(n+1)^{a+(1/p)-1};$
- (ii) $|f(z)| \leq B(p,a)C(1-r)^{-a-(1/p)};$
- (iii) $M_p(r,f') \leq B(p,a)C(1-r)^{a-1};$
- (iv) if $a > 0$, then

$$M_p(r,v) \leq M_p(r,f) \leq B(p,a)C(1-r)^{-a}.$$

Theorem 3.5 will be used to prove some general theorems about h^p .

Theorem 3.6. The spaces h^p , $0 < p < \infty$ are complete.

Proof: If $p > 1$, the result follows from Theorem 3.1 and the completeness of H^p . Assume then that $p \leq 1$. By Theorem 3.5 (ii), we have

$$(3.1) \quad |u(z)| \leq B(p)\|u\|_{h^p}(1-r)^{-1/p}$$

for each $u \in h^p$. It follows from (3.1) that if $\{u_n\}$ is a Cauchy sequence in h^p , then $\{u_n\}$ converges uniformly on compact subsets of D to a harmonic function u . On the other hand, $\{u_n\}$ is a Cauchy sequence in $L^p(D)$ which is complete. Thus, $\{u_n\}$ converges in $L^p(D)$ norm to an $L^p(D)$ function g . But then there exists a subsequence which converges a.e. to g . Thus $u = g$ a.e. So $u \in h^p$ and $\{u_n\}$ converges to u in h^p norm.

Theorem 3.7. If $0 < p \leq 1$, then h^p has enough continuous linear functionals to separate points.

Proof: If u is harmonic in D , then u can be written in the form

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta}.$$

We may assume without loss of generality that u is real-valued.

Let $f(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ be the analytic completion of u . Then

$$(3.2) \quad c_n = \begin{cases} \frac{1}{2} \gamma_n & \text{if } n \geq 0 \\ \frac{1}{2} \overline{\gamma_n} & \text{if } n < 0. \end{cases}$$

It follows from Theorem 3.5 (i) that

$$|c_n| \leq B(p) \|u\|_{h^p} |n|^{(1/p)-1}.$$

For each integer n , define the operator φ_n on h^p by

$\varphi_n(u) = c_n$. φ_n is a continuous linear functional on h^p and

$$\|\varphi_n\| \leq B(p) |n|^{(1/p)-1}.$$

If $u_1 = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$ and $u_2 = \sum_{-\infty}^{\infty} d_n r^{|n|} e^{in\theta}$ are in h^p

and $u_1 \neq u_2$, then there exists n such that $c_n \neq d_n$. Hence $\varphi_n(u_1) \neq \varphi_n(u_2)$.

We will show next that h^p is not locally convex. This is a consequence of a more general theorem about the H.B.E.P.

Theorem 3.8. If S is an F space with the H.B.E.P. and A is a closed subspace of S , then A has the H.B.E.P.

Proof: Assume S has the H.B.E.P. and let B be closed in A . Then B is also closed in S . Thus, if $\phi \in B^*$, then there exists $\phi_S \in S^*$ extending ϕ . Let ϕ_A be the restriction of ϕ_S to A . Then $\phi_A \in A^*$ and ϕ_A extends ϕ to A .

Corollary 3.9. If $0 < p < 1$, then h^p does not have the H.B.E.P. and consequently is not locally convex.

Proof: H^p is a closed subspace of h^p , so the result is immediate from Theorems 1.8 and 3.8.

3. The spaces b^p , $0 < p < 1$.

Recall that for $0 < p < 1$, b^p is the class of functions u harmonic in D for which

$$(3.3) \quad \|u\|_{b^p} = \int_0^1 (1-r)^{(1/p)-2} M_1(r,u) dr < \infty.$$

Because of Theorem 3.4, most of the theorems stated for B^p in Chapter I hold also for b^p . The proofs of these theorems for b^p are either immediate consequences of Theorem 3.4 and the results for B^p or they are along the same lines as the proofs given by Duren, Romberg, and Shields in [3]. The relationships between h^p and b^p , however, are not in general the same as the relationships between the spaces of analytic functions. This is evidenced in the next theorem.

Theorem 3.10. (P. Duren and A. Shields [4], p. 256). If $u \in h^p$ ($0 < p \leq 1$), then $v \in b^q$ for all $q < p$. If $p = \frac{1}{k+1}$, k a positive integer, then $u \in h^p$ does not imply $v \in b^p$.

Hence, if $p = \frac{1}{k+1}$, h^p is not contained in b^p , however, h^p is contained in b^q for all $q < p$.

Corresponding to Theorem 1.9, we have the following theorem for b^p .

Theorem 3.11. The space b^p with norm (3.3) is a Banach space. Furthermore,

- (i) $|u(z)| \leq C(p) \|u\|_{b^p} (1-r)^{-(1/p)}$
for $u \in b^p$ and $u(z) = o((1-r)^{-(1/p)})$;
- (ii) for each $u \in b^p$, $u_\rho \rightarrow u$ in b^p norm as $\rho \rightarrow 1$
($u_\rho(z) = u(\rho z)$);
- (iii) if $p < q \leq \infty$, then h^q is dense in b^p .

Proof: Set $R = \frac{1}{2}(1+r)$, then

$$\begin{aligned}
 (3.4) \quad \|u\|_{b^p} &\geq \int_R^1 (1-\rho)^{(1/p)-2} M_1(\rho, u) d\rho \\
 &\geq M_1(R, u) \left(\frac{1}{p} - 1\right)^{-1} (1-R)^{(1/p)-1}.
 \end{aligned}$$

Hence

$$(3.5) \quad M_1(R, u) \leq \left(\frac{1}{p} - 1\right) \|u\|_{b^p} (1-R)^{1-(1/p)},$$

and since

$$\begin{aligned}
 |u(re^{i\theta})| &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{|Re^{it} - re^{i\theta}|^2} |u(Re^{it})| dt \\
 &\leq \frac{4}{1-r} M_1(R, u),
 \end{aligned}$$

we have

$$|u(re^{i\theta})| \leq C(p) \|u\|_{b^p} (1-r)^{-(1/p)}.$$

To show the "o" condition, note that for $\epsilon > 0$ given, the left hand side of (3.4) may be replaced by ϵ if r is sufficiently close to 1. This proves (i). The proof of completeness follows in the same manner as the proof of Theorem 3.6 using part (i) in place of (3.1) and noting that b^p lies in the L^1 space formed with respect to the measure $\frac{1}{2\pi} (1-r)^{(1/p)-2} dr d\theta$. (ii) follows from Theorem 1.9 since if $u \in b^p$, then its analytic completion $f \in B^p$ and

$$\|u - u_\rho\|_{b^p} \leq \|f - f_\rho\|_{B^p}.$$

Finally, (iii) follows from (ii) since $h^\infty \subset h^q \subset b^p$ and (ii) implies h^∞ is dense in b^p .

The next theorem is an immediate consequence of (3.2), Theorem 3.4 and Theorem 1.10.

Theorem 3.12. If $u(z) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta} \in b^p$, then

$$(3.6) \quad |c_n| \leq C(p) \|u\|_{b^p} |n|^{(1/p)-1}$$

and $c_n = o(|n|^{(1/p)-1})$. Conversely, if $0 < p < 1$ and $c_n = O(|n|^\alpha)$, $\alpha < (1/p)-3/2$, then $u \in b^p$. Furthermore, the $(1/p)-3/2$ is best possible (i.e., there exists $g = \sum_{-\infty}^{\infty} \beta_n r^{|n|} e^{in\theta}$ such that $\beta_n = O(|n|^{(1/p)-3/2})$ and $g \notin b^p$).

Proof: Let $f(z) = \sum \gamma_n z^n = u(z) + iv(z)$. Then by (3.2), Theorem 3.4 and Theorem 1.10,

$$|c_n| = \frac{1}{2} |\gamma_n| \leq \frac{1}{2} K(p) \|f\|_{B^p} |n|^{(1/p)-1}$$

$$\leq C(p) \|u\|_{b^p} |n|^{(1/p)-1}$$

and $|\gamma_n| = o(|n|^{(1/p)-1})$.

If $c_n = O(|n|^\alpha)$, then $\gamma_n = O(|n|^\alpha)$. So, by Theorem 1.10, $f \in B^p$ and hence $u \in b^p$. The $(1/p)-3/2$ is best possible for B^p and consequently for b^p . Alternatively, (3.6)

could have been derived by direct computation using (3.5)

since $c_n = \frac{1}{2} (a_n - ib_n)$, $c_{-n} = \overline{c_n}$ ($n > 0$) where

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} u(\operatorname{Re}^{i\theta}) \cos n\theta d\theta$$

and

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} u(\operatorname{Re}^{i\theta}) \sin n\theta d\theta.$$

Theorem 3.13. Suppose $0 < p < q < 1$ and let $\beta = \frac{1}{p} - \frac{1}{q}$.

- (i) If $u \in b^p$, then $u_{[\beta]} \in b^q$.
- (ii) If $u \in b^q$, then $u^{[\beta]} \in b^p$.

Proof: It is easily verified that if $u = \operatorname{Re} f$, then $u^{[\beta]} = \operatorname{Re} f^{[\beta]}$ and $u_{[\beta]} = \operatorname{Re} f_{[\beta]}$. The theorem then follows immediately from Theorems 3.4 and 1.11.

Theorems 3.11, 3.12 and 3.13 will be used to prove the major results of this section which follow in the next two theorems. Recall that ψ^p denotes the space of all functions g harmonic in D such that

$$\|g\|_{\Psi^p} = \sup_{r < 1} \{M_\infty(r, g^{[1/p]})(1-r)\} < \infty.$$

Theorem 3.14. Let $\varphi \in (b^p)^*$. Then there is a unique $g \in \Psi^p$ such that $\varphi(u) = \langle u, g \rangle$ for each $u \in b^p$. Conversely, if $g \in \Psi^p$, then $\langle u, g \rangle$ exists for all $u \in b^p$ and $\varphi(u) = \langle u, g \rangle$ defines a continuous linear functional on b^p .

Proof: Let φ be a continuous linear functional on b^p and set

$$b_k = \begin{cases} \varphi(z^k) & \text{for } k \geq 0 \\ \varphi(\bar{z}^{|k|}) & \text{for } k < 0, \end{cases}$$

then

$$|b_k| \leq \|\varphi\| \|z^{|k|}\|_{b^p}.$$

Now,

$$\|z^{|k|}\|_{b^p} \leq C(p) \|z^{|k|}\|_{H^p} = C(p).$$

(A more precise estimate of $\|z^{|k|}\|_{b^p}$ will be given in Chapter IV), so that

$$|b_k| \leq C(p) \|\varphi\|.$$

It follows that

$$g(z) = \sum_{-\infty}^{\infty} b_k r^{|k|} e^{ik\theta}$$

is harmonic in D . Let $u(z) = \sum_{-\infty}^{\infty} c_k r^{|k|} e^{ik\theta} \in b^p$. For fixed ρ , $0 < \rho < 1$, let $u_\rho(z) = u(\rho z)$. Since u_ρ is the uniform limit on $|z| = 1$ of the partial sums $(S_N = \sum_{-N}^N)$ of

its power series (hence the limit in the b^p norm) and since φ is continuous, it follows that

$$\begin{aligned}\varphi(u_\rho) &= \lim_{N \rightarrow \infty} \varphi\left(\sum_{-N}^N c_k \rho^{|k|} r^{|k|} e^{ik\theta}\right) \\ &= \sum_{-\infty}^{\infty} c_k b_k \rho^{|k|}.\end{aligned}$$

But $u_\rho \rightarrow u$ in b^p norm as $\rho \rightarrow 1$, so

$$\varphi(u) = \lim_{\rho \rightarrow 1} \sum_{-\infty}^{\infty} c_k b_k \rho^{|k|}.$$

Let $\xi = \lambda e^{i\beta} \in D$ and $h(z) = \frac{1+z}{1-z}$, $h(z) \in B^p$ for all $p < 1$.

Set

$$\begin{aligned}V(z) &= \operatorname{Re}(h(\xi z)) \\ &= \sum_{-\infty}^{\infty} \lambda^{|k|} e^{ik\beta} r^{|k|} e^{ik\theta},\end{aligned}$$

then

$$\begin{aligned}\varphi(V) &= \lim_{\rho \rightarrow 1} \sum_{-\infty}^{\infty} b_k \lambda^{|k|} \rho^{|k|} e^{ik\beta} \\ &= g(\xi).\end{aligned}$$

Thus,

$$|g(\xi)| \leq \|\varphi\| \|V\|_{b^p}$$

$$\leq \|\varphi\| \|h\|_{B^p}$$

so that g is in H^∞ and hence in H^1 . Now,

$$\begin{aligned}
& \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\theta}) g(e^{-i\theta}) d\theta \\
&= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} u(r\rho e^{i\theta}) g(re^{-i\theta}) d\theta \\
&= \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{-\infty}^{\infty} \rho^{|k|} r^{|k|} c_k e^{ik\theta} \right) \left(\sum_{-\infty}^{\infty} b_k r^{|k|} e^{-ik\theta} \right) d\theta \\
&= \sum_{-\infty}^{\infty} \rho^{|k|} c_k b_k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varphi(u) &= \lim_{\rho \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\theta}) g(e^{-i\theta}) d\theta \\
&= \langle u, g \rangle.
\end{aligned}$$

To show $g \in \psi^p$, we consider first the case $\frac{1}{n+1} < p < \frac{1}{n}$.

Let $F(z) = n! [2(1-z)^{-(n+1)} - 1]$ and set $U(z) = \operatorname{Re} F(\xi z)$

where $\xi = \rho e^{i\theta} \in D$. Then

$$U(z) = \sum_{-\infty}^{\infty} \frac{(n+|k|)!}{|k|!} \rho^{|k|} e^{ik\theta} r^{|k|} e^{ik\theta}$$

and hence

$$\varphi(U) = \sum_{-\infty}^{\infty} \frac{(n+|k|)!}{|k|!} \rho^{|k|} e^{ik\theta} b_k.$$

Now

$$g(\xi) = \sum b_k \rho^{|k|} e^{ik\theta}$$

so that

$$g^{[n]}(\xi) = \sum_{-\infty}^{\infty} \frac{(|k|+n)!}{|k|!} b_k \rho^{|k|} e^{ik\theta},$$

i.e.,

$$\varphi(U) = g^{[n]}(\xi).$$

It follows that

$$\begin{aligned} |g^{[n]}(\xi)| &\leq \|\varphi\| \|U\|_{b^p} \\ &\leq \|\varphi\| \|F_\rho\|_{B^p} \\ &\leq K(p) \|\varphi\| \|F_\rho\|_{H^p}, \end{aligned}$$

and by Theorem 1.3

$$\|F_\rho\|_{H^p} = O((1 - \rho)^{(1/p)-n-1}).$$

Let $\beta = \frac{1}{p} - n$ and set $h(z) = g^{[n]}(z)$. Then by Theorem 1.7,

$$(3.7) \quad |g^{[1/p]}(\xi)| = |h^{[\beta]}(\xi)| \leq C(p)((1 - |\xi|)^{-1})\|\varphi\|,$$

so that $g \in \Psi^p$.

If $p = \frac{1}{n+1}$, define

$$F(z) = (n+1)! [2(1 - z)^{-(n+2)} - 1]$$

and set

$$U(z) = \operatorname{Re} F(\xi z), \quad \xi \in D.$$

Then

$$\varphi(U) = g^{[n+1]}(\xi) = g^{[1/p]}(\xi)$$

and hence

$$\begin{aligned}
(3.8) \quad |g^{[1/p]}(\xi)| &\leq \|\varphi\| \|U\|_{b^p} \\
&\leq K(p) \|\varphi\| \|F\|_{H^p} \\
&\leq C(p) ((1 - |\xi|)^{-1}) \|\varphi\|.
\end{aligned}$$

Thus, $g \in \Psi^p$.

To show g is unique, suppose $g_1 = \sum_{-\infty}^{\infty} d_k r^{|k|} e^{ik\theta} \in \Psi^p$ and $\langle u, g_1 \rangle = \langle u, g \rangle$ for each $u \in b^p$. Let $u_k = r^{|k|} e^{ik\theta}$, $u_k \in b^p$ for each k . Furthermore, $b_k = \langle u_k, g \rangle = \langle u_k, g_1 \rangle = d_k$.

Hence $g = g_1$.

For the proof of the converse, assume first that $\frac{1}{n+1} < p < \frac{1}{n}$ and $g(z) = \sum_{-\infty}^{\infty} b_k r^{|k|} e^{ik\theta} \in \Psi^p$ is given. We must show that for every $u(z) = \sum_{-\infty}^{\infty} c_k r^{|k|} e^{ik\theta} \in b^p$, $\delta(r) = \sum_{-\infty}^{\infty} c_k b_k r^{|k|}$ has a limit as $r \rightarrow 1$ and that

$$\lim_{r \rightarrow 1} |\delta(r)| \leq C \|u\|_{b^p},$$

where C depends only on g and p . We will prove the existence of the limit by showing

$$\int_0^1 |\delta'(r)| dr < \infty.$$

Let

$$\begin{aligned}
h(z) &= \frac{\partial}{\partial r} (g^{[n-1]}(re^{i\theta})) \\
&= \frac{\partial}{\partial r} \left(\sum_{-\infty}^{\infty} \frac{(|k|+n-1)!}{|k|!} b_k r^{|k|} e^{ik\theta} \right).
\end{aligned}$$

Then

$$h(re^{-i\theta}) = \sum_{-\infty}^{\infty} \frac{(|k|+n-1)!}{(|k|-1)!} b_k r^{(|k|-1)} e^{-ik\theta}$$

and

$$u_{[n-1]}(re^{i\theta}) = \sum_{-\infty}^{\infty} \frac{|k|!}{(|k|+n-1)!} c_k r^{|k|} e^{ik\theta},$$

so that

$$\begin{aligned} r\delta'(r^2) &= r \sum_{-\infty}^{\infty} c_k b_k |k| r^{2(|k|-1)} \\ &= \sum_{-\infty}^{\infty} c_k b_k |k| r^{2|k|-1} \\ &= \frac{1}{2\pi} \int_0^{2\pi} u_{[n-1]}(re^{i\theta}) h(re^{-i\theta}) d\theta. \end{aligned}$$

By Theorem 3.13, we have that $u_{[n-1]} \in b^\beta$ where $\beta = \frac{p}{1-(n-1)p}$.

Now $g \in \Psi^p$ implies

$$g^{[1/p]}(z) = O((1-r)^{-1})$$

and so by Theorem 1.7,

$$g^{[n]}(z) = O((1-r)^{(1/p)-n-1}),$$

and hence

$$h(z) = O((1-r)^{(1/p)-n-1}).$$

Thus,

$$r|\delta'(r^2)| \leq C(1-r)^{(1/p)-n-1} M_1(r, u_{[n-1]}).$$

Finally, $u_{[n-1]} \in b^\beta$ implies

$$\int_0^1 (1-r)^{(1/\beta)-2} M_1(r, u_{[n-1]}) dr < \infty$$

and $\frac{1}{\beta} - 2 = \frac{1}{p} - n - 1$.

For the case $p = \frac{1}{n+1}$, set $U(z) = u_{[n-1]}(z) = \sum_{-\infty}^{\infty} A_k r^{|k|} e^{ik\theta}$, then $U(z) \in b^{1/2}$ by Theorem 3.13. Let

$$\begin{aligned} G(z) &= \frac{\partial}{\partial r} g^{[n-1]}(z) \\ &= \sum_{-\infty}^{\infty} \frac{(|k|+n-1)!}{(|k|-1)!} b_k r^{|k|-1} e^{ik\theta} \\ &= \sum_{-\infty}^{\infty} B_k r^{|k|} e^{ik\theta}. \end{aligned}$$

Then

$$r\delta'(r^2) = \frac{1}{2\pi} \int_0^{2\pi} U(re^{i\theta}) G(re^{-i\theta}) d\theta.$$

Set

$$J(z) = U_{[1/2]}(z) = \sum_{-\infty}^{\infty} \frac{|k|!}{\Gamma(|k|+3/2)} A_k r^{|k|} e^{ik\theta}.$$

Then by Theorem 3.13, $J(z) \in b^{2/3}$ and hence

$$\int_0^1 (1-r)^{-1/2} M_1(r, J) dr < \infty.$$

Let

$$K(z) = G^{[1/2]}(z) = \sum_{-\infty}^{\infty} \frac{\Gamma(|k|+3/2)}{|k|!} B_k r^{|k|} e^{ik\theta}.$$

It follows from Theorem 1.7 that

$$K(z) = O((1 - r)^{-1/2})$$

(since $G'(z) = O((1 - r)^{-1})$). Now

$$\frac{1}{2\pi} \int_0^{2\pi} U(re^{i\theta})G(re^{-i\theta})d\theta = \frac{1}{2\pi} \int_0^{2\pi} J(re^{i\theta})K(re^{-i\theta})d\theta$$

and hence

$$\int_0^1 \delta'(r^2)dr \leq C \int_0^1 (1 - r)^{-1/2} M_1(r, J)dr < \infty.$$

Finally, we must show that the operator φ , defined by

$$\varphi(u) = \langle u, g \rangle$$

for each $u \in b^p$, is bounded. For fixed $p < 1$, let

$$\varphi_p(u) = \sum_{-\infty}^{\infty} c_k b_k p^{|k|}.$$

Then by Theorem 3.12

$$|\varphi_p(u)| \leq C(p) \|u\|_{b^p} \sum_{-\infty}^{\infty} b_k |k|^{(1/p)-1} p^{|k|},$$

so $\varphi_p \in (b^p)^*$. But for fixed $u \in b^p$,

$$\begin{aligned} \sup_{p < 1} |\varphi_p(u)| &= \lim_{p \rightarrow 1} \left| \sum_{-\infty}^{\infty} c_k b_k p^{|k|} \right| \\ &= |\langle u, g \rangle| < \infty. \end{aligned}$$

Thus, by the Principle of Uniform Boundedness $\varphi \in (b^p)^*$.

Theorem 3.15. The Banach spaces $(b^p)^*$ and ψ^p are equivalent.

Proof: The mapping $T: \varphi \rightarrow g$ defined as in Theorem 3.14 is a one-to-one linear mapping of $(b^p)^*$ onto ψ^p . It follows

from (3.7) and (3.8) that T is bounded. The Open Mapping Theorem implies T^{-1} is bounded.

Theorems 3.14 and 3.15 give a characterization of the dual space of b^p . These results correspond to Theorem 1.12 for B^p . In Theorem 1.14, it was observed that B^p is itself a conjugate space. We will show next, by the same methods used in [3], that b^p is the conjugate of ψ^p .

Given $u \in b^p$, we may define the operator φ_u on ψ^p by

$$(3.9) \quad \varphi_u(g) = \langle u, g \rangle$$

for each $g \in \psi^p$. For fixed $g \in \psi^p$, with $\|g\|_{\psi^p} = 1$, let φ_g be the operator on b^p corresponding to g in Theorem 3.14, then

$$\begin{aligned} |\varphi_u(g)| &= |\varphi_g(u)| \\ &\leq \|\varphi_g\| \|u\|_{b^p}. \end{aligned}$$

By Theorem 3.15, we have

$$\|\varphi_g\| \leq C(p) \|g\|_{\psi^p}$$

so that

$$|\varphi_u(g)| \leq C(p) \|u\|_{b^p}.$$

Hence $\varphi_u \in (\psi^p)^*$ and

$$\|\varphi_u\|_{(\psi^p)^*} \leq C(p) \|u\|_{b^p}.$$

Similarly $\varphi_u \in (\psi^p)^*$ and

$$\|\varphi_u\|_{(\psi^p)^*} \leq C(p) \|u\|_{b^p}.$$

We may now define a new norm on b^p by

$$(3.10) \quad |||u||| = \|\varphi_u\|_{(\psi^p)^*}$$

for each $u \in b^p$.

Lemma 3.16. The norm (3.10) is equivalent to the b^p norm, i.e.,

$$\|u\|_{b^p} \leq K(p) |||u||| \leq C(p) \|u\|_{b^p}$$

for each $u \in b^p$.

Proof: The above remarks give

$$|||u||| \leq C(p) \|u\|_{b^p}.$$

It remains to show

$$\|u\|_{b^p} \leq K(p) |||u|||.$$

Since $(b^p)^*$ is equivalent to ψ^p , it follows that $(b^p)^{**}$ is equivalent to $(\psi^p)^*$. Hence

$$\|u\|_{b^p} = \|\varphi_u\|_{(b^p)^{**}} \leq K(p) \|\varphi_u\|_{(\psi^p)^*}.$$

Thus, it suffices to show

$$\|\varphi_u\|_{(\psi^p)^*} \leq \|\varphi_u\|_{(\psi^p)^*}.$$

Let $\epsilon > 0$ be given and choose $g \in \psi^p$ such that $\|g\|_{\psi^p} = 1$ and

$$|\langle u, g \rangle| > \|\varphi_u\|_{(\psi^p)^*} - \epsilon.$$

Now $g_\rho \in \psi^p$ and

$$\|g_\rho\|_{\psi^p} \leq \|g\|_{\psi^p} = 1$$

so that

$$\begin{aligned} |\langle u, g_\rho \rangle| &\leq \|g_\rho\|_{\psi^p} \|\varphi_u\|_{(\psi^p)^*} \\ &\leq \|\varphi_u\|_{(\psi^p)^*}. \end{aligned}$$

But, $\langle u, g_\rho \rangle \rightarrow \langle u, g \rangle$ as $\rho \rightarrow 1$. Hence

$$\|\varphi_u\|_{(\psi^p)^*} \leq \|\varphi_u\|_{(\psi^p)^*} + \epsilon.$$

Theorem 3.17. If $0 < p < 1$ and $\varphi \in (\psi^p)^*$, then there exists a unique $u \in b^p$ such that $\varphi(g) = \langle u, g \rangle$ for all $g \in \psi^p$. Conversely, each $u \in b^p$ determines a bounded linear functional on ψ^p by the above formula. Finally, the Banach spaces $(\psi^p)^*$ and b^p are equivalent.

Proof: In view of Lemma 3.16 and the remarks preceding it, we need only show that if $\varphi \in (\psi^p)^*$, then there exists $u \in b^p$ such that $\varphi(g) = \langle u, g \rangle$ for each $g \in \psi^p$. For a given $\varphi \in (\psi^p)^*$, define

$$c_k = \begin{cases} \varphi(z^k) & \text{for } k \geq 0 \\ \varphi(\bar{z}^{|k|}) & \text{for } k < 0 \end{cases}$$

and let

$$u(z) = \sum_{-\infty}^{\infty} c_k r^{|k|} e^{ik\theta}.$$

Then

$$\begin{aligned} |c_k| &\leq \|\varphi\| \|z^{|k|}\|_{\psi^p} \\ &\leq C(p) \|\varphi\| |k|^{(1/p)} \end{aligned}$$

(where Stirling's formula ([6], p. xv) has been used to estimate $\|z^{|k|}\|_{\psi^p}$). It follows that u is harmonic in D . Let $g(z) = \sum b_k r^{|k|} e^{ik\theta}$ be in ψ^p and $g_\rho(z) = g(\rho z)$. Since g_ρ is the uniform limit of the partial sums of its power series, we have

$$\begin{aligned} \varphi(g_\rho) &= \lim_{N \rightarrow \infty} \varphi\left(\sum_{-N}^N b_k \rho^{|k|} r^{|k|} e^{ik\theta}\right) \\ &= \sum_{-\infty}^{\infty} c_k b_k \rho^{|k|}. \end{aligned}$$

We claim that $\|g_\rho - g\|_{\psi^p} \rightarrow 0$ as $\rho \rightarrow 1$. Indeed since

$$g^{[1/p]}(z) = o((1 - r)^{-1}),$$

we can choose R sufficiently close to 1 so that

$$(1 - r) |g^{[1/p]}(z)| < \epsilon/2 \quad \text{for } r > R.$$

It follows that

$$(1 - r)|g_{\rho}^{[1/p]}(z)| < \epsilon/2 \quad \text{for } r > R$$

and hence

$$(3.11) \quad (1 - r)|g_{\rho}^{[1/p]}(z) - g^{[1/p]}(z)| < \epsilon \quad \text{for } r < R.$$

Choose ρ_0 such that if $\rho_0 < \rho < 1$ then

$$(3.12) \quad |g_{\rho}^{[1/p]}(z) - g^{[1/p]}(z)| < \epsilon$$

for $|z| \leq R$. It follows from (3.11) and (3.12) that

$$\|g_{\rho} - g\|_{\psi^p} < \epsilon \quad \text{for } \rho > \rho_0.$$

Hence

$$\begin{aligned} \varphi(g) &= \lim_{\rho \rightarrow 1} \varphi(g_{\rho}) \\ &= \lim_{\rho \rightarrow 1} \sum_{k=-\infty}^{\infty} c_k b_k \rho^{|k|}. \end{aligned}$$

To show $u \in b^p$, define

$$\varphi_R(g) = \langle u_R, g \rangle$$

for each $g \in \psi^p$, where $u_R(z) = u(Rz)$, $0 < R < 1$. $\varphi_R \in (\psi^p)^*$ since $u_R \in b^p$. For each fixed $g \in \psi^p$

$$\begin{aligned} \lim_{R \rightarrow 1} \varphi_R(g) &= \lim_{R \rightarrow 1} \langle u_R, g \rangle \\ &= \lim_{R \rightarrow 1} \langle u, g_R \rangle \\ &= \varphi(g). \end{aligned}$$

Therefore, by the Principle of Uniform Boundedness,

$\{\|\varphi_R\|_{(\psi^p)^*}, 0 < R < 1\}$ is uniformly bounded. It follows from Lemma 3.16 that $\{\|u_R\|_{b^p}, 0 < R < 1\}$ is uniformly bounded. However, the b^p norm is an L^1 norm and $u_R \rightarrow u$ pointwise. So by Fatou's Lemma

$$\|u\|_{b^p} \leq \underline{\lim} \|u_R\|_{b^p} < \infty.$$

4. The spaces $h^{p,q}$.

Recall that $h^{p,q}$ is the space of all functions u harmonic in D such that

$$\|u\|_{p,q} = \left\{ \int_0^1 M_p^q(r,u) dr \right\}^{1/q} < \infty$$

and that $h^{p,q}$ is a Banach space if and only if

$$m = \min(p,q) \geq 1.$$

Theorem 3.18. If $m = \min(p,q) \leq 1$ and $u \in h^{p,q}$, then $u \in b^{m/2}$ and

$$\|u\|_{b^{m/2}} \leq C(p,q) \|u\|_{p,q}.$$

Proof: If $u \in h^{p,q}$, then $u \in h^{m,m}$. It follows from Theorem 3.3 that $f = u + iv \in H^{m,m}$ and

$$\begin{aligned} \|f\|_{m,m} &\leq 2^{1/m} (\|u\|_{m,m} + \|v\|_{m,m}) \\ &\leq C(m) \|u\|_{m,m}. \end{aligned}$$

Furthermore, since $f \in H^{m,m}$, we have by Theorem 2.3 that $f \in B^{m/2}$ and

$$\|f\|_{B^{m/2}} \leq K(m) \|f\|_{m,m}.$$

Thus, $u \in b^{m/2}$ and

$$\begin{aligned} \|u\|_{b^{m/2}} &\leq \|f\|_{B^{m/2}} \\ &\leq K(m) \|f\|_{m,m} \\ &\leq C(m) K(m) \|u\|_{m,m} \\ &\leq C(m) K(m) \|u\|_{p,q}. \end{aligned}$$

Corollary 3.19. If $m = \min(p, q) \leq 1$ and

$$u(z) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta} \in h^{p,q}$$

then

$$|u(z)| \leq C(p, q) \|u\|_{p,q} (1 - r)^{-2/m}$$

and

$$|c_n| \leq K(p, q) \|u\|_{p,q} |n|^{(2/m)-1}.$$

Proof: The result follows immediately from Theorems 3.11(i), 3.12 and 3.18.

If $p > 1$ and $u \in h^{p,q}$, then by the remarks preceding Theorem 3.3, we have that $f \in H^{p,q}$ and

$$\|f\|_{p,q} \leq C(p)\|u\|_{p,q}.$$

It follows from Theorem 2.1 that

$$|u(z)| \leq C(p,q)\|u\|_{p,q}(1-r)^{-(1/\sigma)}$$

where $\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q}$.

Theorem 3.20. The spaces $h^{p,q}$ are complete.

Proof: If $p > 1$, then the result follows from the completeness of $H^{p,q}$. If $p \leq 1$, then $m = \min(p,q) \leq 1$ and the proof follows in the same manner as the proof of Theorem 3.6, treating $h^{p,q}$ as a subspace of $L^{p,q}$, and using Corollary 3.19 instead of (3.1).

Theorem 3.21. If $m = \min(p,q) < 1$, then $h^{p,q}$ is not locally convex. If $0 < p < 1$, $0 < q \leq \infty$, then $h^{p,q}$ does not have the H.B.E.P.

Proof: If $h^{p,q}$ is locally convex, then since $H^{p,q}$ is a linear subspace of $h^{p,q}$, it must also be locally convex. But this contradicts Theorem 2.2. The second statement is immediate from Theorems 2.2 and 3.8.

In Chapter II, we studied the relationships between H^σ , $H^{p,q}$, and B^σ . It is natural to ask whether the same relationships hold between h^σ , $h^{p,q}$, and b^σ . It has been observed that in general h^σ is not contained in b^σ . Hardy and Littlewood ([9], p. 416) showed that the function

$$u(z) = \operatorname{Re} f(z) = \operatorname{Re}(e^{(1/2)k\pi i}(1-z)^{-k-1})$$

is in h^σ for $\sigma = \frac{1}{k+1}$ (k a positive integer). However, $f(z)$ is not in B^σ and hence $u(z)$ is not in b^σ (see [4], p. 257). A similar example may be used to show h^σ is not contained in $h^{p,q}$ for the proper choice of p and q .

Theorem 3.22. If $p = q = \frac{1}{k}$, $\sigma = \frac{1}{2k}$, k a positive integer, then h^σ is not contained in $h^{p,q}$.

Proof: Let

$$f(z) = u(z) + iv(z) = e^{i(2k-1)\frac{\pi}{2}}(1-z)^{-2k}.$$

By the above remarks, $u(z) \in h^\sigma$ for $\sigma = \frac{1}{2k}$. On the other hand,

$$(M_p(r, f))^q = \frac{1}{2\pi} \int_0^{2\pi} |1-z|^{-2} d\theta$$

and thus by Theorem 1.3,

$$(M_p(r, f))^q \geq \frac{C}{1-r}.$$

Hence, $f \notin H^{p,q}$ and since $p = q = \frac{1}{k}$, we have by Theorem 3.3, $u \notin h^{p,q}$.

Although in general h^σ is not contained in b^σ , we know that h^σ is contained in b^t for all $t < \sigma$. A similar result holds for $h^{p,q}$.

Theorem 3.23. If $u \in h^\sigma$, $\sigma = (\frac{1}{p} + \frac{1}{q})^{-1} \leq 1$, then $f \in H^{s,t}$ and $u \in h^{s,t}$ where $s \leq p$, $t \leq q$ and $(s,t) \neq (p,q)$. Furthermore,

$$\|u\|_{s,t} \leq \|f\|_{s,t} \leq K(s,t,p,q) \|u\|_\sigma.$$

Proof: We need only consider the cases (i) $t = q$, $s < p$ and (ii) $t < q$, $s = p$. To prove (i), we may assume $\sigma < s < p$.

Then by Theorem 3.5 (iii), we have that $u \in h^\sigma$ implies

$$M_\sigma(r, f') \leq C(\sigma) \|u\|_{h^\sigma} (1 - r)^{-1}$$

and hence by Theorem 1.1

$$M_s(r, f') \leq K(\sigma) \|u\|_{h^\sigma} (1 - r)^{-1 - (1/\sigma) + (1/s)}.$$

Therefore by Theorem 1.2,

$$M_s(r, f) \leq E(\sigma, s) \|u\|_{h^\sigma} (1 - r)^{(1/s) - (1/\sigma)}.$$

Let $\epsilon = \frac{1}{s} - \frac{1}{p} > 0$, then

$$M_s^q(r, q) \leq (E(\sigma, s))^q \|u\|_{h^\sigma}^q (1 - r)^{q\epsilon - 1},$$

and hence

$$\|f\|_{s, q} \leq C(\sigma, s) \|u\|_{h^\sigma}.$$

To show (ii), we have as in the argument above,

$$M_p(r, f') \leq C(\sigma) \|u\|_{h^\sigma} (1 - r)^{-1 - (1/q)}$$

and hence

$$M_p(r, f) \leq K(\sigma, p) \|u\|_{h^\sigma} (1 - r)^{-(1/q)}.$$

Thus,

$$M_p^t(r, f) \leq (K(\sigma, p))^t \|u\|_{h^\sigma}^t (1 - r)^{-(t/q)},$$

so that

$$\|f\|_{p, t} \leq C(p, q, t) \|u\|_{h^\sigma}.$$

We remark that if $\sigma = (\frac{1}{p} + \frac{1}{q})^{-1} > 1$, then $h^\sigma \subset h^{p,q}$ and for $u \in h^\sigma$,

$$\|u\|_{p,q} \leq C(p,q) \|u\|_{h^\sigma}.$$

The containment is strict by the remarks following Corollary 2.12.

We next consider the relationships between $h^{p,q}$ and b^σ . If $q > 1$, we have by Theorem 2.6 that $h^{p,q}$ is not contained in b^σ ($\frac{1}{\sigma} = \frac{1}{p} + \frac{1}{q}$). As a consequence of Theorem 3.4, we have that if $p > 1$, then Theorem 2.8 remains valid if the spaces of analytic functions are replaced by the corresponding spaces of harmonic functions. Similarly, if $p \geq 1$, then Theorem 2.9 holds for harmonic functions. If $m = \min(p,q) \leq 1$, then by Theorem 3.18, $h^{p,q} \subset b^{m/2}$. In particular, $h^{p,p} \subset b^{p/2}$, $0 < p \leq 1$.

Theorem 3.24. If $0 < p \leq 1$, and $g \in \Psi^{p/q}$, then $\langle u, g \rangle$ exists for each $u \in h^{p,p}$ and defines a continuous linear functional on $h^{p,p}$. Conversely, for each continuous linear functional ϕ on $h^{p,p}$, there is a unique $g \in \Psi^{p/2}$ such that $\phi(u) = \langle u, g \rangle$ for each $u \in h^{p,p}$. Moreover,

$$\|g\|_{\Psi^{p/2}} \leq C(p) \|\phi\|.$$

Proof: If $g \in \Psi^{p/2}$, then $\phi(u) = \langle u, g \rangle$ defines a continuous linear functional on $b^{p/2}$. Thus, in view of Theorem 3.18, ϕ restricted to $h^{p,p}$ defines a continuous linear functional on $h^{p,p}$.

Conversely, if $\varphi \in (h^{p,p})^*$, let $u \in h^{p,p}$ and f be its analytic completion. Set $u_\rho(z) = u(\rho z)$ and $f_\rho(z) = f(\rho z)$ for $0 < \rho < 1$. Then

$$\|u - u_\rho\|_{p,p} \leq \|f - f_\rho\|_{p,p}$$

and hence by Theorem 2.1,

$$\lim_{\rho \rightarrow 1} \|u - u_\rho\|_{p,p} = 0.$$

Since φ is continuous,

$$\varphi(u) = \lim_{\rho \rightarrow 1} \varphi(u_\rho).$$

Let

$$b_n = \begin{cases} \varphi(z^n) & \text{for } n \geq 0 \\ \varphi(\bar{z}^{|n|}) & \text{for } n < 0. \end{cases}$$

If $u(z) = \sum c_n r^{|n|} e^{in\theta}$, then it follows as in Theorem 3.14 that

$$\begin{aligned} \varphi(u_\rho) &= \lim_{N \rightarrow \infty} \varphi\left(\sum_{-N}^N c_n (\rho r)^{|n|} e^{in\theta}\right) \\ &= \sum_{-\infty}^{\infty} c_n b_n \rho^{|n|} \end{aligned}$$

and hence

$$\varphi(u) = \lim_{\rho \rightarrow 1} \sum_{-\infty}^{\infty} c_n b_n \rho^{|n|}.$$

Since φ is bounded,

$$\begin{aligned} |b_n| &\leq \|\varphi\| \|z^{|n|}\|_{p,p} \\ &\leq C(p) \|\varphi\| |n|^{-(1/p)}. \end{aligned}$$

So $g(z) = \sum_{-\infty}^{\infty} b_n r^{|n|} e^{in\theta}$ is harmonic in D . Let $F(z) = \frac{1+z}{1-z} \in H^{p,p}$ and set $U(z) = \operatorname{Re} F(\xi z)$ where $\xi = \rho e^{i\alpha}$, $\rho < 1$. Then as in Theorem 3.14, $\varphi(U) = g(\xi)$ and $|g(\xi)| \leq \|\varphi\| \|U\|_{p,p} \leq \|\varphi\| \|F\|_{H^{p,p}}$ so that $g \in H^\infty$. If $\beta > 0$, then

$$\begin{aligned} F^{[\beta]}(\xi z) &= 2 \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} (\xi z)^n - \Gamma(\beta+1) \\ &= \Gamma(\beta+1) [2(1 - \xi z)^{-(\beta+1)} - 1] \end{aligned}$$

and

$$\begin{aligned} U^{[\beta]}(z) &= \operatorname{Re} F^{[\beta]}(\xi z) \\ &= \sum_{-\infty}^{\infty} \frac{\Gamma(n+\beta+1)}{n!} (\rho r)^{|k|} e^{ik\alpha} e^{ik\theta}. \end{aligned}$$

Hence,

$$\varphi(U^{[\beta]}) = g^{[\beta]}(\xi).$$

In particular for $\beta = 2/p$

$$\begin{aligned} |g^{[2/p]}(\xi)| &\leq \|\varphi\| \|U^{[2/p]}\|_{p,p} \\ &\leq \|\varphi\| \|F^{[2/p]}\|_{p,p} \\ &= \Gamma((2/p)+1) \|\varphi\| \|2(1 - \xi z)^{-(2/p)-1} - 1\|_{p,p}. \end{aligned}$$

It follows from Theorem 1.3 that

$$\|2(1 - \xi z)^{-(2/p)-1} - 1\|_{p,p} \leq K(p)(1 - |\xi|)^{-1}.$$

Hence $g \in \psi^{p/2}$ and

$$\|g\|_{\psi^{p/2}} \leq C(p)\|\varphi\|.$$

Corollary 3.25. $(h^{p,p})^*$ and $\psi^{p/2}$ are equivalent.

Proof: The mapping $\varphi \rightarrow g$ defined in Theorem 3.24 is continuous, one-to-one, and onto. Its inverse is continuous by the Open Mapping Theorem.

In the same manner Theorem 2.5 was proved, we may show:

Theorem 3.26. $b^{p/2}$ is equivalent to the closure of $h^{p,p}$ in $(h^{p,p})^{**}$, $0 < p \leq 1$.

Thus, $b^{p/2}$ is the containing Banach space for $h^{p,p}$.

Note, that for $p = 1$, the spaces are identical.

CHAPTER IV
COMPOSITION OPERATORS

Let ϕ be a nonconstant analytic function mapping D into itself. If f is analytic in D , set $C_\phi(f) = f \circ \phi$ where $f \circ \phi(z) = f(\phi(z))$. C_ϕ defines a linear operator on H^p and B^p .

It was shown by J. Ryff [15] that C_ϕ is a bounded operator on H^p , $0 < p < \infty$. In [13], E. Nordgren studied the operators C_ϕ on H^2 for ϕ an inner function. Composition operators on H^p , $1 \leq p \leq \infty$, were studied by H. J. Schwartz [16]. We intend to present a similar study for B^p .

In section 1, it is shown that C_ϕ is a bounded linear operator on B^p . Upper and lower bounds on $\|C_\phi\|$ are given, and a necessary and sufficient condition is determined in order that C_ϕ be an isometry. In section 2, the methods of H. J. Schwartz [16] are used to characterize those operators on B^p which are composition operators. This characterization is then used to determine which composition operators are invertible. In section 3, conditions are given on ϕ in order that C_ϕ be a bounded operator from B^p into H^q , $0 < q \leq \infty$. Finally in section 4, compact composition operators and their spectra are discussed.

1. Bounds on C_ϕ .

It has already been remarked that C_ϕ is a bounded linear operator on H^p . This was shown by J. Ryff [15] as part of the following theorem.

Theorem 4.1. (J. Ryff [15], p. 348). Let $0 < p < \infty$.

Let f be analytic in D and ϕ be an analytic function mapping D into D .

(i) If ϕ maps $|z| \leq r$ into $|z| \leq R$, then

$$M_p(r, f \circ \phi) \leq \left(\frac{R+|\phi(0)|}{R-|\phi(0)|} \right)^{1/p} M_p(R, f).$$

(ii) If $f \in H^p$, then $f \circ \phi \in H^p$. The operator C_ϕ , defined by $C_\phi(g) = g \circ \phi$ for all $g \in H^p$, is a bounded linear operator on H^p and

$$\|C_\phi\| \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{1/p}.$$

(iii) If $\phi(0) = 0$ and for some r , $0 < r < 1$,

$$M_p(r, f \circ \phi) = M_p(r, f), \text{ then either } \phi(z) = \epsilon z, \\ |\epsilon| = 1 \text{ or } f \text{ is constant.}$$

Proof: To prove (i), we will first assume ϕ maps $|z| \leq r$ into $|z| < R$. Let $\alpha_1, \dots, \alpha_n$ be the zeros of f in $|z| \leq R$ where each zero is counted according to its multiplicity. Let

$$b_k(z) = \frac{R(z - \alpha_k)}{R^2 - \alpha_k z}$$

and

$$B(z) = \prod_{k=1}^n b_k(z).$$

It is easily seen that $|B(z)| \leq 1$ in $|z| \leq R$ and $|B(z)| = 1$ if and only if $|z| = R$. If f has no zeros in $|z| \leq R$, set $B(z) \equiv 1$. Note that $(f/B)^p$ is analytic in $|z| \leq R$.

If $|z| \leq r$, then

$$\left[\frac{f(\phi(z))}{B(\phi(z))} \right]^p = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{f(\operatorname{Re}^{it})}{B(\operatorname{Re}^{it})} \right]^p \frac{R^2 - |\phi(z)|^2}{|\operatorname{Re}^{it} - \phi(z)|^2} dt$$

and

$$(4.1) \quad \frac{1}{2\pi} \int_0^{2\pi} |f(\phi(\operatorname{re}^{i\theta}))|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(\phi(\operatorname{re}^{i\theta}))}{B(\phi(\operatorname{re}^{i\theta}))} \right|^p d\theta$$

since $|B(\phi(\operatorname{re}^{i\theta}))| \leq 1$. Thus,

$$(4.2) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(\phi(\operatorname{re}^{i\theta}))|^p d\theta \\ \leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |f(\operatorname{Re}^{it})|^p \frac{R^2 - |\phi(z)|^2}{|\operatorname{Re}^{it} - \phi(z)|^2} dt d\theta \end{aligned}$$

(since $B(\operatorname{Re}^{it}) = 1$). If the order of integration is changed, we may use the well-known property of the Poisson kernel that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |\phi(z)|^2}{|\operatorname{Re}^{it} - \phi(z)|^2} d\theta &= \frac{R^2 - |\phi(0)|^2}{|\operatorname{Re}^{it} - \phi(0)|^2} \\ &\leq \frac{R + |\phi(0)|}{R - |\phi(0)|}. \end{aligned}$$

It follows that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\operatorname{re}^{i\theta})|^p d\theta \leq \left(\frac{R + |\phi(0)|}{R - |\phi(0)|} \right)^{1/p} M_p(r, f).$$

To prove (i) in the case ϕ maps $|z| \leq r$ into $|z| \leq R$, let $R_n = R + \frac{1-R}{n}$, $n \geq 2$. So, ϕ maps $|z| \leq r$ into $|z| < R_n$ and hence

$$M_p(r, f) \leq \left(\frac{R_n + |\phi(0)|}{R_n - |\phi(0)|} \right)^{1/p} M_p(R_n, f).$$

Letting $n \rightarrow \infty$,

$$M_p(r, f) \leq \left(\frac{R + |\phi(0)|}{R - |\phi(0)|} \right)^{1/p} M_p(R, f).$$

To prove (ii), let $f \in H^p$ and assume ϕ maps $|z| \leq r$ into $|z| \leq R$. Then by (i)

$$M_p(r, f \circ \phi) \leq \left(\frac{R + |\phi(0)|}{R - |\phi(0)|} \right)^{1/p} M_p(R, f).$$

Letting $R \rightarrow 1$,

$$M_p(r, f \circ \phi) \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{1/p} \|f\|_{H^p}.$$

Thus, $f \circ \phi \in H^p$ and

$$\|f \circ \phi\|_{H^p} \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{1/p} \|f\|_{H^p}.$$

This proves (ii).

To prove (iii), we note that $\phi(0) = 0$ implies by Schwarz's Lemma that ϕ maps $|z| \leq R$ into $|z| \leq R$. Thus, we may let $r \rightarrow R$ in (4.1) obtaining

$$(4.3) \quad \frac{1}{2\pi} \int_0^{2\pi} |f(\phi(\operatorname{Re}^{i\theta}))|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(\phi(\operatorname{Re}^{i\theta}))}{B(\phi(\operatorname{Re}^{i\theta}))} \right|^p d\theta.$$

If the equality holds in (4.3), then either $|\phi(\operatorname{Re}^{i\theta})| \equiv R$ or $B \equiv 1$. If, however, $|\phi(\operatorname{Re}^{i\theta})| \equiv R$, then the equality case of the Schwarz Lemma implies $\phi = \epsilon z$, $|\epsilon| = 1$.

Assuming equality holds in (4.3), if ϕ is not of the form ϵz , then $B \equiv 1$ and $|\phi(\operatorname{Re}^{i\theta})| < R$, hence (4.2) becomes

$$(4.4) \quad (f(\phi(\text{Re}^{i\theta})))^p = \frac{1}{2\pi} \int_0^{2\pi} (f(\text{Re}^{it}))^p \frac{R^2 - |\phi(\text{Re}^{i\theta})|^2}{|\text{Re}^{it} - \phi(\text{Re}^{i\theta})|^2} dt.$$

Taking absolute values,

$$(4.5) \quad |f(\phi(\text{Re}^{i\theta}))|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\text{Re}^{it})|^p \frac{R^2 - |\phi(\text{Re}^{i\theta})|^2}{|\text{Re}^{it} - \phi(\text{Re}^{i\theta})|^2} dt.$$

Integrating both sides of (4.5) with respect to θ , we get, as in the proof of (i),

$$(4.6) \quad M_p(R, f \circ \phi) \leq M_p(R, f).$$

If equality holds in (4.6), it follows that (4.5) must have been an equality. But then (4.4) and (4.5) imply

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^{2\pi} (f(\text{Re}^{it}))^p \frac{R^2 - |\phi(\text{Re}^{i\theta})|^2}{|\text{Re}^{it} - \phi(\text{Re}^{i\theta})|^2} dt \right| \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(\text{Re}^{it})|^p \frac{R^2 - |\phi(\text{Re}^{i\theta})|^2}{|\text{Re}^{it} - \phi(\text{Re}^{i\theta})|^2} dt. \end{aligned}$$

However, if in general $g = u + iv$ is continuous and $|\int g| = \int |g|$, then $g = \beta u$, $|\beta| = 1$. So f must be of this form on $|z| = R$ and hence f is constant.

It has been shown by E. Nordgren [13] that if ϕ is an inner function, then

$$\|C_\phi\| = \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{1/p}.$$

On the other hand, H. Schwartz [16] has shown that there are functions ϕ mapping D into itself for which

$$\|C_\phi\| < \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{1/p}.$$

Theorem 4.2. If $f \in B^p$, $0 < p < 1$ and $\phi: D \rightarrow D$ is analytic, then $f \circ \phi \in B^p$. The operator C_ϕ defined by $C_\phi(f) = f \circ \phi$ for all $f \in B^p$ is a bounded linear operator on B^p and

$$\|C_\phi\| \leq \begin{cases} 4 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{1/p} & \text{for } p > 1/2 \\ [2 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)]^{1/p} & \text{for } p \leq 1/2. \end{cases}$$

Proof: A general form of Schwarz's Lemma gives the inequality

$$(4.7) \quad \frac{|\phi(z) - \phi(0)|}{|1 - \overline{\phi(0)}\phi(z)|} \leq |z|.$$

It may be verified by elementary methods that if a and b are any two complex numbers such that $|a| < 1$, $|b| < 1$, then

$$\frac{|a| - |b|}{1 - |a||b|} \leq \frac{|a+b|}{|1+\overline{a}b|}.$$

This inequality applied to (4.7) yields

$$\frac{|\phi(z)| - |\phi(0)|}{1 - |\phi(0)||\phi(z)|} \leq |z|.$$

Let

$$\lambda(r) = \frac{|\phi(0)| + r}{1 + |\phi(0)|r}, \quad 0 \leq r < 1.$$

Then $|\phi(re^{i\theta})| \leq \lambda(r)$, $\lambda(r)$ is an increasing function of r , and $\lambda(r) \rightarrow 1$ as $r \rightarrow 1$. Set $R = \frac{1}{2}(1 + \lambda(r))$. Then

$\frac{1}{2}(1 + |\phi(0)|) \leq R < 1$ and ϕ maps $|z| \leq r$ into $|z| < R$.

Furthermore,

$$\begin{aligned} \frac{R+|\phi(0)|}{R-|\phi(0)|} &\leq \frac{1+|\phi(0)|}{\frac{1}{2}(1+|\phi(0)|)-|\phi(0)|} \\ &= 2 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right) \end{aligned}$$

and

$$\frac{dr}{dR} = \frac{2(1-|\phi(0)|^2)}{(1-|\phi(0)|\lambda(r))^2} \leq 2 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right).$$

If $p > \frac{1}{2}$, then

$$(4.8) \quad (1-r)^{(1/p)-2} \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{(1/p)-2} (1-R)^{(1/p)-2}.$$

If $p \leq \frac{1}{2}$, then

$$(1-r)^{(1/p)-2} \leq \left[2 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right) \right]^{(1/p)-2} (1-R)^{(1/p)-2}.$$

It follows from Theorem 4.1 (i) that

$$M_1(r, f \circ \phi) \leq \frac{R+|\phi(0)|}{R-|\phi(0)|} M_1(R, f).$$

Thus,

$$\begin{aligned} \int_0^1 (1-r)^{(1/p)-2} M_1(r, f \circ \phi) dr &\leq K(p, \phi) \int_0^1 (1-R)^{(1/p)-2} M_1(R, f) dR \\ &\quad \frac{1+|\phi(0)|}{2} \\ &\leq K(p, \phi) \|f\|_{B^p}, \end{aligned}$$

where

$$K(p, \phi) = \begin{cases} 4 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right)^{(1/p)} & \text{for } p > \frac{1}{2} \\ \left[2 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right) \right]^{(1/p)} & \text{for } p \leq \frac{1}{2}. \end{cases}$$

Therefore, $f \circ \phi \in B^p$ and

$$\|f \circ \phi\|_{B^p} \leq K(p, \phi) \|f\|_{B^p}.$$

If $p > \frac{1}{2}$ and $|\phi(0)|$ is not too large, we may improve the bound on $\|C_\phi\|$ by a slight alteration to the above proof. The following is then a corollary to the proof of Theorem 4.2.

Corollary 4.3. If $p > \frac{1}{2}$ and $|\phi(0)| < 2^{(2-(1/p))} - 1$, then

$$\|C_\phi\| \leq 2^{1/p} (1+|\phi(0)|) \left(\frac{1-|\phi(0)|}{1+|\phi(0)|} \right)^{1/p}.$$

Proof: In the proof of Theorem 4.2, replace (4.8) by

$$(1-r)^{(1/p)-2} \leq (1+|\phi(0)|) \left[2 \left(\frac{1+|\phi(0)|}{1-|\phi(0)|} \right) (1-R) \right]^{(1/p)-2}.$$

In the next theorem, a lower bound for $\|C_\phi\|$ is given in terms of $\phi(0)$.

Theorem 4.4. If C_ϕ is a composition operator on B^p , then

$$\frac{1}{1-|\phi(0)|^2} \leq \|C_\phi\|.$$

Proof: If $f(z)$ is in B^p , then

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

and hence

$$|f(0)| \|e_0\|_{B^p} \leq \|f\|_{B^p}$$

where $e_0(z) \equiv 1$. Let $g(z) = \left(\frac{1}{1-\phi(0)z}\right)^2$. Now $g \in B^p$, so $g \circ \phi \in B^p$ and

$$\frac{\|e_0\|_{B^p}}{(1-|\phi(0)|^2)^2} = \|e_0\|_{B^p} |g(\phi(0))|$$

$$\leq \|g \circ \phi\|_{B^p}$$

$$\leq \|g\|_{B^p} \|C_\phi\|$$

$$\leq \|e_0\|_{B^p} \|g\|_{H^1} \|C_\phi\|.$$

The conclusion follows since

$$\|g\|_{H^1} = \frac{1}{1-|\phi(0)|^2}.$$

Corollary 4.5. $\|C_\phi\| = 1$ if and only if $\phi(0) = 0$.

Proof: If $\phi(0) = 0$, then Schwarz's Lemma implies

$\phi: |z| \leq r \rightarrow |z| \leq r$. Thus, by Theorem 4.1 (i),

$$M_1(r, f \circ \phi) \leq M_1(r, f)$$

and hence

$$\|f \circ \phi\|_{B^p} \leq \|f\|_{B^p}$$

for each $f \in B^p$. Thus, $\|C_\phi\| \leq 1$. But, $\|C_\phi e_0\|_{B^p} = \|e_0\|_{B^p}$, where $e_0 \equiv 1$, so $\|C_\phi\| = 1$. Conversely, if $\|C_\phi\| = 1$, then Theorem 4.4 implies $\phi(0) = 0$.

The next theorem characterizes those composition operators which are isometries on B^p .

Theorem 4.6. C_ϕ is an isometry if and only if ϕ is a rotation. (i.e., $\phi(z) = \epsilon z$, $|\epsilon| = 1$).

Proof: If ϕ is a rotation, then $M_1(r, f \circ \phi) = M_1(r, f)$ for $f \in B^p$ and hence

$$\|f\|_{B^p} = \|f \circ \phi\|_{B^p} = \|C_\phi(f)\|_{B^p}.$$

On the other hand, if C_ϕ is an isometry, then $\|C_\phi\| = 1$ so that $\phi(0) = 0$. Let $f \in B^p$, f not constant. If $M_1(r, f \circ \phi) < M_1(r, f)$ for each r , $0 < r < 1$, then $\|f \circ \phi\|_{B^p} < \|f\|_{B^p}$ which implies C_ϕ is not an isometry. Therefore, $M_1(r, f \circ \phi) = M_1(r, f)$ for some r . But then Theorem 4.1 (iii) implies ϕ is a rotation.

Results similar to the theorems given in this section have been proved for H^p , $1 \leq p < \infty$ by H. J. Schwartz [16]. The lower bound he obtained for $\|C_\phi\|$ was $(\frac{1}{1-|\phi(0)|^2})^{1/p}$. He showed that C_ϕ is an isometry on H^p if and only if ϕ is

an inner function vanishing at zero, which is quite different from our case.

2. Characterization.

The question of when a bounded operator is a composition operator can be answered in terms of its multiplicative properties.

Lemma 4.7. Let $e_n(z) = z^n$. Then $\|e_n\|_{B^p} \simeq C(p)n^{-\gamma}$ where

$$\gamma = \begin{cases} \frac{1}{p} & \text{if } p \neq \frac{1}{k+1} \\ \frac{1}{p}-1 & \text{if } p = \frac{1}{k+1}, \end{cases}$$

k a positive integer.

Proof: Let

$$\binom{n+\alpha}{\alpha} = \frac{(\alpha+1)(\alpha+2) \dots (\alpha+n)}{n!} = \binom{\alpha+n}{n}$$

and set $\beta = \frac{1}{p} - 2$. It follows from Stirling's formula that

$$\binom{n+\alpha}{\alpha} \simeq \Gamma(\alpha+1)n^\alpha$$

(see [7], p. xv). Now,

$$\|e_n\|_{B^p} = \int_0^1 (1-r)^\beta r^n dr.$$

If $p = \frac{1}{k+1}$, $\|e_n\|_{B^p}$ may be computed using integration by

parts an appropriate number of times. This gives

$$\|e_n\|_{B^p} = \frac{(k-1)!}{(n+1)(n+2)\dots(n+k)} = \frac{1}{n+k} [(\beta+n)]^{-1}.$$

If $p \neq \frac{1}{k+1}$, then repeated integration by parts yields:

$$\|e_n\|_{B^p} = \left(\frac{1}{\beta+n+1}\right)\left(\frac{1}{\beta+n+2}\right) [(\beta+n)]^{-1}.$$

Thus,

$$\|e_n\|_{B^p} \simeq C(p)n^{-\gamma}$$

$$\text{where } C(p) = \frac{1}{\Gamma(\beta+1)} = \frac{1}{\Gamma(\frac{1}{p}-1)}.$$

We will use this lemma in proving our next theorem.

Theorem 4.8. If A is a bounded linear operator other than zero on B^p , $0 < p < 1$, then A is a composition operator if and only if

$$A(e_n) = (A(e_1))^n \quad \text{for } n = 0, 1, 2, \dots,$$

Proof: If A is a composition operator, then $A = C_\phi$ for some ϕ . Hence, $A(e_n) = C_\phi(e_n) = \phi^n = (A(e_1))^n$. Conversely, suppose $A(e_n) = (A(e_1))^n$ for each n . Let $\phi = Ae_1$, then

$$\|\phi^n\|_{B^p} \leq \|A\| \|e_n\|_{B^p}.$$

It follows from Lemma 4.7 that $\|e_n\|_{B^p} \rightarrow 0$ and hence

$\|\phi^n\|_{B^p} \rightarrow 0$ as $n \rightarrow \infty$. Let ρ be fixed, $0 < \rho < 1$ and

set $T_\rho = \{z: |z| = \rho\}$. If $|\phi(z)| \geq 1$ on some subset S of T_ρ of positive measure, then $M_1(\rho, \phi^n) \geq m(S)$ (where m

denotes normalized Lebesgue measure on T_ρ). It follows that

$$\begin{aligned}\|\phi^n\|_{B^p} &= \int_0^1 (1-r)^{(1/p)-2} M_1(r, \phi^n) dr \\ &\geq \int_\rho^1 (1-r)^{(1/p)-2} M_1(\rho, \phi^n) dr \\ &\geq m(S)(1-\rho)^{(1/p)-1} \left(\frac{1}{p} - 1\right)^{-1},\end{aligned}$$

contradicting $\|\phi^n\|_{B^p} \rightarrow 0$. Hence $\phi \in H^\infty$, $\|\phi\|_{H^\infty} \leq 1$ and $\phi \neq 1$. Therefore, $\phi: D \rightarrow D$. Now, $C_\phi(e_n) = \phi^n = (A(e_1))^n = A(e_n)$. Thus, C_ϕ and A are continuous linear operators and they agree on the polynomials (since the polynomials are linear combinations of the e_n 's). But the polynomials are dense in B^p , hence $C_\phi = A$.

This theorem was proved for H^p , $1 \leq p < \infty$, by H. J. Schwartz ([16], p. 8). His proof is also valid for H^p , $0 < p < 1$. Schwartz then showed that the theorem could be restated in terms of the property "almost multiplicative".

Definition 4.1. An operator A on a function space S is almost multiplicative if whenever $f, g, fg \in S$ then $A(fg) = (Af)(Ag)$.

The following corollary was proved for H^p , $1 \leq p < \infty$ by H. J. Schwartz ([16], p. 10). His proof works also for B^p .

Corollary 4.9. If A is a bounded linear operator other than zero on B^p , $0 < p < 1$, then A is a composition operator if

and only if A is almost multiplicative.

Proof: If $A = C_\phi$ and $f, g, fg \in B^p$, then

$$A(fg) = C_\phi(fg) = fg \circ \phi = (f \circ \phi)(g \circ \phi)$$

and

$$(f \circ \phi)(g \circ \phi) = (C_\phi f)(C_\phi g) = (Af)(Ag)$$

so A is almost multiplicative.

Conversely, if A is almost multiplicative, then

$$A(e_n) = A(e_n e_0) = (A(e_n))(Ae_0).$$

If $A(e_0) = 0$, then $A(e_n) = 0$ and A vanishes on the polynomials. But this would imply A is the zero operator, contrary to hypothesis. Therefore $A(e_0) \neq 0$. Furthermore, $A(e_0) = (Ae_0)(Ae_0)$ and hence $A(e_0) = e_0$. Also, $A(e_2) = (A(e_1))(A(e_1)) = (Ae_1)^2$. It follows by induction that $A(e_n) = (Ae_1)^n$ for each n .

This corollary may be used to characterize invertible composition operators. The next theorem was proved by H. J. Schwartz ([16], p. 12) for H^p . Again, his proof is also valid for B^p .

Theorem 4.10. C_ϕ is invertible if and only if ϕ is a conformal map of D onto D , in which case $(C_\phi)^{-1} = C_{\phi^{-1}}$.

Proof: If ϕ is a conformal map of D onto D then

$C_{\phi^{-1}} = (C_\phi)^{-1}$ so that C_ϕ is invertible. Conversely, if C_ϕ is invertible, then there exists

an operator A on B^p such that $AC_\phi = C_\phi A = I$. Choose $f_1, f_2 \in B^p$ such that $f_1 f_2 \in B^p$. Let $g_1 = A(f_1)$, $g_2 = A(f_2)$, and $g_3 = A(f_1 f_2)$. Then $C_\phi g_1 = f_1$, $C_\phi g_2 = f_2$, and $C_\phi g_3 = f_1 f_2$. Also, $C_\phi(g_1)C_\phi(g_2) = f_1 f_2 \in B^p$. Thus, $g_1 g_2 \circ \phi = g_3 \circ \phi$. Now, $g_1 g_2$ and g_3 are both analytic and agree on $\phi(D)$. Hence, $g_1 g_2 = g_3$, i.e., $(Af_1)(Af_2) = A(f_1 f_2)$. So A is almost multiplicative and by Corollary 4.9, $A = C_\varphi$ for some φ . Furthermore,

$$C_\phi C_\varphi = C_\varphi C_\phi = I = C_{e_1}$$

which implies $\varphi \circ \phi = \phi \circ \varphi = e_1$, i.e., $\varphi = \phi^{-1}$.

Corollary 4.11. C_ϕ maps B^p onto B^p if and only if ϕ is a conformal map of D onto itself.

Proof: If ϕ is conformal, then clearly C_ϕ is onto. Conversely, assume C_ϕ is onto. C_ϕ is also one-to-one, for if $C_\phi(f) = C_\phi(g)$, then f and g are analytic functions agreeing on $\phi(D)$ and hence $f = g$. Thus, C_ϕ is one-to-one, continuous and onto. The Open Mapping Theorem implies C_ϕ is invertible.

3. Operators into H^q .

The first theorem in this section gives a sufficient condition for C_ϕ to be a bounded operator from B^p into H^q .

Theorem 4.12. If $(1 - |\phi(z)|)^{-1}$ is in $L^{q/p}(D)$, $0 < p < 1$, $0 < q < \infty$, then $C_\phi: B^p \rightarrow H^q$ and

$$\|C_\phi\| \leq C(p) \|(1 - |\phi(z)|)^{-1}\|_{L^{q/p}},$$

where $\|C_\phi\|$ denotes the norm of C_ϕ as an operator from B^p into H^q .

Proof: If $f \in B^p$, then by Theorem 1.9,

$$|f(z)| \leq C(p) \|f\|_{B^p} (1-|z|)^{-1/p}.$$

Thus,

$$|f(\phi(z))| \leq C(p) \|f\|_{B^p} (1-|\phi(z)|)^{-1/p}$$

so that $f \circ \phi \in H^q$ and

$$\|f \circ \phi\|_{H^q} \leq C(p) \|f\|_{B^p} \|(1-|\phi|)^{-1}\|_{L^{q/p}}.$$

Corollary 4.13. If $(1-|\phi|)^{-1} \in L^1(D)$, then $C_\phi: B^p \rightarrow H^p$ and

$$\|C_\phi\| \leq C(p) \|(1-|\phi|)^{-1}\|_{L^1},$$

where $\|C_\phi\|$ denotes the norm of C_ϕ as an operator from B^p into H^p .

Corollary 4.14. If $(1-|\phi|)^{-1} \in L^{q/p}(D)$, $0 < p < 1$, $q > p$, then $C_\phi: H^p \rightarrow H^q$ and $\|C_\phi\| \leq C(p) \|(1-|\phi|)^{-1}\|_{L^{p/q}}.$

Proof: If $f \in H^p$, then

$$|f(z)| \leq C(p) \|f\|_{H^p} (1-|z|)^{-1/p}$$

(see [2], p. 36). The proof follows in the same manner as the proof of Theorem 4.12.

The next theorem gives a necessary condition for C_ϕ to be a bounded operator from B^p into H^q .

Theorem 4.15. If C_ϕ is a bounded operator from B^p into H^q , $0 < p < 1$, $0 < q < \infty$, then $|\phi(e^{it})| < 1$ a.e.

Proof: If C_ϕ is a bounded operator from B^p into H^q , then $\|C_\phi(e_n)\|_{H^q} \leq \|C_\phi\| \|e_n\|_{B^p}$. Thus, it follows from Lemma 4.7 that $\|C_\phi(e_n)\|_{H^q} \rightarrow 0$ as $n \rightarrow \infty$. Suppose there exists a set $E \subset [0, 2\pi)$ of positive measure such that $|\phi(e^{it})| = 1$ for $t \in E$. Then,

$$\begin{aligned} \|C_\phi(e_n)\|_{H^q} &= \|\phi^n\|_{H^q} \\ &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{it})|^{nq} dt \right\}^{1/q} \\ &\geq \left\{ \frac{1}{2\pi} \int_E |\phi(e^{it})|^{nq} dt \right\}^{1/q} \\ &\geq \left\{ \frac{m(E)}{2\pi} \right\}^{1/q} > 0 \end{aligned}$$

contradicting $\|C_\phi(e_n)\| \rightarrow 0$ as $n \rightarrow \infty$.

If $q \geq p$, the condition $|\phi(e^{it})| < 1$ a.e. is not sufficient. In the case $q > p$, choose s such that $p < s \leq q$ and let $\phi(z) = \frac{1+z}{2}$. The function $f(z) = (1-z)^{-1/s}$ is in H^p and hence in B^p . However, $f(\phi(z)) = 2^{1/s}(1-z)^{-1/s}$ is not in H^q . For the case $q = p$, again choose $\phi(z) = \frac{1+z}{2}$. The function $f_{p,\beta}$ defined by (2.3) is in B^p for $p < \beta < 1$, but $f(\phi(z)) \notin H^p$.

Theorem 4.16. C_ϕ is a bounded operator from B^p into H^∞ if and only if $\|\phi\|_{H^\infty} < 1$.

Proof: If C_ϕ is a bounded operator from B^p into H^∞ , then as in Theorem 4.15,

$$\|C_\phi(e_n)\|_{H^\infty} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

But,

$$\|C_\phi(e_n)\|_{H^\infty} = \|\phi^n\|_{H^\infty} = (\|\phi\|_{H^\infty})^n.$$

Thus, $\|\phi\|_{H^\infty} < 1$. Conversely, if $\|\phi\|_{H^\infty} < 1$, then

$$\begin{aligned} |f(\phi(z))| &\leq C(p) \|f\|_{B^p} (1 - |\phi(z)|)^{-1/p} \\ &\leq C(p) \|f\|_{B^p} (1 - \|\phi\|_{H^\infty})^{-1/p}. \end{aligned}$$

Hence $C_\phi(f) \in H^\infty$ and

$$\|C_\phi\| \leq C(p) (1 - \|\phi\|_{H^\infty})^{-1/p}.$$

4. Compact operators.

A bounded linear operator A on a Banach space X is said to be compact if the image under A of every bounded sequence has a convergent subsequence. The following theorem was proved for H^p , $1 \leq p < \infty$ by H. J. Schwartz. His proof is also valid for B^p and is given below.

Theorem 4.17. C_ϕ is a compact operator on B^p if and only if for every bounded sequence $\{f_n\}$ in B^p such that $f_n \rightarrow f$ uniformly on compact subsets of D , $\|C_\phi f_n - C_\phi f\|_{B^p} \rightarrow 0$ as $n \rightarrow \infty$.

Proof: Assume C_ϕ is compact. Let $\{f_n\}$ be a sequence in B^p such that $\|f_n\|_{B^p} \leq K$ for each n and $f_n \rightarrow f$ uniformly on compact subsets of D . Suppose there exists a subsequence $\{f_{n_k}\}$ such that

$$(4.9) \quad \|C_\phi f_{n_k} - C_\phi f\|_{B^p} \geq \alpha > 0$$

for each k . Since $\|f_{n_k}\|_{B^p} \leq K$ and C_ϕ is compact, there exists a subsequence $\{f_{n_{k_i}}\}$ such that $C_\phi(f_{n_{k_i}}) \rightarrow g$ in B^p norm. But then $C_\phi(f_{n_{k_i}}) \rightarrow g$ uniformly on compact subsets of D . It follows from our hypothesis that $C_\phi(f_{n_{k_i}}) \rightarrow C_\phi(f)$ uniformly on compact subsets of D . Therefore $g = C_\phi(f)$ and $C_\phi(f_{n_{k_i}}) \rightarrow C_\phi(f)$ in B^p norm contradicting (4.9).

Conversely, let $\{f_n\}$ be any bounded sequence in B^p . It follows from Theorem 1.9 (i) that $\{f_n\}$ is a normal family and hence there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow f$ uniformly on compact subsets of D . By our hypothesis, $C_\phi f_{n_k} \rightarrow C_\phi f$ in B^p norm, and hence C_ϕ is compact.

Theorem 4.18. If $(1-|\phi(z)|)^{-1} \in L^1(D)$, then C_ϕ is a compact operator on B^p .

Proof: Let $\{f_n\}$ be a bounded sequence in B^p which converges uniformly on compact subsets of D to a function f . Then

$$\begin{aligned} |f_n(\phi(z))| &\leq C(p) \|f_n\|_{B^p} (1-|\phi(z)|)^{-1/p} \\ &\leq K(p) (1-|\phi(z)|)^{-1/p} \end{aligned}$$

for each n . Letting $n \rightarrow \infty$,

$$|f(\phi(z))| \leq K(p)(1-|\phi(z)|)^{-1/p}.$$

Hence,

$$(4.10) \quad |f_n(\phi(z)) - f(\phi(z))|^p \leq C(p)(1-|\phi(z)|)^{-1}$$

for each n . Let $g_n = C_\phi(f_n) - C_\phi(f)$ for each n . It follows from (4.10) and the Dominated Convergence Theorem that $g_n \in H^p$ for each n and $\|g_n\|_{H^p} \rightarrow 0$ as $n \rightarrow \infty$. Finally, Theorem 1.9 (iv) implies $\|g_n\|_{B^p} \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\|C_\phi(f_n) - C_\phi(f)\|_{B^p} \rightarrow 0$ as $n \rightarrow \infty$. Hence C_ϕ is compact.

Corollary 4.19. If $|\phi(z)| \leq r < 1$ for all $z \in D$, then C_ϕ is compact.

Similar results for H^p , $1 \leq p < \infty$, were given by H. J. Schwartz ([16], p. 26). The proof of Theorem 4.18 is also valid for H^p , $0 < p < 1$. Next we give a necessary condition for C_ϕ to be compact.

Theorem 4.20. If C_ϕ is a composition operator on B^p , then

$$(4.11) \quad \|\phi^n\|_{B^p} = O(n^{-\gamma}),$$

where

$$\gamma = \begin{cases} 1/p & \text{if } p \neq \frac{1}{k+1} \\ (1/p)-1 & \text{if } p = \frac{1}{k+1} \end{cases}$$

and k is a positive integer. If C_ϕ is compact, then

$$(4.12) \quad \|\phi^n\|_{B^p} = o(n^{-\gamma}).$$

Proof: (4.11) follows immediately from Lemma 4.7 since

$$\|\phi^n\|_{B^p} = \|C_\phi e_n\|_{B^p} \leq \|C_\phi\| \|e_n\|_{B^p}.$$

If C_ϕ is compact, let $f_n(z) = \frac{e_n(z)}{\|e_n\|_{B^p}}$ for each n .

$|f_n(z)| \leq C(p)n^\gamma |z|^n$ for each n . Hence $\{f_n\}$ converges uniformly to zero on compact subsets of D . Furthermore,

$\|f_n\|_{B^p} = 1$ for each n . Therefore, since C_ϕ is compact, $\|C_\phi f_n\|_{B^p} \rightarrow 0$. But,

$$\|C_\phi f_n\|_{B^p} = (\|e_n\|_{B^p})^{-1} \|\phi^n\|_{B^p}$$

and

$$(\|e_n\|_{B^p})^{-1} \simeq C(p)n^\gamma$$

hence

$$n^\gamma \|\phi^n\|_{B^p} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As a simple example, consider the function $\phi(z) = \rho z$, $0 < \rho \leq 1$. If $\rho < 1$, then C_ϕ is compact by Corollary 4.19. If $\rho = 1$, then C_ϕ is not compact since $\|\phi^n\|_{B^p} = \|e_n\|_{B^p}$ does not satisfy (4.12).

We next turn our attention to finding the spectrum $\sigma(C_\phi)$ of a compact composition operator C_ϕ . This problem

was investigated for compact operators C_ϕ on H^p by H. J. Schwartz [16]. He relied upon the following theorems by M. Koenig (see [16], p. 72).

Theorem 4.21. (M. Koenig). If $\phi:D \rightarrow D$ is analytic, $\phi(0) = 0, \phi'(0) \neq 0$ then there exists a function $K(z)$ analytic in D such that $K(\phi(z)) = (\phi'(0))K(z)$.

Theorem 4.22. (M. Koenig). Let $\phi:D \rightarrow D$ and $\phi(0) = 0$. Then there exists a non-zero analytic function f , satisfying $f(\phi(z)) = \lambda f(z)$ if and only if $\lambda = 1$ or $\lambda = (\phi'(0))^n$.

We remark that both of these theorems may be stated for any fixed point z_0 of ϕ . Furthermore, if $\phi:D \rightarrow D$ is analytic, then ϕ can have at most one fixed point ([16], p. 74).

Theorem 4.23. (H. J. Schwartz [16], p. 77). If C_ϕ is an operator on H^p , $1 < p < \infty$, $\phi(z_0) = z_0$ and $(1-|\phi|)^{-1} \in L^s(D)$ $s = \max[p, p']$, $(\frac{1}{p} + \frac{1}{p'} = 1)$, then

$$\sigma(C_\phi) = \{\phi'(z_0)^n\}_{n=1}^{\infty} \cup \{1\}.$$

We use Theorem 4.23 to prove the following:

Theorem 4.24. If $\phi(z_0) = z_0$, $(1-|\phi|)^{-1} \in L^2(D)$, and C_ϕ is an operator on B^p , then

$$\sigma(C_\phi)_{B^p} = \{(\phi'(z_0))^n\}_{n=1}^{\infty} \cup \{1\}.$$

Proof: If $(1-|\phi|)^{-1} \in L^2(D)$, then C_ϕ is compact by

Theorem 4.18. Hence $\sigma(C_\phi)_{B^p}$ consists entirely of eigenvalues and by Theorem 4.22, $\sigma(C_\phi)_{B^p} \subset \{(\phi'(z_0))^n\} \cup \{1\}$. On the

other hand, C_ϕ is also a compact operator on H^2 (see [16], p. 26). By Theorem 4.23, the spectrum of C_ϕ as an operator on H^2 is given by

$$\sigma(C_\phi)_{H^2} = \{(\phi'(z_0))^n\} \cup \{1\}.$$

If $\lambda \in \sigma(C_\phi)_{H^2}$, then λ is an eigenvalue and hence there exists $f \in H^2$ such that $f(\phi(z)) = \lambda f(z)$. But $f \in H^2$ implies $f \in B^p$ and hence $\lambda \in \sigma(C_\phi)_{B^p}$.

Corollary 4.25. If $|\phi(z)| \leq r < 1$ and C_ϕ is a composition operator on B^p , then the spectrum of C_ϕ is given by

$$\sigma(C_\phi) = \{(\phi'(z_0))^n\}_{n=1}^{\infty} \cup \{1\}$$

for some $z_0 \in D$.

Proof: $\phi: |z| \leq r \rightarrow |z| \leq r$. Hence ϕ has a fixed point z_0 by the Brouwer fixed point theorem.

The following theorem can be proved in the same manner as Theorem 4.24.

Theorem 4.26. If $0 < p < 2$, $\phi(z_0) = z_0$, $(1-|\phi|)^{-1} \in L^2(D)$, and C_ϕ is an operator on H^p , then

$$\sigma(C_\phi) = \{\phi'(z_0)\}_{n=1}^{\infty} \cup \{1\}.$$

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