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ON THE EXISTENCE AND NONEXISTENCE OF PERIODIC ORBITS IN A NEIGHBOURHOOD OF HOMOCLINIC AND HETEROCLINIC ORBITS

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ON THE EXISTENCE AND NONEXISTENCE OF PERIODIC ORBITS IN A NEIGHBOURHOOD OF HOMOCLINIC AND HETEROCLINIC ORBITS

By

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ABSTRACT

ON THE EXISTENCE AND NONEXISTENCE OF PERIODIC ORBITS IN A NEIGHBOURHOOD OF HOMOCLINIC AND HETEROCLINIC ORBITS

By

Mohammad Riazi-Kermani

We show that the eigenvalues of the linearized system determines the existence or nonexistence of periodic orbits in a neighbourhood of a homoclinic or heteroclinic orbit in $\mathbb{R}^2$ and $\mathbb{R}^3$ for autonomous systems. We also show the existence of periodic orbits in the modified logistic equation corresponding to the central difference scheme by the method of averaging. The existence of a heteroclinic orbit is also proved in the modified logistic equation.

TO

MY PARENTS

AND

MY WIFE

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INTRODUCTION

It is well known that the numerical approximation of the solution of an initial value problem for an ordinary differential equation requires many precautions to be successful. One way of investigating the adequacy of numerical schemes is the introduction of the concept of modified equation. The logistic equation

$$\frac{dy}{dt} = y(1-y) \qquad y(0) = y_0 \tag{1}$$

has been studied by M. Yamaguti and S. Ushiki. In particular, these authors studied the central difference scheme

$$\begin{aligned} &\frac{u_{n+1} - u_{n-1}}{2h} = u_n(1-u_n) \qquad n = 1,2,\ldots \\ &\text{with} \quad u_0 = y_0 \quad \text{and} \quad u_1 = y_0 + hy_0(1-y_0) \end{aligned} \tag{2}$$

The results exhibit the phenomenon of the so called "ghost" solution.

The iterative scheme (2) may be recreated again in continuous form by inserting the taylor expansion

$$u_{n\pm 1} = u((n\pm 1)h) = u(nh) \pm hu'(nh) + \frac{h^2}{2!}u''(nh)$$
$$\pm \frac{h^3}{3!}u'''(nh) + O(h^4) \quad .$$

If we put again $u^{(i)}(nh) = u^{(i)}(t)$, the equation (2) becomes $\frac{h^2}{6}u'''(t) + u'(t) = u(1-u) + O(h^3)$ with $u(0) = y_0$, $u'(0) = y_0(1-y_0)$, $u''(0) = (1-2y_0)(1-y_0)y_0$. We call

$$\varepsilon^2 u'''(t) + u'(t) = u(1-u) \qquad (3)$$

the modified logistic equation.

The first two chapters of this manuscript are devoted to the existence or nonexistence theorems in neighbourhoods of homoclinic and heteroclinic orbits in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The existence of periodic orbits in (3) is shown in Chapter 3 and Chapter 4 is devoted to the existence of a heteroclinic orbit in (3).

Some graphs made by computer are also included.

CHAPTER 1

The existence and nonexistence of periodic orbits in a small neighbourhood of homoclinic and heteroclinic orbits in $\mathbb{R}^2$

Flows on $\mathbb{R}^2$

1.1. Neighbourhoods of homoclinic orbits in $\mathbb{R}^2$

Consider a system of two differential equations

$$\dot{x} = f(x) \tag{1}$$

where f is C^2 and $f(0) = 0$. Assume that the eigenvalues of the matrix $A = f'(0)$ are $-\lambda$ and α where $\alpha > 0$, $\lambda > 0$ and $\alpha - \lambda \neq 0$.

Suppose there exists a homoclinic orbit issuing from 0 and returning to it as shown in Figure 1.

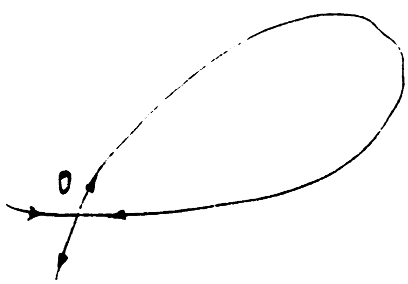


Figure 1

By the Stable Manifold Theorem, in a sufficiently small neighbourhood of O there exists a stable manifold and an unstable manifold corresponding to the negative and positive eigenvalue of A respectively.

Since periodic orbits are topological invariants of the space, following the ideas of the Grobman-Hartman theorem we may assume that the system (1) is linear in a neighbourhood V of the hyperbolic point and also we may assume the stable and unstable manifolds are coordinate axes in that neighbourhood.

The linearized system then has the form:

$$\begin{cases} \dot{x} = -\lambda x \\ \dot{y} = \alpha y \end{cases} \qquad \lambda > 0, \quad \alpha > 0 \qquad \alpha - \lambda \neq 0 \,. \tag{2}$$

Without loss of generality we can assume that the points $(0,1)$ and $(1,0)$ are in V. The lines $x = 1$ and $y = 1$ are transversal to the solution curves of (2) passing nearby (1,0) and (0,1). (See Figure 2)

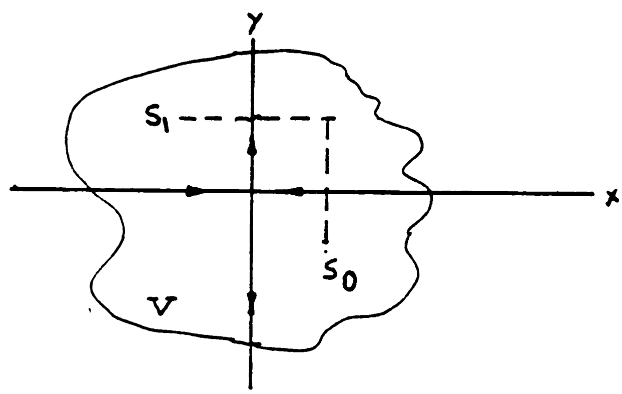


Figure 2

Let us define: $S_0 = \{(1,y) : 0 \leq y \leq 1\}$ and $S_1 = \{(x,1) : 0 \leq x \leq 1\}$. Starting at $t = 0$, from $(1,y_0) \in S_0$ and following the trajectory, we get:

$$\begin{cases} x(t) = e^{-\lambda t} \\ \\ y(t) = y_0 e^{\alpha t} . \end{cases} \tag{3}$$

Therefore at $t = \frac{-1}{\alpha} \ln (y_0)$ the trajectory hits S_1 at the point $x = e^{-\lambda(\frac{-1}{\alpha}) \ln(y_0)} = y_0^{\frac{\lambda}{\alpha}} = y_0^{\gamma}$, where $\gamma = \frac{\lambda}{\alpha} \neq 1$. We define the map $T_0 : S_0 \to S_1$ by $T_0(1,y_0) = (y_0^{\frac{\lambda}{\alpha}},1)$, $y_0 \neq 0$.

Since $\gamma > 0$, $y_0^{\frac{\lambda}{\alpha}}$ tends to 0 as y_0 tends to 0. Therefore if we define $T_0(1,0) = (0,1)$, then T_0 is continuous on S_0. Since the point $B(0,1)$ is mapped by the homoclinic orbit on the point $A(1,0)$, from theorems on the continuous dependence of solutions on initial value and from the transversality of S_1, the correspondence map, T_1 is defined and is as smooth as f in some neighbourhood $S_1^{r_1} = S_1 \cap \{(x,y) : x^2 + (y-1)^2 \leq r_1^2\}$ onto $S_0^{r_2} = S_0 \cap \{(x,y) : (x-1)^2 + y^2 \leq r_2^2\}$.

Consider the composite function $T_1 T_0 : S_0^r \to S_0^r$ and let $T = T_1 T_0$. Then $T(0) = 0$.

Suppose $T_1(x) = \beta_1 x + \beta_2 x^2 + \text{h.o.t.}$ Then $T(y) = T_1 T_0(y) = \beta_1 T_0(y) + \beta_2 (T_0(y))^2 + \text{h.o.t.} = \beta_1 y^{\gamma} + \beta_2 y^{2\gamma} + \text{h.o.t.}$

$\frac{d}{dy}(T(y)) = \beta_1 \gamma y^{\gamma-1} + 2\gamma\beta_2 y^{2\gamma-1} + \text{h.o.t.}$ β_1 is the derivative of the correspondence function. Therefore $\beta_1 \neq 0$.

Since $\gamma \neq 1$, $T'(y)$ tends to zero in case $\gamma > 1$ and goes to infinity in case $\gamma < 1$. Therefore T cannot have any fixed point near zero and different from zero. (See Figure 3)

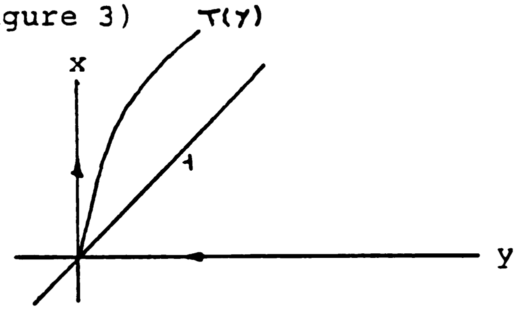


Figure 3-a: $(0 < \gamma < 1)$

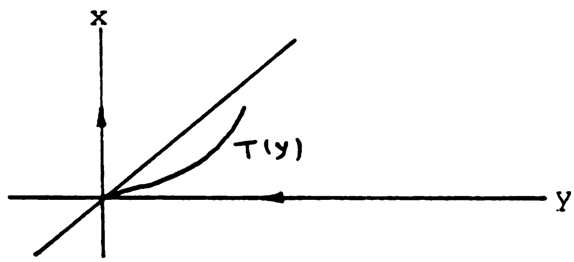


Figure 3-b: $(\gamma > 1)$

The following theorem is the direct result of the above discussions.

Theorem 1.1. If the sum of the eigenvalues of the linearized system at the hyperbolic point is different from zero, then there exists a neighbourhood of the homoclinic orbit without any periodic orbit.

1.2. Neighbourhoods of heteroclinic orbits in $\mathbb{R}^2$

Systems of differential equation in $\mathbb{R}^2$ with heteroclinic orbits joining two critical points appear naturally in science. For example the pendulum equation

$$\ddot{x} + \sin x = 0$$

has its solutions on the level curves of

$$G(x,y) = \frac{1}{2} y^2 - \cos x = K$$

where $y = \dot{x}$. Therefore in the (x,y) plane for $k = 1$ the trajectories are hetroclinic orbits joining the critical points $(-\pi,0)$ and $(\pi,0)$. In the above example the eigenvalues of the linearized system at $p_1(-\pi,0)$ and $p_2(\pi,0)$ are ± 1, and every trajectory starting at a point inside the region bounded by the hetroclinic orbit is a periodic orbit. The existence of periodic orbits in a neighbourhood of the hetroclinic orbit depends on the ratio of the products of positive and negative eigenvalues, namely:

Theorem 1.2. Let

$$\dot{x} = f(x) \tag{1}$$

be a 2-dimensional ordinary differential equation with P_1 and P_2 hyperbolic critical points, and let $\lambda_i < 0$, $\rho_i > 0$ be the eigenvalues of the linearized system at P_i.

If $\gamma = \frac{\lambda_1\lambda_2}{\rho_1\rho_2} \neq 1$, then there exists a neighbourhood of the heteroclinic orbit without any periodic orbits.

Proof: We use the method of point transformation, assuming that the system is linear in a small neighbourhood of P_1.

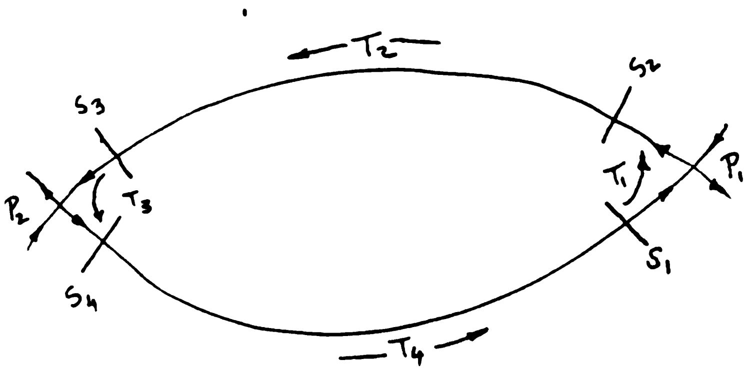


Figure 4

Without loss of generality we assume that $P_1 = 0$ and the stable and unstable manifolds are the coordinate axes.

Let $S_1 = \{(-1,y) : 0 \leq y \leq 1\}$, $S_2 = \{(x,1) : -1 \leq x \leq 0\}$ and $T_1 : S_1 \to S_2$ be defined as follows.

Since S_1 and S_2 are transversal to the trajectories of (1), starting at $(-1,y_0)$ the trajectory will hit S_2 at a point $(x_0,1)$, where $x_0 = y_0^{\frac{\lambda_1}{\rho_1}}$. Define

$$T_1(-1,y) = (x,1) = (y^{\frac{\lambda_1}{\rho_1}},1).$$

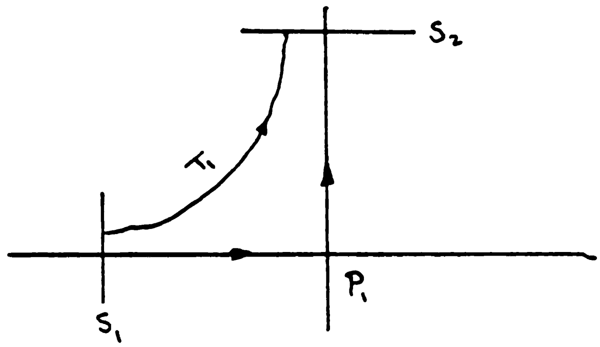


Figure 5

Similarly we define the map $T_3 : S_3 \to S_4$ in a neighbourhood of P_2.

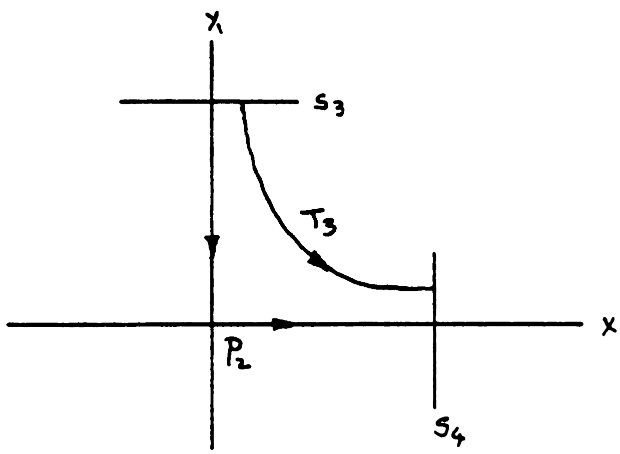


Figure 6

Applying the Hartaman-Grobman theorem in a neighbourhood of P_2 and parametrizing the stable and unstable manifold such that $P_2 = (0,0)$, and the stable manifold is y_1-axis and the unstable manifold the x_1-axis, in that neighbourhood of P_2.

If $S_3 = \{(x_1,1) : 0 \leq x_1 \leq 1\}$, and if $S_4 = \{(1,y_1) : 0 \leq y_1 \leq 1\}$, then the map $T_3 : S_3 \to S_4$ is defined and a similar computation shows that

$$T_3(x_1,1) = (1, x_1^{\frac{\lambda_2}{\rho_2}}).$$

Since the branch of heteroclinic orbit joining P_1 to P_2 takes the O of S_2 on the zero point on S_3, S_2 and S_3 are transversal to the trajectories, the correspondence map

$$T_2 : S_2 \to S_3$$

is defined for a small neighbourhood of O, and it is analytic. Therefore we can assume

$$T_2(x) = \alpha_1 x + \alpha_2 x^2 + \text{h.o.t.} \quad (\alpha_1 \neq 0)$$

Similarly we define the map $T_4 : S_4 \to S_1$ to be the correspondence map and assume

$$T_4(y_1) = \beta_1 y_1 + \beta_2 y_2^2 + \text{h.o.t.}\ . \quad (\beta_1 \neq 0)$$

The map $T = T_4 \circ T_3 \circ T_2 \circ T_1 : S_1 \to S_1$ is defined and continuous on a small neighbourhood of O on that side of S_1 which is the interior of the region bounded by the hetroclinic orbits.

We are interested in the right-hand derivative of T at O. Applying the chain rule, we get

$$T' = T_4' \cdot T_3' \cdot T_2' \cdot T_1' \ .$$

Therefore near zero, $T'(y) \approx \alpha y^{\gamma_1\gamma_2 - 1}$ where $\gamma_1 = \frac{-\lambda_1}{\rho_1}$ and $\gamma_2 = \frac{-\lambda_2}{\rho_2}$. Since we assume $\gamma = \gamma_1\gamma_2 \neq 1$, we get

$$T'(y) \to \begin{cases} 0 & \text{if } \gamma > 1 \\ \infty & \text{if } \gamma < 1 \end{cases} \quad \text{as } y \to 0 .$$

In either case there is no fixed point of $T'y)$ near $y = 0$, there is no periodic orbit in a neighbourhood of the hetro-clinic orbit.

In case $\gamma > 1$, the hetroclinic orbit is attractive from inside. Therefore starting at a point near the hetro-clinic orbit the trajectory will intersect a transversal to the hetroclinic orbit infinitely many times making a sequence which is convergent to the point of intersection.

When the distance between two consecutive terms of the above Cauchy-sequence is less than the round off error of the computer, then solving the differential equation by computational methods results in the so-called ghost solutions. The one dimensional picture looks like the following:

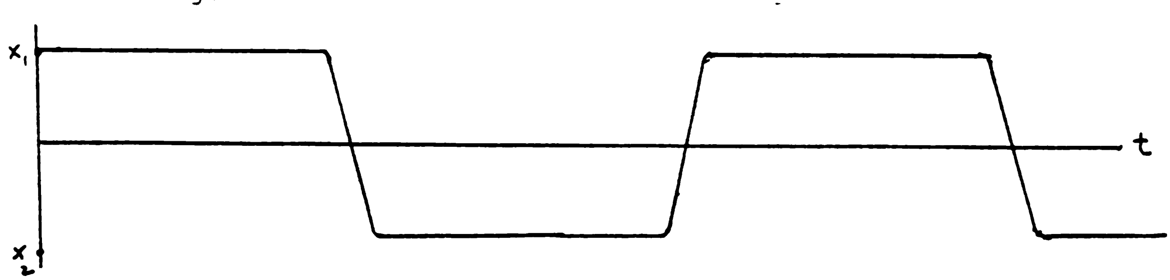


Figure 7

The flat parts of the graph in Figure 7 corresponds to the long time which the trajectory remains in the small neighbourhoods of the critical points and the steep parts correspond to the short time which the trajectory takes to pass nearby the remainder of the heteroclinic orbits.

CHAPTER 2

The existence and nonexistence of periodic orbits in a small neighbourhood of homoclinic and hetroclinic orbits in $\mathbb{R}^3$

Flows on $\mathbb{R}^3$

In the case of three-dimensional systems of autonomous differential equations it has been shown by P.L. Silnikov that if the eigenvalues of the linearized system at the hyperbolic point satisfy $\gamma > -\lambda > 0$ where eigenvalues are: $\gamma > 0$ and $\lambda \pm i\omega$, the existence of a homoclinic orbit issuing from 0 and returning to it results in the existence of a denumerable set of periodic orbits in any neighbourhood of the homoclinic orbit.

We use the Hartman-Grobman theorem to give a more geometrical proof of Silnikov's theorem, showing also the existence of a neighbourhood of the homoclinic orbit without any periodic orbit under the condition $0 < \gamma < -\lambda$ where the eigenvalues of the linearized system are γ and $\lambda \pm i\omega$.

We suppose the system is linear in a small neighbourhood of the hyperbolic point 0, and assume the box $[-1,1]^3$ is contained in that neighbourhood.

The motion is governed by the system:

$$\begin{cases} \dot{x} = \lambda x - \omega y \\ \dot{y} = \omega x + \lambda y \\ \dot{z} = \gamma z \end{cases} \quad . \tag{1}$$

Starting at a point (x_0, y_0, z_0) on the cylinder $x_0^2 + y_0^2 = 1$ the solution curve would be according to (1)

$$\begin{aligned} x(t) &= e^{\lambda t}(x_0 \cos \omega t - y_0 \sin \omega t) \\ y(t) &= e^{\lambda t}(x_0 \sin \omega t + y_0 \cos \omega t) \\ z(t) &= e^{\gamma t} z_0 \quad . \end{aligned} \tag{2}$$

The point of intersection of this trajectory with the plane $z = 1$ is found easily from (2) as follows:

$$z(t) = 1$$

$$t = \frac{1}{\gamma} \ln \left(\frac{1}{z_0}\right)$$

$$e^{\lambda t} = e^{\frac{\lambda}{\gamma} \ln \left(\frac{1}{z_0}\right)} = \left(\frac{1}{z_0}\right)^{\frac{\lambda}{\gamma}} = z_0^{\frac{-\lambda}{\gamma}}$$

$$\begin{cases} x = z_0^{\frac{-\lambda}{\gamma}} \left(x_0 \cos \omega\left(\frac{1}{\gamma} \ln \left(\frac{1}{z_0}\right)\right) - y_0 \sin \omega\left(\frac{1}{\gamma} \ln \left(\frac{1}{z_0}\right)\right)\right) \\ y = z_0^{\frac{-\lambda}{\gamma}} \left(x_0 \sin \omega\left(\frac{1}{\gamma} \ln \left(\frac{1}{z_0}\right)\right) + y_0 \cos \omega\left(\frac{1}{\gamma} \ln \left(\frac{1}{z_0}\right)\right)\right) \\ z = 1 \quad . \end{cases}$$

This point is on a circle of radius $z_0^{\frac{-\lambda}{\gamma}}$; therefore circles on the cylinder $x^2 + y^2 = 1$, $0 < z \leq 1$ will be mapped onto circles on the punctured disc $z = 1$, $0 < x^2 + y^2 \leq 1$. We will call this map T_0.

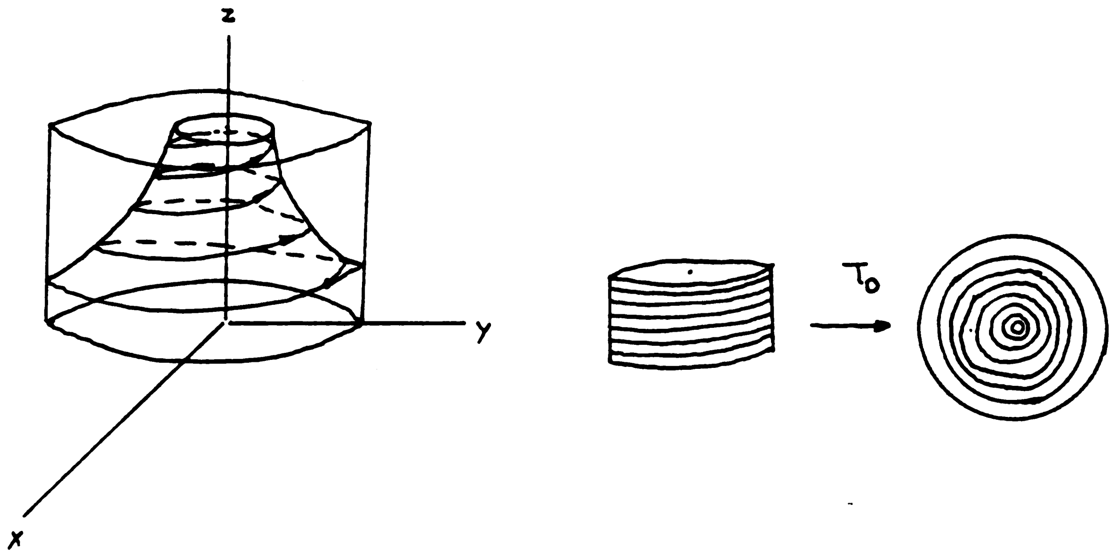


Figure 8

Under T_0 each point of the cylinder will trace a funnel which is invariant for the system (2) in the time interval necessary to reach the plane $z = 1$.

Since $z_0 = e^{-\gamma t}$, $\gamma > 0$ where t is the time necessary for $\mathbb{X}_0 = (x_0, y_0, z_0)$ to hit $z = 1$, as $\mathbb{X}_0$ gets closer to the stable manifold, $T_0(x_0, y_0, z_0)$ gets closer to the unstable manifold in longer time. This correspondence is illustrated in Figure 9.

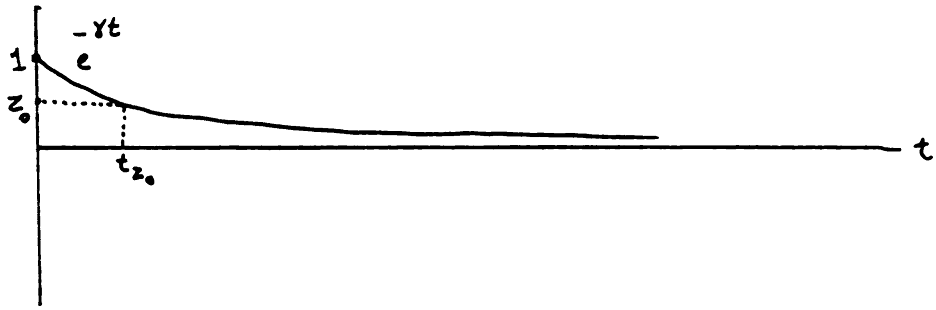


Figure 9

Without loss of generality we may assume that the homoclinic orbit passes through points $(1,0,0)$ and $(0,0,1)$. Therefore the point $(0,0,1)$ maps to $(1,0,0)$ in a finite time t.

By the continuous dependence of solutions on the initial value, there is a neighbourhood of $(0,0,1)$ which will be mapped onto a neighbourhood of $(1,0,0)$ by the trajectories of the system $\dot{x} = f(x)$.

Since the cylinder $x^2 + y^2 = 1$, $|z| \leq 1$ is transversal to the solution curves of (2), we can define the correspondence map from the points on $z = 1$, nearby $(0,0,1)$ and the points of cylinder $x^2 + y^2 = 1$ nearby $(1,0,0)$. We call this map T_1 and by the Implicit Function Theorem T_1 is as smooth as the function f.

We define a new coordinate system on the cylinder as follows:

For any point (x,y,z) near $(1,0,0)$, we define

$$\begin{cases} s = \sin^{-1} y \\ u = z \ . \end{cases}$$

Then $s = \cos^{-1}x$ because $x^2 + y^2 = 1$.

We define T_1 to be the correspondence map from points on $z = 1$, near $(0,0,1)$, and define a new coordinate system by $\Delta x = x$, $\Delta y = y$. The correspondence map T_1 can be expressed as: $T_1 = (f_1, f_2)$, where

$$\begin{cases} f_1(\Delta x, \Delta y) = \alpha_1 \Delta x + \alpha_2 \Delta y + \alpha_3 \Delta x \Delta y + \text{h.o.t.} \\ f_2(\Delta x, \Delta y) = \beta_1 \Delta x + \beta_2 \Delta y + \beta_3 \Delta x \Delta y + \text{h.o.t.} \end{cases}$$

The composite map $T = T_1 T_0$ is defined on a small neighbourhood of 0, and maps the point (s,z) to the point $(f_1(\Delta x,\Delta y),\ f_2(\Delta x,\Delta y))$ where;

$$\begin{cases} \Delta x = \pi_1(T_0(\cos s, \sin s, z)) \\ \quad = z^{-\frac{\lambda}{\gamma}} (\cos s \cos \frac{\omega}{\gamma} \ln(\frac{1}{z}) - \sin s \sin \frac{\omega}{\gamma} \ln(\frac{1}{z})) \\ \Delta y = \pi_2(T_0(\cos s, \sin s, z)) \\ \quad = z^{-\frac{\lambda}{\gamma}} (\cos s \sin \frac{\omega}{\gamma} \ln(\frac{1}{z}) + \sin s \cos \frac{\omega}{\gamma} \ln(\frac{1}{z})) \ . \end{cases}$$

Therefore,

$f_1(\Delta x,\Delta y)$

$$= \alpha_1 z^{\frac{-\lambda}{\gamma}} \left(\cos s \cos \tfrac{\omega}{\gamma} \ln(\tfrac{1}{z}) - \sin s \sin \tfrac{\omega}{\gamma} \ln(\tfrac{1}{z})\right)$$

$$+ \alpha_2 z^{\frac{-\lambda}{\gamma}} \left(\cos s \sin \tfrac{\omega}{\gamma} \ln(\tfrac{1}{z}) + \sin s \cos \tfrac{\omega}{\gamma} \ln(\tfrac{1}{z})\right)$$

$$+ z^{\frac{-2\lambda}{\gamma}} h_1(s,z) \; .$$

$f_2(\Delta x,\Delta y)$

$$= \beta_1 z^{\frac{-\lambda}{\gamma}} \left(\cos s \cos \tfrac{\omega}{\gamma} \ln(\tfrac{1}{z}) - \sin s \sin \tfrac{\omega}{\gamma} \ln(\tfrac{1}{z})\right)$$

$$+ \beta_2 z^{\frac{-\lambda}{\gamma}} \left(\cos s \sin(\tfrac{\omega}{\gamma} \ln(\tfrac{1}{z})) + \sin s \cos(\tfrac{\omega}{\gamma} \ln(\tfrac{1}{z}))\right)$$

$$+ z^{\frac{-2\lambda}{\gamma}} h_2(s,z)$$

where $h_1(s,z)$ and $h_2(s,z)$ are bounded functions near $(0,0)$.

We are looking for the fixed points of the map T near $(0,0)$ that is the solutions of the system

$$\begin{cases} z^{\frac{-\lambda}{\gamma}}\,[(\alpha_1 \cos s + \alpha_2 \sin s)\cos(\frac{\omega}{\gamma}\,\ell n(\frac{1}{z})) \\ \qquad + (\alpha_2 \cos s - \alpha_1 \sin s)\,\sin(\frac{\omega}{\gamma}\,\ell n(\frac{1}{z}))] + z^{\frac{-2\lambda}{\gamma}}\,h_1(s,z) = s \\ z^{\frac{-\lambda}{\gamma}}\,[(\beta_1 \cos s + \beta_2 \sin s)\cos(\frac{\omega}{\gamma}\,\ell n(\frac{1}{z})) \\ \qquad + (\beta_2 \cos s - \beta_1 \sin s)\,\sin(\frac{\omega}{\gamma}\,\ell n(\frac{1}{z}))] + z^{\frac{-2\lambda}{\gamma}}\,h_2(x,z) = z \end{cases}$$

or simply:

$$\begin{cases} f_1^*(s,z) = s \\ f_2^*(s,z) = z \end{cases}.$$

Let $g_1(s,z) = f_1^*(s,z) - s$; we show that the equation $g_1(s,z) = 0$ could be solved for s as a function of z near $z = 0$, for positive values of z.

Let $\varepsilon > 0$ be small enough that the function $T = T_1T_0$ is defined for $(s,z) \in [-\pi,\pi] \times [0,\varepsilon]$.

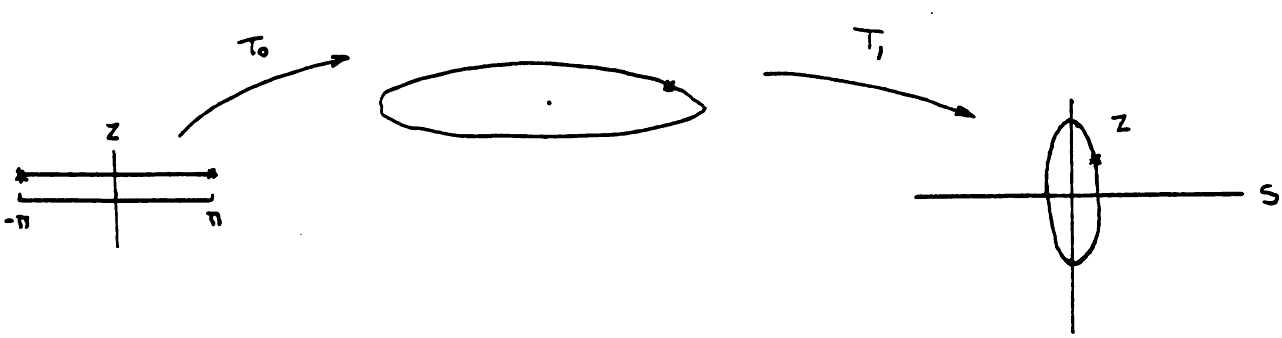


Figure 10

Since $T_1T_0([-\pi,\pi] \times \{z\})$ is homeomorphic to $T_0([-\pi,\pi] \times \{z\})$ which is a circle, identifying $(-\pi,z)$ and (π,z) we get

$$T(-\pi,z) = T(\pi,z)$$

which implies $f_1^*(-\pi,z) = f_1^*(\pi,z)$.

If $f_1^*(-\pi,z) = -\pi$ or $f_1^*(\pi,z) = \pi$, then we define $s(z) = -\pi$ or $s(z) = +\pi$, respectively. Otherwise $g_1(-\pi,z)$ and $g_1(\pi,z)$ have different signs. In this case by the intermediate value theorem $g_1(s,z) = 0$ for some $s \in (-\pi,\pi)$.

Now we show the uniqueness of s. Suppose $f_1^*(s_1,z) = s_1$ and $f_1^*(s_2,z) = s_2$. Then $g_1(s_1,z) = 0$ and $g_1(s_2,z) = 0$. Therefore by the mean value theorem $\frac{\partial g_1(s,z)}{\partial s} = 0$ for some $s_1 < s < s_2$. But $\frac{\partial g(s,z)}{\partial s} = \frac{\partial f_1^*(s,z)}{\partial s} - 1 = z^{\frac{-\lambda}{\gamma}}[B(s,z)] - 1$ where $B(s,z)$ is bounded in s and z. Therefore for z small enough, $|z^{\frac{-\lambda}{\gamma}}[B(s,z)]| < \frac{1}{2}$. Hence $s = s(z)$ for some $|z| \leq \gamma$. Then substituting in $f_2^*(s,z) = z$, we get

$$z^{\frac{-\lambda}{\gamma}}[(\beta_1 \cos s + \beta_2 \sin s)\cos(\tfrac{\omega}{\gamma}\ln(\tfrac{1}{z}))$$
$$+ (\beta_2 \cos s - \beta_1 \sin s)\sin \tfrac{\omega}{\gamma}(\ln(\tfrac{1}{z}))]$$
$$+ z^{\frac{-2\lambda}{\gamma}} h_2(s,z) = z \quad . \tag{4}$$

Substituting $z = e^{-\gamma t}$ in (4) we get

$$e^{\lambda t}\,[(\beta_1 \cos s + \beta_2 \sin s)\cos \omega t + (\beta_2 \cos s - \beta_1 \sin s)\sin \omega t] + e^{2\lambda t}\, h_2(s,t) = e^{-\gamma t}.$$

We also need the continuity of the function $s = s(z)$ in t. Since $z = e^{-\lambda t}$ is continuous in t, it suffices to show that $s(z)$ is continuous in z.

Let $z_n \to z$. We show that $s(z_n) \to s(z)$ by proving that the only limit point of $s(z_n)$ is $s(z)$.

Suppose $s(z_{n_k}) = s_{n_k} \to \alpha$. Since by definition of s_{n_k}, $f(s_{n_k}, z_{n_k}) = s_{n_k}$, and since f is continuous,

$$z_{n_k} \to z,\ s_{n_k} \to \alpha \Rightarrow f(s_{n_k}, z_{n_k}) \to f(\alpha, z).$$

Therefore $f(\alpha,z) = \alpha$. Definition and uniqueness of $s(z)$ imply $\alpha = s(z)$.

Since $s(z_n) \in [-\pi,\pi]$, it has at least one limit point by Boltzano-Weirestrass theorem. The above argument shows

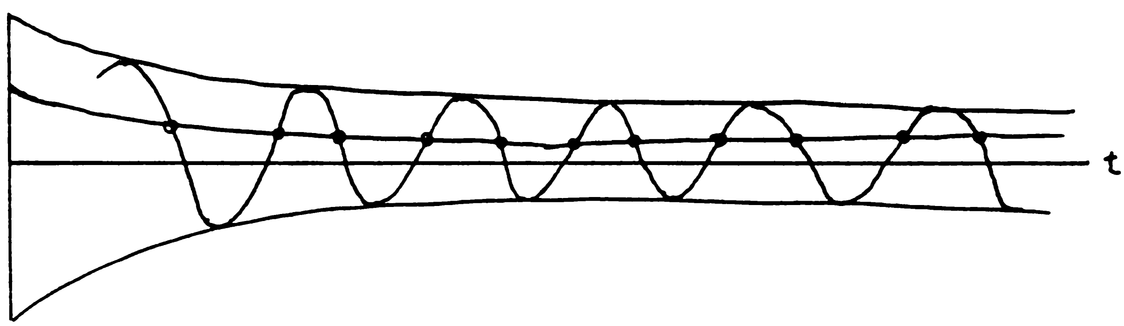
$$\overline{\lim}\, s(z_n) = \underline{\lim}\, s(z_n) = s(z)\ .$$

Let $f(t) = e^{\lambda t}[(\beta_1 \cos s + \beta_2 \sin s)\cos \omega t + (\beta_2 \cos s - \beta_1 \sin s)\sin \omega t + e^{\lambda t}\, h_2(s,t)]$. Since $\lambda < 0$, $e^{\lambda t}\, h_2(s,t) \to 0$ exponentially as $t \to \infty$. Therefore $f(t)$ is an oscillating function, and

$$k_2 e^{\lambda t} \leq |f(t)| \leq k_1 e^{\lambda t}$$

for k_1 and k_2 positive numbers.

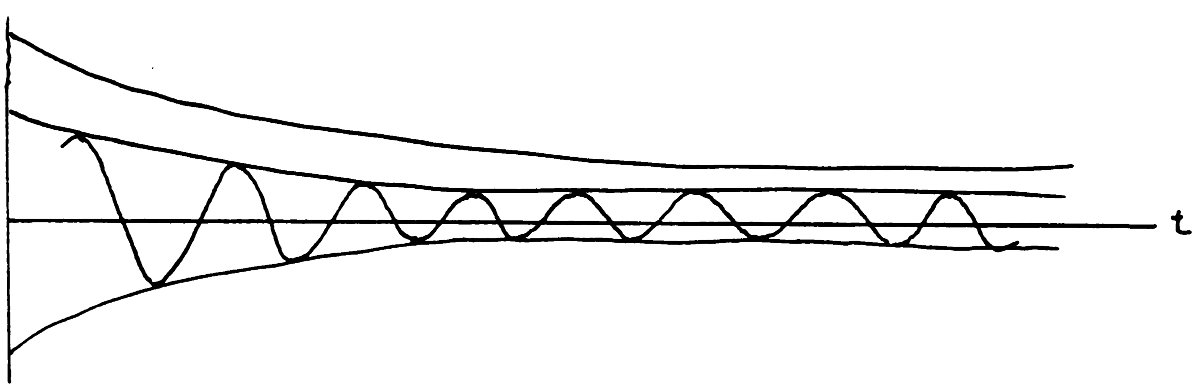
In case $\gamma > -\lambda$ the curves $f(t)$ and $e^{-\lambda t}$ have infinitely many intersection points as $t \to \infty$, and in case $\gamma < -\lambda$, the function $e^{\gamma t}$ dominates the functions $k_1 e^{\lambda t}$ and $k_2 e^{\lambda t}$ eventually and there will be no intersection points for $t \geq t^*$.



Case 1. Infinitely many intersections

($\gamma > -\lambda$)

Figure 11-a



Case 2. No intersection ($\gamma < -\lambda$)

Figure 11-b

The geometry of the transformations T_0 and T_1 and $T = T_1T_0$ is illustrated in Figure 12.

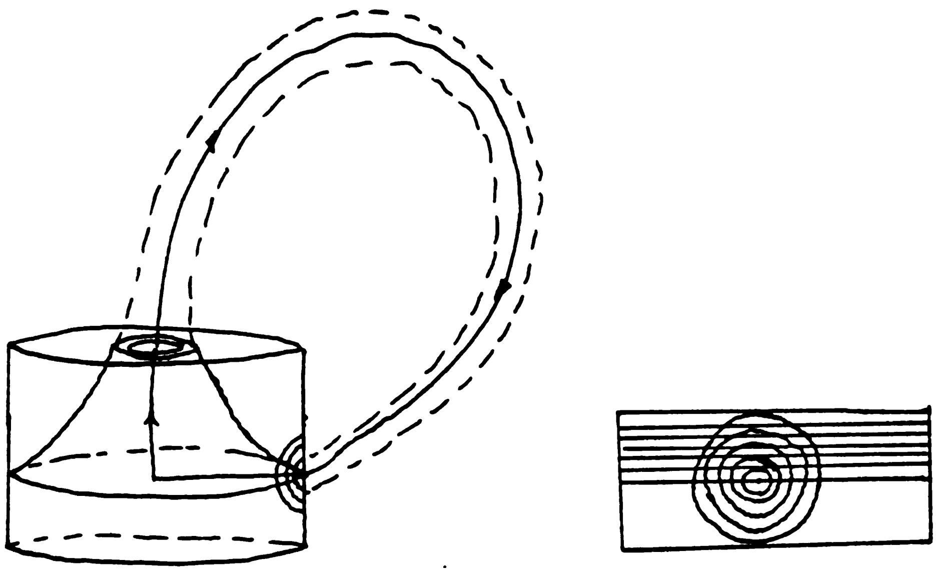


Figure 12

In case 1, eventually the circles will be mapped so that they hit the appropriate levels to create periodic orbits, and in case 2, all circles are eventually mapped lower than the appropriate level and therefore there is no periodic orbit in a small neighbourhood of homoclinic orbit. We can summarize the above results in the following theorem.

Theorem 2.1. Under the above assumptions there are infinitely many periodic solutions in any neighbourhood of the homoclinic orbit in case $\gamma > -\lambda$ and there is a neighbourhood of the homoclinic orbit with no periodic orbit in case $\nu < -\lambda$.

The next section will be devoted to the neighbourhoods of hetroclinic orbits in $\mathbb{R}^3$.

2.2. On the existence of a neighbourhood of a cycle consisting of two hetroclinic orbits without any periodic orbit in $\mathbb{R}^3$

Consider a system of 3 differential equations

$$\dot{x} = f(x) \tag{1}$$

and assume that (1) has two hyperbolic critical points, p_1 and p_2 with eigenvalues of the linearized system $\lambda_i \pm \omega_i$ and γ_i respectively where $\gamma_1\gamma_2 < 0$ and $\lambda_1\lambda_2 < 0$ and also $\lambda_1 < 0$, $\gamma_i\lambda_i < 0$ for $i = 1,2$. In this case the well-known theorem on the stable and unstable manifolds indicates the existence of a two dimensional stable manifold, and a one-dimensional unstable manifold at one point and a 2-dimensional unstable manifold along with a one-dimensional stable manifold at the other point.

We also assume there exists a hetroclinic orbit joining the two critical points from p_1 to p_2 and another one joining p_2 to p_1 making a cycle.

The following illustration shows the complete situation.

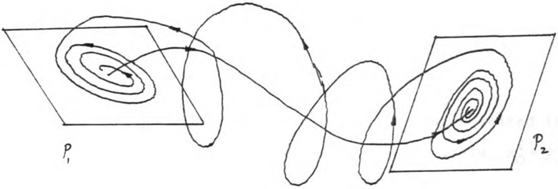


Figure 13

We claim the following:

Theorem 2.2. If $\gamma_1\lambda_2 < \gamma_2\lambda_1$, then there exists a neighbourhood of the cycle without any periodic orbit.

Proof: We use the method of point transformation, assuming the system to be linear in a small neighbourhood of each critical point. Define the maps T_1, T_2, T_3, T_4, consider the composite map $T_4 \circ T_3 \circ T_2 \circ T_1$ and look for its fixed points in a small neighbourhood of the cycle.

a) Construction of the map T_1:

Assume the system to be linear in a small neighbourhood of p_1, and also assume that the motion is governed by the following system in some neighbourhood N of $p_1 = 0$:

$$\begin{cases} \dot{x} = \lambda_1 x - \omega_1 y \\ \dot{y} = \omega_1 x + \lambda_1 y \\ \dot{z} = \gamma_1 z. \end{cases} \tag{2}$$

We also suppose that the cube $[-1,1]^3$ is contained in N. Starting at a point (x_0,y_0,z_0) on the circle $x_0^2 + y_0^2 = 1$ the solution curve would be, according to (2)

$$\begin{cases} x(t) = e^{\lambda_1 t} \; (x_0 \cos \omega_1 t - y_0 \sin \omega_1 t) \\ y(t) = e^{\lambda_1 t} \; (x_0 \sin \omega_1 t + y_0 \cos \omega_1 t) \\ z(t) = e^{\gamma_1 t} \; z_0 \; . \end{cases} \tag{3}$$

The point of intersection of this trajectory with the plane $z = 1$ is found easily from (2) as follows:

$$z(t) = 1 \Rightarrow$$

$$t = \frac{1}{\gamma_1} \ln \left(\frac{1}{z_0}\right) \Rightarrow$$

$$e^{\lambda_1 t} = e^{\frac{\lambda_1}{\gamma_1} \ln \left(\frac{1}{z_0}\right)} = \left(\frac{1}{z_0}\right)^{\frac{\lambda_1}{\gamma_1}} = z_0^{\frac{-\lambda_1}{\gamma_1}}$$

$$\begin{cases} x = z_0^{\frac{-\lambda_1}{\gamma_1}} \left(x_0 \cos \omega \left(\frac{1}{\gamma_1} \ln \left(\frac{1}{z_0}\right)\right) - y_0 \sin \omega \left(\frac{1}{\gamma_1} \ln (z_0)\right)\right) \\ y = z_0^{\frac{-\lambda_1}{\gamma_1}} \left(x_0 \sin \omega \left(\frac{1}{\gamma_1} \ln \left(\frac{1}{z_0}\right)\right) + y_0 \cos \omega \left(\frac{1}{\gamma_1} \ln (z_0)\right)\right) \\ z = 1 \; . \end{cases} \tag{4}$$

This point is on the circle of radius $z_0^{\frac{-\lambda_1}{\gamma_1}}$. Therefore circles on the cylinder $x^2+y^2 = 1$, $0 \leq z \leq 1$ will be mapped onto circles on the punctured disc $z = 1$, $0 < x^2+y^2 \leq 1$. This transformation is illustrated in Figure 14 and we call it T_1.

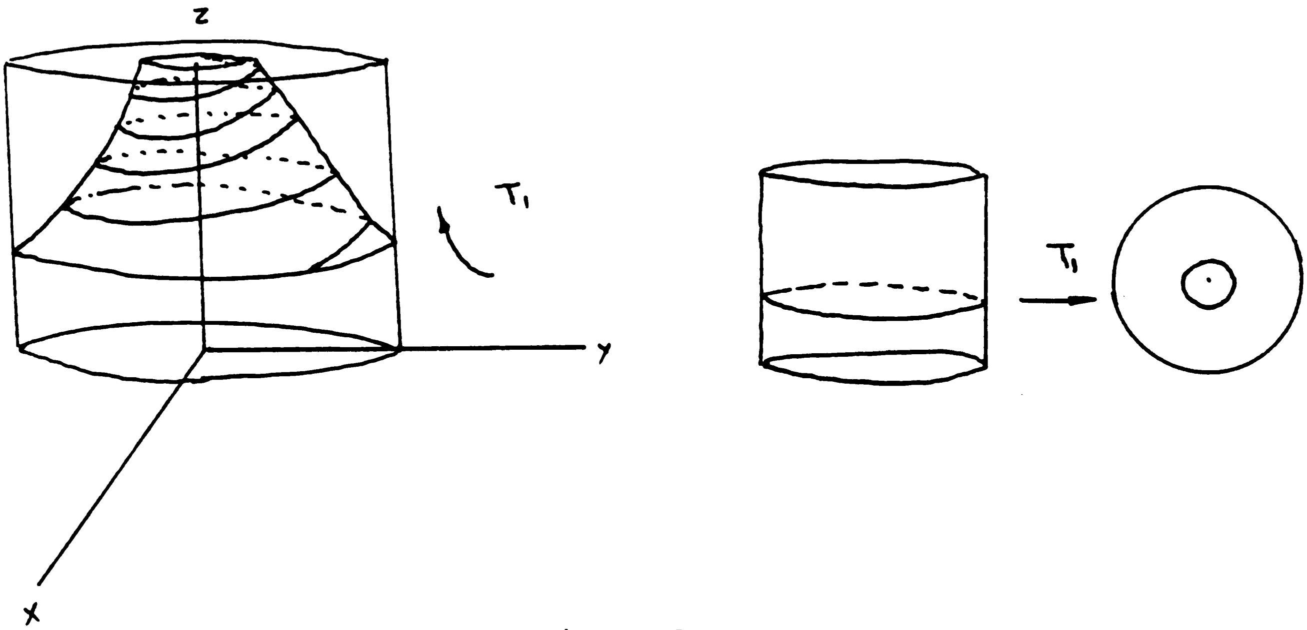


Figure 14

b) Construction of the map T_3:

We assume that the system is linear in a small neighbourhood of P_2, and since we have a one dimensional stable manifold along with a two dimensional unstable manifold, every trajectory starting at a point on the disk $z_1 = 1$, $0 < x_1^2+y_1^2 \leq 1$ will oscillate and hit the cylinder $x_1^2+y_1^2 = 1$, $z \leq 1$ in a point (x_{10}, y_{10}, z_{10}) where the following relation could be easily verified

$$
(5)\quad \begin{cases} x_1 = z_{10}^{\frac{-\lambda_2}{\gamma_2}} (x_{10} \cos \omega(\frac{1}{\gamma_2} \ln(\frac{1}{z_{10}})) \\ \qquad - y_{10} \sin \omega_2(\frac{1}{\gamma_2} \ln(z_{10}))) \\ y_1 = z_{10}^{\frac{-\lambda_2}{\gamma_2}} (x_{10} \sin \omega(\frac{1}{\gamma_2} \ln(\frac{1}{z_{10}})) \\ \qquad + y_{10} \cos \omega_2(\frac{1}{\gamma_2} \ln(z_{10}))) . \end{cases}
$$

Therefore

$$x_1^2 + y_1^2 = z_{10}^{\frac{-2\lambda_2}{\gamma_2}}$$

or

$$z_{10} = (x_1^2 + y_1^2)^{\frac{-\gamma_2}{2\lambda_2}} .$$

Solving the above system for (x_{10}, y_{10}), we get

$$
(6)\quad \begin{cases} x_{10} \cos \omega(\frac{1}{\gamma_2} \ln(\frac{1}{z_{10}})) - y_{10} \sin \omega(\frac{1}{\gamma_2} \ln(\frac{1}{z_{10}})) \\ \qquad = x_1 z_{10}^{\frac{-\gamma_2}{\lambda_2}} \\ x_{10} \sin \omega(\frac{1}{\gamma_2} \ln(\frac{1}{z_{10}})) + y_{10} \cos \omega(\frac{1}{\gamma_2} \ln(\frac{1}{z_{10}})) \\ \qquad = y_1 z_{10}^{\frac{-\gamma_2}{\lambda_2}} \end{cases}
$$

or

$$\begin{cases} x_{10} = z_{10}^{\frac{-\gamma_2}{\lambda_2}} \left(x_1 \cos \omega\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_0}\right)\right)\right. \\ \qquad\qquad \left. + y_1 \sin \omega\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_{10}}\right)\right)\right) \\ y_{10} = z_{10}^{\frac{-\gamma_2}{\lambda_2}} \left(y_1 \cos \omega\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_0}\right)\right)\right. \\ \qquad\qquad \left. - x_1 \sin \omega\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_{10}}\right)\right)\right) \ . \end{cases} \tag{7}$$

Therefore in fact the map $T_3 : (x_1, y_1) \to (x_{10}, y_{10}, z_{10})$ with the following formula maps the punctured disk $z_1 = 1 \quad 0 < x_1^2 + y_1^2 \le 1$ to the cylinder $0 < z_{10} \le 1$, $x_{10}^2 + y_{10}^2 = 1$. The map T_3 is not well-defined at the point $(0,0)$ but we define $T_3(0,0) = M_2$ where M_2 is the point of intersection of the hetroclinic orbit with the cylinder 2.

$$\begin{cases} z_{10} = (x_1^2 + y_1^2)^{\frac{-\gamma_2}{2\lambda_2}} \\ x_{10} = z_{10}^{\frac{-\gamma_2}{\lambda_2}} \left(x_1 \cos \omega_2\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_{10}}\right)\right)\right. \\ \qquad\qquad \left. + y_1 \sin \omega_2\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_{10}}\right)\right)\right) \\ y_{10} = z_{10}^{\frac{-\gamma_2}{\lambda_2}} \left(y_1 \cos \omega_2\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_{10}}\right)\right)\right. \\ \qquad\qquad \left. - x_1 \sin \omega_2\left(\frac{1}{\gamma_2}\ \ln\left(\frac{1}{z_{10}}\right)\right)\right) \ . \end{cases} \tag{8}$$

The map T_3 is illustrated in Figure 15.

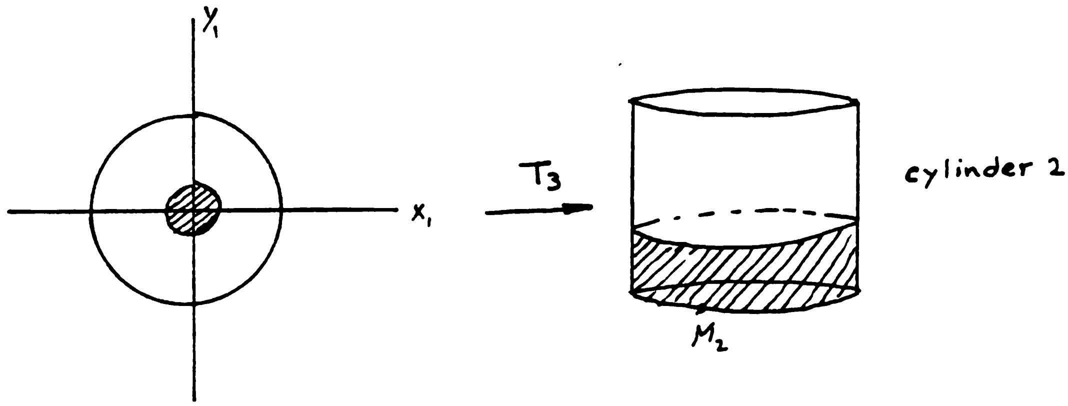


Figure 15

c) Construction of T_2 and T_4:

The map T_2 is defined along the hetroclinic orbit from the disc $z = 1$ to $z_1 = 1$. It is defined by point transformation by following the trajectory and since both discs $z = 1$, $0 \leq x^2 + y^2 \leq 1$ and $z_1 = 1$, $0 \leq x_1^2 + y_1^2 \leq 1$ are transversal to the trajectories of the system, the map T_2 could be expressed as

$$T_2 : \begin{cases} x_1 = a_{11}x + a_{12}y + \text{h.o.t.} \\ y_1 = a_{21}x + a_{22}y + \text{h.o.t.} \end{cases} \tag{9}$$

and the map T_4 is defined similarly but from a neighbourhood of M_1 on cylinder (2) to a neighbourhood of M_2 on cylinder (1) where x_2 and y_2 are new local coordinates of points in a neighbourhood of M_2 on cylinder (1) with $M_2 = (0,0)$.

To have a periodic orbit, we need a fixed point of the composite map $T = T_4T_3T_2T_1$, which means starting at a point on a level $z = z_0$ on cylinder 1, we have to come back to the same level and the same point. We show that if the product $\gamma_1\lambda_2$ is greater than $\gamma_2\lambda_1$, then we can never return to the starting level $z = z_0$, which means no periodic orbit could exist in a small neighbourhood of the cycle.

From (4) it is clear that starting at the level $z = z_0$ we end up at a point on a circle of radius $z_0^{\frac{-\lambda_1}{\gamma_1}}$ via T_1; that is, $x^2 + y^2 = z_0^{\frac{-2\lambda_1}{\gamma_1}}$. From (9) in the definition of T_2, assuming the transversality of the hetroclinic in the sense that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is invertible, we get

$$K_1^2(x^2 + y^2) \leq x_1^2 + y_1^2 \leq K_2^2(x^2 + y^2)$$

where (x,y) and (x_1,y_1) are in small neighbourhoods of $(0,0)$.

Therefore the composite map T_2T_1 transforms the points of level $z = z_0$ to the points on the circle of radius γ_1 where

$$K_1 z_0^{\frac{-\lambda_1}{\gamma_1}} \leq r_1 \leq K_2 z_0^{\frac{-\lambda_1}{\gamma_1}} , \qquad K_1^2 + K_2^2 > 0 .$$

T_3 takes the points on the circle of radius r to the points of level $z_{10} = (r^2)^{\frac{-\gamma_2}{2\lambda_2}}$. Therefore the composite map $T_3T_2T_1$ takes level $z = z_0$ to the level $z_{10} = r_2$ where

$$L_1 z_0^{\frac{\lambda_1\gamma_2}{\gamma_1\lambda_2}} \leq r_2 \leq L_2 z_0^{\frac{\lambda_1\gamma_2}{\gamma_1\lambda_2}} , \qquad L_1^2 + L_2^2 > 0 .$$

Assuming that the matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ in the definition of T_4 is invertible,we get

$$M_1(x_0^2 + y_0^2) \leq x_2^2 + y_2^2 \leq M_2(x_0^2 + y_0^2) , \qquad M_1^2 + M_2^2 > 0 .$$

Therefore $T_4T_3T_2T_1$ takes the level $z = z_0$ to points on circle with radius R, where

$$N_1 z_0^{\frac{\lambda_1\gamma_2}{\gamma_1\lambda_2}} \leq R \leq N_2 z_0^{\frac{\lambda_1\gamma_2}{\gamma_1\lambda_2}} , \qquad N_1^2 + N_2^2 > 0 .$$

Since by our assumption $\frac{\lambda_1 \gamma_2}{\gamma_1 \lambda_2} > 1$ and the equation

$$N z_0^{\frac{\lambda_1 \gamma_2}{\gamma_1 \lambda_2}} = z_0$$

does not have any nonzero solution in a small neighbourhood of zero, the system cannot have any periodic orbit in a small neighbourhood of the cycle.

CHAPTER 3

A singular perturbation problem on the existence of periodic orbits for $\varepsilon^2\dddot{u} + \dot{u} = u(1-u)$

3.1. The differential equation:

$$\varepsilon^2\dddot{u} + \dot{u} = u(1-u) \tag{1}$$

appears naturally as a modified equation for the logistic equation $\dot{u} = u(1-u)$ solved by some numerical methods.

We show that even if the logistic equation does not have any nontrivial,periodic solutions, the modified equation (1) has nontrivial periodic orbits for ε sufficiently small.

The third order differential equation (1) is equivalent to the 3-dimensional system:

$$\begin{cases} \dot{u} = v \\ \dot{v} = w \\ \varepsilon^2\dot{w} = -v + u(1-u) \end{cases} \tag{2}$$

Due to the rapid growth of w and v relative to u, it is appropriate to change the variables and look for the periodic solutions of the new system. The proper change of variable would be

$$\begin{cases} U = u \\ V = \varepsilon v \\ W = \varepsilon^2 w \\ T = \dfrac{1}{\varepsilon}\, t \end{cases}$$

which changes the system (3) into

$$\begin{cases} \dot{U} = V \\ \dot{V} = W \\ \dot{W} = -V + \varepsilon U(1-U) \end{cases}$$

or $\dot{X} = AX + \varepsilon f(X)$ where $X = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$, $f(X) = \begin{pmatrix} 0 \\ 0 \\ U(1-U) \end{pmatrix}$

and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

The eigenvalues of A are $0, \pm i$, with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}$ respectively.

Since $w = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} = u + iv$, where $u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,

the matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ has the property that,

$B = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ is the Jordan Canonical form of

A and the transformation $y = P^{-1}x$ changes the linear system $\dot{X} = AX$ into $\dot{y} = By$.

Let $y = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The transformation $y = P^{-1}x$ has the form:

$$\begin{cases} x = U + W \\ y = -W \\ z = V \end{cases} \tag{4}$$

and the system (3) becomes:

$$\begin{cases} \dot{x} = \varepsilon(x+y)(1-x-y) \\ \dot{y} = z - \varepsilon(x+y)(1-x-y) \\ \dot{z} = -y \ . \end{cases} \tag{5}$$

To show the existence of a periodic solution in (5) we use another change of variable;

$$\begin{cases} y = \gamma \sin \theta \\ z = \gamma \cos \theta \\ x = \rho \end{cases} \tag{6}$$

$$\dot{\gamma} = \frac{1}{r}(y\dot{y} + z\dot{z}) = -\varepsilon \sin \theta(\rho + \gamma \sin \theta) + \varepsilon \sin \theta(\rho + \gamma \sin \theta)^2$$
$$= -\varepsilon \sin \theta(\rho + \gamma \sin \theta)(1 - \rho - \gamma \sin \theta)$$

$$\tan \theta = \frac{y}{z} \Rightarrow \dot{\theta}(1 + \tan^2\theta) = \frac{1}{z^2}(\dot{y}z - y\dot{z})$$

$$\dot{\theta} = \frac{\dot{y}z - y\dot{z}}{z^2(1+\tan^2\theta)} = \frac{\dot{y}z - y\dot{z}}{z^2(1 + \frac{y^2}{z^2})} = \frac{\dot{y}z - y\dot{z}}{z^2 + y^2}$$

$$= \frac{z^2 - \varepsilon z(x+y) + \varepsilon z(x+y)^2 + y^2}{y^2 + z^2} = 1 - \varepsilon \frac{z(x+y)(1-x-y)}{y^2 + z^2} \ .$$

Therefore

$$\dot{\theta} = 1 - \varepsilon \frac{\cos\theta(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta)}{\gamma}$$

and

$$\dot{\rho} = \dot{x} = \varepsilon(x + y) - \varepsilon(x + y)^2$$
$$= \varepsilon(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta) \quad .$$

The new system in (γ,ρ,θ) is

$$\begin{cases} \dot{\rho} = \varepsilon(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta) \\ \dot{\gamma} = -\varepsilon\sin\theta(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta) \\ \dot{\theta} = 1 - \varepsilon \cdot \frac{1}{\gamma}\cos\theta(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta). \end{cases} \tag{7}$$

The existence of a periodic solution for (7) is equivalent to the existence of a 2π periodic solution of

$$\begin{cases} \frac{d\rho}{d\theta} = \frac{\varepsilon\gamma(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta)}{\gamma - \varepsilon\cos\theta(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta)} \\ \frac{d\gamma}{d\theta} = -\varepsilon\frac{\gamma\sin\theta(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta)}{\gamma - \varepsilon\cos\theta(\rho + \gamma\sin\theta)(1 - \rho - \gamma\sin\theta)} \end{cases} \tag{8}$$

for γ in a compact set.

Consider the 2π-periodic change of variable

$$\begin{pmatrix}\gamma\\ \rho\end{pmatrix} = \begin{pmatrix}\gamma_0\\ \rho_0\end{pmatrix} + \varepsilon\begin{pmatrix}\gamma^{(1)}(\gamma_0,\rho_0,\theta)\\ \rho^{(1)}(\gamma_0,\rho_0,\theta)\end{pmatrix} + \varepsilon^2\begin{pmatrix}\gamma^{(2)}(\gamma_0,\rho_0,\theta)\\ \rho^{(2)}(\gamma_0,\rho_0,\theta)\end{pmatrix} + \cdots \tag{9}$$

taking $\begin{pmatrix}\gamma\\ \rho\end{pmatrix}$ into $\begin{pmatrix}\gamma_0\\ \rho_0\end{pmatrix}$ and try to determine $\gamma^1(\gamma_0,\rho_0,\theta)$ $\gamma^{(2)}(\gamma_0,\rho_0,\theta),\ldots,\rho^1(\gamma_0,\rho_0,\theta)$, $\rho^{(2)}(\gamma_0,\rho_0,\theta)$ and functions

$R^1(\gamma_0,\rho_0)$, $R^{(2)}(\gamma_0,\rho_0),\ldots$ so that the differential equation for $\begin{pmatrix}\gamma_0\\ \rho_0\end{pmatrix}$ is autonomous and given by

$$\frac{d}{d\theta}\begin{pmatrix}\gamma_0\\ \rho_0\end{pmatrix} = \varepsilon\, R^1(\gamma_0,\rho_0) + \varepsilon^2\, R^{(2)}(\gamma_0,\rho_0) + \cdots \quad . \tag{10}$$

If such a transformation can be found, then the 2π periodic solutions of (8) coincide with the equilibrium points of (10).

We can find functions $R^1(\gamma_0,\rho_0)$, $R^{(2)}(\gamma_0,\rho_0),\ldots$ $\gamma^{(1)}(\gamma_0,\rho_0,\theta)$, $\gamma^{(2)}(\gamma_0,\rho_0,\theta),\ldots,\rho^{(1)}(\gamma_0,\rho_0,\theta)$, $\rho^{(2)}(\gamma_0,\rho_0,\theta),\ldots$ inductively by the Method of Averaging, using the requirement of 2π-periodicity of $\gamma^{(i)}(\gamma_0,\rho_0,\theta)$ and $\rho^{(i)}(\gamma_0,\rho_0,\theta)$, $i = 1,2,\ldots$. Since $\frac{d}{d\theta}\begin{pmatrix}\gamma\\ \rho\end{pmatrix} = \frac{d}{d\theta}\begin{pmatrix}\gamma_0\\ \rho_0\end{pmatrix} + \varepsilon\frac{d}{d\theta}\begin{pmatrix}\gamma^{(1)}(\gamma_0,\rho_0,\theta)\\ \rho^{(1)}(\gamma_0,\rho_0,\theta)\end{pmatrix} + \cdots$ we get

$$\begin{aligned}\varepsilon\begin{pmatrix}g_1(\gamma,\rho,\varepsilon)\\ g_2(\gamma,\rho,\varepsilon)\end{pmatrix} &= \varepsilon\, R^1(\gamma_0,\rho_0) + \varepsilon^2\, R^2(\gamma_0,\rho_0) + \cdots + \\ &\quad \varepsilon\frac{d}{d\theta}\begin{pmatrix}\gamma^{(1)}(\gamma_0,\rho_0,\theta)\\ \rho^{(1)}(\gamma_0,\rho_0,\theta)\end{pmatrix} + \varepsilon^2\frac{d}{d\theta}\begin{pmatrix}\gamma^{(2)}(\gamma_0,\rho_0,\theta)\\ \rho^{(2)}(\gamma_0,\rho_0,\theta)\end{pmatrix} + \cdots\end{aligned} \tag{11}$$

where

$$g_1(\gamma,\rho,\varepsilon) = \frac{-\gamma(\rho+\gamma\sin\theta)(1-\rho-\gamma\sin\theta)\sin\theta}{\gamma-\varepsilon\cos\theta(\rho+\gamma\sin\theta)(1-\rho-\gamma\sin\theta)}$$

$$g_2(\gamma,\rho,\varepsilon) = \frac{\gamma(\rho+\gamma\sin\theta)(1-\rho-\gamma\sin\theta)}{r-\varepsilon\cos\theta(\rho+\gamma\sin\theta)(1-\rho-\gamma\sin\theta)} \; .$$

From (11) we see that the following equation must be satisfied:

$$R^1(\gamma_0,\rho_0) = \begin{pmatrix} g_1(\gamma_0,\rho_0,0) \\ g_2(\gamma_0,\rho_0,0) \end{pmatrix} + \frac{d}{d\theta}\begin{pmatrix} \gamma^1(\gamma_0,\rho_0,\theta) \\ \rho^1(\gamma_0,\rho_0,\theta) \end{pmatrix}$$

or

$$\frac{d}{d\theta}\begin{pmatrix} \gamma^1(\gamma_0,\rho_0,\theta) \\ \rho^1(\gamma_0,\rho_0,\theta) \end{pmatrix} = R^1(\gamma_0,\rho_0) - \begin{pmatrix} g_1(\gamma_0,\rho_0,0) \\ g_2(\gamma_0,\rho_0,0) \end{pmatrix} . \qquad (12)$$

(12) is equivalent to

$$R^1(\gamma_0,\rho_0) = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} g_1(\gamma_0,\rho_0,0) \\ g_2(\gamma_0,\rho_0,0) \end{pmatrix} d\theta \qquad (12.1)$$

$$\begin{pmatrix} \gamma^{(1)}(\gamma_0,\rho_0,\theta) \\ \rho^{(1)}(\gamma_0,\rho_0,\theta) \end{pmatrix}$$

$$= \int \left(R^{(1)}(\gamma_0,\rho_0) - \begin{pmatrix} g_1(\gamma_0,\rho_0,0) \\ g_2(\gamma_0,\rho_0,0) \end{pmatrix}\right) d\theta . \qquad (12.2)$$

$$\frac{1}{2\pi}\int_0^{2\pi} g_1(\gamma_0,\rho_0,0)d\theta = \frac{1}{2\pi}\int_0^{2\pi} -\sin\theta(\rho_0+\gamma_0\sin\theta)$$

$$(1-\rho_0-\gamma_0\sin\theta)d\theta = \frac{1}{2}\gamma_0(2\rho_0-1)$$

$$\frac{1}{2\pi}\int_0^{2\pi} g_2(\gamma_0,\rho_0,0)d\theta = \frac{1}{2\pi}\int_0^{2\pi} (\rho_0+\gamma_0\sin\theta)(1-\rho_0-\gamma_0\sin\theta)d\theta$$

$$= \rho_0-\rho_0^2-\frac{1}{2}\gamma_0^2.$$

Therefore $R^1(\gamma_0,\rho_0) = \begin{pmatrix} \frac{1}{2}\gamma_0(2\rho_0-1) \\ \rho_0-\rho_0^2-\frac{1}{2}\gamma_0^2 \end{pmatrix}$.

$\gamma^{(1)}(\gamma_0,\rho_0,\theta) = \int \frac{1}{2}\gamma_0(2\rho_0-1)+\sin\theta(\rho_0+\gamma_0\sin\theta)$
$(1-\rho_0-\gamma_0\sin\theta)d\theta = -(\rho_0-\rho_0^2)\cos\theta+\frac{1}{4}\gamma_0(2\rho_0-1)\sin 2\theta -$
$\gamma_0^2(\frac{1}{3}\cos^3\theta-\cos\theta)$

$\rho^{(1)}(\gamma_0,\rho_0,\theta) = \int \rho_0-\rho_0^2-\frac{1}{2}\gamma_0^2-(\rho_0+\gamma_0\sin\theta)$
$(1-\rho_0-\gamma_0\sin\theta)d\theta = \gamma_0(1-2\rho_0)\cos\theta-\frac{1}{4}\gamma_0^2\sin 2\theta$. Therefore

$$\begin{aligned}\frac{d}{d\theta}\begin{pmatrix}\gamma_0\\ \rho_0\end{pmatrix} &= \epsilon\begin{pmatrix}\frac{1}{2}\gamma_0(2\rho_0-1)\\ \rho_0-\rho_0^2-\frac{1}{2}\gamma_0^2\end{pmatrix} + \epsilon^2 R^2(\gamma_0,\rho_0)+\cdots \\ &= \epsilon\left[\begin{pmatrix}\frac{1}{2}\gamma_0(2\rho_0-1)\\ \rho_0-\rho_0^2-\frac{1}{2}\gamma_0^2\end{pmatrix} + \epsilon R^{(2)}(\gamma_0,\rho_0)+\cdots\right] \qquad (13) \\ &= \epsilon\, G(\gamma_0,\rho_0,\epsilon) .\end{aligned}$$

Since

$$G(\gamma_0,\rho_0,0) = \begin{pmatrix}\frac{1}{2}\gamma_0(2\rho_0-1)\\ \rho_0-\rho_0^2-\frac{1}{2}\gamma_0^2\end{pmatrix},$$

the equation $G(\gamma_0,\rho_0,0) = 0$ has solutions

$$(\gamma_0,\rho_0) = (\frac{\sqrt{2}}{2},\frac{1}{2}),\ (0,0),\ (0,1) .$$

We are not interested in the trivial periodic solutions (0,0) and (0,1). Therefore we use The Implicit Function Theorem at $(\gamma_0,\rho_0) = (\frac{\sqrt{2}}{2},\frac{1}{2})$ to show the existence of an equilibrium point near $(\frac{\sqrt{2}}{2},\frac{1}{2})$.

$$\frac{\partial G(\gamma_0,\rho_0,0)}{\partial(\gamma_0,\rho_0)} = \begin{bmatrix} \frac{1}{2}\ (2\ \rho_0 - 1) & \gamma_0 \\ -\gamma_0 & 1 - 2\ \rho_0 \end{bmatrix} .$$

Therefore

$$\det\left(\frac{\partial G(\gamma_0,\rho_0,0)}{\partial(\gamma_0,\rho_0)}\right)\Bigg|_{(\frac{\sqrt{2}}{2},\ \frac{1}{2})} = \frac{1}{2} \neq 0$$

which gives $(\gamma_0(\varepsilon),\rho_0(\varepsilon)$, $G(\gamma_0(\varepsilon),\rho_0(\varepsilon),\varepsilon)) = 0$ for small enough ε and a 2π-periodic solution for (8).

We have proved

Theorem 3.1. The modified equation

$$\varepsilon^2 \dddot{u} + \dot{u} = u(1-u)$$

has a periodic solution for sufficiently small $\varepsilon > 0$.

3.2. The Averaged System

The averaged system

$$\begin{cases} \frac{dr}{d\theta} = r(\rho - \frac{1}{2}) \\ \\ \frac{d\rho}{d\theta} = \rho - \rho^2 - \frac{1}{2}\ r^2 \end{cases} \tag{14}$$

is conservative with the energy function

$$H(r,\rho) = r^2(4\rho - 4\rho^2 - r^2) ,$$

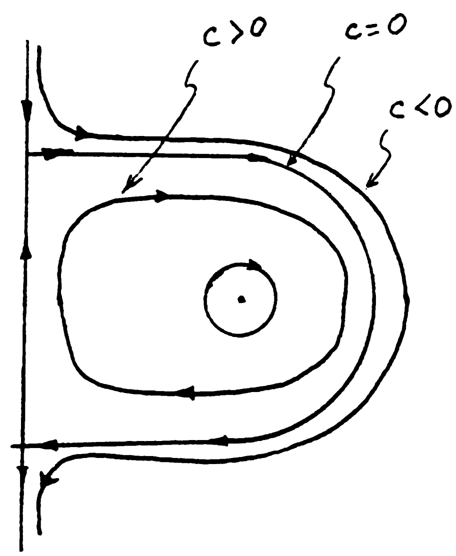
because

$$\frac{\partial H}{\partial \gamma} = 2\gamma(4\rho - 4\rho^2 - \gamma^2) - 2\gamma^3 = 8\gamma(\rho - \rho^2 - \frac{1}{2}\gamma^2)$$

$$\frac{\partial H}{\partial \rho} = \gamma^2(4 - 8\rho) = -8\gamma(\gamma\rho - \frac{1}{2}\gamma).$$

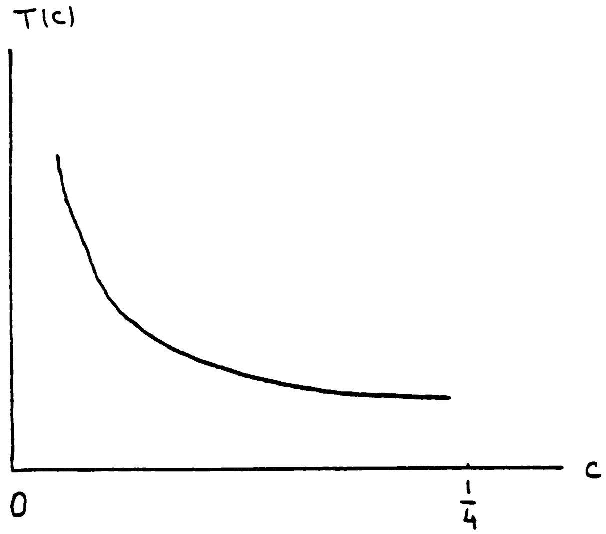
Therefore, $\frac{\partial H}{\partial \gamma}\gamma^{\cdot} + \frac{\partial H}{\partial \rho}\rho^{\cdot} = 0$, which shows the solution curves of the averaged system (14) are on the level curves of $H(\gamma,\rho) = c$.

For $c = 0$, the level curves correspond to $\gamma = 0$ and $\gamma = 2\sqrt{\rho(1-\rho)}$ which are the heteroclinic orbits whose existence was shown in previous chapters by studying the qualitative behavior of the averaged system.



For $c \neq 0$, $H(\gamma,\rho) = c$ implies $4\rho - 4\rho^2 - \gamma^2 = \frac{c}{\gamma^2}$, $(\gamma \neq 0)$. The positive values of c correspond to the periodic orbits inside the heteroclinic. The maximum value of

H is $\frac{1}{4}$ and it is attained at $\gamma = \frac{\sqrt{2}}{2}$, $\rho = \frac{1}{2}$ which is a critical point of the averaged system corresponding to the periodic orbit of the original 3-dimesnional system. For negative values of c, the level curves are unbounded as shown in the graph. We are only interested in the positive values of c, corresponding to the periodic orbits and we would like to investigate the existence of periodic solutions of (8). Since it takes infinite time to trace the heteroclinic orbit, the period map goes to infinity as c tends to 0^+.

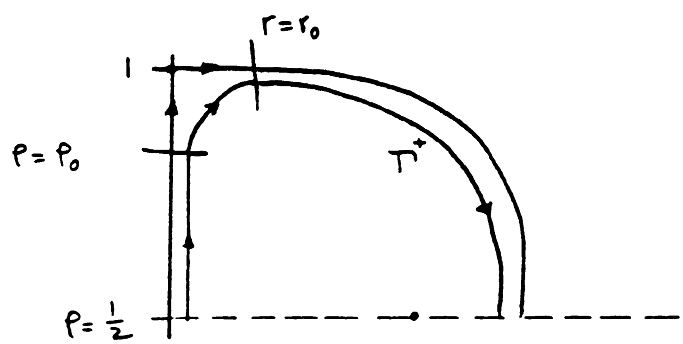


5.2 The existence of periodic orbits in (8)

Since (2) remains unchanged under the transformation

$$\begin{cases} \gamma \to \gamma \\ \rho \to 1-\rho \\ \theta \to -\theta , \end{cases} \tag{15}$$

starting at $\theta = 0$, $\rho(0) = \frac{1}{2}$, $0 < \gamma(0) < \frac{\sqrt{2}}{2}$, the trajectory Γ^+ corresponding to the positive values of θ traces a curve in (γ,ρ) plane symmetric to the trace of the trajectory Γ^- corresponding to the negative values of θ, with respect to the line $\rho = \frac{1}{2}$.



Since the solutions to the system (8) remains close to the solutions of the averaged system for finite time, and due to the 2π periodicity of (8), the stable and unstable manifolds at the hyperbolic points are 2π periodic, and as a point approaches the stable manifold on the line $\rho = \rho_0$, the solution curve passing through that point hits the line $\gamma = \gamma_0$ at a point approaching the unstable manifold, and from the line $\gamma = \gamma_0$ it takes finite time to hit $\rho = \frac{1}{2}$, at a point close to $(1,\frac{1}{2})$. We conclude that starting at $\theta = 0$, $\rho(0) = \frac{1}{2}$, $\gamma(0) > 0$ small, Γ^+ would hit $\rho = \frac{1}{2}$ at some $\gamma(\theta_1)$ near 1. For $\theta_1 = n\pi$ we get n distinct periodic solutions of period $2n\pi$ in (8) because (8) is 2π periodic and

$$\begin{cases} \gamma(-\theta_1) = \gamma(\theta_1) \\ \rho(-\theta_1) = \rho(\theta_1) \end{cases} \qquad 2\theta_1 = 2n\pi \quad .$$

We have proved

<u>Theorem 3.2</u>. For every $\epsilon > 0$ there is $N = N(\epsilon) > 0$ such that for every n, $0 \leq n \leq N$, there are n distinct periodic orbits with period $\approx 2n\pi$ in (8). Furthermore $N(\epsilon) \to \infty$ as $\epsilon \to 0$.

CHAPTER 4

On the Existence of a heteroclinic orbit

4.1. The differential equation

$$\varepsilon^2 \ddot{u} + \dot{u} = u(1-u) \tag{1}$$

has been studied in the previous chapter and the existence of periodic orbits for small ε was proved.

The existence of a heteroclinic orbit from $(1,0,0)$ to $(0,0,0)$ has been observed in computer experiments using the Shampine-Gordon Method to solve the associated system

$$\begin{cases} \dot{u} = v \\ \dot{v} = w \\ \varepsilon^2 \dot{w} = -v + u(1-u) \, , \end{cases} \tag{2}$$

We prove the existence of a spiral heteroclinic orbit using a proper change of variable and the variation of constant formula.

The change of variable which suits our purpose would be:

$$U = u, \quad V = \varepsilon v, \quad W = \varepsilon^2 w, \quad T = \frac{1}{\varepsilon} t \tag{3}$$

which changes the system into

$$\begin{cases} \dot{U} = V \\ \dot{V} = W \\ \dot{W} = -V + \varepsilon\, U(1-U) \end{cases} \tag{4}$$

System (4) could be written in the form $\dot{X} = AX + F(\varepsilon, X)$ where:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad F(\varepsilon, X) = \varepsilon \begin{pmatrix} 0 \\ 0 \\ u(1-u) \end{pmatrix}$$

and $X = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$.

A simple computation will show that $e^{2\pi A} = I$, and the solution to $\dot{X} = AX$ is;

$$\begin{cases} u_\ell(t) = u_0 + v_0 \sin t + w_0(1-\cos t) \\ v_\ell(t) = v_0 \cos t + w_0 \sin t \\ w_\ell(t) = -v_0 \sin t + w_0 \cos t\,, \end{cases} \tag{5}$$

These solutions are 2π-periodic, and satisfy

$$\begin{cases} v^2(t) + w^2(t) = v^2(0) + w^2(0) \\ u(t) + w(t) = u(0) + w(0)\,. \end{cases} \quad t \in \mathbb{R} \tag{6}$$

Solutions to the purturbed system (3) satisfy the variation of constant formula:

$$X(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} F(x(s), \varepsilon)\, ds\,.$$

For $t = 2\pi$,

$$X(2\pi) = X(0) + \int_0^{2\pi} \begin{bmatrix} \varepsilon(1-\cos s)u(s)(1-u(s)) \\ -\varepsilon(\sin s)u(s)(1-u(s)) \\ \varepsilon \cos s\, u(s)(1-u(s)) \end{bmatrix} ds \tag{7}$$

Since the solutions of $\dot{X} = AX + F(\varepsilon, X)$ depend smoothly on ε, and for $\varepsilon = 0$, the solutions satisfy (5), we get

$$u(t) = u_\ell(t) + \varepsilon h$$

where h is bounded and $u_\ell(t) = u_0 + v_0 \sin t + w_0(1-\cos t)$. Therefore,

$$u(t)(1-u(t)) = u_\ell(t)(1-u_\ell(t)) + \varepsilon B .$$

Substituting the above expression for $u(s)(1-u(s))$ in (7) we get

$$\Delta u = \varepsilon \int_0^{2\pi} (1-\cos s)u(s)(1-u(s))\, ds$$

$$= \varepsilon \int_0^{2\pi} (1-\cos s)(u_\ell(s))(1-u_\ell(s)) + \varepsilon^2 B_1\, ds.$$

Computing the integral part in Δu, we get

$$\Delta u = \pi(2u_0 + 3w_0 - 2u_0^2 - 5w_0^2 - v_0^2 - 6u_0w_0)\varepsilon + \varepsilon^2 B_1 .$$

Similarly:

$$\Delta v = -\pi\varepsilon(v_0 - 2u_0v_0 - 2v_0w_0) + \varepsilon^2 B_2$$

$$\Delta w = \pi\varepsilon(-w_0 + 2u_0w_0 + 2w_0^2) + \varepsilon^2 B_3 .$$

For small ε

$$\begin{cases} \Delta_1 u \approx \pi(2u_0 + 3w_0 - 2u_0^2 - 5w_0^2 - v_0^2 - 6u_0w_0) \\ \Delta_1 v \approx -\pi(v_0 - 2u_0v_0 - 2v_0w_0) \\ \Delta_1 w \approx \pi(-w_0 + 2u_0w_0 + 2w_0^2) . \end{cases}$$

Since $w = 0$ is transversal to the trajectories for $v_0 \neq \varepsilon u_0(1 - u_0)$, the Poincare's map is defined on $w = 0$, a compact set disjoint from a neighbourhood of $v_0 = \varepsilon u_0(1 - u_0)$. Therefore for $\tau = \tau(\varepsilon) \approx 2\pi$

$$\begin{cases} \Delta_1 u = \pi(2u_0 - 2u_0^2 - v_0^2) + \varepsilon B_{11} \\ \Delta_1 v = -\pi(v_0 - 2u_0v_0) + \varepsilon B_{21} \\ \Delta_1 w = 0 . \end{cases}$$

We divide the (u_0, v_0) plane into 4 regions, namely

$$R_1 : \begin{cases} \Delta_1 u > 0 \\ \Delta_1 v > 0 \end{cases} , \qquad R_2 : \begin{cases} \Delta_1 u > 0 \\ \Delta_1 v < 0 \end{cases}$$

$$R_3 : \begin{cases} \Delta_1 u < 0 \\ \Delta_1 v > 0 \end{cases} , \qquad R_4 : \begin{cases} \Delta_1 u < 0 \\ \Delta_1 v < 0 . \end{cases}$$

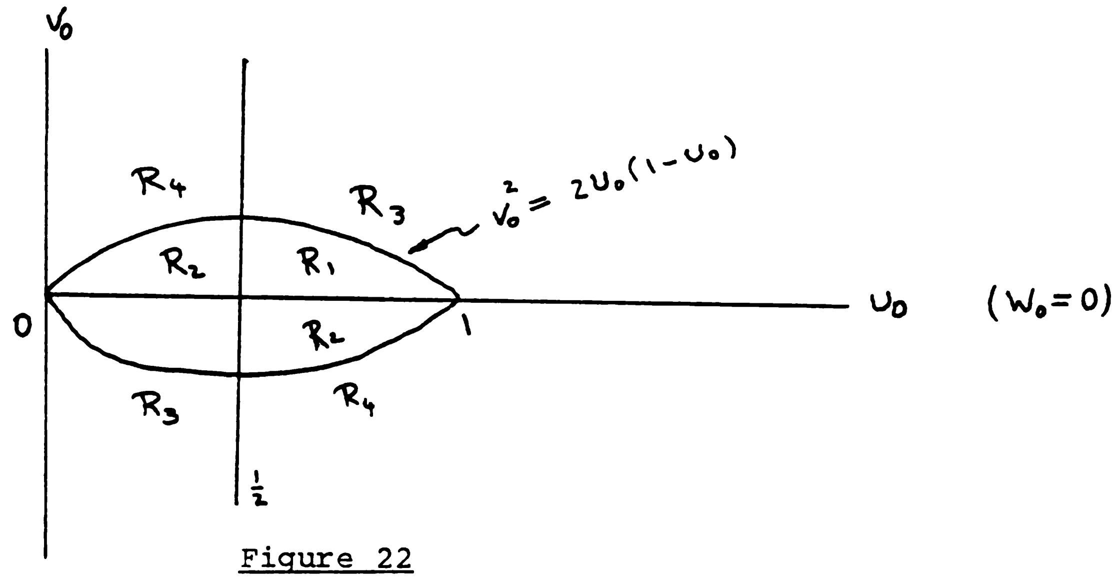


Figure 22

For $\varepsilon = 0$, starting at a point (u_0, v_0), $w_0 = 0$, the trajectory will be periodic in the (u,v) plane. It will trace the circle

$$(u - u_0)^2 + v^2 = R_0^2 \qquad R_0 = |v_0|$$

returning to (u_0, v_0) at $t = 2\pi$. For $\varepsilon > 0$, $t = \tau(\varepsilon)$ the trajectory will be nearby (u_0, v_0) and according to the position of (u_0, v_0) above, below, to the right or to the left of (u_0, v_0). For instance, starting near $(1,0)$ in the (u_0, v_0) plane in R_3, the trajectory will reach a point $(u_0 + \Delta u,\ v_0 + \Delta v)$ which is located to the left and above (u_0, v_0). (See Figure 23.)

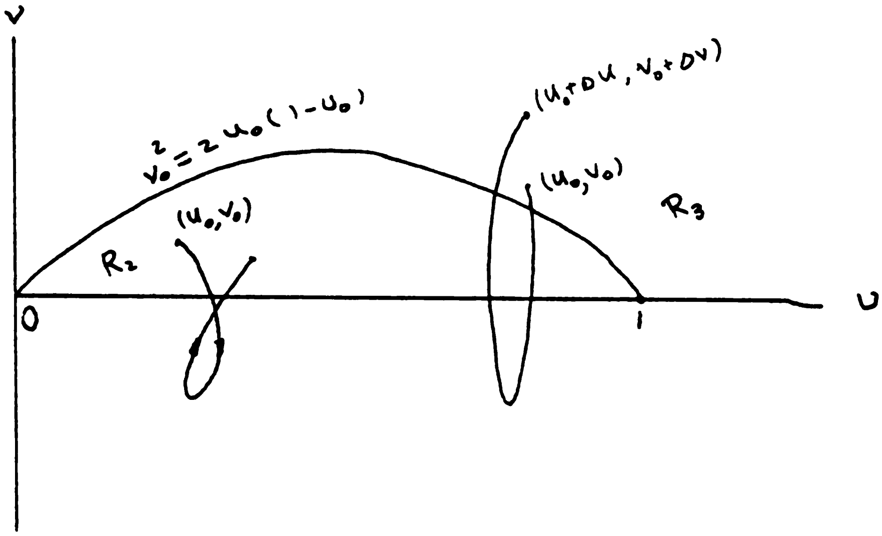


Figure 23

While starting at a point in R_2, the trajectory will reach a point to the right and below (u_0, v_0). Therefore starting in R_3, the trajectory starts to oscillate and move to the left and the radius of oscillation will grow until it reaches $u_0 = 1/2$.

After entering R_4 the trajectory starts to trace smaller and smaller rings until it tends to zero, or escapes to the region $u < 0$. (See Figure 24.)

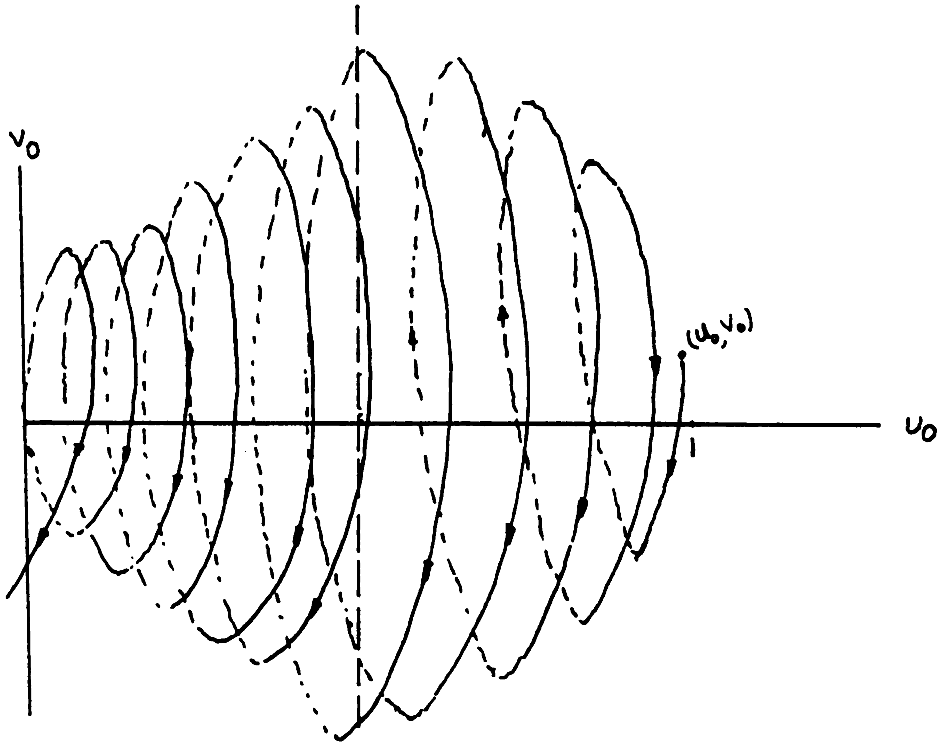


Figure 24

We are looking for a point (u_0, v_0) such that the trajectory starting at (u_0, v_0) moves to the left and tends to $(0,0)$ in positive time and tends to $(1,0)$ in negative time.

Since the 2-dimensional unstable manifold at the critical point $(1,0,0)$ is almost perpendicular to the u-axis, the intersection of that manifold with the plane $w = 0$ is a curve passing through $(1,0)$ almost perpendicular to the u-axis at $u = 1$. (Figure 25)

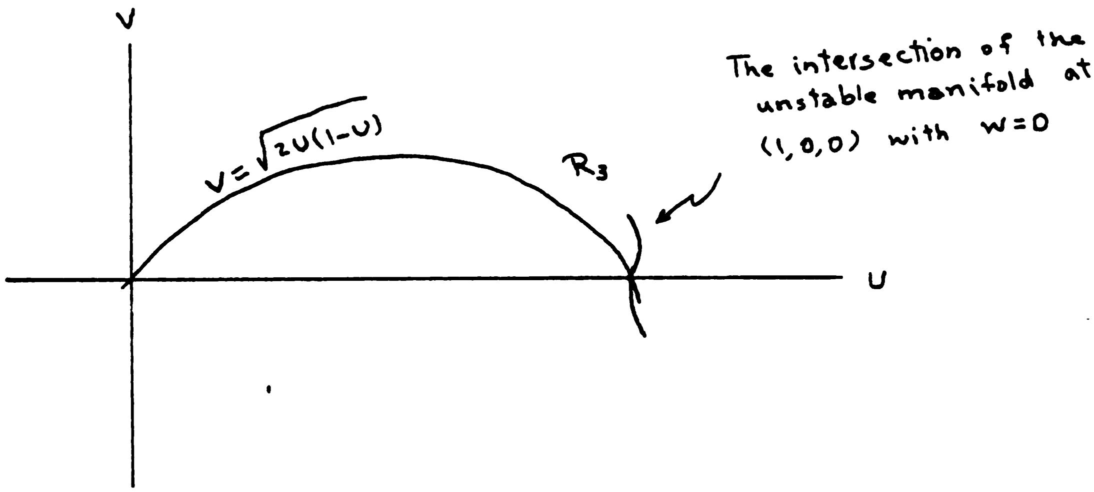


Figure 25

We pick a point (u_0, v_0) on the intersection of $w = 0$ and the unstable manifold in R_3, following the trajectory we get another point of the intersection forward in time, $(t \approx 2\pi)$ say $(u_1, v_1) \in R_3$. Since $(u_1, v_1) \in R_3$, starting at (u_1, v_1) the trajectory will reach a point $(u_2, v_2) \in R_3$ forward in time after $\Delta t \approx 2\pi$, and in this way we define

$$P_0 = (u_0, v_0), \quad P_1 = (u_1, v_1), \quad P_2 = (u_2, v_2), \ \ldots$$

If for some k, P_k enters R_1, from the positivity of Δu in R_1, the trajectory will reenter R_3. Therefore, let us assume the trajectory enters R_4 for some P_k where $P_{k-1} \in R_3$. This simply means $u_{k-1} > 1/2$ and $u_k < 1/2$.

Let $\Delta u_n = u_n - u_{n-1}$. Then we have

$$u_0 + \sum_{n=1}^{k} \Delta u_n < 1/2 \quad \text{and} \quad u_0 + \sum_{n=1}^{k-1} \Delta u_n > 1/2.$$

Since solutions to ODEs are continuously dependent on the initial values (u_0, v_0), u_k is a continuous function of (u_0, v_0).

Let the unstable manifold on $w = 0$ be parametrized by $u = \varphi(s)$, $v = \psi(s)$, $-s_0 \le s \le s_0$. Then as $s \to 0$, $k \to \infty$.

Therefore, k depends on s and u_k is a continuous function of s. We define a step function $k = k(s)$ as follows:

For $0 < s \le s_0$, let $k(s)$ be the number of oscillations required to reach $u_{k(s)} \ge 1/2$ and $u_{k(s)+1} < 1/2$. Then clearly as $s \to 0 : k \to \infty$. (See Figure 26.)

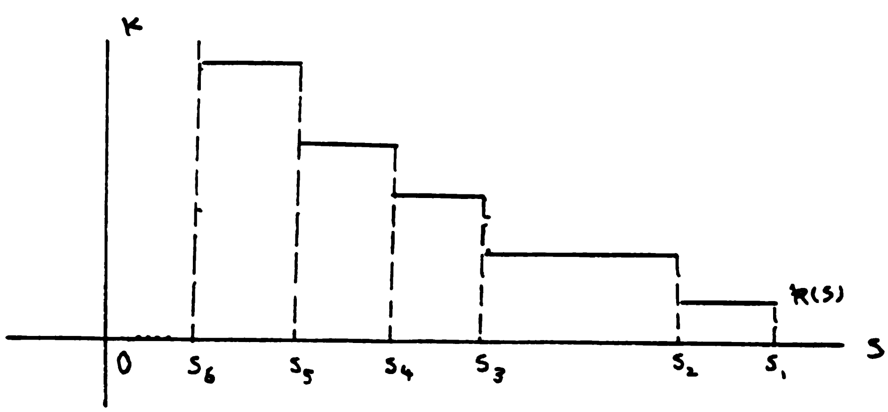


Figure 26

From the definition of $k(s)$, it is clear that for the jumping points, $s_1, s_2, \ldots$, $u_k(s_r) = 1/2$; that is, there are infinitely many points on the unstable manifolds such that the trajectory starting at those points hits the plane $u = 1/2$ after an integral number of oscillations.

Let $s \in (0, s_0]$ be such a value; therefore, $u_{k(s)} = 1/2$. We claim that starting at $u_0 = u_k(s)$, $w_0 = 0$, $v = v_0$, the trajectory will be an oscillating heteroclinic orbit.

Since the system remains unchanged under the change of variable

$$U = 1 - u$$

$$V = v$$

$$W = -w$$

$$T = -T \ ,$$

the trajectories are symmetric with respect to $u = \frac{1}{2}$. The trajectory starting at $u_0 = u_k(s)$, $w_0 = 0$, $v = V_0$ would merge to $(0,0,0)$ as $t \to \infty$.

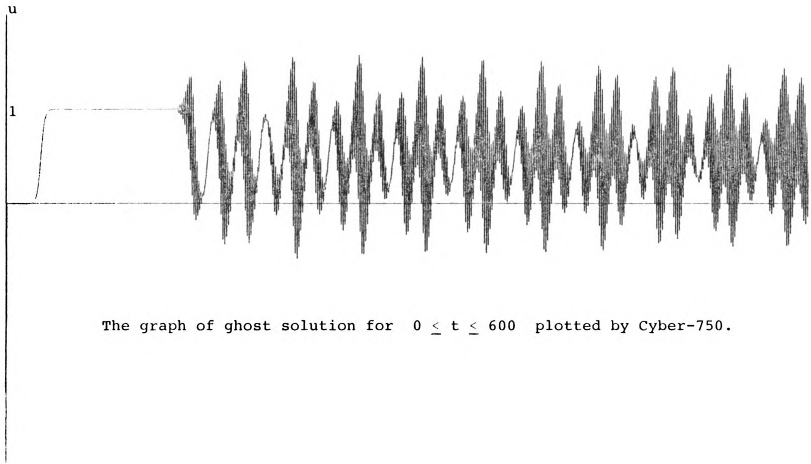
The graphs on the next 3 pages were made by the computer, showing the oscillating heteroclinic orbit with different scalings in (u_0, v_0) plane.

We have proved

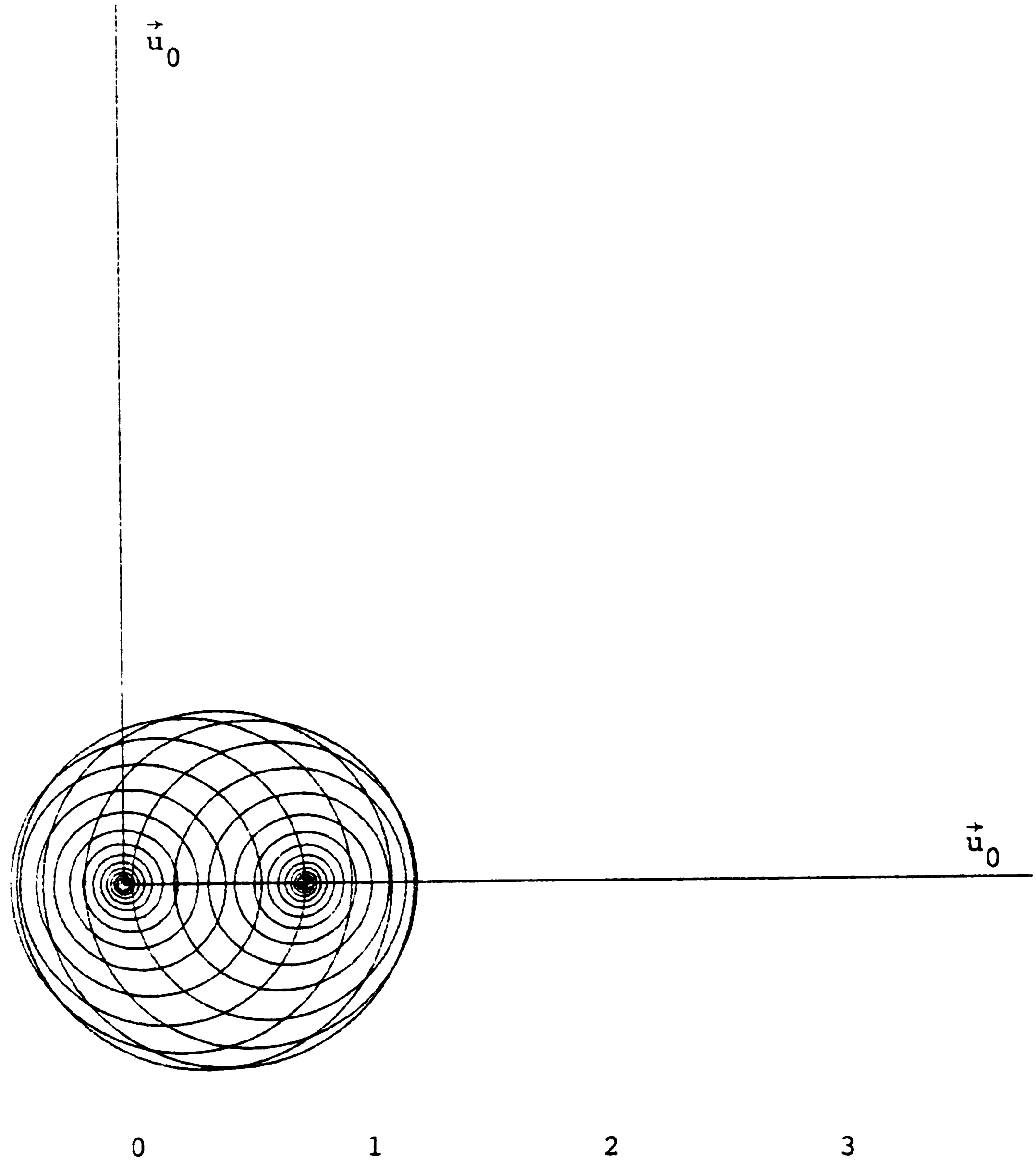
<u>Theorem 4.1</u>. There is a heteroclinic orbit in the modified logistic equation

$$\epsilon^2 \dddot{u} + \dot{u} = u(1 - u)$$

for $\epsilon > 0$ sufficiently small.



The graph of ghost solution for $0 \le t \le 600$ plotted by Cyber-750.



The oscillating heteroclinic orbit connecting two critical points of $\varepsilon\dddot{u} + \dot{u} = u(1-u)$.

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