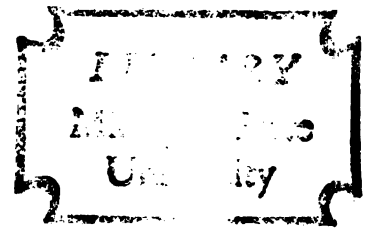


C₁₁-PARTITIONS OF FINITE GROUPS

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THESIS

This is to certify that the
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ABSTRACT

C_π -PARTITIONS OF FINITE GROUPS

by James W. Richards

Often in the study of mathematical systems, information about a given system can be obtained from presupposed properties other than the intrinsic ones. More specifically, in the study of finite groups--with which we shall be concerned--information about a given abstract group sometimes can be extracted if one presumes certain conditions on its lattice structure.

Reinhold Baer [1] has studied finite groups which admit a partition, that is, a family σ of subgroups of G which cover G so that distinct members have trivial intersection. The purpose of this dissertation is to consider more general types of partitions.

Let $\pi [n]$ denote the set of prime divisors of the positive integer n . If π is a given set of primes, then a subgroup H of G is said to be a π -subgroup if $\pi [|H|] \subseteq \pi$ and a π' -subgroup if $\pi [|H|] \cap \pi = \emptyset$. Similarly, an element g of G is said to be a π -element if $\pi [|g|] \subseteq \pi$ and a π' -element if $\pi [|g|] \cap \pi = \emptyset$. If G is a finite group and if $\pi \subseteq \pi [|G|]$, then we define a C_π -partition of G as follows:

a family σ of subgroups of G which cover G such that no member is a proper subgroup of any other and distinct

members intersect as a cyclic π -subgroup.

If $\pi = \{p\}$, then we call σ a cp -partition of G . Also, σ is said to be trivial if $\sigma = \{G\}$.

In chapter I, we prove the following results:

If σ is a $c\pi$ -partition of a finite group and if

the center $Z(G)$ of G contains a π -element of composite order, then σ is trivial.

If G is a finite nilpotent group and if $\pi \subset \pi[|G|]$,

then G admits a non-trivial $c\pi$ -partition

iff G is the direct product of a cyclic π -subgroup

with a Sylow q -subgroup S_q of G where $q \notin \pi$,

$|S_q| \neq q$, and S_q is not generated by all those elements of S_q which do not have order q .

If σ is a non-trivial $c\pi$ -partition of G such

that $\pi \cap \pi[|G:h(G)|] = \emptyset$, then there is a normal

Hall π -subgroup H of G such that $\tau = \{L/H : L \in \sigma\}$ is

a Baer partition of G/H .

In chapter II, we define " σ -admissibility" and derive the following results:

If σ is a non-trivial cp -partition of a finite group G

such that all components are normal, then G is a p -group

or the semi-direct product of a cyclic p -group by an

elementary abelian q -group where $q \neq p$.

If σ is a normal non-trivial cp -partition of a finite group G and if L is a component of σ which is not self-normalizing, then there is a prime divisor q of $[N_G(L):L]$ such that L is an H_n -group where n divides $q \cdot \exp(\sigma)$.

In chapter III, we define a Frobenius cp -partition of G as a normal cp -partition which contains a proper self-normalizing component. We first prove the following result which is a partial generalization of a well-known theorem of Frobenius:

If H is a self-normalizing Hall subgroup of a finite group G such that $g \in G - H \Rightarrow H \cap H^g$ is a cyclic p -group, then H has a normal complement in G if G is solvable or if H is a p -group.

The self-normalizing components of a Frobenius cp -partition are called the cp -complements when the conclusion of the above theorem holds. We then prove the following theorem:

Let σ be a Frobenius cp -partition of a finite group G where one of the self-normalizing components T is a Hall subgroup. Also, assume that G is solvable or T is a p -group. Then, G has a Frobenius cp -kernel K . Moreover, if $Z(K)$ contains a p' -element of composite order, then there is a normal

component L of σ such that the following hold:

- a) $K \subseteq L$;
- b) $L = K$ iff σ is a Frobenius Baer partition;
- c) If $K \subset L$, then L/K is a cyclic p -group;
- d) K is nilpotent unless the Frobenius cp -complements are p -groups.

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By

James W. Richards

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INTRODUCTION

1. Background

R. Baer [4] defines a partition of a finite group G as a family σ of subgroups of G such that each $g(\neq 1) \in G$ is contained in exactly one member (or component) of σ . The problem of determining all groups which admit non-trivial partitions, that is, with $\sigma \neq \{G\}$, has been solved by Baer [1], [2] [4], Kegel and Wall [9], and Kegel [8].

The class of all groups which admit non-simple partitions was characterized by Baer [4]. These non-simple partitions are partitions whose groups possess a proper normal σ -admissible subgroup, i. e., a subgroup K of G satisfying the property that $L \cap K \neq 1 \Rightarrow L \subseteq K$ for all components L of σ . Frobenius partitions are special cases of non-simple partitions. They are partitions which have a proper self-normalizing component. Their structure is described in theorem B.6 of appendix B. The remaining non-simple partitions are described in theorem B.3.

The other class is the class of all groups admitting simple partitions, i. e., partitions which possess no proper normal σ -admissible subgroup. It turns out that the key to the analysis of this class is the socle $S(G)$, the product of all minimal normal non-trivial subgroups of G . Baer shows that $S(G)$ must be either abelian or non-abelian simple if G admits a non-trivial

partition (see theorem B. 7). If $S(G)$ is abelian, then the fitting subgroup $F(G)$, the maximal normal nilpotent subgroup of G , is non-trivial. Baer shows that the only group G which admits a non-trivial simple partition where $F(G) \neq I$ is S_4 (see theorem B. 2).

Some insight into the final determination was given by Baer (see theorem B. 9) when he showed that if $S(G)$ is non-abelian simple, then: $[G:S(G)] = 2$, the Sylow p -subgroups for odd primes are abelian, and the Sylow 2-subgroups are D-groups. The problem was completely solved when Kegel and Wall [9] showed for $S(G)$ non-abelian that: G is $PGL(2, q)$ where q is an odd prime power > 4 if G is not simple, and G is either one of the Suzuki [12] groups or $PSL(2, q)$ where q is given as above, if G is simple.

2. Statement of the Problem

The purpose of this dissertation is to generalize the concept of partition of a finite group. It should be pointed out that there are many possible directions of generalization. The author's choice is to generalize upon the intersection property.

Before we give the formal statement, we need to introduce some notation. If n is a positive integer, then $\pi[n]$ denotes the set of all prime divisors of n . An element g of a group G is said to be a π -element for a given set of primes π if

$\pi[|g|] \subseteq \pi$. A subgroup H of G is said to be a π -subgroup of G if $\pi[|H|] \subseteq \pi$. Note that $\pi[|I|] = \emptyset$ where I is the trivial subgroup of G and hence, I is a π -subgroup for any set of primes.

We now describe the generalized partitions studied in this thesis.

If G is a finite group and if $\pi \subseteq \pi[|G|]$, then a family σ of subgroups of G is said to be a $c\pi$ -partition of G if

(i) σ is a cover of G , (ii) no member of σ is a proper subgroup of any other element of σ , and (iii) distinct members of σ intersect as a cyclic π -subgroup.

The restriction that no member be a proper subgroup of any other is introduced so that redundancy may be eliminated. The generalization is realized when one sees that a partition as defined by Baer is a $c\emptyset$ -partition.

3. Objectives and Synopsis of Results

The original object of this dissertation was to classify groups which admit non-trivial $c\pi$ -partitions. This we have been able to do in certain special cases. In order to obtain a complete classification, two well-known results must be generalized. The first one that must be generalized concerns the H_p -problem, solved by Hughes and Thompson [6]. The needed generalization will be alluded to in the conclusion of theorem 2.2.1. The famous Frobenius

theorem, see theorem A.6, is the other result which must be generalized. As shown in Chapter III, no complete generalization can be found, in the sense which we need. However, a partial generalization is obtained.

After the definition of $c\pi$ -partition in chapter I, we discuss the lattice of all possible $c\pi$ -partitions of a given group. Next, we consider commuting elements in G and develop sufficiency criteria for elements to be contained in a common unique component of σ . These conditions enable us to characterize all nilpotent groups which admit non-trivial $c\pi$ -partitions when π is not too large. We finish the chapter with a structure theorem for some groups which admit non-trivial $c\pi$ -partitions when suitable restrictions are placed on π .

In chapter II, we first define the concept of σ -admissibility in a fashion analogous to that for a Baer partition. We restrict ourselves in this chapter, and throughout the rest of the thesis, to cp -partitions, or $c\pi$ -partitions where $\pi = \{p\}$. After a sequence of lemmas and theorems, we obtain the main result of this chapter which is a structure theorem for all groups admitting a non-trivial cp -partition where all components are normal subgroups. We end the chapter with a theorem which gives some insight into the structure of components which are not self-normalizing subgroups of G .

The last chapter is devoted to Frobenius cp -partitions

or those cp-partitions which are normal and contain a proper self-normalizing component. After obtaining a partial generalization of Frobenius's theorem, we use the result to derive information about some Frobenius cp-partitions.

A list of symbols and notation used in this thesis is given in the index. Also, there are two appendices. Appendix A is for general group theoretic results while Appendix B is reserved explicitly for results on Baer partitions of groups.

CHAPTER I

NOTATION AND BASIC CONCEPTS

All groups hereunder are assumed to be finite.

In this chapter we are concerned with the family of all c^π -partitions of a given group. Also, we consider commutability properties which yield sufficiency criteria for elements to be contained within a unique common component of a given c^π -partition. The latter situation will enable us to characterize all nilpotent groups which admit c^π -partitions when suitable restrictions are placed on π .

Let $\pi [n]$ denote the set of all prime divisors of the positive integer n . If π is a given set of primes, then a group G is said to be a π -group if $\pi [|G|] \subseteq \pi$, and a π' -group if $\pi [|G|] \cap \pi = \emptyset$. Similarly, an element g of a group G is said to be a π -element if $\pi [|g|] \subseteq \pi$, and a π' -element if $\pi [|g|] \cap \pi = \emptyset$. A π -group (π -element) is said to be p -group (p -element) if $\pi = \{p\}$. We note that the only $\emptyset$ -group is I , the trivial group. We shall see that a $c\emptyset$ -partition is precisely a partition as defined by R. Baer [4].

Definition 1.1: Let $\pi \subseteq \pi [|G|]$. A family σ of subgroups of a group G is said to be a c^π -partition of G if the following hold:

- a) $G = \bigcup_{H \in \sigma} H$ such that no member of σ is a proper subgroup

of any other

b) If $H, K (\neq H) \in \sigma$, then $H \cap K$ is a cyclic π -subgroup.

The members of σ will be called components of σ . If $\sigma = \{G\}$, then σ is a $c\pi$ -partition for any given set of primes. This obviously trivial $c\pi$ -partition will be referred to as the trivial $c\pi$ -partition of G . If $\pi = \{p\}$, then σ will be called a cp -partition.

R. Baer [4] defines a partition of G as a family σ of subgroups such that $g (\neq 1) \in G$ implies that g is contained in exactly one member of σ . Then, distinct members of σ intersect as the trivial subgroup I , the cyclic $\emptyset$ -subgroup of G . Also, no member of σ is a proper subgroup of any other. So, σ is a $c\emptyset$ -partition. Conversely, it is easy to see that any $c\emptyset$ -partition of G is a partition as defined by Baer. We henceforth shall refer to these $c\emptyset$ -partitions as Baer partitions.

Let G be any non-cyclic group, τ the family of all maximal cyclic subgroups of G , and $\pi^* [|G|] = \cup \{\pi [|H \cap K|] : H, K (\neq H) \in \tau\}$. Then, τ is a $c\pi$ -partition where $\pi = \pi^* [|G|]$. Hence, if $\pi^* [|G|] \subseteq \pi \subseteq \pi^* [|G|]$ and if G is non-cyclic, then G always admits a $c\pi$ -partition, namely, the one above. Furthermore, it is clear that τ is non-trivial since G is not cyclic. If G is cyclic, then the subgroups of G are well-ordered and thus, it is impossible for G to admit a non-trivial $c\pi$ -partition for any

choice of π . Consequently, in order to restrict our attention to a class of groups, the members of which do not necessarily admit a $c\pi$ -partition, we shall usually assume that $\pi^* [|G|] \not\subseteq \pi$. In particular, we will have $\pi = \{p\}$ in the later sections.

Example 1.1.1: Let $G = A_5$ and σ be the family of all maximal cyclic subgroups. Since A_5 contains no elements of composite order, we have that $\pi^* [|A_5|] = \emptyset$. So, σ is a Baer partition of A_5 .

Example 1.1.2: Let $G = A_7$. The elements of A_7 are as follows:

Type	Order
(1, 2, 3, 4, 5, 6, 7)	7
(1, 2, 3, 4, 5)	5
(1, 2, 3, 4) (5, 6)	4
(1, 2, 3) (4, 5) (6, 7)	6
(1, 2, 3) (4, 5, 6)	3
(1, 2, 3)	3
(1, 2) (3, 4)	2
(1)	1

Now, let τ be the family of all maximal cyclic subgroups of A_7 . It is evident that $\pi^* [|A_7|] = \{2\}$. So, τ is a $c2$ -partition of A_7 . N. Iwahori and T. Kondo-- [7] , theorem 5--have shown that the alternating group A_n on n elements admits a Baer partition if and only if $n = 4, 5$, or 6 . So, A_7 admits a $c2$ -partition, but not a Baer partition.

We shall now obtain a means of comparing two $c\pi$ -partitions of a group G . In particular, we shall introduce a partial ordering on the family of all $c\pi$ -partitions of G , where π ranges over all subsets of $\pi[|G|]$.

Definition 1.2. Let σ_1 and σ_2 be $c\pi_1$ - and $c\pi_2$ -partitions respectively of a group G . Then, σ_1 is said to be a refinement of σ_2 if each component of σ_1 is a subgroup of some component of σ_2 . We write $\sigma_1 \leq \sigma_2$ if σ_1 is a refinement of σ_2 and $\sigma_1 < \sigma_2$ for proper refinement where $\sigma_1 \leq \sigma_2$ but $\sigma_1 \neq \sigma_2$.

It is clear that the concept of refinement defines a partial ordering on the family of all $c\pi$ -partitions of G since inclusion is a partial ordering on the family of all subgroups of G .

If G is not-cyclic, then denote by $\rho_m(G)$ the $c\pi$ -partition of G which consist of all the maximal cyclic subgroups of G and by $\rho_M(G)$, the trivial $c\pi$ -partition of G . (When there is no danger of ambiguity, we simply write ρ_m and ρ_M respectively.) Similarly, write π^* for $\pi^*[|G|]$. Now, if σ is any $c\pi$ -partition of G , then we see that $\rho_m \leq \sigma \leq \rho_M$. So, the family of all $c\pi$ -partitions of a non-cyclic group always has a largest element and a smallest element.

We now proceed to establish that the family of all $c\pi$ -partitions of G is indeed a lattice under our partial ordering. In fact, we

shall describe the meet and join operations in a precise fashion.

Theorem 1.2.1: Let σ_1 and σ_2 be $c\pi_1$ - and $c\pi_2$ -partitions respectively of G . Then, the maximal elements of $S = \{H \cap K : H \in \sigma_1, K \in \sigma_2\}$ determine a $c\pi_3$ -partition, written $\sigma_1 \wedge \sigma_2$, of G where $\pi_1 \cup \pi_2 = \pi_3$, and $\sigma_1 \wedge \sigma_2$ is the largest refinement of both σ_1 and σ_2 .

Proof: Let $\sigma_1 \wedge \sigma_2$ denote the family of all maximal elements of $S = \{H \cap K : H \in \sigma_1, K \in \sigma_2\}$. It is clear that the members of $\sigma_1 \wedge \sigma_2$ are subgroups of G . Let $g \in G$. Since σ_1 and σ_2 are both $c\pi$ -partitions of G , there are subgroups H and K of G such that $g \in H \in \sigma_1$ and $g \in K \in \sigma_2$. Then, $g \in H \cap K \in S$ implies that there is a maximal element $H_1 \cap K_1$ of S such that $g \in H_1 \cap K_1$. But, $H_1 \cap K_1 \in \sigma_1 \wedge \sigma_2$ and hence, $G = \bigcup_{L \in \sigma_1 \wedge \sigma_2} L$. Since $\sigma_1 \wedge \sigma_2$ consists precisely of the maximal elements of S , no member can be a proper subgroup of any other.

Now, let $U, V (\neq U) \in \sigma_1 \wedge \sigma_2$. Then, $U = H_1 \cap H_2$ and $V = K_1 \cap K_2$ where $H_1, K_1 \in \sigma_1$ and $H_2, K_2 \in \sigma_2$. But, $U \neq V$ implies that $H_1 \neq K_1$ or $H_2 \neq K_2$. If $H_1 \neq K_1$, then $U \cap V = (H_1 \cap K_2) \cap (K_1 \cap K_2) \subseteq H_1 \cap K_1$ a cyclic π_1 -subgroup. The other part follows mutatis mutandis. Thus, the intersection of any two distinct members is a cyclic π_3 -subgroup where $\pi_3 = \pi_1 \cup \pi_2$. Therefore, $\sigma_1 \wedge \sigma_2$ is a $c\pi_3$ -partition of G .

Now, let τ be any $c\pi$ -partition which is a refinement of both σ_1 and σ_2 . Let $L \in \tau$. Then, $\tau \leq \sigma_1$ and $\tau \leq \sigma_2$ imply that there are subgroups H_1 and H_2 of G such that $L \subseteq H_1 \in \sigma_1$ and $L \subseteq H_2 \in \sigma_2$. So, $L \subseteq H_1 \cap H_2 \in S$. It then follows that there is a maximal element U of S such that $L \subseteq U$. But, $U \in \sigma_1 \wedge \sigma_2$. So, $L \subseteq U \in \sigma_1 \wedge \sigma_2$, which implies that $\tau \leq \sigma_1 \wedge \sigma_2$. Therefore, $\sigma_1 \wedge \sigma_2$ is the largest refinement of both σ_1 and σ_2 and the proof is completed.

The refinement $\sigma_1 \wedge \sigma_2$ of both σ_1 and σ_2 will be referred to as the meet of σ_1 and σ_2 .

Next, we wish to define the join of two $c\pi$ -partitions. If σ_1 and σ_2 are two given $c\pi_1$ - and $c\pi_2$ -partitions respectively of G , then they both are refinements of ρ_M . So, the obvious approach is to define the join of σ_1 and σ_2 as the meet of all $c\pi$ -partitions, each one of which σ_1 and σ_2 are refinements. This approach is contingent on the fact that the meet operation is associative and hence, our next theorem.

Corollary 1.2.2: If σ_i is a $c\pi_i$ -partition for $i = 1, 2, 3$, then

$$(\sigma_1 \wedge \sigma_2) \wedge \sigma_3 = \sigma_1 \wedge (\sigma_2 \wedge \sigma_3).$$

Proof: Let $W \in (\sigma_1 \wedge \sigma_2) \wedge \sigma_3$. It then follows by theorem 1.2.1 that there are subgroups H_i of G such that $W = (H_1 \cap H_2) \cap H_3$ where $H_i \in \sigma_i$ for $i = 1, 2, 3$ and $H_1 \cap H_2 \in \sigma_1 \wedge \sigma_2$. Since $\sigma_2 \wedge \sigma_3$ consists

of the maximal elements of $S = \{K_2 \cap K_3 : K_2 \in \sigma_2, K_3 \in \sigma_3\}$, we have that there is a $U \in \sigma_2 \wedge \sigma_3$ such that $W = H_1 \cap (H_2 \cap H_3) \subseteq H_2 \cap H_3 \subseteq U$. It now follows that $(\sigma_1 \wedge \sigma_2) \wedge \sigma_3 \leq \sigma_2 \wedge \sigma_3$. We also have that $W = (H_1 \cap H_2) \cap H_3 \subseteq H_1 \in \sigma_1$, which implies that $(\sigma_1 \wedge \sigma_2) \wedge \sigma_3 \leq \sigma_1$. We then conclude that $(\sigma_1 \wedge \sigma_2) \wedge \sigma_3 \leq \sigma_1 \wedge (\sigma_2 \wedge \sigma_3)$ since $\sigma_1 \wedge (\sigma_2 \wedge \sigma_3)$ is the largest refinement of σ_1 and $\sigma_2 \wedge \sigma_3$, and $(\sigma_1 \wedge \sigma_2) \wedge \sigma_3$ is a refinement of σ_1 and $\sigma_2 \wedge \sigma_3$. By repeating the same type of argument as above, we see that $\sigma_1 \wedge (\sigma_2 \wedge \sigma_3) \leq (\sigma_1 \wedge \sigma_2) \wedge \sigma_3$. The proof is then complete.

Since we now know that the meet operation is associative, we may consider without difficulty a refinement of any non-empty collection $\{\sigma_i : 1 \leq i \leq n\}$ of $c\pi_i$ -partitions of G . Thus, we have our next theorem, the proof of which we omit since it is straight forward by induction.

Corollary 1.2.3: Let $\{\sigma_i : 1 \leq i \leq n\}$ be any non-empty collection of

$c\pi_i$ -partitions of G . Then, $\tau = \bigwedge_{1 \leq i \leq n} \sigma_i$ is a $c\pi_s$ -partition of G

where $\pi_s = \bigcup_{i=1}^n \pi_i$ and the components of τ are the maximal elements of $S = \{ \bigcap_{i=1}^n H_i : H_i \in \sigma_i \}$. Moreover, τ is the largest

refinement of the σ_i 's.

We now define the join operation.

Definition 1.2.1: Let τ_1 and τ_2 be $c\pi_1$ - and $c\pi_2$ -partitions of G and Σ be the family of all $c\pi$ -partitions of G . Then,

$$\tau_1 \vee \tau_2 = \wedge \{ \sigma_i \mid \sigma_i \in \Sigma, \tau_j \leq \sigma_i, j = 1, 2 \}.$$

Our next theorem establishes the fact that Σ is a lattice under our partial ordering. We omit its proof.

Theorem 1.2.1: If τ_1 and τ_2 are $c\pi_1$ - and $c\pi_2$ -partitions of G respectively, then $\tau_1 \vee \tau_2$ is the smallest $c\pi$ -partition of which τ_1 and τ_2 are both refinements.

Let σ be a $c\pi$ -partition of a group G and B be a group of automorphisms of G . Let $\beta \in B$ and $H \in \sigma$. Although H^β is a subgroup of G , H^β is not necessarily a component of σ . This suggests the following definition.

Definition 1.3: Let σ be a $c\pi$ -partition of G and B be a group of automorphisms of G . Then, σ is said to be B -invariant if $H^\beta \in \sigma$ whenever $H \in \sigma$ and $\beta \in B$. σ is said to be normal if $B = I(G)$.

Theorem 1.3.1: Let σ be a $c\pi$ -partition of G and B a group of automorphisms of G . Then, σ has a B -invariant refinement which is a $c\pi$ -partition.

Proof: For $\beta \in B$ define $\sigma^\beta = \{H^\beta : H \in \sigma\}$. Now, σ^β is clearly a $c\pi$ -partition of G for each $\beta \in B$. Define $\tau = \wedge_{\beta \in B} \sigma^\beta$. We have

by corollary 1.2.3 that τ consists precisely of the maximal elements of $S = \{ \bigcap_{\beta \in B} H_\beta : H_\beta \in \sigma^\beta \}$. Now, let $H \in \tau$. Then,

$H = \bigcap_{\beta \in B} K_\beta$ where $K_\beta \in \sigma^\beta$. But, $K_\beta \in \sigma^\beta \Rightarrow K_\beta^\mu \in \sigma^{\beta\mu}$ for each

$\mu \in B$. So, $K_\beta^\mu = L_\rho \in \sigma^\rho$ for some $\rho \in B$. Hence

$H^\mu = \bigcap_{\beta \in B} K_\beta^\mu = \bigcap_{\rho \in B} L_\rho$ which implies that $H^\mu \in S$. Since

automorphisms preserve the lattice structure of G , we have that H^μ must be a maximal element of S . Thus, $H^\mu \in \tau$, which establishes that τ is B -invariant.

The fact that τ is a $c\pi$ -partition follows directly from corollary 1.2.3 since $\pi_\beta = \pi$ for all $\beta \in B$. The proof is now complete.

The techniques used to prove the preceding theorems are analogous to the ones used by Baer and credited by him to O. Kegel--see Baer [4], pages 350-351.

We shall now develop some sufficiency criteria for elements of a group to be contained within a common unique component of a $c\pi$ -partition of G . The following theorems in this connection are analogous to Baer's criterion for Baer partitions and in fact the same type of argument is employed for each one--see theorem B.1.

Let g be an element of a group G and $\pi \subseteq \pi[|G|]$. Then,

$|g|$ factors uniquely as mn where $\pi[m] \subseteq \pi$, $\pi[n] \cap \pi = \emptyset$;
consequently, $(m, n) = 1$. It then follows by theorem A.1 that
 $g = g_1 g_2 = g_2 g_1$ uniquely where $|g_1| = m$ and $|g_2| = n$. Also, g_1
and g_2 are both powers of g . g_1 is said to be the π -part of g
and g_2 the π' -part. This decomposition of g is called the
 π -decomposition of g .

Theorem 1.4.1: Let σ be a $c\pi$ -partition of a group G and x, y
be elements of G which commute. Then, there is a unique
component of σ which contains both x and y if one of their
 π' -parts has composite order.

Proof: Let $x_1 x_2 = x$ and $y_1 y_2 = y$ be the decompositions of x
and y into their respective π -parts and π' -parts. Let us first
consider the case where $|y_2| < |x_2|$. Now, $xy = yx$ implies
that $x_i y_j = y_j x_i$ for $i, j = 1, 2$ since powers of commuting elements
commute. Let $m = |x_1 y|$. σ being a $c\pi$ -partition implies that
there are components H and K of σ such that $x \in H$ and
 $xy \in K$. But, $(|x_1 y_1|, |x_2 y_2|) = 1$ and $|y_2| < |x_2|$ together
imply that $1 \neq x_2^m = (xy)^m \in H \cap K$. Then, we conclude that
 $H = K$ since x_2^m is a π' -element. So, $x, xy \in H$. But, H
being a subgroup of G yields that $y = x^{-1}(xy) \in H$. The
uniqueness of H follows from the fact that $x \in H$ and x is
not a π -element. We get the same conclusion when $|x_2| < |y_2|$

mutatis mutandis.

Let us assume now that $|x_2| = |y_2|$. Since x_2 or y_2 has composite order and $|x_2| = |y_2|$, we have that x_2 and y_2 both have composite order. Let q be a prime divisor of $|y_2|$ and $z = y_1 y_2^q$. Then, $z = y_1 z_2$ where $1 \neq z_2 = y_2^q$. Also, $|z_2| < |x_2|$.

It then follows from the previous paragraph that there is a unique component H of σ such that $x, z \in H$. Also, y_2 being of composite order and $|z_2| < |y_2|$ imply by the previous paragraph that there is a unique component K of σ such that $z, y \in K$. Then, $z \in H \cap K$ and hence, $H = K$ since z is not a π -element. The proof is now complete.

The assumption that x_2 or y_2 has composite order in the preceding theorem cannot be removed as the following example shows.

Example 1.4.1: Let G be the abelian group of order 18 whose Sylow 3-subgroup is elementary abelian. The family of all cyclic subgroups of order 6 determines a c2-partition of G . Let $\langle x \rangle$ and $\langle y \rangle$ be distinct components of σ . Now, $xy = yx$, but x and y are contained within distinct components. The 2'-parts of x and y both have order 3. So, theorem 1.4.1 cannot be improved.

The above theorem gives us a sufficiency criterion, in terms of the π' -parts, for elements to be contained in a unique common

component. Our next step is to develop one in terms of the π -parts.

Recall that the exponent of a group G , written $\exp(G)$, is the smallest positive integer such that $g^n = 1$ for all $g \in G$.

Definition 1.4: Let σ be a $c\pi$ -partition of a group G . The exponent of σ is defined to be $\exp(\sigma) = \text{l.c.m.} \{ \exp(H \cap K) : H, K (\neq H) \in \sigma \}$ if σ is non-trivial and $\exp(\sigma) = \exp(G)$ otherwise.

If σ is non-trivial, then it is clear that σ is a Baer partition if and only if $\exp(\sigma) = 1$. So, $\exp(\sigma)$ in some sense measures how close σ is to being a Baer partition.

Theorem 1.4.2: Let σ be a $c\pi$ -partition of G and x, y be commuting elements of G . If $|x_1| > |y_1| \exp(\sigma)$ or $|y_1| > |x_1| \exp(\sigma)$ where x_1 and y_1 are the π -parts of x and y respectively, then there is a unique component H of σ which contains both x and y .

Proof: Let us assume that $|x_1| > |y_1| \exp(\sigma)$ where $x = x_1 x_2$ and $y = y_1 y_2$ are the decompositions of x and y respectively into their π -parts and π' -parts. Let H and K be components of σ such that $x \in H$ and $xy \in K$. Also, let $m = |x_2 y_2|$ and $k = |y_1|$. Now, $(m, |x_1|) = 1$ implies that $|x_1^m| = |x_1|$. This together with $k \leq k \exp(\sigma) < |x_1|$ yields that $1 \neq x_1^{mk} = x^{mk} = (xy)^{mk} \in H \cap K$. However, $(x_1^m)^k \exp(\sigma) \neq 1$ since

$|x_1^m| = |x_1| > k \exp(\sigma)$. Hence, $\exp(H \cap K)$ does not divide $\exp(\sigma)$ which implies that $H = K$. So, $x, xy \in H$ which implies that $x, y \in H$ since H is a subgroup. The argument for the case $|y_1| > |x_1| \exp(\sigma)$ is the same as before.

Assume that there is a component L of σ which contains both x and y and such that $L \neq H$. Then, $\langle x, y \rangle \subseteq H \cap L$, a cyclic π -subgroup. In particular, it follows that $x = x_1$ and $y = y_1$. Let w be a generator of $\langle x_1, y_1 \rangle$. If $|x_1| > |y_1| \exp(\sigma)$, then $|w| \geq |x_1| > \exp(\sigma)$ which is a contradiction to the fact that $\exp(H \cap K)$ divides $\exp(\sigma)$. We get the same contradiction if $|y_1| > |x_1| \exp(\sigma)$. So, we are forced to conclude that H is indeed unique and hence, the proof is complete.

The assumption that $|x_1| > |y_1| \exp(\sigma)$ or $|y_2| > |x_1| \exp(\sigma)$ cannot be weakened as the following example shows.

Example 1.4.2: Let $G = H \times Q$ where $|H| = 2$ and Q is the quaternion group of order 8. $\exp(G) = 4$. Let σ be the family of all subgroups of order 4. Then, σ is a c_2 -partition of G and $\exp(\sigma) = 2$. Let $x \in Q$ have order 4 and $y \in H$ have order 2. Then, $xy = yx$. Now, x and y are contained in distinct components of σ . However, $|x| = |y| \exp(\sigma)$. This shows that the theorem cannot be improved.

We now examine some consequences of the above theorems.

Theorem 1.4.3: Let σ be a $c\pi$ -partition of a group G and H be a subgroup of G . H is then contained in a unique component of σ if $Z(H)$ contains a π' -element of composite order.

Proof: Let $h \in Z(H)$ be a π' -element of composite order and K be the unique component of σ which contains h . Let $x \in H$. Since $h \in Z(H)$, we have that $xh = hx$. Since h has composite order, we conclude from theorem 1.4.1 that x and h are contained within a unique component of σ . But, K is the unique component of σ which contains h . So, $x \in K$. Hence, $H \subseteq K$. The uniqueness of K follows from the fact that it is the only component which contains h . The proof is now complete.

Corollary 1.4.4: Let G be a group and $\pi \subseteq \pi[|G|]$. If $Z(G)$ contains a π' -element of composite order, then G admits only the trivial $c\pi$ -partition.

Proof: Let σ be any $c\pi$ -partition of G and g_0 be a π' -element of G of composite order such that $g_0 \in Z(G)$. It then follows by theorem 1.4.3 that G is contained in a unique component of σ . But, this occurs when and only when $\sigma = \{G\}$. So, the proof is complete.

Our next step is to extend these theorems to a slightly more general situation.

Let G be a group and $Z_0 = I$. We inductively define Z_i by

$Z_i/Z_{i-1} = Z(G/Z_{i-1})$. Then, $1 = Z_0 \leq Z_1 \leq \dots \leq Z_n = Z_{n+1} = Z_{n+2} = \dots$

since G is finite. This series is called the upper central series of G and its largest member is called the hypercenter $h(G)$ of G . Another characterization of $h(G)$ given by Baer--see theorem A.2--is that an element g of G is contained in the hypercenter if and only if each p -part of g is centralized by all p' -elements of G for every prime p . Also, we mention here that G is nilpotent if and only if $G = h(G)$.

Now, let σ be a $c\pi$ -partition of G and U be a subgroup of G . If $S = \{K \cap U : K \in \sigma\}$, then it is clear that the maximal elements of S determine a $c\pi_0$ -partition of U where $\pi_0 \subseteq \pi$. This $c\pi_0$ -partition is called the induced partition on U by σ and is denoted as σ_U . Obviously, if U is not contained in a component of σ , then σ and σ_U both are non-trivial.

We now describe the structure of $h(G)$ if it is not contained in a component of σ .

Theorem 1.4.5: Let σ be a $c\pi$ -partition of a group G and assume that $h(G)$ is not a π -subgroup and is not contained in a component of σ . Then, $h(G)$ is the direct product of a cyclic π -subgroup with a group whose center has exponent q .

Proof: Since $h(G)$ is nilpotent and is not a π -subgroup, we know that $h(G) = H \times K$ where H is a π -subgroup and K is a π' -subgroup. Clearly, σ induces a non-trivial $c\pi_0$ -partition σ_U

on $U = h(G)$. Now, $Z(U)$ is not a π -subgroup since

$\pi[|U|] = \pi[|Z(U)|]$. It then follows by corollary 1.4.4 that $\exp(Z(K)) = q$, a prime.

We assert now that if $k(\neq 1) \in K$ and if $k \in L \in \sigma_U$, then $H \subseteq L$. Let $h \in H$. The conditions: U nilpotent, k a π' -element, and h a π -element, imply that $hk = kh$ and that hk is contained within a unique component T of σ_U . From $(|h|, |k|) = 1$ and $|h| = n$, we conclude that $1 \neq k^n = (hk)^n \in L \cap T$. It then follows that $L = T$ since k^n is not a π -element. So, $k, hk \in L$ and hence, $h = (hk)k^{-1} \in L$ since L is a subgroup of G . Thus, the assertion that $H \subseteq L$ is established.

Now, σ_U is non-trivial and so there is a component M of σ_U such that $M \neq L$. Since $U = H \times K$ where H is a π -subgroup and $\exp(Z(K)) = q$, we have that $M = H_1 \times K_1$ where $H_1 \subseteq H$ and $K_1 \subseteq K$. It then follows from the fact that $H \subseteq L \neq M$ that M contains an element m of order q such that $m \in M - L$. Since $|m| = q$ implies that $m \in K$, we see that $H \subseteq M$. So, $H \subseteq M \cap L$ such that $M \neq L$ implies that H is a cyclic π -subgroup. Therefore, the proof is complete.

In the proof of the next theorem, we use some results on Baer partitions. The needed concepts are the following:

Definition: If σ is a Baer partition, then a subgroup K of G is said to be σ -admissible if $L \cap K \neq I \Rightarrow L \subseteq K$ for all components

L of σ .

Definition: A Baer partition σ of G is said to be non-simple if G contains a proper normal σ -admissible subgroup; otherwise, it is said to be simple.

Definition: A Baer partition σ is said to be Frobenius if σ contains a self-normalizing component which is proper.

We now state the main theorem of this chapter.

Theorem 1.4.6: Let G be a nilpotent group and $\pi \subsetneq \pi[|G|]$. Then G admits a non-trivial $c\pi$ -partition if and only if G is the direct product of a cyclic π -group with a Sylow q -subgroup Q of G such that $H_q(Q) \subsetneq Q$ and $|Q| \neq q$.

Proof: Let us begin by assuming that G admits a non-trivial $c\pi$ -partition σ . By theorem 1.3.1, we may assume that σ is normal. Since $G = h(G)$, it follows by theorem 1.4.5 that $\pi[|G|] = \pi \cup \{q\}$ where $q \notin \pi$, and that $G = H \times Q$ where H is a cyclic π -subgroup and Q is the Sylow q -subgroup of G .

We assert that Q is not contained in any component of σ . Assume that there is a component L of σ such that $Q \subseteq L$. Let $x(\neq 1) \in Q$. As in the proof of theorem 1.4.5, we see that $H \subseteq L$. However, this implies that $G = H \times Q = \langle H, Q \rangle \subseteq L$, a contradiction to σ being non-trivial. Thus, Q is contained in

no component of σ .

It now follows that σ induces a non-trivial $c\pi$ -partition σ_Q on Q . However, Q contains no π -element of G other than 1. So, σ_Q is in fact a Baer partition of Q . Also, σ_Q is normal since σ_Q is the maximal elements of $S = \{Q \cap L : L \in \sigma\}$ and σ is normal. From the fact that $Q = F(Q)$, we conclude by theorem B.2 that σ_Q is not simple. Also, σ_Q is not Frobenius since nilpotent groups contain no proper self-normalizing subgroups. So, Q contains a proper normal σ_Q -admissible subgroup K . If $g \in Q - K$, then it follows by theorem B.3(b) that $|g| = q$. This implies that $H_q(Q) \subseteq K \subset Q$. It is clear that $|Q| \neq q$ since σ_Q is non-trivial on Q . This completes the only if part.

Let us assume now that $G = H \times Q$ where H is a cyclic π -group and $H_q(Q) \subset Q$ where $|Q| \neq q$. By theorem B.4, it follows that Q admits a non-trivial Baer partition σ . Define $\tau = \{HL : L \in \sigma\}$. Now, if K and L are subgroups of Q , then $HL \subseteq HK$ if and only if $L \subseteq K$.

Let $g \in G$ and $g = g_1 g_2$ be the decomposition of g into its respective π -part and π' -part. Since σ is a Baer partition of Q , there is a component L of σ such that $g_2 \in L$. Thus, $g = g_1 g_2 \in HL \in \tau$. If $HL \neq HK$, then it is clear that $H = HL \cap HK$. This completes the if part and hence, the theorem.

We shall now describe the nature of a $c\pi$ -partition σ of G when suitable restrictions are placed on the given set of primes.

Theorem 1.4.7: Let σ be a non-trivial $c\pi$ -partition of G such that $\pi \cap \pi [[G:h(G)]] = \emptyset$. Then, there is a normal Hall π -subgroup H of G such that $\tau = \{L/H: L \in \sigma\}$ is a Baer partition of G/H .

Proof: Let us first consider the case where G is nilpotent.

Then, $G = h(G)$. If G has no π -elements, then σ is a Baer partition of G . So, assume that G contains π -elements and let H be the normal Hall π -subgroup of G . We showed in theorem 1.4.6 that H must be contained in all components of σ . Let $L, K (\neq L) \in \sigma$. Then, $H \subseteq L \cap K$. However, H being the maximal π -subgroup of G and $L \cap K$ being a cyclic π -subgroup imply that $H = L \cap K$. So, $\bar{L} = L \cap K/H = L/H \cap K/H$. Let $\bar{x} = xH \in G/H$. Now, σ being a $c\pi$ -partition of G implies that there is a component S of σ such that $x \in S$. So, $xH \in S/H$ since $H \subseteq S$. Therefore, $\tau = \{L/H: L \in \sigma\}$ is a Baer partition of G/H .

Assume now that G is non-nilpotent. Then, $h(G) \subsetneq G$. Let H be the Hall π -subgroup of $h(G)$. Then, $\pi [[G:h(G)]] \cap \pi = \emptyset$ implies that H is the Hall π -subgroup of G . Also, G contains a π' -element $x \neq 1$. Let L be the component of σ which contains x . Let $h \in H \subseteq h(G)$. It then follows by theorem A.2 that $xh = hx$. If K is the component of σ which contains hx ,

we then have for $|h| = n$ that $\langle x \rangle = \langle x^n \rangle = \langle (xy)^n \rangle \subseteq L \cap K$ since x and h are coprime elements. But $x (\neq 1)$ being a π' -element implies that $L = K$. So, $x, xh \in L$ which yields that $h = x^{-1}(xh) \in L$. So, $H \subseteq L$. Let L and K be distinct components of σ . Using the same argument as in the first paragraph, we have that $\bar{I} = L \cap K/H = L/H \cap K/H$ and hence, $\tau = \{L/H: L \in \sigma\}$ is a Baer partition of G/H . This completes the proof.

If the restriction placed upon the given set of primes is removed, then the theorem is false as the following example shows.

Example 1.4.3: Let $H = \langle a, b, c: a^2 = b^3 = c^3 = 1, ab = ba, ac = bc \rangle$.

Let θ be the automorphism of H defined by:

$a^\theta = a, b^\theta = b^{-1}, c^\theta = c^{-1}$. Let G be the extension of H by θ .

Since $|\theta| = 2$, we have that $H = h(G)$. We define a c_2 -partition σ of G to be the family of all Sylow 2-subgroups together with all the cyclic groups of order 6.

We see that $\{2\} = \pi = \pi[[G:h(G)]]$. Since G does not have a normal Sylow 2-subgroup, the conclusion of theorem 1.4.7 is not satisfied.

CHAPTER II

σ -ADMISSIBLE SUBGROUPS

OF CP-PARTITIONS

In this chapter, and throughout the rest of the thesis, we consider cp-partitions where p is a fixed but arbitrary prime.

Definition 2.1: Let σ be a $c\pi$ -partition of a group G . Then, a subgroup H of G is said to be σ -admissible if for each component L of σ either $L \cap H$ is a cyclic π -subgroup with $\exp(L \cap H)$ dividing $\exp(\sigma)$ or $L \subseteq H$.

All non-cyclic components of a $c\pi$ -partition are σ -admissible subgroups but the converse is not true as the following example shows.

Example 2.1.1: Let $A = \langle a \rangle$ where $|a| = 4$ and $G = A \times S_3$.

The family σ of subgroups of order 4 of G together with the subgroups of order 3 determine a $c2$ -partition of G where $\exp(\sigma) = 2$. The Sylow 2-subgroups of G are σ -admissible but are not components of σ .

If σ is a normal $c\pi$ -partition of G and if H is a σ -admissible subgroup of G , then any conjugate of H is σ -admissible. However, $H \neq H^g$ does not imply that $H \cap H^g$ is a cyclic π -subgroup as the

following example shows.

Example 2.1.2: Let $G = S_4$ and define σ to be all the conjugates of S_3 in S_4 together with all the cyclic subgroups of orders equal to 4. σ is a c2-partition of G and $\exp(\sigma) = 2$. The Sylow 2-subgroups of G are σ -admissible but intersect in the Klein 4-group which is not cyclic.

We now begin with a sequence of lemmas and theorems which will terminate with a structure theorem for those groups which admit non-trivial cp-partitions having all components normal.

Theorem 2.1.1: Let σ be a cp-partition of a group G , K a σ -admissible subgroup of G , U a component of σ , and $u \in U - K$. Let $u = u_1 u_2$ be the decomposition of u into its respective p -part and p' -part. Then, the following hold:

- a) If $u_2 = 1$, then $\exp(C_K(u))$ divides $|u_1| \exp(\sigma)$;
- b) If $|u_2| = q$, a prime, then $C_K(u)$ is the semi-direct product of a subgroup of $U \cap K$ by a group of exponent q or 1 ;
- c) If $|u_2|$ is composite, then $C_K(u) \subseteq U \cap K$.

Proof: Let us first consider the case where $u_2 = 1$. Then, $u = u_1$ which implies that $u_1 = u \in U - K$. Let $k \in C_K(u)$ and $k = k_1 k_2$ be the decomposition of k into its respective p -part and p' -part.

Since k_1 and k_2 are both powers of k , it follows that $k_1, k_2 \in K$. If $k_2 \neq 1$, then there is a unique component S of σ which contains k_2 . But, K being σ -admissible and $k_2 \in K \cap S$ imply that $S \subseteq K$ since $k_2 \neq 1$ is a p' -element. Since k centralizes u , we have that k_2 centralizes u because k_2 is a power of k . Also, k_2 and u are coprime elements of G . Let $n = |u|$ and W be the unique component of σ which contains uk_2 . Now, $\langle k_2 \rangle = \langle k_2^n \rangle = \langle (uk_2)^n \rangle \subseteq W \cap K$. Then, K being σ -admissible and k_2 being p' -element imply that $W \subseteq K$. However, if $m = |k_2|$, then $\langle u \rangle = \langle u^m \rangle = \langle (uk_2)^m \rangle \subseteq K$, a contradiction to our assumption that $u \in U - K$. Hence, $k_2 = 1$.

Assume now that $|k| = |k_1| > |u_1| \exp(\sigma)$. It then follows by theorem 1.4.2 that there is a unique component R of σ which contains both u_1 and k_1 . But, $k_1 \in R \cap K$ and $\exp(\sigma) < |u_1| \exp(\sigma) < |k_1|$ imply that $\exp(R \cap K)$ does not divide $\exp(\sigma)$ and hence, $R \subseteq K$. Then, it follows that $u_1 = u \in R \subseteq K$, a contradiction to our assumption that $u \in U - K$. Therefore, we see that $\exp(C_K(u))$ does indeed divide $|u_1| \exp(\sigma)$ which completes part a.

Assume that $|u_2| = q$, a prime. It is clear that $C_K(u) \subseteq C_K(u_2)$. Let $k \in C_K(u_2)$. If k is a p -element, then let W be the component of σ which contains $ku_2 = u_2k$. Let $m = |k|$. Then, we have $\langle u_2 \rangle = \langle u_2^m \rangle = \langle (ku_2)^m \rangle \subseteq W \cap U$. But, $W \cap U$ is not a cyclic p -group and so $W = U$. Since $k \in \langle ku_2 \rangle \subseteq U$, we conclude

that $k \in U \cap K$. It then follows that $C_K(u_2)$ has a unique cyclic Sylow p -subgroup contained in $K \cap U$. Now, let k be any p' -element of $C_K(u_2)$. Since $U \cap K$ is a cyclic p -group, we know that $k \notin U$. So, it follows by theorem 1.4.1 that $|k| = r$, r a prime. If $r \neq q$, then, ku_2 has composite order and centralizes k . But, this implies by theorem 1.4.1 that $ku_2 \in U$ and hence, $k = (ku_2)u_2^{-1} \in U$, a contradiction. So, $|k| = q$. Thus, we have that $C_K(u)$ is the semi-direct product of a subgroup of $U \cap K$ by a group of exponent q .

Finally, assume that $|u_2|$ is composite. It then follows by theorem 1.4.1 that $C_K(u_2) \subseteq U$ and hence, $C_K(u) \subseteq K \cap U$. However, $C_K(u) \subseteq C_I(u_2)$ since u_2 is a power of u . Hence, $C_K(u) \subseteq K \cap U$ and the proof of the theorem is complete.

Corollary 2.1.2: Let σ be a cp -partition of a group G , K a σ -admissible subgroup of G , and U a component of σ . Let $u \in U - K$ such that u is neither a p -element nor a p' -element. Then, $C_K(u) \subseteq U \cap K$.

Proof: Let $u = u_1 u_2$ be the decomposition of u into its respective p -part and p' -part. Then, u being neither a p -element nor a p' -element implies that $u_1 \neq 1 \neq u_2$. If $|u_2|$ is not a prime, then we have by theorem 1.2.1 (c) that $C_K(u) \subseteq U \cap K$. So, assume that $|u_2| = q$, q a prime. It then follows by theorem 2.1.1 (b) that $C_K(u)$ is the semi-direct product of a subgroup of $U \cap K$ by a

group of exponent q or l . But, we have by theorem 2.1.1 (a) that $\exp(C_K(u_1))$ divides $|u_1| \exp(\sigma)$ which is a power of p . Since $C_K(u) \subseteq C_K(u_1)$, we now conclude that $C_K(u) \subseteq U \cap K$ which completes the proof.

The next two lemmas which we prove are well known facts about automorphisms.

Lemma 2.1.3: Let γ be an automorphism of order $n > 1$ of a group G and assume that $H \neq I$ is a normal γ -invariant subgroup such that $\gamma_H = \text{id}_H$ and γ induces the identity automorphism on G/H . Then, there is an $h (\neq 1) \in H$ such that $h^n = 1$.

Proof: Let $\{g_1 = 1, g_2, \dots, g_t\}$ be a complete set of coset representatives for H in G . Since $n > 1$, there is a g_i such that $\gamma(g_i) \neq g_i$. Since γ induces the identity automorphism on G/H , it follows $\gamma(g_i) = g_i h$ where $h (\neq 1) \in H$. Since $\gamma(h) = h$, $g_i = \gamma^n(g_i) = g_i h^n$ which implies that $h^n = 1$ and hence, the proof is complete.

Lemma 2.1.4: Let γ be an automorphism of order $n > 1$ of G and $G_\gamma = \{g \in G: \gamma(g) = g\}$ be the stability group of G relative to γ . If G_γ is normal and if n is coprime to $|G_\gamma|$, then γ induces a fixed-point-free automorphism on G/G_γ .

Proof: Since $\gamma \neq \text{id}_G$, it follows that $G_\gamma \subset G$. Assume that

$\gamma(g)G_\gamma = gG_\gamma$ for some $g \in G - G_\gamma$ and define $H = \langle g, G_\gamma \rangle$.

Clearly, H is a γ -invariant subgroup of G and $\gamma_H \neq \text{id}_H$ since

$G_\gamma \subset H$. But, γ_H induces the identity automorphism on H/G_γ . So,

we have by Lemma 2.1.3 that there is a $g_0 (\neq 1) \in G_\gamma$ such that $g_0^m = 1$

where m is the order of γ_H . However, m divides n which is a

contradiction to the assumption that n and $|G_\gamma|$ are coprime.

Thus, γ induces a fixed-point-free automorphism on G/H and the proof is complete.

Lemma 2.1.5: Let σ be a cp-partition of a group G , U a component of σ , and K a σ -admissible subgroup of G with $U \not\subseteq K$. Let $T = K \cap U$. If $x (\neq 1) \in U$ has order $q \neq p$ and if M is an x -invariant subgroup of K which contains T as a proper normal subgroup such that x induces the identity automorphism on M/T , then q divides $[M:T]$.

Proof: Assume that q does not divide $[M:T]$ as described in the hypotheses. Let $C = C_K(x)$ where $x \in U$ and $|x| = q$, a prime $\neq p$. We have by theorem 2.1.1 (b) that C is the semi-direct product of a subgroup of $T = U \cap K$ by a group of exponent q or 1. Hence, $C_M = C \cap M$ is the semi-direct product of a subgroup of T by a group of exponent q or 1. Since T is a cyclic p -group and $q (\neq p)$ does not divide $[M:T]$, q does not divide $|M|$. Hence, $C_M \subseteq T$. Since T is a cyclic p -group normal in M and $C_M \subseteq T$, it follows that C_M is normal in M since it is characteristic in T .

Now, $|x|$ coprime to $|C_M|$ implies by lemma 2.1.4 that x induces a fixed-point-free automorphism $\bar{x}$ on M/C_M . Since T is a normal x -invariant subgroup of M , it follows that T/C_M is a normal $\bar{x}$ -invariant subgroup of M/C_M . So, $\bar{x}$ induces a fixed-point-free automorphism φ on $M/C_M/T/C_M$ by theorem A.4 where $\varphi(mC_M T/C_M) = \bar{x}(mC_M)T/C_M = m^x C_M T/C_M$.

Now, by theorem A.5, the mapping $f: M/T \rightarrow M/C_M/T/C_M$ defined by $f(mT) = mC_M T/C_M$ is an operator isomorphism of M/T onto $M/C_M/T/C_M$. Let $m \in M - T$. Since x induces the identity automorphism on M/T , it follows that $m^x T = mT$. But, $f(m^x T) = \varphi f(mT) = \varphi(mC_M)/T/C_M \neq mC_M/T/C_M = f(mT)$ since φ is a fixed-point-free automorphism of $M/C_M/T/C_M$. We then have that $f(mT) \neq f(m^x T) = f(mT)$ a contradiction. Hence, we are forced to conclude that indeed q divides $[M:T]$ and the proof is complete.

Theorem 2.1.6: Let σ be a cp-partition of a group G , K a σ -admissible subgroup of G , and U a component of σ such that $U \not\subseteq K$, but $U \subseteq N_G(K)$. Then, the following hold:

- a) $[U, N_K(U)] \subseteq K \cap U$
- b) $T = K \cap U$ is a normal cyclic p -subgroup of both U and $N = N_K(U)$
- c) U/T and N/T are either both p -groups or q -groups of exponent $q \neq p$ if $T \subseteq N$.

Proof: Let $u \in U$ and $k \in N = N_K(U)$. Then, $u^k \in U$ which implies that $[u, k] = u^{-1}u^k \in U$. Since $U \subset N_G(K)$, we have that $u^{-1}k^{-1}u \in K$ and hence, $[u, k] = u^{-1}k^{-1}uk \in K$. By combining our results, we have that $[U, N] \subseteq U \cap K$ which completes part a.

Since K is a σ -admissible subgroup of G and does not contain U , it follows that $T = U \cap K$ is a cyclic p -group. It follows directly from $U \subset N_G(K)$ that $T = U \cap K$ is a normal subgroup of U . Consider $t \in T$ and $k \in N$. By part a it follows that $[t, k] \in T$. Thus, $t[t, k] = k^{-1}tk \in T$. Thus, T is also normal in N . This establishes part b.

Let us assume now that $T \subset N = N_K(U)$. We shall show that N/T and U/T are both p -groups if p divides $[N:T]$ and then show that they are both q -groups of exponent $q \neq p$ if p does not divide $[N:T]$.

We first consider the case where p divides $[N:T]$. We assert that U/T is a p -group. Assume to the contrary and let $u \in U$ have order $q \neq p$. Let S_p be a Sylow p -subgroup of N . Then, p dividing $[N:T]$ implies that $T \subset S_p$. Since $[U, N] \subseteq T$, it follows that S_p is a u -invariant subgroup of N and also that S_p contains T as a normal u -invariant subgroup. But, $[N, T] \subseteq T$ implies that u induces the identity automorphism on S_p/T . We then have by lemma 2.1.5, since $u \notin K$, that $q(\neq p)$ divides $[S_p:T]$, a contradiction. So, U/T is a p -group as asserted.

We assert next that N/T is a p -group. Assume the contrary and let $n \in N$ have order $q \neq p$. Let W be the component of σ which contains n . Then, $n \in N \cap W \subseteq K \cap W$ which implies that $W \subseteq K$ since K is σ -admissible and n is a p' -element. So, $n \notin U$. Since $n \in N = N_K(U)$, it follows that U is a n -invariant subgroup of G . Now, T is a normal n -invariant subgroup of U since it is normal in both N and U . We conclude from $[U, N] \subseteq T$ that n induces the identity automorphism on U/T and hence, it follows from lemma 2.1.5 that q divides $[U:T] = p^a$, a contradiction to $q \neq p$. Hence, it follows that U/T is a p -group.

Assume now that p does not divide $[N:T]$. Let $n \in N$ have order q , a prime, and W be a component of σ which contains n . Then, $n \in N \cap W \subseteq K \cap W$ which implies that $W \subseteq K$ since K is σ -admissible and n is a p' -element. Now, $W \cap U \subseteq K \cap U = T$ a cyclic p -group yields that $n \notin U$. Let H be any subgroup of U such that $T \subseteq H$. Since $[N, U] \subseteq T$, we have that H is a n -invariant subgroup of U and that n induces the identity automorphisms on U/T and hence on H/T . From lemma 2.1.5 we conclude that q divides $[H:T]$. Since H is arbitrary, it follows that U/T is a q -group.

Now, let $u \in U$ have order q . Since $U \cap K = T$ a cyclic p -group, we see that $u \notin K$. Let M be any subgroup of N such that $T \subseteq M$. Now, M is a u -invariant subgroup of N since $[N, U] \subseteq T$. Also, u induces the identity automorphism on

M/T since $[N, U] \subseteq T$. Thus, it follows by lemma 2.1.5 that q divides $[M:T]$. Therefore, N is a q -group since M is arbitrary.

We assert now that $\exp(N/T) = \exp(U/T) = q$. Let $n \in N$ such that $|n| = q^2$. It then follows by theorem 2.1.1 (c) that $C = C_U(n) \subseteq T$. Since T is cyclic and is normal in U we have that C is characteristic in T and hence, normal in U . But, $|n|$ is coprime to $|C|$. So, we have by lemma 2.1.4 that n induces a fixed-point-free automorphism of q power order on U/T . However, it is well-known that a fixed-point-free automorphism of prime power order has coprime order to that of the group, which is a contradiction. So, $\exp(N/T) = q$. Now, let $u \in U$ have order q^2 . Again, we have by theorem 2.1.1 (c) that $C = C_K(u) \subseteq K \cap U = T$. So, u induces a fixed point free automorphism on N/C . But this is a contradiction since q divides $[N:C]$ and the fixed-point-free automorphism is of q power order. Thus, $\exp(U/T) = q$ and the proof is complete.

We now prove the main result of this chapter.

Corollary 2.1.7: Let σ be a non-trivial cp-partition of G such that all components are normal subgroups of G . Then, G is either a p -group or the semi-direct product of a cyclic p -group by an elementary abelian q -group.

Proof: If G is a p -group, then there is nothing to prove. So, assume that G is not a p -group and let $g \in G$ have order $q \neq p$. Let L be the component of σ which contains g . Since σ is non-trivial, there is a component W of σ such that $W \neq L$. So, $W \cap L = T$ is a proper subgroup of both W and L . Since W and L are normal subgroups of G , it follows that $W \cap L \subseteq L = N_L(T)$ and $W \cap L \subseteq W = N_W(T)$. Since q divides $|L|$, we conclude from theorem 2.1.6 (c) that W/T and L/T are both groups of exponent q .

We now separate the argument into two cases; (i) $W \cap L = I$ and (ii) $W \cap L \neq I$. If $W \cap L = I$, then W and L are both q -groups of exponent q . Let R be any component of σ such that $R \neq L$. Then, $R \cap L$ being a cyclic p -group and L having exponent q yield that $R \cap L = I$ and hence R has exponent q by theorem 2.1.6 (c). So, all components of σ have exponent q . This implies that σ is a Baer partition of G since distinct components intersect as cyclic p -groups and G has no non-trivial p -elements. It then follows by theorem B.5 that G is an elementary abelian q -group.

Assume now that $W \cap L = T \neq I$. Let R be any component of σ where $R \neq L$. By theorem 2.1.6 (c) $R/R \cap L$ and $L/R \cap L$ are both groups of exponent q . But, this implies that $T \subseteq R \cap L$. Then, we conclude that $T = R \cap L$ since $\exp(L/T) = q$. We thereupon have established that T is the Sylow p -subgroup of G and is contained

in all components of σ .

Now, define $\tau = \{U/T : U \in \sigma\}$. Let $\bar{g} = gT \in G/T$. Then, $g \in G$ implies that there is a component S of σ such that $g \in S$. Since $T \subseteq S$, it then follows that $\bar{g} = gT \in S/T$. Let U/T and S/T be distinct elements of τ . Then $U/T \cap S/T = U \cap S/T = T/T = \bar{1}$ and thus, τ is a Baer partition of G . Now, $U/T \in \tau$ is normal in G/T since U is normal in G . Hence all components of τ are normal subgroups of G . It then follows from theorem B.5 that G/T is an elementary abelian q -group which completes the proof.

We now wish to say something about the components of σ -partition of G . Due to the combined results of Baer, Kegel, and Suzuki, the structures of all components are essentially determined. To be more specific, Let H be component of a Baer partition which is not self-normalizing. Then, $H \subset G$. If H is normal in G , then H is a proper normal σ -admissible subgroup of G . If H is not normal, then let N be the normalizer of H in G . Then, $H \subset N$ implies that σ induces a non-trivial Baer partition σ_N on N . Then, H is a proper normal σ_N -admissible subgroup of N . If N is a q -group, then N and hence H is nilpotent which we wish to show. So, assume that N is not a q -group. It then follows by theorem B.3 (c) that H is generated by all those elements of N which do not have order p .

So, $g \in N - H$ implies that $|g| = q$. Let $h \in H$. Then, $hg^{-1} \notin H$ implies that $(hg^{-1})^q = 1$. So, this together with $g^q = 1$ yields that $1 = (hg^{-1})^q = hh^g h^{g^2} \dots h^{g^{q-1}} = 1$ which implies that H admits an H_q -automorphism. It then follows by theorem A.7 that H is nilpotent. This now brings us to our next definition.

Definition 2.2: An automorphism γ of order n is said to be an H_n -automorphism of a group G if $g \cdot g^\gamma \cdot g^{\gamma^2} \dots g^{\gamma^{n-1}} = 1$ for all $g \in G$. A group G is said to be an H_n -group if G admits an H_n -automorphism.

Remark 2.2.1: A fixed-point-free automorphism of order n of a group G is an H_n -automorphism, but not conversely. The converse, however, is true if n is coprime to $|G|$. It is still an open question as to whether a group G which admits a fixed-point-free automorphism is solvable or not.

We now state our result for proper normal σ -admissible subgroups of a cp-partition.

Theorem 2.2.1: Let σ be a normal cp-partition of a group G and K a proper normal σ -admissible subgroup of G . Then, there is a prime divisor q of $[G:K]$ such that K is an H_n -group where n divides $q \exp(\sigma)$, K is a p -group, or the semi-direct product of a cyclic p -group by a group of exponent q .

Proof: We use induction on $|G|$. S_3 is the ground case.

Let $g \in G - K$ such that gK has order q in G/K . Define $U = \langle g, K \rangle$. If $U \subset G$, then U not contained within a component of σ implies that σ induces a normal non-trivial cp -partition σ_U on U . Now, K is clearly a proper normal σ_U -admissible subgroup of U . Hence, it follows by induction that there is a prime divisor q of $[U:K]$ such that K is an H_n -group where n divides $q \exp(\sigma_U)$. Then, $\exp(\sigma_U)$ dividing $\exp(\sigma)$ yields that n divides $q \exp(\sigma)$. So, let us assume that $G = \langle K, g \rangle$.

Now, let g be any element of $G - K$. Then, G/K being cyclic of order q implies that $g^q \in K$. Since, $g \notin K$, there is a component L of σ such that $L \not\subseteq K$ and $g \in L$. But, $g^q \in L \cap K$ a cyclic p -group such that $\exp(L \cap K)$ divides $\exp(\sigma)$. Hence, $|g^q|$ divides $\exp(\sigma)$. We thereupon have that $|g|$ divides $q \exp(\sigma)$.

Assume first that $p = q$. If K is a p -group, then we are through. So, assume that K is not a p -group. Since $|g|$ divides $p \exp(\sigma)$, g is a p -element. Now, $C_K(g)$ is a p -group by theorem 2.1.1 (a). Then, K not being a p -group implies that g induces a non-trivial automorphism on K . Assume now that $q \neq p$. By the previous paragraph $|g|$ divides $q \exp(\sigma)$. Hence $C_K(g)$ is the semi-direct product of a cyclic p -group by a group of exponent q by theorem 2.1.1 (b). If K is the semi-direct product of a cyclic p -group by a group of exponent q , then we are through. So, assume to the contrary. Then g induces a non-trivial automorphism on K .

Now, let $g \in G - K$ and $k \in K$. Then, $kg^{-1} \in G - K$ implies that $(kg^{-1})^{q \exp(\sigma)} = 1$. Let $m = q \exp(\sigma)$. It then follows that $1 = (kg^{-1})^m = k k^g k^{g^2} \dots k^{g^{m-1}} g^m = k k^g \dots k^{g^{m-1}}$; and hence g induces an H_n -automorphism on K where n divides $q \exp(\sigma)$, and the proof is complete.

Corollary 2.2.2: Let σ be a normal non-trivial cp-partition of a group G and L a component of σ which is not self-normalizing. Then, there is a prime divisor q of $[N_G(L):L]$ such that L is an H_n -group where n divides $q \exp(\sigma)$.

Proof: Since L is not self-normalizing in G , we have that $L \subset N_G(L) = N$; hence, N is not contained within a component of σ . If $N = G$, then the result follows directly from theorem 2.2.1. So, assume that $N \subset G$. Then, σ induces a non-trivial normal cp-partition σ_N on N and L is a proper normal σ_N -admissible subgroup of N . Thus, it follows by theorem 2.2.1 that there is a prime divisor q of $[N:L]$ such that L is an H_n -group where n divides $q \exp(\sigma_N)$. But, $\exp(\sigma_N)$ divides $\exp(\sigma)$ which yields that n divides $q \exp(\sigma)$, and the proof is complete.

This last theorem reduces to question of the structure of non-self-normalizing components of cp-partitions to that of the structure of H_n -groups.

CHAPTER III

FROBENIUS CP-PARTITIONS

In this chapter, we shall generalize the concept of Frobenius partitions to cp-partitions. After establishing a partial generalization of Frobenius's theorem, we shall use this result to obtain information about Frobenius cp-partitions in a limited scope.

Baer [4] defines a Frobenius partition of a group G as a normal partition σ where one of the components, say H , is a proper self-normalizing subgroup of G . By a well-known Frobenius result, theorem A.6, H has a normal complement in G , say K . K is referred to as the Frobenius kernel of G and the complements to K in G are called the Frobenius complements. J. Thompson [15] established that the Frobenius kernel is always nilpotent. Baer, theorem B.6, uses Thompson's result to show that $F(G)$ is the Frobenius kernel of G and that it is a proper σ -admissible Hall subgroup of G . So, the Frobenius complements are Hall subgroups of G .

The author has not been able to decide whether or not a self-normalizing component of a cp-partition is a Hall subgroup of G . Our discussion will be limited in this respect.

Definition 3.1: A normal cp-partition σ of a group G is said to

be a Frobenius cp-partition if one of the components is a proper self-normalizing subgroup of G .

One readily sees that if H is a self-normalizing component of σ , then H has the property that $g \in G - H \Rightarrow H \cap H^g$ is a cyclic p -group since H^g is a component distinct from H . In order to completely generalize the situation of Frobenius Baer partitions to that of Frobenius cp-partitions, we must at least show that H has a normal complement in G . However, this is impossible as the following example shows.

Example 3.1.1: Consider A_5 , the alternating group on five letters. Let σ consist of the isomorphic copies of A_4 in A_5 together with the cyclic subgroups of order 5. A_4 has the property that it is a self-normalizing Hall subgroup of A_5 and $g \in A_5 - A_4 \Rightarrow A_4 \cap A_4^g$ is a cyclic 3-group. However, A_4 has no normal complement in A_5 since A_5 is simple.

We now proceed to show that Frobenius's theorem can be generalized in a limited situation.

Let S_p be a Sylow p -subgroup of G and Z its center. G is said to be p -normal if $Z^g \subseteq S_p \Rightarrow Z^g = Z$ for all $g \in G$.

Lemma 3.1.1: Let S_p be a Sylow p -subgroup of a group G such that $g \in G - N_G(S_p) \Rightarrow S_p \cap S_p^g$ is cyclic. Then, G is p -normal.

Proof: Assume that G is not p -normal and let Z be the center of a Sylow p -subgroup of G . Then, there is a $g \in G$ such that

$Z \neq Z^g \subseteq S_p$. Let $N = N_G(Z^g)$. By theorem A.9, Z^g is not normal

in S_p and hence, S_p is not contained in N . So, there is a Sylow

p -subgroup S_p^h of G such that $S_p^h \subseteq N$ and $S_p \cap N \subseteq S_p^h$. It

then follows that $S_p \cap N \subseteq S_p \cap S_p^h \subseteq S_p \cap N$. So, $S_p \cap N = S_p \cap S_p^h$.

Since $S_p^h \neq S_p$, it follows that $h \in G - N_G(S_p)$. Thus,

$S_p \cap N = S_p \cap S_p^h$ is cyclic.

Now, we have that $Z^g \subseteq S_p \cap N$. It then follows that

$N_{S_p}(S_p \cap N) \subseteq N_{S_p}(Z^g)$ since $S_p \cap N$ is cyclic and all subgroups of

cyclic groups are characteristic. But, $N_{S_p}(Z^g) = S_p \cap N$. Hence,

$N_{S_p}(S_p \cap N) = S_p \cap N$. Now, Z^g not normal in S_p implies that

$S_p \cap N \subsetneq S_p$. But, this is contrary to the fact that no proper subgroup

of a p -group is equal to its normalizer. We are, therefore, forced

to conclude that G is indeed p -normal, which completes the proof.

We now give our partial generalization of Frobenius's theorem.

Theorem 3.1.2: Let H be a self-normalizing Hall subgroup of a

group G with the property that $g \in G - H \Rightarrow H \cap H^g$ is a cyclic

p -group, p a fixed prime. Then, H has a normal complement in

G if G is solvable or if H is a p -group.

Proof: We use induction on $|G|$. S_3 is the ground case.

Case I: $\text{Cor}_G(H) \neq I$

Let $K = \text{Cor}_G(H)$. Since K is a normal subgroup of G contained in H , we have that $K \subseteq H \cap H^g$ for all $g \in G$. In particular, if $g \in G - H$, then $K \subseteq H \cap H^g$, a cyclic p -group. Choose $S \subseteq K$ so that $|S| = p$. It is clear that S is normal in G and that H/S is a Hall subgroup of G/S . Let $\bar{g} = gS \in G/S - H/S$. Then, $g \in G - H$. So, $H/S \cap (H/S)^{\bar{g}} = H/S \cap H^g/S = H \cap H^g/S$, a cyclic p -group. If $\bar{g} = gS \in N_{G/S}(H/S)$, then $H/S = H/S \cap (H/S)^{\bar{g}} = H/S \cap H^g/S = H \cap H^g/S$ which implies that $H = H \cap H^g$, that is $g \in N_G(H) = H$. Thus, H/S is self-normalizing in G/S . Finally G solvable or H a p -group implies that G/S is solvable or H/S is a p -group. It then follows by induction that H/S has a normal complement, say W/S , in G/S .

Now, $\bar{I} = W/S \cap H/S = W \cap H/S$ which implies that $S = W \cap H$. Also, $|HW|/|S| = |H| \cdot |W| / |H \cap W| |S| = |H| \cdot |W| / |S|^2$
 $= [H:S] [W:S] = |H/S| |W/S| = |H/S| |W/S| / |H/S \cap W/S|$
 $= |H/S W/S| = |G/S| = [G:S] = |G|/|S|$. Hence, $G = HW$.

Now, $[W:S] = [W:H \cap W] = [WH:H] = [G:H]$. Then, $|H|$ coprime to $[G:H]$ and $S \subseteq H$ imply that $|S|$ is coprime to $[W:S]$. So, S is a Sylow p -subgroup of W . If S has a normal complement, say R , in W , then R being a Hall subgroup of W implies that R is characteristic in W . But, W is normal in G . So,

R is normal in G . Then, $HR = HSR = HW = G$. Also, $R \cap H \subseteq W \cap H = S$ and $R \cap S = I$ yield that $R \cap H = I$. Therefore, it suffices to show that S has a normal complement in W .

Assume now that H is not a p -group and let $h \in H$ have order $q \neq p$. If $w \in W$ so that $wh = hw$, then $h \in H \cap H^w$. But, h is not a p -element. It then follows that $w \in N_G(H) = H$ and hence, $w \in W \cap H = S$. Thus, if h does not centralize S , then h induces a fixed-point-free automorphism of prime order on W . We then have by Thompson's result, theorem A.10, that W is nilpotent. Then, S being the Sylow p -subgroup of W implies that S has a normal complement in W and we are through.

Assume now that h does centralize S and consider $H_1 = \langle S, h \rangle$ and $G_1 = WH_1$. If $g_1 \in G_1 - H_1$, then $g_1 \in G - H$ and so $H_1 \cap H_1^{g_1} \subseteq H \cap H^{g_1}$, a cyclic p -group. If $g_1 \in N_{G_1}(H_1)$, then $H_1 = H_1 \cap H_1^{g_1} \subseteq H \cap H^{g_1}$. But, H_1 is not a cyclic p -group. Hence, we have that $g_1 \in N_G(H) = H$. So, $g_1 \in H \cap G_1 = H \cap WH_1 = H_1$ which yields that H_1 is self-normalizing in G_1 . We also have that $[G_1 : H_1] = |WH_1 : H_1| = [W : W \cap H_1] = [W : S]$ which is coprime to $|H|$ and consequently to $|H_1|$. Finally, if $H_1 \subset H$, then $[G : G_1] = [WH : WH_1] = [H : H_1] > 1$. It then follows by induction that S has a normal complement in W . So, let us assume that $H = H_1$.

Now, $H = H_1 = \langle S, h \rangle$ which is abelian. Then $H \subseteq C = C_G(S)$.

But, S normal in G implies that C is normal in G . This together with the fact that H is self-normalizing in G yields that $H \subseteq C$. We assert now that $C = G$. If $C \subsetneq G$, then we have by induction that H has a normal complement U in C . Let $g \in G - C$. C normal in G implies that $H^g \subseteq C$. But, H^g is a complement to U in C . It then follows by the Schur-Zassenhaus result, theorem A. 8, that all complements to U in C are conjugate. So, there is a $c \in C$ such that $H^g = H^c$. This implies that $g c^{-1} \in H_G(H) = H \subseteq C$. Then, $g = (g c^{-1})c \in C$, a contradiction to $g \in G - C$. We are thus forced to conclude that $C = G$. It then follows in particular that S is in the center of W . Using Burnside's result, theorem A.12, we are able to conclude that S has a normal complement in W and thus, H has a normal complement in G .

The remaining subcase is where H is a p -group. But, any normal subgroup of order p of a p -group is necessarily in the center of the group. So, $H \subseteq C_G(S)$. By repeating the above arguments we get that $C = G$ and hence, H has a normal complement in G .

Case II: $\text{Cor}_G(H) = I$

If G is solvable, then the commutator subgroup G' is proper in G . We assert that $HG' = G$. Since $G' \subseteq HG'$ and G/G' is abelian, it follows that HG' is normal in G . If $HG' \subsetneq G$, then we have by induction that H has a normal complement U in HG' . Let $g \in G - HG'$. Then, HG' normal in G yields that $H^g \subseteq HG'$.

Again, we have by the Schur-Zassenhaus result that there is an $x \in HG'$ such that $H^g = H^x$ and hence, $gx^{-1} \in N_G(H) = H \subseteq HG'$. So, $g = (gx^{-1})x \in HG'$, a contradiction. Thus, we are able to conclude that $G = HG'$.

Let $H_2 = H \cap G'$. It is clear that H_2 is a Hall subgroup of G' . Assume that H_2 is not a cyclic p -group. If $g \in G' - H_2$, then $g \in G - H$ and we have that $H_2 \cap H_2^g \subseteq H \cap H^g$, a cyclic p -group. If $g \in N_{G'}(H_2)$, then $H_2 = H_2 \cap H_2^g \subseteq H \cap H^g$. Then, H_2 not a cyclic p -group implies that $g \in N_G(H) = H$. So, $g \in G' \cap H = H_2$ implies that H_2 is self-normalizing in G' . Then, $G' \subseteq G$ implies by induction that H_2 has a normal complement V in G' . But, V is a Hall subgroup of G' and consequently is normal in G . Then, $G = HG' = HH_2V = HV$, where $H \cap V = I$. Thus, H has a normal complement in G .

Assume now that $H_2 = H \cap G'$ is a cyclic p -group. If $H_2 = I$, then G' is a normal complement to H in G . So, assume that $H_2 \neq I$. Now, $\text{Cor}_G(H) = I$ implies that $N = N_G(H_2) \subseteq G$. But, G' normal in G yields that $H_2 = H \cap G'$ is normal in H . So, $H \subseteq N$. If $H = N$, then $N_{G'}(H_2) = H_2$ which implies by Burnside's theorem that H_2 has a normal complement in G' , say V . Then, $G = HG' = HH_2V = HV$ where $H \cap V = I$. So, assume that $H \subset N$. By induction, it follows that H has a normal complement U in N . Then, U being the normal p -complement to H in N and H_2 being

normal in N imply that $U \subseteq C_N(H_2)$. In particular, $H_2 \subseteq Z(N_G(H_2))$. Therefore, it follows that H_2 has a normal complement M in G' by Burnside's theorem. Finally, we have that $G = HG' = HH_2M = HM$ where $H \cap M = I$. This completes the subcase where G is solvable.

Assume now that H is a p -group. We then have by lemma 3.1.1 that G is p -normal. Let Z be the center of H and N the normalizer in G of Z . Now, $N \subseteq G$ since $\text{Cor}_G(H) = I$. Also, $H \subseteq N$. If $H \subset N$, then it follows by induction that H has a normal complement in N and hence, H is isomorphic to the largest factor group of N which is a p -group. The same conclusion follows if $H = N$. But, it follows by the second Hall-Greun theorem, theorem A.11, that the largest factor group of G which is a p -group is isomorphic to the same for N . Therefore, H has a normal complement in G and whence, the proof is complete.

The normal complement described in the above theorem is unique when it exists since it is a Hall subgroup of G . This normal complement shall be referred to as the Frobenius cp -kernel of G . The complements to it in G shall be called the Frobenius cp -complements.

We now state our main result of this chapter.

Theorem 3.1.3: Let σ be a Frobenius cp -partition of a group G where one of the self-normalizing components H is a Hall sub-

group. Then, G has a Frobenius cp-kernel K if G is solvable or if H is a p -group. Moreover, if $Z(K)$ contains a p' -element of composite order, then there is a normal component L of σ such that the following hold:

- a) $K \subseteq L$;
- b) $[L:K] = 1$ iff σ is a Frobenius Baer partition;
- c) If $[L:K] > 1$, then L/K is a cyclic p -group;
- d) K is nilpotent unless the Frobenius cp-complements are p -groups.

Proof: Let H be a self-normalizing Hall component of σ . Since Frobenius cp-partitions are normal, it follows that H^g is a component of σ for all $g \in G$. Then, $g \in G - H \Rightarrow H \neq H^g$ since H is self-normalizing. So, $H \cap H^g$ is a cyclic p -group for all $g \in G - H$. We then have by theorem 3.1.2 that G has a Frobenius cp-kernel, say K , since G is solvable or H is a p -group.

Assume now that $Z(K)$ contains a p' -element x of composite order. It then follows by theorem 1.4.3 that there is a component L of σ such that $K \subseteq L$. Since K is normal in G , we have that $K \subseteq L \cap L^g$ for all $g \in G$. But, K is not a cyclic p -group and so, $L = L^g$ for all $g \in G$. Thus, part a is established.

Assume now that $[L:K] = 1$. Then, $L = K$. First, we assert that the elements of H different from 1 induce fixed-point-free automorphisms on K . Let $h (\neq 1) \in H$. If h is neither a

p -element nor a p' -element, then it follows by corollary 2.1.2 that $C_K(h) \subseteq H \cap K = I$. So, assume that h is either a p -element or a p' -element. If h is a p -element, then we conclude from theorem 2.1.1 (a) that $\exp(C_K(h))$ divides $|h| \cdot \exp(\sigma)$. Since H is a Hall subgroup of G , $h(\neq 1) \in H$ is a p -element, and since K is a normal complement to H in G , it follows that K is a p' -subgroup of G and hence, $C_K(h)$ is a p' -subgroup. But, $|h| \cdot \exp(\sigma)$ is a power of p . We then must conclude that $C_K(h) = I$. So, assume that h is a p' -element. Choose n so that $|h^n| = q$, a prime. Then, $C_K(h) \subseteq C_K(h^n)$. But, $|h^n| = q$ where $q \neq p$ implies by theorem 2.1.1 (b) that $C_K(h^n)$ is the semi-direct product of a subgroup of $H \cap K = I$ by a group of exponent q . Then, $|H|$ and $|K|$ being coprime yield that $C_K(h) = I$. This establishes that $C_K(h) = I$ and hence, h induces a fixed-point-free automorphism on K .

Our next assertion is that $C_G(k) \subseteq K$ for all $k(\neq 1) \in K$. Let $g \in C_G(k)$ where $k(\neq 1) \in K$. Then, $G = HK$ implies that $g = hk_o$ where $h \in H$ and $k_o \in K$. We then have that $k_o^{hk_o} = k$ or that $k_o^h = k_o k_o^{-1}$. So, h induces a fixed-point-free automorphism on K and fixes the conjugate class of K which contains k if $h \neq 1$. Since h induces a fixed-point-free automorphism on K , k_o can be expressed uniquely in the form $k_o = k_1^h k_1^{-1}$. So,

$k_o^h = k_1^h k_1^{-1} k k_1 (k_1^{-1})^h$ or $(k_1^{-1} k k_1)^h = k_1^{-1} k k_1$. But, this is a

contradiction to the fact that h induces a fixed-point-free automorphism on K . Thus, $h = 1$ and hence, $g = k_0 \in K$ as asserted.

Now, K is nilpotent by Thompson's result, theorem A.10. We also have by the Schur-Zassenhaus result, theorem A.8, that all complements to K in G are conjugate. By theorem A.13 we conclude K has a complement M such that $M \cap M^g = I$ for all $g \in G - M$. Then, all complements being conjugate implies that $M = H^x$ for some $x \in G$. Hence, all complements have trivial intersection.

Now, let L be any component of σ such that $L \not\leq K$. We first assert that $L \cap K = I$. Assume to the contrary. Then, $L \cap K$ is a cyclic p -group. Now, $[L:L \cap K] = [LK:K]$. Then, K being a Hall subgroup of K implies that $1 = ([LK:K], |K|) = ([L:L \cap K], |K|)$. Also, $L \cap K$ is a Hall subgroup of L . Let $x \in L$ have order $q \neq p$, where q divides $[L:L \cap K]$. Then H being a Hall subgroup of G such that $(|H|, |K|) = 1$ implies that q divides $|H|$ and hence, there is a conjugate H^x of H such that $x \in H^x$. It then follows that $x \in L \cap H^x$. But x is a p' -element and so, $L = H^x$. However, this implies that $H^x \cap K = L \cap K \neq I$, a contradiction. So, we are forced to conclude that $L \in \sigma$ such that $L \not\leq K \Rightarrow L \cap K = I$. Since $K = F(G)$ and $L \cap K = I$, it follows by theorem B.6 (c) that there is a conjugate H^y of H such that $L \subseteq H^y$ and hence, $L = H^y$. It now follows that σ is indeed a

Frobenius Baer partition of σ . This establishes the only if part of b.

Assume now that $K \subset L$. Then, $G = HK$ implies that $L \cap H \neq I$ and hence, σ is not a Frobenius Baer partition since H and L are components of σ . This completes the if part of a.

Again, if $K \subset L$, then $G = HK$ implies that there is a subgroup H_1 of H such that $L = H_1 K$. So, $L/K = H_1 K/K \cong H_1$. But $H_1 = L \cap H$ a cyclic p -group since L and H are distinct components.

Finally, if H is not a p -group, then let $h \in H$ have order $q \neq p$. We have by theorem 2.1.1(b) that $C_K(h)$ is the semi-direct product of a subgroup of $H \cap L$ by a group of exponent q . But, $|H|$ and $|K|$ being coprime implies that $C_K(h) = I$. Thus, h induces a fixed-point-free automorphism of prime order on K and so, K is nilpotent by theorem A.10. The theorem is now proved.

The conclusion that K is contained in a component of σ is false if $Z(K)$ does not contain an element of composite order as the following example shows.

Example 3.1.2. Let $K = \langle a, b : a^5 = b^5 = 1 \rangle$. Consider the group of automorphisms of K defined by $H = \langle \alpha, \beta : a^\alpha = b^4, b^\alpha = ab^4, a^\beta = a, b^\beta = ab^4 \rangle$. One can easily verify that $\alpha^3 = \beta^2 = 1$ and that $\beta \alpha \beta = \alpha^2$. So, $|H| = 6$. Let G be the holomorph of K by H . If we define σ to be all the conjugates of H in G together with the cyclic subgroups of order 10, then σ is a Frobenius c_2 -partition with $\exp(\sigma) = 2$.

Clearly, K is not contained in any component of σ .

APPENDIX A

Theorem A.1: Let x be an element of order mn in a group G where $(m, n) = 1$. Then, $x = yw = wy$ uniquely where $|y| = n$ and $|z| = m$. Both y and w are powers of x .

This is lemma 3.2.1 in M. Hall's Group Theory.

Theorem A.2: Let G be a group and $g \in G$. Then, $g \in \mathcal{H}(G)$ if and only if each q -part-- q a prime--is centralized by all q' -elements.

This was proved by Baer [3], page 38.

Theorem A.3: Let A , B , and C be subgroups of G such that $B \subseteq A$ and $BC = CB$. Then, $A \cap \langle B, C \rangle = \langle B, A \cap C \rangle$.

This is theorem 8.4.1 in M. Hall's Group Theory.

Let φ be an automorphism of a group G and H a subgroup, then H is said to be φ -invariant if $H^\varphi = H$. If H is a normal φ -invariant subgroup of G , then φ induces an automorphism $\bar{\varphi}$ on G/H in the following natural fashion: $\bar{\varphi}(gH) = g^\varphi H$. Before we state the next theorem, an automorphism φ of G is said to be fixed-point-free if $g^\varphi = g$ implies that $g = 1$.

Theorem A.4: Let φ be a fixed-point-free automorphism of G and H a normal φ -invariant subgroup of G . Then, φ induces a fixed-point-free automorphism on G/H .

The statement of this theorem is given in Schenkman, page 279, and its proof is left as an exercise.

Now, if S is a non-empty set and if G is a group, then G is said to be an S -group if there is a function $*$ from $G \times S$ into G such that $(g_1 g_2) * s = (g_1 * s) (g_2 * s)$ for all $g_1, g_2 \in G, s \in S$. A particular case of this is where S is a group of automorphisms of G . Two S -groups G_1 and G_2 are said to be S -isomorphic if there is an isomorphism φ from G_1 onto G_2 such that $\varphi(g * s) = \varphi(g) * s$. Now, a subgroup H of an S -group G is said to be an S -subgroup if $h * s \in H$ for all $h \in H$.

Theorem A. 5: Let G be an S -group and $K \subseteq H \subseteq G$ normal S -subgroups of G . Then, the relation $U = \{(gH, gK(H/K)) | g \in G\}$ is an S -isomorphism of G/H onto $G/K/H/K$.

This is theorem 2.9.4 in W. Scott and its proof is left as an exercise.

Theorem A. 6: Let G be a finite group and H a subgroup of G such that $g \in G - H$ implies that $H^g \cap H = I$. Then, H has a normal complement in G .

This is a well-known Frobenius theorem--see W. Scott, theorem 12.5.11. The normal complement to H is referred to as the Frobenius kernel of G and has been shown to be nilpotent by Thompson [15].

Now, a group G is said to be an H_q -group if it admits an automorphism γ of order q such that $g \cdot g^\gamma \cdot g^{\gamma^2} \cdots g^{\gamma^{q-1}} = 1$ for all $g \in G$.

Theorem A. 7: All H_q -groups are nilpotent.

This theorem was proved by O. Kegel, Math A. 75, 373-376 (1961).

A subgroup H of G is said to be Hall if $|H|$ and $[G:H]$ are coprime.

Theorem A. 8: If K is a normal Hall subgroup of a group G such that K or G/K is solvable, then any two complements of K are conjugate.

This result was proved by H. Zassenhaus and is theorem 9.3.9 in W. Scott's book.

Theorem A. 9: If P_1 and P_2 are Sylow p -subgroups of G such that $Z(P_1)$ is normal in P_2 , then $Z(P_1) = Z(P_2)$.

This is theorem 13.5.3 in W. Scott's book.

An automorphism γ of a group G is said to be fixed-point-free if $g^\gamma = g$ implies that $g = 1$.

Theorem A. 10: Any group G which admits a fixed-point-free automorphism of prime order is nilpotent.

This classic result was first proved by Thompson [15] , and its proof may be found in Schenkman, theorem IX. 4. h.

Theorem A.11: If G is p -normal, then the largest factor group of G which is a p -group is isomorphic to the same for the normalizer of the center of a Sylow p -subgroup.

This is known as the second theorem of Hall-Grün and is theorem 14. 4. 6 in M. Hall's book.

Theorem A.12: If P is an abelian Sylow p -subgroup of G which is in the center of its normalizer, then P has a normal complement in G .

This is known as Burnside's theorem and is theorem 14. 3. 1 in M. Hall's Book.

W. Scott in his book defines a Frobenius group as one which contains a proper normal subgroup K (called the Frobenius kernel) such that if $k(\neq 1) \in K$, then $C_G(k) \subseteq K$, see page 348. He then proves the following theorem, theorem 12. 6. 1.

Theorem A.13: If G is a Frobenius group with Frobenius kernel K , then there is a Hall subgroup H such that $G = HK$ and $H \cap H^g = I$ for all $g \in G - H$.

APPENDIX B

Theorem B.1: If σ is a Baer partition of a group G and $a, b \in G$ are commutable elements of G , then a and b are contained within a common component unless they both have order q for some prime q .

For the proof of this theorem see Baer [4] , lemma 2.1.

Theorem B.2: If σ is a normal non-trivial Baer partition of a non-simple group G , then the following are equivalent:

- a) σ is simple
- b) If $F(G) \neq I$, then G is isomorphic with S_4
- c) If G is not isomorphic with S_4 , then $S(G)$ is simple and non-abelian.
- d) G contains S_4 as a subgroup.

For the proof, see Baer [2] , page 2.

Theorem B.3: If σ is a normal Baer partition and K a proper normal σ -admissible subgroup of G , then the following hold:

- a) σ is not Frobenius if and only if $[G:K]$ and $|K|$ are not coprime.
- b) If σ is not Frobenius, then there is a prime p such that $g \in G - K$ implies $|g| = p$.
- c) If σ is not Frobenius and G is not a p -group, then $[G:K] = p$

where p is a prime. Also, K and G both are extensions of nilpotent groups by a p -group. Finally, K is a component which is generated by all $g \in G$ such that $|g| \neq p$.

For the proof, see Baer [4], theorem 5.1.

Theorem B.4: A p -group G where $|G| \neq p$ admits a non-trivial partition if $H_q(G) \subset G$.

This was proved by O. Kegel [8].

Theorem B.5: If σ is a non-trivial Baer partition of a group G and each component of σ is a normal subgroup of G , then G is an elementary abelian q -group.

For the proof of this theorem see O. Kegel [8], pages 172-173.

Theorem B.6: Every Frobenius partition of a group G has the following properties.

- a) $F(G)$ is a proper normal σ -admissible Hall subgroup of G .
- b) All self-normalizing components are conjugate.
- c) A subgroup U of G is contained within a self-normalizing component if and only if $U \cap F(G) = I$.
- d) Every normal subgroup not contained in $F(G)$ contains $F(G)$; and the proper normal σ -admissible subgroups are contained in $F(G)$.
- e) The self-normalizing components faithfully induce on $F(G)$

Frobenius groups of automorphisms and do not contain any elementary abelian subgroups of order p^2 .

f) If a subgroup U of G contains a self-normalizing component, then U is self-normalizing.

This was proved by Baer [4], theorem 4.1.

Theorem B. 7: If the socle $S(G)$ of a finite group is neither abelian nor simple, then any normal partition on G is trivial.

This was proved by Baer [4], theorem 3.6.

Theorem B. 8: If σ is a normal non-trivial partition of a finite group G and $F(G) \neq 1$, σ is then and only then simple when the following occur:

- a) G is isomorphic with S_4 ;
- b) $S(G)$ is elementary abelian of order 4;
- c) σ consists precisely of those cyclic subgroups of G which are not contained in $S(G)$.

This is a result of Baer [1], theorem A.

Theorem B. 9: Let σ be a non-trivial partition of a finite group G and assume that $S(G)$ is non-abelian simple. Then, the following hold:

- a) $[G:S(G)] = 2$
- b) The Sylow p -subgroups of odd order are abelian
- c) The Sylow 2-subgroups are D-groups, that is, non-abelian groups

which contain an abelian subgroup H of index 2 and an involution g_o outside of H such that $(g_o h)^2 = 1$ for all $h \in H$.

This was proved by Baer [2] , Hauptsatz, page 1.

INDEX OF NOTATION

I. Relations and Sets

$\subseteq$	Is a subset of
$\subset$	Is a proper subset of
$\leq$	Is less than or equal to
$<$	Is less than
$\in$	Is an element of

II. Groups

I	The trivial group
$S(G) =$	$\langle H: H \neq I \text{ is a minimal normal subgroup of } G \rangle$
$F(G) =$	$\langle H: H \text{ is nilpotent normal subgroup of } G \rangle$
$GL(n, q) =$	The group of $n \times n$ non-singular matrices over $GF(q)$
$PGL(n, q) =$	$GL(n, q) / Z(GL(n, q))$
$SL(n, q) =$	$\{A \in GL(n, q): \det A = 1\}$
$PSL(n, q) =$	$SL(n, q) / Z(SL(n, q))$
$ G $	the order of G
$ g $	the order of g
π	a set of primes
$\pi[n]$	the set of all prime divisors of $n \geq 1$.
π -subgroup	$\pi[H] \subseteq \pi$

π -element	$\pi [g] \subseteq \pi$
π' -subgroup	$\pi [H] \cap \pi = \phi$
π' -element	$\pi [g] \cap \pi = \phi$
h^β	The image of h under β
H^β	The image of H under β
$I(G)$	The inner automorphism group of G
$\exp(G) =$	smallest positive integer n such that $g^n = 1 \text{ for all } g \in G$
$\langle \dots \rangle$	The group generated by ...
$Z(G)$	$\langle x \in G : xg = gx \text{ for all } g \in G \rangle$
$H_n(G) =$	$\langle g \in G : g \text{ does not divide } n \rangle$
$h(G)$	Largest element of the ascending central series of G .
$C_K^{(H)} =$	$\{k \in K : kh = hk \text{ for all } h \in H\}$
$C_K^{(h)} =$	$\{k \in K : kh = hk\}$
γ -invariant	$H^\gamma = H$
γ is fixed-point-free	$g^\gamma = g \Rightarrow g = 1$
$G_\gamma =$	$\{g \in G : g^\gamma = g\}$
$N_K^{(L)} =$	$\{k \in L : L^k = L\}$
$[x, y] =$	$x^{-1}y^{-1}xy$
$[H, K] =$	$\langle [h, k] : h \in H, k \in K \rangle$

H_n -automorphism $|\gamma| = n$ and $g \cdot g^\gamma \cdot g^{\gamma^2} \dots g^{\gamma^{n-1}} = 1$ for

all $g \in G$

H is Hall in G $|H|$ is coprime to $[G:H]$

p -normal $Z(S_p)^g \subseteq S_p \Rightarrow Z(S_p)^g = Z(S_p)$ for all $g \in G$

$G' =$ $\langle [g_1, g_2] : g_1 \in G, g_2 \in G \rangle$

$\text{Cor}_G(H) =$ $\bigcap_{g \in G} H^g$

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