AN INVESTIGATION OF UNIFORMLY CONVERGENT POWER SERIES ON THE CLOSED UNIT DISK

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

LOUIS THURMAN RICHARDS

1970

This is to certify that the

thesis entitled

AN INVESTIGATION OF UNIFORMLY
CONVERGENT POWER SERIES ON THE
CLOSED UNIT DISK

presented by

Louis T. Richards

has been accepted towards fulfillment
of the requirements for

Ph.D degree in Mathematics

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Major professor

Date July 17, 1970

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ABSTRACT

AN INVESTIGATION OF UNIFORMLY CONVERGENT POWER SERIES ON THE CLOSED UNIT DISK

By

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Two facts are immediately known about a given power series with radius of convergence $R \geq 1$:

(1) the series converges absolutely for $|z| < 1$, and

(2) the series converges uniformly on $|z| \leq \rho < 1$.

Included in the class of all such power series are two subclasses:

(1') those power series which converge absolutely on $|z| = 1$, and

(2') those power series which converge uniformly on $|z| \leq 1$.

The class of all power series obeying (1') has been extensively investigated. However, the class of all power series obeying (2') has not been adequately investigated.

After showing that U, the space of all power series obeying (2'), is a Banach algebra, this paper investigates some of the functional analysis properties of the space. The investigator was also interested in finding classes of functions, Φ, such that the composition of any power series in U with Φ would again be in U.

The following are typical results from the study:

Theorem: If $\{a_k\}$ is a sequence such that $\sum a_k z^k$ is in U, and if λ is a complex number such that $\lambda \neq 0$, $\lambda \neq a_k$ $k = 0,1,2,\ldots$,

then $\{\frac{a_k}{a_k-\lambda}\}$ is a sequence whose terms are coefficients of an element in U.

Theorem: If X is a sequence space which is a Banach algebra under coordinate-wise multiplication, and has a Schauder basis, then projections into the coordinates are the only non zero homomorphisms on X.

Theorem: If $\Phi(z) = \frac{\alpha + (1-\alpha-\beta)z}{1-\beta z}$, $0<\alpha<1$, $0<\beta<1$, then for any f in U, $f\circ\Phi$ is in U.

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1970

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ACKNOWLEDGMENTS

The investigator wishes to acknowledge the suggestions and guidance which Dr. William Sledd rendered in the preparation of this manuscript. The investigator is also very thankful to Miss Linda Taylor who typed the manuscript and proved to be very helpful in proofreading the final copy.

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"INTRODUCTION"

If $\sum a_n z^n$ is a given power series with radius of convergence greater than or equal to 1, then

(1) $\sum a_n z^n$ converges absolutely for $|z| < 1$, and

(2) $\sum a_n z^n$ converges uniformly on $|z| \leq \rho < 1$

Now if $\sum a_n z^n$ has radius of convergence $R > 1$, then

(1') $\sum a_n z^n$ converges absolutely on $|z| = 1$, and, hence

(2') $\sum a_n z^n$ converges uniformly on $|z| \leq 1$.

However, a series need not have radius of convergence greater than 1 in order to satisfy (1'). The series $\sum \frac{z^n}{n^2}$ has radius of convergence 1 [since $\lim \left(\frac{1}{n^2}\right)^{1/n} = 1$], and it is absolutely convergent on $|z| = 1$.

The space of all power series obeying (1') can easily be identified with ℓ_1. In fact, letting f be a typical power series which obeys (1') and denoting the norm of f by $\|f\| = \sum |a_n|$, the map $\{a_n\} \to f$ is an isometric isomorphism from ℓ_1 onto this space. The space of all power series obeying (1') is generally given as an example of a Banach algebra, and theorems are proved about it in books dealing with functional analysis and Banach algebras.

The space of all power series obeying (2') is not so very well known. This paper will deal with the space of all power series obeying (2') as a Banach algebra, its dual space, its Gelfand transform, and continuous linear operators mapping the space into itself.

Kahane and Katznelson [14] proved that the space of functions satisfying (2') is not an algebra under pointwise multiplication. Although the investigator was well into this paper before seeing their results, the notation for the space is nearly identical [they denoted the space of series obeying (2') by U^T, and it is denoted in this paper by U], and the norm used is the same:

$$||f|| = \sup_{p} \sup_{|z|\leq 1} \left|\sum_{k=0}^{p} a_k z^k\right| . \tag{3}$$

The only mention which the investigator has seen of the space of series obeying (2') as a Banach space occurred in the above paper.

It will be shown in this paper that U is a semi-simple commutative Banach algebra under coordinate-wise multiplication, and, hence (3) is essentially the only norm that can be used.

Another author who has written about series obeying (2') is Alpar [3], and he has proved the following three theorems:

<u>Theorem 1</u>: Given a fixed point α $(0<|\alpha|<1)$, then one can always find a function $f_1(z)$, which is holomorphic in $|z|<1$ and which has an absolutely convergent power series such that the power series defined by $f_1\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) = f_2(z) = \sum b_k(\alpha)z^k$

is not absolutely convergent on $|z| = 1$.

Theorem 2: Let $f_1(z) = \sum a_k z^k$ be holomorphic in $|z|<1$, $\sum |a_k| < \infty$, α $(0<|\alpha|<1)$ a fixed point, and $|z_1| = 1$, $|z_2| = 1$ two points related by

$$z_1 = \frac{z_2 - \alpha}{1 - \bar{\alpha} z_2} .$$

Then the power series obtained by the transformation

$$f_2(z) = f_1\left(\frac{z - \alpha}{1 - \bar{\alpha} z}\right) = \sum b_k(\alpha) z^k$$

is uniformly convergent on the circumference $|z|=1$ and

$$f_1(z_1) = \sum a_k z^k = f_2(z_2) = \sum b_k(\alpha) z^k .$$

Theorem 3: There exist functions $f_1(z)$ holomorphic in $|z|<1$, whose power series development $\sum a_k z^k$ converges uniformly but not absolutely on $|z| = 1$ and which are changed by the transformation $f_1\left(\frac{z - \alpha}{1 - \bar{\alpha} z}\right)$ into a function $f_2(z)$ whose power series $\sum b_k(\alpha) z^k$ is not absolutely convergent on the circumference $|z| = 1$ for no value of α $(0<|\alpha|<1)$.

There are both propositions and theorems in this paper. Propositions will refer to those mathematical truths whose proofs are fairly elementary. Theorems will refer to those truths which demand some care in proving. In each chapter, lemmas, theorems, propositions, and corollaries have been numbered consecutively without regard to their special characters.

Chapter I deals principally with the functional analysis consequences of U being a Banach algebra. In order to facilitate the investigation, U will be identified with three spaces:

(4) $U_1 = \left\{ f(t) = \sum a_n e^{int} : \sum a_n e^{int} \text{ converges uniformly} \right\}$

(5) $U_2 = \left\{ x = \{a_n\} : \sum a_n e^{int} \text{ is a function in } U_1 \right\}$

(6) $U_3 = \left\{ S(t,n) = \sum_{k=0}^{n} a_k e^{ikt} : \sum a_k e^{ikt} \text{ is in } U_1 \right\}$.

Under appropriate norm, all of these spaces are isomorphic-isometric to U.

Chapter II deals with continuous linear operators on U. The major theorem of Chapter II states that Karamata type functions, i.e., functions of the type

$$\varphi(z) = \frac{\alpha + (1 - \alpha - \beta) z}{1 - \beta z} ; \ (0 < \alpha < 1), \ (0 < \beta < 1),$$

operate on U under composition of functions. This result is used to show that certain Bajanski [4] type functions also operate on U.

The following usages and notations have been employed throughout the investigation:

a) infinite series and sequences whose indices begin with 0 or an appropriate positive integer have been written without the index

b) $\widetilde{\mathcal{L}}_1$ will denote the space of functions obeying (1')

c) B[X,Y] will denote the space of all continuous linear operators from the Banach space X to the Banach space Y

d) X' will denote the dual space of X

e) coordinate-wise multiplication will be denoted by " $\star$ ".

CHAPTER ONE

UNIFORMLY CONVERGENT POWER SERIES ON $|z| \leq 1$

1. The Banach Space U

In this chapter, some basic theory concerning the structure of the class of uniformly convergent power series on $|z| \leq 1$ will be developed. It will be shown that this class forms a Banach space under appropriate definitions of addition, scalar multiplication, and norm. Furthermore, it will be shown that the space forms a semi-simple Banach algebra under coordinate-wise multiplication. The Gelfand transform of the space will also be investigated, and additional information about its structure will be obtained. The dual space will also be briefly investigated.

The goal of this section is to establish that the class,

(1) U = the set of all uniformly convergent power series on $|z| \leq 1$,

is a Banach space under very natural conditions. Since each $\sum a_k z^k$ which belongs to U defines a unique function, f, which is holomorphic in $|z| < 1$ and whose power series development is precisely $\sum a_k z^k$, the elements of U will be denoted by these f's.

On the set U, addition is defined by the rule: if f_1, f_2 are elements of U, then $f_1 + f_2$ is the element of U defined by

$$(f_1 + f_2)(z) = f_1(z) + f_2(z) \quad \text{for all } |z| \leq 1.$$

If α is any complex number, and $f \in U$, then αf is the function defined by $(\alpha f)(z) = \alpha \cdot f(z)$ for all $|z| \leq 1$. The set U clearly becomes a linear space over the complex numbers under these definitions.

If $f \in U$, $f(z) = \sum a_k z^k$, define $||f||$ by

$$(2) \qquad ||f|| = \sup_n \sup_{|z| \leq 1} |S_n(z)| \quad , \text{ where } S_n(z) = \sum_{k=0}^{n} a_k z^k.$$

Lemma 1.1: The function, $|| \ ||$, defined by (2), is a norm on U.

Proof: Let $f,g \in U$, $\alpha \in C$, [C denotes the field of complex numbers]. It must be shown that:

i) $||f|| < \infty$.

ii) $||f|| = 0$ iff f is the zero function.

iii) $||\alpha f|| = |\alpha| \, ||f||$.

iv) $||f+g|| < ||f|| + ||g||$.

Since ii), iii), and iv) involve only direct calculations, only i) will be verified.

Verification (i): By definition, $||f|| = \sup_n \sup_{|z| \leq 1} |S_n(z)|$ where $S_n(z) = \sum_{k=0}^{n} a_k z^k$. Also S_n converges uniformly to f on $|z| \leq 1$.

We note that S_n is an entire function for n= 0,1,2,.... Hence, $\sup_{|z|<1} |S_n(z)| = S_n(\zeta)$, for some ζ satisfying $|\zeta| = 1$. Now for each n, choose a point on $|z| = 1$ where S_n attains its supremum and call it ζ_n. Since f is continuous on $|z| \leq 1$, there exists M such that $\sup_{|z| \leq 1} |f(z)| < M$. Since S_n converges uniformly to f on $|z| \leq 1$, there exists N such that n>N implies $|S_n(z) - f(z)| < 1$ for all $|z| \leq 1$.

Hence, $n>N$ implies that $|S_n(z)|<1+|f(z)|$ for all $|z|\leq 1$. Therefore, $\sup_{|z|=1}|S_n(z)| = |S_n(\zeta_n)|<1+M = M'$, if $n>N$. Hence, $\sup_{n>N}\sup_{|z|\leq 1}|S(z)|<\infty$. It follows easily now that $||f|| = \sup_n \sup_{|z|\leq 1}|S(z)|<\infty$ since there exists M'' such that $\sup_{n<N}|S(\zeta_n)|<M''$.

In addition to proving that $||\ ||$ is a norm on U, the verification of i) has also shown

(3) $$||f|| = \sup_n \sup_{|z|=1}|S_n(z)|.$$

Theorem 1.2: The linear space U is a Banach space under $||\ ||$ as norm.

Proof: Since $(U, ||\ ||)$ is a normed linear space, it suffices to show that every Cauchy sequence in $(U, ||\ ||)$ converges. Let $\{f^p\}$ be a Cauchy sequence in U. Then, given $\varepsilon>0$, there is an N such that

(4) $$||f^p - f^q|| = \sup_n \sup_{|z|=1}\left|\sum_{k=0}^{n}(a_k^p - a_k^q)z^k\right|<\varepsilon \quad \text{if } p, q>N.$$

Hence, for any n, $\left|\sum_{k=0}^{n}(a_k^p - a_k^q)\right| \leq ||f^p - f^q||<\varepsilon$ if $p,q>N$.

Given m, we have,

$|a_m^p - a_m^q| - \left|\sum_{k=0}^{m-1}a_k^p - a_k^q\right| \leq \left|\sum_{k=0}^{m}a_k^p - a_k^q\right|<\varepsilon$ if $p, q>N$, and therefore,

(5) $$|a_k^p - a_k^q|\leq 2\varepsilon \text{ if } p,q>N.$$

We now have $\{a_k^p\}_{p=0}^{\infty}$ is a Cauchy sequence of complex numbers for each k. Hence, there is an a_k such that

(6) $$\lim_p a_k^p = a_k \quad \text{(Uniformly in k).}$$

Now define f by $f(z) = \sum a_k z^k$. We claim that $\sum a_k z^k$ converges for all $|z| < 1$. To see this, we note that if $\{a_k\}$ is bounded, say by M, then for any $|z_0|<1$, $\sum |a_k z_0^k| < M \sum |z_0^k| < \infty$; hence, $\sum a_k z^k$ converges for any $|z|<1$. To show that $\{a_k\}$ is bounded, observe that $\{a_k^p\}_{k=0}^{\infty}$ is bounded for any p. According to 5, given $\varepsilon = \frac{1}{2}$, there exists N such that $p'>N$ implies that

$$|a_k| = |a_k^{p'} + a_k - a_k^{p'}|$$

$$\leq |a_k^{p'}| + |a_k - a_k^{p'}|$$

$$< |a_k^{p'}| + 1.$$

Since $\sup_k |a_k^{p'}|$ is finite, $\sup_k |a_k|$ is finite.

Thus f is a candidate for the limit function of f^p. To see that f^p converges to f, observe that by (4),

$$\sup_n \sup_{|z|=1} \left|\sum_{k=0}^{n} (a_k^p - a_k^q) z^k\right| < \varepsilon \quad \text{if } p,q>N.$$

Fix z, n, and $p>N$ and let $q\to\infty$. Then

$$\left|\sum_{k=0}^{n} (a_k^p - a_k) z^k\right| < \varepsilon .$$

Since this is true for any $|z| \leq 1$, n, and p,

(7) $$||f^p - f|| = \sup_n \sup_{|z|=1} \left|\sum_{k=0}^{n} (a_k^p - a_k) z^k\right| < \varepsilon \quad \text{if } p > N.$$

It must now be shown that $f \varepsilon U$. But,

$$\left|\sum_{k=n}^{m} a_k z^k\right| = \left|\sum_{k=n}^{m} (a_k - a_k^p) z^k + \sum_{k=n}^{m} a^p z^k\right|$$

$$\leq \left|\sum_{k=n}^{m} (a_k - a_k^p) z^k\right| + \left|\sum_{k=n}^{m} a_k z^k\right| .$$

Let $\varepsilon > 0$ be given, then, by (7), there is an N such that $p > N$ implies that

$$\sup_{r, |z|=1} \left|\sum_{k=0}^{r} (a_k - a_k^p) z^k\right| < \frac{\varepsilon}{3} .$$

Choose $p > N$ and fix it. Then

$$\sup_{|z|=1} \left|\sum_{k=n}^{m} (a_k - a_k^p) z^k\right| \leq \sup_{|z|=1} \left|\sum_{k=0}^{n-1} (a_k - a_k^p) z^k\right|$$

$$+ \sup_{|z|=1} \left|\sum_{k=0}^{m} (a_k - a_k^p) z^k\right|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} .$$

Since $f^p \varepsilon U$, there exists an M such that m, n>m implies that

$$\sup_{|z|=1} \left|\sum_{k=n}^{m} a_k^p z^k\right| \leq \frac{\varepsilon}{3} .$$

Therefore, if m, n > M, then

$$\sup_{|z|=1} \left|\sum_{k=n}^{m} a_k z^k\right| < \varepsilon.$$

Hence, $f \varepsilon U$. This finishes the proof of Theorem 1.2.

Corollary 1.3: Define the collection $\{p_k\}$ by $p_k(f) = a_k$, where a_k is the kth coordinate of f, then the set $\{p_k\}$ is an equicontinuous family of functionals on U. Hence, in particular, for each k, p_k is a continuous linear functional on U.

Proof: It is obvious that each p_k is a linear functional. It follows from (4) and (5) that given $\varepsilon>0$, we can choose $\delta = \frac{\varepsilon}{2}$ so that if $||f-g||<\delta = \frac{\varepsilon}{2}$, then $|p_k(f) - p_k(g)| = |a_k-b_k| < \varepsilon$ for each k.

Proposition 1.4: Let e_k, k = 0,1,2,... be the functions defined by $e_k(z) = z^k$. Then $\{e_k\}$ is a Schauder-basis for U.

Proof: Let $f\varepsilon U$, $f(z) = \sum a_k z^k$. Let $\varepsilon>0$ be given. Since $f\varepsilon U$, there is an N such that

$$\sup_{n>p} \sup_{|z|=1} \left| \sum_{k=p}^{n} a_k z^k \right| < \varepsilon \quad \text{if } p>N.$$

Therefore,

$$||f - \sum_{k=0}^{p-1} a_k e_k|| = \sup_{n>p} \sup_{|z|=1} \left| \sum_{k=p}^{n} a_k z^k \right| < \varepsilon$$

Hence,

$$f = \sum a_k e_k.$$

It is interesting to note that one merely observed the behavior of f at z=1 in some of the most crucial steps in the proof of Theorem 1.2, and that the norm used on U is analogous to the norm used on the space of convergent series. Hence, a natural question to ask is, "If $\sum a_k$ converge, does $\sum a_k z^k$ converge uniformly on $|z|\leq 1$?" Although $\sum a_k$ converges implies that $\sum a_k z^k$ converges uniformly on $|z|\leq p<1$, it does not imply uniform convergence on $|z|\leq 1$ as the following simple example shows:

(8) $$\sum \frac{(-1)^k}{k+1}$$

The series defined by (8) converges; however, the series $\sum \frac{(-1)^k z^k}{k+1}$ does not converge at $z = -1$; a fortiori, it cannot be uniformly convergent on $|z| \leq 1$.

Since, the elements of U converge uniformly on $|z| \leq 1$, one may naturally ask, "Does $f \varepsilon U$ imply that the radius of convergence of f is $R > 1$?" The answer to this question is, "No, there are elements in U whose radius of convergence is 1." One way to see this is to observe that $\sum \frac{z^k}{k^2}$ converges absolutely, and hence, uniformly on $|z| \leq 1$, but the radius of convergence is 1.

2. Some Spaces Equivalent To U

If X is a Banach space, then we will say that X is equivalent to U provided there exists a continous linear operator Λ such that it maps U one-to-one and onto X, and such that $||\Lambda(f)||_X = ||f||$. That is, X is equivalent to U iff there exists an isomorphism between X and U which is also an isometry.

The purpose of this section is to develop some theory about U by looking at different ways in which it is possible to describe the class of uniformly convergent power series on $|z| \leq 1$ as a Banach space while maintaining the norm which was defined by (2). Of course, if X is equivalent to U, then any information which is obtained about X can be easily translated via Λ to information concerning U. Since this can obviously be accomplished with no effort, there will not be a need to specifically translate anything.

Since by (3), if f belongs to U, then $||f|| = \sup_n \sup_{|z|=1} |S_n(z)|$, the norm will be restricted to the unit circle T.

The spaces which follow are all Banach spaces under the norm defined by (3).

(9) U_1 = the set of all f such that $f(t) = \sum a_k e^{ikt}$, and $\sum_{k=0}^{n} a_k e^{ikt}$ converges uniformly to f.

(10) U_2 = the set of all $x = \{a_k\}$ such that $\sum a_k e^{ikt}$ is a function in U_1.

Let N denote the non-negative integers with the discrete topology. Let $\overline{N}$ denote the one-point compactification of N.

(11) U_3 = the set of all functions, S, on $T \times \overline{N}$ of the form

$$S(t,n) = \sum_{k=0}^{n} a_k e^{ikt} \text{ , and } \sum a_k e^{ikt}$$

is a function in U_1.

Define $\Gamma: U \to U_1$ by $\Gamma(f) = g$, where $f(z) = \sum a_k z^k$ and $g(t) = f(e^{it})$; $\tau: U_1 \to U_2$ by $\tau(g) = x$, where $g(t) = \sum a_k e^{ikt}$ and $x = \{a_k\}$; and $\psi: U_1 \to U_3$, by $\psi(g) = S$, where $g(t) = \sum a_k e^{ikt}$ and

$$S(t,n) = \sum_{k=0}^{n} a_k e^{ikt}.$$

It is a trivial matter to check that Γ, τ, and ψ are isometric isomorphisms. Hence, the following proposition is stated without proof:

<u>Proposition 1.5</u>: U_1, U_2, and U_3 are equivalent to U.

Proposition 1.6: U_1, U_2, and U_3 have Schauder-bases.

Proof: The proof follows immediately from Propositions 1.4 and 1.5.

From the mappings defined prior to Proposition 1.5, the following facts are obvious:

1. A Schauder basis for U_1 is given by the set of functions defined by $e_k(t) = e^{ikt}$.
2. A Schauder basis for U_2 is given by the set of sequences of the form $e_k = (0,0,\ldots,0,1,0,0,\ldots)$ with 1 in the k^{th} coordinate.
3. A Schauder basis for U_3 is given by the set of functions defined by

$$e_k(t,n) = \begin{cases} 0 & \text{if } k > n \\ e^{ikt} & \text{if } k \leq n \end{cases}.$$

Proposition 1.7: If $\{a_k\} \varepsilon U_2$, then $\{\overline{a}_k\} \varepsilon U_2$, where $\overline{a}_k$ denotes the complex conjugate of a_k.

Proof: Since $\{a_k\} \varepsilon U_2$, given $\eta > 0$, there is an N such that

$$\sup_t \left|\sum_{k=p}^{q} a_k e^{ikt}\right| < \eta \qquad \text{if} \quad p,q > N.$$

Choose $p,q > N$, and fix them. Then

$$\sup_t \left|\sum_{k=p}^{q} \overline{a}_k e^{ikt}\right| = \sup_t \left|\overline{\sum_{k=p}^{q} a_k e^{-ikt}}\right|$$

$$= \sup_t \left|\overline{\sum_{k=p}^{q} a_k e^{ikt}}\right|.$$

Since $|\bar{z}| = |z|$, Proposition 1.7 now follows from the last equality.

It is well known that the complex series $\sum a_k + ib_k$ converges (converges absolutely) iff the two real series $\sum a_k$ and $\sum b_k$ converge (converge absolutely). The following corollary follows directly from Proposition 1.7 and the fact that U_2 is a vector space over C; hence, it is stated without proof:

<u>Corollary 1.8</u>: $\{a_k + ib_k\} \varepsilon U_2$ iff $\{a_k\}$ and $\{b_k\} \varepsilon U_2$.

U_3 is an interesting space and seems to be the "natural" space which one should use in investigating uniformly convergent power series. Note that $T\times\bar{N}$ is a compact Hausdorff space, and that U_3 is at least a subset of $B(T\times\bar{N})$, where $B(T\times\bar{N})$ is the class of bounded functions on $T\times\bar{N}$.

<u>Theorem 1.9</u>: U_3 is a closed subspace of $C(T\times\bar{N})$, where $C(T\times\bar{N})$ is the Banach algebra of continuous functions on $T\times\bar{N}$ with sup-norm.

<u>Proof</u>: Since U_3 is a Banach space under sup-norm on $T\times\bar{N}$, and, therefore, is a closed subspace of $B(T\times\bar{N})$, it suffices to show that U_3 is a subset of $C(T\times\bar{N})$.

Let $S\varepsilon U_3$, and let Ω be any open set in C. It must be shown that $S^{-1}[\Omega]$ is open in $T\times\bar{N}$, where $S^{-1}[\Omega] = \{(t,n): S(t,n)\varepsilon\Omega\}$.

Case 1. $S^{-1}[\Omega] = \phi$; whichis open, and we are done.

Case 2. $(t_0,n_0)\varepsilon S^{-1}[\Omega]$ with $n_0 \neq \infty$. In this case,

$S(t_0,n_0) = \sum_{k=0}^{n_0} a_k e^{ikt_0}$. Since $S(t,n_0)$ is continuous in t, there is a neighborhood V_{t_0} about t_0 such that $S(t,n_0)\varepsilon\Omega$ for all $t \varepsilon V_{t_0}$. Hence, $V_{t_0} \times n_0$ is an open set about (t_0,n_0) which is contained within $S^{-1}[\Omega]$.

Case 3. $(t_0,\infty) \varepsilon S^{-1}[\Omega]$; $S(t_0,\infty)\varepsilon \Omega$. Since Ω is open, there exists a $\delta>0$ such that $V_\delta = \{\zeta: |S(t_0,\infty) - \zeta|<\delta\}$ is a subset of Ω. Since $S(t,\infty)$ is continuous in t, there is an open neighborhood V_{t_0} about t_0 such that $|S(t,\infty) - S(t_0,\infty)|< \frac{\delta}{2}$ for all $t\varepsilon V_{t_0}$. Since $S(t,n)$ converges uniformly to $S(t,\infty)$, there is an N such that, for all n>N, $\sup_t|S(t,\infty) - S(t,n)|< \frac{\delta}{2}$. A fortiori, $|S(t,\infty) - S(t,n)|< \frac{\delta}{2}$ for all $t\varepsilon V_{t_0}$ and n>N. Finally, if $t \varepsilon V_{t_0}$ and n>N, we have

$$|S(t_0,\infty) - S(t,n)|\leq |S(t_0,\infty) - S(t,\infty)| + |S(t,\infty) - S(t,n)|$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, $(t,n) \varepsilon V_{t_0} \times \{n\}_{n=N+1}^{\infty}$ implies that $S(t,n)\varepsilon V_\delta$ which is a subset of Ω. Therefore, $V_{t_0} \times \{n\}_{n=N+1}^{\infty}$ is an open neighborhood about (t_0,∞) which is contained in $S^{-1}[\Omega]$.

From the above three cases, it follows that if Ω is any open set in C, and if $S \varepsilon U_3$, then $S^{-1}[\Omega]$ is open in $T\times\overline{N}$. Hence, U_3 is a subset of $C(T\times\overline{N})$.

<u>Corollary 1.10</u>: The norm, $\|\ \|$, originally defined on U as $\sup_n \sup_{|z|\leq 1} |S_n(z)|$ can now be redefined as

(12) $$\|f\| = \max_{(n,t)} |\sum_{k=0}^{n} a_k e^{ikt}|,$$ where maximum is used

in the sense that the value $n = \infty$ is admitted.

Proof: Since $T \times \bar{N}$ is compact, and since continuous functions on compact sets attain their supremum, we have for $S \varepsilon U_3$, there is a point (t_0, n_0) such that $\sup_{(t,n)} |S(t,n)| = |S(t_0, n_0)|$.

3. The Banach Algebra U

Since $C(T \times \bar{N})$ is a Banach algebra under pointwise multiplication of functions, one is tempted to conclude that U_3 is also a Banach algebra under pointwise multiplication. However, this is not the case. Kahane and Katznelson [14] proved that U_1 is not an algebra under pointwise multiplication; hence, U_3, U_2, and U are not algebras under this definition of multiplication. In fact, this seems to be an unnatural way to define multiplication on U_2.

In this section, it will be shown that U_2 is a Banach algebra under coordinate-wise multiplication. This multiplication corresponds to convolutions on U_1.

Lemma 1.11: Let f, g ε U_1, and define $f*g$ by

$$f*g(t) = \frac{1}{2\pi} \int_0^{2\pi} f(t-u)g(u)du. \tag{13}$$

Then the function $f*g \ \varepsilon \ U_1$.

Proof: It is well known--and it is easily proved via Fubini's theorem--that the Fourier coefficients of $h(t) = f*g(t)$ are given by $\hat{h}(n) = c_n = \hat{f}(n)\hat{g}(n) = a_n b_n$. Now $a_n = 0$ for $n<0$ implies that $c_n = 0$

for $n<0$. Hence, to show that $h \in U_1$, it suffices to show that $\sum a_k b_k e^{ikt}$ is uniformly Cauchy.

$$(14) \qquad \left|\sum_{k=p}^{q} a_k b_k e^{ikt}\right| = \left|\sum_{k=p}^{q} b_k e^{ikt} \frac{1}{2\pi}\int_0^{2\pi} f(u)e^{-iku}du\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(u)| \left|\sum_{k=p}^{q} b_k e^{ik(t-u)}\right| du$$

$$\leq \sup_u |f(u)| \sup_u \left|\sum_{k=p}^{q} b_k e^{ik(t-u)}\right| .$$

Since $\sup_u \left|\sum_{k=p}^{q} b_k e^{ik(t-u)}\right| = \sup_u \left|\sum_{k=p}^{q} b_k e^{iku}\right|$, the right hand side can be made arbitrarily small by choosing p sufficiently large. Hence, it is immediate that $h \in U_1$.

Corollary 1.12: Let μ be a complex Borel measure on T. Define the sequence $\{c_k\}$ by

$$c_k = \int_T e^{-ikt}\, d\mu(t), \qquad k=0,1,\ldots.$$

Then $\{a_k c_k\} \in U_2$ for all $\{a_k\} \in U_2$.

Proof: Since μ is a complex Borel measure on T, the total variation of μ, $|\mu|(T)$ is finite. Repeating the argument used in (14) yields

$$\left|\sum_{k=p}^{q} a_k c_k e^{iku}\right| \leq |\mu|\,(T) \sup_t \left|\sum_{k=p}^{q} a_k e^{ik(u-t)}\right|$$

Corollary 1.12 follows immediately from the above inequality.

From Corollary 1.12, one has the fact that if $f \in U_1$ and $g \in L^1$, then $f*g \in U_1$. This follows from the fact that the measure, μ, defined by

$$\mu(E) = \int_E g(t)\,dt$$

is a complex Borel measure and that $d\mu = g\,dt$.

Corollary 1.13: If $f,g \in U_1$, then $||f*g|| \leq ||f||\ ||g||$.

Proof: In (14), let $p = 0$. Then

$$\left|\sum_{k=0}^{q} a_k b_k e^{ikt}\right| \leq \sup_u |f(u)| \sup_u \left|\sum_{k=0}^{q} b_k e^{ikt}\right|$$

$$\leq ||f||\ ||g||.$$

Hence $\sup_{(q,t)} \left|\sum_{k=0}^{q} a_k b_k e^{ikt}\right| = ||f*g|| \leq ||f||\ ||g||$.

Since it is obvious that U_2 is a commutative ring under addition and coordinate-wise multiplication, we have proved:

Theorem 1.14: U_2 is a commutative Banach algebra under coordinate-wise multiplication.

Corollary 1.15 If $\{a_k\} \in U_2$, then $\{|a_k|^2\} \in U_2$.

Proof: This follows immediately from Proposition 1.7 and Theorem 1.14 since $a_k \bar{a}_k = |a_k|^2$.

The converse of Corollary 1.15 is not true since the sequence defined by $a_k = \frac{(-1)^k}{k}$ does not belong to U_2 while $\{|a_k|^2\} \in U_2$.

Definition 1.1: If R is a commutative ring without identity, then an ideal, I, of R is called a regular ideal provided there exists $u \in R$ such that $ux - x \in I$ for all x in R.[u is called an identity modulo I].

Definition 1.2: If R is a commutative ring, then it is semi-simple iff the intersection of all its maximal regular ideals is zero.

A well known result is that every regular maximal ideal in a commutative ring R is the kernel of some non-zero homomorphism from R to the complex numbers.

Theorem 1.16: If X is a space of sequences which is a commutative Banach algebra under coordinate-wise multiplication and has the set $\{e_k\}$ as a Schauder basis, then projections into the coordinates are the only homomorphisms, and, therefore, X is semi-simple.

Proof: Since the system $\{e_k\}$ is a Schauder basis for X, X contains the set of all finite sequences.

Let h be any non-zero homomorphism of X into C. Then h is a continuous linear functional, and $h(x*y) = h(x)h(y)$ for all $x,y \in X$. Since $x = \{a_k\}$, $y = \{b_k\}$ can be written as

$x = \sum a_k e_k$ and $y = \sum b_k e_k$, we obtain

$$\sum a_k b_k h(e_k) = \sum a_k h(e_k) \sum b_k h(e_k) \tag{15}$$

for all $x,y \in X$.

Let $h(e_k) = \lambda_k$, $k = 0,1,\ldots$. Given n, let $x_n = y_n = (1,1,1,\ldots,1,0,0,0,\ldots)$, where 0 is in the n+p th coordinate for $p = 1,2,3,\ldots$. Hence, by (15)

$$\sum_{k=0}^{n} \lambda_k = \sum_{k=0}^{n} \lambda_k \sum_{k=0}^{n} \lambda_k .$$

Therefore, for any n, $\sum_{k=0}^{n} \lambda_k = 0$ or $\sum_{k=0}^{n} \lambda_k = 1$. Now $\sum_{k=0}^{n} \lambda_k = 0$

for all n would imply that h was the zero homomorphism. Hence, there exists n such that

$$\sum_{k=0}^{n} \lambda_k = 1.$$

Let

$$N = \min\{n: \sum_{k=0}^{n} \lambda_k = 1\} .$$

Then $\lambda_N = 1$, and $\lambda_k = 0$ for $k < N$. It will be shown by induction

that $\lambda_k = 0$ for $k > N$.

Let x = y be the sequence defined by

$$a_k = \begin{cases} 1 & \text{if } k = N \\ 1/2 & \text{if } k = N+1 \\ 0 & \text{otherwise} \end{cases} .$$

For the sequence x = y above, (15) yields $-3\lambda_{N+1} = (\lambda_{N+1})^2$. Since λ_{N+1} must be -1 or 0, the above equality yields that it must equal zero. Assume that $\lambda_{N+q} = 0$ for $q < p$. Then λ_{N+p} must be -1 or 0. Let x = y be defined by

$$a_k = \begin{cases} 1 & \text{if } k = N \\ 1/2 & \text{if } k = N+p \\ 0 & \text{otherwise} \end{cases} .$$

Using (15) again, we obtain $1 + \frac{1}{4}\lambda_{N+p} = (1 + \frac{1}{2}\lambda_{N+p})^2$. Hence, $\lambda_{N+p} = 0$. Since

$$h(e_k) = \begin{cases} 0 & \text{if } k \neq N \\ 1 & \text{if } k = N \end{cases},$$

$h(x) = P_N(x) = a_N$. Projections into the coordinates are clearly homomorphisms. It has now been shown that these are the only ones. Define M_k by

$$M_k = \{\, x\varepsilon X\colon a_k = 0\}.$$

Obviously, M_k is the kernel of P_k. Hence, M_k is a regular maximal ideal for $k = 0,1,2,\ldots$. The intersection of all regular maximal ideals is clearly the zero sequence. Hence X is semi-simple. This completes the proof of Theorem 1.16.

It should be noted that one need only to have shown that the set of projections was a subset of the set of homomorphisms on X in order to have the fact that X was semi-simple. Hence, the interesting part of the theorem is the fact that a characterization of the homomorphisms on this class of Banach algebras is obtained.

Corollary 1.17: U_2 is semi-simple.

Proof: The proof follows immediately from Proposition 1.4 and Theorem 1.16.

Corollary 1.18: The set ℓ_1 is an ideal in U_2, but it is not contained in any regular ideal.

Proof: If $x,y\varepsilon\ell_1$, then $x - y\varepsilon\ell_1$, and if $x\varepsilon\ell_1$ and $y\varepsilon U_2$, then $x*y = \{a_kb_k\}\varepsilon\ell_1$. Hence ℓ_1 is an ideal. To show that ℓ_1 is not contained in any regular ideal, merely note that $\{1/k!\}\varepsilon \mathcal{l}_1$ and that no term of the sequence is zero; it now follows that $\mathcal{l}_1$ is not in any regular ideal since all such ideals are contained in maximal regular ideals.

From the fact that U_2 is semi-simple, one can obtain more information about the norm which has been used. In a paper on F-H spaces, Wilansky and Zeller [25] gave a very short proof of the fact that a commutative semi-simple Banach algebra, A, has an essentially unique norm under which A is a Banach algebra. Hence, the norm, $|| \ ||$, defined on U_2 is essentially the only one that could be used.

<u>Definition 1.3</u>: In a commutative ring, an element x is said to have an adverse y iff $x + y - xy = 0$.

It is a well known result that an element x in a commutative ring has an adverse in the ring iff it is not an identity modulo any regular maximal ideals. Using this fact, one obtains:

<u>Proposition 1.19</u>: If $\{a_k\}\varepsilon U_2$, $a_k \neq 1$, for $k = 0,1,2,\ldots$, then $\left\{\frac{a_k}{a_k - 1}\right\} \varepsilon\ U_2$.

Proof: For $x = \{a_k\}$ to be an identity modulo M_n, $x{*}y - y$ must be in M_n for all $y\varepsilon U_2$. But this means that $a_n b_n - b_n = 0$ for all $y\varepsilon U_2$, and this is true iff $a_n = 1$. Since $a_k \neq 1$ for any k, x is not an identity modulo any regular maximal ideal. Hence, there is an element $y\varepsilon U_2$ such that

$$x + y - x{*}y = (0,0,0,0,0,\ldots)$$

Hence, $a_k + b_k - a_k b_k = 0$ for $k = 0,1,2,\ldots$. Therefore, **y** is given by the sequence

$$b_k = \frac{a_k}{a_k - 1} .$$

Let $x \varepsilon U_2$. The Gelfand transform of x, denoted by $\hat{x}$, is a function defined on, $\Delta = \{ M_k : k = 0, 1, \ldots\}$, the collection of maximal regular ideals of U_2 by

(16) $\hat{x}(M_k) = \eta$ where η is a co-set of M_k and $x \varepsilon \eta$.

It is well known that if M is a regular maximal ideal in a commutative Banach algebra X, then the quotient algebra, X/M , is isomorphic to the field of complex numbers,C. In the particular case $X = U_2$ and $M = M_k$, an isomorphism can be exhibited explicitly.

<u>Proposition 1.20</u>: Let M_k be a regular maximal ideal of U_2. Let $\Pi_\eta^k = \{x \varepsilon U_2 : a_k = \eta\}$. Then $U_2/M_k = \{ \Pi_\eta^k : \eta \varepsilon C \}$, and the mapping $\Pi_\eta^k \to \eta$ is an isomorphism of U_2/M_k onto C.

<u>Proof</u>: Elements x,y clearly belong to the same co-set of M_k iff $a_k = b_k$, and it is equally clear that for any complex number η there is an $x \varepsilon U_2$ such that $a_k = \eta$. Hence, it follows that $\{\Pi_\eta^k : \eta \varepsilon C\}$ is precisely the collection of co-sets of M_k.

From the way addition and multiplication are defined on U_2/M_k, and from the fact that the co-sets of M_k are disjoint, it follows that the mapping $\Pi_\eta^k \to \eta$ is an isomorphism.

The Gelfand transform of an element $x \varepsilon U_2$ can now be described more completely by using Proposition 1.20:

(17) $\hat{x}(M_k) = a_k$ where a_k is the <u>kth</u> coordinate of x.

<u>Definition 1.4</u>: Let T_1 and T_2 be two topologies on a set X. Then T_1 is coarser than T_2 if T_1 is a subset of T_2.

Let X be a commutative Banach algebra without identity, and let ¶ denote the collection of all maximal regular ideals of X. Let T be the coarsest topology on ¶ such that all the Gelfand transforms are continuous on ¶. A subbase for the topology is given by the sets

(18) $\hat{x}^{-1}[\Omega]$ for all open Ω in $\mathbb{C}$, and for all x.

This topology makes ¶ a locally compact Hausdorff space. The functions $\hat{x}$ have the property that given $\varepsilon>0$, there is a compact subset K of ¶ such that $|\hat{x}(M)|<\varepsilon$ for M not in K. (such functions are said to vanish at infinity). Let C_0(¶) denote the collection of all continuous functions on ¶--with the above topology--which vanish at infinity.

Applying the above facts to U_2 yields:

<u>Proposition 1.21</u>: $C_0(\Delta)$ is isomorphic to the space of null sequences.

<u>Proof</u>: Since $C_0(\Delta)$ consists of all continuous functions which vanish at infinity, it suffices to show that the topology which is defined on Δ must be the discrete topology. If the topology on Δ is the discrete topology, then Δ is homeomorphic to the natural

numbers, N, with the discrete topology, and the continuous functions on N which vanish at infinity is precisely the collection of null sequences. To show that the topology on Δ is the discrete topology, it suffices to show that each M_k is open. From (18)

$$\hat{x}^{-1}[\Omega] = \{ M_k : \hat{x}(M_k) \in \Omega\}$$

will be an open set--since it is a member of the subbase--for each x in U_2 and open set Ω in C. Let $M_n \in \Delta$. Then for $e_n \in U_2$, let Ω be an open set in C such that 1 belongs to Ω and 0 does not belong to Ω. Then,

$$\begin{aligned} \hat{e}_n^{-1}[\Omega] &= \{ M_k : \hat{e}_n(M_k) \in \Omega\} \\ &= \{ M_k : a_k \in \Omega \ , e_n = \{a_m\} \} \text{ by(17)} \\ &= \{M_n\} \ , \end{aligned}$$

since $e_n = (0,0,0,\ldots,0,1,0,0,\ldots)$ with 1 in the n th coordinate. Hence, the coarsest topology on Δ for which all the $\hat{x}$'s are continuous is the discrete topology.

One of the cases which one looks for in the Gelfand representation of a commutative Banach algebra X is the case when the collection of transforms equals $C_0(\Delta)$. Letting $\hat{U}_2$ denote the collection of all Gelfand transforms of members of U_2, (17) and Proposition 1.21 yield the fact that $\hat{U}_2$ is a proper subset of $C_0(\Delta)$.

Definition 1.5: In a commutative Banach algebra, A, without identity, the spectrum of an element $x \in A$, $\sigma(x)$, is defined by

(19) $\sigma(x) = \{\lambda \in C: \lambda \neq 0, \text{ and } (1/\lambda)x \text{ does not have an adverse}\} \cup \{0\}$.

It is well known that the range of $\hat{x}$ is either identical to the spectrum of x, $\sigma(x)$, or it is $\sigma(x)$ with the value zero removed. For U_2, this means that $\sigma(x) = \{a_k\}$. This yields the following proposition which is an improvement upon Proposition 1.19:

Proposition 1.22: If $\{a_k\} \varepsilon U_2$, and $\lambda \varepsilon C, \lambda \neq 0$, and $\lambda \neq a_k$ for $k = 0,1,2,\ldots$, then

$$\left\{\frac{a_k}{a_k - \lambda}\right\} \varepsilon U_2 .$$

Proof: $x = \{a_k\}$, and $\lambda \neq 0$, $\lambda \neq a_k$ implies that λ does not belong to $\sigma(x)$. Hence, by (19), this means that $(1/\lambda)x$ has an adverse in U_2. This adverse is obviously given by

$$\left\{\frac{a_k}{a_k - \lambda}\right\}.$$

Definition 1.6: The multipliers on U_1, $M(U_1)$, is the set of all g such that the pointwise product, $f(t)g(t)$, is in U_1 for all f in U_1.

Let $\tilde{\ell}_1$ denote those elements of U_1 which have absolutely convergent series. Kahane and Katznelson [14] have proved that $\hat{\ell}_1$ is a proper subset of $M(U_1)$. Nevertheless, the following theorem yields a more direct proof that $\tilde{\ell}_1$ is a subset of $M(U_1)$.

Theorem 1.23: $\tilde{\ell}_1$ is a subset of $M(U_1)$.

Proof: Let $f(t) = \sum a_k e^{ikt}$ be in $\tilde{\ell}_1$, and $g(t) = \sum b_k e^{ikt}$ be in U_1. To show that $f(t)g(t)$ is in U_1, it suffices to show that given $\varepsilon > 0$, there is an M such that $p,q > M$ implies that

$$\sup_t |S(t;p,q)| = \sup_t \left|\sum_{n=p}^{q} e^{int} \sum_{k=0}^{n} a_k b_{n-k}\right| < \varepsilon .$$

$$S(t;p,q) = \sum_{n=p}^{q} \sum_{k=0}^{n} a_k e^{ikt} b_{n-k} e^{i(n-k)t}$$

$$= \sum_{k=0}^{p} a_k e^{ikt} \sum_{n=p-k}^{q-k} b_n e^{int} + \sum_{k=p+1}^{q} a_k e^{ikt} \sum_{n=0}^{q-k} b_n e^{int}$$

$$= \Sigma_1 + \Sigma_2 .$$

$$\Sigma_1 = \sum_{k=0}^{N} a_k e^{ikt} \sum_{n=p-k}^{q-k} b_n e^{int} + \sum_{k=N+1}^{q} a_k e^{ikt} \sum_{n=p-k}^{q-k} b_n e^{int}$$

$$= \Sigma_{11} + \Sigma_{12}$$

Without loss of generality, one can assume that neither f nor g is zero. Then there exists M_1 such that $N>M_1$ implies that

$$\sum_{k=N+1}^{\infty} |a_k| < \varepsilon/3 \|g\| .$$

There exists M_2 such that $m > M_2$ implies that

$$\sup_t \left| \sum_{n=m}^{m'} b_n e^{int} \right| < \varepsilon/3 \|f\|_1 \qquad (\|\ \|_1 \text{ is } \ell_1 \text{ norm})$$

Choose $M_3 = \max(M_1, M_2)$. Choose $N_0 > M_3$ and fix it. Let $M = M_3 + N_0$. Now if $p > M$, then

$$|\Sigma_2| \leq \sup_{p+1 \leq k \leq q} \left| \sum_{n=0}^{q-k} b_n e^{int} \right| \sum_{k=p+1}^{q} |a_k|$$

$$\leq \|g\| \sum_{k=p+1}^{\infty} |a_k|$$

$$< \varepsilon/3 ,$$

since $p+1 > M > N_0 > M_1$.

Since $N_0 > M_1$,

$$|\Sigma_{12}| \leq \sup_{N_0+1 \leq k \leq p} \left|\sum_{n=p-k}^{q-k} b_n e^{int}\right| \sum_{k=N_0+1}^{p} |a_k|$$

$$\leq ||g|| \sum_{k=N_0+1}^{\infty} |a_k|$$

$$< \quad \varepsilon/3 \ .$$

Now, if $K < N_0$, then $-k > -N_0$. Hence, $p-k > p-N_0$. But $p-N_0 > M_3 > M_2$. Hence, $k < N_0$ implies that $p - k > M_2$. Therefore,

$$\sup_{0 \leq k \leq N_0} \ \sup_t \left|\sum_{n=p-k}^{q-k} b_n e^{int}\right| < \ \varepsilon/3||f||_1 .$$

Hence,

$$|\Sigma_{11}| \leq \sup_{0 \leq k \leq N_0} \ \sup_t \left|\sum_{n=p-k}^{q-k} b_n e^{int}\right| \sum_{k=0}^{N_0} |a_k|$$

$$\leq \sup_{0 \leq k \leq N_0} \ \sup_t \left|\sum_{n=p-k}^{q-k} b_n e^{int}\right| \ ||f||_1$$

$$< \quad \varepsilon/3$$

Since $S(t;p,q) = \Sigma_{11} + \Sigma_{12} + \Sigma_2$,

$$\sup_t |S(t;p,q)| < \varepsilon \qquad \text{if } p > M.$$

This completes the proof of Theorem 1.23.

4. The Dual Space

Since U is isomorphic-isometric to U_1,U_2, and U_3, the dual space of U, U', is isomorphic-isometric to the dual spaces of U_1, U_2, and U_3—denoted respectively by U_1', U_2', and U_3'. That is, if X is U_1, U_2, or U_3 and Γ is an isomorphic-isometric mapping of U onto X, then the map, Ψ, defined by

(20) $\Psi(L) = T$ where T is defined by $T(f) = L(\Gamma(f))$

is an isomorphic-isometric mapping of X' onto U'.

<u>Proposition 1.24</u>: The members of U_2' are uniquely determined by sequences $\{c_k\}$ which have the property $\sum a_k c_k$ converges for all $\{a_k\}$ in U_2.

<u>Proof</u>: Let L be a continuous linear functional on U_2. Let x be any member of U_2. Since U_2 has a Schauder basis, $x = \sum a_k e_k$. Since L is a continuous linear functional, $L(x) = \sum a_k L(e_k)$. Hence, the sequence defined by $L(e_k) = c_k$ uniquely determines L. Conversely, let $\{c_k\}$ be a sequence such that $\sum a_k c_k$ converges for all $\{a_k\}$ in U_2. Now $P_k(x) = a_k$ is a continuous linear functional on U_2 for each k. Hence, for each n,

$$S_n(x) = \sum_{k=0}^{n} P_k(c_k x) = \sum_{k=0}^{n} a_k c_k$$

is a continuous linear functional on U_2. The assumption that $\sum a_k c_k$ converges for all $\{a_k\}$ in U_2 implies that S_n converges weakly to S, where $S(x) = \sum a_k c_k$. Hence, S is a continuous linear functional.

An immediate result from Proposition 1.24 is that if z_0 is such that $|z_0| \leq 1$, then $\{z_0^k\}$ defines a continuous linear functional on U_2, In particular, the sequences $(1, i, 1, \ldots)$ and $\{(-1)^k\}$ are continuous linear functionals on U_2.

For each fixed t, $0 \leq t \leq 2\pi$, define the class B.V.(t) by

(21) $$B.V.(t) = \{\{c_k\} : \sum |c_k - e^{-it}c_{k+1}| < \infty\}.$$

For t=0, the above class is simply the sequences of bounded variations.

Proposition 1.25: If $\{c_k\} \in B.V.(t)$, then $\lim c_k e^{-ikt}$ exists.

Proof: $\{c_k\} \in B.V.(t)$ implies that $\sum |c_k - e^{-it}c_{k+1}| < \infty$. But

$$\sum |e^{-ikt}c_k - e^{-i(k+1)t}c_{k+1}| = \sum |c_k - e^{-it}c_{k+1}|.$$

Hence, $\sum \left(c_k e^{-ikt} - c_{k+1}e^{-i(k+1)t}\right)$ converges. But

$$S_n = \sum_{k=0}^{n} \left(c_k e^{-ikt} - c_{k+1}e^{-i(k+1)t}\right)$$

$$= c_0 - c_{n+1}e^{-i(n+1)t}$$

Hence, $\lim c_n e^{-int}$ exists.

Now if $C = \lim c_n e^{-int}$, then $|C| = \lim |c_n|$ since $\left| |c_n| - |C| \right| \leq |c_n e^{-int} - C|$.

Proposition 1.26: B.V.(t) is a subset of U_2' for each t.

Proof: Let $\{c_k\} \in B.V.(t)$. Then for any n and any $\{a_k\} \in U_2$

$$\sum_{k=0}^{n} a_k c_k = \sum_{k=0}^{n-1} S_k(t)e^{-ikt}[c_k - c_{k+1}e^{-it}] + c_n e^{-int}S_n(t),$$

where $S_k(t) = \sum_{m=0}^{k} a_m e^{imt}$. But $\lim c_n e^{-int}S_n(t)$ exists, and

$\sum S_k(t)e^{-ikt}[c_k - c_{k+1}e^{-it}]$ is aboslutely convergent. Hence, $\sum a_k c_k$ converges for each $\{a_k\} \varepsilon U_2$. By Proposition 1.24, $\{c_k\} \varepsilon U_2'$

Corollary 1.27: Let $L = \{c_k\} \varepsilon B.V.(t)$. Then

$$||L|| = \sup_{||x||=1} |L(x)|$$

$$\leqq \sum |c_k - c_{k+1}e^{-it}| + \lim |c_k| .$$

Proof: From the proof of Proposition 1.26, it follows that

$$|L(x)| \leqq \sup_k |S_k(t)|\left(\sum |c_k - c_{k+1}e^{-it}| + \lim |c_k|\right)$$

$$\leqq ||x||\left(\sum |c_k - c_{k+1}e^{-it}| + \lim |c_k|\right).$$

Hence, Corollary 1.27 follows.

Proposition 1.28: There exist $\{c_k\} \varepsilon U_2'$ such that $\{c_k\}$ does not belong to B.V.(t) for any t.

Proof: Since $\{(-1)^k\}$ and $(1,1,1,1,\ldots)$ are in U_2', the sequence $\{c_k\}$ defined by $c_k = 1 + (-1)^k$ is in U_2'. However, for any t,

$$\sum_{k=0}^{m} |c_k - c_{k+1}e^{-it}| = \sum_{k=0}^{m} |[1 + (-1)^k] - [1 + (-1)^{k+1}]e^{-it}|$$

$$= 2\sum_{k=0}^{m} 1 = 2(m+1) \to \infty .$$

From the fact that ℓ_1 is a subset of U_2, the following proposition is easily obtained, and, is therefore stated without proof:

Proposition 1.29: As sets, U_2' is a subset of m, where m is the collection of bounded sequences.

Corollary 1.30: There exists f in U such that the derivative of f, f', does not belong to U.

Proof: If $\sum a_k z^k \varepsilon$ U implied that $\sum k a_k z^{k-1} \varepsilon$ U, then $\sum k a_k$ converges for all $\{a_k\}$ in U_2. Hence, by Proposition 1.24, $\{k\}$ would be in U_2', but this contradicts Proposition 1.29.

Corollary 1.30 is a prefectly natural result since there are power series in U whose radius of convergence is precisely 1, and, hence, have a singularity on $|z| = 1$.

Theorem 1.31: $\{c_k\} \varepsilon U_2'$ iff $\{a_k c_k\} \varepsilon U_2$ for all $\{a_k\} \varepsilon U_2$.

Proof: Assume that $\{a_k c_k\} \varepsilon U_2$ for all $\{a_k\} \varepsilon U_2$. Then the series $\sum a_k c_k z^k$ converges on $|z| \leqq 1$ for all $\{a_k\} \varepsilon U_2$. In particular, it converges at $z = 1$. Hence, by Proposition 1.24 $\{c_k\} \varepsilon U_2'$.

Now assume that $L = \{c_k\} \varepsilon U_2'$. Let $\{a_k\} \varepsilon U_2$. To show that $\{a_k c_k\} \varepsilon U_2$, it suffices to show that $\sum a_k c_k e^{ikt}$ is uniformly Cauchy. Let e^{is} be a point on T and fix it. Then $y = \{a_k e^{iks}\}$ is in U_2, and

$$\sup_t \left|\sum_{k=p}^{n} a_k e^{iks} e^{ikt}\right| = \sup_t \left|\sum_{k=p}^{n} a_k e^{ikt}\right|.$$

Hence,
$$\left|\sum_{k=p}^{n} a_k c_k e^{iks}\right| = \left| L\left(\sum_{k=p}^{n} a_k e^{iks} e_k\right)\right|$$

$$\leqq ||L|| \; \left|\left|\sum_{k=p}^{n} a_k e_k\right|\right| .$$

By choosing p sufficiently large, the right hand side of the inequality can be made as small as desired. Therefore, $\sum a_k c_k e^{ikt}$ is uniformly Cauchy.

Corollary 1.32: If $L = \{c_k\}$, $M = \{u_k\} \varepsilon U_2'$, then $L * M = \{c_k u_k\} \varepsilon U_2'$.

Proof: Let $x = \{a_k\} \varepsilon U_2$. Then $y = \{a_k c_k\} \varepsilon U_2$. But $M(y) = \sum a_k c_k u_k$. Hence, $\sum a_k (c_k u_k)$ converges for all $\{a_k\} \varepsilon U_2$. Therefore, $\{c_k u_k\} \varepsilon U_2'$.

Theorem 1.33: U_2' is a commutative Banach algebra with identity under coordinate-wise multiplication.

Proof: Let $L = \{c_k\}$, $M = \{u_k\} \varepsilon U_2'$. Let $x = \{a_k\} \varepsilon U_2$. Then $y = \{a_k c_k\} \varepsilon U_2$. For fixed s, $\{a_k e^{iks}\} \varepsilon U_2$. Hence,

$$\left| \sum_{k=0}^{n} a_k c_k e^{iks} \right| = \left| L\left(\sum_{k=0}^{n} a_k e^{iks} e_k \right) \right|$$

$$\leqq ||L|| \; \left|\left| \sum_{k=0}^{\infty} a_k e_k \right|\right|$$

$$= ||L|| \; ||x||.$$

Since the right-hand side is independent of n and s,

$$||y|| \leqq ||L|| \; ||x||.$$

Hence, $|M*L(x)| = |M(y)| \leq ||M|| \; ||y|| \leqq ||M|| \; ||L|| \; ||x||$.

Therefore, $||M*L|| \leqq ||M|| \; ||L||$.

The ring structure of U_2' is obvious, and $(1,1,1,\ldots)$ is the identity. Hence U_2' is a commutative Banach algebra with identity.

In order to find the precise dual space of U_2, we will examine the dual space of U_3. From the fact that U_3' and U_2' are equivalent, we will be able to deduce the manner in which the sequence space, U_2', is generated.

Definition 1.7: Let B be the sigma-algebra generated by the open sets of a topological space X. Let $E \varepsilon B$. Then a collection of sets, $\{E_n\}$, in B is said to partition E iff E is equal to the union of the E_n, and the collection is pair-wise disjoint. A complex Borel measure, μ, on B is a complex-valued function on B such that for each $E \varepsilon B$,

$$\mu(E) = \sum_{n=0}^{\infty} \mu(E_n) \quad \text{for every partition } \{E_n\} \text{ of } E,$$

and the above series always converges absolutely.

Definition 1.8: If μ is a complex Borel measure on B, the total variation of μ is the finite, positive Borel measure $|\mu|$ defined on B by

$$|\mu|(E) = \sup \{\sum |\mu(E_n)| : \{E_n\} \text{is a partition of } E\}.$$

Definition 1.9: If μ is a complex Borel measure on B, then it is regular iff for every $E \varepsilon B$,
$|\mu|(E) = \inf\{|\mu|(V)$: E is a subset of V, and V is open$\}$, and
$|\mu|(E) = \sup\{|\mu|(K)$: K is a subset of E, and K is compact$\}$.

The form of the Riesz Representation theorem which we need states:

If X is a compact Hausdorff space, then to each continuous linear functional L on C(X) there corresponds a unique complex

Borel measure, μ, such that

$$(21) \qquad L(f) = \int_X f \, d\mu \ , \quad \text{for all } f \in C(X),$$

and $||L|| = |\mu|(X)$.

By the Hahn-Banach Theorem, if $L \in U_3'$, then there exists a continuous linear functional H on $C(Tx\overline{N})$ such that $L(S) = H(S)$ for all S in U_3. Since $Tx\overline{N}$ is a compact Hausdorff space, there exists a unique complex regular Borel measure, μ, such that

$$H(S) = \int_{Tx\overline{N}} S \, d\mu \quad .$$

Hence, for each L in U_3', L can be represented by a complex regular Borel measure on the Borel sets in $Tx\overline{N}$. Moreover, L can be represented by one whose total variation on $Tx\overline{N}$ is equal to the norm of L. However, it is not true that L can be represented by a unique Borel measure since the extension of L to H is not unique.

Now the system $\{e_k\}$ defined by

$$e_k(t,n) = \begin{cases} 0 & \text{if } n < k \\ e^{ikt} & \text{if } n \geqq k \end{cases}$$

is a Schauder basis for U_3. If $L \in U_3'$, and $S \in U_3$, then there is a complex regular Borel measure, μ, on $Tx\overline{N}$ such that

$$L(S) = \sum_{k=0}^{\infty} a_k L(e_k)$$

$$= \sum_{k=0}^{\infty} a_k \int_{T \times \bar{N}} e_k(t,m)\, d\mu(t,m) \ .$$

Finally, if $L = \{c_k\}$ belong to U_2', then it is immediately seen that there is a complex regular Borel measure on $T\times\bar{N}$ which generates the c_k by the rule

$$\text{(22)} \qquad c_k = \int_{T \times \bar{N}} e_k(t,m)\, d\mu(t,m) \quad \text{and}$$

conversely, any complex regular Borel measure will generate a sequence--when defined by (22)--which belongs to U_2'.

CHAPTER TWO

CONTINUOUS LINEAR OPERATORS ON U

1. Concepts From Summability

Let $A=(a_{nk})$ be an infinite matrix. A sequence $x = \{x_k\}$ is said to be A - limitable provided the sequence $y = \{y_n\}$ defined by $y_n = \sum_{k=0}^{\infty} a_{nk}x_k$ converges. If every convergent sequence is A - limitable, then the matrix A is said to be conservative. A conservative matrix $A = (a_{nk})$ is clearly a linear operator on the Banach space of convergent sequences. A conservative matrix is called a regular matrix provided A - limit $x = \lim_{n\to\infty} x_n$ for every convergent sequence $x = \{x_n\}$. Toeplitz' Theorem gives the following necessary and sufficient conditions for a matrix to be conservative:

i) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$

ii) $\lim_{n\to\infty} a_{nk} = a_k$ exists for each k,

iii) $\lim_n \sum_{k=0}^{\infty} a_{nk} = a$ exists.

If $a_k = 0$ for each k and a=1, then the above conditions become necessary and sufficient for A to be regular. Toeplitz' Theorem also shows that a conservative matrix is a bounded operator, i.e.,

$$\sup_n \left|\sum_{k=0}^{\infty} a_{nk}x_k\right| \leq \sup_k |x_k| \sup_n \sum_{k=0}^{\infty} |a_{nk}| .$$

Let Φ be a holomorphic function in $|z|<R$, $R>1$. Then by taking powers of Φ, $[\Phi(z)]^n = \sum_{k=0}^{\infty} a_{nk}z^k$, a matrix $A = (a_{nk})$ is obtained where $a_{00} = 1$, and $a_{0k} = 0$ for $k = 1, 2, 3, \ldots$. If

$\Phi(z) = \dfrac{\alpha + (1-\alpha-\beta)z}{1-\beta z}$, then the resulting matrix is called a Karamata matrix.

Bajanski [4] has proved that if

i) Φ is holomorphic in $|z|<R$, $R>1$

ii) $|\Phi(z)|<1$ for $|z|\leq 1$. $z \neq 1$

iii) $\Phi(1) = 1$, and

iv) Re $A \neq 0$, where

$$\Phi(z) - z^{\alpha} = i^p A(z-1)^p + o(1)(z-1)^p \text{ as } z \to 1, \quad A \neq 0, \text{ and } \alpha = \Phi'(1),$$

then the matrix defined by $[\Phi(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k$ is regular. In this paper, he also has shown that necessary and sufficient conditions for a Karamata matrix to be regular (for real α, β) are $\alpha < 1$, $\beta < 1$, and $\alpha + \beta > 0$, or $\alpha = \beta = 0$.

Notation 1: If X and Y are Banach spaces, let $B[X,Y]$ denote the Banach space of all continuous linear operators from X to Y.

2. Bounded Linear Operators From A Banach Space X (With Schauder-Basis) To U.

<u>Theorem 2.1</u> In order for an operator A to belong to $B[X,U]$ (where X is any Banach-space with a Schauder-basis $(\hat{e}_0, e_1, \ldots)$) it is necessary and sufficient that

i) A is uniquely determined by a matrix (a_{nk}) satisfying

ii) $f_k(z) = \sum_{n=0}^{\infty} a_{nk}z^n$ belongs to U for each k, and

iii) $L_n = \{a_{nk}\}_{n=0}^{\infty}$ belongs to X' for each n and the continuous linear functionals defined by $F_{p,t} = \sum_{n=0}^{p} e^{int} L_n$ satisfy $\sup_{p,t} ||F_{p,t}||_{X'} < \infty$.

Proof: Assume that A belongs to $B[X,U]$. Let x belong to X. Then x can be written as $x = \sum_{k=0}^{\infty} x_k \hat{e}_k$. Since A is continuous and linear, we have $A(x) = \sum_{k=0}^{\infty} x_k A(\hat{e}_k)$. But $A(\hat{e}_k)$ belongs to U for each k. Hence,

(1) $A(\hat{e}_k) = \sum_{n=0}^{\infty} a_{nk}e_n$ where $e_n(z) = z^n$, and we obtain

(2) $A(x) = \sum_{k=0}^{\infty} x_k \sum_{n=0}^{\infty} a_{nk} e_n$. Since $A(x)$ belongs to U, we have

(3) $A(x) = \sum_{n=0}^{\infty} b_n e_n$.

We will show that $b_m = \sum_{k=0}^{\infty} a_{mk}x_k$. Without loss of generality, we can assume that $|||A||| = \sup_{||x||=1} ||A(x)||_U \neq 0$. Given $\varepsilon > 0$, there is an N such that $r > N$ implies that $||\sum_{k=r}^{\infty} x_k\hat{e}_k||_X < \frac{\varepsilon}{|||A|||}$.

Hence r>N implies that

$$\Big|\Big|\sum_{k=r}^{\infty} x_k\, A(\hat{e}_k)\Big|\Big|_U = \Big|\Big|\, A\Big(\sum_{k=r}^{\infty} x_k\, \hat{e}_k\Big)\Big|\Big|_U$$

$$\leq |||A||| \quad \Big|\Big|\sum_{k=r}^{\infty} x_k\, \hat{e}_k\Big|\Big|_X < \varepsilon \;.$$

Hence, $\sum_{k=0}^{r-1} x_k \sum_{n=0}^{\infty} a_{nk} e_n$ converges in U-norm to

$\sum_{k=0}^{\infty} x_k \sum_{n=0}^{\infty} a_{nk} e_n$. Let P_m be the m^{th} projection. Then, from (3),

$p_m(A(x)) = b_m$. From (2)

$$P_m(A(x)) = P_m\Big(\sum_{k=0}^{\infty} x_k \sum_{n=0}^{\infty} a_{nk} e_n\Big)$$

$$= P_m\Big(\lim_{r\to\infty} \sum_{k=0}^{r} x_k \sum_{k=0}^{\infty} a_{nk} e_n\Big)$$

$$= \lim_{r\to\infty} \sum_{k=0}^{r} x_k P_m \Big(\sum_{n=0}^{\infty} a_{nk} e_n\Big)$$

$$= \sum_{k=0}^{\infty} x_k\, a_{mk} \;.$$

Hence, $b_m = \sum_{k=0}^{\infty} a_{mk} x_k$, and

(4) $$A(X) = \sum_{n=0}^{\infty} e_n \sum_{k=0}^{\infty} a_{nk}\, x_k$$

We now have that A is completely determined by the matrix (a_{nk}) and from (1), we have $f_k = A(\hat{e}_k) = \sum_{n=0}^{\infty} a_{nk}e_n$ belongs to U for each k.

Moreover, from (4), we have that $\sum_{k=0}^{\infty} a_{nk}x_k$ must converge for each n and arbitrary x in X.

Hence, $L_n = \{a_{nk}\}_{k=0}^{\infty}$ is a continuous linear functional on X for each n. Now let p and t be given. Let x be an arbitrary member of X. Then

$$||A(x)||_U = \left|\left|\sum_{n=0}^{\infty} e_n \sum_{k=0}^{\infty} a_{nk}x_k\right|\right|_U$$

$$= \left|\sum_{n=0}^{p'} e^{int_0} \sum_{k=0}^{\infty} a_{nk}x_k\right|$$

(by Corollary 1.10 where p' may be ∞).

Hence,

$$||A(x)||_U = \left|\sum_{n=0}^{p'} e^{int_0} L_n(x)\right| \geq \left|\sum_{n=0}^{p} e^{int} L_n(x)\right| = \left| F_{p,t}(x)\right|.$$

Hence, $|||A||| = \sup_{||x||=1} ||A(x)|| \geq \sup_{||x||=1} |F_{p,t}(x)| = ||F_{p,t}||.$

Since $|||A||| < \infty$, and p and t are arbitrary, we have

$$\sup_{p,t} ||F_{p,t}||_{X'} \leq |||A||| < \infty .$$

Hence, conditions i), ii) and iii) are necessary.

Assume that (a_{nk}) is a matrix which satisfies ii) and iii). To show that $\sum_{n=0}^{\infty} e_n \sum_{k=0}^{\infty} a_{nk}x_k$ belongs to U for each x in X, we need to show that $\sum_{n=0}^{\infty} z^{n}L_n(x)$ converges uniformly on $|z|\leq 1$.

But this is equivalent to showing that $\sum_{n=0}^{\infty} e^{int}L_n(x)$ converges uniformly in t for each x. Hence, we need to show that the continuous linear functionals $\{F_{p,t}\}$ converges weakly and uniformly in t. As a consequence of the Uniform Boundness Principle $\{F_{p,t}\}$ will converge weakly and uniformly in t, provided

$$(5) \qquad F_{p,t}(\hat{e}_k) = \sum_{n=0}^{p} e^{int} L_n(\hat{e}_k) = \sum_{n=0}^{p} a_{nk}e^{int} \text{ converges}$$

uniformly in t for each k, and

$$(6) \qquad \sup_{p,t} ||F_{p,t}||_{X'} < \infty .$$

But conditions (5) and (6) are satisfied by (a_{nk}). Hence, $\sum_{n=0}^{\infty} e_nL_n(x)$ is in U for each x in X. Letting A denote the operator defined by (a_{nk}), we clearly have $A(x_1 + x_2) = A(x_1) + A(x_2)$ and $A(\alpha x) = \alpha A(x)$. To show that A belongs to $B[X,U]$, it suffices to show that A is bounded.

But
$$||A(x)||_U = ||\sum_{n=0}^{\infty} e_nL_n(x)||_U$$

$$= \sup_{p,t} |\sum_{n=0}^{p} e^{int}L_n(x)|$$

$$= \sup_{p,t} |F_{p,t}(x)|$$

$$\leq \sup_{p,t} ||F_{p,t}||_{X'} \; ||x||_X .$$

Hence, A is bounded, and $|||A||| = \sup_{p,t} ||F_{p,t}||_{X'}$.

Therefore, conditions ii and iii are sufficient.

For $X = \ell_1$, conditions ii and iii become $f_k = \sum_{n=0}^{\infty} a_{nk}e_n$ is in U for each k and $\sup_k ||f_k||_U < \infty$. This follows from the fact that

$$F_{p,t}(x) = \sum_{n=0}^{p} e^{int} \sum_{k=0}^{\infty} a_{nk}\, x_k$$

$$= \sum_{k=0}^{\infty} x_k \sum_{n=0}^{p} a_{nk}e^{int}$$

and $\qquad ||F_{p,t}||_{\ell_1'} = \sup_k |\sum_{n=0}^{p} a_{nk}e^{int}|$. Hence,

$$\sup_{p,t} ||F_{p,t}||_{\ell_1'} = \sup_k ||f_k||_U .$$

Hence, if an f in U is chosen such that $||f|| \leq 1$ and if we define $a_{nk} = a_n^k \quad k=1, 2, 3, \ldots$ where $f = \sum a_n e_n$, $a_{no} = 0$ for all n,

then the matrix (a_{nk}) is a continuous linear operator from ℓ_1 into U. Here, $f_k(z) = \sum_{n=0}^{\infty} a_n^k z^n$, and since U is a Banach algebra, we have $\sup_k ||f_k|| = \sup_k ||f^k|| \leq 1$. Where f^k denotes $f*f*f*\ldots*f$.

3. Karamata Type Operators on U.

If Φ is holomorphic in $|z| < R$, $R > 1$, and if $|\Phi(z)| \leq 1$ for $|z| \leq 1$, then for f belonging to U, we can consider the composition of f with Φ, $f(\Phi)$. $f(\Phi)$ will certainly be holomorphic in $|z| < 1$, and will be continuous on $|z| = 1$. The power series coefficients for $f \circ \Phi$ will be given by

$$b_n = \frac{1}{2\pi i} \int_{|z|=1} f(\Phi(z)) \frac{1}{z^{n+1}} \, dz$$

$$= \frac{1}{2\pi i} \int_{|z|=1} \sum_{k=0}^{\infty} a_k \Phi^k(z) \frac{1}{z^{n+1}} \, dz$$

$$= \sum_{k=0}^{\infty} a_k \frac{1}{2\pi i} \int_{|z|=1} \Phi^k(z) \frac{1}{z^{n+1}} \, dz \quad .$$

Letting $a_{nk} = \frac{1}{2\pi i} \int_{|z|=1} \Phi^k(z) \frac{1}{z^{n+1}} \, dz$, we have that

$$b_n = \sum_{k=0}^{\infty} a_{nk} a_k \quad .$$

Hence, if we define Γ by $\Gamma(f) = f \circ \Phi$, we can ask if Γ belongs to $B[U,U]$.

The following lemmas whose proofs are well known will be used extensively:

Lemma 2.2: If $0 \leq t < 1$, then for any $p \geq 0$

$$\frac{t^p}{(1-t)^{p+1}} = \sum_{k=p}^{\infty} \binom{k}{p} t^k \quad ; \quad \binom{k}{p} = \frac{k!}{p!(k-p)!} .$$

Lemma 2.3: If m and n are nonnegative integers with $m \geq n$, then $\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$.

Lemma 2.4: Abel summation:

$$\sum_{k=p+1}^{m} a_k b_k = A_m b_m - A_p b_{p+1} + \sum_{k=p+1}^{m-1} A_k (b_k - b_{k+1})$$

where $A_k = \sum_{r=0}^{k} a_r$.

Notation 2: $[x]$ will denote the greatest integer which is less than or equal to x.

Theorem 2.5: If $\Phi(z) = \alpha + (1-\alpha)z$, $0<\alpha<1$, then the operator Γ, defined by $\Gamma(f) = f\circ\Phi$, belongs to B[U,U].

Proof: For each k, we have $\Gamma(e_k) = (\alpha + (1-\alpha)e_1)^k = \Phi^k(z)$ which certainly belongs to U. Now $\Phi^k(z) = \sum_{n=0}^{k} \binom{k}{n} \alpha^{k-n}(1-\alpha)^n z^n$,

Hence,
$$a_{nk} = \begin{cases} \binom{k}{n}\left(\frac{1-\alpha}{\alpha}\right)^n \alpha^k & \text{if } n \le k \\ 0 & \text{if } n>k \end{cases}$$

Now $\sum_{k=0}^{\infty} a_{nk} a_k = \sum_{k=n}^{\infty} \binom{k}{n} \alpha^k a_k \left(\frac{1-\alpha}{\alpha}\right)^n$

$$= \left(\frac{1-\alpha}{\alpha}\right)^n \sum_{k=n}^{\infty} \binom{k}{n} \alpha^k a_k \text{ is absolutely convergent}$$

for each $f = \sum a_k e_k$ in U. Hence, the linear functional $L_n = \{a_{nk}\}_{k=0}^{\infty}$ belongs to U. We must now show that the continuous linear functionals $F_{p,t} = \sum_{n=0}^{p} e^{int} L_n$ satisfy $\sup_{p,t} ||F_{p,t}|| < \infty$.

Let f belong to U, $f = \sum a_k e_k$.

Then $F_{p,t}(f) = \sum_{n=0}^{p} e^{int}\left(\frac{1-\alpha}{\alpha}\right)^n \sum_{k=n}^{\infty} \binom{k}{n} \alpha^k a_k$.

Since the expression on the right is absolutely convergent, we can change the order of summation to obtain:

$$F_{p,t}(f) = \sum_{k=0}^{p} a_k \alpha^k \sum_{n=0}^{k} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n + \sum_{k=p+1}^{\infty} a_k \alpha^k \sum_{n=0}^{p} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n$$

$$= \Sigma_1 + \Sigma_2 \quad .$$

Now,

$$|\Sigma_1| = \left|\sum_{k=0}^{p} a_k \left[1 + \frac{1-\alpha}{\alpha} e^{it}\right]^k \alpha^k \right|$$

$$= \left|\sum_{k=0}^{p} a_k (\alpha + (1-\alpha) e^{it})^k \right|$$

$$\leq ||f|| \quad .$$

Let $m = \left[\frac{p}{1-\alpha}\right]$. Then we write

$$\Sigma_2 = \sum_{k=p+1}^{m} a_k \alpha^k \sum_{n=0}^{p} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n + \sum_{k=m+1}^{\infty} a_k \alpha^k \sum_{n=0}^{p} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n$$

$= \Sigma_{21} + \Sigma_{22}$. If $m = p$, then $\Sigma_{21} = 0$.

If $m = p+1$, then $|\Sigma_{21}| = \left|a_{p+1} \alpha^{p+1} \sum_{n=0}^{p} \binom{p+1}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n \right|$

Hence, if $m=p+1$,

$$|\Sigma_{21}| \leqq ||f|| \ \alpha^{p+1} \sum_{n=0}^{p+1} \binom{p+1}{n} \left[\frac{1-\alpha}{\alpha}\right]^n = ||f|| \ .$$

If $m>p+1$, then we write Σ_{21} as

$$\Sigma_{21} = \sum_{k=p+1}^{m} a_k \alpha^k \sum_{n=0}^{k} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n - \sum_{k=p+1}^{m} a_k \alpha^k \sum_{n=p+1}^{k} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n$$

$$= \Sigma_{211} - \Sigma_{212} \ .$$

$$|\Sigma_{211}| = \left|\sum_{k=p+1}^{m} a_k \left[\alpha + (1-\alpha)e^{it}\right]^k\right|$$

$$\leqq 2||f|| .$$

We will use Abel summation on Σ_{212}. Letting

$$b_k = \alpha^k \sum_{n=p+1}^{k} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n ,$$

we obtain $\Sigma_{212} = S_m b_m - S_p b_{p+1} + \Sigma_{2121}$

where

$$\Sigma_{2121} = \sum_{k=p+1}^{m-1} S_k \left[\alpha^k \sum_{n=p+1}^{k} \binom{k}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n - \alpha^{k+1} \sum_{n=p+1}^{k+1} \binom{k+1}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n\right]$$

$$= \sum_{k=p+1}^{m-1} S_k \alpha^k \sum_{n=p+1}^{k} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n \left(\binom{k}{n} - \alpha \binom{k+1}{n}\right) - \sum_{k=p+1}^{m-1} S_k \left((1-\alpha)e^{it}\right)^{k+1}$$

$$= \Sigma_{21211} - \Sigma_{21212} \ ,$$

$$|S_m b_m| = \left| S_m \alpha^m \sum_{n=p+1}^{m} \binom{m}{n} \left[\frac{1-\alpha}{\alpha} e^{it}\right]^n \right|$$

$$\leq |S_m| \alpha^m \sum_{n=0}^{m} \binom{m}{n} \left[\frac{1-\alpha}{\alpha}\right]^n$$

$$= |S_m| \quad \text{since} \quad \sum_{n=0}^{m} \binom{m}{n} \left(\frac{1-\alpha}{\alpha}\right)^n = \left(1 + \frac{1-\alpha}{\alpha}\right)^m$$

$$\leq ||f||.$$

Similarly,

$$|S_p b_{p+1}| \leq ||f||.$$

$$\left|\sum\nolimits_{21212}\right| \leq ||f|| \sum_{k=p+1}^{m-1} (1-\alpha)^{k+1}.$$

We observe that $\binom{k}{n} \geq \alpha \binom{k+1}{n}$ iff $k+1 \geq \frac{n}{1-\alpha}$.

Now in $\sum_{21211}$, $m-1 \geq k$. Hence, $\frac{p}{1-\alpha} \geq m \geq k+1$. But in $\sum_{21211}$, we have $n > p$. Hence, $\frac{n}{1-\alpha} > k+1$. Therefore,

$$\sum_{n=p+1}^{k} \left| \left(\frac{1-\alpha}{\alpha} e^{it}\right)^n \left(\binom{k}{n} - \alpha\binom{k+1}{n}\right) \right| = \sum_{n=p+1}^{k} \left(\frac{1-\alpha}{\alpha}\right)^n \left(\alpha\binom{k+1}{n} - \binom{k}{n}\right)$$

$$= \sum_{n=p+1}^{k} \left(\frac{1-\alpha}{\alpha}\right)^n \left(\alpha\binom{k}{n-1} - (1-\alpha)\binom{k}{n}\right)$$

$$= \sum_{n=p+1}^{k} \left(\frac{(1-\alpha)^n}{\alpha^{n-1}} \binom{k}{n-1} - \frac{(1-\alpha)^{n+1}}{\alpha^n} \binom{k}{n}\right)$$

$$= \frac{(1-\alpha)^{p+1}}{\alpha^p} \binom{k}{p} - \frac{(1-\alpha)^{k+1}}{\alpha^k} .$$

Thus,

$$\left|\Sigma_{21211}\right| \leqq ||f|| \left(\frac{(1-\alpha)^{p+1}}{\alpha^p} \sum_{k=p+1}^{m-1} \alpha^k \binom{k}{p} - \sum_{k=p+1}^{m-1} (1-\alpha)^{k+1} \right).$$

$$\begin{aligned} \left|\Sigma_{2121}\right| &\leqq \left|\Sigma_{21211}\right| + \left|\Sigma_{21212}\right| \\ &\leqq ||f|| \frac{(1-\alpha)^{p+1}}{\alpha^p} \sum_{k=p+1}^{m-1} \alpha^k \binom{k}{p} \\ &\leqq ||f|| \frac{(1-\alpha)^{p+1}}{\alpha^p} \sum_{k=p}^{\infty} \binom{k}{p} \alpha^k \\ &= ||f|| \text{ by Lemma 1.} \end{aligned}$$

Thus,

$$\begin{aligned} \left|\Sigma_{212}\right| &\leqq |S_m b_m| + |S_p b_{p+1}| + \left|\Sigma_{2121}\right| \\ &\leqq 3||f||. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\Sigma_{21}\right| &\leqq \left|\Sigma_{211}\right| + \left|\Sigma_{212}\right| \\ &\leqq 5||f||. \end{aligned}$$

For Σ_{22}, we observe that Abel summation, with

$$b_k = \alpha^k \sum_{n=0}^{p} \binom{k}{n} \left(\frac{1-\alpha}{\alpha} e^{it}\right)^n \quad \text{, gives us}$$

$$\sum_{k=m+1}^{r} a_k b_k = S_r b_r - S_m b_{m+1} + \sum_{k=m+1}^{r-1} S_k \alpha^k \sum_{n=0}^{p} \left(\frac{1-\alpha}{\alpha} e^{it}\right)^n \left(\binom{k}{n} - \alpha \binom{k+1}{n} \right).$$

We will show that the last sum on the right converges absolutely and since the sum on the left converges, we must have $\lim_{r \to \infty} S_r b_r$ existing. Hence,

$$\Sigma_{22} = \lim_{r} S_r b_r - S_m b_{m+1} + \Sigma_{221},$$

where $\Sigma_{221} = \sum_{k=m+1}^{\infty} S_k \alpha^k \sum_{n=0}^{p} (\frac{1-\alpha}{\alpha} e^{it})^n [\binom{k}{n} - \alpha\binom{k+1}{n}]$.

In Σ_{221}, $k \geqq m+1$. Hence, $k+1 \geqq m+2 > \frac{p}{1-\alpha} + 1 > \frac{n}{1-\alpha}$.

Hence,

$$\sum_{n=0}^{p} |(\frac{1-\alpha}{\alpha} e^{it})^n| \, |\binom{k}{n} - \alpha\binom{k+1}{n}| = \sum_{n=0}^{p} (\frac{1-\alpha}{\alpha})^n [\binom{k}{n} - \alpha\binom{k+1}{n}]$$

$$= \sum_{n=0}^{p} [\frac{(1-\alpha)^{n+1}}{\alpha^n} \binom{k}{n} - \frac{(1-\alpha)^n}{\alpha^{n-1}} \binom{k}{n-1}]$$

$$= \frac{(1-\alpha)^{p+1}}{\alpha^p} \binom{k}{p} .$$

Hence, $|\Sigma_{221}| \leqq ||f|| \frac{(1-\alpha)^{p+1}}{\alpha^p} \sum_{k=m+1}^{\infty} \binom{k}{p} \alpha^k$

$$\leqq ||f|| \frac{(1-\alpha)^{p+1}}{\alpha^p} \sum_{k=p}^{\infty} \binom{k}{p} \alpha^k$$

$$= ||f||.$$

$$|\lim_{r\to\infty} S_r b_r| \leqq \sup_{r>m} |S_r| \, |b_r|$$

$$\leqq ||f|| \sup_{r>m} \alpha^r \sum_{n=0}^{p} \binom{r}{n} (\frac{1-\alpha}{\alpha})^n$$

$$\leqq ||f||.$$

In like manner, $|S_m b_{m+1}| \leqq ||f||$. Hence, $|\Sigma_{22}| \leqq 3||f||$.

Hence, $|\Sigma_2| \leqq |\Sigma_{21}| + |\Sigma_{22}|$

$$\leqq 8||f||.$$

Finally,

$$|F_{p,t}(f)| \leqq |\Sigma_1| + |\Sigma_2| \leqq 9||f||.$$

Hence, $||F_{p,t}|| \leqq 9$. Since p, and t are arbitrary, we have

$$\sup_{p,t} ||F_{p,t}|| \leqq 9.$$

Hence, Γ belongs to $B[U,U]$.

In connection with the proof of Theorem 2.5, we observe that our linear functionals, L_n, are also given by

(7) $$L_n(f) = \frac{1}{2\pi i} \int_{|\phi(z)|=1} f(\phi(z)) \frac{1}{z^{n+1}} dz.$$

This follows from the fact $L_n(f) = \sum_{k=0}^{\infty} a_{nk} a_k = b_n$ where b_n is the n-th coefficient in the power series expansion of $f \circ \phi$. In fact, letting $w = \phi(z)$, we obtain $z = H(w) = \frac{w-\alpha}{1-\alpha}$, $dz = \frac{dw}{1-\alpha}$, and integrating around $|w| = 1$, we have

$$L_n(f) = \frac{1}{2\pi i} \int_{|w|=1} f(w) \frac{(1-\alpha)^n}{(w-\alpha)^{n+1}} dw.$$

By the calculus of residues, we have

$$L_n(f) = \frac{(1-\alpha)^n}{n!} \lim_{w \to \alpha} f^{(n)}(w)$$

$$= \left(\frac{1-\alpha}{\alpha}\right)^n \sum_{k=n}^{\infty} \binom{k}{n} a_k \alpha^k \quad \left(\text{since } f^{(n)}(\alpha) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k \alpha^{k-n}\right)$$

which is precisely the value that we should obtain.

<u>Theorem 2.6</u>: If $\lambda(z) = \frac{(1-\beta)z}{1-\beta z}$, $0 < \beta < 1$, then the operator Λ, defined by $\Lambda(f) = f \circ \lambda$, belongs to $B[U,U]$.

Proof: We have $\Lambda(e_k) = \left(\frac{(1-\beta)e_1}{1-\beta e_1}\right)^k$ belongs to U for each k.

$$a_{nk} = \frac{1}{2\pi i}\int_{|\lambda(z)|=1} \lambda^k(z)\frac{1}{z^{n+1}}\,dz \quad .$$

Let $w = \lambda(z)$. Then $z = \pi(w) = \frac{w}{1-\beta+\beta w}$, $dz = \frac{1-\beta}{(1-\beta+\beta w)^2}\,dw$.

Hence, for $n \geq 1$

$$a_{nk} = \frac{1-\beta}{2\pi i}\int_{|w|=1} \frac{w^k(1-\beta+\beta w)^{n-1}}{w^{n+1}}\,dw$$

$a_{00}=1 \qquad a_{n0}=0 \qquad n=1,2,\ldots.$

Using the calculus of residues, we obtain

$$a_{nk} = \begin{cases} \binom{n-1}{k-1}\beta^n\left(\frac{1-\beta}{\beta}\right)^k & n \geq k \quad n \neq 0 \quad k \neq 0 \\ 0 & n < k \\ 1 & n = 0,\ k = 0 \\ 0 & n \geq 1,\ k = 0 \quad . \end{cases}$$

Hence, $L_0(f) = a_0$, and $L_n(f) = \sum_{k=1}^{n} \binom{n-1}{k-1}\beta^n\left(\frac{1-\beta}{\beta}\right)^k a_k$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k}\beta^n\left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1}$$

for $n \geq 1$.

Let q and t be given, and let f be an element of U.

Then

$$F_{q,t}(f) = a_0 + \sum_{n=1}^{q}(\beta e^{it})^n \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1}$$

$$= a_0 + \sum_{k=0}^{q-1}\left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1} \sum_{n=k+1}^{q}\binom{n-1}{k}(\beta e^{it})^n$$

$$= a_0 + \sum_{k=0}^{q-1}\left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1} \sum_{n=k+1}^{\infty}\binom{n-1}{k}(\beta e^{it})^n$$

$$- \sum_{k=0}^{q-1}\left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1} \sum_{n=q+1}^{\infty}\binom{n-1}{k}(\beta e^{it})^n$$

$$= a_0 + \Sigma_1' - \Sigma_2'.$$

$$a_0 + \Sigma_1' = a_0 + \sum_{k=0}^{q-1}\left[\frac{(1-\beta)e^{it}}{1-\beta e^{it}}\right]^{k+1} a_{k+1}$$

$$= \sum_{k=0}^{q}\left[\frac{(1-\beta)e^{it}}{1-\beta e^{it}}\right]^{k} a_k$$

$$= \Sigma_1 .$$

Changing the order of summation in Σ_2' yields

$$\Sigma_2' = \sum_{n=q+1}^{\infty}(\beta e^{it})^n \sum_{k=0}^{q-1}\binom{n-1}{k}\left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1} .$$

Let $m = [\frac{q+1}{1-\beta}]$. If $m = q+1$, then Σ_2' remains as it is and nothing is lost. For $m > q+1$, we write Σ_2' as

$$\Sigma_2' = \sum_{n=q+1}^{m-1} (\beta e^{it})^n \sum_{k=0}^{q-1} \binom{n-1}{k} \left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1}$$

$$+ \sum_{n=m}^{\infty} (\beta e^{it})^n \sum_{k=0}^{q-1} \binom{n-1}{k} \left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1}$$

$$= \Sigma_2'' + \Sigma_3 .$$

We write Σ_2'' as

$$\Sigma_2'' = \sum_{n=q+1}^{m-1} (\beta e^{it})^n \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1}$$

$$- \sum_{n=q+1}^{m-1} (\beta e^{it})^n \sum_{k=q}^{n-1} \binom{n-1}{k} \left(\frac{1-\beta}{\beta}\right)^{k+1} a_{k+1}$$

$$= \Sigma_2''' - \Sigma_2 .$$

Hence,

$$F_{q,t}(f) = \Sigma_1 - \Sigma_2''' + \Sigma_2 - \Sigma_3 .$$

But,

$F_{q,t}(f) + \Sigma_2''' = F_{m-1,t}(f)$. Therefore,

$$F_{m-1,t}(f) = \Sigma_1 + \Sigma_2 - \Sigma_3 .$$

$$|\Sigma_1| = \left| \sum_{k=0}^{q} a_k \left[\frac{(1-\beta)e^{it}}{1-\beta e^{it}} \right]^k \right| \leq ||f||.$$

We use Abel summation on the inner sum of Σ_2. Letting

$$b_k = \binom{n-1}{k}\left(\frac{1-\beta}{\beta}\right)^{k+1},$$

we obtain

$$\sum_{k=q}^{n-1} a_{k+1} b_k = S_n b_{n-1} - S_q b_q + \sum_{k=q}^{n-2} S_{k+1}(b_k - b_{k+1})$$

$$= S_n\left(\frac{1-\beta}{\beta}\right)^n - S_q\binom{n-1}{q}\left(\frac{1-\beta}{\beta}\right)^{q+1}$$

$$+ \sum_{k=q}^{n-2} S_{k+1}\left(\frac{1-\beta}{\beta}\right)^{k+1}\left[\binom{n-1}{k} - \frac{1-\beta}{\beta}\binom{n-1}{k+1}\right].$$

Hence, we can write Σ_2 as

$$\Sigma_2 = \Sigma_{21} - \Sigma_{22} + \Sigma_{23}.$$

$$|\Sigma_{21}| = \left|\sum_{n=q+1}^{m-1} (\beta e^{it})^n S_n\left(\frac{1-\beta}{\beta}\right)^n\right| \leq ||f|| \sum_{n=q+1}^{m-1} (1-\beta)^n$$

$$|\Sigma_{22}| = \left|S_q\left(\frac{1-\beta}{\beta}\right)^{q+1} \sum_{n=q+1}^{m-1} \binom{n-1}{q}(\beta e^{it})^n\right|$$

$$\leq ||f|| \left(\frac{1-\beta}{\beta}\right)^{q+1} \sum_{n=q+1}^{m-1} \binom{n-1}{q} \beta^n .$$

We observe that

$$\binom{n-1}{k} \geq \frac{1-\beta}{\beta}\binom{n-1}{k+1} \quad \text{iff} \quad (k+1) \geq (1-\beta)n.$$

In Σ_{23}, $n \leq m-1 \leq \frac{q+1}{1-\beta} - 1 = \frac{q+\beta}{1-\beta}$.

Hence, $q+\beta \geq (1-\beta)n$. But in Σ_{23}, $k \geq q$. Hence, $(k+1) \geq q+\beta \geq (1-\beta)n$.

Therefore,

$$|\Sigma_{23}| \leqq ||f|| \sum_{n=q+1}^{m-1} \beta^n \sum_{k=q}^{n-2} [(\tfrac{1-\beta}{\beta})^{k+1} \binom{n-1}{k} - (\tfrac{1-\beta}{\beta})^{k+2} \binom{n-1}{k+1}]$$

$$= ||f||[(\tfrac{1-\beta}{\beta})^{q+1} \sum_{n=q+1}^{m-1} \binom{n-1}{q} \beta^n - \sum_{n=q+1}^{m-1} (1-\beta)^n].$$

Finally, we obtain

$$\begin{aligned} |\Sigma_2| &\leqq |\Sigma_{21}| + |\Sigma_{22}| + |\Sigma_{23}| \\ &\leqq 2||f|| (\tfrac{1-\beta}{\beta})^{q+1} \sum_{n=q+1}^{m-1} \binom{n-1}{q} \beta^n \\ &\leqq 2||f|| \end{aligned}$$

since $(\frac{1-\beta}{\beta})^{q+1} \sum_{n=q+1}^{m-1} \binom{n-1}{q} \beta^n = \frac{(1-\beta)^{q+1}}{\beta^q} \sum_{n=q}^{m-2} \binom{n}{q} \beta^n \leqq 1.$

We use Abel summation on the inner sum of Σ_3. Letting

$$b_k = \binom{n-1}{k} (\tfrac{1-\beta}{\beta})^{k+1} ,$$

we obtain

$$\sum_{k=0}^{q-1} a_{k+1} b_k = S_q b_{q-1} - a_0(\tfrac{1-\beta}{\beta}) + \sum_{k=0}^{q-2} S_{k+1}[(\tfrac{1-\beta}{\beta})^{k+1} \binom{n-1}{k} - (\tfrac{1-\beta}{\beta})^{k+2} \binom{n-1}{k+1}].$$

Therefore, we can write Σ_3 as

$\Sigma_3 = \Sigma 31 - \Sigma 32 + \Sigma 33$

$$|\Sigma_{31}| = |S_q(\frac{1-\beta}{\beta})^q \sum_{n=m}^{\infty} (\beta e^{it})^n \binom{n-1}{q-1}|$$

$$\leq ||f|| \; (\frac{1-\beta}{\beta})^q \sum_{n=m}^{\infty} \beta^n \binom{n-1}{q-1}$$

$$|\Sigma_{32}| = |a_0(\frac{1-\beta}{\beta}) \sum_{n=m}^{\infty} (\beta e^{it})^n|$$

$$\leq ||f|| \; (\frac{1-\beta}{\beta}) \sum_{n=m}^{\infty} \beta^n \quad .$$

In Σ_{33}, we have $(1-\beta)n > k+1$. Hence,

$$|\Sigma_{33}| \leq ||f|| \sum_{n=m}^{\infty} \beta^n \sum_{k=0}^{q-2} \left((\frac{1-\beta}{\beta})^{k+2} \binom{n-1}{k+1} - (\frac{1-\beta}{\beta})^{k+1} \binom{n-1}{k} \right)$$

$$= \; ||f|| \left((\frac{1-\beta}{\beta})^q \sum_{n=m}^{\infty} \binom{n-1}{q-1} \beta^n - (\frac{1-\beta}{\beta}) \sum_{n=m}^{\infty} \beta^n \right) .$$

Finally, we have

$$|\Sigma_3| \leq |\Sigma_{31}| + |\Sigma_{32}| + |\Sigma_{33}|$$

$$\leq 2||f|| \; (\frac{1-\beta}{\beta})^q \sum_{n=m}^{\infty} \binom{n-1}{q-1} \beta^n$$

$$\leq 2||f|| \quad .$$

Hence, $|F_{m-1,t}(f)| \leq |\Sigma_1| + |\Sigma_2| + |\Sigma_3|$

$$\leq 5||f||.$$

We now have that for any positive integer q,

$$|F_{m-1,t}(f)| \leq 5||f||, \text{ where } m = [\frac{q+1}{1-\beta}] .$$

We now show that there exists an integer h such that if q is any integer, then there are at most h integers between

$$m_1 = [\frac{q+1}{1-\beta}] \text{ and } m_2 = [\frac{q+2}{1-\beta}].$$

Let $h = [\frac{1}{1-\beta}] + 2$. Then

$$\begin{aligned} h + m_1 &= \left([\tfrac{1}{1-\beta}] + 2\right) + [\tfrac{q+1}{1-\beta}] \\ &\geqq [\tfrac{1}{1-\beta}] + 2 + \tfrac{q+1}{1-\beta} - 1 \\ &= [\tfrac{1}{1-\beta}] + 1 + \tfrac{q+1}{1-\beta} \\ &\geqq \frac{1}{1-\beta} + \frac{q+1}{1-\beta} \\ &= \frac{q+2}{1-\beta} \\ &\geqq [\tfrac{q+2}{1-\beta}] = m_2. \end{aligned}$$

Hence, $h \geqq m_2 - m_1$ and h is independent of q.

Now for any q and associated m, we have

$$\begin{aligned} |F_{m,t}(f)| &= |F_{m-1,t}(f) + (\beta e^{it})^m \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\tfrac{1-\beta}{\beta}\right)^{k+1} a_{k+1}| \\ &\leqq |F_{m-1,t}(f)| + \beta^m ||f|| \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\tfrac{1-\beta}{\beta}\right)^{k+1} \\ &\leqq 5||f|| + ||f|| (1-\beta) \\ &= ||f|| (5 + (1-\beta)). \end{aligned}$$

$$|F_{m+1,t}(f)| = |F_{m,t}(f) + (\beta e^{it})^{m+1} \sum_{k=0}^{m} \binom{m}{k} \left(\tfrac{1-\beta}{\beta}\right)^{k+1} a_{k+1}|.$$

Therefore,

$$|F_{m+1,t}(f)| \leq |F_{m,t}(f)| + \beta^{m+1}||f|| \sum_{k=0}^{m} \binom{m}{k} \left(\frac{1-\beta}{\beta}\right)^{k+1}$$

$$\leq ||f|| \ (5 + (1-\beta)) + ||f||(1-\beta)$$

$$= ||f|| \ (5 + 2(1-\beta)) .$$

Continuing in this manner, we obtain

$$|F_{m+h-2,t}(f)| \leq ||f|| \quad (5 + [\tfrac{1}{1-\beta}] \ (1-\beta)) .$$

Since we have at most h points between m_1-1 and m_2-1, $m_1+h-1 \geq m_2-1$. Hence, we have $|F_{m_1+k,t}(f)| \leq ||f|| \ (5 + k(1-\beta))$ where $1 \leq k \leq [\frac{1}{1-\beta}]$. Hence, $k \leq \frac{1}{1-\beta}$ and $k(1-\beta) \leq 1$. Therefore, for any integer p and given t, $|F_{p,t}(f)| \leq 6||f||$. Finally, we have $\sup_{p,t} ||F_{p,t}|| \leq 6$ and Theorem 2.6 is proved.

Corollary 2.7: If $\Psi(z) = \frac{\alpha + (1-\alpha-\beta)z}{1-\beta z}$, $0<\alpha<1$, $0<\beta<1$, then the operator T defined by $T(f) = f\circ\Psi$ belongs to B[U,U].

Proof: Let $\Phi(z) = \alpha+(1-\alpha)z$, $\lambda(z) = \frac{(1-\beta)z}{1-\beta z}$

Then Γ and Λ are members of B[U,U]. Since B[U,U] is a Banach algebra under compositions, we have $\Lambda\circ\Gamma$ belongs to B[U,U], but

$$\Lambda(\Gamma(f)) = \Lambda \ (f\circ\Phi)$$

$$= (f\circ\Phi)\circ\lambda$$

$$= f\circ(\Phi\circ\lambda)$$

But $\Phi(\lambda(z)) = \alpha + (1-\alpha)\frac{(1-\beta)z}{1-\beta z}$

$$= \frac{\alpha + (1-\alpha-\beta)z}{1-\beta z} = \Psi(z)$$

Hence $T \in$ B[U,U].

4. Bajanski Type Operators On U

The goal of this section is to show that certain Bajanski type functions operate on U under composition of functions. The method used consists of comparing a given Bajanski type operator with a Karamata type operator.

Theorem 2.8: If

i) Φ is holomorphic in $|z| < R$, $R>1$

ii) $|\Phi(z)| < 1$ for $|z|\leqq 1$ $z\neq 1$

iii) $\Phi(1) = 1$

iv) Re $A\neq 0$, where

$$\Phi(z) - z^{\gamma} = -A(z-1)^2 + o(1)\ (z-1)^2 \quad \text{as } z\to 1,$$

$$\gamma = \Phi'(1)$$

v) $\Phi'(1) > 0$, $\Phi''(1) > 0$, and $(\Phi'(1))^2 < \Phi'(1) + \dfrac{\Phi''(1)}{2}$,

then the operator Λ defined by $\Lambda(f) = f\circ\Phi$ belongs to B[U,U].

One should note that the first four hypotheses serve to identify the Bajanski type functions,and that the Karamata type functions with $0<\alpha< 1$, $0<\beta<1$, are included in the Bajanski type functions. The fifth hypothesis is added for comparison purposes with Karamata-type functions.

It will facilitate the proof of Theorem 2.8 to first prove a few lemmas.

Lemma 2.9: If Φ obey the hypotheses of Theorem 2.8, then Φ has a local inverse Ψ in a neighborhood of z=1, such that for $|\theta|$ sufficiently small,

$$|\Psi(e^{it}) - e^{i\theta}|^2 \geqq k(\theta^4 + (t-\tau)^2)$$

where $\tau = \arg(\Phi(e^{i\theta}))$ and k is a constant independent of θ .

Proof: Since $\phi'(1)>0$, a local inverse to ϕ, ψ, exists in a neighborhood of $z=1$, and $\psi'(1)>0$. One can clearly choose a neighborhood N about $z=1$ such that

$$\mathrm{Re}\ \psi'(z) > \frac{\Psi'(1)}{2} > 0 \quad \text{if } z \in N.$$

Let ζ be a number such that $\Phi(\zeta)$ is in N. Integrating along the line segment joining $\Phi(\zeta)$ and z in N, one obtains

$$\Psi(z) - \zeta = \int_{\Phi(\zeta)}^{z} \Psi'(\xi)d\xi \quad .$$

Let $\xi = \Phi(\zeta) + \lambda(z-\Phi(\zeta))$, then $d\xi = (z-\Phi(\zeta))d\lambda$ and

$$|\Psi(z) - \zeta| = |z - \Phi(\zeta)| \left| \int_0^1 \Psi'(\Phi(\zeta) + \lambda(z - \Phi(\zeta))d\lambda \right|$$

$$\geqq |z - \Phi(\zeta)| \left| \int_0^1 \mathrm{Re}\ \Psi' d\lambda \right|$$

$$\geqq \frac{\Psi'(1)}{2} |z - \Phi(\zeta)|.$$

Hence,

(8) $\qquad |\Psi(z) - \zeta|^2 \geqq C|z - \Phi(\zeta)|^2$ where $C = \dfrac{\Psi'(1)}{2}$ is independent of ζ.

Let $g(t) = |e^{it} - \Phi(\zeta)|^2$, and let τ denote the point where g attains a minimum.

Since

$$|e^{it} - \Phi(\zeta)| \geqq 1 - |\Phi(\zeta)| ,$$

it is clear that $\tau = \arg \Phi(\zeta)$. Let $\zeta = e^{i\theta}$. Then,

$$1 - |\Phi(e^{i\theta})| = \frac{1 - |\Phi(e^{i\theta})|^2}{1 + |\Phi(e^{i\theta})|} .$$

From hypothesis iv) of Theorem 2.8 ,

$$\Phi(z) = z^{\gamma} - A(z-1)^2 + o(1)(z-1)^2 \quad \text{as } z \to 1$$

$$\frac{\Phi(z)}{z^{\gamma}} = 1 - A\,[1 + (z-1)]^{-\gamma}(z-1)^2 + o(1)(z-1)^2 \quad \text{as } z \to 1$$

$$= 1 - A(z-1)^2 + o(1)(z-1)^2 \quad \text{as } z \to 1 .$$

Hence,

$$|\Phi(e^{i\theta})|^2 = |1 - A(e^{i\theta} - 1)^2 + \ldots|^2 \quad \text{as } \theta \to 0$$

$$= |1 + A\theta^2 + \ldots|^2 \qquad \text{as } \theta \to 0$$

$$= 1 + 2 \operatorname{Re} A\theta^2 + \text{larger powers of } \theta .$$

Hence,

$$1 - |\Phi(e^{i\theta})| = \frac{1 - |\Phi(e^{i\theta})|^2}{1 + |\Phi(e^{i\theta})|}$$

$$= \frac{-\theta^2 2 \operatorname{Re} A + \text{large powers of } \theta}{1 + |\Phi(e^{i\theta})|} .$$

Therefore ,

$$\frac{1 - |\Phi(e^{i\theta})|}{\theta^2} \to -\operatorname{Re} A \quad \text{as } \theta \to 0 \ (\operatorname{Re} A < 0).$$

Hence, $\dfrac{1 - |\Phi(e^{i\theta})|}{\theta^2}$ is bounded in a neighborhood of $\theta = 0$. Hence, there exists $C_1 > 0$ such that

$$1 - |\Phi(e^{i\theta})| \geqq C_1 \theta^2.$$

Hence,

$$g(\tau) = (1 - |\Phi(e^{i\theta})|)^2 \geqq C_1^2 \theta^4. \tag{9}$$

Expanding g in a neighborhood of $t=\tau$, one obtains

$$g(t) = g(\tau) + \frac{g''(T)}{2} (t-\tau)^2 , \tag{10}$$

where T is between t and τ, and

$$g''(T) = 2\mathrm{Re}\,\Phi(e^{i\theta}) \cos T + 2 \,\mathrm{Im}\, \Phi(e^{i\theta}) \sin T.$$

For $e^{i\theta}$ in N, there exists $\eta > 0$ such that

(11) $g''(T) \geqq \eta > 0$ independent of θ.

From 8, 9, 10, and 11, one obtains the existences of a $k > 0$ such that

$$|\Psi(e^{it}) - e^{i\theta}|^2 \geqq k(\theta^4 + (t-\tau)^2.$$

Before the next Lemma is stated, observe that for $h(z) = \dfrac{\alpha + (1-\alpha-\beta)z}{1 - \beta z}$, the power series development is given in a neighborhood of $z=1$ by

$$h(z) = 1 + (A+B) \sum_{k=1}^{\infty} B^{k-1} (z-1)^k ,$$

where $A = \dfrac{1-\alpha-\beta}{1-\beta}$, $B = \dfrac{\beta}{1-\beta}$.

Hence, $h'(1) = A+B = \dfrac{1-\alpha}{1-\beta}$, $\dfrac{h''(1)}{2} = B(A+B) = \dfrac{\beta(1-\alpha)}{(1-\beta)^2}$.

Lemma 2.10: If a, b>0 with $\frac{a^2}{a+b} < 1$, then there exist real numbers α and β such that

i) $0<\alpha<1$, $0<\beta<1$ and

ii) $\frac{1-\alpha}{1-\beta} = a$, and $\frac{\beta(1-\alpha)}{(1-\beta)^2} = b$.

Proof: Let $\alpha = 1 - \frac{a^2}{a+b}$, $\beta = \frac{b}{a+b}$.

If Φ is a function which obey the hypotheses of Theorem 2.8, then from Lemma 2.10, there exists α, β, such that

$$h(z) = \frac{\alpha + (1-\alpha-\beta)z}{1 - \beta z}$$

satisfies $h(1) = \Phi(1) = 1$; $h'(1) = \Phi'(1)$, and $h''(1) = \Phi''(1)$. Also, the local inverse to Φ,Ψ, in a neighborhood of z=1 satisfies

$$\Psi(1) = 1, \quad \Psi'(1) = \frac{1}{\Phi'(1)}, \quad \text{and} \quad \Psi''(1) = -\frac{\Phi''(1)}{(\Phi'(1))^3}.$$

If H is the local inverse to h in a neighborhood about z=1,

$H(1) = \Psi(1)$, $H'(1) = \Psi'(1)$, and $H''(1) = \Psi''(1)$.

Therefore,

(12) $$|H(e^{it}) - \Psi(e^{it})| = 0(t^3) \quad \text{as } t\to 0$$

(13) $$|H'(e^{it}) - \Psi'(e^{it})| = 0(t^2) \text{ as } t\to 0$$

Lemma 2.11: If τ is a function of θ such that $\frac{\tau}{\theta} = 0(1)$ as $\theta \to 0$, then

$$\int_a^b \frac{t^2\,dt}{\theta^4 + (t-\tau)^2} = 0(1) \qquad \text{as } \theta \to 0.$$

Proof: Let $u = t-\tau$, then

$$\int_a^b \frac{t^2\,dt}{\theta^4 + (t-\tau)^2} = \int_{a-\tau}^{b-\tau} du + 2\tau \int_{a-\tau}^{b-\tau} \frac{udu}{u^2+\theta^4} + (\tau^2-\theta^4)\int_{a-\tau}^{b-\tau} \frac{du}{u^2+\theta^4}.$$

$$\int_{a-\tau}^{b-\tau} du = (b-a) = 0(1) \qquad \text{as } \theta \to 0.$$

$$2\tau \int_{a-\tau}^{b-\tau} \frac{udu}{u^2+\theta^4} = \tau \log \frac{(b-\tau)^2+\theta^4}{(a-\tau)^2+\theta^4} = 0(1) \qquad \text{as } \theta \to 0.$$

$$(\tau^2-\theta^4)\int_{a-\tau}^{b-\tau} \frac{du}{u^2+\theta^4} = \frac{\tau^2-\theta^4}{\theta^2}\left[\tan^{-1}\left(\frac{b-\tau}{\theta^2}\right) - \tan^{-1}\left(\frac{a-\tau}{\theta^2}\right)\right].$$

Since $\tan^{-1}x$ is bounded as $x\to\infty$, and since $\frac{\tau}{\theta} = 0(1)$ as $\theta\to 0$, the last integrand is $0(1)$ as $\theta \to 0$.

Corollary 2.12: If τ is a function of θ such that $\frac{\tau}{\theta} = 0(1)$ as $\theta \to 0$, then

$$\int_a^b \frac{|t|^n\,dt}{[\theta^4 + (t-\tau)^2]^{1/2}} = 0(1) \qquad \text{as } \theta \to 0, \quad \text{for } n \geq 1.$$

Proof:

$$\int_a^b \frac{|t|^n dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} \leq \sup_{a\leq t\leq b} |t|^{n-1} \int_a^b \frac{|t|\,dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} .$$

$\sup_{a\leq t\leq b} |t|^{n-1} = 0(1)$, and the Cauchy-Schwarz inequality

yields

$$\left(\int_a^b \frac{|t|\,dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}}\right)^2 \leq \int_a^b dt \int_a^b \frac{t^2 dt}{\theta^4 + (t-\tau)^2} = 0(1)$$

as $\theta \to 0$.

Lemma 2.13: If τ is a function of θ such that $\frac{\tau}{\theta} = 0(1)$ as $\theta\to 0$, and $c > 0$ a constant independent of θ, then

$$I(p,\theta) = \int_a^b \frac{|t|^3 pe^{-pct^2}\,dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} = 0(1) \quad \text{as } \theta \to 0,\ p \to \infty.$$

Proof: If 0 is between a and b,

$$I(p,\theta) = \int_a^0 \frac{-t^3 pe^{-cpt^2}\,dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} + \int_0^b \frac{t^3 pe^{-cpt^2}\,dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}}$$

$$= I_1(p,\theta) + I_2(p,\theta).$$

In $I_1(p,\theta)$, let

$$u = \frac{t^2}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} \qquad dv = -pte^{-cpt^2}dt.$$

Then

$$du = \frac{2t[\theta^4 + (t-\tau)^2] - t^2(t-\tau)}{[\theta^4 + (t-\tau)^2]^{3/2}}\, dt \text{ and } v = \frac{1}{2c}e^{-cpt^2} .$$

Hence,

$$I_1(p,\theta) = \left.\frac{t^2e^{-cpt^2}}{2c[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}}\right|_a^0$$

$$- \frac{1}{2c}\int_a^0 e^{-pct^2}\, \frac{2t[\theta^4 + (t-\tau)^2] - t^2(t-\tau)}{[\theta^4 + (t-\tau)^2]^{3/2}}\, dt$$

$$= -\frac{a^2e^{-cpa^2}}{2c[\theta^4 + (a-\tau)^2]^{\frac{1}{2}}} - I_{11}(p,\theta).$$

The first expression on the right is clearly bounded as $\theta \to 0$, $p \to \infty$. The integral on the right can be expressed as

$$I_{11}(p,\theta) = \frac{1}{2c}\int_a^0 c^{-pct^2}\, \frac{t^2(t-\tau)}{[\theta^4 + (t-\tau)^2]^{3/2}}\, dt - \frac{1}{2c}\int_a^0 e^{-pct^2}\, \frac{2t\quad dt}{[\theta^4+(t-\tau)^2]^{\frac{1}{2}}}$$

$$= \frac{1}{2c}\, I_{111}(p,\theta) - \frac{1}{2c}\, I_{112}(p,\theta) .$$

Now $$|I_{112}(p,\theta)| \leq 2\int_a^0 \frac{|t|\, dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} = O(1) \text{ as } \theta \to 0,\ p \to \infty$$

by Corollary 2.12.

$$|I_{111}(p,\theta)| \leq \int_a^0 \frac{t^2 |t-\tau|\, dt}{[\theta^4 + (t-\tau)^2]^{3/2}} \quad .$$

Assume that $a<\tau<0$. If this does not prevail then nothing is lost. Then

$$\int_a^0 \frac{t^2\ |t-\tau| dt}{[\theta^4 + (t-\tau)^2]^{3/2}} = \int_a^\tau \frac{t^2 |t-\tau| dt}{[\theta^4 + (t-\tau)^2]^{3/2}} + \int_\tau^0 \frac{t^2\ |t-\tau| dt}{[\theta^4 + (t-\tau)^2]^{3/2}}$$

$$= -\int_a^\tau \frac{t^2(t-\tau)\ \ dt}{[\theta^4 + (t-\tau)^2]^{3/2}} + \int_\tau^0 \frac{t^2(t-\tau)dt}{[\theta^4 + (t-\tau)^2]^{3/2}}$$

$$= -I'_{111}(\theta) + I''_{111}(\theta).$$

It suffices to examine $I''_{111}(\theta)$.

Let $u = t^2$ $\quad dv = \dfrac{t-\tau\ dt}{[\theta^4 + (t-\tau)^2]^{3/2}}$. Then, $du = 2tdt$

$v = \dfrac{-1}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}}$. Hence,

$$I''_{111}(\theta) = \frac{\tau^2}{\theta^2} + 2 \int_\tau^0 \frac{t \quad dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} \quad .$$

Now, $\dfrac{\tau^2}{\theta^2} = 0(1)$ as $\theta \to 0$, and

$$\left| \int_\tau^0 \frac{t \quad dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} \right| \leq \int_\tau^0 \frac{|t| \quad dt}{[\theta^4 + (t-\tau)^2]^{\frac{1}{2}}} = 0(1) \quad \text{as } \theta \to 0$$

by Corollary 2.12.

Hence, $\quad |I_{111}(p,\theta)| = O(1) \quad$ as $\theta \to 0$, $p \to \infty$.

Hence, $|I_{11}(p,\theta)| = O(1)$ as $\theta \to 0$, $p \to \infty$.

Therefore, $|I_1(p,\theta)| = O(1)$ as $\theta \to 0$, $p \to \infty$.

In like manner,

$$|I_2(p,\theta)| = O(1) \quad \text{as } \theta \to 0, \ p \to \infty.$$

This proves Lemma 2.13.

Lemma 2.14: If Φ obey the hypotheses of Theorem 2.8, then $\tau = \arg \Phi(e^{i\theta})$ obeys $\frac{\tau}{\theta} = O(1)$ as $\theta \to 0$.

Proof:

$$\Phi(re^{i\theta}) = u(r,\theta) + iv(r,\theta),$$ and u and v are continuously differentiable. Moreover, $v(1,0) = 0$.

$$\arg (\Phi(e^{i\theta})) = \tan^{-1} \frac{v(1,\theta)}{u(1,\theta)},$$

By L'Hopitals rule,

$$\lim_{\theta \to 0} \frac{1}{\theta} \tan^{-1} \frac{v(1,\theta)}{u(1,\theta)} = \frac{\partial v}{\partial \theta}(1,0).$$

Hence, $\quad \frac{\tau}{\theta} = O(1) \quad$ as $\theta \to 0$.

Lemma 2.15: If Φ obey the hypotheses of Theorem 2.8, and if $|t|$ is sufficiently small, there exists a constant $c>0$ such that the inverse function to Φ, H, obeys,

$$|H(e^{it})| \geqq e^{ct^2}.$$

Proof: Let $\gamma = \Phi'(1)$. Then

$$z^{1/\gamma} = (1 + (z-1))^{1/\gamma}$$

$$= 1 + \frac{1}{\gamma}(z-1) + \frac{1}{2\gamma}(\frac{1}{\gamma} - 1)(z-1)^2 + \ldots .$$

Observe that from hypothesis v) that $\Phi'(1)>0$, $\Phi''(1)>0$, and $(\phi(1))^2 < \Phi'(1) + \frac{\Phi''(1)}{2} < \Phi'(1) + \Phi''(1)$.

Hence,

let $$c_1 = \frac{\Phi'(1) + \Phi''(1) - (\Phi'(1))^2}{2(\Phi'(1))^3} > 0.$$

$$H(z) - z^{1/\gamma} = 1 + H'(1)(z-1) + \frac{H''(1)}{2}(z-1)^2 + \ldots.$$

$$- [1 + H'(1)(z-1) + \frac{1}{2} H'(1)(H'(1) - 1)(z-1)^2 + \ldots]$$

$$= \frac{H''(1) - (H'(1))^2 + H'(1)}{2}(z-1)^2 + \ldots .$$

$$H''(1) - (H'(1))^2 + H'(1) = -\frac{\Phi''(1)}{(\Phi'(1))^3} - \frac{1}{(\Phi'(1))^2} + \frac{1}{\Phi'(1)}$$

$$= \frac{-\Phi''(1) - \Phi'(1) + (\Phi'(1))^2}{(\Phi'(1))^3}$$

$$= -2c_1 .$$

Hence,

$$H(e^{it}) - e^{it/\gamma} = -c_1(e^{it}-1)^2 + o(1)(e^{it}-1)^2 \quad \text{as } t \to 0 .$$

$$\frac{H(e^{it})}{e^{it/\gamma}} = 1 - c_1(e^{it}-1)^2 + o(1)(e^{it}-1)^2 \quad \text{as } t \to 0.$$

Since $(e^{it}-1)^2 = [it + \frac{(it)^2}{2} + \frac{(it)^3}{3!} + \ldots]^2$,

$$|H(e^{it})| \geqq 1 + c_1 t^2 + \ldots$$

$$\geqq 1 + \frac{c_1}{2} t^2 \quad \text{for } |t| \text{ sufficiently small.}$$

Now, $$e^{(c_1 t^2)/4} = 1 + \frac{c_1}{4} t^2 + \left(\frac{c_1}{4}\right)^2 \frac{t^4}{2} + \ldots .$$

For $|t|$ sufficiently small,

$$1 + \frac{c_1}{2} t^2 \geq 1 + \frac{c_1}{4} t^2 + (\frac{c_1}{4})^2 \frac{t^4}{2} + \ldots, \text{ since}$$

$$c_1 \geq \frac{c_1}{2} + 2(\frac{c_1}{4})^2 \frac{t^2}{2} + 2(\frac{c_1}{4})^3 \frac{t^4}{3!} + \ldots,$$

since $2(\frac{c_1}{4})^2 \frac{t^2}{2} + 2(\frac{c_1}{4})^3 \frac{t^4}{3!} + \ldots$ can be made arbitrarily small by choosing t sufficiently small. Hence, letting $c = \frac{c_1}{4}$, one obtains $|H(e^{it})| \geq e^{ct^2}$ for $|t|$ sufficiently small.

<u>Proof of Theorem 2.8</u>: Since $|\Phi(z)|<1$ for $|z|\leq 1$, $z\neq 1$, a curve Γ' can be chosen such that Γ' surrounds the unit circle and touches it only at $z=1$, and Γ' is inside the set of points where $|\Phi(z)| = 1$ $(z \neq 1)$, and $\Gamma'=\{z:|\Phi(z)|=1\}$ for z close to 1.

Since $\Lambda(e_k) = \Phi^k$ is in U for each k, it suffices to show that

$$\sup_{p,\theta} |G(f,p,\theta)| = \sup_{p,\theta} \left| \sum_{k=0}^{p} e^{ik\theta} \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\Phi(z))}{z^{k+1}} dz \right|$$

is finite for each $f \varepsilon U$.

It will facilitate the following discussion to assume that we have chosen a neighborhood, N, about z=1 so small that all the assertions which follow hold.

Denote the part of the curve, Γ', inside N by γ, and the part outside by Γ. In N, $\gamma = \{z:|\Phi(z)|=1\}$.

$$G(f,p-1,\theta) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\Phi(z))}{z} \sum_{k=0}^{p-1} (\frac{e^{i\theta}}{z})^k dz + \frac{1}{2\pi i}\int_{\gamma} \frac{f(\Phi(z))}{z} \sum_{k=0}^{p-1} (\frac{e^{i\theta}}{z})^k dz.$$

Hence, $G(f, p-1, \theta) = I_\Gamma + I_\gamma$.

Now, there exists $\delta > 0$ such that $|z| > 1 + \delta$ on Γ. Hence,

(14) $$\sup_{p,\theta} |I_\Gamma| < \infty.$$

Let $w = \Phi(z)$ for z on γ, so that $z = \Psi(w)$ [assuming that N is small enough so that Φ has Ψ as a local inverse in N], and γ is mapped onto an arc, c, of the unit circle. Hence,

$$I_\gamma = \frac{1}{2\pi i} \int_c \frac{f(w)}{\Psi(w) - e^{i\theta}} \left(1 - \frac{e^{ip\theta}}{\Psi^p(w)}\right) dw.$$

Clearly if $|\theta| > \eta > 0$, then

(15) $$\sup_{p, |\theta| > \eta} |I_\gamma| < \infty.$$

Hence, only θ's sufficiently close to zero need be considered. By Lemma 2.10, there exist α, β such that $h(z) = \dfrac{\alpha + (1-\alpha-\beta)z}{1-\beta z}$ satisfies $h(1) = \Phi(1)$, $h'(1) = \Phi'(1)$ and $h''(1) = \Phi''(1)$. By Corollary 2.7,

$$\sup_{p,\theta} \left| \frac{1}{2\pi i} \int_{|h(z)|=1} \frac{f(h(z))}{z} \sum_{n=0}^{p-1} \frac{e^{in\theta}}{z^n} dz \right| < \infty.$$

Let γ' denote the part of the curve $|h(z)| = 1$ inside of N, and let Γ denote the remainder. Then one easily obtains

$$\sup_{p,\theta} \left| \frac{1}{2\pi i} \int_{\gamma'} \frac{f(h(z))}{z} \sum_{n=0}^{p-1} \frac{e^{in\theta}}{z^n} dz \right| < \infty.$$

Without loss of generality, it can be assumed that the mapping $w=h(z)$ maps γ' onto the arc of the unit circle c. One then obtains

$$(16)\qquad \sup_{p,\theta}\left|\frac{1}{2\pi i}\int_c \frac{f(w)\,H'(w)}{H(w)-e^{i\theta}}\left(1-\frac{e^{ip\theta}}{H^p(w)}\right)dw\right| < \infty .$$

Now write I_γ as

$$I_\gamma = I_1 + I_2 \text{ , where}$$

$$I_1 = \frac{1}{2\pi i}\int_c f(w)\left[\frac{\Psi'(w)}{\Psi(w)-e^{i\theta}}\left(1-\frac{e^{ip\theta}}{\Psi^p(w)}\right) - \frac{H'(w)}{H(w)-e^{i\theta}}\left(1-\frac{e^{ip\theta}}{H^p(w)}\right)\right]dw$$

and

$$I_2 = \frac{1}{2\pi i}\int_c \frac{f(w)\,H'(w)}{H(w)-e^{i\theta}}\left(1-\frac{e^{ip\theta}}{H^p(w)}\right)dw .$$

From (16) $\quad \sup_{p,\theta}|I_2| < \infty$.

[Notation: For the remainder of the proof, it will be convient to denote all constants which are independent of θ by $O(1)$]

From (12) and (13), we have

$$|\Psi(e^{it}) - H(e^{it})| \leq O(1)|t^3|$$

$$|\Psi'(e^{it}) - H'(e^{it})| \leq O(1)t^2.$$

From Lemma 2.9,

$$\frac{1}{|H(e^{it}) - e^{i\theta}|} \leq \frac{O(1)}{[\theta^4 + (t-\tau_1)^2]^{\frac{1}{2}}} \qquad \text{where}$$

$\tau_1 = \arg h(e^{i\theta})$; and

$$\frac{1}{|\Psi(e^{it}) - e^{i\theta}|} \leq \frac{O(1)}{[\theta^4 + (t-\tau_2)^2]^{\frac{1}{2}}}$$

where $\tau_2 = \arg \Phi(e^{i\theta})$.

[In order to have adequate space, in the remainder of the proof, functions in the integrands will be written without the variable in those cases where possible. For example we will write $\psi(e^{it})$ as ψ.]

Let $w = e^{it}$; c being given by $a \leqq t \leqq b$. Write I_1 as

$$I_1 = I_{11} + I_{12}$$

where

$$I_{11} = \frac{1}{2\pi}\int_a^b \frac{e^{it}f[\psi' - H']}{H - e^{i\theta}}\left(1 - \frac{e^{ip\theta}}{H^p}\right)dt$$

and

$$I_{12} = \frac{1}{2\pi}\int_a^b e^{it}f\psi'\left(\frac{1 - \frac{e^{ip\theta}}{\psi^p}}{\psi - e^{i\theta}} - \frac{1 - \frac{e^{ip\theta}}{H^p}}{H - e^{i\theta}}\right)dt \ .$$

Now

$$|I_{11}| \leqq \sup_t \left| f(e^{it}) \left(1 - \frac{e^{ip\theta}}{H^p(e^{it})}\right)\right| O(1)\int_a^b \frac{t^2 dt}{[\theta^4 + (t-\tau_1)^2]^{\frac{1}{2}}}$$

$$= O(1)\int_a^b \frac{t^2 \ dt}{[\theta^4 + (t-\tau_1)^2]^{\frac{1}{2}}}.$$

By Corollary 2.12

(17) $$\sup_{p,|\theta| \leq \eta} |I_{11}| < \infty \ .$$

Write I_{12} as

$$I_{12} = I_{121} + I_{122} \qquad \text{where}$$

$$I_{121} = \frac{1}{2\pi}\int_a^b e^{it} f \psi' \frac{H - \psi}{(\psi - e^{i\theta})(H - e^{i\theta})}\, dt \quad \text{, and}$$

$$I_{122} = \frac{e^{ip\theta}}{2\pi}\int_a^b e^{it} f \psi' \left(\frac{1}{\psi^p[\psi - e^{i\theta}]} - \frac{1}{H^p[H - e^{i\theta}]}\right) dt .$$

Now write I_{122} as

$$I_{122} = I_{1221} + I_{1222} \qquad \text{where}$$

$$I_{1221} = \frac{e^{ip\theta}}{2\pi}\int_a^b \frac{e^{it} f \psi'}{\psi^p} \frac{H - \psi}{(\psi - e^{i\theta})(H - e^{i\theta})}\, dt \text{ , and}$$

$$I_{1222} = \frac{e^{ip\theta}}{2\pi}\int_a^b \frac{e^{it} f \psi'}{H - e^{i\theta}}\left(\frac{1}{\psi^p} - \frac{1}{H^p}\right) dt.$$

Now

$$|I_{121}| \leq O(1) \sup_t |f(e^{it})\ \psi'(e^{it})| \int_a^b \frac{|t|\, t^2 dt}{[\theta^4 + (t - \tau_1)^2]^{\frac{1}{2}}[\theta^4 + (t - \tau_2)^2]^{\frac{1}{2}}}$$

$$= O(1)\int_a^b \frac{|t|}{[\theta^4 + (t - \tau_1)^2]^{\frac{1}{2}}} \frac{t^2 dt}{[\theta^4 + (t - \tau_2)^2]^{\frac{1}{2}}} .$$

But

$$\left(\int_a^b \frac{|t|}{[\theta^4 + (t-\tau_1)^2]^{1/2}} \frac{t^2\ dt}{[\theta^4 + (t-\tau_2)^2]^{1/2}}\right)^2 \leq \int_a^b \frac{t^2\ dt}{\theta^4 + (t-\tau_1)^2}\int_a^b \frac{t^4\ dt}{\theta^4+(t-\tau_2)^2}.$$

From the proof of Lemma 2.11,

(18) $$\sup_{p,|\theta|\leq n} |I_{121}| < \infty .$$

In like manner, we have

(19) $$\sup_{p,|\theta|\leq n} |I_{1221}| < \infty .$$

By Lemma 2.15, there exists $c>0$ such that

$$|H(e^{it})| \geq e^{ct^2}$$

and

$$|\Psi(e^{it})| \geq e^{ct^2} \quad \text{for each point in N.}$$

Then

$$|I_{1222}| \leq O(1) \sup_t |f(e^{it})|\ |\Psi'(e^{it})| \int_a^b \frac{|t^3| \sum_{k=1}^{p} \frac{1}{|H|^k|\Psi|^{p+1-k}}}{[\theta^4 + (t-\tau_1)^2]^{1/2}} dt .$$

But

$$\int_a^b \frac{|t^3|}{[\theta^4 + (t-\tau_1)^2]^{1/2}} \sum_{k=1}^{p} \frac{1}{|H|^k|\Psi|^{p+1-k}} dt \leq \int_a^b \frac{|t^3| Pe^{-c(p+1)t^2}}{[\theta^4 + (t-\tau_1)^2]^{1/2}} dt$$

$$\leq \int_a^b \frac{|t^3| pe^{-cpt^2}}{[\theta^4 + (t-\tau_1)^2]^{1/2}} dt.$$

$$= O(1) \quad \text{as } p\to\infty, \theta \to 0,$$

by Lemma 2.13. Hence,

$$(20) \qquad \sup_{p,|\theta|\leq n} |I_{1222}| < \infty .$$

From (20), (19), (18), (17), and (16), we obtain

$$\sup_{p,|\theta|\leq n} |I_{\gamma}| < \infty .$$

Finally, from (15) and (14), we obtain

$$\sup_{p,\theta} |G(f;p-1,\theta)| < \infty .$$

OPEN QUESTIONS

The aim of this section is to present a series of questions concerning U, its dual space and operators on U:

(1) As sets, ℓ_1 is a proper subset of U_2, and U_2 is a proper subset of ℓ_2. Does there exist p such that $1<p<2$ and ℓ_p is a subset of U_2 or U_2 a subset of ℓ_p?

(2) If $\{c_k\} \in U_2'$, then a necessary condition for $\{\frac{1}{c_k}\} \in U_2'$ is that there exists $\eta>0$ such that $|c_k|> \eta$ for $k = 0,1,2,\ldots$ Sufficient conditions for $\{\frac{1}{c_k}\} \in U_2'$ when $\{c_k\} \in U_2'$ are (1) there exists $\eta>0$ such that $|c_k|> \eta$ for $k = 0,1,\ldots$ and (2) $\{c_k\}$ is of bounded variation. Establish necessary and sufficient conditions for a sequence to have an inverse in U_2'.

(3) A question related to (2) is "What are the homomorphisms on U_2'?"

(4) If f belong to U, with partial sums $S_n(z)$ obeying,

$$\inf_{n,|z|\leq 1} |S_n(z)|\geq\delta>0, \quad \text{will } \frac{1}{f} \text{ belong to U?}$$

(5) A more difficult question is "If f belong to U, $|f|>0$ on $|z|\leq 1$, will $\frac{1}{f}$ belong to U?"

(6) Given $\alpha(0<|\alpha|<1)$, will $f\circ\Phi$ be in U for all f in U, where

$$\Phi(z) = \frac{z - \alpha}{1-\bar{\alpha}z}?$$

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