THE MAXIMUM GENUS OF A GRAPH

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This is to certify that the

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E. A. Nordhaus
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ABSTRACT

THE MAXIMUM GENUS OF A GRAPH

By

Richard Delose Ringeisen

In 1965, R. Duke considered imbeddings of graphs which were not necessarily minimal, but which were cellular (2-cell) imbeddings. Such considerations led to the concept of the maximum genus of a connected graph.

The maximum genus, $\gamma_M(G)$, of a connected graph G is the largest integer $\gamma(N)$ for compact orientable 2-manifolds N in which G has a cellular imbedding, where $\gamma(N)$ is the genus of the surface N. In 1969, Nordhaus, Stewart, and White made the above definition and studied some of the properties of maximum genus. It is the purpose of this thesis to extend the knowledge of this concept. Some new maximum genus formulas are obtained and an investigation is made of graphs which do or do not attain certain bounds for the maximum genus.

Two techniques to obtain imbeddings of graphs are frequently used in the thesis, one of which is Edmonds' permutation technique, and the other an edge adding or ribbon technique which provides a cellular imbedding for a graph from a cellular imbedding of a suitably chosen subgraph.

The first three chapters of the thesis introduce the subject, define basic terms and techniques, and survey known results pertinent to the study of genus problems.

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In Chapter 4, the maximum genus of certain planar graphs is determined. This class includes the wheel graphs, the ladder graphs, the standard maximal planar graphs, and the regular polyhedral graphs.

In Chapter 5 some general properties of maximum genus are presented. A major problem concerns the determination of those connected graphs for which the maximum genus is the sum of the maximum genera of its components. Some results pointing toward a possible complete solution to this problem are given. In particular, if G_1 and G_2 are two connected disjoint graphs which are joined by a single edge to form a new graph G, we show that $\gamma_M(G) = \gamma_M(G_1) + \gamma_M(G_2)$. A complete characterization of those graphs for which the maximum genus and the minimum genus are equal is presented. Those graphs with maximum genus one are also determined.

Chapter 6 treats those graphs which we designate as either upper or lower imbeddable. A conjecture made by Duke is proven true for lower imbeddable graphs, and for certain other classes of graphs.

In Chapter 7, the maximum genus of the complete bipartite graph $K_{m,n}$ is determined, and the formula, $\gamma_M(K_{m,n}) = \left[\frac{1}{2}(m-1)(n-1)\right]$, is proven. The maximum genus of the complete graph, although known previously, is determined in a different way using an edge adding technique.

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For Carolyn,

Mom and Dad

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CHAPTER 1

INTRODUCTION

It was only in recent years that graph theorists began to concern themselves with the topological properties of a graph. The early considerations of such concepts were a direct result of the famous four color conjecture and its graph theoretical interpretation. However, mathematicians soon began to consider the genus of a graph primarily for the properties of the graph which it might yield.

In 1965, Richard Duke [4] considered imbeddings of graphs which were not necessarily minimum but which have each of their faces homeomorphic to an open disk. In particular, he raised the following question: Upon which compact orientable two manifolds might a given graph have such a cellular imbedding? Such considerations led to the concept of the maximum genus of a connected graph. Is there a surface of maximum genus upon which a given graph G has a cellular imbedding? It was soon discovered that the well known Euler Polyhedral formula provides an upper bound. In 1969, Nordhaus, Stewart, and White [10] defined the maximum genus of a graph and studied some of its properties.

It is the purpose of this thesis to extend the knowledge of this concept. Some maximum genus formulas are obtained and an investigation is made of graphs which do or do not attain maximum genus.

The definitions, notations, and techniques used in this thesis are stated in Chapter 2. Of particular interest is the development of an edge adding technique, which is widely employed throughout the thesis. Chapter 3 makes a survey of results of importance known prior to this thesis. Both pertinent results concerning ordinary genus and those relative to maximum genus are discussed.

In Chapter 4 the maximum genera of some well known planar graphs are found. Formulas for the standard maximal planar graph, the wheel graph, the ladder graph, and the polyhedral graphs are given.

In Chapter 5, some of the main results of this thesis are presented. An important question concerns the determination of those connected graphs for which the maximum genus is the sum of the maximum genera of the components. Some results pointing toward a possible complete solution to this problem are given. A complete characterization of those graphs for which maximum genus and ordinary genus are equal is presented. Those graphs which have maximum genus one are also characterized. Chapter 6 concerns itself with those graphs which we designate as either upper or lower imbeddable. Some additional genera formulas are given.

Chapter 7 contains two results of special interest. The maximum genus formula for the complete bipartite graph is derived. The maximum genus of the complete graph is also given, although this result was established by different procedures prior to this thesis [10].

CHAPTER 2

DEFINITIONS, NOTATIONS, AND TECHNIQUES

This chapter defines those terms which are fundamental to the study of topological properties in graph theory. We also present those techniques which are often employed in the later chapters of this thesis.

A graph G is a set of vertices V(G) and a set of unordered pairs of vertices E(G), called edges. If the elements of E(G) are ordered pairs, G is a directed graph. For two vertices x and y in V(G), (x,y) represents the corresponding edge (if any) in E(G). If G is a directed graph, the corresponding edge directed from x to y is denoted [x,y]. (This notation is that which is used by Youngs [15].) An edge of the form (x,x) is called a loop. When an edge appears more than once in E(G), G is called a multigraph. A graph in which both V(G) and E(G) are finite is called finite. A graph is connected if every pair of vertices is joined by a path of edges and vertices. A cutpoint of a graph is a vertex whose removal disconnects the graph. A connected graph is nonseparable if it is connected and has no cutpoints. A block of a connected graph is a maximal nonseparable subgraph. Henceforth, we shall only consider graphs which are finite, undirected, connected, and without loops or multiple edges.

The first Betti number (or Betti number) of a graph G is, by definition, $\beta(G) = E - V + 1$, where E is the number of edges of G and V is the number of vertices. This number is also known as the cyclomatic number of G. A graph H is a subgraph of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The degree of a vertex is the number of edges to which it belongs.

The least integer greater than or equal to a real number z is written $\{z\}$. The greatest integer smaller than or equal to z is written $[z]$.

Other terms which are not given here may be found in Harary [6]. Topological terms may be found in either Dugundji [3] or Massey [8].

A graph can be thought of in a strictly set-theoretic sense as a collection of vertices and edges. However, one can also regard graphs in a geometric sense. It is when one attempts such a geometric realization that imbedding problems are encountered. Although any graph can be realized in Euclidean 3-space, the realization need not be in a 2-manifold. By placing this 2-manifold restriction on our imbeddings we encounter some very interesting problems.

The graph G is said to be imbedded in the 2-manifold M if the geometric realization of G as a one-dimensional simplicial complex is homeomorphic to a subspace of M. We can state this condition in a more graph theoretic form. Let G be a graph where $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. An imbedding of G in M is a subspace G(M) of M such that

$$G(M) = \bigcup v_i(M) \cup \bigcup e_j(M),$$

where

(i) $v_1(M), \ldots, v_n(M)$ are distinct points of M.

(ii) $e_1(M), \ldots, e_m(M)$ are m mutually disjoint open arcs in M.

(iii) $e_j(M) \cap v_i(M) = \phi$, $i = 1, \ldots, n$; $j = 1, \ldots, m$.

(iv) if $e_j = (v_{j_1}, v_{j_2})$ then the open arc $e_j(M)$ has $v_{j_1}(M)$ and $v_{j_2}(M)$ as end points; $k = 1, \ldots, m$.

An arc in M is a homeomorphic image of the closed unit interval; an open arc is an arc less its two end points, the images of 0 and 1. The surfaces considered in this thesis will be compact orientable 2-manifolds. We will use the term "surface" to refer to compact orientable 2-manifolds.

Given an imbedding of a graph G on a surface N, each component in the complement of G in N is called a face of the imbedding. Whenever a face is homeomorphic to the open unit disk, it is said to be a 2-cell. If an imbedding is so that each of its faces is a 2-cell, the imbedding is called a cellular imbedding. The total number of faces in an imbedding is denoted by F. In a cellular imbedding, F_i denotes the number of faces having i sides, and V_i the number of vertices of degree i.

When a graph can be imbedded in a surface of genus n but cannot be imbedded in any surface of lower genus, the imbedding is called minimal, and the genus of the graph is defined to be n; we use the notation $\gamma(G) = n$. A graph is planar if and only if $\gamma(G) = 0$. The number of faces in a minimal imbedding is denoted $\delta(G)$.

Define the maximum genus, $\gamma_M(G)$, of a connected graph G to be the largest number $\gamma(N)$ for compact orientable 2-manifolds N in which G has a cellular imbedding. Such an imbedding is called a maximum imbedding. The number of faces in a maximum imbedding for a graph G is denoted $\delta_M(G)$. Note that if G is not connected, then no imbedding is cellular. The restriction of consideration to cellular imbeddings is essential, since otherwise there is no upper bound to the genera of possible surfaces upon which a graph may be imbedded. The upper bound which can be determined is given in Chapter 3.

We now turn our attention to the consideration of certain techniques employed throughout the thesis. The first is commonly called Edmonds' Permutation Technique.

Suppose a connected graph G has n vertices; we write $V(G) = \{1,\dots,n\}$. Let $V(i) = \{k: (i,k) \in E(G)\}$. Let $p_i: V(i) \to V(i)$ be a cyclic permutation of $V(i)$ of length $n_i = |V(i)|$, where $i = 1,\dots,n$. The following theorem of Edmonds [5] (also see Youngs [15]) indicates the correspondence between cellular imbeddings and choices of the p_i.

Theorem: Each choice $(p_1,\dots,p_n)$ determines a cellular imbedding G(M) of G in a compact orientable 2-manifold M, such that there is an orientation on M which induces a cyclic ordering of the edges (i,k) at i in which the immediate successor to (i,k) is $(i,p_i(k))$, $i = 1,\dots,n$. In fact, given $(p_1,\dots,p_n)$, there is an algorithm which produces the determined imbedding. Conversely, given a cellular imbedding G(M) in a compact orientable 2-manifold M with a given orientation, there is a corresponding $(p_1,\dots,p_n)$ determining that imbedding.

Now, let $D = \{[a,b] : (a,b) \in E(G)\}$, and define $P: D \to D$ by: $P([a,b]) = [b,p_b(a)]$. Then P is a permutation on the set D of directed edges of G (where each edge of G is associated with two oppositely-directed directed edges), and the orbits under P determine the faces of the corresponding imbedding.

From the Edmonds' technique one can conclude that the genus of any graph and the maximum genus of any connected graph can be computed. It is a matter of choosing the correct permutation schemes from the $\prod_{i=1}^{n}(n_i-1)!$

possible ones. Of course, the difficulty is in the choosing, because of the usual enormous number of permutations possible.

To illustrate this concept, we present an imbedding of the graph K_4, the complete graph having 4 vertices, on the torus. The imbedding is that shown in Figure 2.1. The parameters for this graph are:

$$V(K_4) = \{1,2,3,4\}$$

$$V(i) = \begin{cases} \{2,3,4\} & \text{for } i = 1 \\ \{1,3,4\} & \text{for } i = 2 \\ \{1,2,4\} & \text{for } i = 3 \\ \{1,2,3\} & \text{for } i = 4 \end{cases}$$

$$n(i) = 3,\ i = 1,2,3,4.$$

The vertex permutations are seen to be:

$$P_1: (2\ 4\ 3)$$
$$P_2: (3\ 1\ 4)$$
$$P_3: (1\ 4\ 2)$$
$$P_4: (3\ 2\ 1).$$

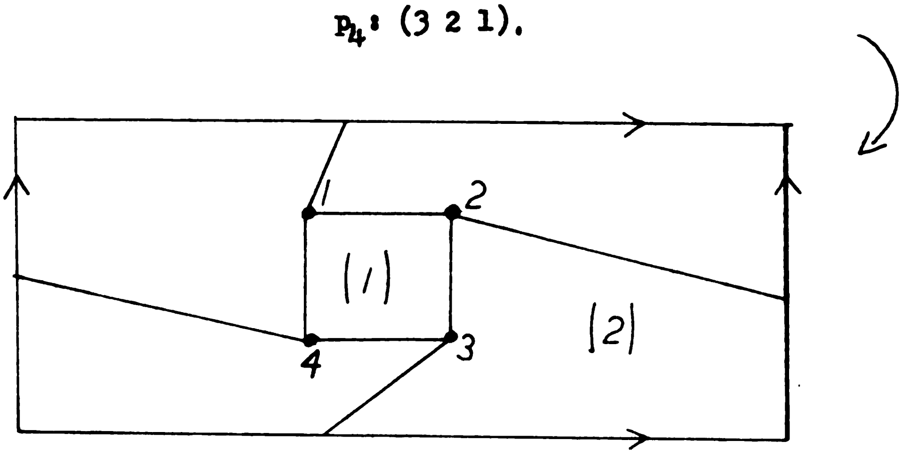


Figure 2.1 A Cellular Imbedding of K_4 on the Torus

This imbedding is cellular, as guaranteed by Edmonds' theorem. We will prove later that it is a maximum imbedding. Hence we have $\gamma_M(K_4) = 1$. We notice that the orbits under P determine the faces of the imbedding, as required. We observe that face (2) has each vertex of K_4 repeated, and consequently, its boundary cannot be a simple closed curve. The faces are as follows:

(1) $[2,1], [1,4], [4,3], [3,2]$

(2) $[1,2], [2,4], [4,1], [1,3], [3,4], [4,2], [2,3], [3,1]$.

These faces correspond to orbits of P and each edge appears as two directed edges in D, so that the sum of the orbit lengths is 2E.

Henceforth we will use a shorter form to designate an orbit. For example orbit (1) above will be represented by -2-1-4-3-. This notation implies that $p_3(4) = 2$ and $p_2(3) = 1$.

As a further illustration of Edmonds' scheme, we determine the maximum genus of the Petersen graph P.

<u>Remark 2.1</u>: The maximum genus of the Petersen graph is three.

<u>Proof</u>: By a theorem which will be given in Chapter 3, we see that 3 is an upper bound for possible genera. Hence, if we can produce an imbedding of P on a surface of genus 3, the remark is established. We label the ten vertices of P as in Figure 2.2.

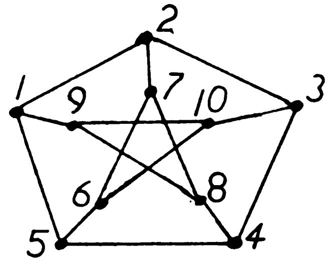


Figure 2.2 The Petersen Graph

Choose the following permutations for the vertices of P:

p_1: (5 2 9)	p_6: (7 10 5)
p_2: (1 7 3)	p_7: (2 8 6)
p_3: (10 4 2)	p_8: (4 7 9)
p_4: (5 8 3)	p_9: (8 1 10)
p_5: (1 4 6)	p_{10}: (6 3 9).

We then have a cellular imbedding of P with the following single orbit of length 30:

-1-2-7-8-9-1-5-4-8-7-6-10-3-4-5-6-7-2-3-10-9-8-4-3-2-1-9-10-6-5- ,

Since this must be a cellular imbedding, Euler's polyhedral formula applies to yield the fact that the genus of the surface upon which the graph is imbedded is 3. The remark is thus established. Notice that Edmonds' technique does not necessarily yield a unique imbedding of genus 3. However, in problems of this kind, to produce one imbedding suffices.

The Edge Adding Technique offers an alternative to the technique of Edmonds. It could be characterized as a constructive procedure. The value of the method lies in its allowing one to construct an imbedding for a graph from an imbedding for a suitable subgraph. We will state a theorem which describes the technique, indicates its proof, and illustrates its use.

Theorem 2.2: (The Edge Adding Technique). Let G be a connected graph which is not complete and i and j denote nonadjacent vertices of G. Let T be a cellular imbedding of G which has vertex i appearing in a face F_i and vertex j appearing in a face F_j. Let G' be the graph G with the edge (i,j) added. Then:

a.) If $F_i \neq F_j$, G' has a cellular imbedding with one less face than T.

b.) If $F_i = F_j$, then G' has a cellular imbedding with one more face than T. Furthermore, the directed edges $[i,j]$ and $[j,i]$ appear in different faces of this imbedding for G'.

<u>Proof</u>: [10] Let p_i and p_j be the vertex permutations at i and j which correspond to the imbedding T. Then p_i and p_j may be represented as follows: (Assume T has F_i, $\ldots x_n$-i-x_1-... and F_j, $\ldots y_m$-j-$y_1\ldots$.)

$$p_i: (x_1,x_2,\ldots,x_n); \quad p_j: (y_1,y_2,\ldots,y_m).$$

Form an imbedding for G' by leaving all vertex permutations for T unchanged, with the exception of those at p_i and p_j. In place of these two permutations we define:

$$p_i': (x_1,x_2,\ldots,x_n,j); \quad p_j': (y_1,y_2,\ldots,y_m,i).$$

Under this new permutation scheme for G' the two faces of a.) become one or the one face of b.) becomes two. Hence, the the theorem follows. Schematically, the imbedding for G' is formed as shown in Figure 2.3.

As further illustration of the manner in which the edge adding technique may be employed, we produce an imbedding of the graph G = K_4-x, where x is an arbitrary edge of K_4. We do so by building this graph from the complete graph K_3. We begin with the graph K_3 imbedded in the plane.

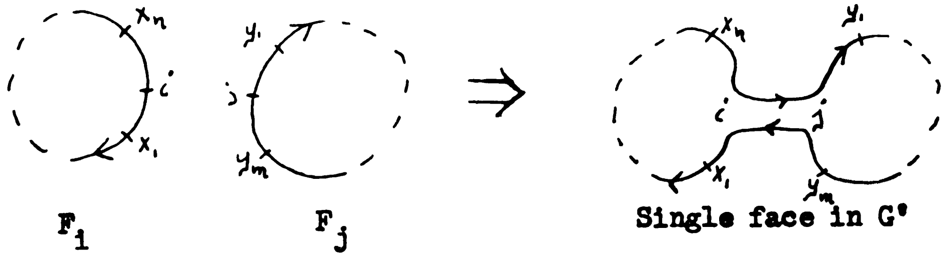


Figure 2.3.a

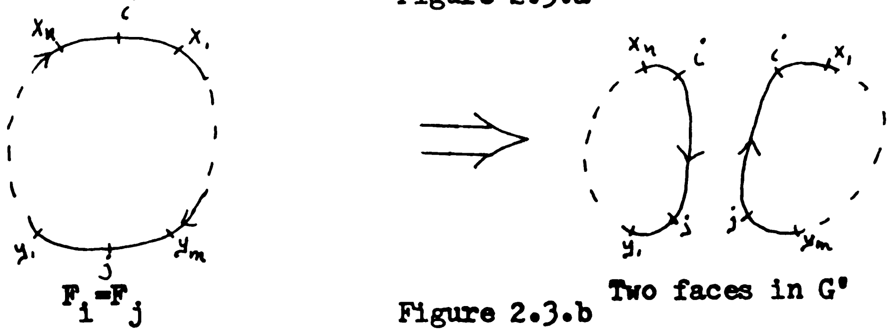
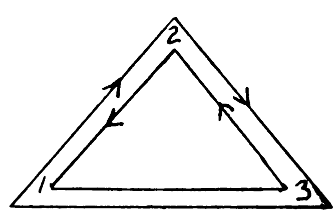


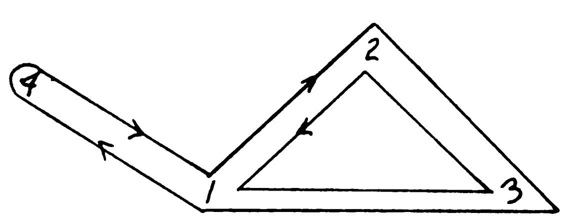
Figure 2.3.b

Figure 2.3 Illustration of the Edge Adding Technique

Let K_3 be imbedded in the plane and imagine its inside face and its outside face each consisting of directed edges as follows:



We add a vertex, labeled 4, and the edge (1,4) so that both the directed edge $[1,4]$ and the directed edge $[4,1]$ are in the outside face:



We now add the edge (4,2) in such a way that the resulting graph has only one face in the new imbedding. (That is, the inside and outside faces become one. We are using the edge adding technique with the vertex 2 appearing in the inside face and the vertex 4 appearing in the outside face.) The final result of this edge adding is shown in Figure 2.4.

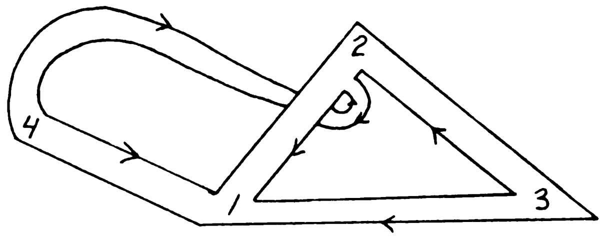


Figure 2.4 K_3 with a Vertex and Two Edges Added.

The edge adding technique will prove useful in finding the maximum genus of graphs whose maximum imbeddings have exactly one or two faces. One can build such graphs by adding one edge at a time to a subgraph which is of the same type. Such a building procedure often makes a difficult problem more amenable.

The edge adding technique was first employed by Nordhaus, Stewart, and White [10] as a proof technique in one of their theorems. It wasn't until closer scrutiny that we discovered its generalization to the technique as stated above. This technique is sometimes referred to as the "ribbon" technique.

CHAPTER 3

A SURVEY OF KNOWN RESULTS

In this chapter, the known results concerning imbeddings of graphs are given. In addition we give the known genus formulas as well as the known maximum genus formulas for those graphs which are discussed in this thesis.

All 2-manifolds upon which graphs are imbedded are considered to be compact orientable 2-manifolds. For this reason, the following characterization theorem is most important:

Classification Theorem: [8] A compact orientable 2-manifold is homeomorphic to a sphere or to a connected sum of tori.

The genus of a compact orientable 2-manifold is the number of connected tori of which it is the sum. [8]

Throughout this thesis we will use the word "surface" to refer to a compact orientable 2-manifold. A surface of this type may be thought of as a sphere or a sphere with handles. The genus, γ, of such a surface is then the number of its handles.

As indicated in the last chapter, the imbeddings considered in this thesis will be 2-cell imbeddings on such surfaces. We will call such imbeddings cellular imbeddings.

The well known Euler Polyhedral formula can be generalized to the following theorem about cellular imbeddings of graphs on a surface.

<u>Euler's</u> <u>Theorem</u>: Let F be the number of faces into which a surface of genus γ is separated by a cellular imbedding of a graph G, where V and E are the number of vertices and edges of G, respectively. Then
$F + V = E + 2(1-\gamma)$.

It is necessary to emphasize that we are making certain assumptions about our graph G. It is assumed that G is finite, connected, undirected, and has no loops or multiple edges. The fact that we are restricting ourselves to cellular imbeddings makes Euler's formula applicable.

The following two theorems concerning ordinary genus are both important and helpful.

The first theorem, due to Battle, Harary, Kodama, and Youngs [1] reduces the problem of the genus of a connected graph to that of finding the genus of a block.

<u>Theorem</u>: If G is a connected graph having k blocks $B_1, \ldots, B_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(B_i)$. Furthermore, in any minimal imbedding of G, $F_G = 1 - k + \quad F_{B_i}$, where F_G and F_{B_i} denote the number of faces for G and for B_i respectively.

The following famous theorem of Kuratowski [7] completely characterizes planar graphs:

<u>Theorem</u>: A graph G is planar if and only if G does not contain a subgraph isomorphic, to within vertices of degree two, to either K_5 or $K_{3,3}$.

There is no such characterization for graphs of higher genus. However, in 1968, Vollermaus [14] stated that the set of exceptional graphs (corresponding to K_5 or $K_{3,3}$) is finite for any sphere with a prescribed number of handles. Nothing further has been established on the subject.

We now list the known genus formulas for graphs which we consider. In 1965 Ringel [12] showed that $\gamma(K_{p,q}) = \left\{\frac{(p-2)(q-2)}{4}\right\}$, where the minimum imbeddings produced have quadrilateral faces (with at most one exception). In 1968, after considerable effort, Ringel and Youngs [13] established that

$$\gamma(K_p) = \left\{\frac{(p-3)(p-4)}{12}\right\}.$$

This result settled the famous Heawood map coloring conjecture in the affirmative. The genus formulas for the generalized n-cube and certain special cases of the complete tripartite graph have been discovered by A. T. White [16] . However, such graphs are not considered in this thesis.

We now turn our attention to the results concerning maximum genus which were discovered prior to this thesis. The first result does not pertain only to maximum genus. However, it was instrumental in inspiring the concept of maximum genus, and such considerations by Duke led to our definition [4] .

<u>Theorem</u>: If there exist cellular imbeddings of the graph G in orientable 2-manifolds of genera m and n, then for any integer k, $m \leq k \leq n$, there exists a cellular imbedding of G in the orientable 2-manifold of genus k.

Once the maximum genus and the ordinary genus of a connected graph G are known, the above theorem completely determines all orientable 2-manifolds upon which G has cellular imbeddings.

The next few theorems are all contained in the paper [10] which introduced the concept of maximum genus.

<u>Theorem</u>: A connected graph G has a cellular imbedding in a surface S_k, of genus k, if and only if $\gamma(G) \leq k \leq \gamma_M(G)$.

We now list some theorems which establish some bounds for the maximum genus of a connected graph.

<u>Theorem</u>: If H is a connected subgraph of a connected graph G, then $\gamma_M(H) \leq \gamma_M(G)$.

<u>Corollary</u>: If G is a connected graph with $\gamma(G) = 1$, then $\gamma_M(G) \geq 2$.

The next theorem insures that the maximum genus of a connected graph is always finite. The fact that the definition of maximum genus only considers cellular imbeddings is of utmost importance. In fact, without such cellular considerations, a theorem similar to the next one simply does not exist.

<u>Theorem</u>: An upper bound for the maximum genus of an arbitary connected graph is given by $\gamma_M(G) \leq \left[\frac{\beta(G)}{2}\right]$. Equality holds if and only if the imbedding has one or two faces according as $\beta(G)$ is even or odd, respectively.

A connected graph G is <u>upper imbeddable</u> if $\gamma_M(G) = \left[\frac{1}{2}\beta(G)\right]$.

The inequality stated in this theorem can be arbitrarily inaccurate, as the following graph G_n shows. It is clear that $\frac{1}{2}\beta(G_n) = n$, but as we will see later, $\gamma_M(G_n) = 0$.

G_n G

Figure 3.1 Two Graphs which are not Upper Imbeddable.

The second graph G of Figure 3.1 shows that the inequality of the last theorem may be strict, even for blocks, since $\gamma_M(G) = 1 < \left[\frac{1}{2}\beta(G)\right] = 2$.

As we stated earlier, it is quite convenient that the genus of a connected graph is the sum of the genera of its components. However, as the graph H below illustrates, such is not the case for maximum genus; $\gamma_M(H) = 1 \neq 0 + 0 = \gamma_M(\text{triangle}) + \gamma_M(\text{triangle})$.

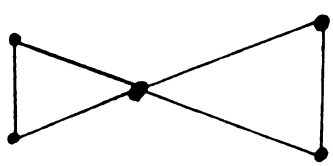


Figure 3.2 The Graph H

We do, however, have the following inequality.

__Theorem:__ If G is a connected graph with blocks H_i, $i=1,\ldots,n$, then $\gamma_M(G) \geq \sum_{i=1}^{n} \gamma_M(H_i)$.

__Corollary:__ If G is a connected graph with n blocks H_i, $i=1,\ldots,n$, then $\sum_{i=1}^{n} \gamma_M(H_i) \leq \gamma_M(G) \leq \left[\frac{1}{2}\beta(G)\right]$.

Corollary: If G is a connected graph with n blocks H_i, such that $\gamma_M(H_i) = \frac{1}{2}\beta(H_i)$, i=1,...,n, then $\gamma_M(G) = \sum_{i=1}^{n}\gamma_M(H_i) = \frac{1}{2}\beta(G)$.

This thesis extends this last theorem in the sense that it discloses some more cases in which the maximum genera of the blocks may be added to obtain the maximum genus of the graph.

The following theorem is the only maximum genus formula known prior to this thesis: [10]

Theorem: The maximum genus of the complete graph on n vertices is given by the formula,

$$\gamma_M(K_n) = \left[\tfrac{1}{4}(n-1)(n-2)\right].$$

The following theorem of Duke [4] is often useful in determining that a certain imbedding is not a maximum imbedding. It is helpful to note that an imbedding is maximum if and only if the number of faces in the imbedding is a minimum.

Theorem: Let G be a connected graph with an imbedding T and b a vertex of G. If there is a vertex a of G, which is adjacent to b so that [a,b] is in one face, [b,a] in another face, and b appears in a third face, then G has an imbedding with two fewer faces than the imbedding T had.

CHAPTER 4

THE MAXIMUM GENUS OF CERTAIN PLANAR GRAPHS

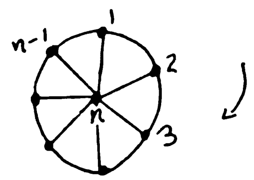
This chapter will determine the maximum genus of four major classes of planar graphs. The wheel graphs, the standard maximal planar graphs, the "rounded" ladder graphs, and the regular polyhedral graphs are examined. With the exception of the finite class of regular polyhedral graphs, each of the above classes contains graphs with arbitrarily large maximum genus.

A possible conjecture which occurred rather early in the study of maximum genus is also settled in the negative by these planar graphs. Is it possible that the maximum genus of a graph is somehow limited by the magnitude of its ordinary genus? Since the classes of planar graphs studied in this chapter contain graphs of arbitrarily large maximum genus, this question is clearly answered in the negative.

All of the graphs examined in this chapter are upper imbeddable. This need not be true for planar graphs in general, however. Theorem 5.11 of Chapter 5 presents an infinite number of planar graphs which are not upper imbeddable.

The maximum genera of the above listed graphs are given in a series of theorems. The proofs of the theorems offer good examples of the application of Edmonds' technique and of the edge adding technique.

By the wheel graph on n vertices, $n \geq 4$, we mean $W_n = K_1 + C_{n-1}$, where the addition means that each vertex of the one graph is joined to each vertex of the other. The cycle with n-1 vertices is represented by C_{n-1} and the complete graph with one vertex by K_1. We label W_n as the following diagram illustrates:-

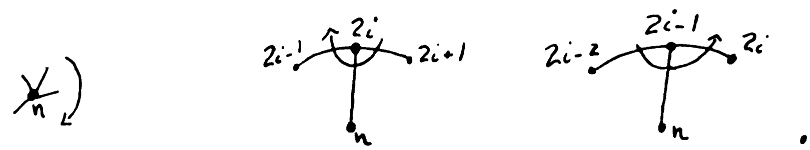


As is customary, we let E be the number of edges of W_n, V the number of vertices, and β the Betti number. Then for the wheel graph, we have: $E = 2n-2$, $v = n$, and $\beta = n-1$.

<u>Theorem 4.1</u>: The maximum genus of the wheel graph W_n on n vertices is given by the formula:

$$\gamma_M(W_n) = \left[\tfrac{1}{2}\beta(W_n)\right] = \begin{cases} \tfrac{1}{2}(n-2) & \text{if } n \text{ is even.} \\ \tfrac{1}{2}(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

<u>Proof</u>: We use Edmonds' permutation technique. The permutations we employ may be schematically represented in the following manner:



Precisely, the permutations which we use are: (addition mod(n-1))

P_n: $(1\ 2\ 3\ \ldots n-1)$.

P_{2i}: $((2i+1)\ n\ (2i-1))$ $\begin{cases} i=1,\ldots,\tfrac{1}{2}(n-2) \text{ if } n \text{ is even,} \\ i=1,\ldots,\tfrac{1}{2}(n-1) \text{ if } n \text{ is odd.} \end{cases}$

P_{2i-1}: $((2i-2)\ n\ 2i)$ $\begin{cases} i=1,\ldots,\tfrac{1}{2}n \text{ if } n \text{ is even,} \\ i=1,\ldots,\tfrac{1}{2}(n-1) \text{ if } n \text{ is odd.} \end{cases}$

Since the Betti number is odd when n is even and even when n is odd, W_n will be upper imbeddable if we exhibit a two face imbedding when n is even and a one face imbedding when n is odd. We consider the cases n even and n odd separately.

Case I: Let n be even

We show that the above listed permutations give an imbedding having two faces; one face has 2E - 8 directed edges, the other has 8 directed edges.

For the first of these faces (orbits), we begin with the directed edge $[1,2]$ and trace its orbit. The orbit begins -1-2-3- and there then follows a section of the orbit best given by an iterative process. We indicate the orbit as -1-2-3-L and proceed to describe that section of the orbit indicated by L. The section L is gotten by placing $j=2,3,\ldots,\frac{1}{2}n-1$ into the following scheme, each number $j \geq 3$ put into the iteration following j-1 and preceding j+1, noticing that the end of the scheme for $j=j_o$ is the beginning of the scheme for $j=j_o+1$. The scheme is:

$$-n-2j-(2j-1)-(2j-2)-n-(2j-1)-2j-(2j+1)-n- \; .$$

We first notice that since $j \neq 1$, the permutations as given do indeed describe this scheme. We remark that it was very important to begin with j=2, however. The section of the scheme -(2j-1)-(2j-2)- n- is dependent upon 2j-2 being even. Remember, however, that addition is done modulo (n-1). Consequently, were j=1, then $2j-2 = 0 \equiv (n-1)(\text{mod}(n-1))$, and in the case we are considering, (n-1) is odd! The scheme does appear as part of the orbit for $j=2,\ldots,\frac{1}{2}n-1$. Notice that $p_n(2j+1) = (2j+2)$ and thus $j=j_o$ is clearly followed by $j=j_o+1$ in the scheme.

Also notice that n appears only when explicitly shown in the scheme written above.

We now devote our attention to showing that all directed edges given in L are distinct. We consider the possibilities of an odd numbered vertex appearing in the same edge twice and an even numbered vertex doing so separately. We begin by considering an even number k, $2 \leq k \leq n-2$. Then k either appears as some $2j_o$ or $2j_1-2$, for some j_o or j_1. We consider these possibilities.

If $k=2j_o$, the possible directed edges in L where k might appear are:

$$[n,k]\ ;\ [k,(k-1)]\ ;\ [(k-1),k]\ ;\ [k,(k+1)].$$

If $k=2j_1-2$, we have the following possible directed edges in L:

$$[k+1,k]\ ;\ [k,n].$$

Since none of these directed edges is the same, no even number appears in a repeated directed edge.

Thus, the only possibility for a repeated directed edge is one involving an odd numbered vertex. Then the odd number r is of one of the forms $r=2j_2-1$ or $r=2j_3+1$. The possible directed edges, then, are:

$$[n,r] \text{ and } [r,n].$$

Thus, no directed edge appears more than once in L.

We now count the total number of directed edges used in L. For each j used in L, 8 directed edges are introduced. (Recall that the last "n" for j is also the first "n" for j+1.) Hence, the number of directed edges in L is: $8(\frac{1}{2}n-2) = 4n-16$.

Notice that the orbit beginning with the directed edge $[1,2]$ terminates with $j=\frac{1}{2}n-1$ placed in the scheme: $2(\frac{1}{2}n-1) + 1 = n-1$ and we have $p_n(n-1) = 1$ and $p_1(n) = 2$. Thus, with the exception of the directed edges represented in -1-2-3-, all the directed edges in the orbit are in L. Hence, the total number of directed edges in the orbit we are enumerating is: $(4n-16) + 4 = 4n - 12 = 2E - 8$. Hence, the imbedding has yielded one face of length 2E - 8.

Since $j \neq 1$ in the scheme displayed in the face above, the directed edge $[2,1]$ does not appear in the face. We now trace the orbit in which this directed edge appears. Using the permutations as given, we have:

$$-2-1-(n-1)-(n-2)-n-(n-1)-1-n- .$$

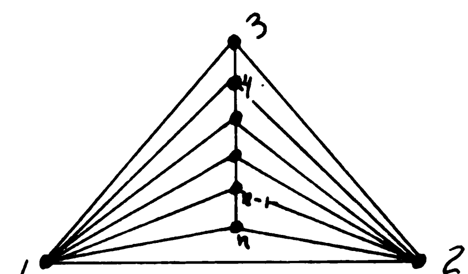
Thus, this orbit contains 8 distinct directed edges. We now have two faces in our imbedding, one of length 2E-8 and the other of length 8. Hence, the number of directed edges used in both faces is 2E. However, this is all the possible directed edges of W_n and consequently the imbedding has exactly these two faces. The case for n even is now completed.

Case II: Let n be odd

This case is much the same as case I with one very notable exception. If we consider the iteration scheme L as given in case 1, we notice that the problem described there for j=1, no longer exists when n is odd. Hence, we may use the scheme -n-2j-(2j-1)-(2j-2)-n-(2j-1)-2j-(2j+1)-n- for all $j=1,2,\ldots,\frac{1}{2}(n-1)$. (Notice that the only possibility for "trouble" where a necessarily odd number might be even is when $j=\frac{1}{2}(n-1)$ and for a necessarily even number to be odd for this same value of j when placed into the expression for 2j. However, $2(\frac{1}{2}(n-1)) = n-1$, which is even, and $2(\frac{1}{2}(n-1)) + 1 = n$, which is odd).

All directed edges in the iteration are distinct just as before in case I. Thus the total number of edges used in the iterated path is $8(\frac{1}{2}(n-1)) = 4n - 4 = 2E$. Thus all the possible directed edges are used in this single face, and W_n has an imbedding with one face. The theorem is now proven.

We next compute the maximum genus of what we call the "standard" maximal planar graph with n vertices. The graph may be described as $K_2 + P_{n-2}$, where addition is as described before, K_2 is the complete graph on 2 vertices, and P_{n-2} is a path with n-2 vertices, of length n-3, $n \geq 4$. We let $M_n = K_2 + P_{n-2}$ and refer to it as the "standard" maximal planar graph on n vertices. The graph M_n has the following parameters: $E = 3n - 6$; $V = n$; $\beta(M_n) = 2n - 5$. We draw M_n with its planar imbedding as shown below, with the vertices numbered as indicated.



M_n

From the theorem given in Chapter 3, we see that $\gamma_M(M_n) \leq \left[\frac{2n-5}{2}\right] = n-3$. Consequently, if we can find a cellular imbedding of M_n with two faces, we will have shown that equality holds in the above inequality. We have the

<u>Theorem 4.2</u>: The standard maximal planar graph M_n has maximum genus n-3.

Proof: Let M_n be labeled as in the illustration above. Since $M_4 = K_4$ and since the maximum genus of K_4 is later shown to be one, we assume $n \geq 5$. We again employ the Edmonds technique.

Let the vertex permutations be as follows:

$$p_1: \ (2\ 3\ 4\ .\ .\ .\ n)$$
$$p_2: \ (1\ 3\ 4\ .\ .\ .\ n)$$
$$p_3: \ (4\ 1\ 2)$$
$$p_n: \ ((n-1)\ 1\ 2)$$
$$p_j: \ ((j-1)\ (j+1)\ 1\ 2)$$
$$j = 4,\ldots,(n-1).$$

When these permutations are used to trace an orbit we notice the following scheme developing:

$$-1-k-2-(k+1)-k- .$$

where $k = 3,4,\ldots,(n-1)$. Since $p_k(k+1) = 1$, and $p_1(k) = (k+1)$, we see that $k = j_o$ is followed by $k = j_o + 1$ in the scheme. When $k = n-1$, we have $-1-(n-1)-2-n-(n-1)-$, which is then followed by $-1-n-2-$, and since $p_2(n) = 1$ and $p_1(2) = 3$, the face is completed. The number of directed edges in the face is $5(n-3) + 3 = 5n - 12$.

Because the above scheme begins with $k = 3$, the directed edge $[1,2]$ does not appear. Thus, we begin with this directed edge and trace its orbit. Following the direction of the permutations we have:

$$-1-2-3-\ .\ .\ .\ -(n-1)-n- .$$

Since $p_n(n-1) = 1$ and $p_1(n) = 2$, the face is completed. Thus we have a face with n directed edges.

The total number of directed edges used in the two faces is then $5n-12 + n = 6n - 12 = 2E$. Hence, all the directed edges are used and M_n has a cellular imbedding with two faces. (It is clear that all the directed edges in each of the two faces are distinct.) Thus, we have the formula, $\gamma_M(M_n) = \left[\frac{1}{2}\beta(M_n)\right] = n - 3$.

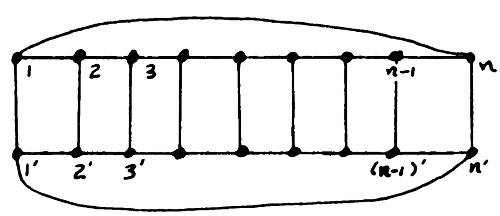
It has now become clear that if one is successful, Edmonds' technique provides a very clever way in which to produce cellular imbeddings. In the next theorem, we use both this technique and the edge adding technique as described in Chapter 3. The next planar graph which we consider is the graph which we call the "rounded" ladder graph.

Let $L_{2n} = C_n \times K_2$, where C_n is the cycle on n vertices $(n \geq 3)$. We call L_{2n} the "rounded" ladder graph on 2n vertices. It has the following parameters: $E = 3n$, $V = 2n$, $\beta(L_{2n}) = n + 1$.

<u>Theorem 4.3</u>: The maximum genus of L_{2n} is given by the formula

$$\gamma_M(L_{2n}) = \left[\tfrac{1}{2}\beta(L_{2n})\right] = \left[\tfrac{1}{2}(n+1)\right].$$

<u>Proof</u>: We use induction on n. The following illustration indicates the labeling we impose on L_{2n}:



Let $n = 3$, then $\beta(L_6) = 4$. We are thus looking for a one face cellular imbedding. We use the following permutations at the vertices of L_6:

p_1: (2 1' 3)

p_2: (1 2' 3)

p_3: (2 3' 1)

$p_{1'}$: (2' 1 3')

$p_{2'}$: (1' 2 3')

$p_{3'}$: (3 2' 1'),

We use these permutations to trace the orbit which contains the directed edge $[1,3]$:

$$-1-3-2-1-1'-3'-3-1-2-2'-3'-1'-2'-2-3-3'-2'-1'-,$$

Hence, all 18 directed edges of L_6 are used in this single face. The imbedding is then the desired cellular imbedding with one face. Our induction is now anchored.

We now suppose that n = k+1 and that the graph L_{2k} has an imbedding on a surface of genus $\left[\frac{1}{2}\beta(L_{2k})\right]$. The cases n = k+1 is odd and n = k+1 is even are most easily done separately.

Case I: Let n = k+1 be an even integer.

Since $\beta(L_{2k})$ = k+1, L_{2k} must have a cellular imbedding with one face because it is upper imbeddable by hypothesis and because its Betti number is even. L_{2k} is labeled as indicated in the above diagram.

Between the vertices (k-1) and k insert a new vertex "a" on the edge ((k-1),k). Similarly, insert a vertex a' on the edge ((k-1)',k'). Form an imbedding for this new graph by changing the vertex permutations at k, (k-1), k', and (k-1)' and leaving all other permutations as they were in the imbedding for L_{2k}. The changes are made as follows:

substitute a for k in p_{k-1}

substitute a for k-1 in p_k

substitute a' for k' in $p_{(k-1)'}$

substitute a' for (k-1)' in $p_{k'}$.

and employ the new vertex permutations at a and a':

$$p_a: \ (k \ (k-1))$$
$$p_{a'}: \ (k' \ (k-1)'),$$

This permutation scheme now gives a cellular imbedding with one face for the new graph $L_{2k} + a + a'$. We now employ the edge adding technique to form L_{2k+2} from this new graph by adding the edge (a,a'). By the theorem for the edge adding technique, we may add the edge (a,a') in such a way that the resulting graph, L_{2k+2}, has a cellular imbedding with two faces. Since $\beta(L_{2k+2}) = (k+1) + 1 = k + 2$, which by assumption is an odd integer, this imbedding has the desired number of faces. Case I follows by mathematical induction.

<u>Case II</u>: Let n = k+1 be an odd integer.

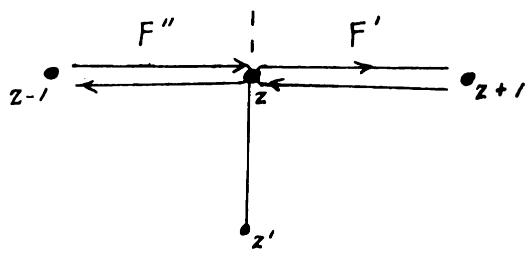
By the induction hypothesis, L_{2k} is upper imbeddable and has a cellular imbedding with two faces. We intend to employ the edge adding technique once again. However, the choice of the edges upon which a and a' may be inserted must be more judiciously chosen. We must join two vertices which are in different faces in order to have an imbedding of L_{2k+2} with one face. We begin with the

<u>Claim</u>: There is a vertex "a" in L_{2k} such that the directed edges $[a,a+1]$ and $[a+1,a]$ are in different faces for the cellular

imbedding of L_{2k} with two faces. (Where $(k+1)\equiv 1 (\mathrm{mod}\ k)$).

Proof of Claim: We employ the indirect method of proof. Suppose that for each vertex $b \in V(L_{2k})$, the directed edges $[b,b+1]$ and $[b+1,b]$ are in the same face.

Since not all edges in a particular orbit can be of the form $[i,i']$ or $[i',i]$, there is a vertex "z" so that $[z,z+1]$ and $[z-1,z]$ are in different faces, say F' and F" respectively. If we let the arrows in the following schematic diagram indicate faces, we have the representation:



By assumption, $[z,z-1]$ is also in F" and $[z+1,z]$ is also in F'. Consequently, the vertex permutation at z must have $p_z(z-1)=z'$ because $[z-1,z]$ and $[z,z+1]$ cannot be in the same face. Likewise, $p_z(z+1) = z'$, since $[z+1,z]$ and $[z,z-1]$ cannot be in the same face. By Edmonds' theorem, the given imbedding can only be cellular if the vertex z' has degree one. However, each vertex of L_{2k} is of degree three. Hence, we have our contradiction and the claim is established.

Without loss of generality, we assume that the vertex "a" of the claim is one of the vertices whose label is not a primed number. Let $[a', (a+1)']$ be in some face of the imbedding, say F'. By the above claim, the directed edges $[a,a+1]$ and $[a+1,a]$ are in each of the two faces of the imbedding.

As in case I we insert the vertex b on the edge (a,a+1) and the vertex b' on the edge (a',(a+1)') and form a new imbedding of this graph exactly as was done in case I. We then have an imbedding of the resulting graph which is cellular with two faces. Furthermore, two faces may be chosen so that b and b' are in different ones. The edge adding technique then applies to allow us to add the edge (b,b') in such a way that these two faces become one. However, when the edge (b,b') is added, the resulting graph is the graph L_{2k+2}. Hence, the graph L_{2k+2} has a cellular imbedding with two faces. Since $\beta(L_{2k+2}) = k+1$, which is odd, we have an imbedding with precisely the number of faces desired for the graph to be upper imbeddable.

By induction, the graph L_{2n} is an upper imbeddable graph for all natural numbers n. The theorem is proven.

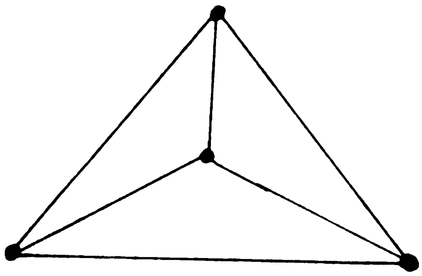
The proof of the above theorem illustrates the manner in which the Edmonds' technique and the edge adding technique may be employed in conjunction in order to obtain a desired result.

We now turn our attention to five planar graphs which are historically important, namely the regular polyhedral graphs. The tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron graphs are shown with their planar imbeddings in Figure 4.1. [11]

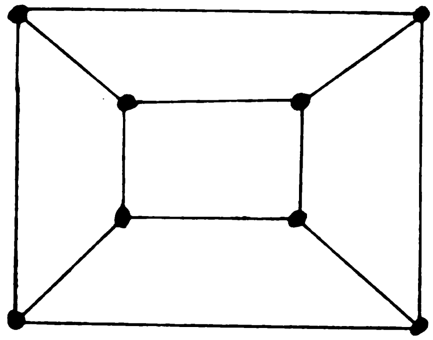
We find the maximum genus of these five graphs in the

<u>Theorem 4.4</u>: All of the regular polyhedral graphs are upper imbeddable.

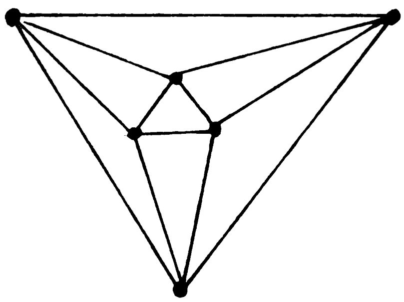
<u>Proof</u>: The tetrahedral graph is the graph K_4, which is studied



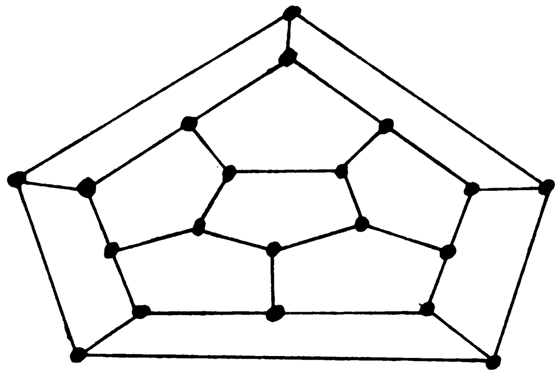
Tetrahedron β = 3



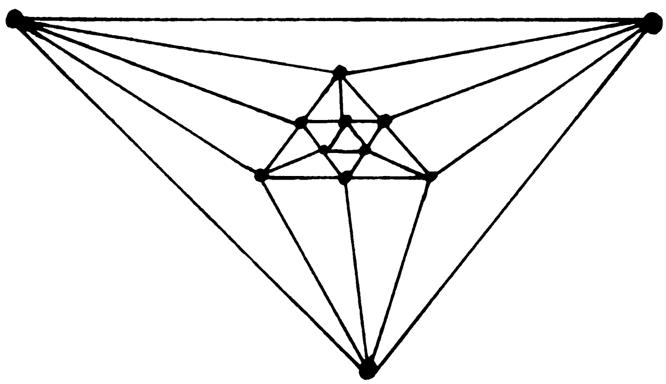
Cube β = 5



Octahedron β = 7



Dodecahedron β = 11



Icosahedron β = 19

Figure 4.1 The Regular Polyhedral Graphs in Chapter 7. The graph of the cube is L_8, and is thus upper imbeddable by Theorem 4.3.

The dodecahedral graph has a cellular imbedding with two faces which can be traced using the cyclic permutations indicated in Figure 4.2 (i.e. an arrow at a vertex means that the permutation at that vertex acts cyclically as the arrow indicates).

Both the octahedral graph and the icosahedral graphs are most easily handled by the edge adding technique, beginning with the innermost triangle. We illustrate the procedure for the icosahedral graph in Figure 4.3. At each step we add three vertices and the number on the edge indicates the number of faces the new imbedding for the graph has after that edge is added. The darkened vertices are those just added. The edge adding is always done beginning with the new vertex farthest to the left and adding edges in a counter-clockwise manner about the original figure. By this edge adding, then, the icosahedral and octahedral graphs are also upper imbeddable. This completes the theorem.

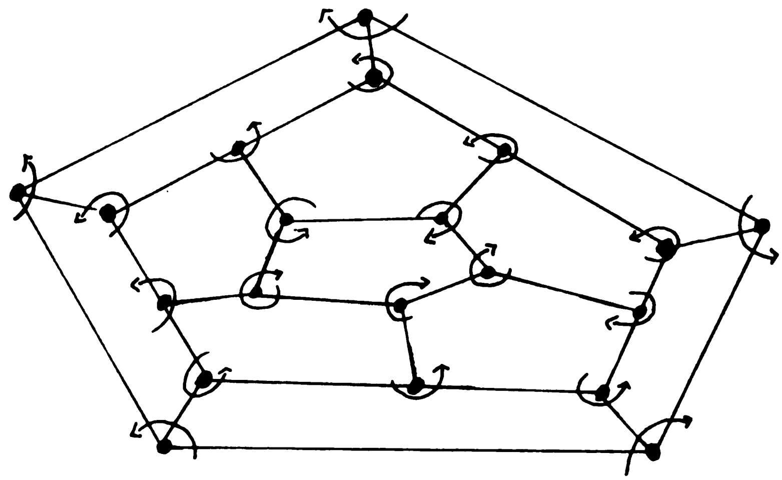


Figure 4.2 The Dodecahedral Graph with Permutations

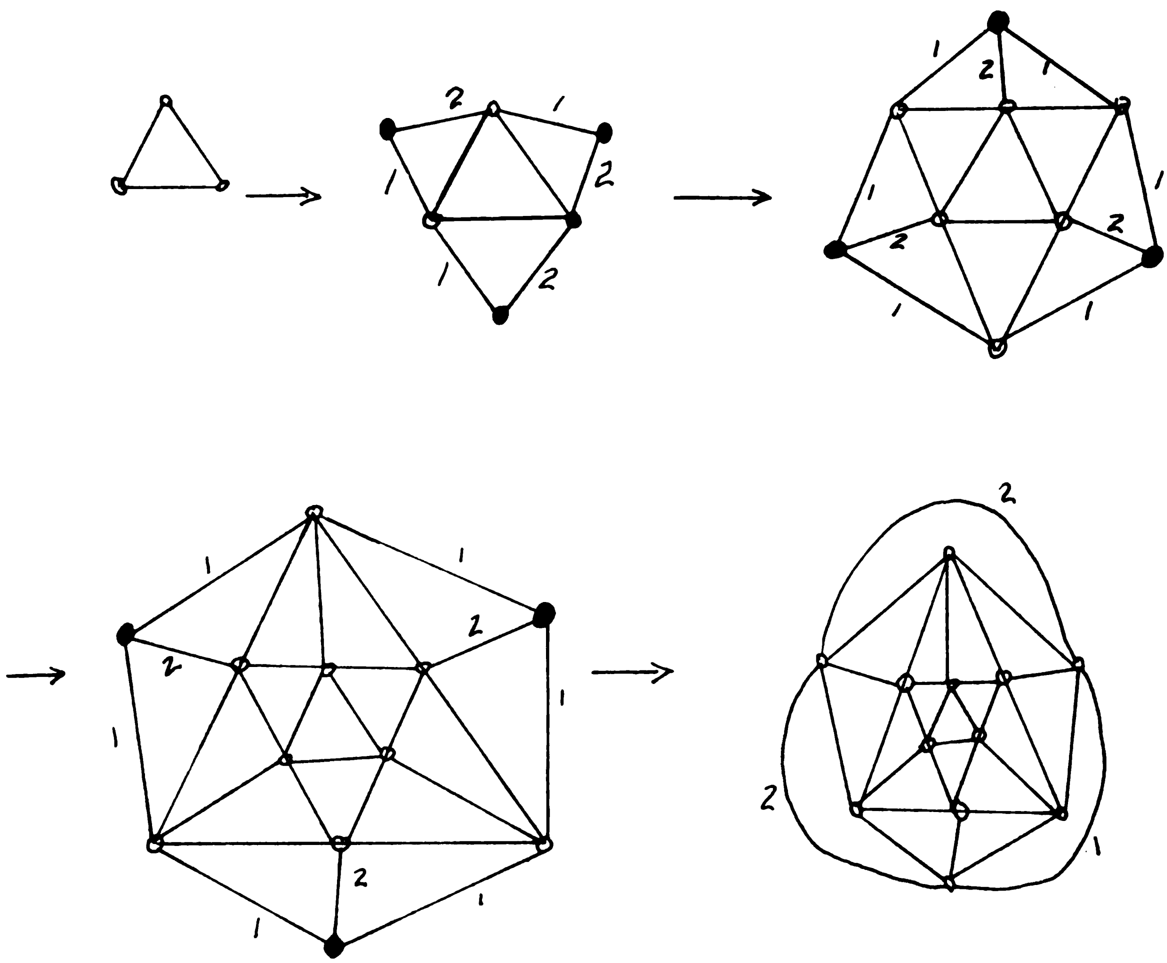


Figure 4.3 Adding Edges for the Icosahedral Graph

CHAPTER 5

PROPERTIES OF THE MAXIMUM GENUS OF A GRAPH

This chapter will investigate some of the properties of maximum genus. Three problems of interest are discussed. As was mentioned in Chapter 3, a major difficulty occurring with computations of maximum genus is the fact that the sum of the maximum genera of the blocks of a connected graph G may be less than the maximum genus of G. The first section of this chapter seeks to give some insight into this problem.

Since two extreme kinds of genera for a connected graph are defined, the question arises of finding graphs for which these two genera are equal. In the second section of this chapter all graphs for which equality holds are determined.

The third part of the chapter deals with characterizing those planar graphs whose maximum genus is one. Such a characterization theorem completes the chapter.

We first determine some connected graphs which have the maximum genus summability property in the

Theorem 5.1: Let G_1 and G_2 be vertex disjoint connected graphs, and $G = G_1(e)G_2$ be the connected graph formed from G_1 and G_2 by adding one edge $e = xy$, joining a vertex x in G_1 to a vertex y in G_2. Then

$$\gamma_M(G) = \gamma_M(G_1) + \gamma_M(G_2).$$

Proof of Theorem 5.1: Let T_1 be any cellular imbedding of G_1 with permutation scheme P^1. The permutation scheme at x is given by p_x^1: $(x_1\ x_2 \ldots)$, where x_1 may equal x_2 if the degree of x is one. Let T_2 be any cellular imbedding of G_2 with permutation scheme P^2. The permutation scheme at y is given by p_y^2: $(y_1\ y_2\ \ldots)$ where y_1 may equal y_2 if y has degree one. We may then form a cellular imbedding T of G by leaving $p_v = p_v^1$ for v in G_1, $v \neq x$; leaving $p_u = p_u^2$ for u in G_2, $u \neq y$; then define p_x: $(x_1\ y\ x_2\ \ldots)$ and p_y: $(y_1\ x\ y_2\ \ldots)$.

There is a face f' in T_1 with f': $-x_1-x-x_2-(q')-$ and a face f" in T_2 with f": $-y_1-y-y_2-(q")-$, where q' and q" are just notational conveniences to indicate the portion of the faces not explicitly given. These two faces are combined in T for G into one face given by f: $-x_1-x-y-y_2-(q")-y_1-y-x-x_2-(q')-$. This one face uses all of the innovations in T as compared with T_1 and T_2, so all the other faces of T are the same as faces in T_1 and T_2. Hence we have

$$F(T) = F(T_1) + F(T_2) - 1\,, \tag{1}$$

where F(s) indicates the number of faces in the imbedding s, s = T, T_1, or T_2.

Conversely, if we are given T for G, we reverse the above definitions and arguments to find T' and T" for G_1 and G_2 respectively, with the property

$$F(T') + F(T") = F(T) + 1. \tag{2}$$

Suppose T* is a maximum cellular imbedding of G corresponding to $\gamma_M(G)$. Then $F(T^*) \leq F(T)$ for every cellular imbedding T of G. Using

(2) we know there exists a cellular imbedding $(T^*)'$ for G_1 and a cellular imbedding $(T^*)''$ for G_2 with $F(T^*) = F((T^*)') + F((T^*)'') - 1$. However, we may also begin with maximum imbeddings of G_1 and G_2.

Let T_1^* be a cellular imbedding of G_1 corresponding to $\gamma_M(G_1)$, so that $F(T_1^*) \leq F(T_1)$ for every cellular imbedding T_1 of G_1. Similarly, let T_2^* be a cellular imbedding of G_2 corresponding to $\gamma_M(G_2)$, so that $F(T_2^*) \leq F(T_2)$ for every cellular imbedding T_2 of G_2. Using (1) we know there exists a cellular imbedding T of G with $F(T) = F(T_1^*) + F(T_2^*) - 1$.

By the minimal properties of $F(T_1^*)$ and $F(T_2^*)$ it follows that

$$F(T) = F(T_1^*) + F(T_2^*) - 1 \leq F((T^*)') + F((T^*)'') - 1 = F(T^*).$$

Should strict inequality hold in the above relation, the minimal property of $F(T^*)$ would be contradicted. Hence equality holds and we make the following computation:

$$\begin{aligned} 2\gamma_M(G) &= 2 + E - V - F(T^*) \\ &= 2 + (E_1 + E_2 + 1) - (V_1 + V_2) - (\, F(T_1^*) + F(T_2^*) - 1\,) \\ &= (\,2 + E_1 - V_1 - F(T_1^*)\,) + (\,2 + E_2 - V_2 - F(T_2^*)\,) \\ &= 2\,\gamma_M(G_1) + 2\,\gamma_M(G_2). \end{aligned}$$

Division by 2 then completes the proof of the theorem.

<u>Corollary 5.2</u>: Let G_i, $i = 1,2,\ldots,n$, be connected graphs with all of the G_i pairwise vertex disjoint. Define $G = G_1(e_1)G_2(e_2)G_3(e_3)\ldots(e_{n-2})G_{n-1}(e_{n-1})\,G_n$, where $e_i = x_ix_{i+1}$ is an edge joining $x_i \in G_i$ to $x_{i+1} \in G_{i+1}$. Then

$$\gamma_M(G) = \sum_{i=1}^{n} \gamma_M(G_i).$$

Proof of Corollary 5.2: We use induction on the number of connected graphs n. The case n = 1 is clear and Theorem 5.1 makes the induction step.

We now turn our attention to connected graphs composed of blocks which do not have K_2 so conveniently located. Let G_1 and G_2 be two connected graphs which share exactly one vertex. We intend to determine the conditions necessary in order to add the maximum genera of these two graphs and obtain the maximum genus of the newly formed graph. We begin with a lemma.

Lemma 5.3: Let T be a cellular imbedding of a connected graph G such that there is a vertex c of G which is contained in the boundary of at least two faces of the imbedding T. Then in G there exists a vertex b, which is adjacent to c, such that the directed edges $[b,c]$ and $[c,b]$ occur in different faces of T.

Proof: Let V(c) denote the set of vertices of G adjacent to c, and let F' be a face of T containing c in its boundary. We define $V' \subseteq V(c)$ as follows:

$$V' = \left\{ x \in V(c) \;\middle|\; \text{either } [x,c] \text{ or } [c,x] \text{ is in } F' \right\}.$$

Then $V' \neq \emptyset$ since c is in F'.

If $V' = V(c)$, let $[y,c]$ be a directed edge in some face $F'' \neq F'$; by assumption, such a vertex y exists. Then $y \in V(c) = V'$, and thus $[c,y]$ is in F'. Thus, y is a vertex of the desired kind.

Assume $V' \subsetneqq V(c)$. Let p_c be the vertex permutation at c of the permutation scheme corresponding to the imbedding T. By Edmonds' theorem,

p_c is a cyclic permutation. Hence, there is a vertex $x_o \in V'$ so that $p_c(x_o) \notin V'$. Thus, $[c, p_c(x_o)] \notin F'$ and hence $[x_o, c] \notin F'$.

Since $x_o \in V'$, we must then have $[c, x_o] \in F'$. Thus x_o is the desired vertex.

Theorem 5.4: Let T be a cellular imbedding of a connected graph G. Suppose there is a vertex c of G which is contained in at least three faces of T. Then there is a cellular imbedding T' of G which has two fewer faces than the imbedding T.

Proof: By Lemma 5.3, we may choose two faces of T, F' and F", along with a vertex b $\in V(c)$, so that $[b,c] \in F'$ and $[c,b] \in F"$.

However, by hypothesis there is a third face F in which c is also contained.

The vertex c and the faces F',F",F serve to make the imbedding T satisfy the hypothesis of the face reduction theorem of Duke discussed in Chapter 3.

Hence, by that theorem, G has an imbedding with two fewer faces than T.

Corollary 5.5: Let T be a maximum cellular imbedding of a connected graph G. Then there is no vertex of G which occurs in the boundary of more than two faces of T.

Proof: By the theorem a vertex contained in three or more faces would permit an imbedding with fewer faces, so the imbedding could not be maximum.

In order to facilitate the statement of the next theorem we introduce some additional notation. Let c be a cutpoint of a connected graph

G. We <u>separate G at c into two connected graphs G_1 and G_2</u> as follows: Let H_i, $i = 1,\ldots,m$, be the blocks of G which contain c, and let k be any integer $1 \leq k \leq m$. Let $G_1 = \bigcup_{i=1}^{k} H_i$ and $G_2 = \bigcup_{i>k}^{m} H_i$, where we intend for the union to be done so that the blocks of both G_1 and G_2 share the vertex c. G is said to be <u>separated at c</u> into the connected graphs G_1 and G_2. We are now prepared to state the theorem.

<u>Theorem 5.6</u>: Let G be a connected graph which is separated at a cutpoint c into two connected graphs G_1 and G_2. Let T_i be a cellular imbedding of G_i having M_i faces, $i = 1$ or 2. Suppose c is contained in at least n faces for each of T_1 and T_2, where n is a positive integer. Then there is a cellular imbedding of G with $M_1 + M_2 - (2n - 1)$ faces.

<u>Proof</u>: We first note that $n \geq 1$. Let P_1 and P_2 be the Edmonds' permutation schemes corresponding to T_1 and T_2 respectively. Suppose that one of the n faces, F^i, chosen from T_i, is of the form

$$F^i: \left[\left[c,x_2^i\right] \cdots \left[x_1^i,c\right] \right], \ i = 1 \text{ or } 2.$$

We mean for the notation to imply that <u>two</u> faces have been chosen, one from the imbedding T_1, the other from the imbedding T_2. The vertex permutations p_c^i, $i = 1$ or 2, can be written:

$$p_c^1: \ (x_2^1 \cdots x_1^1); \ p_c^2: \ (x_2^2 \cdots x_1^2).$$

We form an imbedding T of G as follows:

$$p_x = p_x^i \text{ if } x \in V(G_i), \ x \neq c, \ i = 1 \text{ or } 2$$
$$p_c: \ (x_2^1 \cdots x_1^1 x_2^2 \cdots x_1^2).$$

Then the faces F^1 and F^2 become one face in this new imbedding of G, while all other faces of T_1 and T_2 are now faces of T.

The 2n faces containing c which were originally chosen from T^1 and T^2 now number only 2n - 2. We also have the above new face containing c. Hence, we now have c contained in at least 2n - 1 faces of T.

If n = 1, the above work completes the proof. If $n \geq 2$, then $(2n - 1) \geq 3$, and the T formed above is an imbedding for which c is contained in at least 3 faces. By Theorem 5.4, G has an imbedding T* with two fewer faces than T. It is clear that if there are now three or more faces of T* in which c is contained, we may again reduce the number of faces in an imbedding of G, by again applying Theorem 5.4, this time to T*.

The question now becomes that of knowing how many times such a reduction of faces can occur before we obtain an imbedding of G for which c is contained in fewer than three faces. We cannot determine the exact number of times such an occurrence can happen. However, we do establish the claim which follows.

<u>Claim</u>: We may apply Theorem 5.4, to reduce the number of faces in an imbedding of G, at least n-1 times.

<u>Proof of Claim</u>: Suppose we apply Theorem 5.4 the first time as shown above; i.e. after two of the chosen 2n faces have been combined into a single face. The imbedding T still has the other 2n-2 faces which we originally chose from T_1 and T_2. In addition, T has the face which combined F^1 and F^2. Hence, c is contained in at least 2n-1 faces of T.

Now suppose we have done a reduction, using Theorem 5.4, k times. Each such application results in the loss of two faces in which c was contained. We need only ask how large k may be and there still be at least 3 faces containing c in the new imbedding. We need the inequality $(2n-1) - 2k \geq 3$ to be true in order to reduce at least one more time. (i.e. $(2n-1)$ is the largest possible number of the originally chosen 2n faces which we may have used; it is possible that we haven't used this many of these chosen faces, since some of the reductions might have involved faces which were not among the ones originally chosen). The above inequality is true as long as $k \leq (n-2)$. Thus, we can safely apply the reduction of faces theorem n-2 times and still have c contained in at least three faces of the last imbedding obtained.

Under these circumstances, the reduction of faces theorem may be applied at least one more time. Thus, we can safely apply the Theorem 5.4 at least (n-1) times. The claim is established.

We now count the total number of faces which have been lost from the original $M_1 + M_2$ faces of T_1 and T_2 by these (n-1) applications of Theorem 5.4. Two such faces were lost each time the theorem was applied. Hence, repeated applications of the theorem resulted in the loss of 2(n-1) faces. However, one face was lost before such applications of the theorem were begun. Hence, we have lost a total of 2n-1 faces. Thus, we have obtained an imbedding of G which has $M_1 + M_2 - (2n-1)$ faces, as desired. This completes the proof of the theorem.

We now give some corollaries which give insight into the problem of adding the maximum genera of two connected graphs which share a single vertex. An important consequence of these corollaries is to notice that whether or not one can add the maximum genera of two such graphs is in one sense dependent upon the differences $\left[\frac{1}{2}\beta(G_i)\right] - \gamma_M(G_i)$ for the two graphs G_1 and G_2, $i = 1$ or 2.

<u>Corollary 5.7</u>: Let G be a graph as described in Theorem 5.6, with each of the T_i a maximum imbedding, $i = 1$ or 2. Then $\gamma_M(G) \geq \gamma_M(G_1) + \gamma_M(G_2) + (n-1)$.

<u>Proof</u>: Let T_1 and T_2 in the theorem be imbeddings on compact orientable manifolds of genus $\gamma_M(G_1)$ and $\gamma_M(G_2)$ for the graphs G_1 and G_2 respectively.

Using Euler's formula and the following inequality derived from Theorem 5.6,

$$\delta_M(G) \leq \delta_M(G_1) + \delta_M(G_2) - (2n-1),$$

we obtain the desired result.

<u>Corollary 5.8</u>: Let G be a connected graph with a cutpoint c. Suppose that when G is separated at c, the separation is done in such a way that both G_1 and G_2 are upper imbeddable. Further, suppose that if $\beta(G_i)$ is odd, then c is involved in each of the two faces of the maximum imbedding for G_i, $i = 1$ or 2. Then G is also an upper imbeddable graph.

<u>Proof</u>: If either of the Betti numbers for the two graphs G_1 and G_2 is even, then the theorem gives a cellular imbedding with either one or two faces, depending upon the parity of the other Betti number. In either case, the graph G is upper imbeddable.

Suppose that both the Betti numbers are odd. Then we place $n = 2$ into Corollary 5.7 to obtain the inequality $\gamma_M(G) \geq \gamma_M(G_1) + \gamma_M(G_2) + 1$. However, since both Betti numbers are odd, the following relation holds:

$$\left[\tfrac{1}{2}\beta(G)\right] = \left[\tfrac{1}{2}(\beta(G_1) + \beta(G_2))\right] = \left[\tfrac{1}{2}\beta(G_1)\right] + \left[\tfrac{1}{2}\beta(G_2)\right] + 1.$$

Since G_1 and G_2 are both upper imbeddable, we have the inequality $\gamma_M(G) \leq \gamma_M(G_1) + \gamma_M(G_2) + 1$, from the upper bound property of maximum genus. Hence, the corollary is established.

We are now in a position to give a necessary condition for the maximum genus of a connected graph to be the sum of the maximum genera of two connected graphs into which it is separated.

Theorem 5.9: Let G be a connected graph which is separated at a cutpoint c into two connected graphs G_1 and G_2. If $\gamma_M(G) = \gamma_M(G_1) + \gamma_M(G_2)$, then either G_1 or G_2 has a maximum imbedding in which c is contained in exactly one face.

Proof: We prove the contrapositive of the theorem. Suppose that all maximum imbeddings for both G_1 and G_2 have c involved in two faces. We then place n = 2 into Corollary 5.7 and obtain the inequality $\gamma_M(G) \geq \gamma_M(G_1) + \gamma_M(G_2) + 1$. This disallows the possibility for summing the maximum genera, and the theorem is completed.

We now present a corollary which is just a special case of Theorem 5.9. However, since strongly regular graphs are becoming ever more frequently studied, we state the corollary.

Corollary 5.10: Let G be a connected graph which separates at a cutpoint c into two strongly regular graphs G_1 and G_2. If $\gamma_M(G) = \gamma_M(G_1) + \gamma_M(G_2)$, then either G_1 or G_2 is upper imbeddable with even Betti number.

Proof: If both G_1 and G_2 have at least two faces in their maximum imbeddings, then some vertex is contained in two faces. Since all the vertices of a strongly regular graph have exactly the same adjacency properties, the vertex c may be considered to be that vertex. Corollary 5.7 would then apply to destroy the possibility for summability.

We next turn our attention to the determination of those graphs whose maximum genus and ordinary genus are equal. We first define two very special graphs, which play an important role in the characterization theorem which follows.

The graph H is the graph which is obtained from the wheel graph W_5 by deleting a one factor from C_4. The graph Q is a graph to which $K_{2,3}$ is homeomorphic (i.e. $K_{2,3}$ is isomorphic to a subdivision of Q.) Figure 5.1 illustrates the two graphs.

A <u>cactus</u> is a connected graph each of whose blocks is either a cycle or K_2.

Figure 5.1 The Graphs H and Q

<u>Theorem 5.11</u>: Let G be a connected graph. Then the following statements are equivalent:

i.) $\gamma_M(G) = \gamma(G)$

ii.) G does not contain a subgraph homeomorphic to either H or Q.

iii.) G is a cactus all of whose cycles are vertex disjoint.

iv.) $\gamma_M(G) = \gamma(G) = 0$.

<u>Proof of Theorem 5.11</u>: We use the circular set of implications i.) ⇒ ii.) ⇒ iii.) ⇒ iv.) ⇒ i.).

<u>i.) ⇒ ii.)</u>

The proof of this portion of the theorem can be found in the paper by Nordhaus, Ringeisen, Stewart, and White. [9]

<u>ii.) ⇒ iii.)</u>

We first show that under condition ii.), the graph G can have no blocks which are not cycles. Secondly, we show that the condition forces all the cycles of G to be pairwise vertex disjoint.

Let B be any block which is not a cycle. We show that B contains a subgraph which is homeomorphic to Q. Let a and b be any two vertices of the block B. By a well known property of blocks, there is a cycle C through a and b. Since we are assuming that B is a block which is not a cycle, there is at least one vertex of B which is not on C. Consequently, there is such a vertex, call it d, which is adjacent to a vertex on C, which we name c (see Figure 5.2). By a characterization of a block, there is a cycle, say C_1, which contains both the undirected edges (c,d) and (k,c), where k is some vertex of C which is adjacent to c. Then C and C_1 are cycles which share at least one edge. Hence the graph composed of these two cycles contains a subgraph homeomorphic to Q, contrary to the assumption ii.).

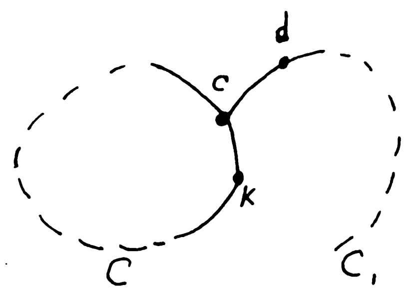


Figure 5.2 Obtaining the Graph Q

Thus, G can have no block which is not a cycle or K_2. Hence, G is a cactus.

Suppose further that G now contains two cycles which share a vertex. Clearly, G then contains a subgraph which is homeomorphic to H, contrary to condition ii.).

Consequently, G is a cactus all of whose cycles are disjoint.

iii.)$\Longrightarrow$iv.)

Since G is a cactus all of whose cycles are disjoint, G is clearly a graph of the kind described in Corollary 5.2. We let each of the G_i in this corollary be either a cycle or K_2. We can then compute the maximum genus of G by adding the maximum genera of the G_i. Since the Betti numbers of all such blocks are zero, $\gamma_M(G) = 0$. Because the maximum genus of a connected graph is always at least as large as the ordinary genus, the ordinary genus of G is also zero.

iv.)$\Longrightarrow$i.)

This is an obvious implication.

By the circular nature of our proof, the theorem is proven, and all the above statements are equivalent.

We now introduce some notations and definitions, along with some theorems concerning them. Although the graphs defined are themselves of interest, our purpose in introducing them now is to lead ultimately to the characterization of graphs whose maximum genus is one. We define various graphs which we shall call chain graphs, butterfly graphs, and rose graphs.

Let G be a connected graph. The chain graph of length m with base G is defined inductively, and denoted $C_m(G)$, where $C_1(G) = G$. To define

$C_2(G)$ from $C_1(G)$, choose two not necessarily distinct vertices of $C_1(G)$, a and b. Form $C_2(G)$ by adding a path from a to b. To avoid loops or multiple edges, if a = b, we insist that the path be of length at least three; if a and b are adjacent the path must be of length at least two. To define $C_{k+1}(G)$ from $C_k(G)$, $k \geq 2$, we choose two not necessarily distinct vertices of $C_k(G)$, which were not vertices of $C_{k-1}(G)$. Add a path between the chosen vertices, subject to the same restrictions on length given when forming $C_2(G)$ from $C_1(G)$. Figure 5.3a gives some examples of chain graphs.

The <u>butterfly graph with n wings</u> is denoted B_n. It is formed by beginning with nontrivial path P of any length and adding n pairwise vertex disjoint paths of arbitrary length between the endpoints of P. In order that B_n not be a multigraph, we insist that not more than one of the n+1 paths (including P) be of length one. Figure 5.3b gives some examples of butterfly graphs.

The <u>rose graph with n petals</u> is the graph obtained by attaching n cycles to a vertex b, in such a way that no two of the cycles share more than the vertex b. A rose graph with n petals will be denoted by R_n. Figure 5.3c gives some examples of rose graphs.

We first remark that all the butterfly graphs and rose graphs are planar. Furthermore, the chain graphs have the same genus as that of their particular base graph. To see this one need only imagine adding the new path of $C_{k+1}(G)$ within a face of $C_k(G)$, an operation which leaves the genus unchanged.

Further notice that if one uses the theorems and corollaries of the earlier portions of this chapter, one can find the maximum genus of a graph which is a combination of chain, rose, or butterfly graphs, by knowing the maximum genus of these basic graphs. We thus turn to the task of finding the maximum genus of these newly defined graphs.

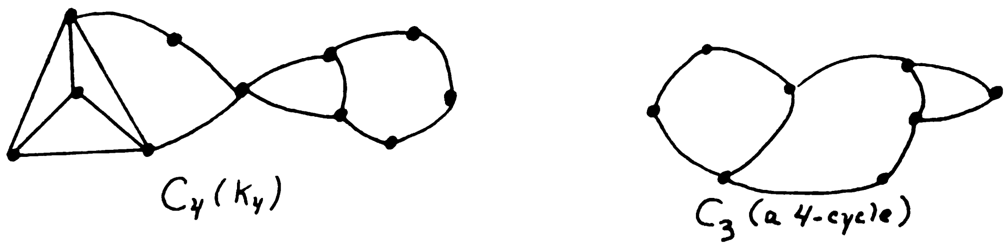


Figure 5.3a

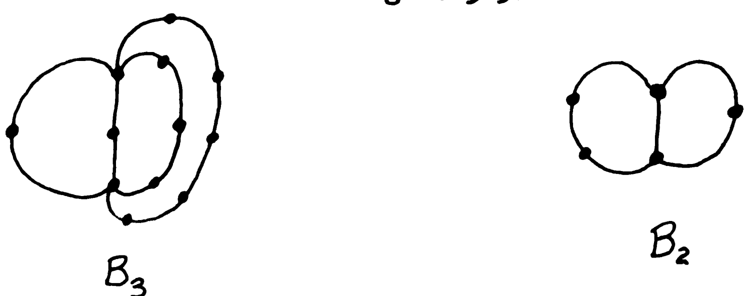


Figure 5.3b

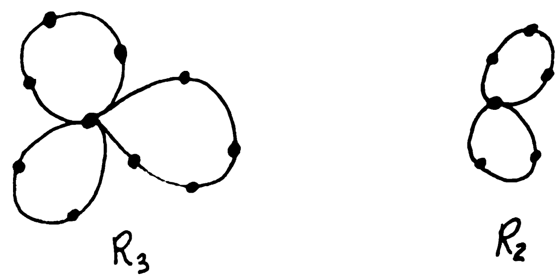


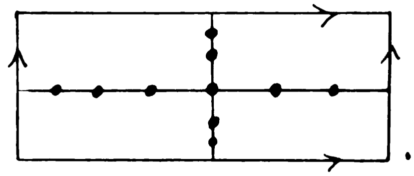
Figure 5.3c

Figure 5.3 Three New Kinds of Graphs

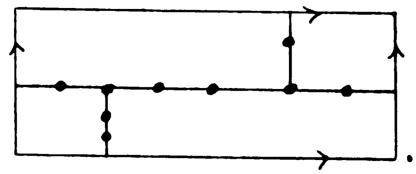
<u>Lemma 5.12</u>: Let G be either one of the two graphs R_2 or B_2. Then $\gamma_M(G) = 1$. (Notice that if K is any cycle, we have $B_2 = C_2(K)$.)

<u>Proof</u>: Each of the two given graphs have Betti number two. Hence, by the upper bound theorem, each has maximum genus no larger than one. Thus, we display cellular imbeddings of each graph on the torus and complete the proof.

An imbedding for R_2 is:



An imbedding for B_2 is:



Notice that each of the two imbeddings is indeed cellular.

The next two theorems will determine the maximum genera of the butterfly graphs, the rose graphs, and some chain graphs.

Theorem 5.13: Let G be an upper imbeddable graph. If $\beta(G)$ is odd, form $C_2(G)$ from $C_1(G) = G$ by adding a path between two vertices of G which occur in two different faces of the maximum cellular imbedding of G. Then $C_k(G)$ is upper imbeddable for all k.

Proof: We use mathematical induction on n, the length of the chain graph. Notice that $\beta(C_n(G)) = \beta(G) + (n-1)$.

If n = 1, then the chain is only the base graph itself, which is assumed to be upper imbeddable. If n = 2, then $C_2(G)$ is upper imbeddable because the edge adding technique may be applied to give either a one (if $\beta(G)$ was odd) or a two (if $\beta(G)$ was even) face cellular imbedding of $C_2(G)$.

Suppose that any chain graph of length k is upper imbeddable and let C be a chain graph of length k+1, where we are assuming that $k+1 \geq 3$.

We employ the edge adding technique and the argument is done in two cases. Without loss of generality, assume that $\beta(G)$ is even.

Case 1: Assume k+1 is even.

Consider the chain graph C' of length k with base G, which is a subgraph of C. This graph differs from C only by the lacking of the path which was added to form C from C'. By induction, C' is an upper imbeddable graph. Because k is odd, and because $\beta(C') = (k-1) + \beta(G)$, C' must have a cellular imbedding with one face.

The procedure now becomes clear. We add the path needed to form C in such a manner that C obtains a two face cellular imbedding from the one face cellular imbedding for C'. Because a path may be obtained by adding an edge and then inserting vertices of degree two on the path, the imbedding problem can be regarded as that of adding an edge. That the desired imbedding is obtained is guaranteed by the edge adding technique. Hence, C has a cellular imbedding with two faces. Case 1 is completed.

Case 2: Assume k+1 is odd.

Let C' be the chain graph as defined in Case 1. Consider the chain graph C" of length k-1 with base G which is a subgraph of C'. C" then has even Betti number, since $\beta(C") = (k-2) + \beta(G)$.

Form C' from C" in a manner analogous to that used in Case 1 to form C from C'. By the edge adding technique, each edge of the newly added path has its two directed edges in different faces of the two face cellular imbedding of C'. We form C from C' as was done in Case 1. Because the endpoints of the path to be added may be considered to be involved in different faces, the path may be added so that a cellular imbedding of C with one face results. Thus, C has a cellular imbedding with one face and Case 2 is completed.

The proof is completed when we realize that if the base graph would have had odd Betti number, the proofs of Cases 1 and 2 would have been reversed.

The theorems for the rose and butterfly graphs are done similarly. The two kinds of graphs are similar enough in their structure that we find their maximum genera simultaneously in the

Theorem 5.14: The rose and butterfly graphs are upper imbeddable.

Proof: We use induction of n, the number of petals (wings) which the given graph has.

Lemma 5.12 establishes the result for $n = 2$ and the case $n = 1$ is a cycle and is thus upper imbeddable.

The induction argument is done in two cases and exactly as was done in the proof of Theorem 5.13. Consequently, we do not give any further details of the argument here.

We now put these theorems to use in characterizing those connected graphs which have maximum genus one.

Since maximum genus and genus can be equal only when both are zero, graphs of maximum genus one must necessarily be planar. (Theorem 5.11).

One will notice in what follows that we have made frequent use of what we have learned concerning the structure of graphs of maximum genus zero. We would also hope that graphs with maximum genus one would have a relatively simple structure as did those of maximum genus zero.

The procedure which we follow is to introduce and prove several lemmas, and to then summarize their results in a characterization theorem.

Graphs of maximum genus one are first characterized to within blocks of maximum genus one. Some comments concerning blocks of maximum genus one are then made to complete the chapter.

Throughout what follows, a <u>nontrivial block</u> will be any block which is not the graph K_2. Corollary 5.2 allows us to confine our study to such blocks.

<u>Lemma 5.15</u>: Let G be a connected graph with $\gamma_M(G) = 1$. Then G has at most one nontrivial block.

<u>Proof</u>: We use a proof by contradiction, by supposing that G has more than one nontrivial block.

Since any such block must contain a subgraph of the form of a 2-winged butterfly, B_2, Lemma 5.12 applies to yield that such a block must have maximum genus at least one. Since the maximum genus of a connected graph must be at least as large as the sum of the maximum genera of its blocks, the maximum genus of G would have to be at least two. We have our contradiction, and the lemma follows.

We find it necessary to define the nontrivial multiplicity of a cutpoint of a graph. The <u>nontrivial multiplicity of a cutpoint c</u>, denoted m(c), is the number of nontrivial blocks in which c is contained.

<u>Lemma 5.16</u>: Let G be a connected graph with $\gamma_M(G) = 1$. Then the following are true:

i.) For any cutpoint c of G, $m(c) \leq 3$.

ii.) If c is a cutpoint with $m(c) = 3$, then each block at c is a cycle. Furthermore, no other cutpoint of G has nontrivial multiplicity larger than one.

Proof: Suppose that G has a cutpoint whose nontrivial multiplicity is at least four.

By a well known theorem about blocks, c must be involved in at least one cycle of each of the blocks of which it is a vertex. Hence, G has a subgraph which is a rose with four petals, R_4. However, Theorem 5.14 then applies to give $\gamma_M(R_4) = 2$. Since R_4 is a subgraph of G, we must then have that $\gamma_M(G) \geq 2$, a contradiction. Hence, i.) is proven.

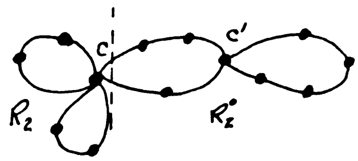
Now suppose that c is a cutpoint with nontrivial multiplicity three. Let one block B containing c be a graph which is not a cycle. Lemma 5.12 implies $\gamma_M(B) \geq 1$. The cutpoint c is contained in a cycle for each of the other two nontrivial blocks containing c. These two cycles may be described as a rose with center c and with two petals. This graph R_2 has been shown to have maximum genus one. Let G' be the subgraph of G composed of R_2 and B, sharing the cutpoint c. By Corollary 5.7 we have $\gamma_M(G') \geq \gamma_M(B) + \gamma_M(R_2) \geq 2$. Since $\gamma_M(G) \geq \gamma_M(G')$, this is a contradiction. Hence, each block at this cutpoint is a cycle.

Now further suppose that there is some other cutpoint of G which has nontrivial multiplicity at least two. Let c' be one such cutpoint so that $d(c',c) = \min_{k \neq c} d(c,k)$, where $d(x,y)$ is the distance in G between two vertices of G and k is any cutpoint with nontrivial multiplicity at least two.

Then either c and c' share a nontrivial block or they do not.

Case 1: Suppose c and c' share a nontrivial block.

Because c has nontrivial multiplicity three, this shared block must be a cycle. Since c' is involved in at least one other nontrivial block, G has a subgraph which can be regarded as two roses R_2 and R_2^0 sharing the vertex c, as the following illustration shows.



By Corollary 5.7 we have $\gamma_M(G) \geq \gamma_M(R_2) + \gamma_M(R_2^0)$, a contradiction.

Case 2: Suppose c and c' do not share a nontrivial block.

Again we carefully choose a subgraph of G. Notice that c and c' must in this case be joined by a path of blocks all of which are copies of the graph K_2. There is a subgraph R_3 at the vertex c and a subgraph R_2 at the vertex c'. By Theorem 5.1, the maximum genus of the subgraph G' consisting of R_3 at c, R_2 at c', and the path of bridges from c to c' is given by the formula $\gamma_M(G') = \gamma_M(R_3) + \gamma_M(R_2) = 2$. Thus, we know $\gamma_M(G) \geq 2$, which is a contradiction. This completes the proof.

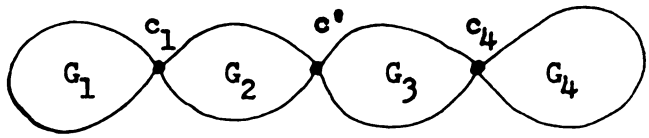
We have considered what could happen in a graph of maximum genus one if one cutpoint of the graph has nontrivial multiplicity three. We now consider what might happen if there is no such vertex.

Lemma 5.17: Let G be a connected graph with $\gamma_M(G) = 1$. Suppose that G has no cutpoint of nontrivial multiplicity three. Then G has no more than two cutpoints of multiplicity two. Furthermore, there are exactly two such cutpoints only if they share a nontrivial block, and this block is either a cycle or a block of maximum genus one.

Proof: We first show that if there are two cutpoints of nontrivial multiplicity two, they must share a nontrivial block. After having shown this, we will show that there are no more than two such cutpoints. This rather inverse order of proof is required because the first given is needed in proving the second.

Let two cutpoints of nontrivial multiplicity two or more be c and c'. Then G contains subgraphs R_2 and R_2' with centers at c and c' respectively. There is a path in G from c to c'. Let G' be the subgraph of G composed of R_2, R_2', and this path. Then, again by Theorem 5.1, $\gamma_M(G') = \gamma_M(R_2) + \gamma_M(R_2') = 2$, which, since G' is a subgraph of G, is a contradiction. Hence, if two such cutpoints exist, they must share a nontrivial block.

Now suppose that G has three or more cutpoints of nontrivial multiplicity two. By the above argument, there must be one, say c', which shares a nontrivial block with each of the two other cutpoints. Hence, we have four blocks appearing in a "chain", each sharing a vertex with the next. An illustration of the manner in which the blocks must be distributed follows.



For blocks G_2 and G_3, there are two cutpoints involved in the "chain" arrangement. Choose a cycle in each block which contains both of these vertices. Choose a cycle in G_1 through c_1 and a cycle in G_4 through c_4. Then, we have a subgraph of G which is a chain graph of length four with base a cycle. By Theorem 5.13, this graph is upper imbeddable and consequently has maximum genus two. This is the desired contradiction of the maximum genus of G. The lemma is proven.

We are now ready to summarize these lemmas and to give a theorem which is a characterization of connected graphs of maximum genus one.

Theorem 5.18: Any connected graph with maximum genus one is a graph of the kind described in Corollary 5.2. Furthermore, exactly one of the graphs G_1, described in that corollary, is not a cycle or K_2. This exceptional graph is either a rose graph with two petals, a rose graph with three petals, a non-block chain graph of length three with a cycle as its base, or is a block of maximum genus one.

Proof: If a connected graph were not one of these kinds, one or more of Lemmas 5.13-5.17 would be contradicted.

Since the graphs described in Theorem 5.18 are of maximum genus one, that theorem is a necessary and sufficient condition for a graph to have maximum genus one.

We close this chapter with a brief discussion of blocks of maximum genus one.

As was mentioned earlier, any block with maximum genus one is necessarily planar. Theorem 5.11 indicates that a block of this type must contain a subgraph homeomorphic to either H or Q, as described in that theorem.

To determine the "kinds" of blocks which might have maximum genus one, one would only need to begin with H or Q (or a subdivision of one of them) and add vertices and edges subject to two conditions. First the graph obtained must be a planar block. Secondly, its maximum genus must be exactly one.

The problem of determining whether a given block has maximum genus one is made easier by using the above processes. At each new addition of vertices or edges, one looks for butterfly, rose, or chain subgraphs. If one obtains a rose with four petals, a butterfly with four wings, or a chain graph of length four, he has added vertices and edges in an improper manner. Any of the graphs just listed has maximum genus two.

The theorems in this chapter are all the equipment which is necessary to examine blocks of maximum genus one. Figure 5.4 shows the manner in which a block of maximum genus one is obtained from a butterfly B_2, and also shows a graph which is obtained which does not have maximum genus one.

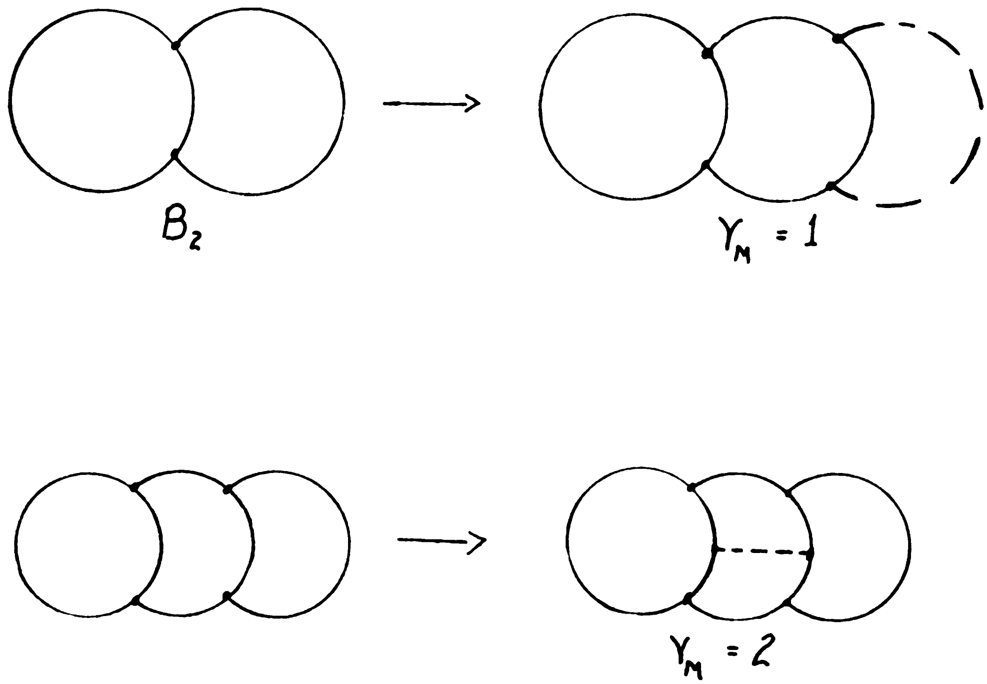


Figure 5.4 Obtaining a Block of Maximum Genus One.

CHAPTER 6

UPPER AND LOWER IMBEDDABLE GRAPHS

We now turn our attention to the examination of graphs which have the property of being upper imbeddable. However, it is also convenient to consider simultaneously those graphs for which their ordinary genus attains a certain lower bound. We will call such graphs lower imbeddable.

We first discuss a conjecture made by Duke. We show that his conjecture is true for all graphs which are lower imbeddable.

Our next consideration is to discuss graphs of each of the two kinds, and then we will display examples of graphs with various combinations of these two properties.

We then devote some attention to the problem of determining when a connected graph is upper imbeddable, given that some or all of its blocks are upper imbeddable. We conclude the chapter by building an infinite chain of upper imbeddable graphs. A subclass of these graphs, called the equilateral triangle graphs, are then upper imbeddable.

We reiterate that a graph is <u>upper imbeddable</u> if its maximum genus is the upper bound of possible genera, namely $\left[\frac{1}{2}\beta(G)\right]$.

To simplify the notation throughout this section, we define two new parameters for a connected graph G, which we designate as $X_1(G)$ and $X_2(G)$.

$$\chi_1(G) = E/6 - \tfrac{1}{2}(V - 2)$$
$$\chi_2(G) = E/4 - \tfrac{1}{2}(V - 2)$$

If the connected graph K_2 is excluded from consideration, Beineke and Harary [6] have shown the following inequalities:

$\gamma(G) \geq \{\chi_1(G)\}$ if G is any connected graph,
$\gamma(G) \geq \{\chi_2(G)\}$ if G is a connected graph without triangles.

Because of these inequalities, it seems appropriate to make the following definition.

A connected graph G of order $p \geq 3$ will be called <u>lower imbeddable</u>

i.) if G has triangles and $\gamma(G) = \{\chi_1(G)\}$ when $\chi_1(G) > 0$ or $\gamma(G) = 0$ when $\chi_1(G) \leq 0$.

ii.) if G has no triangles and $\gamma(G) = \{\chi_2(G)\}$ when $\chi_2(G) > 0$ or $\gamma(G) = 0$ when $\chi_2(G) \leq 0$.

The above definition consists of two parts because the inequality given by Beineke and Harary has two parts. The definition is made so that those graphs which have no triangles may also be lower imbeddable. Had the definition not been made in this manner, a graph without triangles could be lower imbeddable only in those cases where $\{\chi_1(G)\} = \{\chi_2(G)\}$.

The definitions imply that any planar graph is lower imbeddable. We also remark that K_p, $p \geq 3$, and $K_{m,n}$ are lower imbeddable graphs.

The following remark clarifies the definition.

<u>Remark 6.1</u>: Let G be a connected graph without triangles. If $\chi_1(G) > 0$, then $\{\chi_1(G)\} \neq \{\chi_2(G)\}$.

Proof: It is obvious that $\{X_2(G)\} \geq \{X_1(G)\}$ for all connected graphs. The proof need only show that equality cannot hold if $X_1(G) > 0$.

Since $X_2(G) - X_1(G) = \frac{E}{12}$, then if $\{X_1(G)\} = \{X_2(G)\}$, clearly $E < 12$ since $X_2(G) - X_1(G) < 1$. Because $X_1(G) > 0$, then $V < 6$.

The connected graphs without triangles having fewer than six vertices may be observed to have X_1 less than or equal to zero.

Hence, there are no graphs without triangles for which $\{X_1(G)\} = \{X_2(G)\}$ and for which $X_1(G) > 0$.

In the last chapter we noticed the applicability of the edge adding technique in finding the maximum genus of upper imbeddable graphs. These successes might cause us to turn to the same technique to establish the ordinary genus of lower imbeddable graphs. However, the number of orbits in the minimal imbedding of a lower imbeddable graph is not as easily handled as was the number of orbits in a maximum imbedding of an upper imbeddable graph. The following remark formalizes this fact.

Remark 5.2: Let G be a connected lower imbeddable graph with E edges, and let F_M be the number of faces in a minimal imbedding of G. Then

i.) $\left\{\frac{2E}{3} - 2\right\} \leq F_M \leq \left[\frac{2E}{3}\right]$ if G has triangles, and

ii.) $\left\{\frac{1}{2}E - 2\right\} \leq F_M \leq \left[\frac{1}{2}E\right]$ if G has no triangles.

Proof: This result is a simple application of Euler's formula, obtained by placing either $\{X_1(G)\}$ or $\{X_2(G)\}$ for the genus.

As one can see from this remark, to "build" one lower imbeddable graph from another apparently demands different methods than those used for upper imbeddable graphs. As of this time, the author has been unable to apply a technique successfully to such graphs.

We would next like to discuss a conjecture made by Duke [4]. He conjectured that the Betti number and the genus of any graph are related by the inequality $\beta(G) \geq 4\gamma(G)$. We will show that this conjecture is true for planar and toroidal graphs and for all connected graphs which are lower imbeddable.

<u>Theorem 6.3</u>: If G is a connected toroidal graph or if G is any connected lower imbeddable graph, then $\beta(G) \geq 4\gamma(G)$.

<u>Proof</u>: If G is a planar graph, then $4\gamma(G) = 0$. For any connected graph we have $E \geq V - 1$, thus $\beta \geq 0 = 4\gamma(G)$.

Suppose $\gamma(G) = 1$, then it follows from Kuratowski's Theorem that $\gamma_M(G) \geq 2$. Now, $\beta(G) \geq 2\,\gamma_M(G) \geq 4$. Hence, $\beta(G) \geq 4\gamma(G)$ for toroidal graphs.

Now assume $\gamma(G) \geq 2$ and G is lower imbeddable. Since $\gamma(K_7) = 1$, then $V \geq 8$ for all graphs being considered. Hence, $E + 3V \geq 31$ for all such graphs.

<u>Case 1</u>: Suppose G has triangles.

Thus, $\gamma(G) = \{X_1(G)\}$. Then $\gamma(G) < X_1(G) + 1$. So, $4\gamma(G) < \frac{2E}{3} - 2V + 8$ and $\frac{2E}{3} - 2V + 8 \leq \beta(G) = E - V + 1$ if and only if $E + 3V \geq 21$. Case 1 is proven.

<u>Case 2</u>: Suppose G has no triangles.

Thus, $\gamma(G) = \{X_2(G)\}$. Hence, $\gamma(G) < X_2(G) + 1 = \frac{1}{4}E - \frac{1}{2}V + 2$. So, $4\gamma(G) < E - 2V + 8$ and $E - 2V + 8 \leq \beta(G) = E - V + 1$ if and only if $V \geq 7$.

This completes the theorem.

Those graphs which we have just considered are of a very special kind. It seems appropriate to discuss graphs with regard to comparison of genus and maximum genus. Any connected graph belongs to exactly one of the following sets:

(1.) Graphs which are both upper and lower imbeddable.

(2.) Graphs which are upper, but NOT lower imbeddable.

(3.) Graphs which are lower, but NOT upper imbeddable.

(4.) Graphs which are neither upper nor lower imbeddable.

Theorem 6.4: Each of the sets of graphs described above are nonempty sets of connected graphs. Furthermore, each set contains graphs where the appropriate parameter X_i, $i = 1$, or 2, is positive as well as non-positive.

Proof: To prove the theorem we need to display at least one graph with X_i, $i = 1$ or 2, negative or zero and one with the parameter positive in each of the four sets.

Set (1.) Upper and Lower Imbeddable Graphs

Throughout this thesis we discovered more examples in this set than any of the other three. By the work in other chapters of this thesis, the complete graph, the complete bipartite graph, the wheel graphs, and the standard maximal planar graphs, are all examples of graphs in this set.

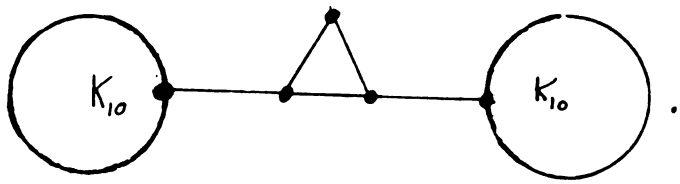
Set (2.) Upper, but not Lower Imbeddable Graphs

a.) We first display a graph which has no triangles and has $X_2(P) < 0$.

The Petersen graph P was shown to have maximum genus three in Remark 2.1. It is well known that the genus of this graph is one.

However, $\chi_2(P) = 15/4 - 8/2 < 0$. Thus, to be lower imbeddable, it would have to be a planar graph. Hence, P is not lower imbeddable. Furthermore, $\beta(P) = 15 - 10 + 1 = 6$. Thus, P is upper imbeddable.

b.) We now display a graph which has triangles for which $\chi_1(G) > 0$. Let G be the following graph:



The following parameters are those of G, where we make use of the well known theorem concerning the genus of a graph and the genera of its components. We also use Theorem 5.1 of Chapter 5. The parameters are:

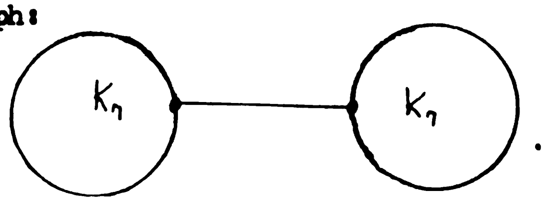
$$\gamma(G) = \gamma(K_{10}) + \gamma(K_{10}) + 0 = 8 > 6 = \left\{\chi_1(G)\right\},$$
$$\gamma_M(G) = \gamma_M(K_{10}) + \gamma_M(K_{10}) + 0 = 36 = \left[\tfrac{1}{2}\beta(G)\right].$$

Hence, G is a graph which is contained in Set (2.).

Set (3.) Graphs which are Lower, but not Upper Imbeddable

Let G be any cactus with disjoint cycles, numbering at least three. Then the Betti number of G is at least three, but G is a planar graph with maximum genus zero as shown in Theorem 5.11. Thus G is not upper imbeddable. However, because every planar graph is lower imbeddable, G is a lower imbeddable graph. The parameter $\chi_i(G)$, $i = 1$ or 2, is nonpositive for this example.

We now display an example of a graph G with $\chi_1(G) > 0$. Let G be the following graph:



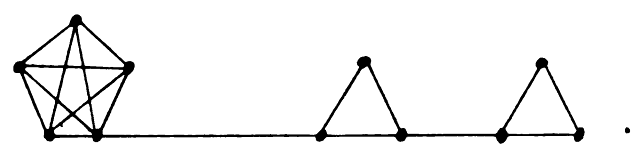
Then $\chi_1(G) = 7/6$ and thus $\{\chi_1(G)\} = 2$. However, it is clear that the genus of G is given by $\gamma(G) = \gamma(K_7) + \gamma(K_7) = 2$. Hence, G is lower imbeddable.

The Betti number of G is 30, and thus $\frac{1}{2}\beta(G) = 15$. However, by Theorem 5.1, $\gamma_M(G) = \gamma_M(K_7) + \gamma_M(K_7) = 14$. Thus G is not upper imbeddable.

<u>Set (4.)</u> Graphs which are neither Lower nor Upper Imbeddable.

a.) $\chi_1(G) < 0$.

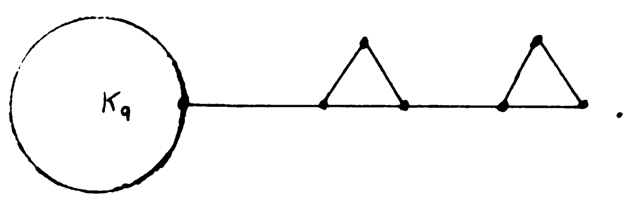
Let G be the following graph:



Then the Betti number of G is eight, but its maximum genus is three. Although $\chi_1(G) = -3/2$, the genus of G is one. G is in Set (4.).

b.) $\chi_1(G) > 0$.

Let G be the following graph:



Then the Betti number of G is 30, while the maximum genus is 14. Hence, G is not upper imbeddable. We have $\chi_1(G) = 5/6$, while the genus of G is 3. Thus G is not lower imbeddable. G is a graph in Set (4.).

By generalizing the examples in Sets (2.) and (4.), it is clear that each of the four sets is infinite. This completes the theorem.

Theorem 6.4 indicates that there is no dependence between the concepts of upper and lower imbeddability. We have established the conjecture of Duke for all planar and toroidal graphs and for all graphs which are in either Set (1.) or (3.) above. If there does indeed exist an example proving the conjecture to be false, a likely place for it to appear would be in Set (4.) of the last theorem. It is in this set that the difference between the maximum and ordinary genus is most likely to be less than the conjecture allows.

Throughout the remainder of the chapter, we devote our attention to upper imbeddable graphs. We begin with a theorem which is in reality a corollary to Theorem 5.6. We use the theorem to establish some results concerning graphs with certain upper imbeddable blocks. We first introduce some necessary notation. Let G be a connected graph and G' be a subgraph of G which contains a cutpoint c of G. By G - G' we mean the graph defined by: $V(G - G') = V(G) \cap C(V(G')) \cup \{c\}$ and $E(G - G') = E(G) \cap C(E(G'))$, where C(S) is the complement of the set S, $S = V(G')$ or $E(G')$.

<u>Theorem 6.5</u>: Let G be a connected graph with a subgraph G' which contains a cutpoint c of G. Suppose that G - G' is also connected. Let F be the number of faces in an imbedding of G' and F" be the number of faces in some imbedding of G - G'. Then there is an imbedding of G with F + F' - 1 faces.

<u>Proof</u>: If n = 1 in Theorem 5.6 and $G_1 = G'$, $G_2 = G - G'$, the theorem is an immediate consequence.

The above theorem yields the following corollary which is often helpful in determining whether a graph is upper imbeddable. In order to facilitate the statement of the theorem, we make a definition.

If G is a connected graph which is not a block, an endblock of G is a block which contains exactly one cutpoint of G. If G is a block, G is considered to be its own endblock.

Corollary 6.6: Let G be a connected graph with an endblock H which is upper imbeddable. Suppose further that G - H is also upper imbeddable. If either H or G - H has even Betti number, then G is upper imbeddable.

Proof: Let F and F' be the number of faces in maximum imbeddings of H and G - H, respectively. Then, by Theorem 6.5, we have $\delta_M(G) \leq F + F' - 1$. However, since either H or G - H has even Betti number, $F + F' \leq 3$, and thus $\delta_M(G) \leq 2$, and G is upper imbeddable.

The following corollary was done with a counting argument by Nordhaus, Stewart, and White [10]. The proof given is independent of the earlier proof.

Corollary 6.7: If G is a connected graph with all of its blocks being upper imbeddable with even Betti number, then G is upper imbeddable with even Betti number.

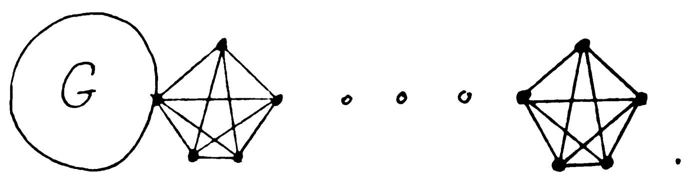
Proof: The proof follows directly from Corollary 6.6.

Corollary 6.8: Let G be a connected graph which is upper imbeddable. Then G is contained in an upper imbeddable graph with an arbitrarily large number of blocks, all but one of which are isomorphic.

Proof: Choose an arbitrary upper imbeddable graph H which has even Betti number. Construct a graph G' by attaching H at any vertex of G, the given graph. We then attach another copy of H to H, etc.

We use induction on the number of copies of H and Corollary 6.6 applies to make any such graph upper imbeddable. It obviously satisfies the isomorphism property stated.

The following is an example of such a graph where the graph H is K_5:



We now consider the more general problem of the relation of the upper imbeddability of a connected graph given that all of its blocks are upper imbeddable. We obtain one negative result and state a conjecture.

Remark 6.9: A connected graph all of whose blocks are upper imbeddable need not be upper imbeddable.

Proof: We simply display an example of a graph whose blocks are all upper imbeddable and which is not itself upper imbeddable.

We see that $\gamma_M(G) = 1 \neq 2 = \frac{1}{2}\beta(G)$, by Corollary 5.2.

We complete this section of the chapter by stating a conjecture.

Conjecture: If G is a connected graph none of whose blocks is K_2 and each of whose blocks is upper imbeddable, then G is upper imbeddable.

We conclude chapter 6 with a theorem which presents an infinite chain of upper imbeddable graphs, with each a subgraph of the one following it and a supergraph of the one preceding it. We obtain the maximum genus of the equilateral triangle graphs as a corollary.

Let G be a connected graph and label a path in G of length (n-1) by 1, 2, ..., n. Define the graph G_n as follows (where + means adding the indicated edges):

$$G_n = G + \sum_{j=1}^{n} (j',j) + \sum_{j=1}^{n} (j,(j+1)') + \sum_{j=1}^{n} (j',(j+1)'),\text{ where}$$

1',2', ..., (n+1)' are pairwise nonadjacent vertices which are not in G. (See Figure 6.1).

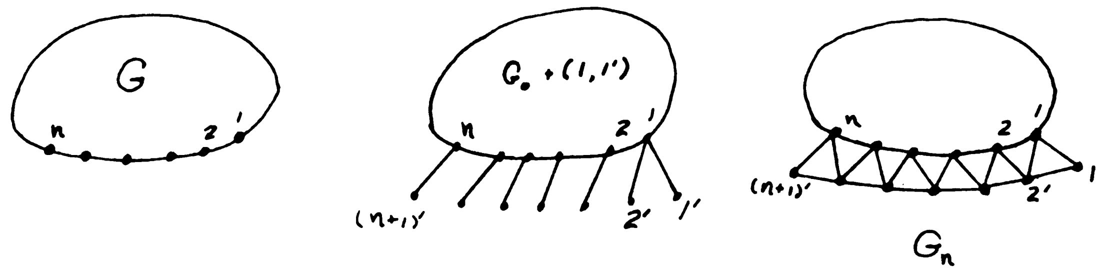


Figure 6.1 The Graph G_n

<u>Theorem 6.10</u>: Let G be a connected graph with an i-face imbedding, i = 1 or 2, and G_n be as defined above. If i = 2, assume that the vertex labeled "1" is involved in the two faces of the imbedding. Then G_n has a (3-i)- face cellular imbedding for each n and is thus upper imbeddable. Furthermore, if i = 1, then the two face imbedding of G_n has (-n'-(n+1)'-n-) as one of its two faces.

<u>Proof</u>: We begin with the graph G and add nodes and edges in order to form G_n.

We first add the nodes $2',\ldots,(n+1)'$ and the edges $(j,(j+1)')$, where $j = 1,2,\ldots,n$. For this new graph G_o we also have a cellular imbedding with i faces.

It is now most easily done by considering two cases. We consider the cases $i = 1$ and $i = 2$.

<u>Case 1.</u>) Let $i = 1$.

We then have an imbedding with one face for G_o. We write it

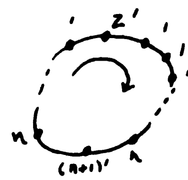
$$(\ldots-1-2'-1-\ldots-2-3'-2-\ldots-j-(j+1)'-j-\ldots-n-(n+1)'-n-\ldots),$$

where $j = 3,\ldots(n-1)$. The proof now proceeds by induction on the number m, where we form $G_m = G_o + \sum_{j=1}^{m} (j',j) + \sum_{j=1}^{m} (j',(j+1)'; 1 \leq m \leq n.)$

<u>Let m = 1</u>.

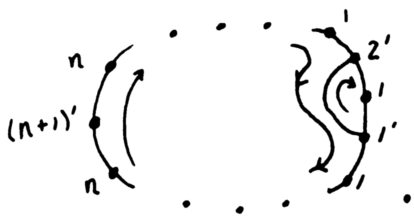
We add the edge $(1',1)$ into the face listed above as follows:

$$(\ldots-1-2'-1-1'-1\ldots-j-(j+1)'-j-\ldots), \text{ where } j = 2,\ldots,n.$$

Schematically, we represent the face as follows: 

.

Because this new graph has a cellular imbedding with only one face, we may add the edge $(1',2')$ as is indicated in the following diagram and have a cellular imbedding of the new graph with two faces. Schematically,



The faces for this new imbedding are as given at the top of page 70.

$$(\ldots-1-2'-1'-1-\ldots-j-(j+1)-j-\ldots), \text{ where } j = 2,\ldots,n\ ,$$

and

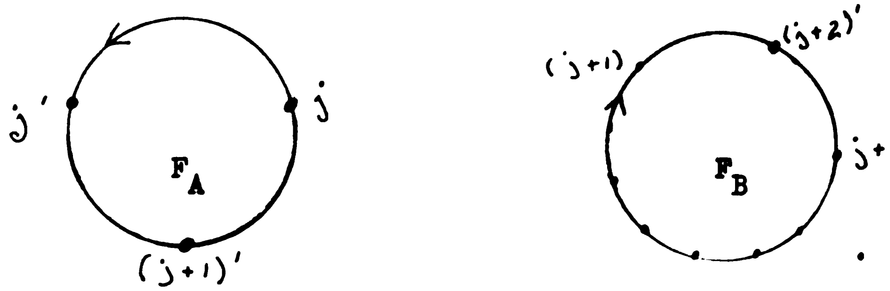
$$(-1'-2'-1-)\ .$$

Consequently, the theorem is true if $m = 1$.

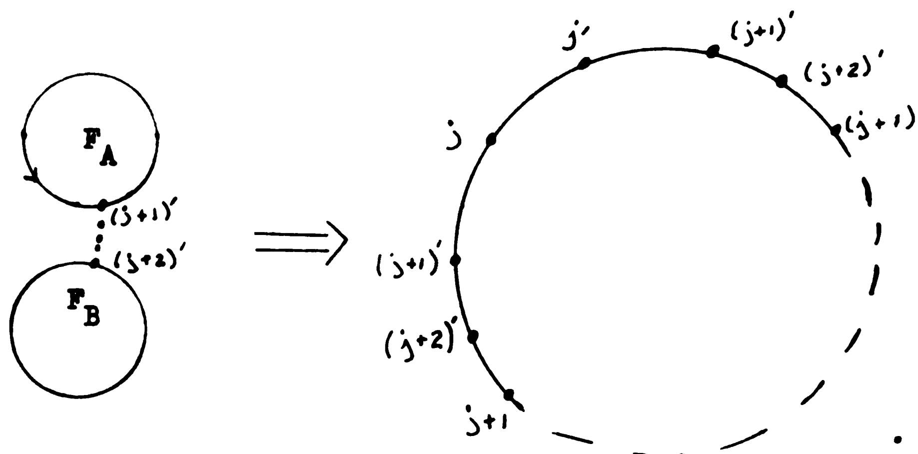
Now let $m = j+1$ and suppose G_j has a two face imbedding where one of the faces F_A is $-j'-(j+1)'-j-$. Then the edge $((j+1), (j+2)')$ has both of its corresponding directed edges in the other face F_B. Because the vertex $(j+2)'$ has degree one, the face F_B must have the following appearance:

$$(\ldots-(j+1)-(j+2)'-(j+1)-\ldots),$$

We write the two orbits of the imbedding schematically,



We now add the edge $((j+1)', (j+2)')$ and combine the faces in the following manner:



So that we now have a one face imbedding of the newly formed graph. We now proceed exactly as in the case m = 1, adding the edge ((j+1)', (j+1)), yielding a cellular imbedding of the graph for n = j+1 with two faces, which are:

$$(-j'-(j+1)'-j+1)-\ldots(j+1)-(j+2)'-(j+1)'-j-),$$

and

$$(-(j+1)'-(j+2)'-(j+1)-),$$

By induction, Case 1 is completed.

<u>Case 2</u>. Let i = 2.

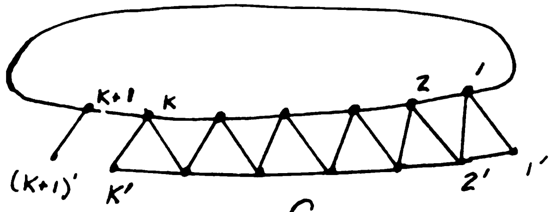
After we have formed G_0, we now have a graph which has a cellular imbedding with two faces. We use induction on m as in Case 1.

Let m = 1. By assumption, the original imbedding of G was so that the vertex labeled 1 was involved in each of the two faces of the imbedding. This will still be the case for the two face imbedding of G_0. Let F_B be the face of the imbedding which contains the directed edges $[1,2']$, $[2',1]$, which necessarily abut since 2' has degree one.

The vertex "1" must also be involved in another face, which we call F_A. We add the directed edges $[1,1']$ $[1',1]$ into this face, again so that they necessarily abut.

Then since $[1,2']$ and $[1,1']$ are in different faces, the edge adding technique allows us to add the edge $[2',1']$ so as to have a cellular imbedding of the resulting graph with one face. The induction is anchored.

Now let m = k+1 and suppose the graph G_k shown below has a one face cellular imbedding.

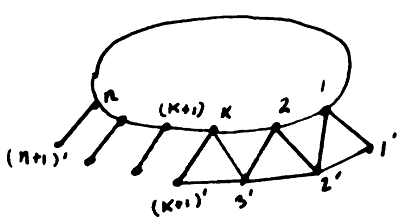


G_K

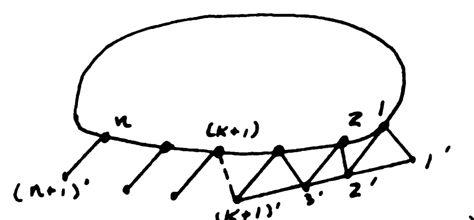
We add the edge $((j+1)', (j+1))$ to G_k. By the edge adding technique, we may add the two corresponding directed edges so that they are in different faces of the new two face imbedding.

Since the degree of $(j+1)'$ is now at least two and since $(j+1)'$ is in both faces of the new imbedding, there is some vertex p in $V(G_k)$ so that the directed edge $[p,(j+1)']$ is in a different face from the directed edge $[(j+1), (j+2)']$. By the edge adding technique we may add the edge $((j+1)', (j+2)')$ so as to form a cellular imbedding with one face for the newly formed graph.

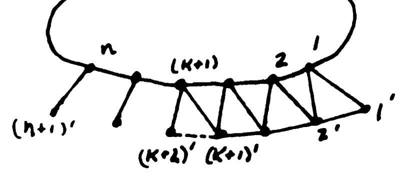
By induction, the proof is completed. We close with an illustration of the way in which the edges were added in Case 2.



G_K



$G_K + (K+1, (K+1)')$



G_{K+1}

Corollary 6.11: Let G be an upper imbeddable graph. Then there is a chain of upper imbeddable graphs,

$$G = G_1 \subseteq G_2 \subseteq G_3 \subseteq \ldots G_k \subseteq G_{k+1} \subseteq \ldots .$$

Furthermore, for each i, $G_i - G$ is a planar graph so that in its planar imbedding all but one of its faces are triangles.

Proof: To form G_{i+1} from G_i we do exactly as in the theorem if $i = 1$; if $i \neq 1$ we use G_i as the graph in the theorem and the path consisting of the triangles added to G_{i-1} to form G_i as the path in the theorem. The theorem assures that if G_i has a two face imbedding, there is a vertex in this path which is involved in each of the two faces. Hence, the theorem constructs the chain of graphs as desired.

Corollary 6.12: We call a graph which is obtained from K_3 by n applications of the building procedure described in the theorem, the equilateral triangle graph with n layers, and denote it by E_n. Then E_n is upper imbeddable.

Proof: This is a restatement of the theorem with $G = K_3$.

CHAPTER 7

THE MAXIMUM GENUS OF THE COMPLETE BIPARTITE GRAPH $K_{m,n}$ AND THE COMPLETE GRAPH K_n

After investigating upper imbeddable graphs in general in Chapter 6, we now turn to two important special graphs of that kind, namely the complete graph K_n and the complete bipartite graph $K_{m,n}$. The determination of the maximum genus of these graphs is made by an application of the edge adding technique. It should be remarked that the maximum genus of the complete graph K_n was first determined by Nordhaus, Stewart, and White [10]. The proof given there is based completely on Edmonds' vertex permutation technique, while that given here is essentially an edge adding technique.

The method of proof is to state and prove a general theorem concerning this edge adding technique and to show its application to the particular graphs mentioned above. It is rather surprising that the resulting theorem applies to both classes of graphs. This theorem deals with adding a single point to the vertex set of an upper imbeddable graph. We then add edges from this vertex to each vertex in a prescribed subset of the vertices of the original graph.

<u>Theorem 7.1</u>: Let G be a connected graph with an i-face cellular imbedding, where $i = 1$ or 2, so that G is an upper imbeddable graph. Let S be a nonempty subset of V(G), with $|S| = n$. If $i = 2$, assume that

S contains two vertices s and t with s appearing in one face and t appearing in the other face. Further assume that the degrees d(s) and d(t) both exceed one. Let S_j be a subset of S with j elements and let p be a vertex not in the vertex set of G. We assume $S = S_n \supset S_{n-1} \supset \ldots \supset S_1 = \{s_1\}$, j = 1,...,n. We define graphs G_r, r = 1,...,n, as follows (where $+\Sigma$ means adding the indicated edges to the graph):

$$G_r = G + \sum_{s \in S_r} (p,s).$$

The vertex set of G_r is composed of the vertex p and the vertices of G. Then, for r = 1,..., n,

G_r has a cellular imbedding with i faces for odd r,

G_r has a cellular imbedding with (3-i) faces for even r.

<u>Proof</u>: We consider the cases i = 1 and i = 2.

<u>Case 1</u>. Let i = 1. Then there is a cellular imbedding of G with one face on some compact orientable 2-manifold.

Label the vertices of S with the integers 1,...,n in a manner so that $S_r = \{1,\ldots,r\}$ for r = 1,...,n. Since a vertex of degree one may be added within a face, we add the undirected edge (p,1) so that the resulting graph G_1 has a cellular imbedding with one face. If $|S| = 2$, the edge adding technique implies that G_2 has a cellular imbedding with two faces and the theorem is thus proven. Hence, we assume that S has three or more elements.

We now proceed inductively. Suppose that we have formed G_i and that G_i has a cellular imbedding with one face. We show that G_{i+1}

has a cellular imbedding with two faces and that G_{i+2} has a cellular imbedding with one face. Since G_i has an imbedding with only one face, the edge adding technique implies that the undirected edge $(p,i+1)$ can be added in only one way. Namely, so that the resulting graph G_{i+1} has a cellular imbedding with two faces. The technique also describes a manner in which $(p,i+1)$ may be added so that the directed edges $[p,i+1]$ and $[i+1,p]$ are in different faces in the resultant imbedding. We so add the edge $(p,i+1)$.

We now form an imbedding for G_{i+2} from that of G_{i+1}. The imbedding formed will have one face. By the edge adding technique, we can obtain such an imbedding for G_{i+2} if we can establish the following fact: There is a vertex v in $V(G_{i+1})$ and a vertex j in S_{i+1} so that the directed edges $[j,p]$ and $[v,i+2]$ are in different faces of the two face imbedding of G_{i+1}.

We find these vertices by considering all the directed edges of the form $[k,p]$, $k = 1,\ldots,i+1$. Since the imbedding of G_{i+1} has only two faces there are vertices v and j as desired if not all edges of this form are in the same face. This is because the vertex $i+2$ must occur in some face of the imbedding. We choose some k_o in $V(G_{i+1})$ so that $[k_o,p]$ is in a face different from the one chosen for $i+2$. Then $k_o = j$ above and any adjacency of $i+2$ with its edge directed into $i+2$ in the chosen face could serve as v above.

Recall that the edge adding technique was applied to insure that the directed edges $[i+1,p]$ and $[p, i+1]$ are in different faces for the two face imbedding of G_{i+1}. Furthermore, p must have degree at least two. Thus, there is some $m \in V(G_{i+1})$ so that the directed edge $[m,p]$ precedes the directed edge $[p,i+1]$ in some face; $m \neq i+1$ since p has degree at least two. Hence $[m,p]$ and $[i+1,p]$ are in different faces. Thus, not all edges directed into p are in the same face; as stated before, we then have the vertices v and j as desired above. We thus add the edge $(i+2,p)$ so that G_{i+2} has a cellular imbedding with one face. We have completed the induction.

Notice that we now have G_i, G_{i+1}, and G_{i+2} with cellular imbeddings of one, two, and one faces respectively. Since we began with G_1 having an imbedding with one face, the number of faces in the imbeddings for the G_r, $r = 1,\ldots,n$ are exactly as described in the theorem. This completes Case 1.

Case 2. Let $i = 2$. Then there is a cellular imbedding of G with two faces on some compact orientable 2-manifold.

We begin by using the additional hypothesis we are given when G has a two face imbedding. Thus, we may assume that there are directed edges $[m,s]$ and $[c,t]$ which are in different faces of the imbedding. Call the faces F' and F" respectively. We relabel s with the label "1" and t with the label "2".

We add the edge $(p,1)$ so that the directed edges $[1,p]$ and $[p,1]$ are both in the face F'. We have then formed G_1 and have established a cellular imbedding of it with two faces. By the edge adding technique, we may add the edge $(p,2)$ so that G_2 has a cellular imbedding with one face. (We choose the face F" at the vertex 2 and are assured that it is different from F'. The edge adding technique can then be applied so that we obtain an imbedding with one less face.)

To complete the proof of the theorem, one need only substitute G_2 for G_1 in the argument for Case 1. Since, however, we began with G_1

having an imbedding with two faces, the imbeddings have the number of faces described in the theorem.

This completes the proof of the theorem.

It is clear that the above theorem is designed to give us a method of building upper imbeddable graphs from a given upper imbeddable graph. It has the following corollary, analogous to Corollary 6.11.

Corollary 7.2: Let G be an upper imbeddable graph. Then there is a chain of upper imbeddable graphs $G_1 \subset G_2 \subset \ldots$ so that $|V(G_i)| - |V(G_{i-1})| = 1$.

The purpose of Theorem 7.1 is to aid in the induction argument for the determination of the maximum genus of both the complete graph and the complete bipartite graph. We now undertake this task.

Theorem 7.3: The complete graph K_n on n vertices is an upper imbeddable graph, and

$$\gamma_M(K_n) = \left[\tfrac{1}{4}(n-1)(n-2)\right].$$

Proof: We use mathematical induction on the number of vertices n.

Let $n = 1$. Then $\beta(K_1) = 0$; thus since $\gamma_M(K_1) \leq \left[\tfrac{1}{2}\beta(K_1)\right]$, we have $\gamma_M(K_1) = 0$. The case $n = 1$ is completed.

We now notice that we have our induction anchored at a number which is congruent to one modulo four. We proceed inductively.

Let $i \equiv 1 (\mathrm{mod}\ 4)$ and assume the induction hypothesis that K_i has a one face cellular imbedding. We show that each of the four graphs K_{i+1}, K_{i+2}, K_{i+3}, K_{i+4} is upper imbeddable. We do so by showing that the four graphs have cellular imbeddings of one, two, two, and one face respectively. Before going further with the actual arguments, we show that we are indeed looking for cellular imbeddings with the listed number of faces. Recall that a graph with even Betti number is upper imbeddable if and only if it has a cellular imbedding with one face. Similarly, one with odd Betti number is upper imbeddable if and only if it has a cellular imbedding with two faces. We examine the parity of the Betti numbers of the four graphs listed above. We first note that for any complete graph K_p, $\beta(K_p) = \frac{1}{2}(p-1)(p-2)$. For the graphs in question we have the following Betti numbers and their parities:

$\beta(K_{i+1}) = \frac{1}{2}(i-1)i$, which is even since i-1 is divisible by four.
$\beta(K_{i+2}) = \frac{1}{2}(i+1)i$, which is odd since both i and $\frac{1}{2}(i+1)$ are odd.
$\beta(K_{i+3}) = \frac{1}{2}(i+2)(i+1)$, which is odd because both i+2 and $\frac{1}{2}(i+1)$ are odd.
$\beta(K_{i+4}) = \frac{1}{2}(i+3)(i+2)$, which is even since i+3 is divisible by 4.

Hence, the imbeddings must have the number of faces listed above. We now proceed with the proof.

The proof consists of repeated applications of Theorem 7.1. At each step we will indicate which set we adopt for S and which of the G_r in the theorem is the desired new graph.

To form K_{i+1} from K_i we let $S = G = K_i$ and label the vertex to be added with the label i+1. Since K_i meets the hypothesis of the theorem, the conclusion applies. Hence the graph G_i has a cellular imbedding with one face, since i is odd and since K_i has a cellular imbedding with one face. It is clear that $K_{i+1} = G_i$, and the case K_{i+1} is complete.

For the case K_{i+2}, we let $S = K_{i+1}$ and label the new vertex i+2. Then $K_{i+2} = G_{i+1}$ in the theorem. Since i+1 is even, the theorem implies that K_{i+2} has a cellular imbedding with $(3-1) = 2$ faces. K_{i+2} is as desired.

For K_{i+3}, we let $S = K_{i+2}$. Since K_{i+2} has an imbedding with two faces, there must be two vertices which are in different faces. In the notation of the theorem $G_{i+2} = K_{i+3}$. Since i+2 is odd, the theorem implies that K_{i+3} has a cellular imbedding with two faces, as desired.

To complete the induction, we need only display an imbedding of K_{i+4} which has one face. The procedure is the same, letting $S = K_{i+3}$. Then $K_{i+4} = G_{i+3}$. Since i+3 is even, K_{i+4} has a cellular imbedding with $(3 - 2) = 1$ face.

This completes the theorem.

Before proceeding to the next theorem, which establishes the maximum genus of the complete bipartite graph, we make the following remarks concerning the Betti number of such a graph.

The Betti number of the complete bipartite graph $K_{m,n}$ is given by the formula $\beta(K_{m,n}) = (n-1)(m-1)$, which is odd only if both m and n are even. Hence, in the proof of the theorem which follows we will be looking for a cellular imbedding in which the number of faces is one in all cases except where both m and n are even. In this exceptional case, we need to find a cellular imbedding with exactly two faces.

<u>Theorem 7.4</u>: The complete bipartite graph $K_{m,n}$ is upper imbeddable for all m and n. The maximum genus is given by the formula:

$$\gamma_M(K_{m,n}) = \left[\tfrac{1}{2}(n-1)(m-1)\right].$$

Proof of Theorem 7.4: We prove the theorem by fixing n and using induction on m. The Betti number of the graph $K_{1,n}$ is zero, thus by the upper bound theorem we must have $\gamma_M(K_{1,n}) = 0$, and the theorem is true for m = 1.

Assume as our induction hypothesis that the graph $K_{i,n}$ is upper imbeddable. We show that $K_{i+1,n}$ is upper imbeddable.

As one might suppose, our method is to apply Theorem 7.1 to the imbedding of $K_{i,n}$ and get the desired imbedding for $K_{i+1,n}$. The proof is done in two cases depending on the parity of n.

Case 1. Let n be odd.

By the remarks preceding this theorem, the Betti number of $K_{i,n}$ is even, and thus $K_{i,n}$ has a cellular imbedding with one face regardless of the parity of i. In Theorem 7.1, we let $G = K_{i,n}$ and $S = \overline{K}_n$, the part of $K_{i,n}$ with n vertices. In the notation of Theorem 7.1, we have $G_n = K_{i+1,n}$. Hence, since n is odd, G_n has a cellular imbedding which agrees with that of $K_{i,n}$ in its number of faces. Thus $K_{i+1,n}$ has a cellular imbedding with one face. Since $K_{i+1,n}$ has even Betti number, this is exactly what we desired.

Case 2. Let n be even.

The number of orbits in the maximum imbedding of $K_{i,n}$ now depends on the parity of i. We consider the two subcases:

Case a. Let i be odd.

We then know that $K_{i,n}$ has a cellular imbedding with one face. We form $K_{i+1,n}$ from $K_{i,n}$ exactly as was done in Case 1. However, since n is now even G_n has a cellular imbedding with 3-1 = 2 faces. Thus, $K_{i+1,n}$ has a cellular imbedding with two Since $\beta(K_{i+1,n})$ is odd, this is as desired.

Case b. Let i be even

In this case we know that $i \geq 2$ and that $K_{i,n}$ has a cellular imbedding with two faces. Consequently, before we can apply Theorem 7.1 we must find s and t in $\bar{K}_n$ as described in that theorem. Since $i \geq 2$, and since we may assume $n \geq 2$ by using an argument analogous to that for the case $m = 1$, any face in an imbedding of $K_{i,n}$ must involve at least two vertices of $\bar{K}_n$, as the illustration

shows. So, let s be any vertex of $\bar{K}_n$. We add the undirected edge $(i+1,s)$ inside a face of the imbedding for $K_{i,n}$. Then the directed edges $[i+1,s]$ and $[s,i+1]$ are both in the same face F. However, since $K_{i,n}$ has two faces in this imbedding, there must be a vertex of $\bar{K}_n$ which appears in some other face F'. We label this vertex t. The vertices s and t are then as desired in Theorem 7.1, with $S = \bar{K}_n$.

In the notation of Theorem 7.1, we have $G_n = K_{i+1,n}$. Since n is even, G_n has an imbedding with $3-2 = 1$ face. Because $i+1$ and n are both odd, this is as desired.

Case 2 is now completed. By induction the graph $K_{m,n}$ is upper imbeddable for all m and n, since we could now repeat the above arguments with m fixed.

The theorem of Duke [4] given in Chapter 3 which concerns the determination of those compact orientable 2-manifolds upon which a given graph has a cellular imbedding, can now be applied to give the following corollaries.

Corollary 7.5: The complete graph K_n has a cellular imbedding on an orientable surface S_k of genus k if and only if:

$$\left\{\frac{(n-3)(n-4)}{12}\right\} \leq k \leq \left[\tfrac{1}{4}(n-1)(n-2)\right].$$

Corollary 7.7: The complete bipartite graph $K_{m,n}$ has a cellular imbedding on an orientable surface S_k of genus k if and only if:

$$\left\{\tfrac{1}{4}(m-2)(n-2)\right\} \leq k \leq \left[\tfrac{1}{2}(m-1)(n-1)\right].$$

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