THE UNSYMMETRICALLY FED PROLATE SPHEROIDAL ANTENNA

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THE UNSYMMETRICALLY FED PROLATE SPHEROIDAL ANTENNA

By

Hugo A. Myers

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ABSTRACT

The problem studied in this thesis is the unsymmetrically-fed prolate spheroidal transmitting antenna. The prolate spheroidal functions are expressed in the form of power series and Laurent series. Radiation patterns have been obtained for antennas of three different lengths up to about one wave length long, for length/thickness ratios of about 5/1, 10/1, 22/1, and 316/1, and for nine unsymmetrical gap locations as well as for the symmetrically-fed cases. Two methods have been developed for computing antenna impedances that take into consideration the width of the gap and the geometry of the transmission line feeding the antenna. One method is based on the usual assumption that the current is driven by the field in the gap. The other method is based on the assumption that the current in the antenna elements is driven by the component of the electric field of the transmission line that is tangential to the elements. Further refinements in the impedance theory will probably have to depend upon experimental evidence.

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INTRODUCTION

The problem of the radiating antenna when studied mathematically becomes an exterior boundary value problem. By this the following is meant: Given a region R with finite or infinite boundary S, a solution of a differential equation or, as in the case of the antenna, a system of differential equations is sought which 1) takes on given values on the boundary S, 2) is regular everywhere exterior to S, and 3) satisfies some special regularity condition at infinity. In the case of the antenna, the system of differential equations is Maxwell's electromagnetic equations, the solution of which will give the electric field $\overline{E}$ and/or the magnetic field $\overline{H}$. The boundary conditions that must be satisfied on S are well known[1] and will be discussed for the particular problem later. The regularity of the solution at infinity is usually described for radiation problems as the radiation condition. In contrast to potential problems, vanishing at infinity is not sufficient for solutions of radiation problems, whether electrodynamic or acoustic in origin.[2]

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[1] J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, pp. 34-37, 1941.

[2] A. J. W. Sommerfeld, Partial Differential Equations in Physics, Tr. by E. G. Straus, Academic Press, New York, p. 189, 1949.

For those cases when Maxwell's equations can be reduced to finding the solution of the scalar equation

$$\nabla^2 \phi + k^2 \phi = 0,$$

the radiation condition for time variation of the form $e^{-i\omega t}$ is

$$\lim_{r \to \infty} r\left(\frac{\partial \phi}{\partial r} - ik\phi\right) = 0.$$

This condition can also be stated more generally for the $\bar{E}$ and $\bar{H}$ fields themselves as has been shown by Synge.[a]

For an arbitrary finite surface S the exterior, or for that matter the corresponding interior, boundary value problem presents great difficulties. In fact, for a simple finite cylinder with ends, so that S is composed of the lateral surface plus the ends, no solution has so far been found for the exterior problem except by numerical approximations. The number of choices that can be made for S for which explicit solutions can be found is extremely limited. Conditions can be stated that will determine surfaces S for which solutions can be found. For example, let $u_1(x,y,z) = c_1$, $u_2(x,y,z) = c_2$, $u_3(x,y,z) = c_3$ be triply orthogonal families of surfaces. Then 1) if the scalar wave equation $\nabla^2\phi + k^2\phi = 0$ when written in the coordinate system u_1, u_2, u_3 has a separable solution $\phi = f(u_1)g(u_2)h(u_3)$ and 2) if the

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[a] J. L. Synge and G. E. Albert, The General Problem of Antenna Radiation and the Fundamental Integral Equation, with Application to an Antenna of Revolution Part I, Quarterly of Applied Math. vol. VI, no. 2, p. 119, (1948).

surface S is one of the three surfaces $u_1 = c_1$, $u_2 = c_2$, or u_3 , c_3, then the exterior boundary value problem can be solved. For regions with finite boundaries this restricts the possibilities to such simple surfaces as spheres and spheroids.[4] Of these a thin prolate spheroid, which in the limiting case becomes a rod, or a segment of a straight line, offers an excellent approximation to the circular cylindrical antenna.

Problems related to the free oscillations of prolate spheroids have been discussed frequently in the literature.[5] The problem of the forced oscillations of a prolate spheroid was first considered by Page and Adams.[6] They treated the case of the thin spheroid driven by a plane wave whose electric field was parallel to the major axis. This would correspond to the case of the receiving antenna with its gap or feed point shorted out. Some of the ideas used here in constructing radial spheroidal functions were first presented by Page and Adams.

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[4] Exterior problems for other finite surfaces have been solved for Laplace's equation; see, e.g., S. N. Karp, Separation of Variables and Wiener-Hopf Techniques, Research Report No. EM-25, New York Univ., New York, 1950.

[5] See, e.g., M. Abraham, Die electrischen Schwingungen um einen stabformigen Leiter, behandelt nach der Maxwell'schen Theorie, Ann. der Physik 66, pp. 435-472, 1898. J. Meixner has recently published many papers on spheroidal functions. See Math. Nachrichten, Band 3, Heft 4, April (1950), Band 5, Heft 1, March (1951), & Band 5, Heft 6, August (1951); Archiv der Math., Band 1, Heft 3, (1948/49) & Band 1, Heft 6, (1948/49); Ann. der Physik, Band 6, (1949) & Band 7, Heft 3-4, (1950); Zeitschrift fur angewandte Physik, Band 1, Heft 12, Dec. (1949) & Band 3, Heft 5, (1951); Zeitschrift fur angewandte Mathematik und Mechanik, Band 28, Heft 10, October 1948.

[6] L. Page and N. I. Adams, The Electrical Oscillations of a Prolate Spheroid, I, Phys. Rev., 53, pp. 819-831, (May 15, 1938).

Chu and Stratton[7] attacked the case of the center-fed prolate spheroidal transmitting antenna by quite different methods and obtained curves from which the impedance at the gap could be estimated. The basic electromagnetic theory used in this thesis follows the analysis of Chu and Stratton as elaborated by Schelkunoff.[8]

R. M. Ryder[9] extended the work of Page and Adams by investigating the behavior of the harmonics due to forced oscillations by perturbation methods. In a second and third paper Page[10] treated the more general vector wave equation and extended his previous results for the thin receiving antenna.

The problem of determining the current distribution and impedance of an unsymmetrically driven cylindrical antenna was formulated by Ronold King.[11] He made use of an integral equation which he solved by the method of successive approximations to obtain general expressions for the current and the impedance. He was able to find a simple approximate expression for the impedance of the unsymmetrically driven antenna

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[7] L. J. Chu and J. A. Stratton, Forced Oscillations of a Prolate Spheroid, Jour. Appl. Phys., 12, pp. 241-248, March (1941).

[8] S. A. Schelkunoff, Advanced Antenna Theory, John Wiley & Sons, New York, pp. 111-125, (1952).

[9] R. M. Ryder, The Electrical Oscillations of a Perfectly Conducting Prolate Spheroid, Jour. Appl. Phys., 13, pp. 327-343, May (1942).

[10] L. Page, The Electrical Oscillations of a Prolate Spheroid, II and III, Phys. Rev., 65, pp. 98-117, February 1 and 15, (1944).

[11] Ronold King, Asymmetrically Driven Antennas and the Sleeve Dipole, unclassified technical report no. 93 for Office of Naval Research, Cruft Lab., Harvard, 1949.

involving a series combination of the known impedances of symmetrically driven antennas. The impedance and current distribution for a cylindrical antenna of length 3 $\lambda/4$ driven $\lambda/4$ from one end were evaluated and the broad-band properties were discussed.

The theory and tables of prolate spheroidal functions was first published by Stratton, Morse, Chu, and Hutner,[12] hereafter referred to as Stratton et al. The equations they solved were more general than the Maxwell equations in spheroidal coordinates in that they include the latter as a special case. This work was extended by Spence[13] and most recently by Hatcher.[14] Many of the values for the norms and the radial functions used in this thesis were taken from the tables that had been worked out by Spence and Hatcher.

The particular problem studied in this thesis is the unsymmetrically-fed prolate spheroidal transmitting antenna. Radiation patterns have been obtained for antennas of three different lengths up to about one wave length long, for length/thickness ratios of about 5/1, 10/1, 22/1, and 316/1, and for nine unsymmetrical gap locations as well as for the symmetrically-fed cases. Two methods have been developed for

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[12] J. A. Stratton, P. M. Morse, L. J. Chu, and R. A. Hutner, Elliptic Cylinder and Spheroidal Wave Functions, John Wiley and Sons, New York, 1941.

[13] R. D. Spence, The Scattering of Sound From Prolate Spheroids, Final Report, Office of Naval Research, NONR-02400, 1951.

[14] E. C. Hatcher, Jr., Radiation of a Point Dipole Located at the Tip of a Prolate Spheroid, M. S. thesis, Michigan State College, 1953. This thesis has recently been published in the August 1954 issue of the Journal of Applied Physics.

computing antenna impedances which take into consideration the width of the gap and the geometry of the transmission line feeding the antenna. Further refinements in the impedance theory will probably have to depend upon experimental evidence.

The reason for the choice of this thesis topic was a very practical one. It was desired to design and construct a TV antenna which would receive* Detroit on channels two and four and Kalamazoo on channel three without having to rotate the antenna. From radiation pattern measurements of various unsymmetrically-fed cylindrical antennas it was found that an antenna very similar to the one analyzed by King had a radiation pattern consisting of two lobes whose directions of maximum radiation were about 130 degrees apart. This was the pattern that was desired. The integral equation which King used in this problem is extremely complicated and a large amount of computing is required to find even a first approximation to the solution. From a purely mathematical point of view it has not even been proved that successive approximations will converge to a solution of the integral equation. Substituting a prolate spheroid for the circular cylinder permits one to proceed directly from Maxwell's equations and use the exact spheroidal functions. Moreover the approximation involved in replacing the cylinder by the spheroid is probably much better than the approximation one must use in solving the integral equation.[15] In addition, prolate spheroids are good approximations to the bodies of missiles and so

[15] For a discussion of this see Ryder, op. cit., p. 327.

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have a physical counterpart in their own right. The problem of end effects which enters in the cylindrical antenna and is of considerable importance is eliminated by going to the prolate spheroid. Also, eliminating these end effects permitted more attention to be given to the unsymmetrical effects and the spheroidal functions themselves.

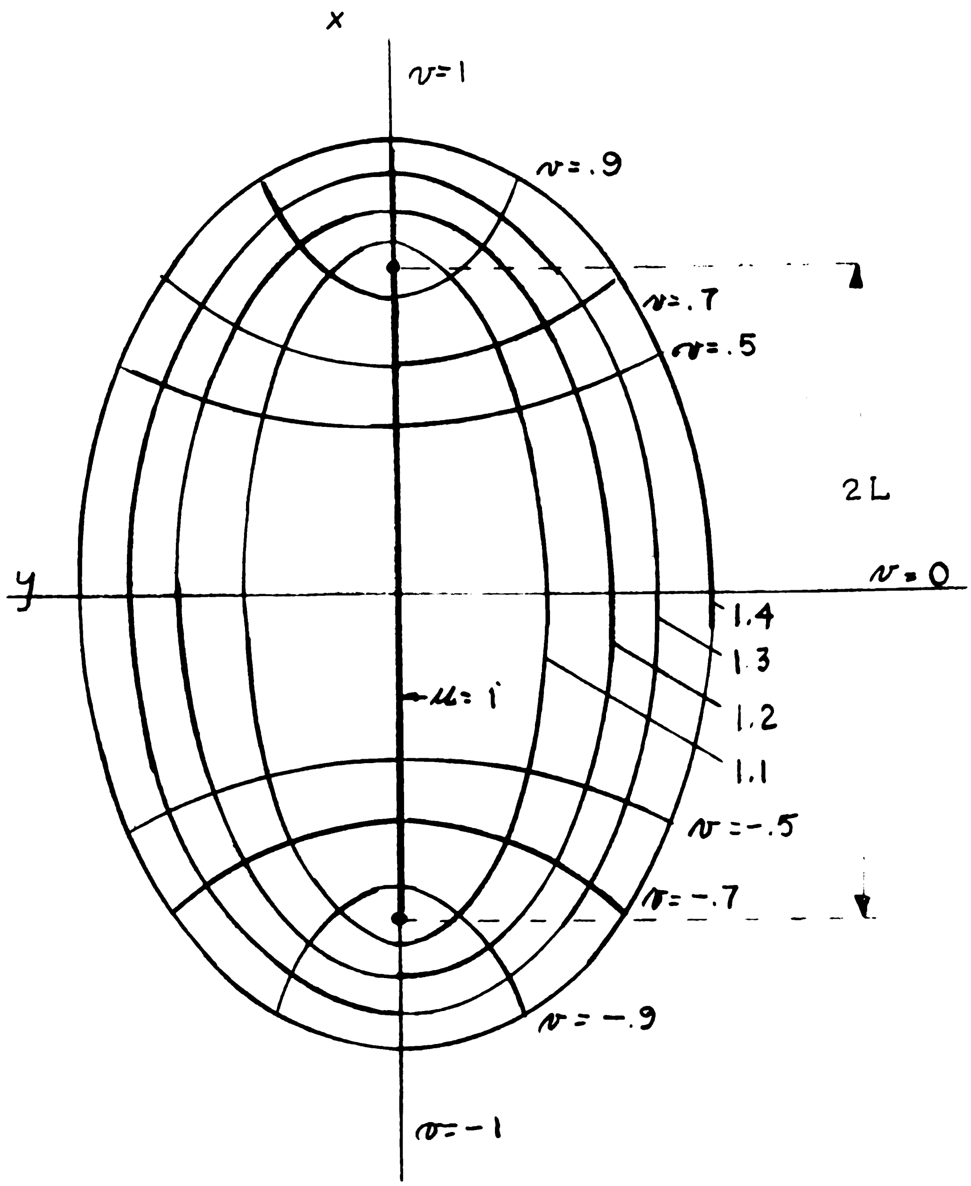

Figure 1. The Prolate Spheroidal Coordinate System

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II

PROLATE SPHEROIDAL FUNCTIONS

The prolate spheroidal coordinate system is shown in Figure 1. The radial and angular variables are taken as u, v, and ϕ, after Schelkunoff.[16] The relations between the spheroidal variables and the cartesian variables x, y, and z are

$$x = Luv, \quad y = L(u^2-1)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}\cos\phi, \quad z = L(u^2-1)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}\sin\phi.$$

L is the semifocal distance. The metric coefficients are

$$e_1 = L(u^2-v^2)^{\frac{1}{2}}(u^2-1)^{-\frac{1}{2}}, \quad e_2 = L(u^2-v^2)^{\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}, \quad e_3 = L(u^2-1)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}.$$

For a homogeneous medium Maxwell's equations are

$$\operatorname{curl} \bar{\mathcal{E}} = -\mu \frac{\partial \bar{\mathcal{H}}}{\partial t}, \qquad \operatorname{curl} \bar{\mathcal{H}} = \varepsilon \frac{\partial \bar{\mathcal{E}}}{\partial t}. \tag{1}$$

$\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ are the electric and magnetic field vectors; μ and ε are the permeability and dielectric constant of the medium. For sources and fields whose time variation is harmonic, it is possible and convenient to eliminate the time variable by the use of complex vectors.[17] Then $\bar{\mathcal{E}} = \mathrm{Re}(\bar{E}\epsilon^{i\omega t})$, and $\frac{\partial \bar{\mathcal{E}}}{\partial t} = \mathrm{Re}(i\omega \bar{E}\epsilon^{i\omega t})$. $\omega = 2\pi f$, where f is the oscillation frequency of the source. The factors Re and $\epsilon^{i\omega t}$ are

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[16] S. A. Schelkunoff, op. cit., p. 111.

[17] J. Aharoni, Antennae, Clarendon Press, Oxford, England, pp. 12 & 13, 1946.

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then dropped by convention, because they appear in each term of the equation. Thus equations (1) become

$$\operatorname{curl} \bar{E} = -i\omega\mu \bar{H}, \qquad \operatorname{curl} \bar{H} = i\omega\varepsilon \bar{E}. \tag{2}$$

Rotational symmetry is assumed. The basis of this assumption is that for thin antennas the diameter is small compared with one wave length, and in this case the potential of any point on a given circumference is essentially the same as that of any other because the antenna elements are good conductors. In other words, the assumption is that the current that flows around the elements is negligible in comparison to the current that flows along the elements, and that the latter current is distributed uniformly around the surface of the elements. This is a good approximation for the antennas treated in this thesis, and the thinner the antenna, the better the approximation will be. With the assumption of rotational symmetry the equations and their solutions are independent of the ϕ coordinate, and ϕ can be taken as any convenient value; ϕ is taken equal to zero here, after Schelkunoff.

In prolate spheroidal coordinates equations (2) become

$$e_2E_u = \frac{1}{i\omega\varepsilon y}\frac{\partial(yH_\phi)}{\partial v}, \qquad e_1E_v = -\frac{1}{i\omega\varepsilon y}\frac{\partial(yH_\phi)}{\partial u},$$
$$\frac{\partial(e_2E_v)}{\partial u} - \frac{\partial(e_1E_u)}{\partial v} = -i\omega\mu\, e_1e_2H_\phi. \tag{3}$$

The field intensities may thus be expressed in terms of the auxiliary wave function, $A = yH_\phi$. Thus where $\beta = 2\,\pi$

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$$E_u = -\frac{i\eta}{\beta L^2}\left[(u^2-1)\ (u^2-v^2)\right]^{-\frac{1}{2}}\frac{\partial A}{\partial v},$$

$$E_v = \frac{i\eta}{\beta L^2}\left[(1-v^2)(u^2-v^2)\right]^{-\frac{1}{2}}\frac{\partial A}{\partial u}, \qquad (4)$$

$$H_\phi = \frac{1}{L}\left[(u^2-1)(1-v^2)\right]^{-\frac{1}{2}} A, \qquad \eta = \sqrt{\mu/\varepsilon}$$

where $\beta = 2\pi/\lambda$; λ is the wave length corresponding to the frequency of the source, and where A satisfies the following differential equation,

$$(u^2-1)\frac{\partial^2 A}{\partial u^2} + (1-v^2)\frac{\partial^2 A}{\partial v^2} + \beta^2 L^2(u^2-v^2)A = 0. \quad (5)$$

It is convenient to make the substitution $\beta L = \underline{c}$.

Ampere's circuital law,[18] equation (6), states that the line integral of magnetic intensity around a closed path is equal to the current enclosed by the path.

$$\oint \bar{H}\cdot d\bar{l} = I, \qquad (6)$$

or

$$\int_0^{2\pi} H_\phi\, d\phi = I. \qquad (7)$$

Hence the current in the antenna is

$$I(v) = 2\pi\ A(u_0,v). \qquad (8)$$

The substitution

$$A = U(u)V(v) \qquad (9)$$

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[18] A. B. Bronwell, & R. E. Beam, _Theory and Application of Microwaves_, McGraw-Hill, New York, p. 248, 1947.

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separates the variables in equation (5), and the differential equations for U and V are

$$(u^2-1)\frac{d^2U}{du^2} + (c^2u^2-k)U = 0, \tag{10}$$

$$(1-v^2)\frac{d^2V}{dv^2} + (k-c^2v^2)V = 0. \tag{11}$$

Both U and V satisfy the same equation. There are three singularities: $u = \pm 1$ and $u = \infty$. The singularity at infinity is irregular, but the ones at ± 1 are regular.[19] Therefore there exists one solution which is regular everywhere and one which is not.[20] Substituting $U(u) = (u^2-1)\bar{U}(u)$, $V(v) = (1-v^2)\bar{V}(v)$, gives

$$(u^2-1)\frac{d^2\bar{U}}{du^2} + 4u\frac{d\bar{U}}{du} + (c^2u^2-k+2)\bar{U} = 0, \tag{12}$$

$$(1-v^2)\frac{d^2\bar{V}}{dv^2} - 4v\frac{d\bar{V}}{dv} + (k-2-c^2v^2)\bar{V} = 0. \tag{13}$$

This form of the equation for spheroidal wave functions was used by Stratton *et al*. Page and Adams and Ryder used the form (11).

At the ends ($v = \pm 1$) of the spheroid the current vanishes; hence,

$$V(\pm 1) = 0. \tag{14}$$

This condition also follows from the fact that at the ends the field components must be finite. For most values of *k*, equation (11)

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[19] E. L. Ince, Ordinary Differential Equations, Dover Publications, New York, p. 160, 1944.

[20] *Ibid*., pp. 400-402.

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has no solutions satisfying the above condition; thus, equation (11) defines the proper or eigenvalues of k. The corresponding set of eigenfunctions are designated by V_k and U_k. At large distances from the spheroid u is large, and equation (10) becomes

$$\frac{d^2U}{du^2} + c^2U = 0. \qquad (15)$$

At these distances also

$$r = (z^2 + y^2)^{\frac{1}{2}} = Lu. \qquad (16)$$

Because the time variation is of the form $\epsilon^{i\omega t}$ and a diverging wave solution is desired, at such distances the field must be proportional to $\epsilon^{-i\beta r} = \epsilon^{-icu}$. Hence, the proper $U_k(u)$ functions are those solutions of equation (10) that satisfy the following condition:

$$U_k(u) \backsim \epsilon^{-icu} \text{ as } u \rightarrow \infty. \qquad (17)$$

Thus, the general solution satisfying the requirements at the ends of the spheroid and at infinity is

$$A = \sum_k a_k U_k(u) \; V_k(v). \qquad (18)$$

The solution of the angular equation (11) was obtained in the form of a power series about the origin.

$$V_k(v) = \sum_{n=0,1}^{\infty}{}' c_n \, v^n \qquad (19)$$

The prime on the summation sign indicates that the summation is taken over even powers of v if k's with even subscripts are used, and over odd powers

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odd powers of _v_ if k's with odd subscripts are used. Substitution of (19) into (11) leads to the recursion formula

$$-(n+2)(n+1)c_{n+2} + [n(n-1) - k]c_n + c^2 c_{n-2} = 0 \qquad (20)$$

This plus the boundary condition (14) then leads to a transcendental equation (i.e., an equation with an infinite number of positive powers of k) in _k_, the roots of which are the eigenvalues k_ℓ . Thus

$$c_0 + c_2 + \ldots + c_{2n} + c_{2n+2} + \ldots = 0 \qquad (21)$$
$$\text{for } k = k_0,\ k_2,\ k_4, \text{ etc.}$$

and

$$c_1 + c_3 + \ldots + c_{2n+1} + c_{2n+3} + \ldots = 0 \qquad (22)$$
$$\text{for } k = k_1,\ k_3,\ k_5, \text{ etc.}$$

The reason for this particular choice of subscripts for the k's is that the roots of equations (21) and (22) are all positive,[21] and the numerical value of k_1 lies between the values of k_0 and k_2, k_3 between k_2 and k_4, etc. - i.e., the sequence $\{k_\nu\}$ is monotonically increasing with ν . Now it happens that by taking the first seven terms of (21) or (22) one obtains values for the k's which are accurate to from six to nine places, the higher accuracy obtaining for the k's with lower subscripts. This is to say that if one were to take eight terms of (21) as the approximation, the k_2 so obtained would be the same as the k_2 obtained with seven terms to eight place accuracy - at least

[21] R. Courant and D. Hilbert, Methods of Mathematical Physics, Inter-Science Publishers, New York, v. 1, p. 294, 1953.

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for the case $c = 1$, for this was the case that was computed in detail during the period of preliminary investigation. Thus for the proper values of k, the contribution of c_{14} and higher terms to the value of $\sum_{n=0}^{\infty}{}' c_n v^n$ is very small indeed, and so the physical boundary condition (14) leads to the mathematical condition that the c_n approach zero very rapidly for the proper values of k.

Actually, since $V_k(u)$ is a solution of the radial equation for $u > 1$ (see below), equation (19) converges for all v, and this implies that $\lim_{n\to\infty} \sqrt[n]{|c_n|} = 0$. Otherwise stated, given $\varepsilon > 0$, $|c_n| < \varepsilon^n$ for $n > N_\varepsilon$. By summing the equations (20) for $n = 0, 2, \ldots, 2n-2$, assuming $c_{-2} = 0$, and defining Θ_{2m} by

$$\Theta_{2m} c_{2m} = \sum_{r=m}^{\infty} c_{2r} = -\sum_{r=0}^{m-1} c_{2r} \tag{23}$$

one finds

$$c^2 c_{2m-2} + [2m(2m-1) - \Theta_{2m}(k - c^2)]\, c_{2m} = 0. \tag{24}$$

If, as it appears for $k = k_\ell$ and for $r \geq m > \ell$, the coefficients c_{2r} decrease in absolute value and alternate in sign, then Θ_{2m} in (24) is less than unity and

$$c_{2m} = \frac{-c^2 c_{2m-2}}{(2m)(2m-1) - \theta_{2m}(k_\ell - c^2)} \doteq \frac{-c^2 c_{2m-2}}{2m(2m-1)} \doteq \frac{K(-c^2)^m}{(2m)!} . \qquad (25)$$

This result is consistent with the assumed monotone decreasing character of $(-1)^m c_{2m}$ for $r \geqslant m$. A similar argument holds for the c's with odd subscripts.

When c is zero the values of $k_{\ell-2}$ are exactly $\ell(\ell-1)$, and when c is small they are found to be

$$k_{-2} = \ell(\ell-1) + c^2 \left[\frac{1}{2} - \frac{3}{2(2\ell+1)(2\ell-3)} \right] + O(c^4), \qquad (26)$$

$$\text{for} \quad = 2, 3, \ldots$$

However, small inaccuracies in the values of k_ℓ completely upset the assumed rapid convergence of c_m. The values of the separation constants k_ℓ used in this thesis were taken from the eleven place tables recently computed by the Bureau of Standards.

The power series representation (19) of the angular function $V_k(v)$ was chosen for three reasons. First, it led to the relatively simple recursion formula (20). Second, the $V_k(v)$ can be easily evaluated for <u>any</u> particular value v_1, whereas tables of Associated Legendre Polynomials, as used by Stratton <u>et al</u>., are published only for a limited number of values of the argument. Finally, the power series can be easily multiplied by an expression for the applied electric field intensity and integrated by elementary methods to yield a value for the coefficient a_k in (18). This representation has the disadvantage that values for the normalizations can be more easily computed by other methods. (See Section III).

The solution of the radial equation (10) subject to the condition at infinity (17) can be expressed as

$$U(u) = \epsilon^{-icu} \sum_{n=0}^{\infty} a_n (iu)^{-n}, \quad a_0 = 1. \tag{27}$$

The argument in the power series was taken as (iu) so that the a_n would all be real. Substitution of this expression in (10) leads to the recursion formula for the a_n:

$$2c(n+1)a_{n+1} + [(n+1)n + c^2 - k]a_n + 2c(n-1)a_{n-1} + (n-1)(n-2)a_{n-2} = 0. \tag{28}$$

McCrea and Newing[22] have proven the existence of series solutions satisfying the boundary conditions for the generalized spheroidal wave equation, and therefore the expression (27) can converge and represent $U(u)$ for $|u| \geq 1$. Computations indicated that the series did in fact converge in every case for $k = k_\ell$.

For thin antennas it is necessary to evaluate the radial functions $U(u)$ for values of u_0 in the neighborhood of one - i.e., $u_0 = 1.020$,

[22] W. H. McCrea and R. A. Newing, "Boundary Conditions for the Wave Equation", Proc., London Math. Soc., v. 37, London, pp. 520-534, (1934).

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1.005, 1.001, and 1.00001 - and this is just the region for which the convergence of (27) is very slow. The antennas are shown in Figure 2. One solution of the radial equation (10) can be obtained in the form

$$\overset{(1)}{U}(u) = \sum_{n=0}^{\infty} b_n (u^2-1)^{n+1}, \quad \text{for } k_0, k_2, k_4, \ldots \tag{29}$$

The solution of the second kind can then be written as[23]

$$\overset{(2)}{U}(u) = \overset{(1)}{U}(u)\ \ln(u^2-1) + \sum_{n=0}^{\infty} d_n (u^2-1)^n. \tag{30}$$

The general solution is then

$$U(u) = C_1 \overset{(1)}{U}(u) + C_2 \overset{(2)}{U}(u), \tag{31}$$

and the C_1 and C_2 must be determined so that the condition at infinity is satisfied. This was accomplished by expressing both (27) and (31) as Laurent series expansions in powers of u and then comparing coefficients.

The expansion of (27) was perfectly straightforward. One merely multiplies the power series expansion for ϵ^{-icu} by the series $\sum_{n=0}^{\infty} a_n (iu)^{-n}$ to obtain

$$U(u) = \ldots - iu^{-1}\left(a_1 - ca_2 + \frac{c^2a_3}{2!} - \frac{c^3a_4}{3!} + \frac{c^4a_5}{4!} - \ldots\right) \tag{32}$$

$$+ \quad \left(a_0 - ca_1 + \frac{c^2a_2}{2!} - \frac{c^3a_3}{3!} + \frac{c^4a_4}{4!} - \ldots\right)$$

[23] E. L. Ince, op. cit., p. 164.

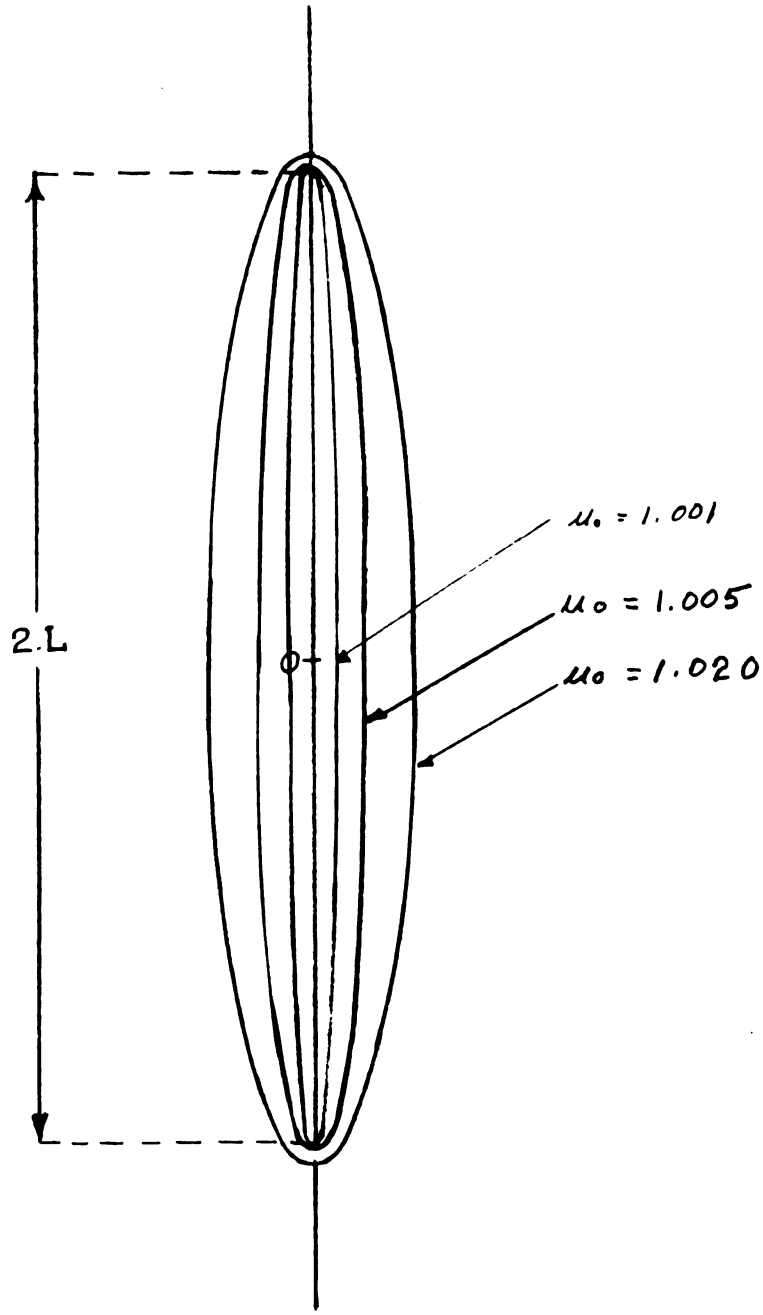

Figure 2. Prolate Spheroids

$$+ \; iu\left(- ca_0 + \frac{c^2}{2!}a_1 - \frac{c^3}{3!}a_2 + \frac{c^4}{4!}a_3 - \frac{c^5}{5!}a_4 + \ldots\right)$$

$$- \; u^2\left(\frac{c^2}{2!}a_0 - \frac{c^3}{3!}a_1 + \frac{c^4}{4!}a_2 - \frac{c^5}{5!}a_3 + \ldots\right) + \ldots \quad .$$

Thus, although the a_n themselves do not converge rapidly, the $\frac{c^m}{m!}$ factors do, and so each term can be computed readily. However, the coefficients of the u's with negative exponents converge very slowly, and so the computation of U(u) can not be carried out directly using (32).

For the purposes of expanding $U^{(1)}(u)$ (eq. 29) in powers of u it is important to note the following fact. $U^{(1)}(u)$ is the only solution of (10) that is regular in the neighborhood of the singular point u = 1. Similarly, $V_k(v)$ is the only solution of (11) that is regular in the neighborhood of the singular point v = 1. But (10) and (11) are the same equation. Therefore $U^{(1)}(u)$ and $V_k(v)$ represent the same function, and by the identity theorem[24] for power series the series representations of $U^{(1)}(u)$ and $V_k(v)$ must be the same (for the same k, of course). When the $(u^2-1)^{n+1}$ terms are expanded and multiplied by the b_n to obtain a series in powers of u for $U^{(1)}(u)$ it is found that the resulting coefficients are exactly proportional to the c_n for the corresponding $V_k(v)$.

Still another way of obtaining an expression for the radial function is to write it in the form of a Laurent series:

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[24] K. Knopp, Theory of Functions, Tr. by F. Bagemihl, part I, Dover Publications, p. 81, New York, 1945.

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$$U(u) = \sum_{n=-\infty}^{+\infty} f_n u^n. \tag{33}$$

Substitution of this expression in (10) leads again to the recursion formula (23). The coefficients of the terms with positive exponents converge very rapidly - in fact the series of terms with positive even exponents is just a multiple of $V_k(v)$ for k_0, k_2, ... - but the coefficients of the terms with negative exponents converge very slowly. This is the result that was obtained also for the expressions (27) and (31), since they represent the same function. The problem was to find an expression of U(u) that would not involve the coefficients of the terms with negative exponents directly. This was done in the following way.

Considering U(u) as written in the form (31), we notice that the only place powers of u with negative exponents can appear is in the $U^{(1)}(u) \ln(u^2-1)$ term. Now

$$\ln(u^2-1) = \ln\frac{(u-1)}{(u+1)} + 2\ln(u+1) \tag{34}$$

and

$$\ln(u+1) = \ln(2+u-1) = \ln 2 + \ln(1+\frac{u-1}{2}),$$

or

$$\ln(u+1) = \ln(2) + \frac{(u-1)}{2} - \frac{1}{2}\frac{(u-1)^2}{4} + \frac{1}{3}\frac{(u-1)^3}{8} - \dots \tag{35}$$

$$\text{for } -1 < u \leq 3.$$

Therefore, ln(u+1) can be expanded in positive powers of u in a neighborhood of u = 1. Also,

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$$\ln\frac{u-1}{u+1} = -2(1/u + 1/3u^3 + 1/5u^5 + 1/7u^7 + \ldots) \qquad (36)$$

for $u > 1$.

Therefore, the part of $\ln(u^2-1)$ that contains the terms with negative exponents is the $\ln\frac{u-1}{u+1}$ term. Remembering that $U^{(1)}(u) = V_k(u) = \sum_{n=0}^{\infty}{}' c_n u^n$, we see that the part of $U(u)$ containing terms with negative exponents is given by $\sum_{n=0}^{\infty}{}' c_n u^n . \ln\frac{u-1}{u+1}$. This expression can be readily computed because the c_n converge very rapidly and the value of the logarithm can be obtained from tables.

The series that results as the product of $\sum_{n=0}^{\infty}{}' c_n u^n$ and the series (36) contains both positive and negative powers of u and, as was to be expected, the coefficients of the terms with negative exponents converge very slowly. However, the part of the product containing the positive powers of u is given by

$$\left(\sum_{n=0}^{\infty}{}' c_n u^n \cdot \ln\frac{u-1}{u+1}\right)^{+} = -2\,[\,u(c_2 + \frac{c_4}{3} + \frac{c_6}{5} + \frac{c_8}{7} + \frac{c_{10}}{9} + \ldots)$$

$$+ u^3(c_4 + \frac{c_6}{3} + \frac{c_8}{5} + \frac{c_{10}}{7} + \ldots)$$

$$+ u^5(c_6 + \frac{c_8}{3} + \frac{c_{10}}{5} + \ldots) + \ldots$$

$$+ u^{11}(c_{12} + \frac{c_{14}}{3} + \frac{c_{16}}{5} + \ldots) + \ldots]$$

$$= -2\sum_{n=1}^{\infty}{}' D_n u^n; \qquad (37)$$

and these terms converge rapidly because the c_n do. Therefore, the part of $U(u)$ containing terms with negative exponents can be evaluated by <u>subtracting</u> the above expression from $\sum_{n=0}^{\infty}{}' c_n u^n \ln\frac{u-1}{u+1}$.

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$$[U(u)]^{-} = \sum_{n=0}^{\infty}{}' c_n u^n \ln\frac{u-1}{u+1} + 2\sum_{n=1}^{\infty}{}' D_n u^n. \qquad (38)$$

The minus and plus signs in the superscripts indicate that only the part of the series containing the terms with negative or non-negative exponents is meant.

Thus $U(u)$ for $k = k_0, k_2, k_4, \ldots$ can be evaluated in the neighborhood of $u = 1$ in the following way: From (33) the terms with positive even exponents can be expressed as a multiple of V_k - i.e., as $A\sum_{n=0}^{\infty}{}' c_n u^n$. The constant $\underline{A}$ can be evaluated and checked by comparing the coefficient Ac_n with the coefficient of the corresponding positive even power of $\underline{u}$ in (32). Thus, since c_0 was taken as unity for convenience,

$$Ac_0 = A = (a_0 - ca_1 + \frac{c^2a_2}{2!} - \frac{c^3a_3}{3!} + \frac{c^4a_4}{4!} - \frac{c^5a_5}{5!} + \ldots) \qquad (39)$$

and a check is[25]

$$Ac_2 = (\frac{c^2a_0}{2!} - \frac{c^3a_1}{3!} + \frac{c^4a_2}{4!} - \frac{c^5a_3}{5!} + \frac{c^6a_4}{6!} - \ldots). \qquad (40)$$

It is almost a necessity to have a simple and accurate check at every step in numerical computations. Again from (33) the coefficients of the terms with positive odd exponents can be obtained by evaluating three corresponding coefficients in (32) and checking that they satisfy

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[25] In some cases, for k's with higher subscripts - e.g. k_6 - it may be necessary to equate corresponding coefficients of higher powers of $\underline{u}$ - u^6 and u^8, say - in order to maintain the desired accuracy, but the procedure is the same. Since, for k_0, k_2, etc. there are no negative even exponents in (33), a check on the a_n is to evaluate the coefficient of u^{-2} in (32) to be sure that it is zero.

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the recursion formula (23). The remaining coefficients of terms with positive odd exponents can be evaluated by means of the recursion formula. Thus the complete expression for U(u) for k_0, k_2, k_4, ... can be written as

$$U(u) = A\sum_{n=0}^{\infty}{}' c_n u^n + \sum_{n=1}^{\infty}{}' c_n u^n + B\left(\sum_{n=0}^{\infty}{}' c_n u^n \ln\frac{u-1}{u+1} + 2\sum_{n=1}^{\infty}{}' D_n u^n\right). \quad (41)$$

The first two terms on the right represent the terms in U(u) with non-negative exponents, and the last expression represents the terms with negative exponents. The constant B can be evaluated and checked by expanding $\left(\sum_{n=0}^{\infty}{}' c_n u^n \ln\frac{u-1}{u+1}\right)^-$ and comparing the coefficients of $1/u^n$ so obtained with the corresponding coefficients from (32). Thus

$$B\left(\sum_{n=0}^{\infty}{}' c_n u^n \ln\frac{u-1}{u+1}\right)^- = -2B\left[\frac{1}{u}\left(c_0 + \frac{c_2}{3} + \frac{c_4}{5} + \frac{c_6}{7} + \dots\right) + \frac{1}{u^3}\left(\frac{c_0}{3} + \frac{c_2}{5} + \frac{c_4}{7} + \dots\right) + \dots\right] \quad (42)$$

Equating the coefficients of $1/u$ in (32) and (42), one expression for B is

$$B = \frac{i\left(a_1 - ca_2 + \frac{c^2 a_3}{2!} - \frac{c^3 a_4}{3!} + \dots\right)}{2\left(c_0 + \frac{c_2}{3} + \frac{c_4}{5} + \frac{c_6}{7} + \frac{c_8}{9} + \dots\right)}, \quad (43)$$

and by equating the coefficients of $1/u^3$ a check is

$$B = \frac{i\left(a_3 - ca_4 + \frac{c^2 a_5}{2!} - \frac{c^3 a_6}{3!} + \dots\right)}{-2\left(\frac{c_0}{3} + \frac{c_2}{5} + \frac{c_4}{7} + \frac{c_6}{9} + \dots\right)}. \quad (44)$$

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To obtain the derivative of the radial function $U'(u)$, it is only necessary to differentiate each term in (41). Thus

$$U'(u) = A\sum_{n=0}^{\infty}{}' nc_n u^{n-1} + \sum_{n=1}^{\infty}{}' nc_n u^{n-1} + 2B\sum_{n=1}^{\infty}{}' nD_n u^{n-1} + B\sum_{n=0}^{\infty}{}' nc_n u^{n-1}\cdot \ln\frac{u-1}{u+1}$$

$$+ \; B\sum_{n=0}^{\infty}{}' c_n u^n \cdot 2/(u^2-1) \qquad (45)$$

In this expression all the constants have already been determined, and so only the series need to be summed again because of the factor of <u>n</u> that was introduced by the differentiation.

The radial functions for the k's with odd subscripts are computed in exactly the same way, the only difference being that the summations over the odd powers in the previous case are now taken over the even powers, and <u>vice-versa</u>. Thus, for k_1, k_3, k_5, etc.,

$$U(u) = A\sum_{n=1}^{\infty}{}' c_n u^n + \sum_{n=0}^{\infty}{}' c_n u^n + B\left(\sum_{n=1}^{\infty}{}' c_n u^n \ln\frac{u-1}{u+1} + 2\sum_{n=0}^{\infty}{}' D_n u^n\right). \qquad (46)$$

where

$$Ac_1 = A = i\left(-ca_0 + \frac{c^2}{2!}a_1 - \frac{c^3}{3!}a_2 + \frac{c^4}{4!}a_3 - \ldots\right), \qquad (47)$$

and a check is

$$Ac_3 = i\left(\frac{c^3}{3!}a_0 - \frac{c^4}{4!}a_1 + \frac{c^5}{5!}a_2 - \frac{c^6}{6!}a_3 + \ldots\right). \qquad (48)$$

$$B = \frac{a_2 - ca_3 + c^2a_4/2! - c^3a_5/3! + c^4a_6/4! - \ldots}{2(\,c_1/3 + c_3/5 + c_5/7 + c_7/9 + \ldots)} \qquad (49)$$

and a check is

$$B = \frac{-\,(a_4 - ca_5 + c^2a_6/2! - c^3a_7/3! + c^4a_8/4! -}{2\,(c_1/5 + c_3/7 + c_5/9 + c_7/11 + \ldots)}\,. \qquad (50)$$

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$$D_0 = (c_1 + c_3/3 + c_5/5 + c_7/7 + \ldots), \tag{51}$$

$$D_2 = (c_3 + c_5/3 + c_7/5 + c_9/7 + \ldots), \tag{52}$$

$$D_4 = (c_5 + c_7/3 + c_9/5 + c_{11}/7 + \ldots), \text{ etc.} \tag{53}$$

The derivative of the radial function $U_k'(u)$ for this case is then

$$\begin{aligned} U'(u) = {} & A\sum_{n=1}^{\infty}{}' nc_n u^{n-1} + \sum_{n=0}^{\infty}{}' nc_n u^{n-1} + 2B\sum_{n=0}^{\infty}{}' nD_n u^{n-1} \\ & + B\sum_{n=1}^{\infty}{}' nc_n u^{n-1} \cdot \ln\frac{u-1}{u+1} + B\sum_{n=1}^{\infty}{}' c_n u^n . 2/(u^2-1) \end{aligned} \tag{54}$$

for $k = k_1, k_3, k_5, \ldots$

Finally, <u>if</u> sufficient accuracy has been maintained in the computation of the radial function and its derivative, a good check on the accuracy of $U(u)$ and $U'(u)$ can be obtained by the use of the Wronskian. If one writes

$$U(u) = U_1 + U_2 \tag{55}$$

where U_1 is the real part and U_2 is the imaginary part, including the <u>i</u>, substitution of U_1 and U_2 into (10), multiplying the first equation by U_2 and the second by U_1, and subtracting the two resulting equations leads to

$$U_2U_1'' - U_1U_2'' = 0, \tag{56}$$

or,

$$U_1'U_2 - U_1U_2' = C, \tag{57}$$

where C

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where $\underline{C}$ is a constant that can be determined from the behavior of the radial function at infinity. Thus, as $\underline{u}$ approaches infinity

$$U_1 \backsim \cos(cu), \quad U_2 \backsim - i \sin(cu)$$

$$U_1' \backsim -c \sin(cu), \quad U_2' \backsim -ic \cos(cu),$$

and therefore

$$\underline{U_1'U_2 - U_1U_2' = ic.} \tag{58}$$

III

IMPEDANCE CALCULATIONS

and

RADIATION PATTERNS

From equations (4), the electric intensity tangential to a typical spheroid confocal with the given spheroid is

$$E_v(u,v) = \frac{i\eta}{\beta L^2} \left[(1-v^2)(u^2-v^2)\right]^{-\frac{1}{2}} \sum_k a_k U_k'(u) V_k(v). \tag{59}$$

At the perfectly conducting surface of the given spheroid $u = u_0$ the total electric intensity should vanish;

$$E_v(u_0,v) + E_v^a(u_0,v) = 0, \tag{60}$$

where $E_v^a(u_0,v)$ is the tangential component of the applied electric field. Therefore,

$$\frac{i\eta}{\beta L^2} \left[(1-v^2)(u_0^2-v^2)\right]^{-\frac{1}{2}} \sum_k a_k U_k'(u_0) V_k(v) = - E_v^a(u_0,v). \tag{61}$$

The orthogonality of the V functions is easily proven:[26]

$$\int_{-1}^{1} (1-v^2)^{-1} V_k V_n \, dv = 0 \qquad \text{for } k \neq n. \tag{62}$$

Multiplying (61) by $(\beta L^2/i\eta)(u_0^2-v^2)^{\frac{1}{2}} V_n(v)$, integrating, and using (62) yields

[26] S. A. Schelkunoff, op. cit., p. 115.

$$\underline{a}_n = \frac{\beta L^2}{i\eta N_n U_n'(u_0)} \int_{-1}^{1} E_v^a(u_0,v)(u_0^2-v^2)^{\frac{1}{2}}(1-v^2)^{-\frac{1}{2}}V_n(v)\,dv. \quad (63)$$

N_n is the normalising factor[27]

$$N_n = \int_{-1}^{1}(1-v^2)^{-1}V_n^2(v)\,dv. \quad (64)$$

Thus, once the angular functions and their norms and the radial functions and their derivatives have been obtained, one must evaluate the expression (63) to obtain the $\underline{a}_n$, and thereby the auxiliary wave function A from (18). But, in order to do this, the exact nature of the driving field that is applied to the antenna must be known. However, the exact solution of the field around an infinite two-wire transmission line in free space has not yet been obtained,[28] much less the problem of the field of a transmission line in the neighborhood of conductors such as the elements of an antenna. Therefore, one must decide upon an approximation to the applied field, and the validity of the approximation will have to be checked indirectly by experimental evidence.

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[27] Although it is possible to compute the norms directly by squaring the series (19), dividing by $(1-v^2)$, and integrating, it is easier to compute them by Stratton's method (Stratton et al., op. cit., p. 63). One must be careful, of course, to divide his values of the norms by the square of the ratio $S_{1\ell}^{(1)}(c,0)/V_\ell(0)$ for even values of ℓ, and by the square of the ratio $\left.\frac{dS_{1\ell}^{(1)}(c,v)}{dv}\right|_{v=0} \Big/ \left.\frac{dV_\ell(v)}{dv}\right|_{v=0}$ for odd values of ℓ.

Thus the ratios are:

ℓ:	0	1	2	3	4	5	6
r^2:	1	9	9/4	225/4	225/64	11025/64	11025/2304

[28] A. Sommerfeld, Electrodynamics, Academic Press, p. 199, 1952.

If the impedance values computed from the approximation agree very closely with experimentally obtained values over a fairly wide range of frequencies, antenna thicknesses, and gap widths, then it is probably true that the correct approximation was used. A common assumption is that the tangential component of the applied electric field intensity has a non-zero value only at the gap or, equivalently, that the static field that would exist due to a charged transmission line is a good approximation to the actual time-varying applied field. However, there are several objections to this assumption,[29] and the computed values of impedance often do not agree with experiment.[30]

Some authors[31] write that the gap is the only source of the radiated energy. One objection to this statement is that the actual radiators are the oscillating atoms on the surface of the antenna elements. Another is that if one shields the gap, experimentally the result is that the radiation pattern is essentially the same as the one with the unshielded gap, but if the elements are shielded, the radiation decreases in proportion to the length that is shielded.[32] Another objection is that in the case of the delta-fed antenna there is no "gap", and the static field approximation breaks down completely. The applied

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[29] H. A. Myers, Fundamentals of Antenna Radiation, Unpublished M. S. thesis, Michigan State College, 1950.

[30] R. W. Beatty, Experimental Check at 3,000 Megacycles of Calculated Antenna Impedance, M. S. thesis, Mass. Inst. of Tech., 1943.

[31] J. L. Synge & G. E. Albert, Quart. Appl. Math., 6, p. 127, (1948). Also see A. B. Bronwell & R. E. Beam, op. cit., pp. 432-433.

[32] H. A. Myers, op. cit.

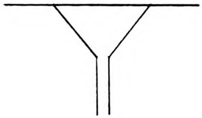

Figure 3. Delta-fed Antenna

electric field must have a tangential component along the antenna in this case, and the current must be induced in the antenna in such a manner as to create a field whose tangential component is equal and opposite to that of the applied field so that the boundary condition on the surface is satisfied.

Reasoning along these lines led to the idea that a better approximation to the applied field would be to take into consideration the tangential component that exists "outside" the wires of the transmission line as well as the field that exists between the wires.

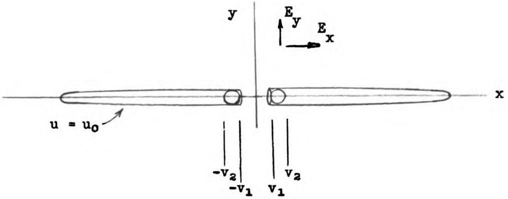

Figure 4. Spheroidal Antenna at the end of a Two Wire Line

(View along the wires)

The electric field intensity $\bar{E}$ outside the wires has three components: the transverse components E_x and E_y due to the (alternating) charges on the wires, and the longitudinal component due to resistance in the wires and to the alternating magnetic field of the current.

According to Sommerfeld[33] the longitudinal component E_z is small for wires of high conductivity. It is easily shown the $E_y \doteq 0$ for thin spheroids. In addition, the field strength decreases approximately as the second power of x. Since E_y and E_z can have appreciable components tangent to the surface of the spheroid <u>only</u> near the ends of the spheroid (i.e., relatively far away from the <u>z</u> axis) these components can be neglected in comparison with E_x, whose tangential component is large near the gap.

Since E_z is small, the electric field of the transmission line carrying alternating current is, at any instant, approximately the same as the field in the static case.

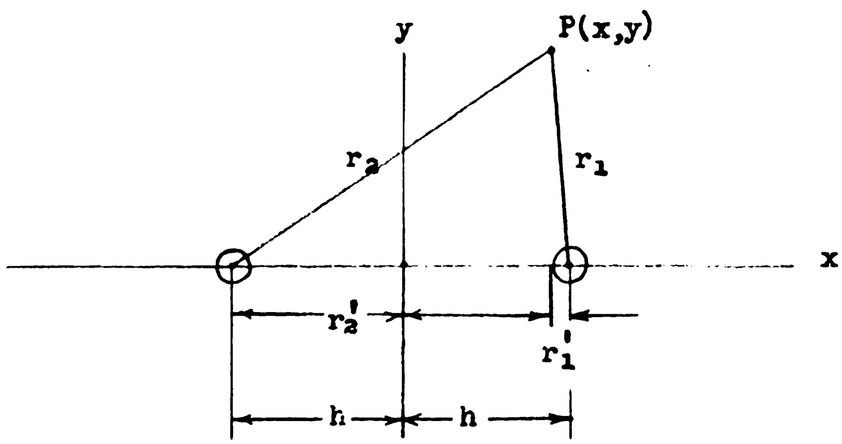

Figure 5. Two Wire Transmission Line

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[33] A. Sommerfeld, <u>op. cit</u>., p. 200.

From d.c. theory[34] the potential at a point P due to two equally and oppositely charged parallel cylindrical wires is, referring to Figure 5,

$$V_p = \frac{2\tau}{\varepsilon}\ln(r_2/r_1) = \frac{2\tau}{\varepsilon}\ln\left[\frac{(x+h)^2+y^2}{(x-h)^2+y^2}\right]^{\frac{1}{2}} \qquad (65)$$

where τ = charge/unit length and ε is the dielectric constant of the surrounding medium. The applied voltage V^a between the wires of the transmission line is

$$V^a = \frac{4\tau}{\varepsilon}\ln(r_2'/r_1') \qquad (66)$$

where r_2' is the perpendicular distance from the axis of one conductor to a point on the surface of the second conductor, and r_1' is the distance from the axis on the second conductor to the same point on its surface. The electric field vector is

$$\bar{E}_p = -\nabla V_p = -\bar{i}\frac{\partial V_p}{\partial x} - \bar{j}\frac{\partial V_p}{\partial y} = \bar{i}\,E_x + \bar{j}\,E_y\,, \qquad (67)$$

where $\bar{i}$ and $\bar{j}$ are the unit vectors along the x and y axes respectively. Then

$$E_x = -\frac{4\tau h}{\varepsilon}\,\frac{(y^2-x^2+h^2)}{[(x+h)^2+y^2][(x-h)^2+y^2]}\;. \qquad (68)$$

For thin spheroids $y \doteq 0$ on the surface and

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[34] S. S. Attwood, Electric and Magnetic Fields, John Wiley & Sons, p. 78, 1948.

$$E_x\Big|_{y=0} = \frac{4\,\tau\,h}{\varepsilon\,(x^2-h^2)} = \frac{V^a h}{(x^2-h^2)\ln(r_2'/r_1')} \qquad (69)$$

For a particular spheroid $u = u_o$:

$$E_x\Big|_{y=0} = \frac{V^a h}{(L^2u_o^2v^2 - h^2)\ln(r_2'/r_1')}. \qquad (70)$$

The component of E_x tangential to the surface of the spheroid is called the applied field E_v^a (the "feeding" field). It is (Figure 6.)

$$E_v^a = (E_x\Big|_{y=0})\cos\theta \qquad (71)$$

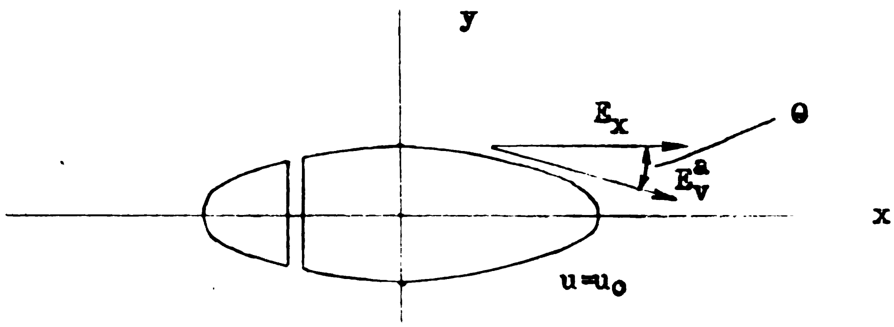

Figure 6. Tangential Component of the Applied Field

Transforming to spheroidal coordinates:

$$E_v^a = \frac{V^a h}{(h^2-L^2u_o^2v^2)\ln(r_2'/r_1')} \cdot \frac{u_o(1-v^2)^{\frac{1}{2}}}{(u^2-v^2)^{\frac{1}{2}}} \qquad (72)$$

Inserting this expression in (63 yields

$$a_n = \frac{\beta L^2}{i\eta N_n U_n'(u_o)} \int_{-1}^{+1} \frac{V^a h u_o V_n(v)\,dv}{\ln(r_2'/r_1').(h^2-L^2u_o^2v^2)} \qquad (73)$$

These ideas were applied to the spheroidal antenna and their effects studied in some detail. The assumptions made above can in principle be applied no matter where the gap is located. Their effect was studied in the special case of symmetrical (center) feeding, since this simplified the mathematical details. In this case, the a_n with odd subscripts are all zero, and the work is cut almost in half.

Three cases were treated: First, if a very small gap is assumed, the electric field across it can be assumed constant, and the expression (55) for a_n becomes[35]

$$a_{ns} = \frac{\beta L\, V_n(v_0)}{i\eta N_n U_n'(u_0)} \; V^a \qquad (74)$$

In this case, the subscript s on the a_n indicates that this is the approximation for a "small" gap. This is the approximation used by Stratton and Chu in their impedance calculations.

Second, if the width of the gap and the thickness of the wires in the transmission line are taken into consideration, but the applied field is still assumed to exist only in the gap, the $E_v^a(u_0,v)$ in (55) would be zero everywhere except in the gap, and the expression for a_n would become

$$a_{ng} = \frac{\beta L^2}{i\eta N_n U_n'(u_0)} \int_{-v_1}^{+v_1} \frac{V^a h u_0 V_n(v) \quad dv}{(h^2 - L^2 u_0^2 v^2)\ \ln(r_2'/r_1')} \qquad (75)$$

---

[35] S. A. Schelkunoff, op. cit., p. 116.

In this case, the subscript g indicates that the applied field is assumed to exist only in the gap. The integral can be evaluated by elementary methods because the $V_n(v)$ is in the form of a power series.

In the third case, a radically different assumption about the nature of the applied field is made. It is assumed that the effective applied field exists only along the elements of the antenna. It is true of course that a fairly strong field exists in the gap, but the ordinary capacitive displacement current that corresponds to this field is small for practical gap widths, and in this case it is assumed that this current contributes a negligible amount to the radiated energy. This theory must be only tentative, of course; it is based on the objections mentioned earlier to the theory of the applied field being located entirely in the gap. With this assumption the a_n become

$$a_{ne} = \frac{2\,\beta L^2}{i\eta N_n U_n'(u_0)} \int_{v_2}^{1} \frac{V^2 h u_0\, V_n(v)\,dv}{(h^2 - L^2 u_0{}^2 v^2)\ln(r_2'/r_1')} \tag{76}$$

In this case, the subscript e indicates that the effective applied field is assumed to exist along the elements. The 2 appears because the integral from -1 to $-v_2$ is the same as the integral from v_2 to 1. Here again the integral is easily evaluated. If an unsymmetrical feed point were being considered, two integrals would have to be evaluated.

Calculations were carried out for the case $c = 2$, $u_0 = 1.00001$, and for two gap widths:

Narrow Gap	Wide Gap
$v_1 = .008$	$v_1 = .049$
$v_2 = .012$	$v_2 = .051$

This very thin antenna was chosen because for the thinner antennas the ratio $U_k(u_o)/U_k'(u_o)$ is smaller. Even for this case, the fourth term in the expansion (18) for $\underline{I}$ was about three percent of the first, and so the numerical values obtained for the impedances are probably only accurate to within four or five percent. However, for smaller values of $\underline{c}$, the successive terms in the series decrease much more rapidly, and it is expected that more accurate values for the impedances can be obtained. Values for the various $\underline{a_n}$ in terms of the $\underline{a_{ns}}$ are given in Table I.

TABLE I

IMPEDANCE CONSTANTS

$c = 2$, $u_o = 1.00001$	
Narrow Gap	Wide Gap
$a_{og} = .99996 a_{os}$, $a_{oe} = .98080 a_{os}$	$a_{og} = .99804 a_{os}$, $a_{oe} = .94959 a_{os}$
$a_{2g} = .99981 a_{2s}$, $a_{2e} = .95042 a_{2s}$	$a_{2g} = .99005 a_{2s}$, $a_{2e} = .86521 a_{2s}$
$a_{4g} = .99956 a_{4s}$, $a_{4e} = .92466 a_{4s}$	$a_{4g} = .97711 a_{4s}$, $a_{4e} = .78888 a_{4s}$
$a_{6g} = .99920 a_{6s}$, $a_{6e} = .89838 a_{6s}$	$a_{6g} = .95853 a_{6s}$, $a_{6e} = .70758 a_{6s}$

Thus for the narrow gap, where the gap width is about 1/100 of the length of the antenna, the more exact expression (75) yields values for the $\underline{a}_n$ that are almost identical to the values obtained by (74). Even for the wide gap, where the gap width is 1/20 of the length of the antenna, the agreement is very good. The conclusion to be drawn from this is that a refinement of the theory which assumes that the applied field is located in the gap does not lead to any significant changes in the impedance values or curves of current distribution. This is the reason that I_s and ϕ_s were not plotted in Figures (8) and (9). They did not differ significantly from I_g and ϕ_g.

However, under the assumption that the effective applied field exists only along the elements we find that the contribution of the higher order terms in the expansion for the current is somewhat less, and that this contribution decreases as the width of the gap is increased. The result of this can be seen in the curves for currents and phase angles, Figures (8) and (9). The current at the gap for this last assumption is less, but the phase angle of the current is slightly greater. Beatty[36] found experimentally that the values of resistance were actually somewhat higher than those predicted by Stratton and Chu, but that the reactance values were about the same. Because of the poor convergence of the series for $\underline{A}$, any conclusions given here must be only tentative, but it appears that impedances computed under the assumption of a distributed applied field will be slightly higher than those computed under the assumption that the applied field is concentrated in the gap. The impedance values are given by

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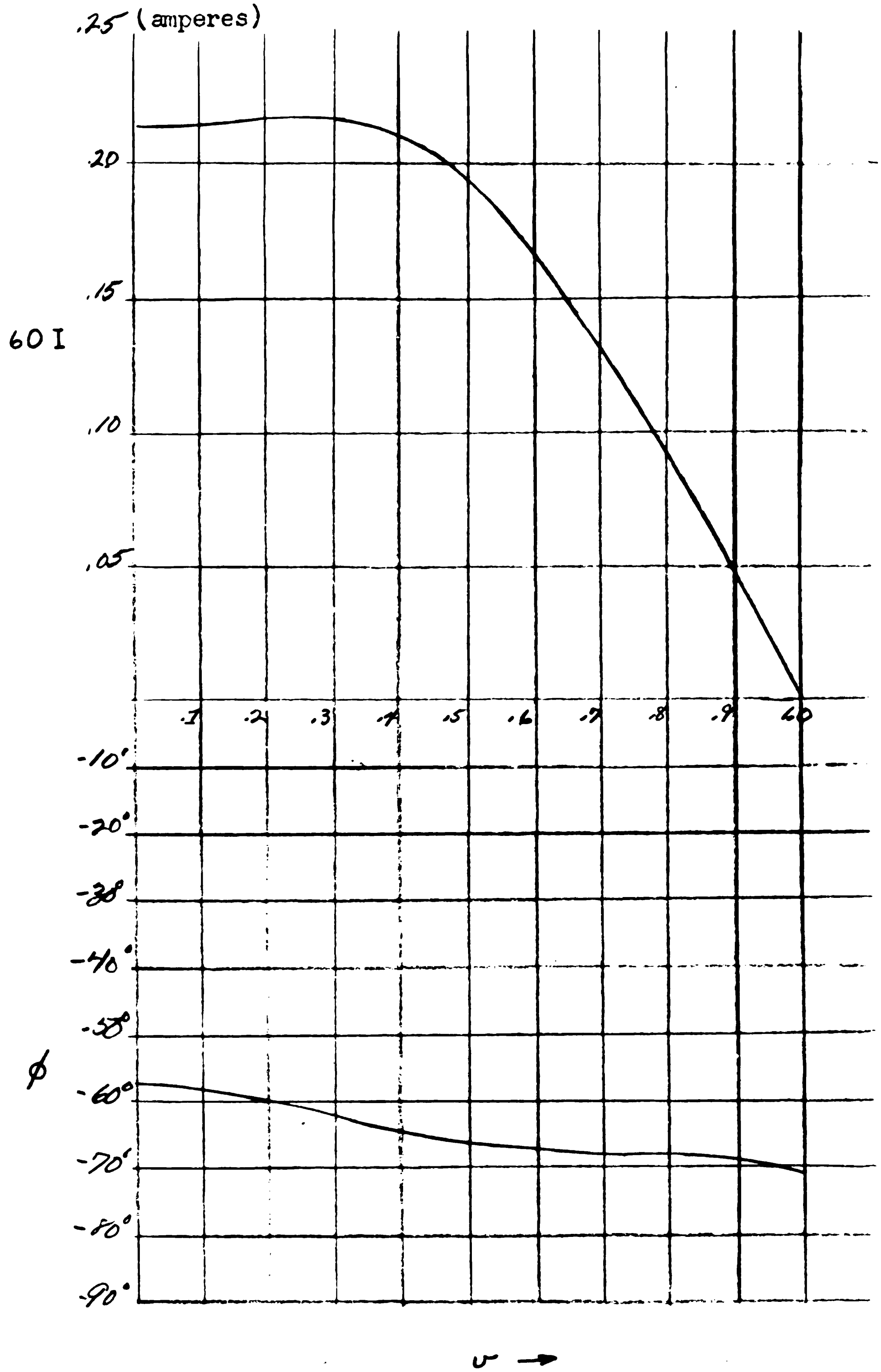
[36] R. W. Beatty, op. cit., p. 32.



(Narrow gap: gap feed, c = 2, u_0 = 1.00001)

Figure 7. Typical Plot of Current Distribution and Phase Angle

Figure 8. Plot of Current Distribution and Phase Angle

(Narrow gap: gap and element feed, c = 2)

$u_0 = 1.00001$

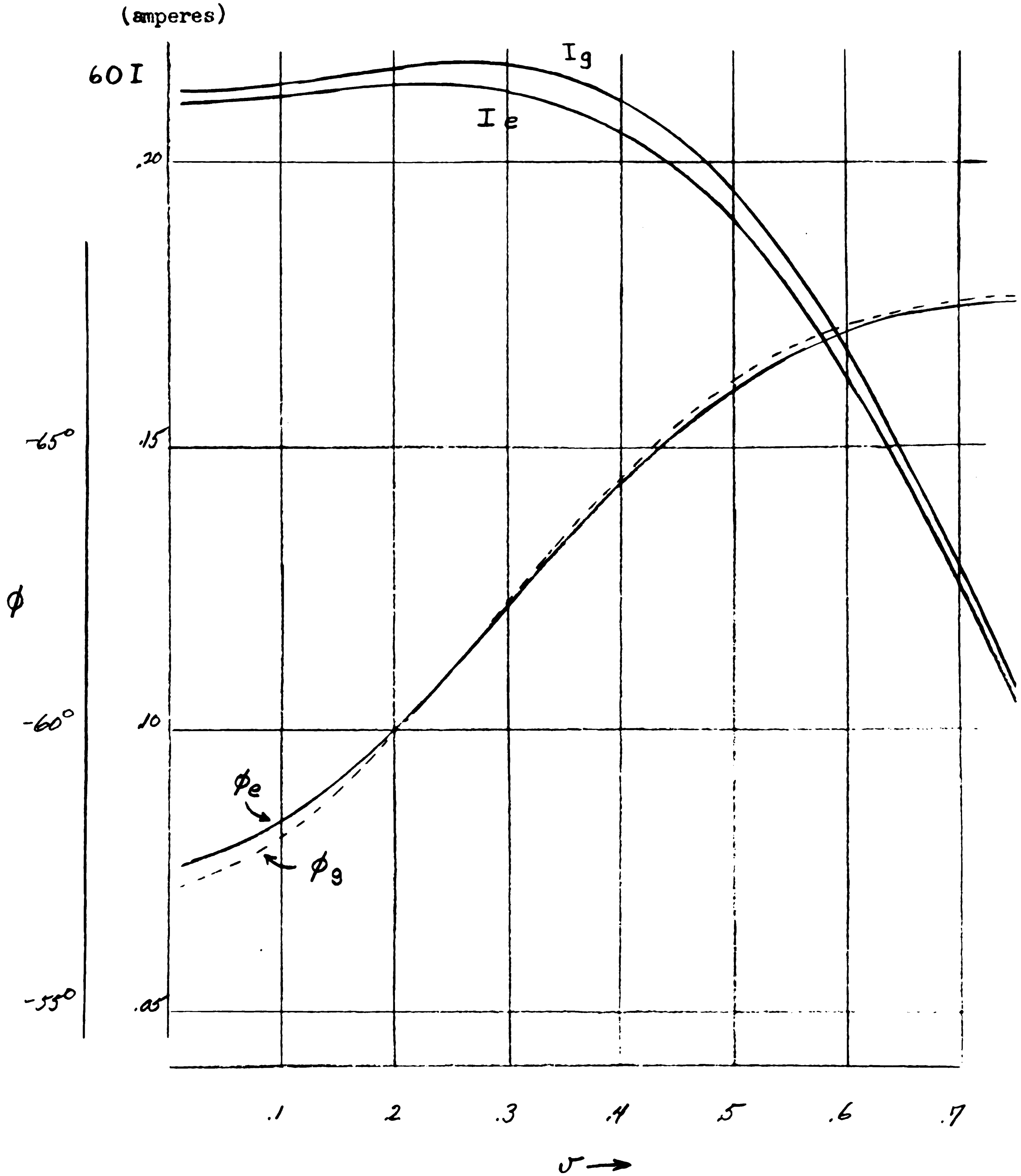
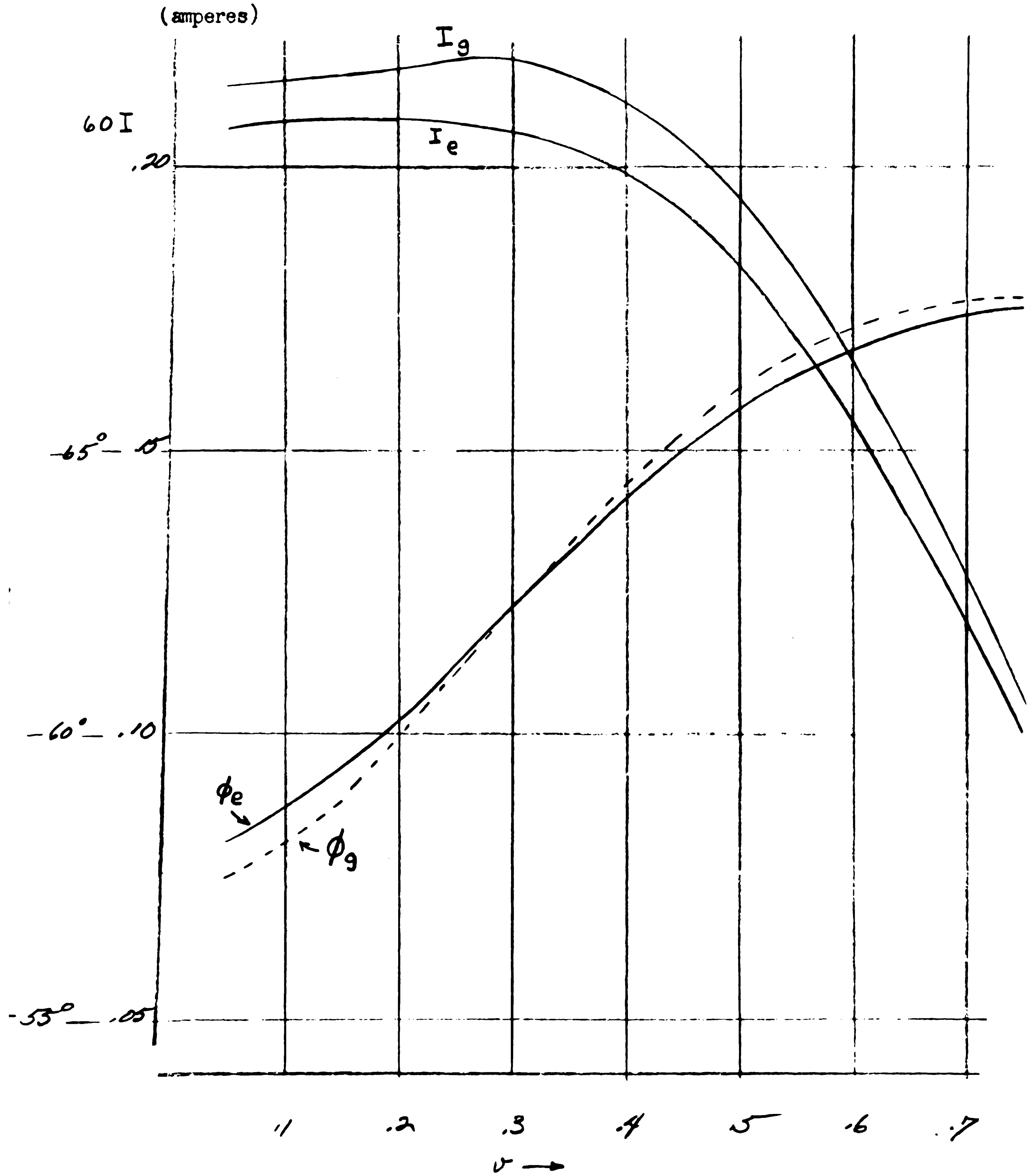


Figure 9. Plot of Current Distribution and Phase Angle

(Wide gap: gap and element feed, c = 2)

$u_o = 1.00001$



$$Z = V^{a}/I(u_0, v_1) \tag{77}$$

The impedances computed in the various cases were:

Narrow gap	Wide gap
$Z_s = Z_g \doteq 151 + i\ 236$ ohms	$Z_s = Z_g \doteq 151 + i\ 236$ ohms
$Z_e \doteq 152 + i\ 240$ ohms	$Z_e \doteq 153 + i\ 246$ ohms

One other general conclusion is that the impedances for antennas with wide gaps should be slightly greater than those for the same antennas with narrow gaps. To determine just what the best assumption for the applied field is very careful experiments must be carried out for spheroidal antennas up to about a half wave-length long ($c = 1.54$), and impedance calculations must be carried out for the same antennas over the same range. Beatty was forced to use changing L/D ratios because it was inconvenient to vary the frequency of the source at 3000 megacycles. However, for the best correlation between the calculated and experimentally obtained values, it would be better to use a fixed spheroid and vary the frequency of the source. It would be very desirable, of course, if an expression for the current or impedance could be obtained which would converge more rapidly than the one used here. Then electrically longer antennas could be used, and experiment and theory could be compared over a wider range.

The radiation patterns are shown in Figures 11 to 54. The mean intensity of energy flow in a harmonic electromagnetic field is[37]

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[37] J. A. Stratton, op. cit., p. 137.

$$\overline{S} = \tfrac{1}{2}\mathrm{Re}(\overline{E}\times\tilde{\overline{H}}). \qquad \tilde{\overline{H}} \text{ is the conjugate of } \overline{H}. \tag{78}$$

From (4), E_u is negligible at infinity because it decreases as $1/u^2$, and so

$$S = \tfrac{1}{2}\mathrm{Re}(E_v\tilde{H}_\phi). \tag{79}$$

But at infinity, $U(u) = \epsilon^{-icu}$, and E_v becomes

$$E_v = \frac{\beta L \eta}{\beta L^2}\left[(1-v^2)(u^2-v^2)\right]^{-\frac{1}{2}} A = \eta H_\phi. \tag{80}$$

Therefore

$$S = \frac{\eta}{2} H_\phi \tilde{H}_\phi \tag{81}$$

For convenience in computation, $2\eta S$ was actually plotted, and so to obtain a numerical value for the radiation intensity for a particular antenna in a given direction, one must divide the plotted value by[38] $240\pi r^2$ (for free space) and multiply by the voltage at the gap squared, since the applied voltage was taken as unity for these patterns. The units for S will then be in watts per square meter.

A step function of applied field intensity across a "small" gap was assumed for computational convenience. This is a fair approximation, because the ratio of the successive a_n, even for the assumption of a distributed applied field, does not vary greatly for the first few terms from the ratio in the step function case. It was found that a significant change in this ratio - of the order of 1.5/1 or more - was

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[38] r is the distance from the center of the spheroid.

necessary to produce a significant change in the radiation pattern. Thus, the radiation patterns, like the impedances, are relatively insensitive to the nature of the applied field or the gap width. The radiation patterns do vary, however, with the width of the antenna, the electrical length, and the location of the gap.

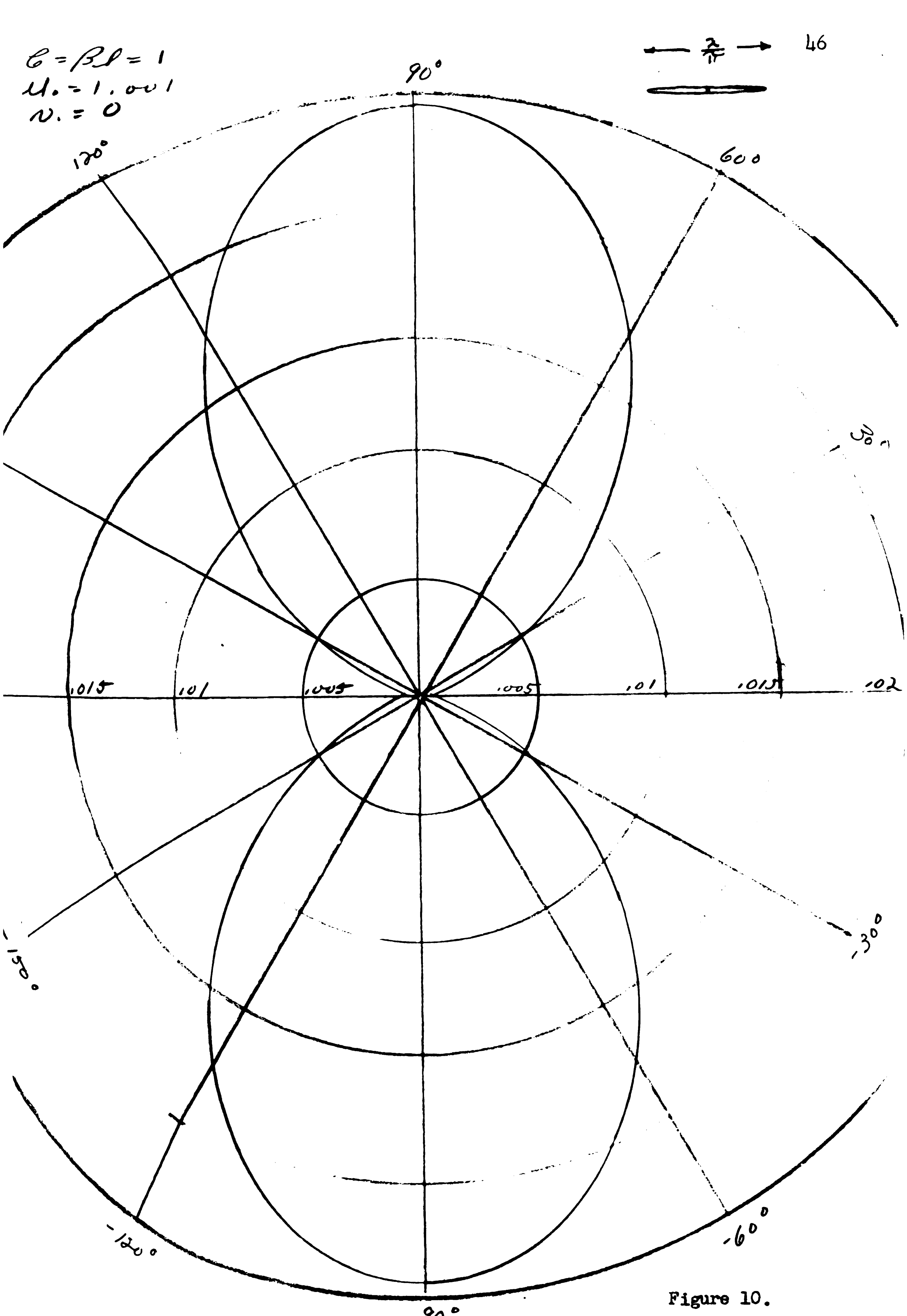
For the thicker antennas the derivatives of the radial functions are not as large, other parameters held constant, and so the higher order $\underline{a_n}$ are somewhat larger, and therefore the unsymmetrical effects are slightly greater. This effect is most easily seen for the case $\underline{c} = 2$ Figures (20) to (38), in which radiation patterns are plotted for three rather widely varying L/D ratios - 10/1 ($u_o = 1.005$), 22/1 ($u_o = 1.001$), and 316/1 ($u_o = 1.00001$).

As the antennas become longer electrically, the unsymmetrical effects are more pronounced. Thus, for $c = 1$ the unsymmetrical effect is negligible for almost any gap location, whereas for $c = 2$ it becomes quite noticeable, and for $c = 3$ it is very pronounced, with small lobes also occurring for the unsymmetrical feeds. For a further investigation the most interesting cases would occur between $c = 2$ and $c = 3$ - interesting in the sense that for certain gap locations these cases would produce the most unsymmetrical patterns with negligible spurious lobes. These patterns could be obtained relatively easily, because the convergence of the expression (18) for A is very good in the far zone.

The location of the gap, of course, contributes greatly to the asymmetry of the radiation pattern. The size of the radiation pattern is an indication of the impedance at the gap, because the same voltage

was assumed in every case. Thus, for gap locations near one end of the antenna the impedance is higher, and therefore the current and the radiated power are lower.

The small antennas drawn in the upper right hand corner of each of the radiation patterns indicate the orientation of the antenna for the given pattern, the location of the gap, the electrical length of the antenna, and the approximate shape.



C = βl = 1
u₀ = 1.001
v₀ = 0
2/π
90°
120°
60°
30°
-30°
-60°
-90°
-120°
-150°
.1015
.101
.1005
.1005
.101
.1015
.102

Figure 10.

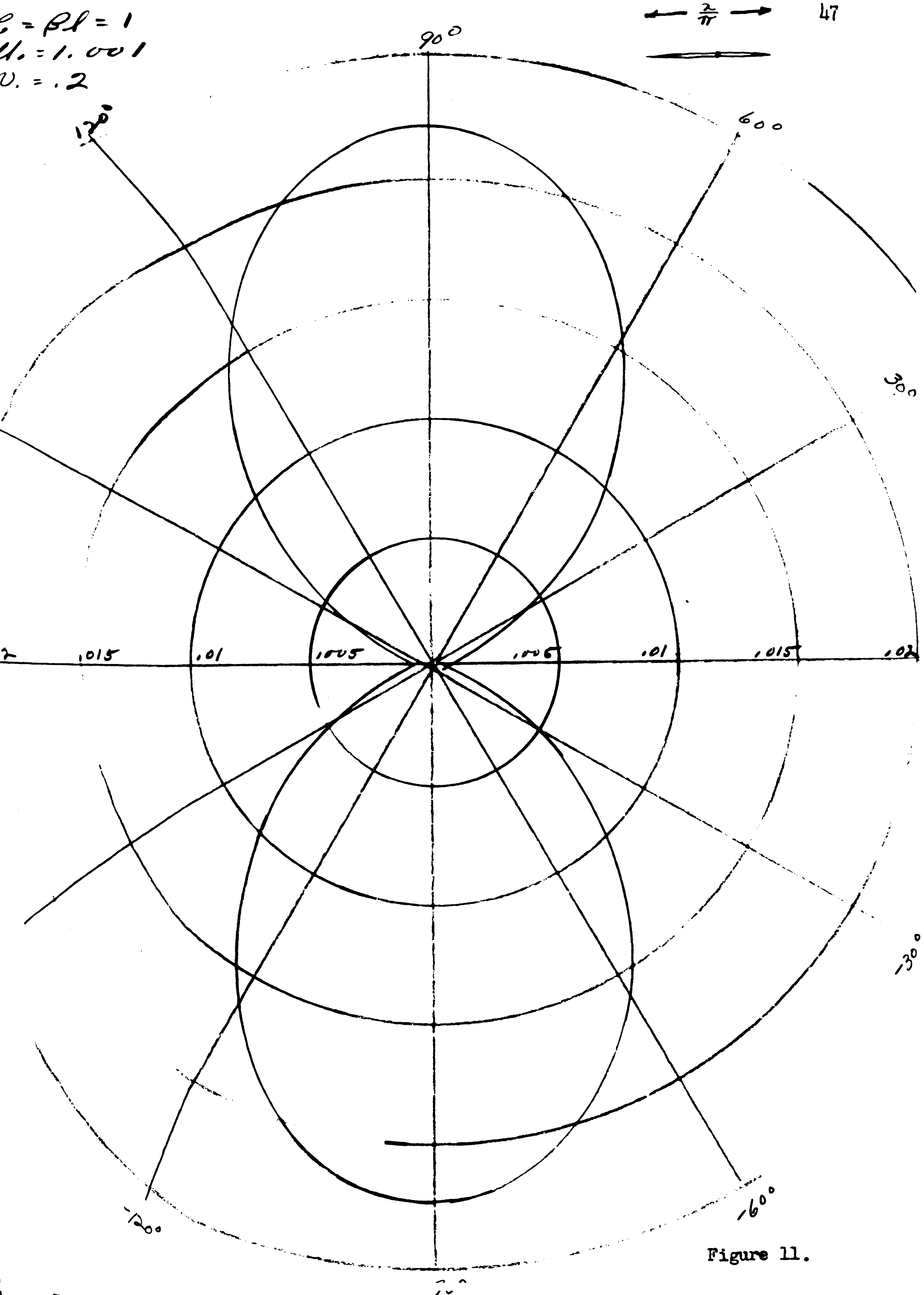


Figure 11.

$\theta = \beta l = 1$
$u_0 = 1.001$
$v_0 = .4$

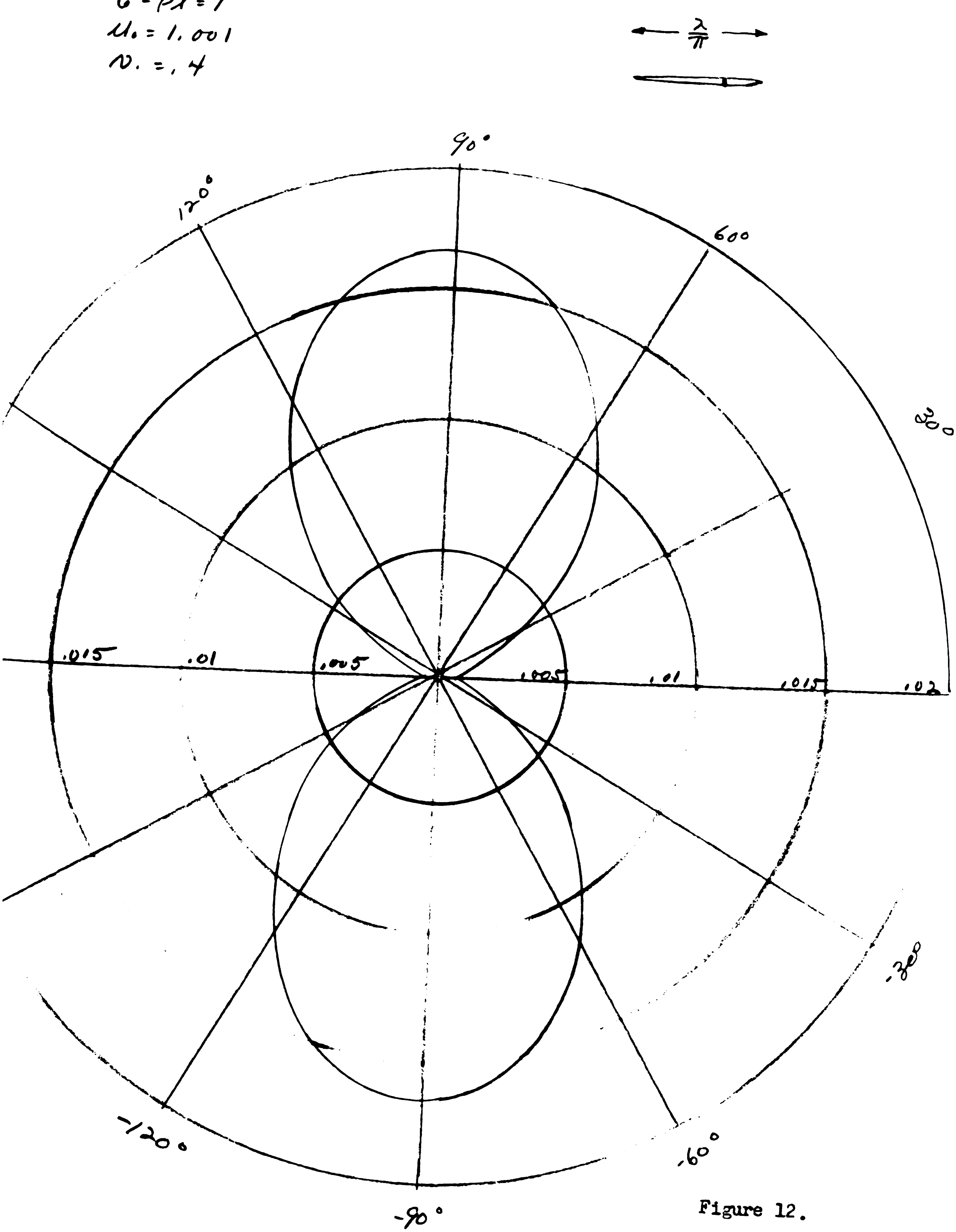


Figure 12.

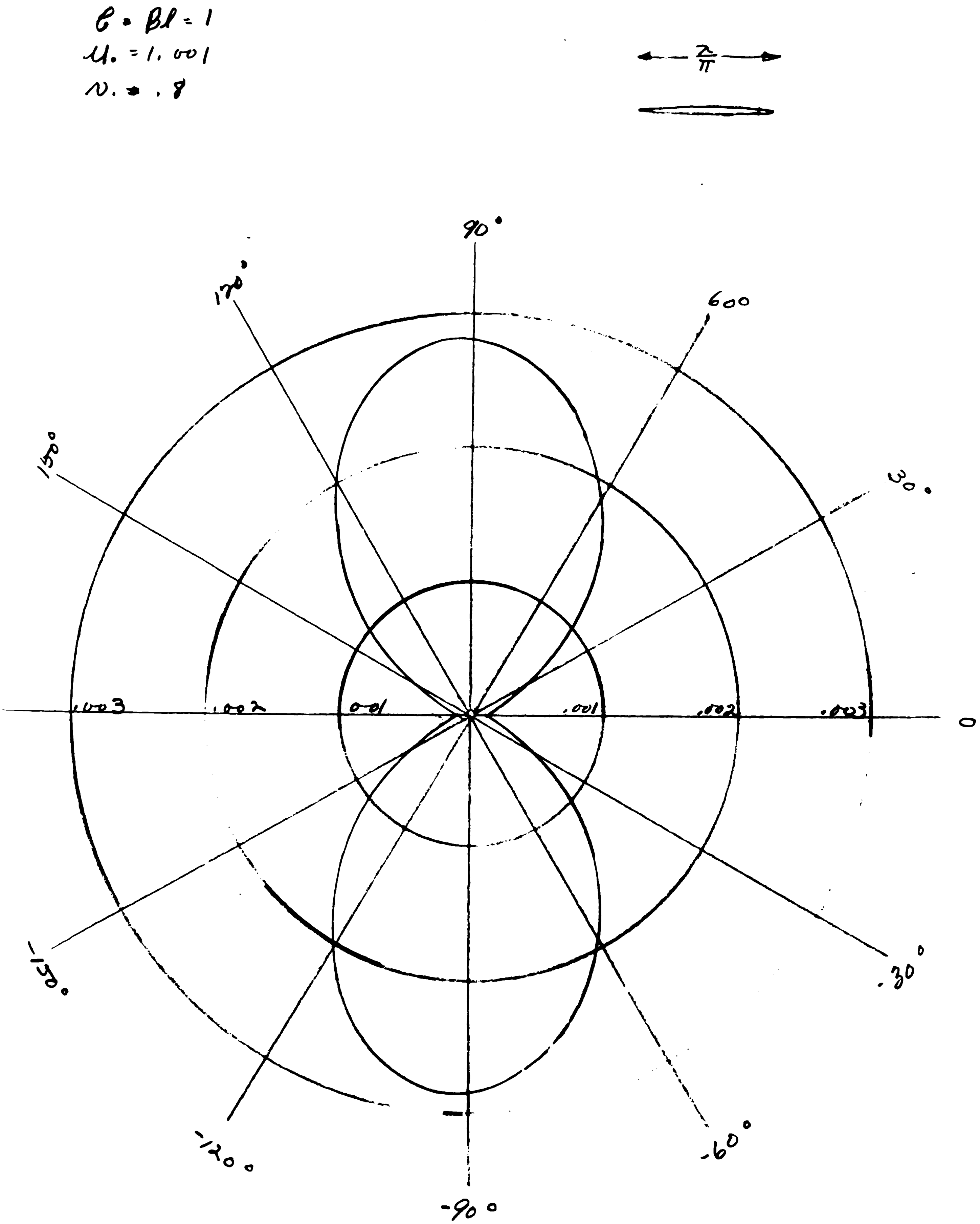

θ = βl = 1
u₀ = 1.001
v₀ = .8
90°
120°
60°
150°
30°
.003
.002
.001
.001
.002
.003
0
-30°
-60°
-90°
-120°
-150°

Figure 14.

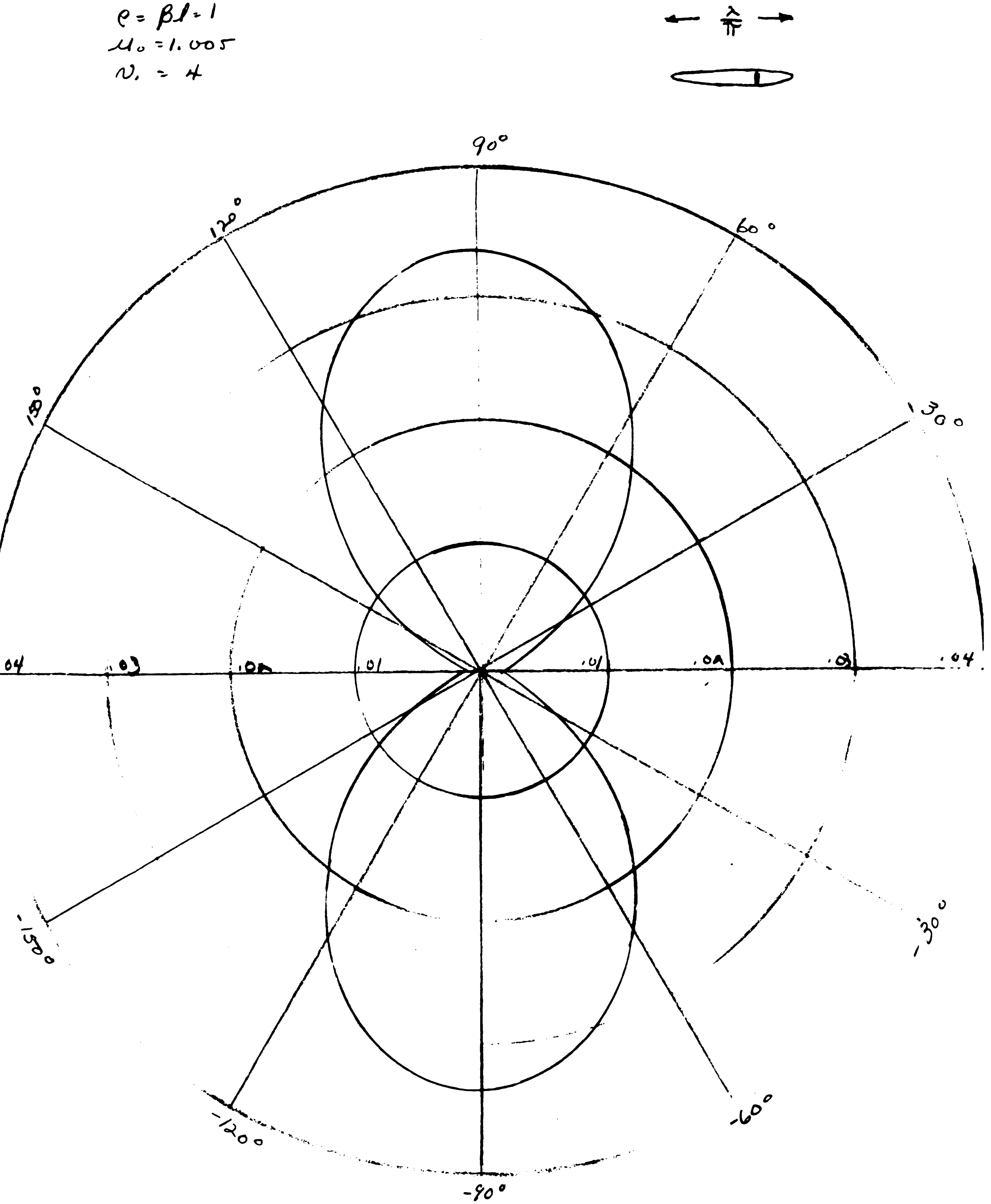

$\rho = \beta l = 1$
$\mu_0 = 1.005$
$\nu_1 = 4$
90°
120°
60°
150
30°
.04
.03
.02
.01
.01
.02
.03
.04
-150°
-30°
-120°
-60°
-90°

Figure 17.

$\ell = \beta l = 1$
$u_1 = 1.005$
$v_1 = .6$

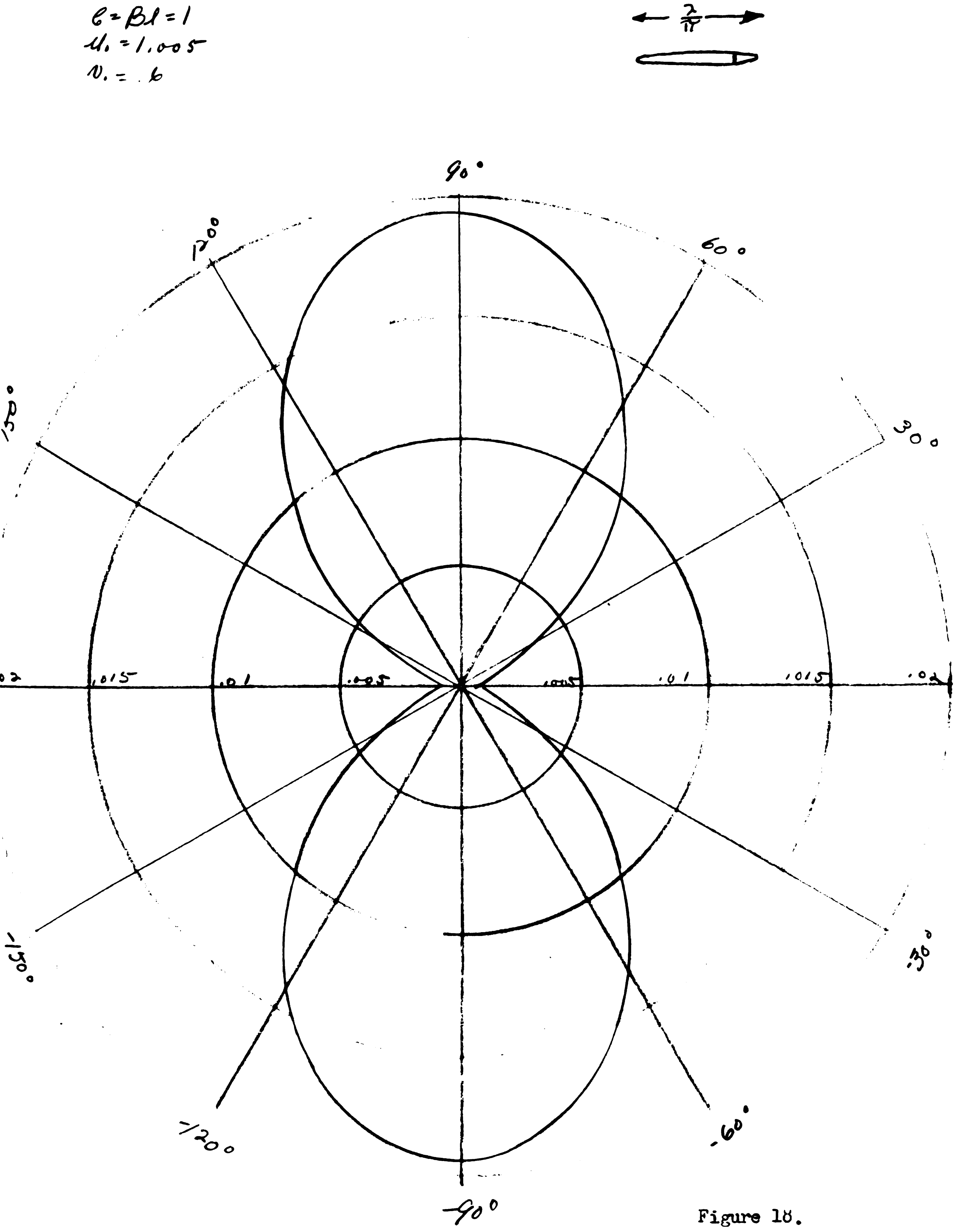
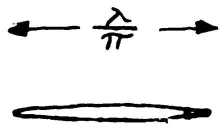


Figure 18.

$c = \beta l = 1$
$\mu_1 = 1.005$
$\nu_1 = .8$



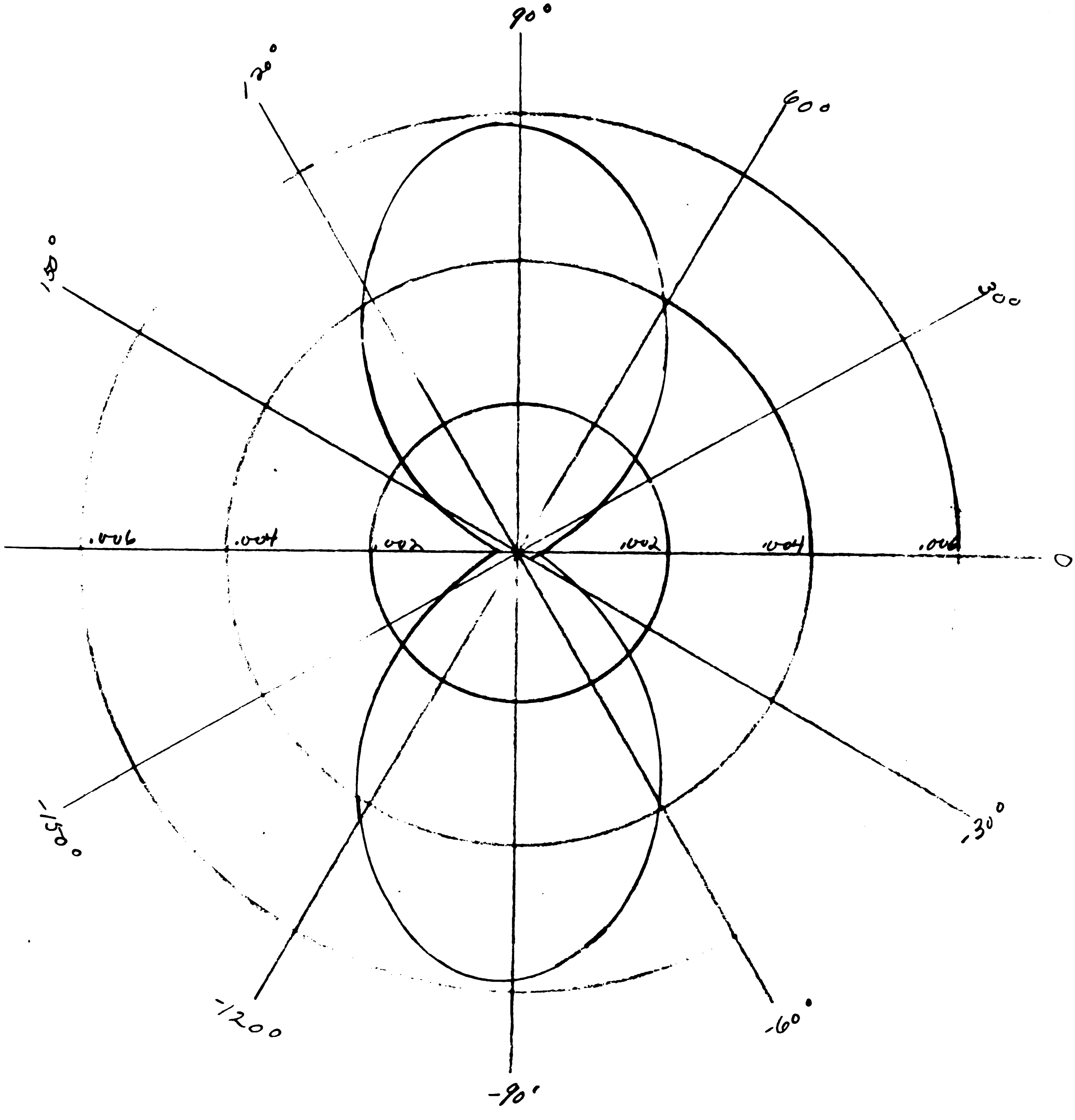
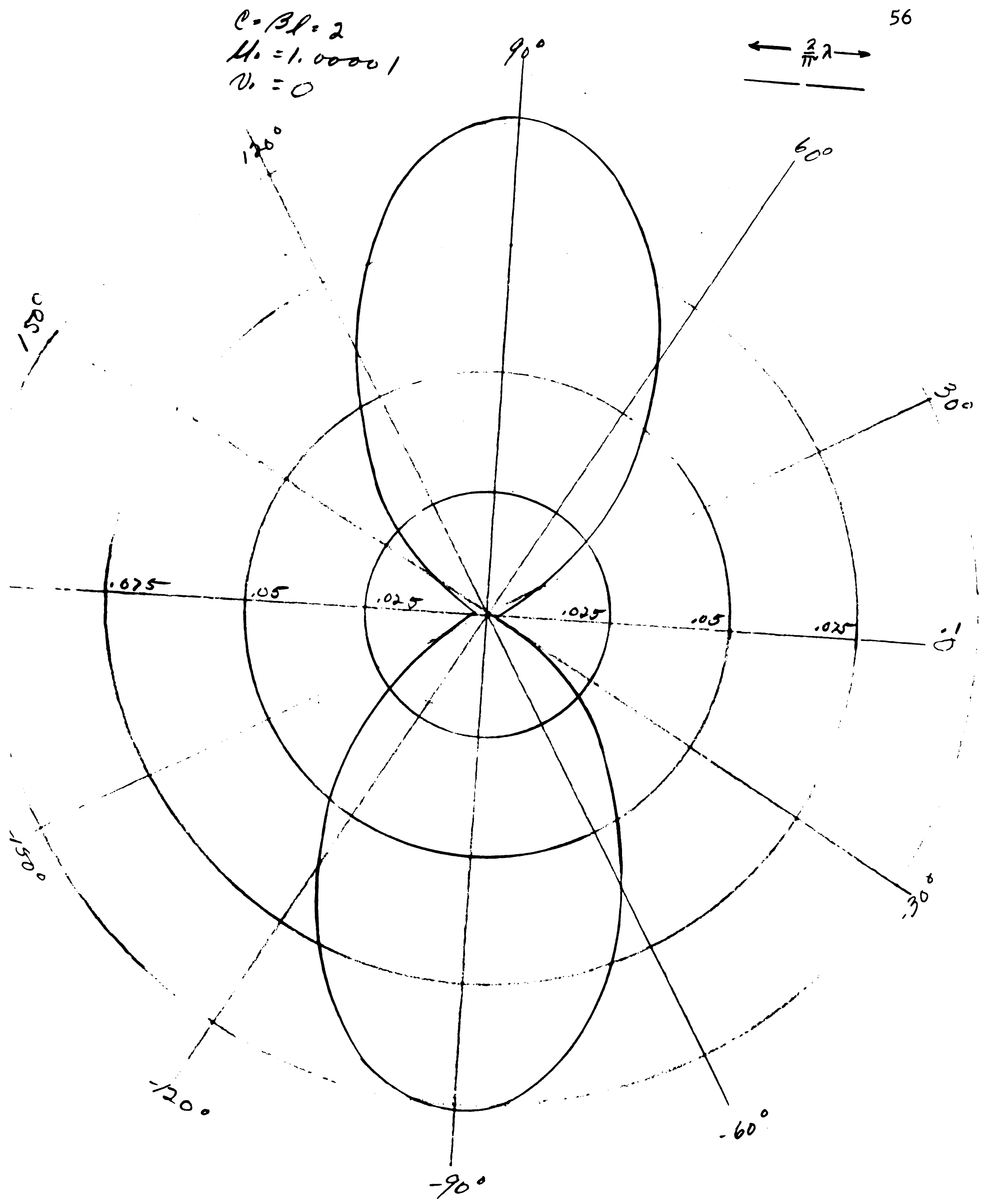


Figure 19.



$C = \beta l = 2$
$\mu_0 = 1.00001$
$\nu_0 = 0$
$\frac{2}{\pi}\lambda$
90°
120°
60°
150°
30°
.075
.05
.025
.025
.05
.075
0°
-150°
-30°
-120°
-60°
-90°

Figure 20.

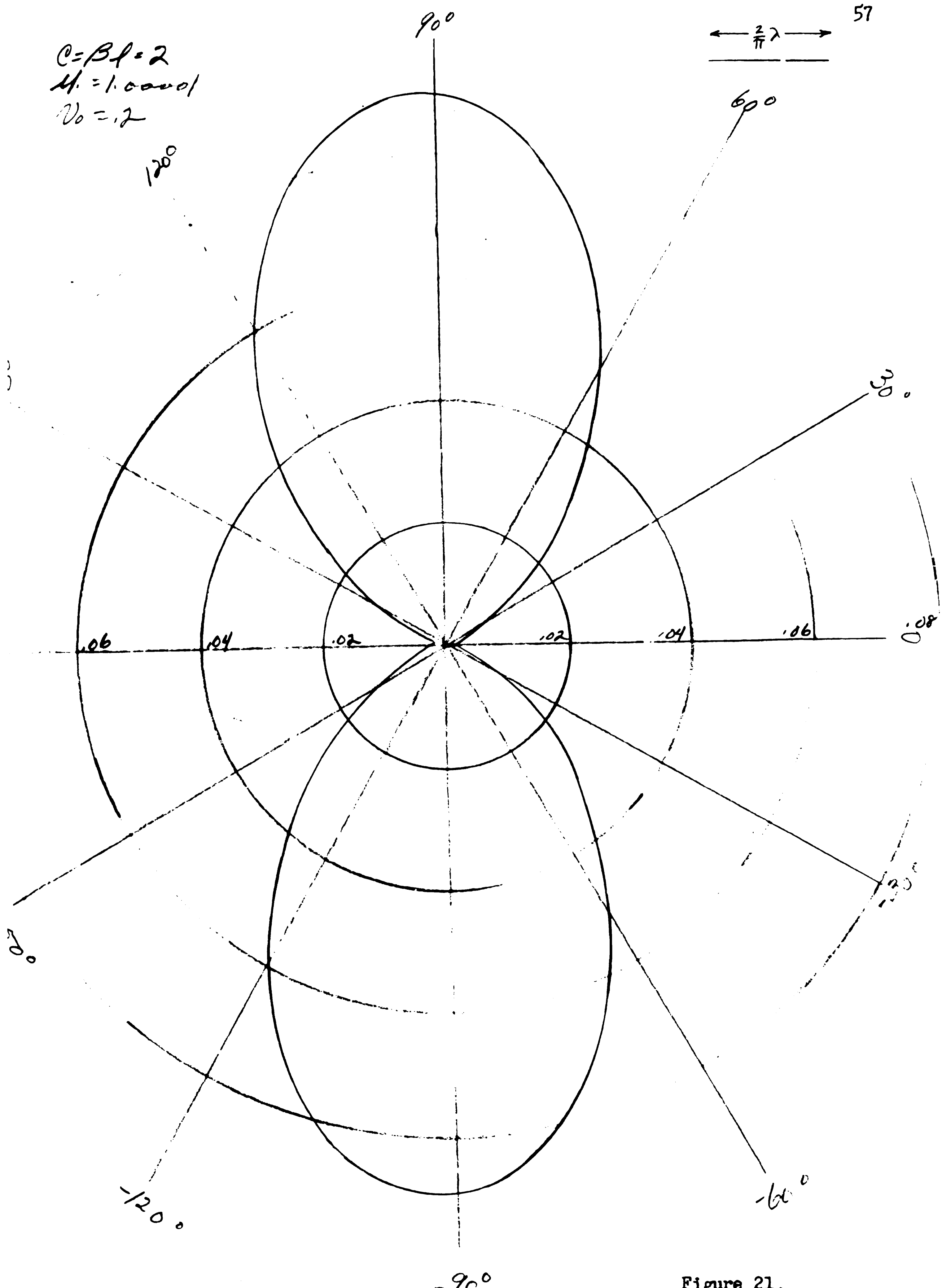

C=βl=2
μ=1.00001
v₀=.2
$\frac{2}{\pi}\lambda$
90°
60°
30°
0°
120°
-60°
-90°
-120°
.06
.04
.02
.02
.04
.06
.08

Figure 21.

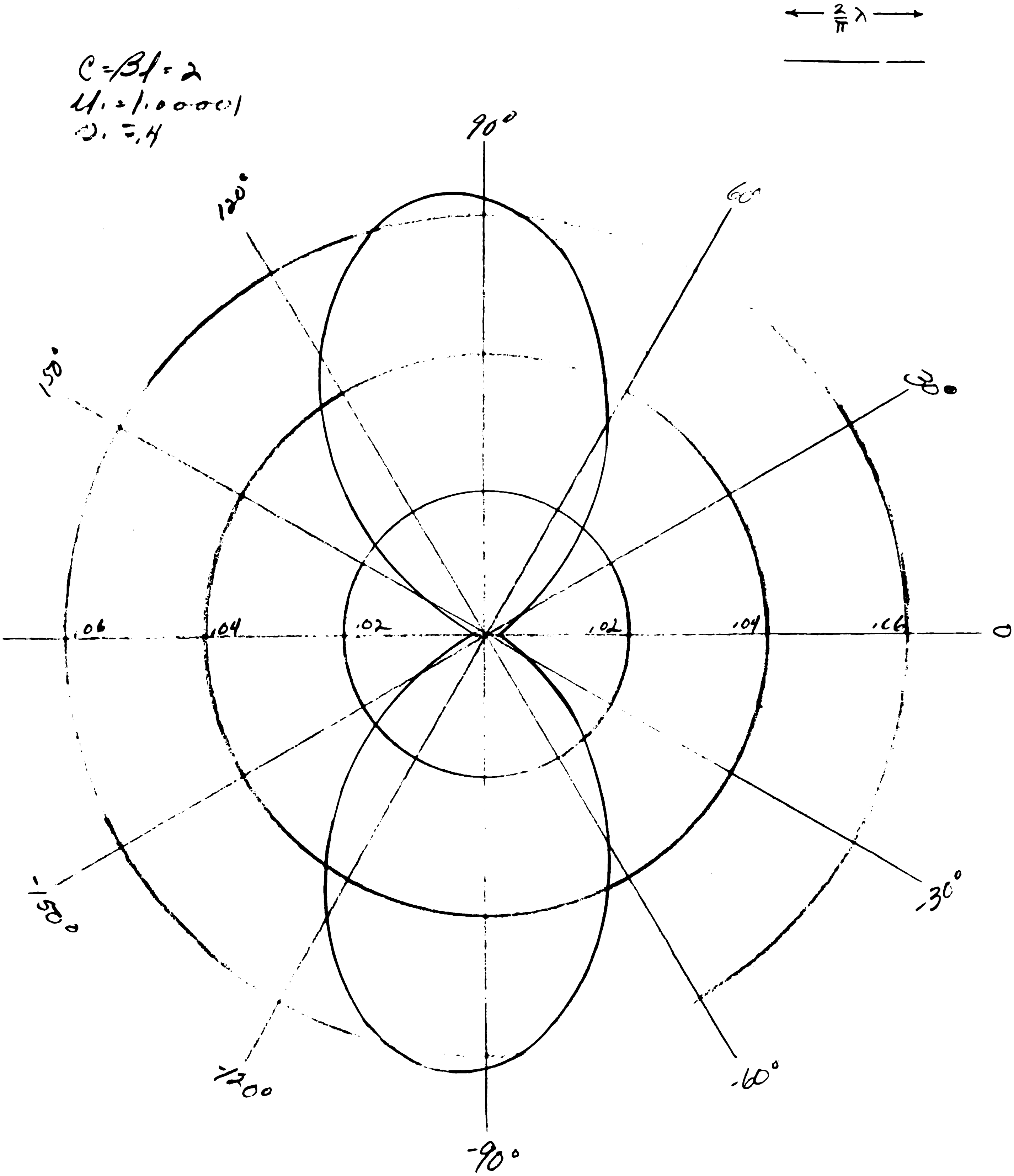
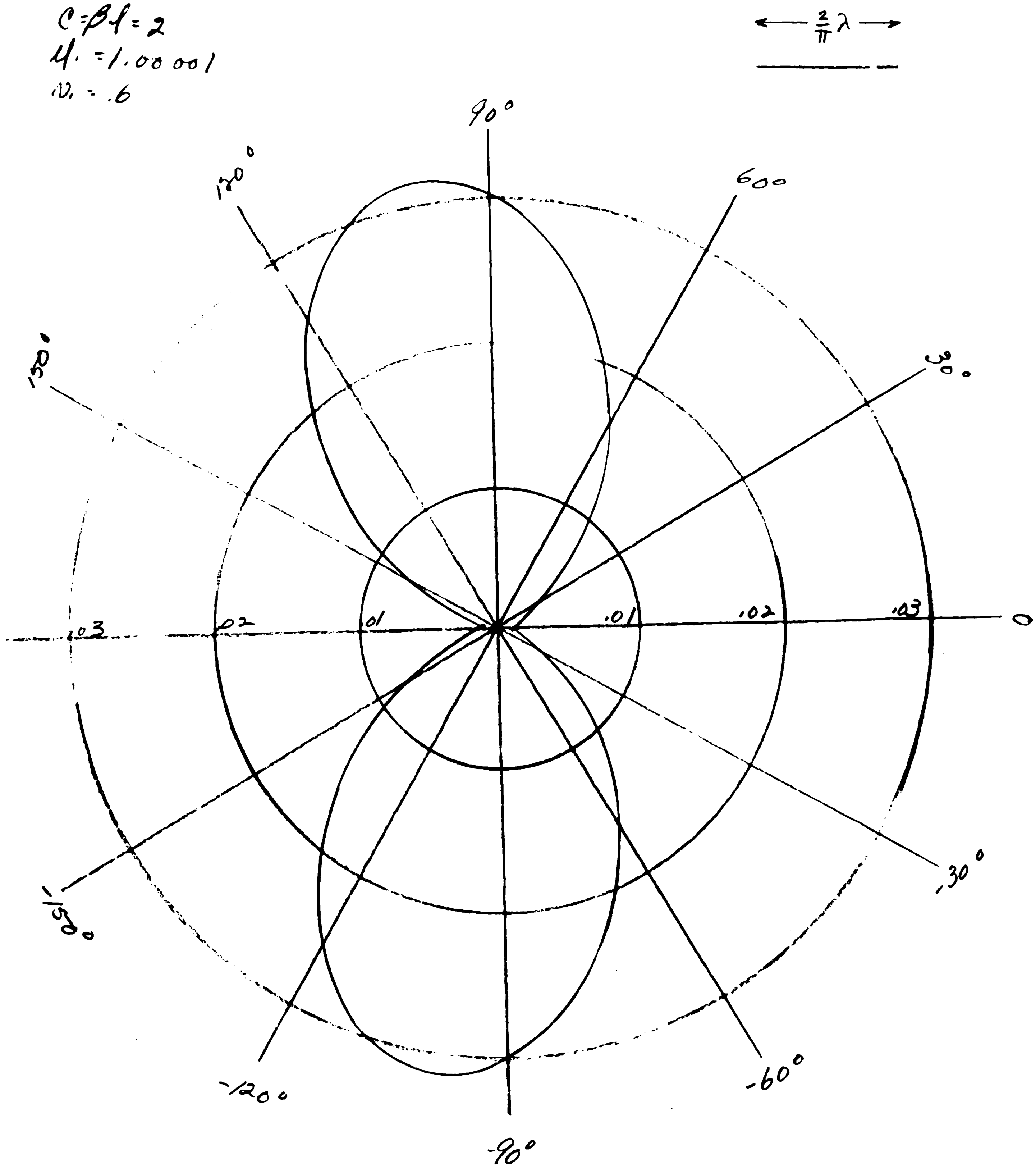
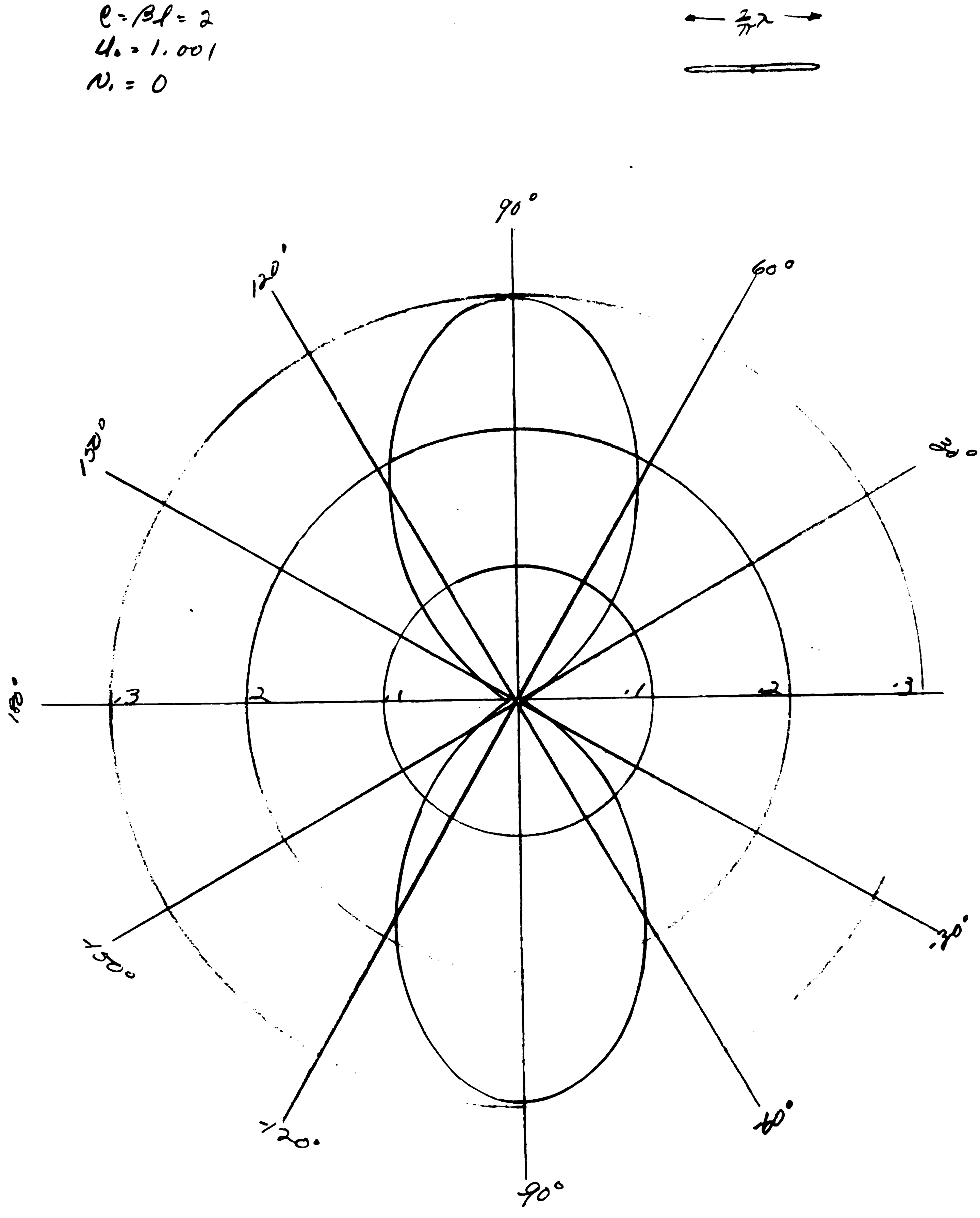

$\longleftarrow \frac{2}{\pi}\lambda \longrightarrow$
$C = \beta l = 2$
$U_1 = 1.00001$
$\Omega_1 = .4$
90°
120°
60°
150°
30°
.06
.04
.02
.02
.04
.06
0
-150°
-30°
-120°
-60°
-90°

Figure 22.



C = βl = 2
μ = 1.00001
ν = .6
← 2/π λ →
90°
120°
60°
150°
30°
0
.03
.02
.01
.01
.02
.03
-30°
-150°
-60°
-120°
-90°

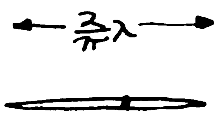
Figure 23.



$e = \beta l = 2$
$u_0 = 1.001$
$v_1 = 0$
$\frac{2}{\pi}\lambda$
90°
120°
60°
150°
30°
180°
.3
.2
.1
.1
.2
.3
-150°
-30°
-120°
-60°
90°

Figure 25.

$C = \beta l = 2$
$u_0 = 1.001$
$v_0 = .2$



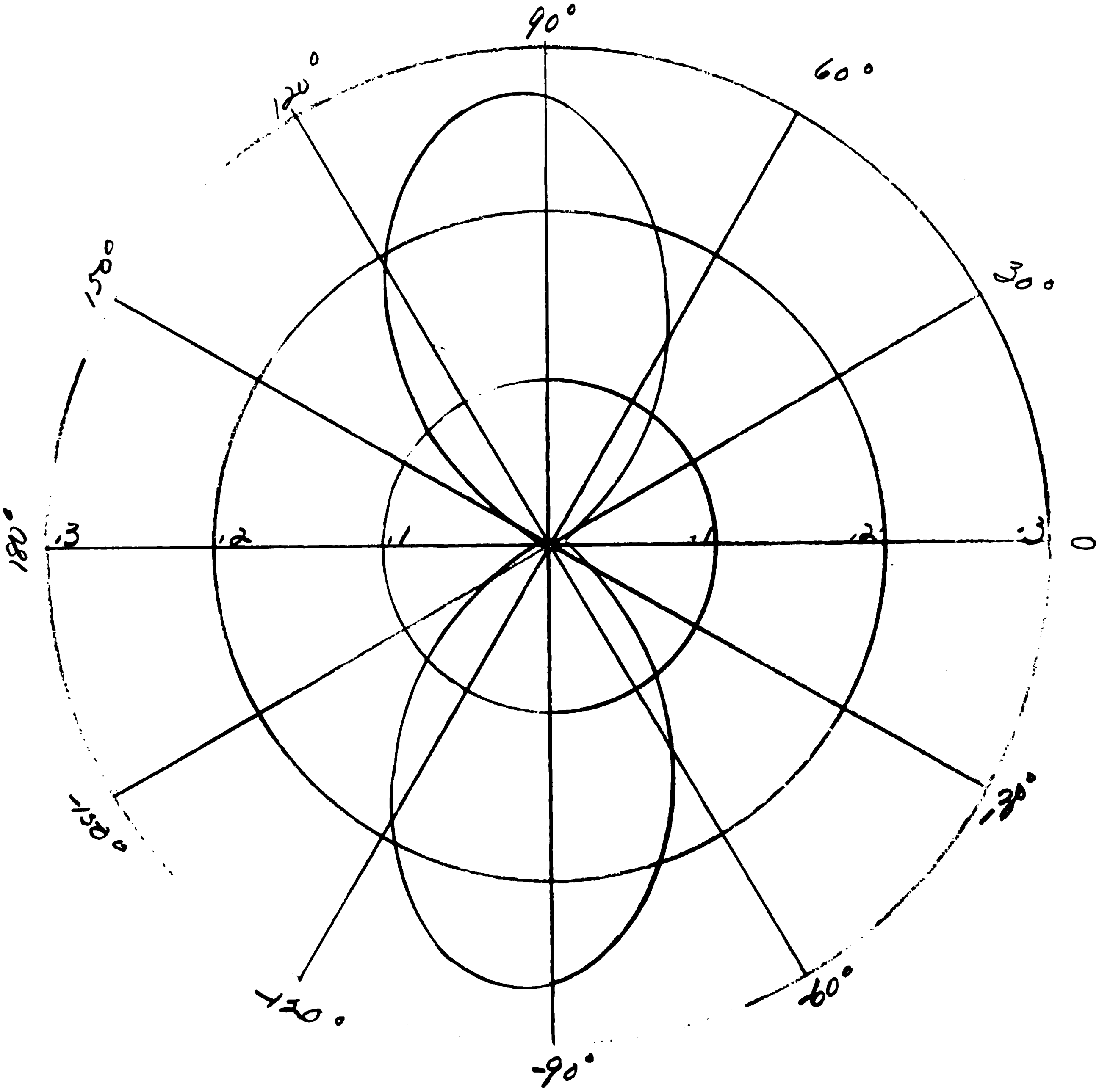


Figure 26.

$C = \beta l = 2$
$\mu_0 = 1.001$
$\nu_0 = .4$

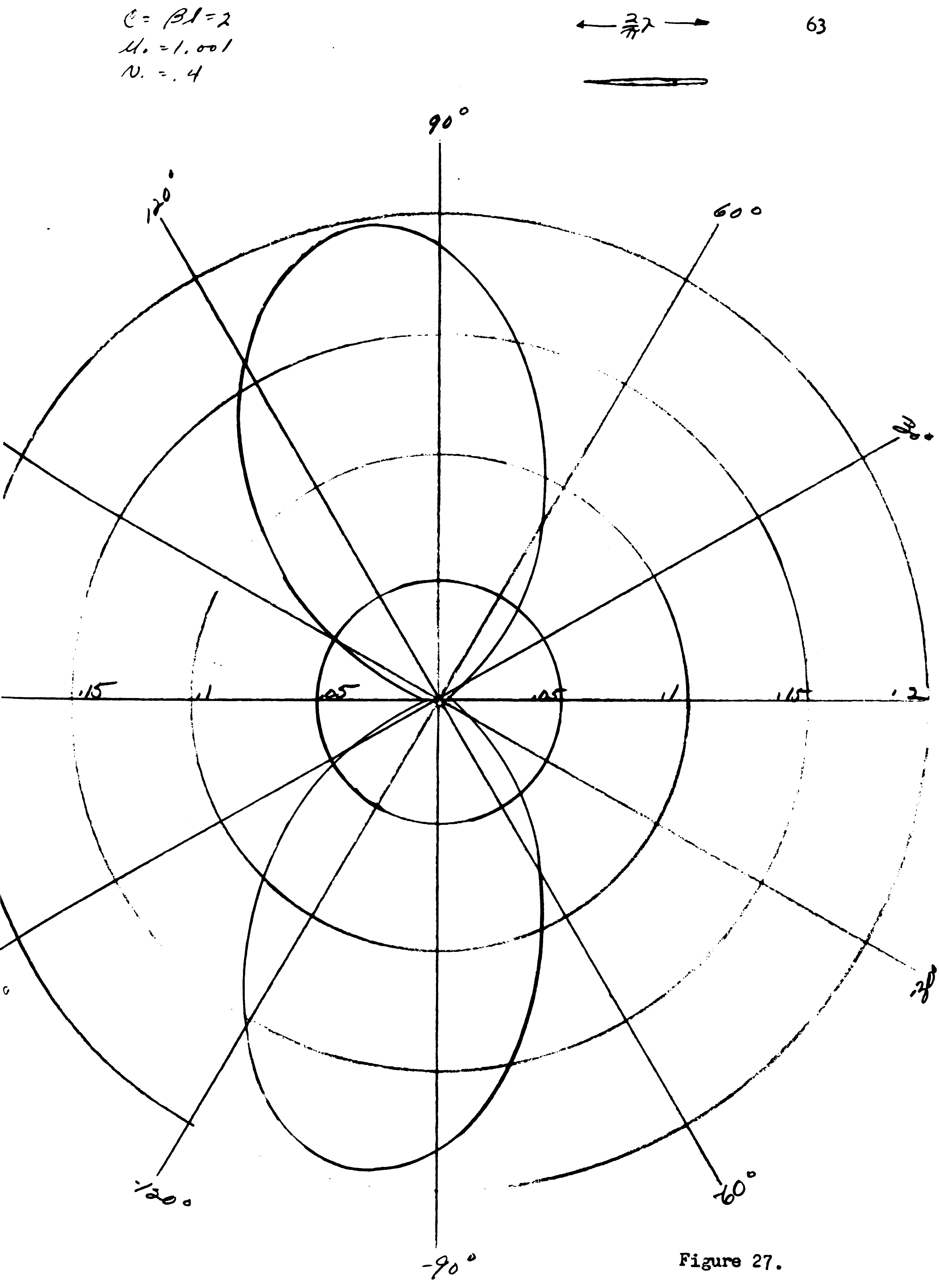
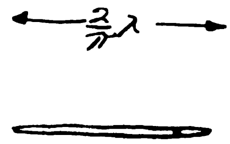


Figure 27.

$C = \beta l = 2$
$U_0 = 1.001$
$N = .8$



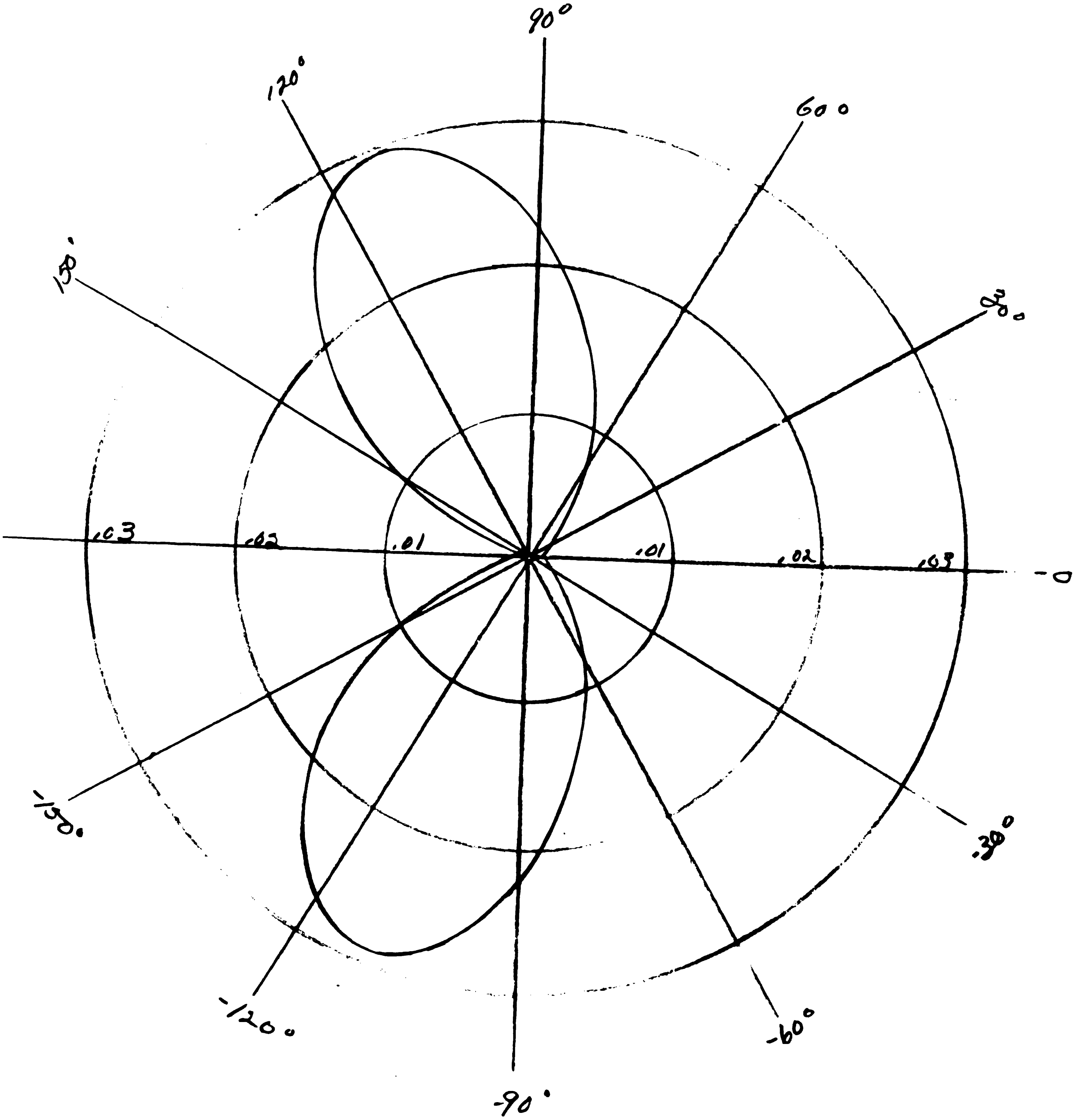


Figure 29.

$c = \beta l = 2$
$u_0 = 1.005$
$v_0 = 0$

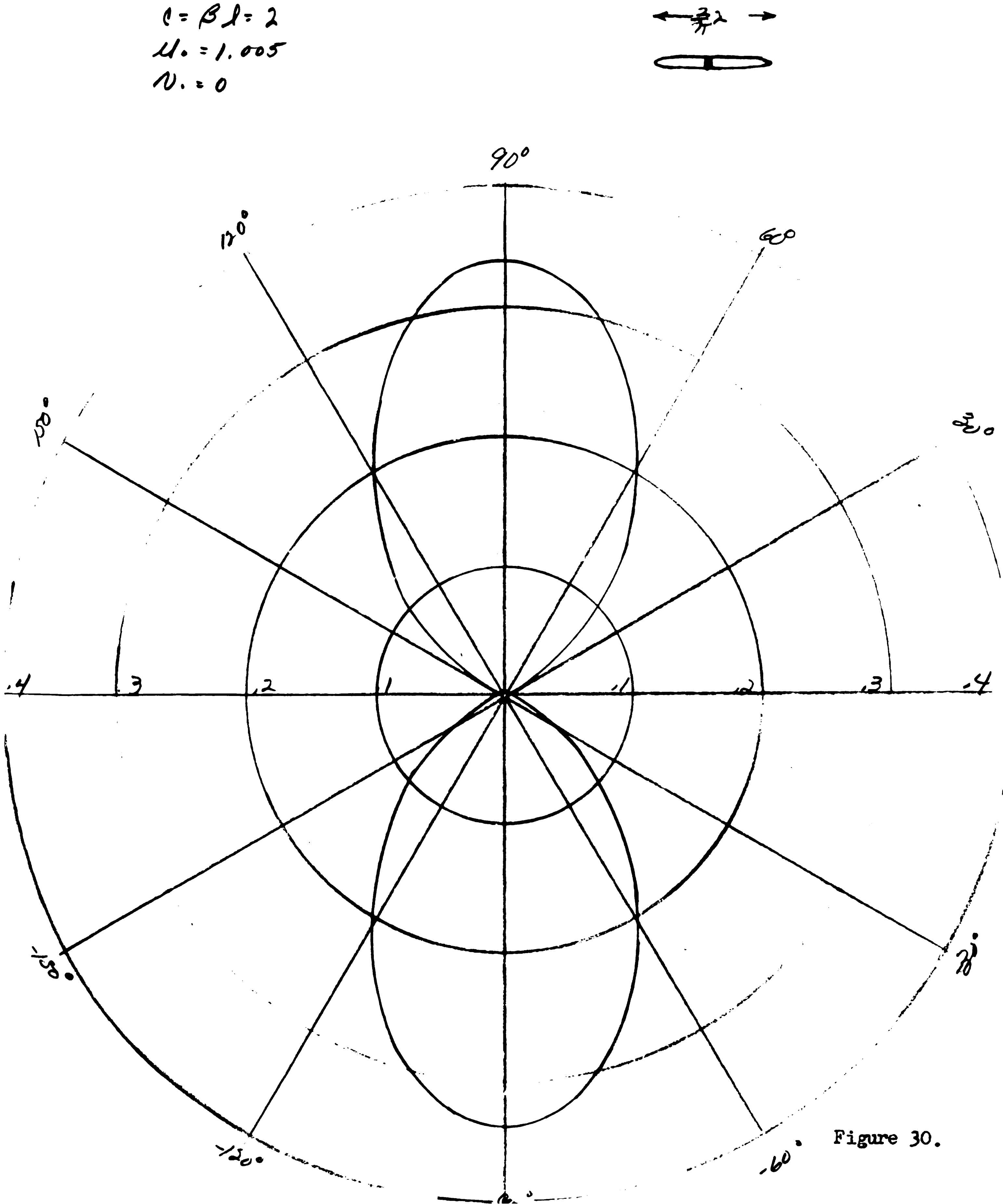
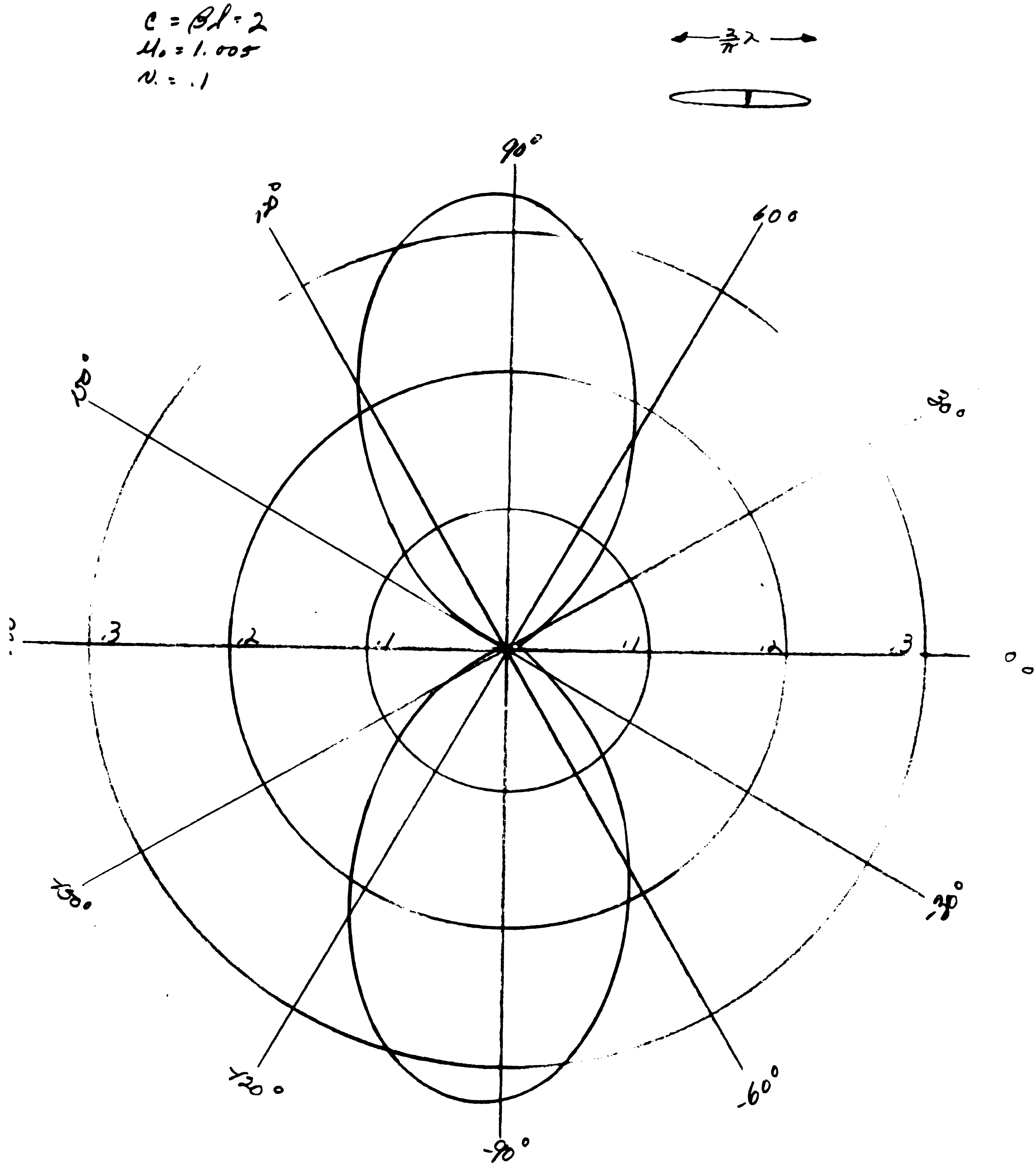


Figure 30.



$C = \beta l = 2$
$\mu_0 = 1.005$
$\nu = .1$
$\frac{2}{\pi}\lambda$
90°
120°
60°
150°
30°
0°
-30°
-60°
-90°
-120°
-150°
.3
.2
.1
.1
.2
.3

Figure 31.

$C = \beta l = 2$
$U_0 = 1.005$
$V_0 = .2$

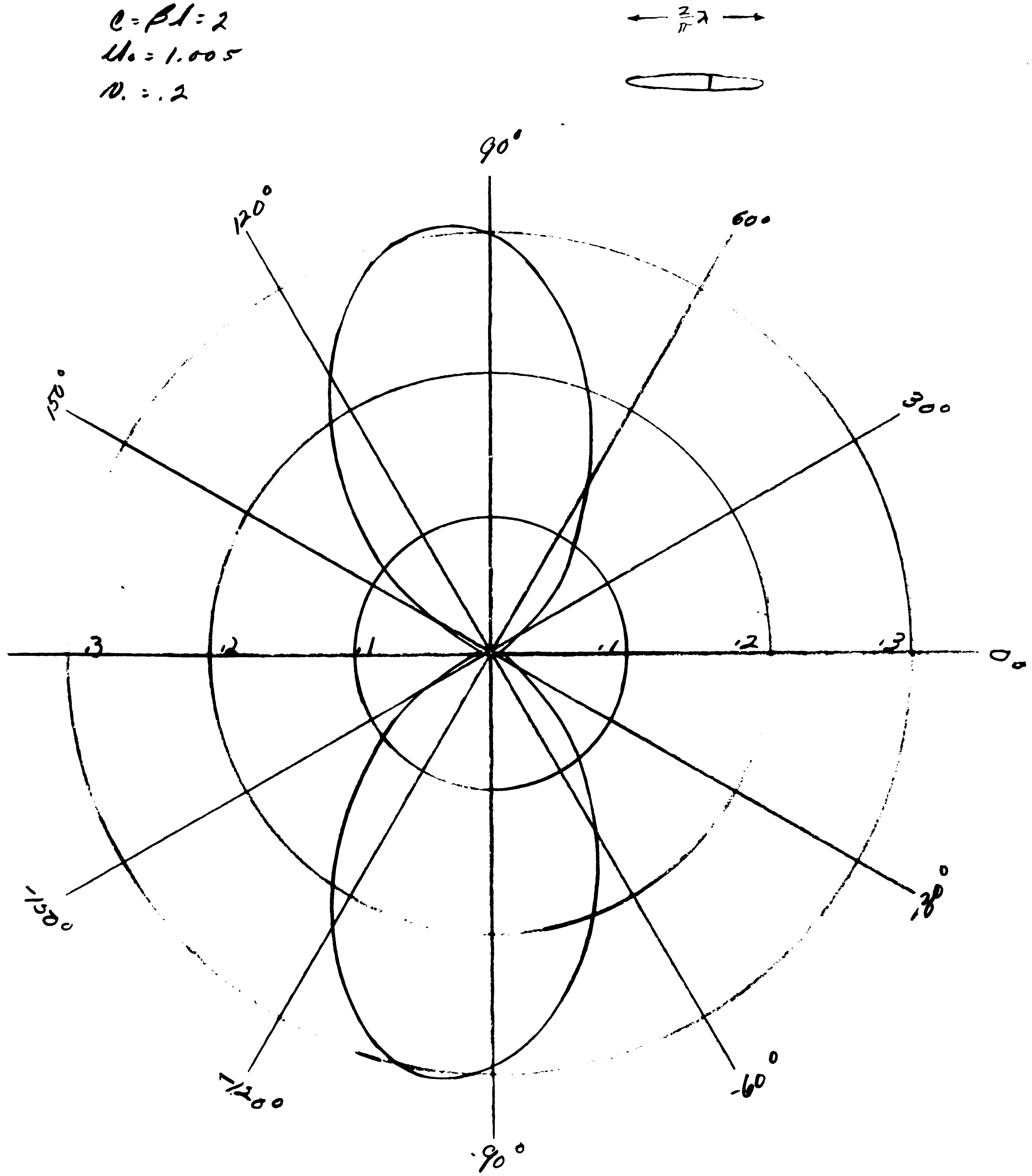
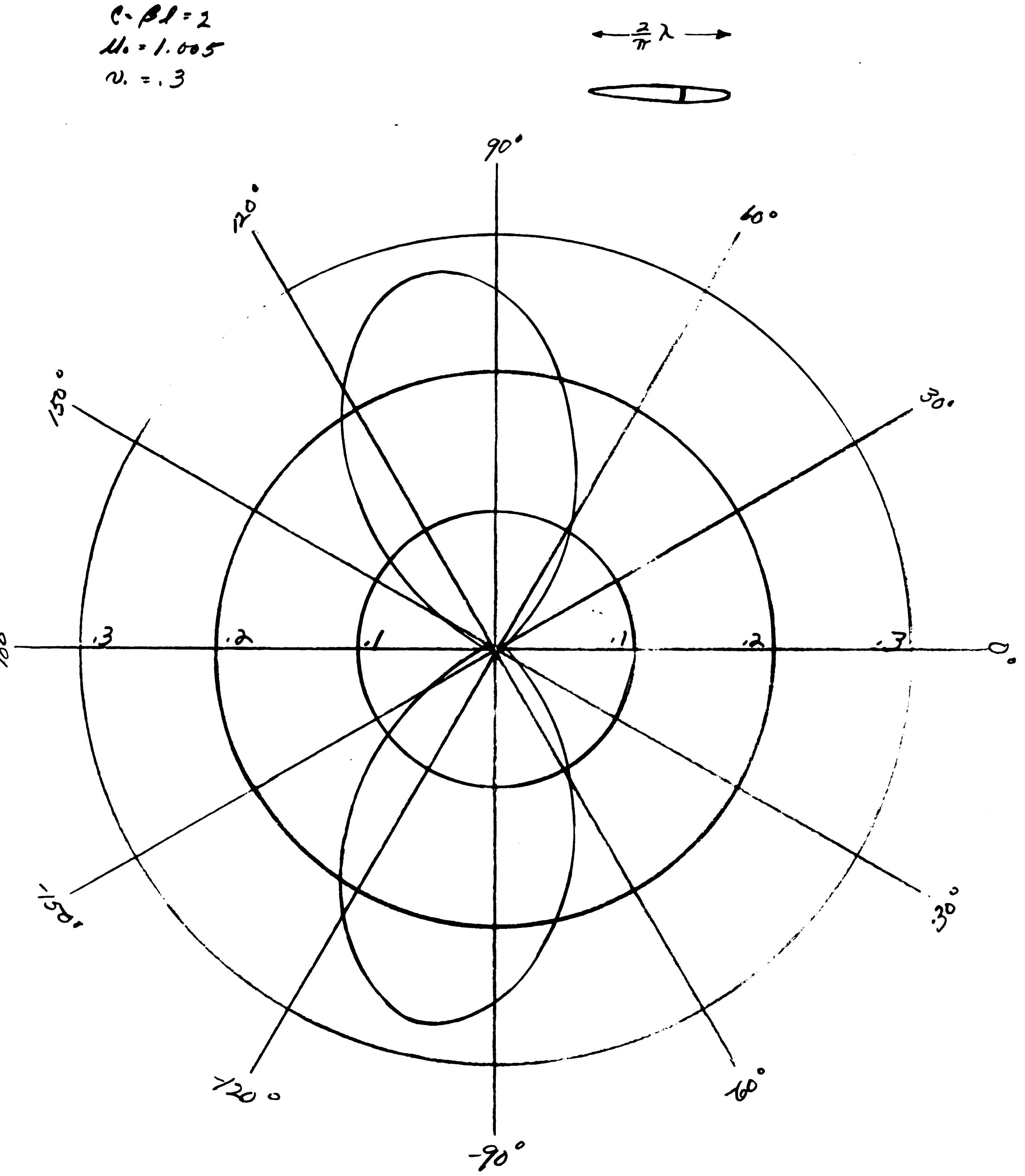


Figure 32.



C = βl = 2
μ₀ = 1.005
ν₀ = .3
2/π λ
90°
120°
60°
150°
30°
180°
.3
.2
.1
.1
.2
.3
0°
-150°
-30°
-120°
-60°
-90°

Figure 33.

$C = \beta l = 2$
$u_0 = 1.005$
$v_0 = .4$

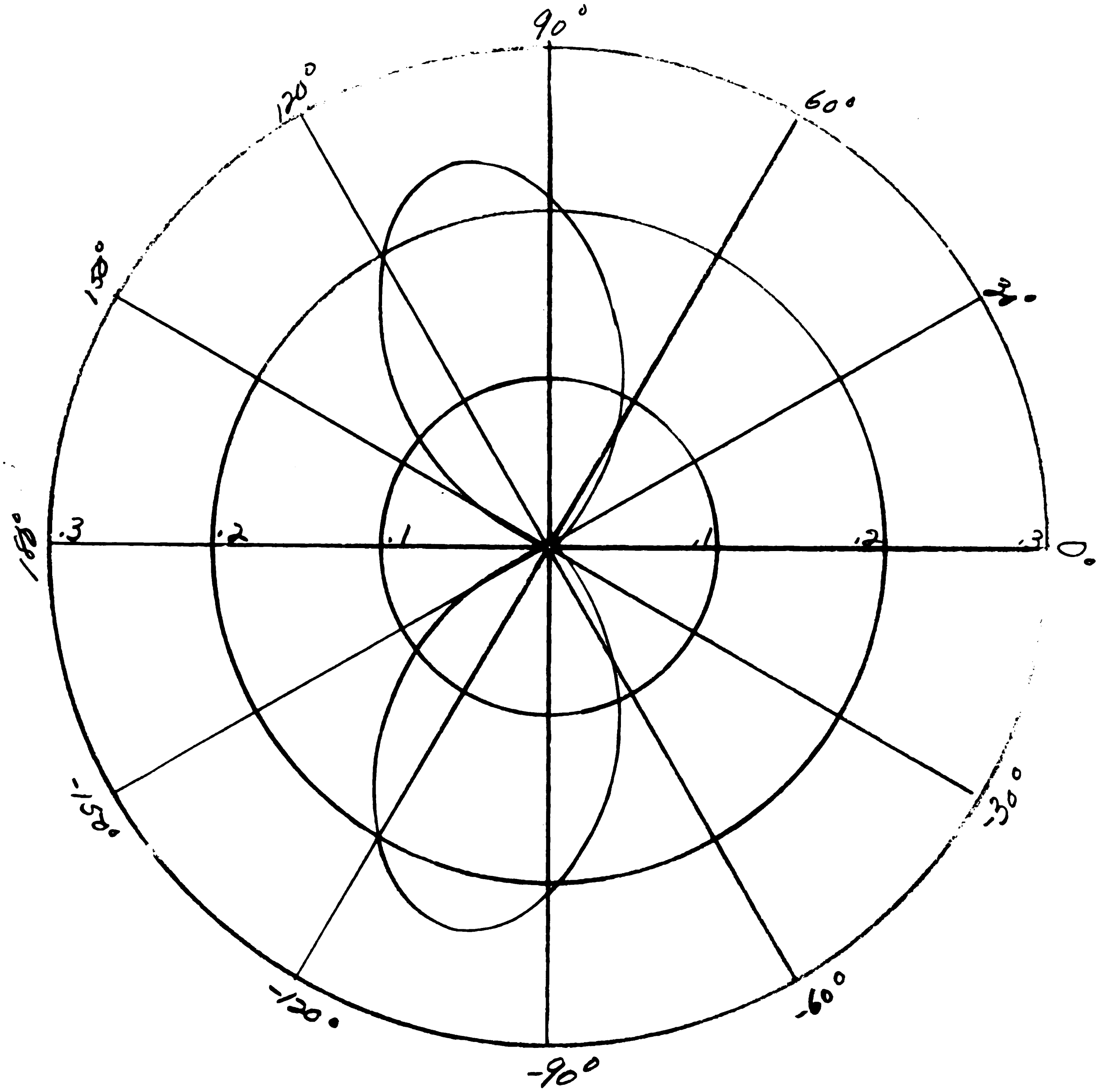
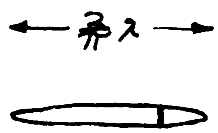


Figure 34.

$C = \beta l = 2$
$\mu_0 = 1.005$
$\nu_0 = .5$



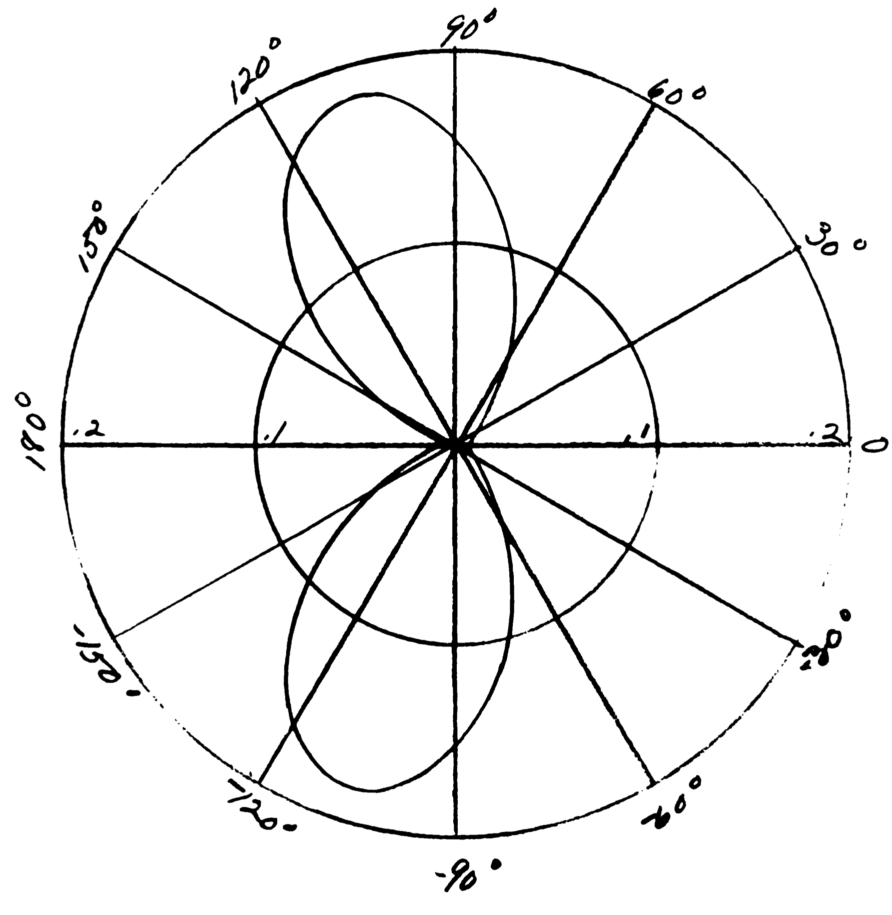


Figure 35

$C = \beta l = 2$
$u_o = 1.605$
$v_o = .6$

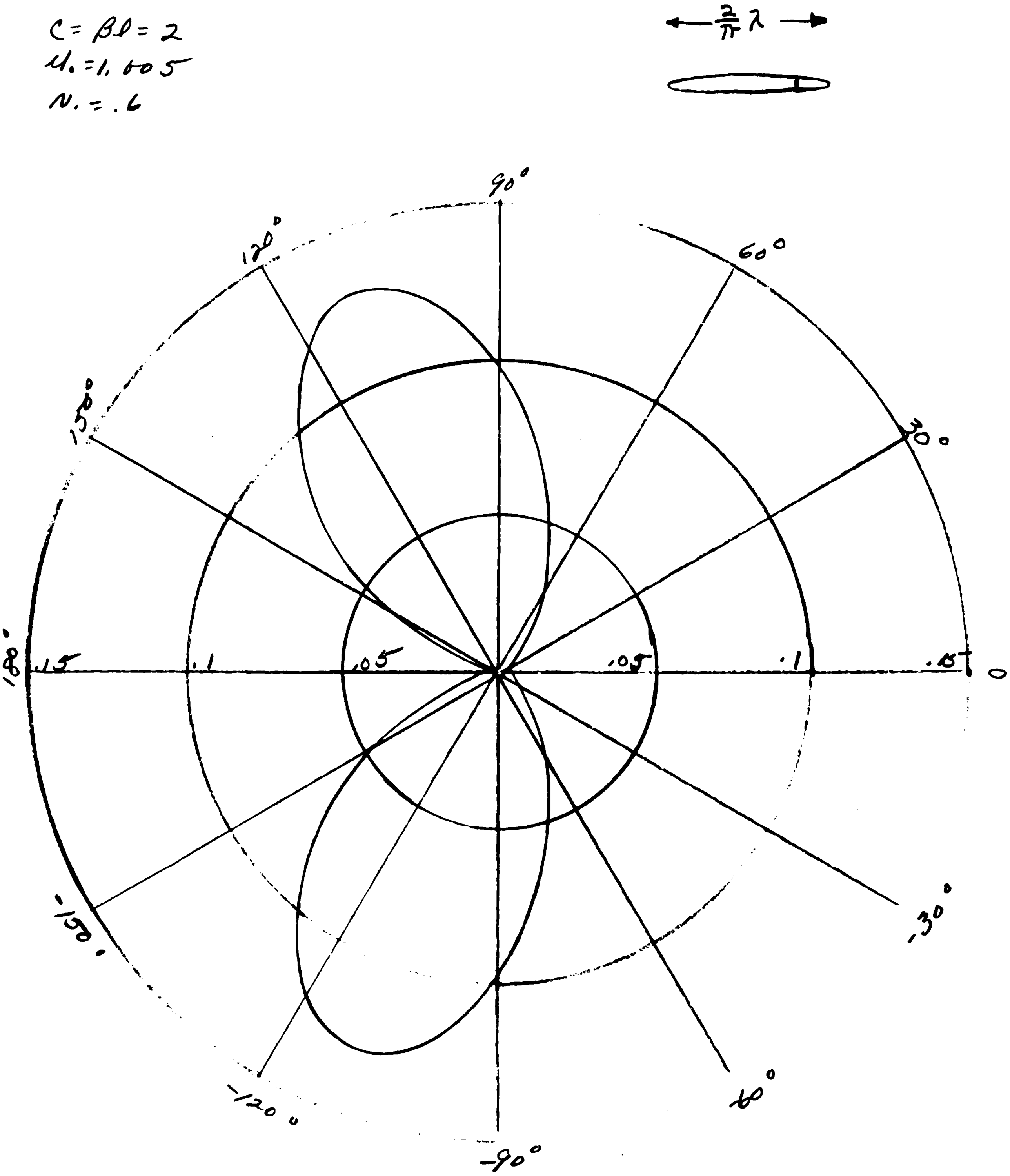


Figure 36.

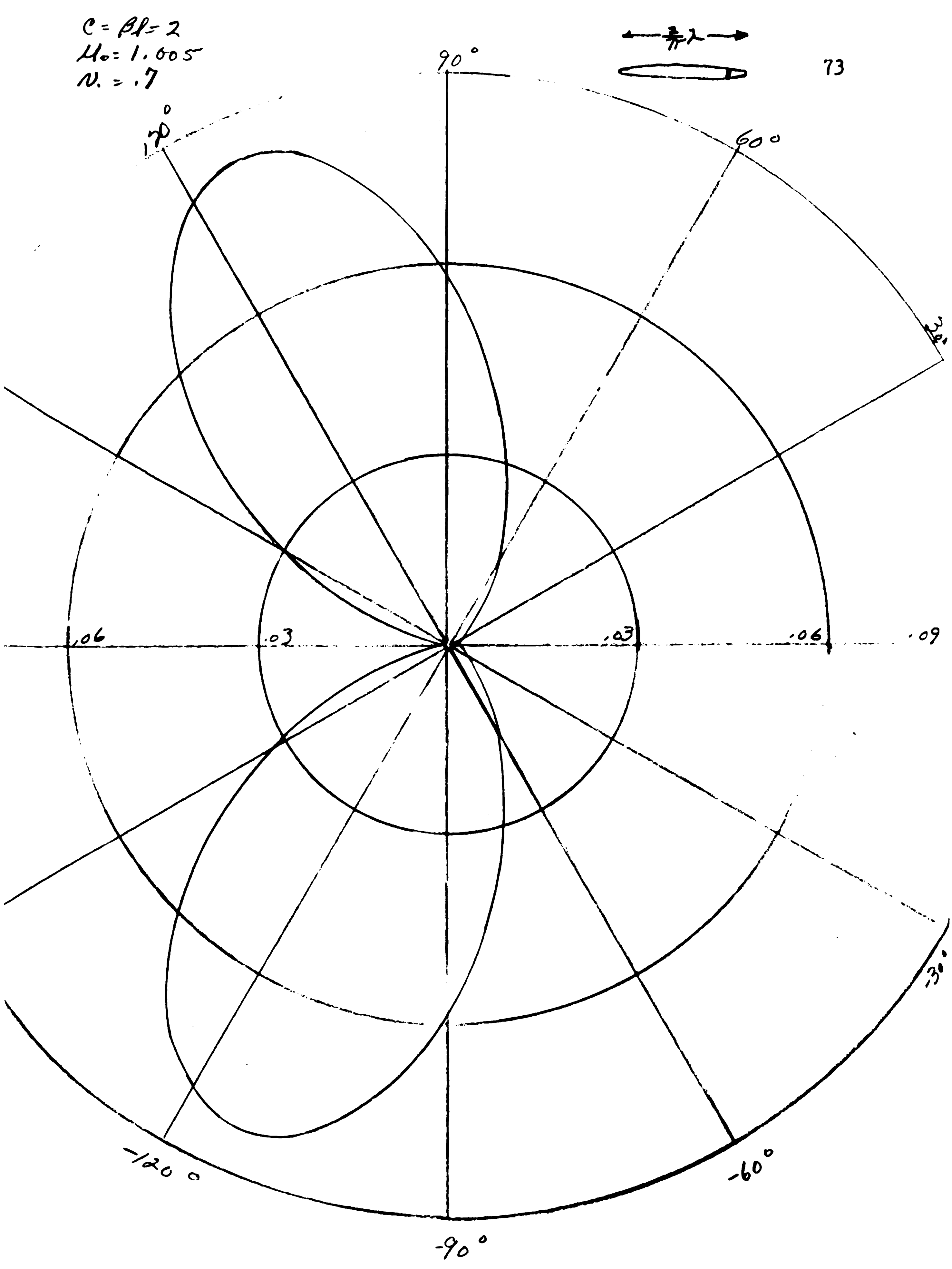
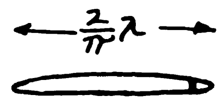


Figure 37.

$C = \beta l = 2$
$\mu_0 = 1.005$
$\nu = .8$



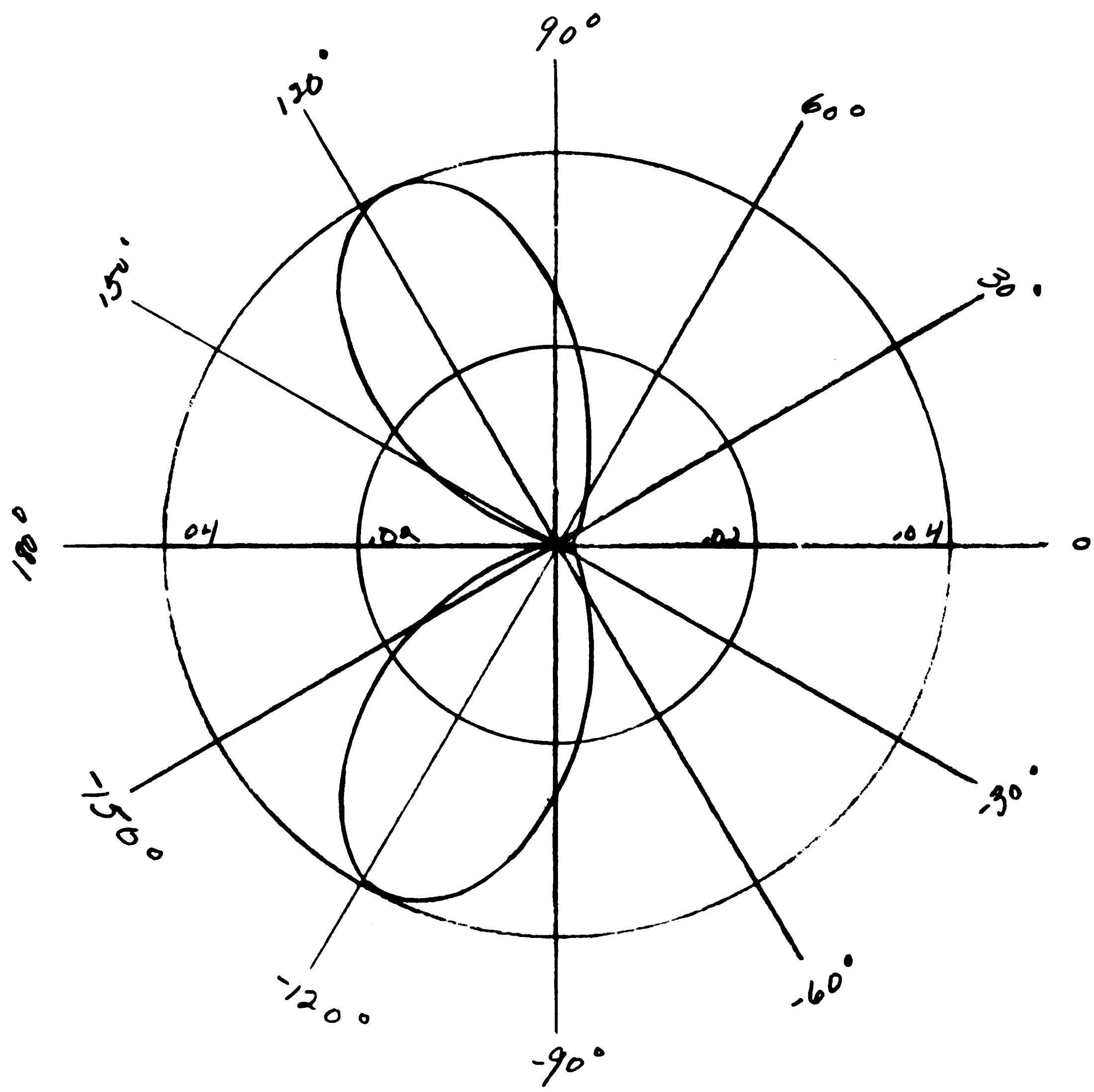
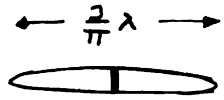


Figure 38.

$c = \beta l = 3$

$\mu = 1.005$

$\nu = 0$



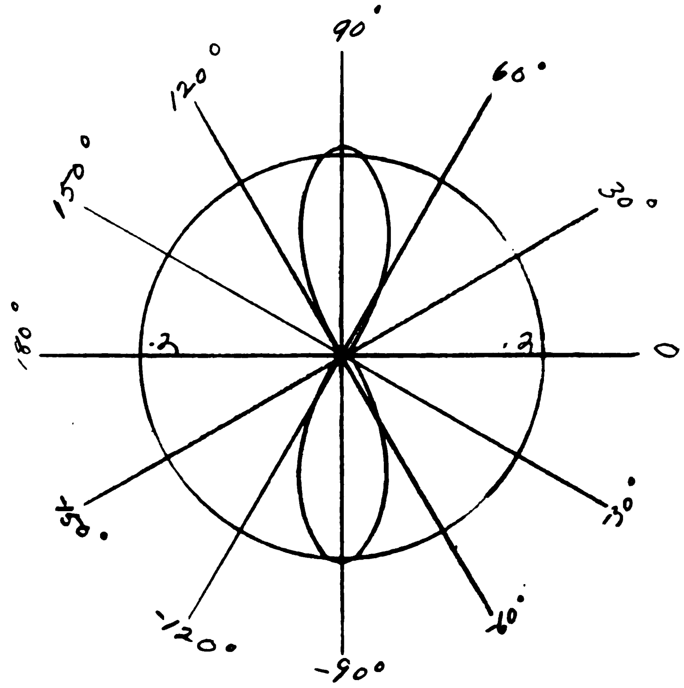


Figure 39.

$c = \beta l = 3$

$u_\circ = 1.005$

$v_\circ = .1$

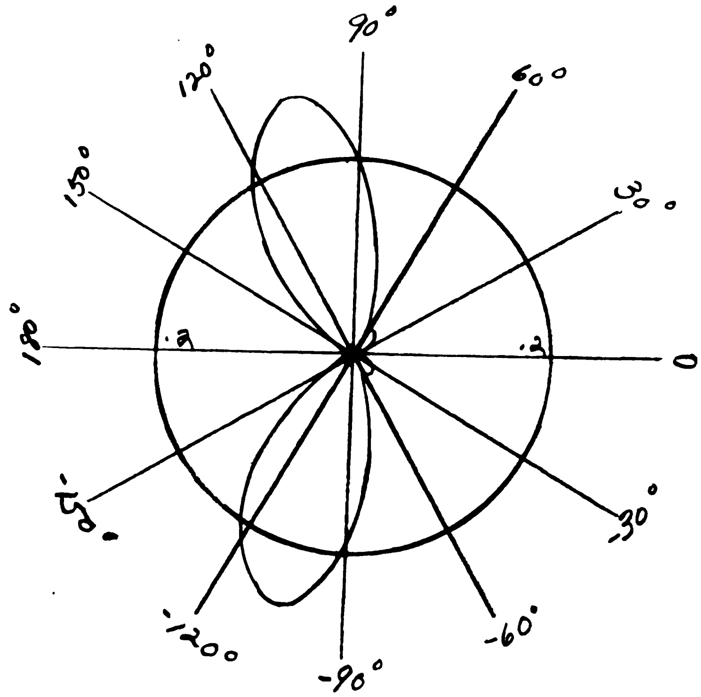
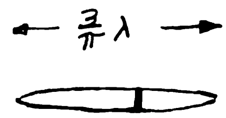


Figure 40.

$c = \beta l = 3$

$u = 1.005$

$\nu = .2$



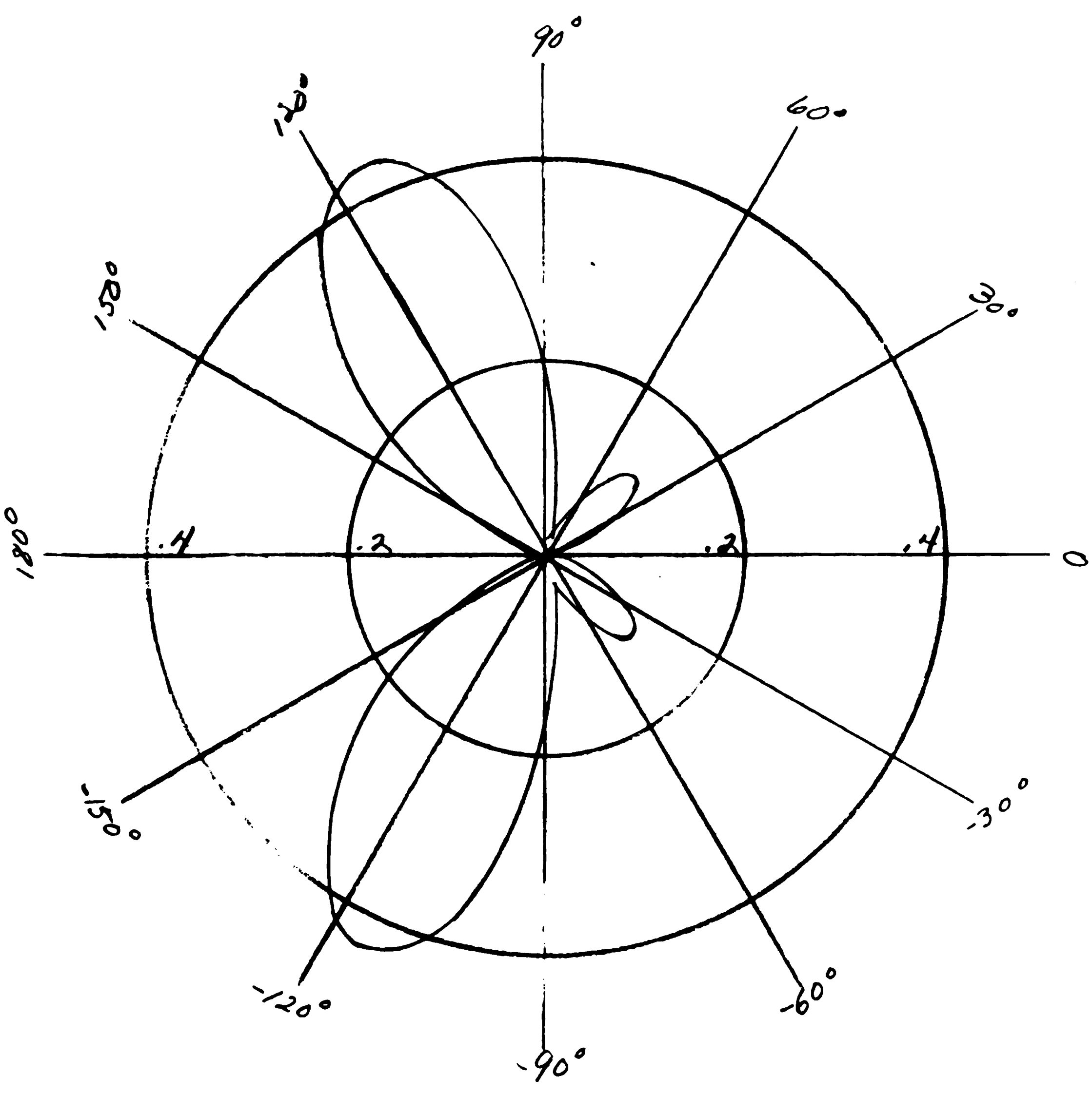


Figure 41.

$C = \beta l = 3$
$\mu = 1.005$
$\nu = .3$

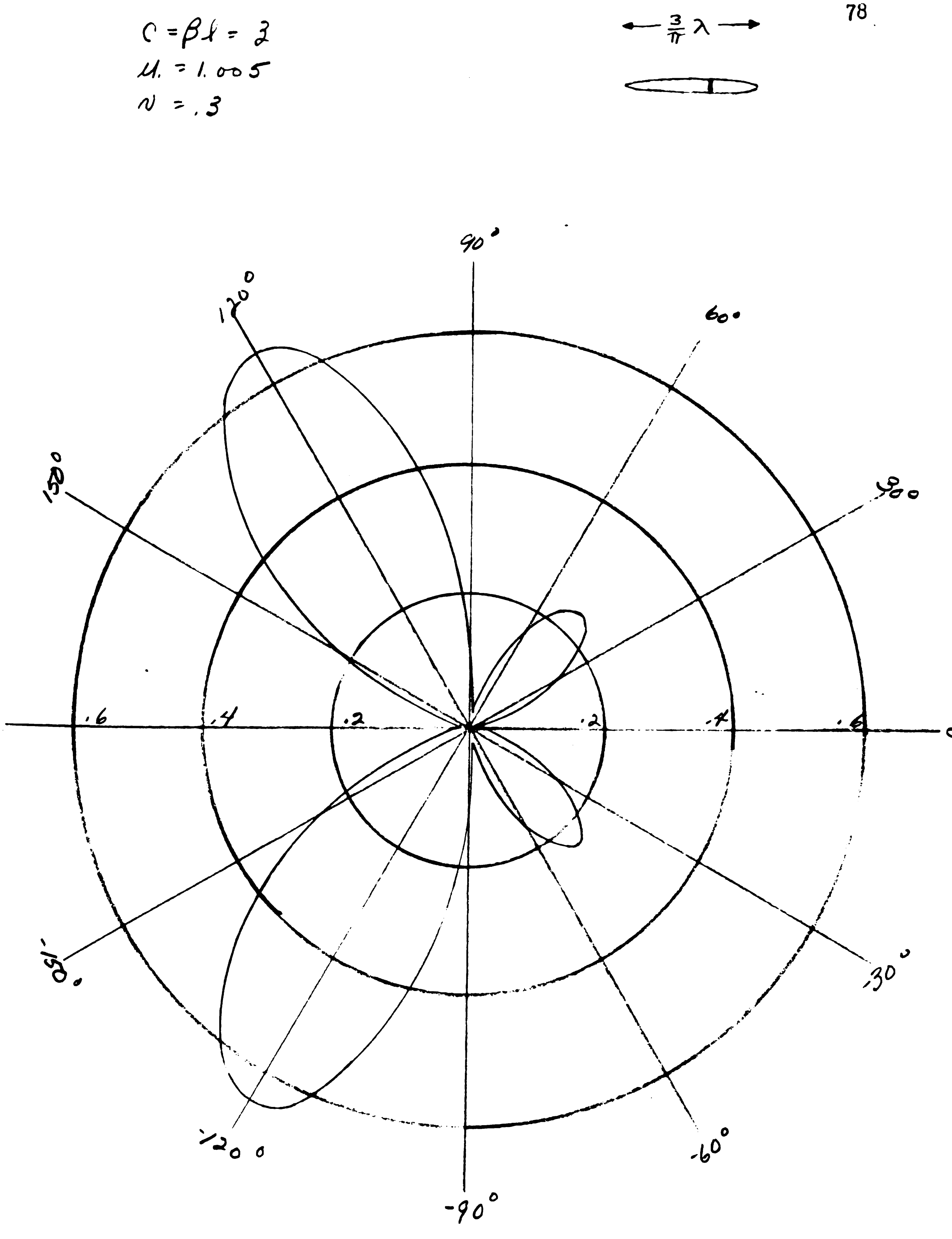


Figure 42.

$C = \beta l = 3$
$\mu_1 = 1.005$
$\nu_1 = .4$

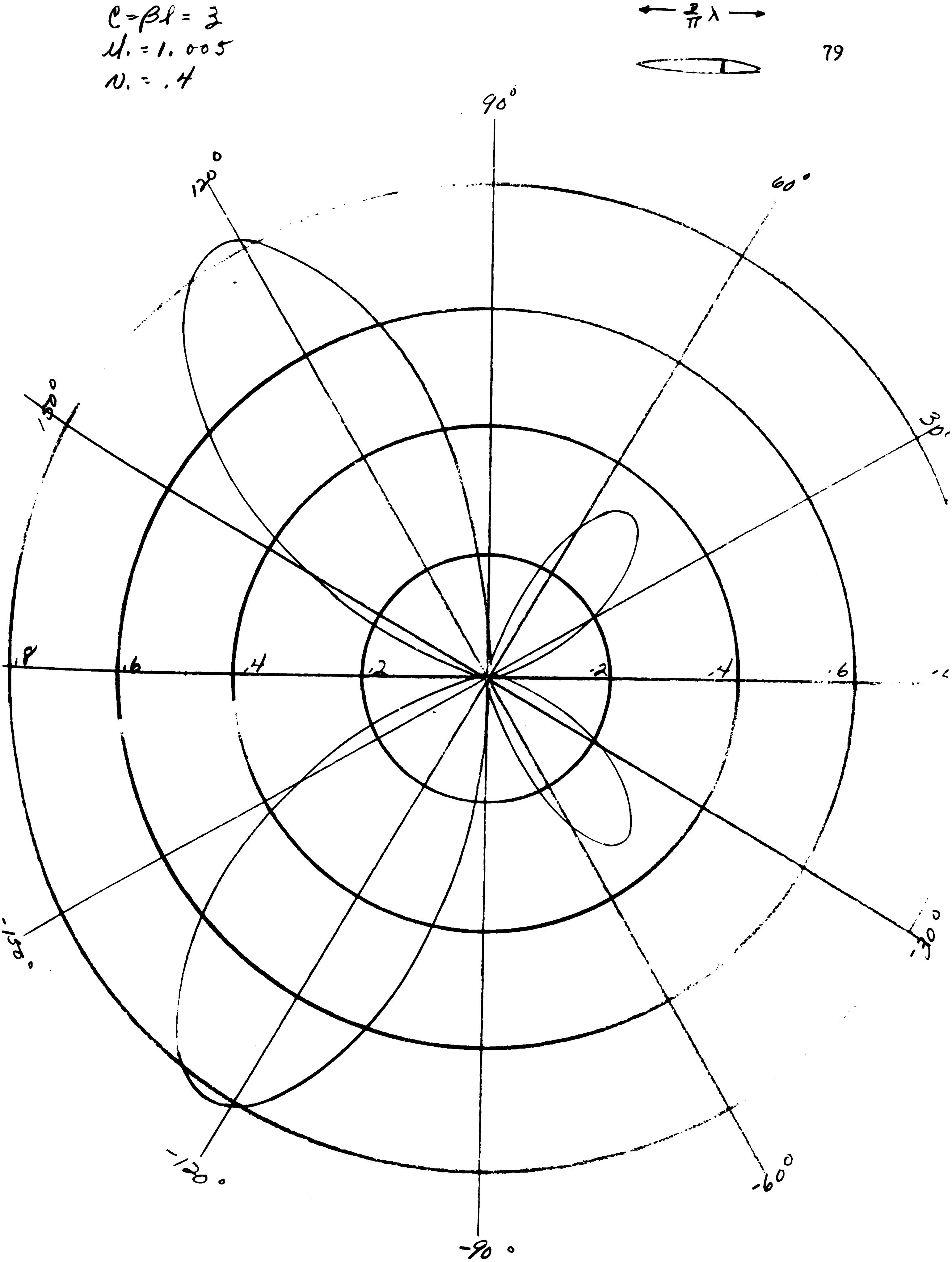


Figure 43.

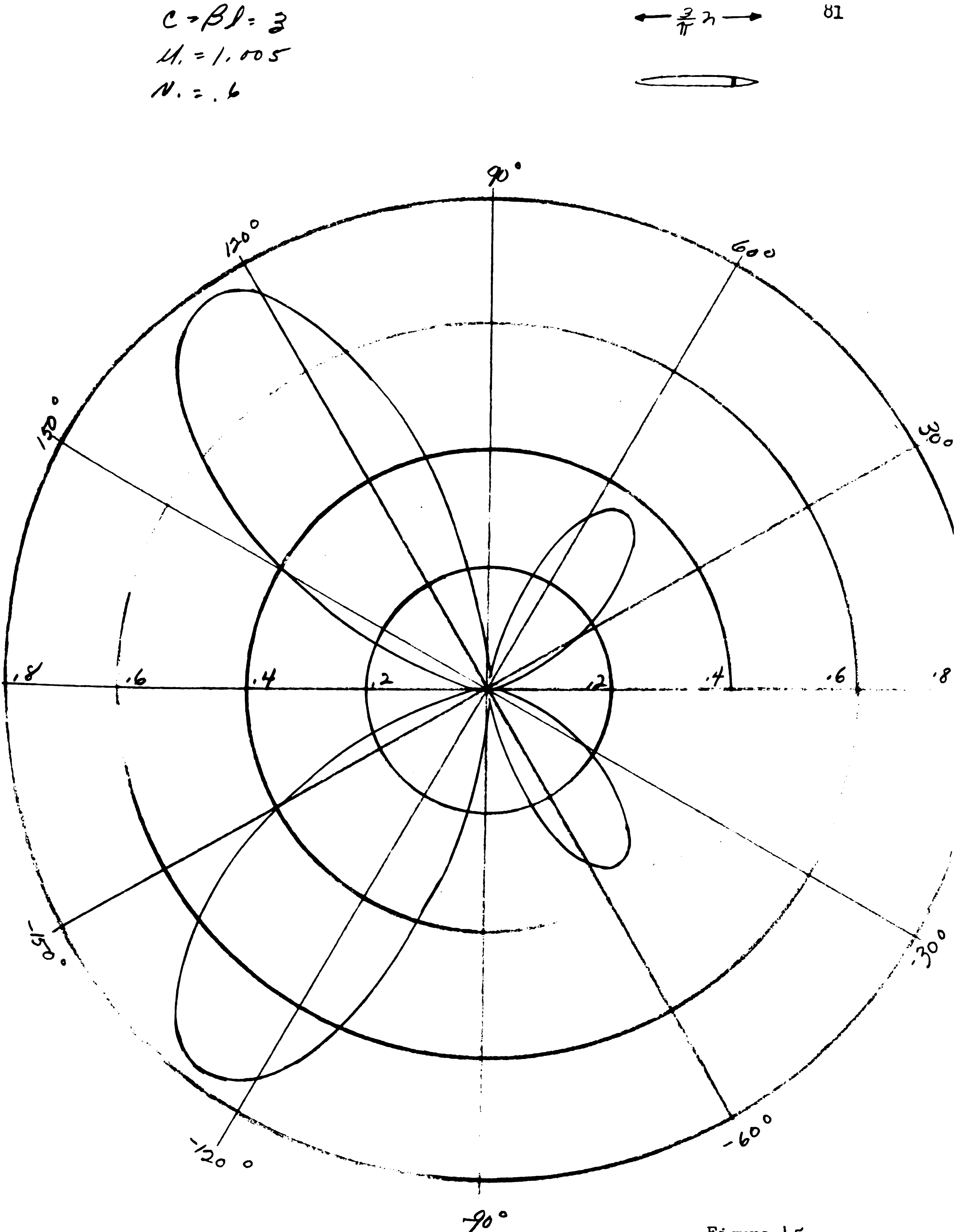


Figure 45.

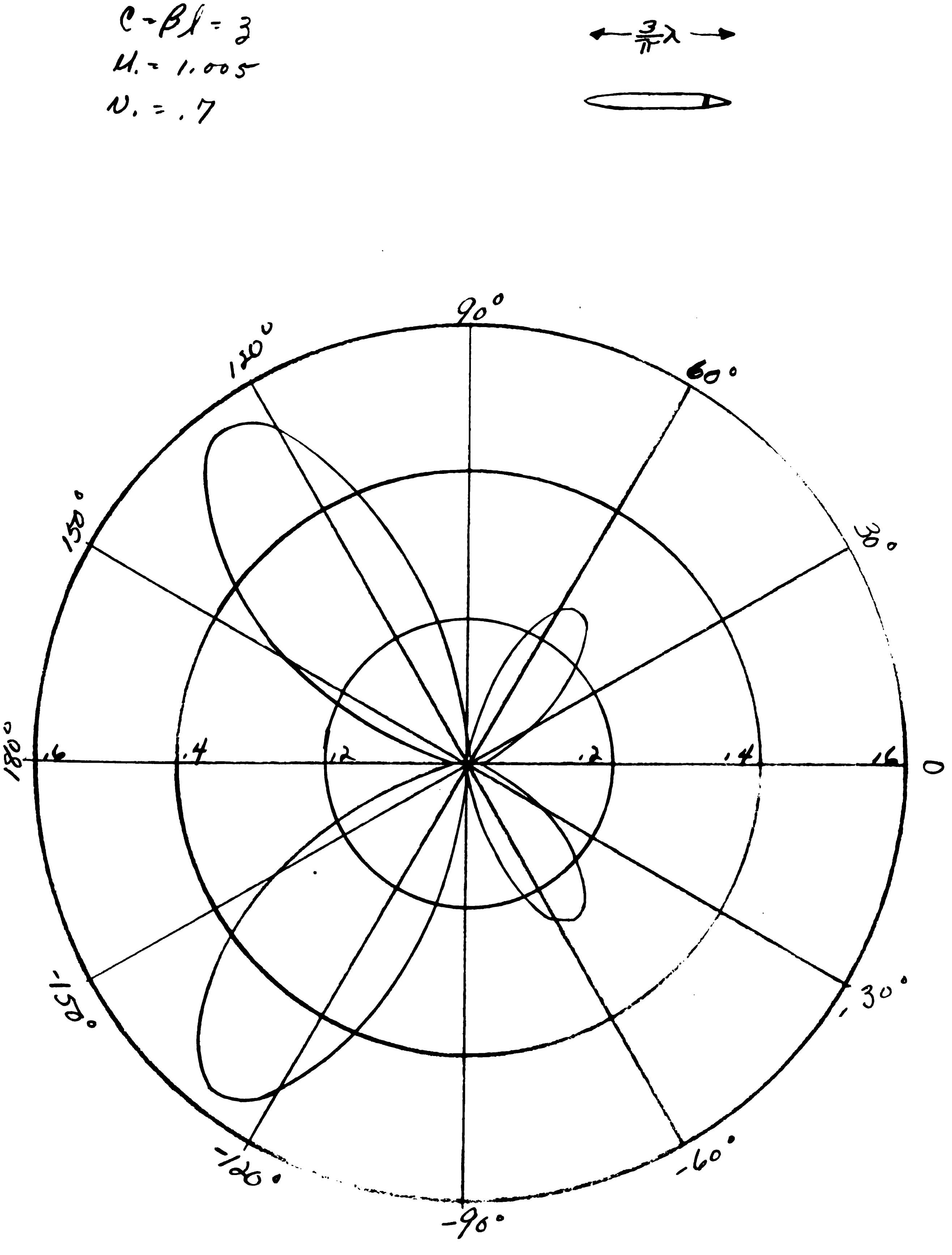
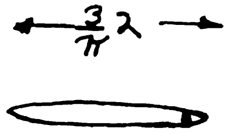

$C = \beta l = 3$
$u_o = 1.005$
$v_o = .7$
$\frac{3}{\pi}\lambda$
90°
120°
60°
150°
30°
180°
0
-150°
-30°
-120°
-60°
-90°
.6
.4
.2
.2
.4
.6

Figure 46.

$C = \beta l = 3$
$u_1 = 1.005$
$v_1 = .8$



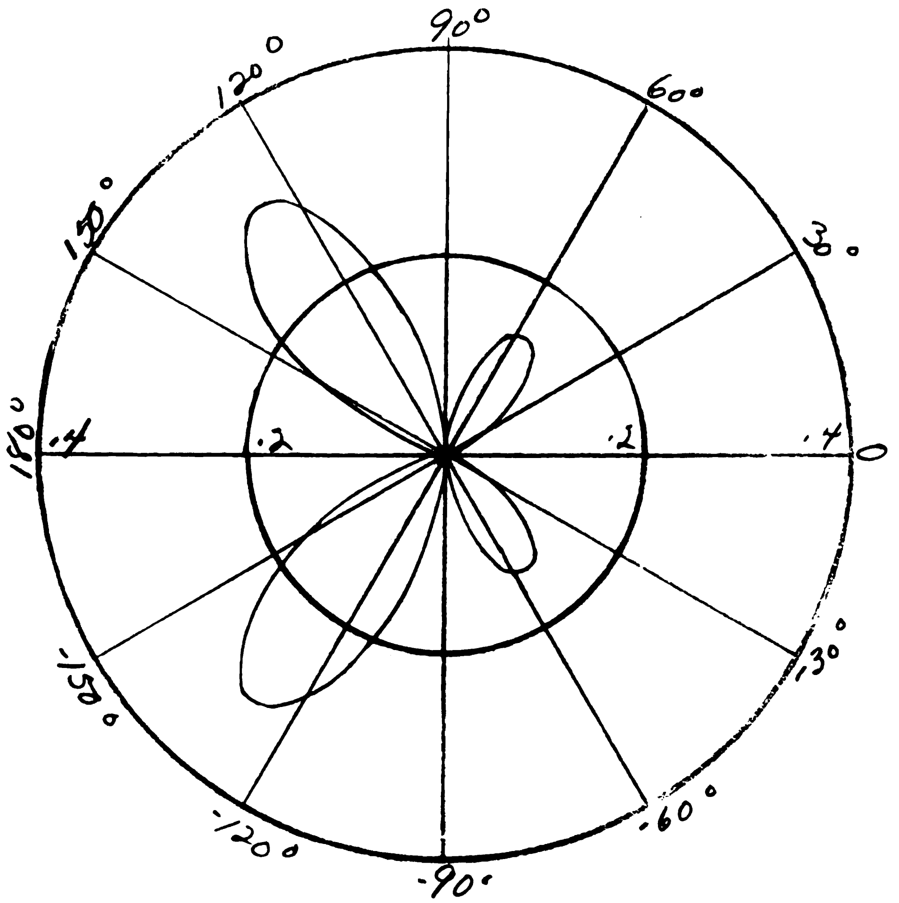
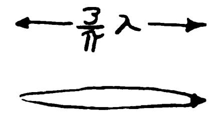


Figure 47.

$C = \beta l = 3$
$u_{\cdot} = 1.005$
$v_{\cdot} = .9$



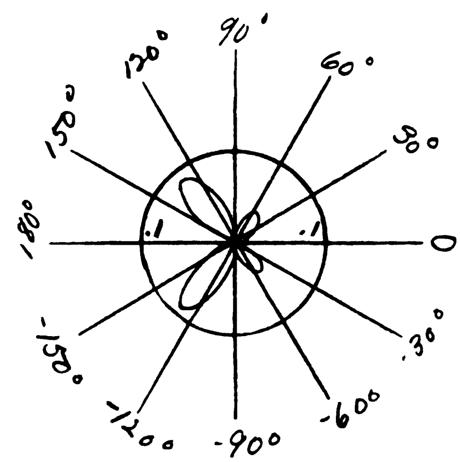
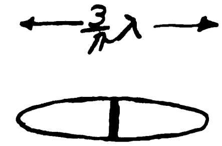


Figure 48.

$C = \beta l = 3$
$u_0 = 1.020$
$v_0 = 0$



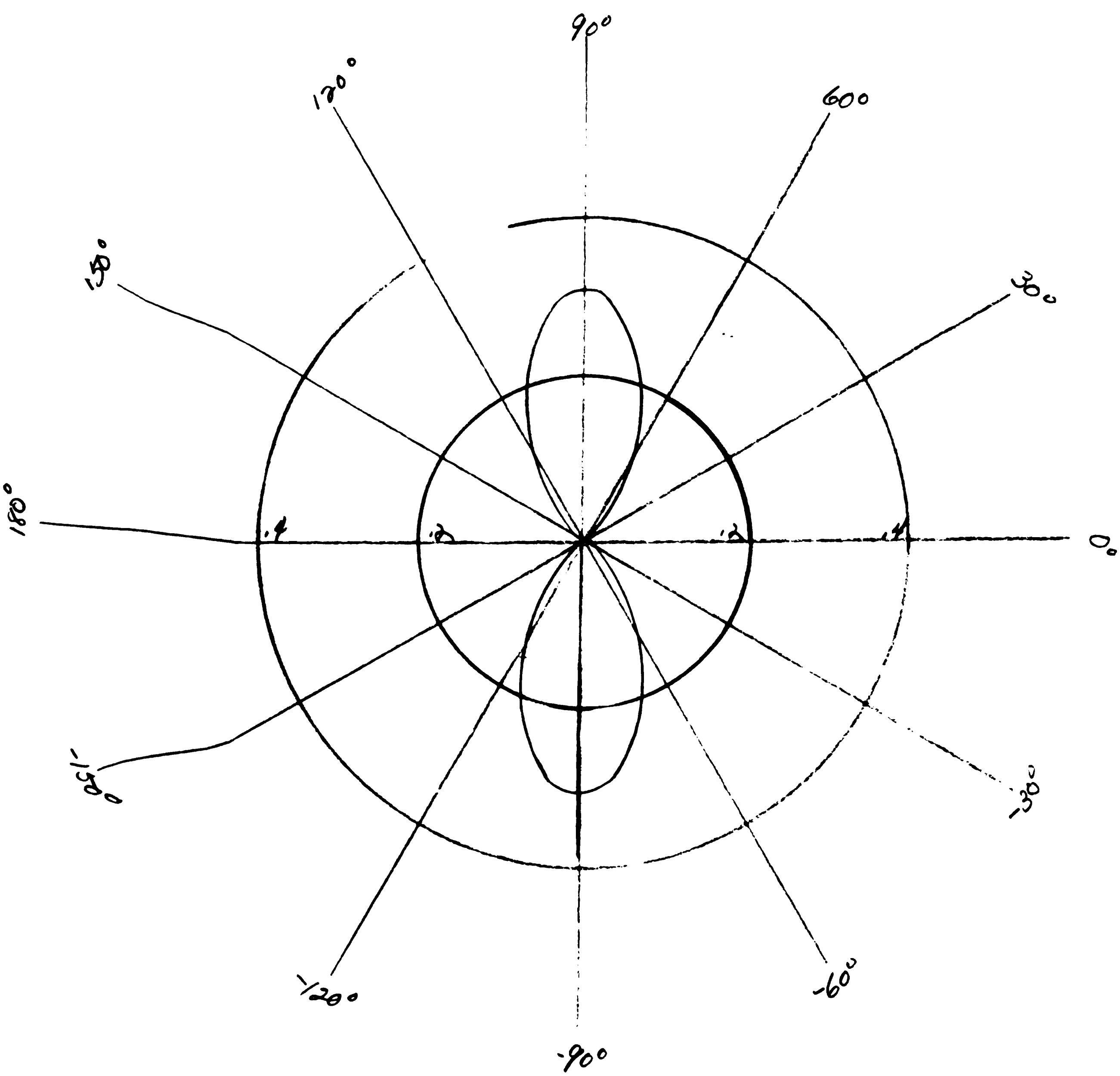


Figure 50.

$b = \beta l = 3$
$u_0 = 1.020$
$N_1 = .1$

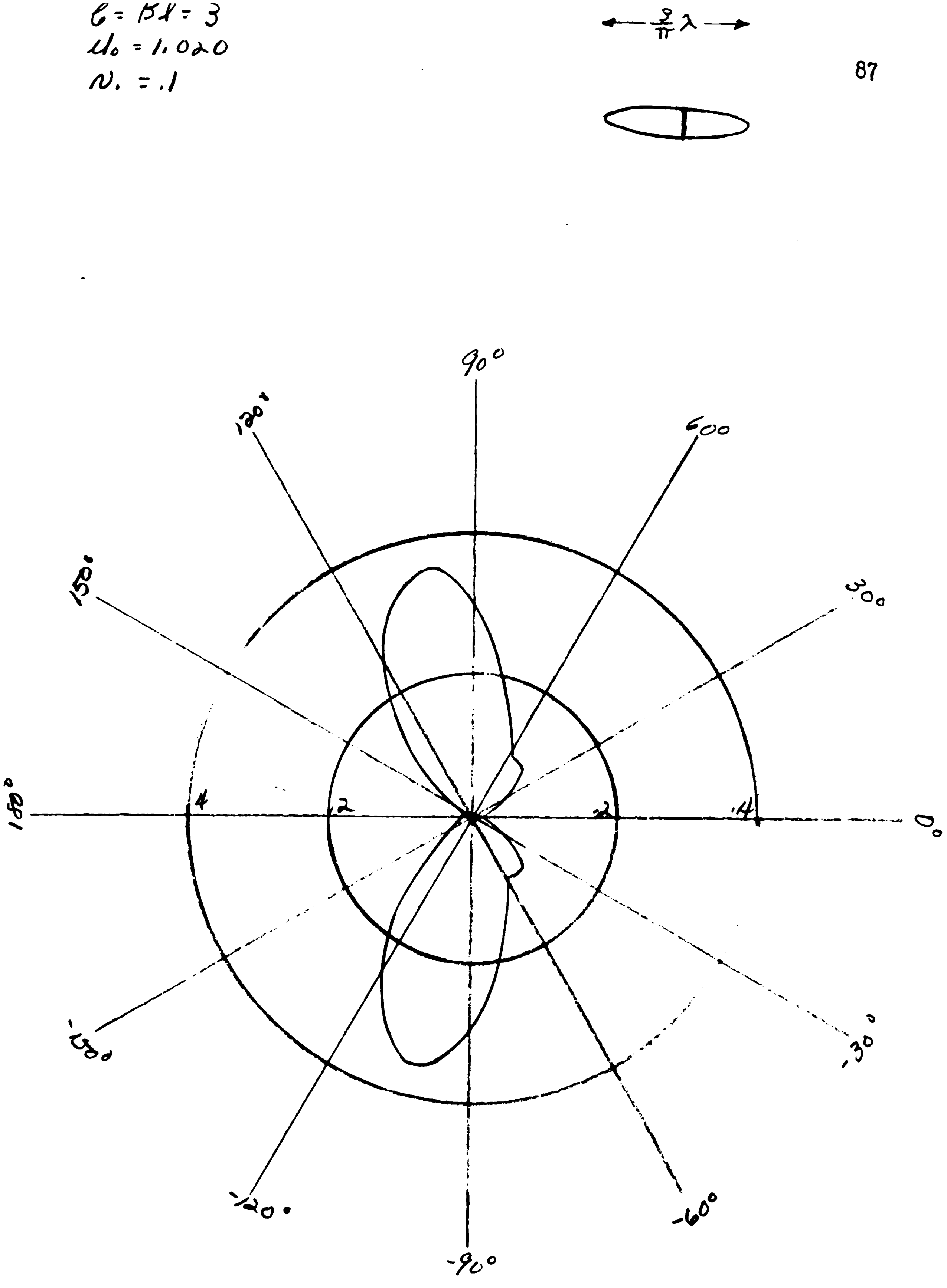


Figure 51.

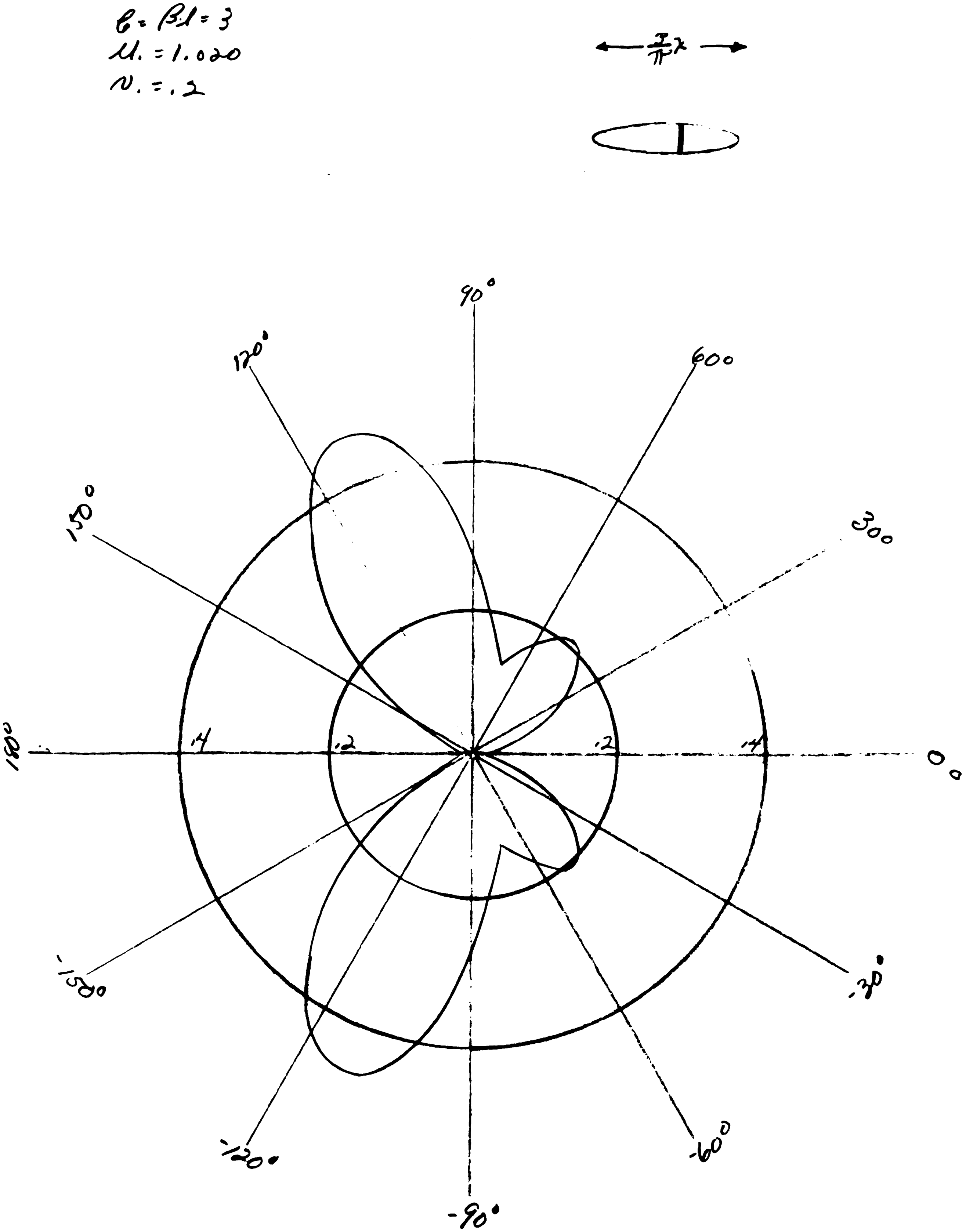
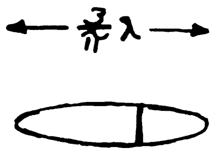


Figure 52.

$C = \beta l = 3$
$u. = 1.020$
$v. = .3$



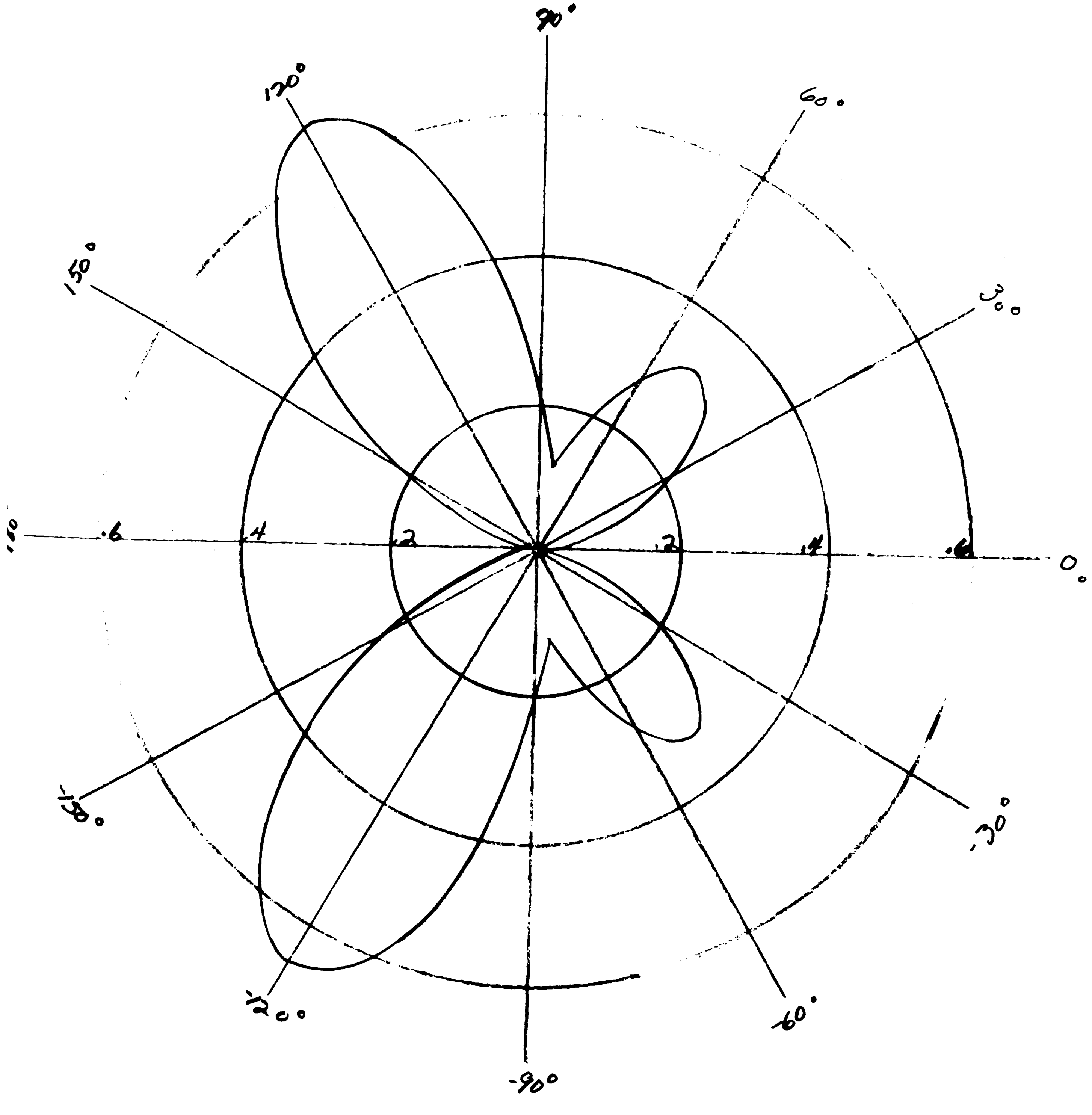
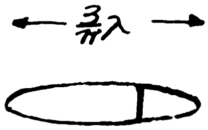


Figure 53.

$C = \beta l = 3$

$u_0 = 1.020$

$N = .4$



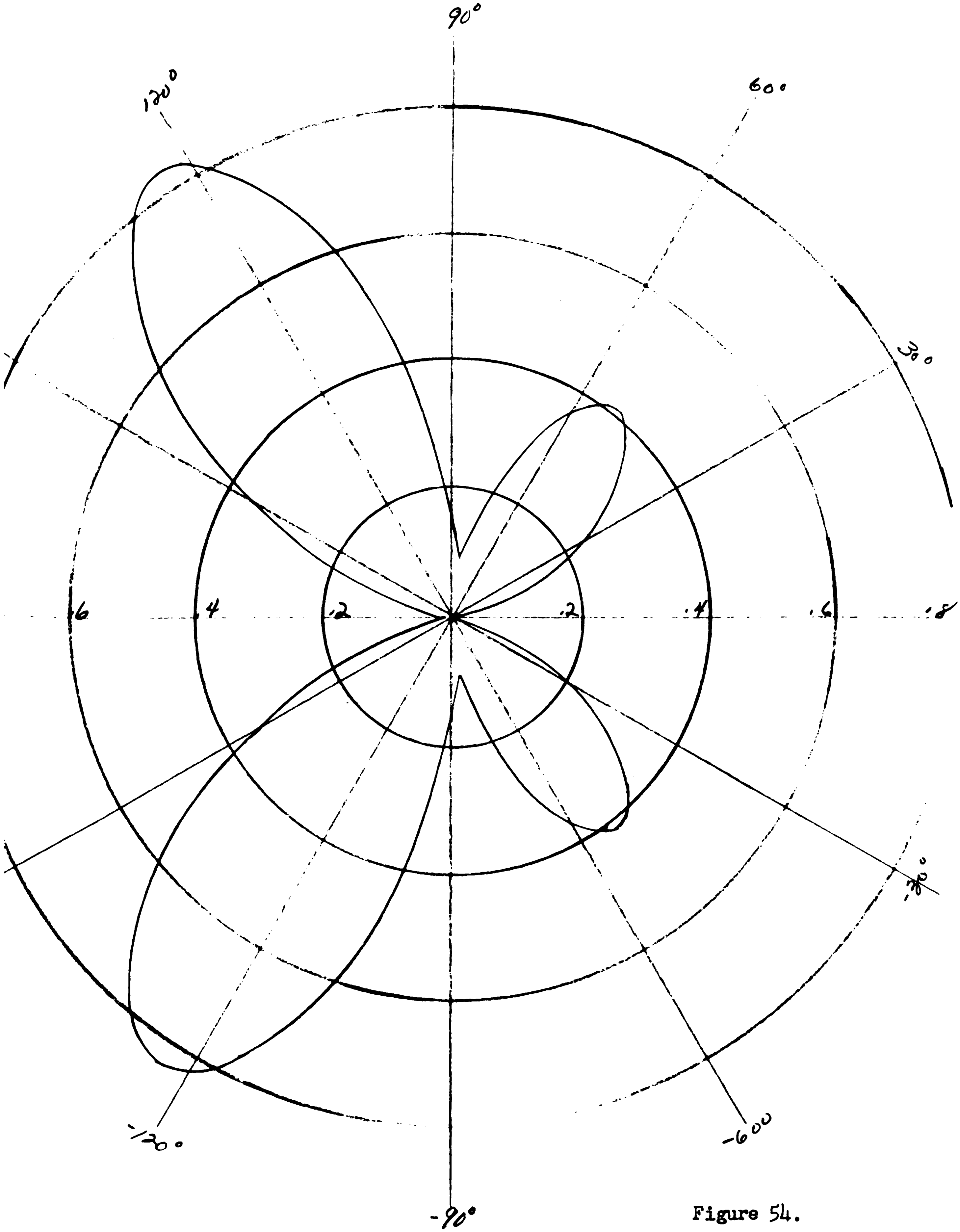


Figure 54.

$C = \beta r = 3$
$\mu_0 = 1,020$
$\nu_0 = .5$

$\frac{3}{\pi}\lambda$

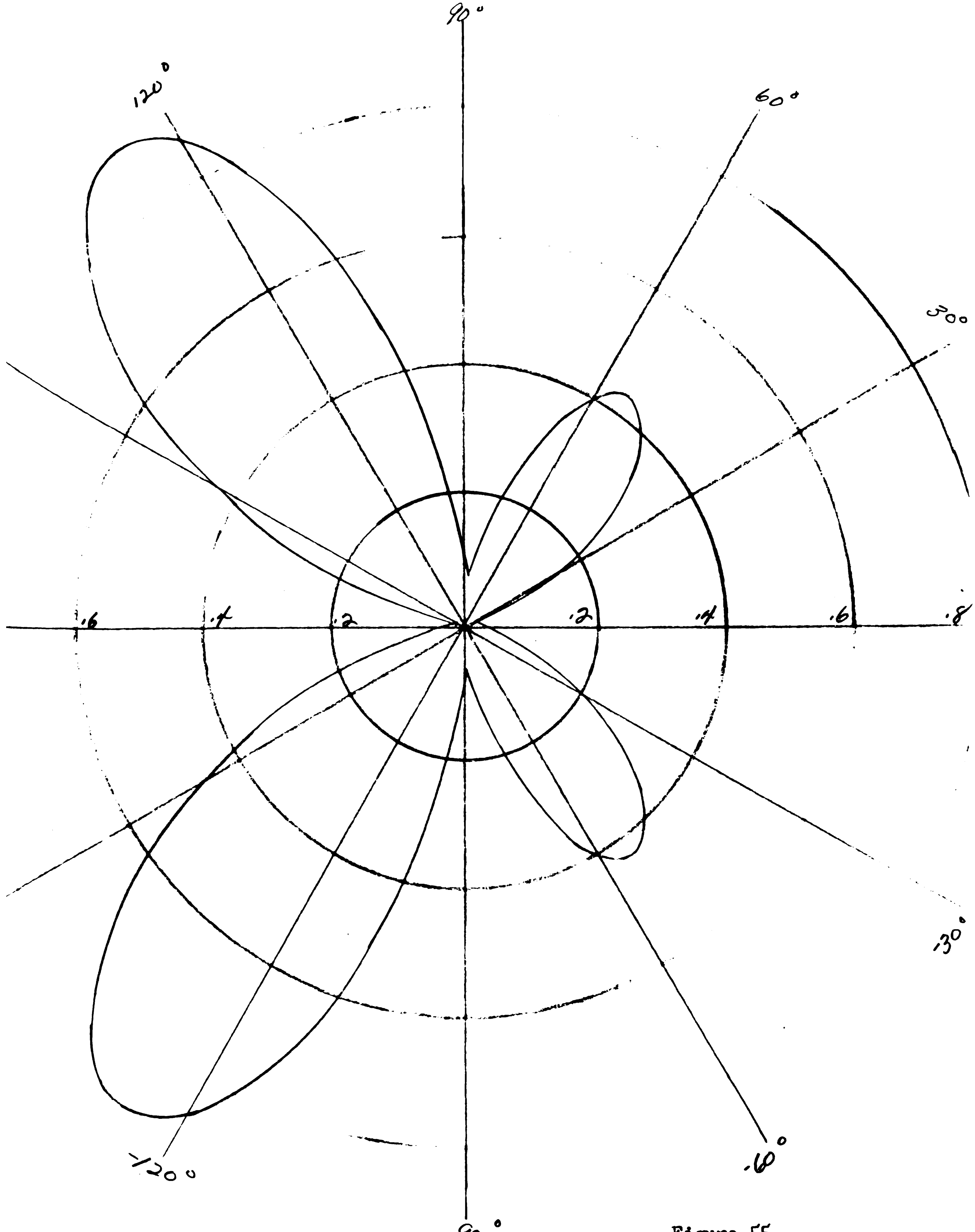


Figure 55.

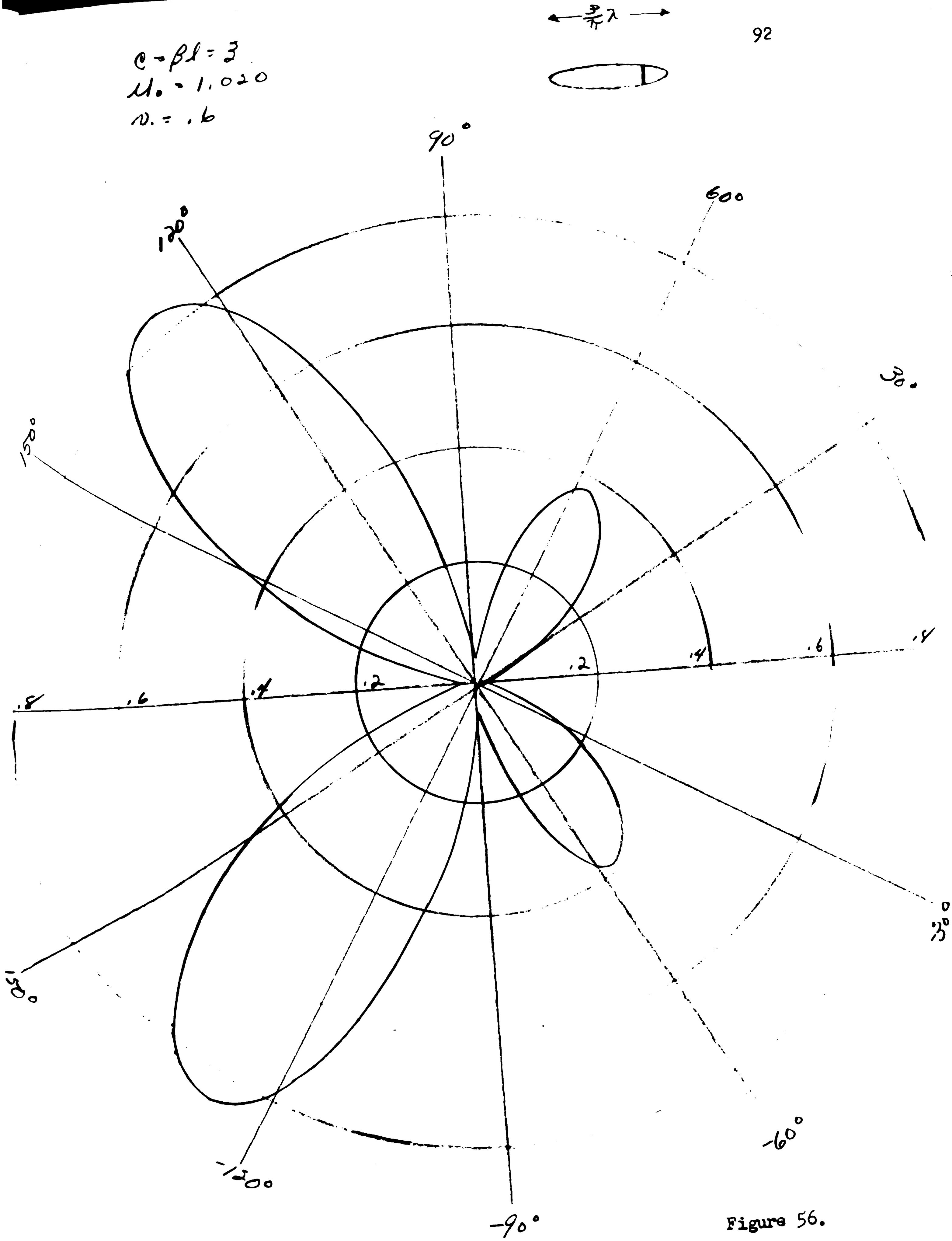
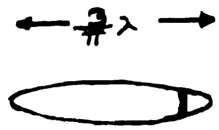

$c = \beta l = 3$
$\mu_0 = 1.020$
$\nu_0 = .6$
$\frac{3}{\pi}\lambda$
90°
60°
30°
120°
150°
0°
-30°
-60°
-90°
-120°
-150°
.2
.4
.6
.8

Figure 56.

$C = \beta l = 3$
$\mu_\circ = 1.020$
$v_\circ = .7$



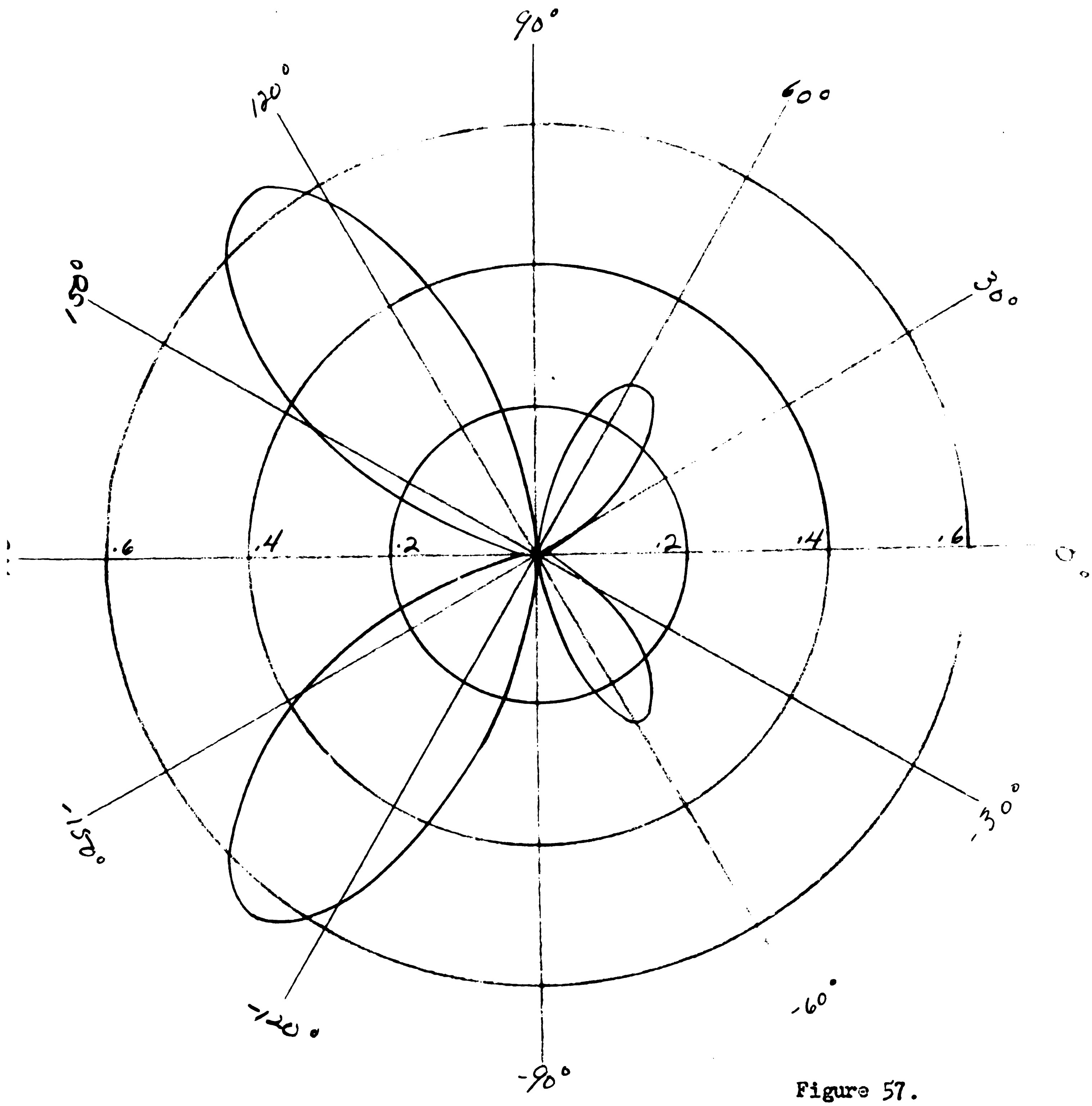
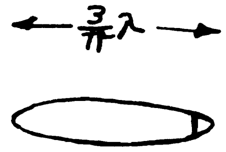


Figure 57.

$C = \beta l = 3$
$u_0 = 1.020$
$\nu = .8$



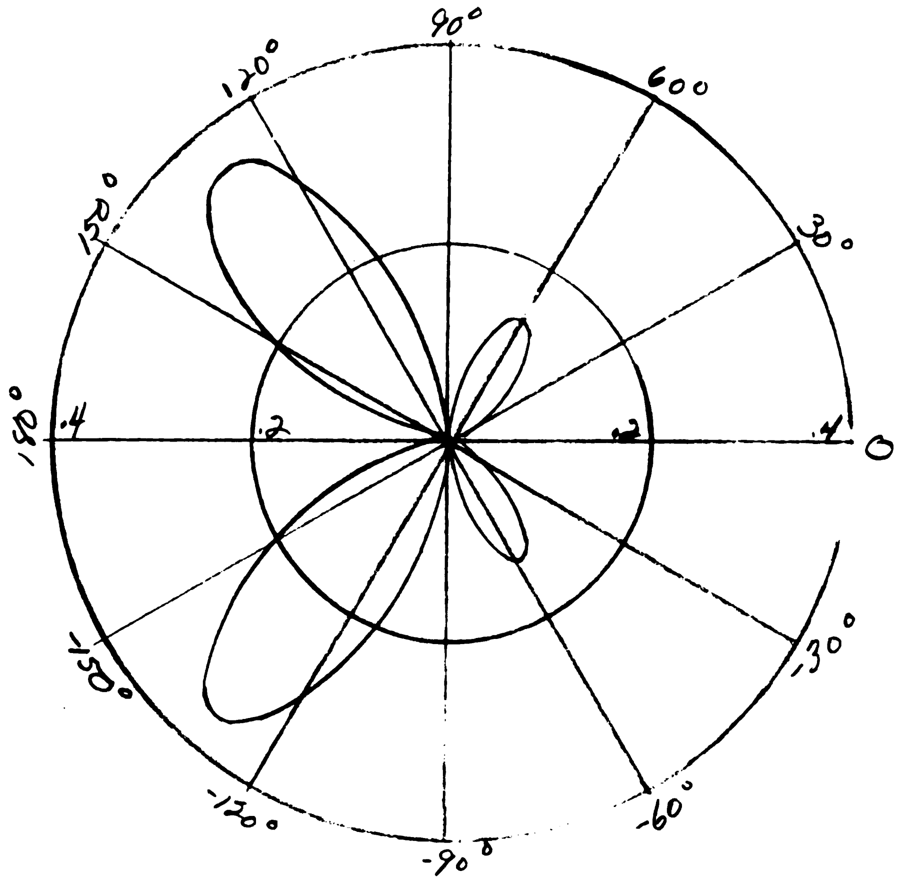
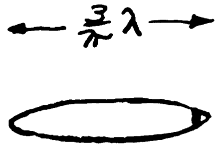


Figure 58.

$c = \beta l = 3$
$u. = 1.020$
$v. = .9$



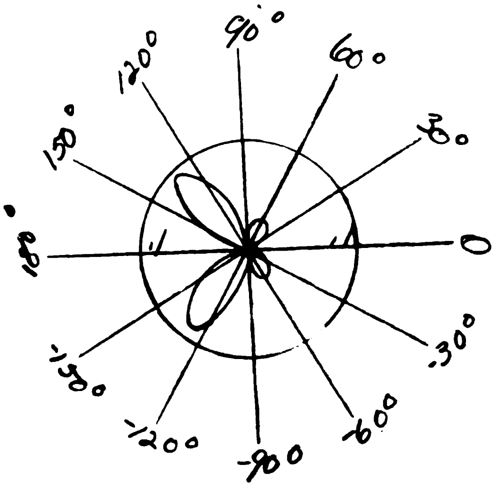


Figure 59.

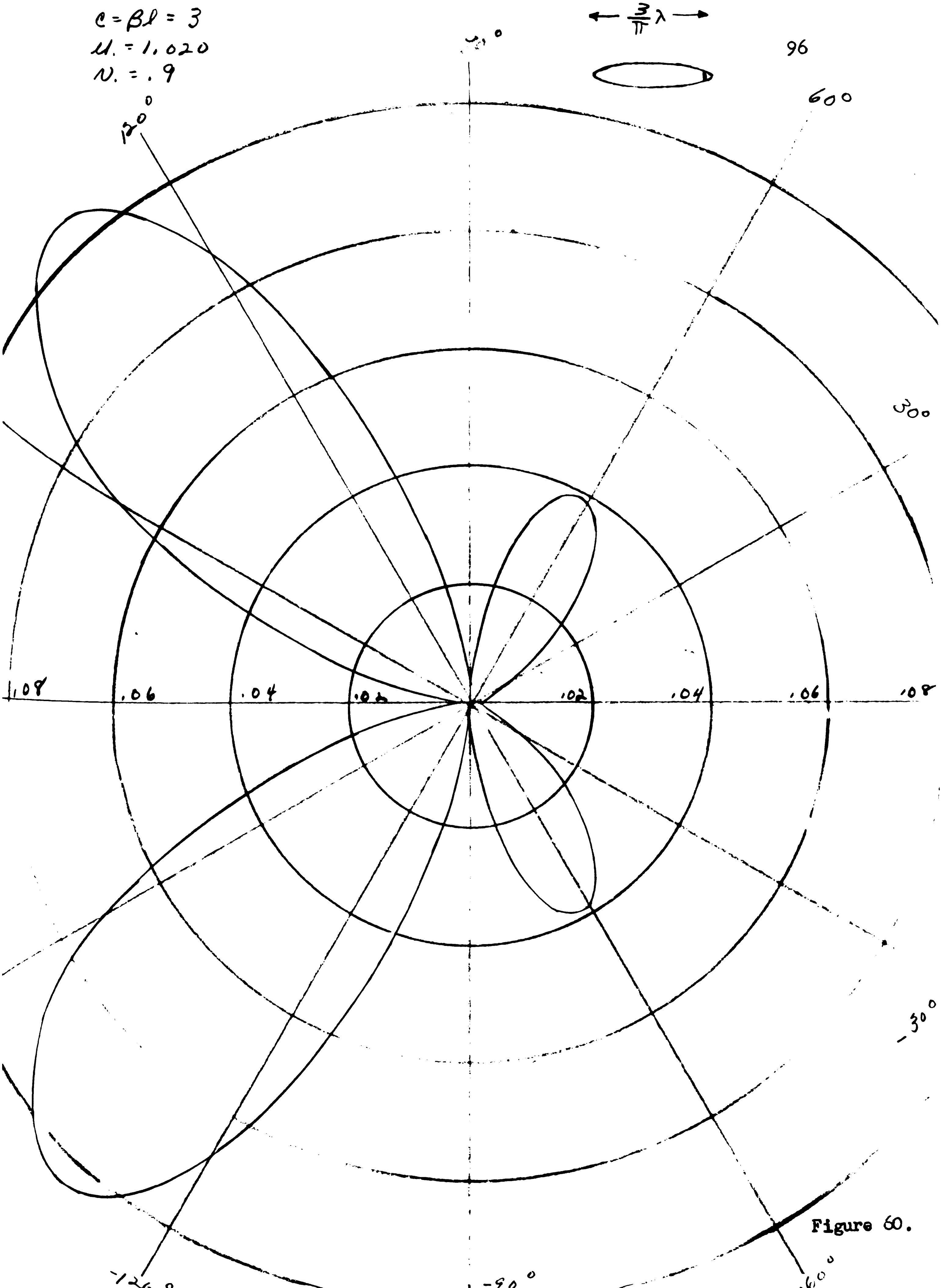

$C = \beta l = 3$
$u. = 1.020$
$v. = .9$
$\leftarrow \frac{3}{\pi}\lambda \rightarrow$
$90°$
$60°$
$30°$
$120°$
$.08$
$.06$
$.04$
$.02$
$.02$
$.04$
$.06$
$.08$
$-30°$
$-60°$
$-90°$
$-120°$

Figure 60.

IV

CONCLUSIONS

The angular functions were obtained in the form of a power series expanded about the origin. This form has the following advantages: 1) The angular functions in this form have a simple recursion formula and are therefore easily computed. 2) The functions can be easily evaluated for any value of the variable, in contrast to angular functions that depend upon tables of associated Legendre polynomials. 3) If the applied field can be expressed in any of a number of simple closed forms, use of the orthogonality properties leads to an integral expression for the coefficients in the series expansion of the solution that can be evaluated by elementary methods. The disadvantage of the power series representation is that the norms are not as easily computed as they are by other methods.

The radial functions were expressed in the form of a Laurent series. This method is valuable in that it furnishes an independent check on the values of the radial functions obtained by other methods. Its main advantage over other methods is that there is a complete check at almost every step of the computations. A disadvantage is that, in its present form, it is not as convenient to compute radial functions for a large number of L/D ratios for the same $\underline{c}$ and $\underline{\ell}$ values. However, usually only two or three different L/D ratios are required, and the computation of radial functions for extra L/D values would require only a small fraction of the time spent in computing the radial constants.

Two methods have been developed for computing antenna impedances that take into consideration the width of the gap and the geometry of the transmission line feeding the antenna. One method is based on the usual assumption that the current is driven by the field in the gap. The other method is based on the assumption that the current in the antenna elements is driven by the component of the electric field of the transmission line that is tangential to the elements. Using the first method, the result is that, even for fairly wide gaps, the impedance values obtained are essentially the same as those obtained under the assumption of a uniform step-function applied field over a "small" gap. Using the second method, the impedance values obtained are slightly higher (on the order of 5%) than those calculated by the first method. In this case, the wider gap yielded a slightly higher value of input impedance than the narrow gap. To find out what the best approximation to the applied field is, calculations and experimental evidence will have to be compared for antennas less than or equal to a half wave length long.

Radiation patterns have been obtained for antennas of three different lengths up to about one wave length long, for length/thickness ratios of about 5/1, 10/1, 22/1, and 316/1, and for nine unsymmetrical gap locations as well as for the symmetrically-fed cases. It was found that the radiation patterns, like the impedances, were relatively insensitive to gap widths or methods of feeding, but that they depended primarily on the frequency of the source, gap location, and length/thickness ratio.

The tables for the angular and radial functions and their related constants will be included in a forthcoming report for the Office of Ordnance Research.

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