PL INVOLUTIONS OF SOME 3-MANIFOLDS

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ABSTRACT

PL INVOLUTIONS OF SOME 3-MANIFOLDS

By

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Let h_1 and h_2 be PL involutions of connected, oriented, closed 3-manifolds M_1 and M_2, respectively. Let a_1 and a_2 be fixed points of h_1 and h_2, respectively, such that near a_i the fixed point sets of h_i are of the same dimension. Taking the connected sum of M_1 and M_2 along neighborhoods of a_i, one can define a PL involution $h_1 \# h_2$ of $M_1 \# M_2$ induced by h_i. Let M_1 and M_2 now be irreducible in addition. The question that under what condition a PL involution h on $M_1 \# M_2$ is of the form $h_1 \# h_2$ and related questions are studied when M_1 and M_2 are lens spaces (not necessarily having the natural orientations). Henceforth assume that M_1 and M_2 are lens spaces. Then the main results are the following:

<u>Theorem 1</u>: Let h be a PL involution of $M_1 \# M_2$. If the fixed point set F contains an orientable surface, than F is a 2-sphere and $M_2 = - M_1$, h being the obvious involution in this case.

<u>Theorem 2</u>: Let h be a PL involution of $M_1 \# M_2$. If the fixed point set F contains a projective plane, then $M_1 = M_2$ is a projective 3-space.

The case $M_1 = M_2$ is a projective 3-space P_3 is separately studied.

<u>Theorem 3</u>: Let h be a PL involution of $P_3 \# P_3$ with 2-dimensional fixed point set F. Then F is a 2-sphere, the disjoint union of two projective planes, or the disjoint union of a Klein bottle and two points.

<u>Theorem 4</u>: In Theorem 3, if F is the disjoint union of two projective planes, then h is unique and $h = h_1 \# h_2$ where h_i is the unique involution on M_i with a projective plane and a point a_i as the fixed point set.

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INTRODUCTION

Let M be a closed, orientable 3-manifold which is the connected sum $M_1 \# M_2$ of two irreducible 3-manifolds M_1 and M_2 and let h be a PL involution of M with a fixed point set F containing a non-orientable surface F_o. Since F_o is one-sided, it would seem that h cannot interchange M_1-part and M_2-part and h must be obtained from involutions h_1 and h_2 of M_1 and M_2, respectively, by attaching two involutions along invariant neighborhoods of fixed points a_1 and a_2 of h_1 and h_2, respectively, where near a_i the fixed point sets of h_i are of the same dimension.

Fremon [2] completely determined all possible fixed point sets of a PL involution of $S^1 \times S^2$ and all PL involutions of $S^1 \times S^2$. Kwun [4] proved that no lens space except the real projective 3-space P_3 admits orientation reversing PL involutions and in case of P_3 there exists a unique PL involution up to PL equivalences.

Motivated by the fact that $S^1 \times S^2$ covers $P_3 \# P_3$, in this thesis, we consider the possibility of the above question, when M_1 and M_2 are isomorphic to lens spaces.

CHAPTER I

CODIMENSION ONE EMBEDDING OF MANIFOLDS

Let N be a connected n-manifold and M a connected $(n - 1)$-manifold. Corresponding to an embedding $M \subset N$, we have a double covering $p\colon N_1 \to N$ having the properties that (1) each component of $p^{-1}(M)$ is two sided and $p^{-1}(M)$ separates N_1, (2) $p^{-1}(M)$ is connected if and only if M is one-sided, (3) N_1 is connected if and only if M does not separate N, and (4) if $N - M$ is connected, then $N_1 - p^{-1}(M)$ has two components each of which maps homeomorphically onto $N - M$ under p. Since we use the technique of "cutting along a submanifold" to obtain N_1, we will call $p\colon N_1 \to N$ the double covering obtained from N by cutting along M.

In this chapter, we will show that no $k + 1$ non-orientable, connected $(n - 1)$-manifolds can be disjointly embedded in an orientable, connected n-manifold N whose homology group $H_1(N; Z_2)$ with coefficient Z_2 is a finitely generated group of rank k using the double covering, and examples will follow the theorem.

An embedding will mean an embedding as a closed subset. We frequently identify an embedding with its image. Suppose that a connected $(n-1)$-manifold M is embedded in a connected n-manifold N. We say that M is one-sided if M does not separate any connected neighborhood of M. Otherwise, M is two-sided. It can be shown that every embedding is one-sided if M is non-orientable and N is orientable, every embedding is two-sided if M and N are orientable, and if M is one-sided, then $N - M$ is connected.

Theorem 1.1: Suppose that N is an orientable, connected n-manifold such that its homology group $H_1(N; Z_2)$ is a finitely generated group of rank k. Then no $k+1$ non-orientable, connected $(n-1)$-manifolds can be disjointly embedded in N.

Proof: Let A_i be a non-orientable, connected $(n-1)$-manifold, $f_i\colon A_i \to N$ an embedding, and $p_i\colon N_i \to N$ the double covering obtained from N by cutting along $f_i(A_i)$. Since $f_i(A_i)$ is non-orientable, $f_i(A_i)$ is one-sided, and hence $p_i^{-1}(f_i(A_i))$ is connected. Therefore f_i cannot be lifted with respect to p_i. By the lifting theorem $f_{i_\#}\pi_1(A_i)$ is not contained in $p_{i_\#}\pi_1(N_i)$. Consequently, the composite

$$\pi_1(A_i) \xrightarrow{f_{i_\#}} \pi_1(N) \xrightarrow{g_i} \pi_1(N)/p_{i_\#}\pi_1(N_i) = Z_2$$

$(\pi_1(N)$ has different base point for different i's.)

is an epimorphism, where g_i is the projection to the quotient group. Now g_i can be factored as

$$\pi_1(N) \xrightarrow{h} H_1(N) \xrightarrow{g_i'} Z_2$$

where h is the Hurewicz homomorphism and g_i' is an epimorphism. Hence we have a commutative diagram:

$$\begin{tikzcd}
\pi_1(A_i) \arrow[r, "f_{i\#}"] \arrow[d] & \pi_1(N) \arrow[r, "g_i"] \arrow[d] & Z_2 \\
H_1(A_i) \arrow[r, "F_{i*}"] \arrow[d] & H_1(N) \arrow[ur, "g_i'"'] \arrow[d] & \\
H_1(A_i ; Z_2) \arrow[r, "f_{i*}"] & H_1(N ; Z_2) \arrow[uur, "g_i''"'] &
\end{tikzcd}$$

where $g_i'' f_{i*}$ is an epimorphism.

In order to complete the proof of the theorem, we need the following lemma:

<u>Lemma 1.2</u>: Suppose that N is an orientable, connected n-manifold such that its homology group $H_1(N; Z_2)$ is a finitely generated group of rank k, and suppose that $f_1(A_1)$, $f_2(A_2)$, ..., $f_t(A_t)$ are mutually disjoint, non-orientable, connected $(n - 1)$-manifolds embedded in N. Let K_i be the kernel of g_i'' where $g_i'' : H_1(N; Z_2) \to Z_2$ is the epimorphism which makes the following diagram commute:

$$\begin{tikzcd}
\pi_1(A_i) \arrow[r, "f_{i\#}"] \arrow[d] & \pi_1(N) \arrow[r, "g_i"] \arrow[d] & Z_2 \\
H_1(A_i) \arrow[r, "f_{i*}"] \arrow[d] & H_1(N) \arrow[ur, "g_i'"] \arrow[d] & \\
H_1(A_i;\ Z_2) \arrow[r, "f_{i*}"] & H_1(N;\ Z_2) \arrow[uur, "g_i''"'] &
\end{tikzcd}$$

Then $K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_s}$, $1 \leq i_1, i_2, \ldots, i_s \leq t$, $s \leq k$, and $i_p \neq i_q$ for $p \neq q$, is of rank $k - s$.

Proof: We use induction on s. Suppose $s = 1$. Let L_i be the image of f_{i*} and K_i the kernel of g_i''. Then $L_i \not\subset K_i$ and $L_i \neq 0$, for all i, since $g_i'' f_{i*}$ is an epimorphism. On the other hand, since $N - f_i(A_i)$ is connected, $N_i - p_i^{-1}(f_i(A_i))$ has two components each of which maps homeomorphically onto $N - f_i(A_i)$. Hence all f_j, $j \neq i$, can be lifted with respect to p_i, as $f_j(A_j) \subset N - f_i(A_i)$. By the lifting theorem, $L_j \subset K_i$ and $K_i \neq K_j$ for all $j \neq i$. Since the exact sequence

$$0 \to \ker g_i'' \to H_i(N;\ Z_2) \to Z_2 \to 0$$

splits, $H_1(N;\ Z_2) = \ker g_i'' \oplus Z_2 = K_i \oplus Z_2$. Hence the rank of K_i is $k - 1$ for all i.

Suppose that $K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_r}$, $1 \leq i_1, i_2, \ldots, i_r \leq t$, $r \leq k$, and $i_p \neq i_q$ for $p \neq q$, is of rank $k - r$. Consider the isomorphism

$$\frac{K_{i_{r+1}} \cdot (K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r})}{K_{i_{r+1}}} \cong \frac{K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r}}{K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_{r+1}}}$$

where $K_{i_{r+1}} \cdot (K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r})$ is the smallest subgroup of $H_1(N; Z_2)$ containing $K_{i_{r+1}}$ and $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r}$. Since $L_{i_{r+1}} \not\subset K_{i_{r+1}}$ but $L_{i_{r+1}} \subset K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r}$, $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r} \not\subset K_{i_{r+1}}$. Hence $K_{i_{r+1}} \cdot (K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r})$ must be of rank k. Since the rank of $K_{i_{r+1}}$ is $k - 1$, the quotient group

$$\frac{K_{i_{r+1}} \cdot (K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r})}{K_{i_{r+1}}}$$

is of rank 1. Therefore the above isomorphism and the fact that the rank of $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_r}$ is $k - r$ imply that $K_{i_1} \cap K_{i_2} \cap \dots \cap K_{i_{r+1}}$ is of rank $k - (r + 1)$. This proves the lemma.

We now return to the proof of the theorem.

Case I: $k = 1$. Since f_{1_*} is non-trivial, $H_1(N; Z_2) = L_1$ and g_1'' is an isomorphism. Hence if there were another $f_2(A_2)$ in N disjoint from $f_1(A_1)$, then $L_2 \subset K_1 = 0$, which contradicts $L_2 \neq 0$. Therefore no two non-orientable, connected $(n - 1)$-manifolds can be disjointly embedded in N.

Case II: $k \geq 2$. Suppose that there are $k + 1$ mutually disjoint, non-orientable, connected $(n - 1)$-manifolds $f_1(A_1)$, $f_2(A_2)$, ..., $f_{k+1}(A_{k+1})$ in N. Since $L_i \subset K_1 \cap K_2 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_k$ and the rank of $K_1 \cap K_2 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_k$ is 1, we have $0 \neq L_i = K_1 \cap K_2 \cap \cdots \cap K_{i-1} \cap K_{i+1} \cap \cdots \cap K_k$ for all i and $L_i \cap L_j = 0$ for all $i \neq j$. The fact that L_1 and L_2 are contained in $K_3 \cap K_4 \cap \cdots \cap K_k$, the rank of $K_3 \cap \cdots \cap K_k$ is 2, and $L_1 \cap L_2 = 0$ imply $K_3 \cap K_4 \cap \cdots \cap K_k = L_1 \oplus L_2$. Repeating this process, we obtain $K_k = L_1 \oplus L_2 \oplus \cdots \oplus L_{k-1}$. Since $L_k \not\subset K_k$ and the rank of L_k is 1, $L_k \cap K_k = 0$, which gives $H_1(N;\ Z_2) = K_k \oplus Z_2 = K_k \oplus L_k$, since K_k is of rank $k - 1$. Therefore $H_1(N;\ Z_2) = L_1 \oplus L_2 \oplus \cdots \oplus L_{k-1} \oplus L_k$, and so $L_{k+1} = 0$, which contradicts the fact that the rank of L_{k+1} is 1. This completes the proof.

Neuwirth [11] proved a stronger version in case embeddings are nice, namely, if the disjoint union of k closed, non-orientable $(n - 1)$-manifolds M^n can be semilinearly embedded in a closed, orientable n-manifold, then there exists a homomorphism of $\pi_1(M^n)$ onto the free product of k copies of Z_2.

<u>Example 1</u>: No two disjoint copies of P_2 can be embedded in P_3. This will be used later.

<u>Example 2</u>: If $H_1(N; Z_2) = 0$, then no non-orientable codimension one manifold can be embedded in N, since $H_1(N; Z_2) = 0$ implies that N is orientable.

<u>Example 3</u>: No three disjoint copies of P_2 can be embedded in $P_3 \# P_3$.

CHAPTER II

PL INVOLUTIONS OF SOME 3-MANIFOLDS

Let h_1 and h_2 be piecewise linear (PL) involutions of connected, oriented, closed 3-manifolds M_1 and M_2, respectively. Let a_1 and a_2 be fixed points of h_1 and h_2, respectively, such that near a_i the fixed point sets of h_i are of the same dimension. Taking connected sum of M_1 and M_2 along invariant neighborhoods of a_i, one can define a PL involution $h_1 \# h_2$ of $M_1 \# M_2$ induced by h_i. Let h be a PL involution of a manifold M, where M is isomorphic to the connected sum $M_1 \# M_2$ of two connected, oriented, closed, irreducible 3-manifolds, with a fixed point set F containing a non-orientable surface F_o. Since F_o is one-sided, it would seem that h cannot interchange M_1-part and M_2-part and that h must be of the form $h_1 \# h_2$. Therefore, in this chapter, we will study under what conditions a PL involution h on $M = M_1 \# M_2$ is of the form $h_1 \# h_2$ and related questions when M_1 and M_2 are isomorphic to lens spaces (not necessarily having the natural orientation).

This work was suggested by Kwun [6], who considered orientation reversing PL involutions of lens spaces, and proved that no lens space except the projective 3-space P_3

admits an orientation reversing PL involution and there exists exactly one orientation reversing PL involution on P_3 up to PL equivalences. In this case, the fixed point set is a projective plane P_2 plus a point.

Definition 2.1: The connected sum $M_1 \# M_2$ of two oriented 3-manifolds is obtained by removing the interior of a nice 3-cell from each, and then matching the resulting boundaries using an orientation reversing homeomorphism.

Definition 2.2: A manifold M is isomorphic to a Manifold M' if there is a piecewise linear, orientation preserving homeomorphism between them.

Definition 2.3: A manifold M is non-trivial if M is not isomorphic to a 3-sphere S^3.

Definition 2.4: A non-trivial manifold P is prime if there is no decomposition $P = M_1 \# M_2$ where M_1 and M_2 are non-trivial.

Milnor [10] has shown that every connected, oriented, non-trivial, compact 3-manifold M is isomorphic to a sum $P_1 \# P_2 \# \ldots \# P_k$ of prime manifolds and the summands P_i are uniquely determined up to order and isomorphism.

Definition 2.5: A 3-manifold M is irreducible if every nice 2-sphere in M bounds a 3-cell.

Milnor [10] also proved that with the exception of manifolds isomorphic to $S^1 \times S^2$ a manifold is prime if and only if it is irreducible and $S^1 \times S^2$ is prime, but it is not irreducible.

From now on, we assume that M_1 and M_2 are lens spaces and h is a PL involution on $M_1 \# M_2$ with a 2-dimensional fixed point set.

Remark: Since the fixed point set F is two-dimensional, any PL involution h on M has the property that near each point of F it maps one side of F to the other side of F. For, if this were not true, one could find a small invariant 2-sphere S near F such that $h|_S$ has a 2-cell as fixed point set. But this is impossible. Hence near each point of F, h reverses the orientation, and therefore h reverses the orientation globally.

Theorem 2.1: Let h be a PL involution of $M = M_1 \# M_2$. If the fixed point set F contains an orientable surface, then F is a 2-sphere and $M_2 = -M_1$, h being the obvious involution in this case.

Proof: We first show that if F contains an orientable surface, then F is a 2-sphere. Let S be an orientable surface contained in F. Then by the Alexander duality theorem [12], over the rationals Q

$$H_2(M_1 \# M_2 - F; Q) = H^0(F; Q) .$$

Hence F separates $M_1 \# M_2$ into two parts U and V. Since h has to interchange those two parts, $M_1 \# M_2 = 2\bar{U}$ and we have

$$\bar{U} \xrightarrow{i} M_1 \# M_2 \xrightarrow{r} \bar{U}$$

such that ri is the identity, where i is the inclusion and r is a retraction defined as follows:

$$r(x) = \begin{cases} x & \text{if } x \in \bar{U} \\ h^{-1}(x) & \text{if } x \in h(U). \end{cases}$$

Therefore we obtain the exact sequence

$$H_i(\bar{U};\ Q) \xrightarrow{i_*} H_i(M_1 \# M_2;\ Q) \xrightarrow{r_*} H_i(\bar{U};\ Q)$$

such that r_*i_* is the identity. Since $H_i(M_1 \# M_2) = 0$ for $i = 1, 2$, $H_i(\bar{U};\ Q) = 0$ for $i = 1, 2$. Therefore F must be a 2-sphere.

We now prove that $M_2 = -M_1$. Since F is a 2-sphere, F separates M. Let U and V be the two complementary domains of $M - F$. Attaching a 3-cell to each of U and V to eliminate the boundaries, we obtain two connected, orientable manifolds U' and V', and $M = U' \# V'$. By the unique decomposition theorem for 3-manifolds [10], either U' is isomorphic to a 3-sphere or U' is isomorphic to one of M_1 and M_2. But U' cannot be a 3-sphere. If it were, U would be a 3-sphere minus a 3-cell and V would be $M_1 \# M_2$ minus a 3-cell, and hence h would not be able to interchange U and V. Hence U' is isomorphic to M_1 or M_2. We may assume that U' is isomorphic to M_1. Similarly, it follows that V' is isomorphic to M_2. Since U and V must be interchanged by h, $V' = -U'$. Therefore, $M_2 = -M_1$, and h is the obvious involution. This completes the proof.

Theorem 2.2: Let M be a connected sum $M_1 \# M_2$, where M_1 and M_2 are isomorphic to lens spaces and h is a PL involution on M. If a real projective plane P_2 is contained in the fixed point set F of h, then $M = P_3 \# P_3$.

Proof: Suppose that h fixes a real projective plane A and assume that M has been triangulated so that h is simplicial and the simplicial neighborhood U of A is an invariant regular neighborhood of A. Moreover, we may assume that $h|_{U - A}$ is fixed point free. Since A is 2-dimensional, near each point of A h maps one side of A to the other side. Hence $h|_{\overline{U}}$ reverses orientation. Since U is orientable, but A is not, A is one-sided in U. Consider the double covering $p\colon M_1 \to M$ obtained from M by cutting along A. Then $p^{-1}(A)$ is connected. Therefore, $p^{-1}(A)$ is isomorphic to a 2-sphere. $p^{-1}(A)$ separates M_1 and $M_1 - p^{-1}(A)$ has two components each of which maps homeomorphically onto

$M - A$. Therefore, $p^{-1}(U)$ is a two-side collar neighborhood of $p^{-1}(A)$, and each component of its boundary, which is isomorphic to $p^{-1}(A) = S^2$, maps homeomorphically onto the boundary of U. Hence (U, A) is homeomorphic to (N, A) where N is the mapping cylinder of a double covering $S^2 \to A$.

Let U' and $(N - U)'$ be the connected manifolds obtained from U and $N - U$ by attaching a 3-cell to each. Then $M = U' \# (N - U)'$. By the unique decomposition theorem for 3-manifolds [10], U' is isomorphic to S^3, M_1, or M_2. But U' cannot be isomorphic to a 3-sphere, since the fundamental group $\pi_1(U')$ of U' is Z_2, but $\pi_1(S^3)$ is trivial. Therefore we may assume that U' is isomorphic to M_1. Now $h|_U$ can be extended to an orientation reversing PL involution h' of $U' \cong M_1$, since $U' - U$ is a 3-cell and $h|_{Bd(U)} : Bd(U) \to Bd(U)$ is fixed point free, and hence $h'|_{U' - U}$ can be defined by the cone over $h|_{Bd(U)}$. Since no lens space except the projective 3-space P_3 admits an orientation reversing PL involution, $U' = M_1$ must be P_3.

From $U' = M_1$ and the unique decomposition theorem, we get $(N - U)' = M_2$. Since $(N - U)' - (N - U)$ is a 3-cell and $h|_{Bd(N - U)}$ is fixed point free, letting $h''|_{(N - U)' - (N - U)}$ be the cone over $h|_{Bd(N - U)}$, $h|_{N - U}$ can be extended to a PL involution h'' of $(N - U)'$, and moreover h'' is orientation reversing, since

$h|_{Bd(N - U)}$ reverses orientation. Therefore $(N - U)'$ $= M_2$ must be P_3. Hence $M_1 = M_2 =$ a real projective 3-space P_3. This proves the theorem.

Henceforth assume that M_i, $i = 1, 2$, is a manifold isomorphic to a projective 3-space P_3 and h is a PL involution on $P_3 \# P_3$ with a 2-dimensional fixed point set. Since the case where a 2-dimensional component is an orientable surface has been taken care of, we have only to consider the case where each 2-dimensional component is non-orientable.

<u>Lemma 2.3</u>: Let h be a PL involution of $P_3 \# P_3$. Then there exists a PL involution $h'\colon S^1 \times S^2 \to S^1 \times S^2$ such that the following diagram commutes

$$\begin{tikzcd} S^1 \times S^2 \arrow[r, dash, "h'"] \arrow[d, dash, "p'"] & S^1 \times S^2 \arrow[d, dash, "p'"] \\ P_3 \# P_3 \arrow[r, dash, "h"] & P_3 \# P_3 \end{tikzcd}$$

where $p'\colon S^1 \times S^2 \to P_3 \# P_3$ is a 4-to-1 covering projection.

<u>Proof</u>: Consider the covering space $S^1 \times S^2$ of $P_3 \# P_3$ and the usual 2-to-1 covering map $p\colon S^1 \times S^2 \to P_3 \# P_3$. Let $H = p_{\#}\pi_1(S^1 \times S^2)$ and $G = \pi_1(P_3 \# P_3)$. Then the index $[G\colon H]$ of G modulo H is 2, since p is 2-to-1. Suppose $h_{\#}H \neq H$. Since $[G\colon H] = [G\colon h_{\#}H]$ $= 2$, neither $h_{\#}H$ contains H nor H contains $h_{\#}H$, and

moreover H and $h_{\#}H$ are normal subgroup of G. Let $L = H \cap h_{\#}H$. Then L is a normal subgroup of G, since L is the intersection of two normal subgroups, and $[G: L] = [G: H][H: L]$. We now show that $[G: L] = 4$. Let $H' = h_{\#}H$. Clearly $HH' = G$ and $H/L = H/H \cap H' = HH'/H'$ $= G/H' = Z_2$, which shows $[H: L] = 2$. Consequently $[G: L]$ $= 4$. Furthermore, $h_{\#}L = h_{\#}(H \cap H') = H' \cap H = L$. Hence by the lifting theorem, there is a PL involution h' on $S^1 \times S^2$ such that $p'h' = hp'$, where $p'\colon S^1 \times S^2 \to P_3 \# P_3$ is a 4-to-1 covering projection. This proves the lemma.

We now show that the possible 2-dimensional non-orientable surfaces which can be fixed under a PL involution of $P_3 \# P_3$ are a projective plane and a Klein bottle. Consider S^n as a suspension of S^{n-1} and let k_n be the simplicial involution of S^n that leaves S^{n-1} pointwise fixed and interchanges the suspension vertices, and define two involutions h_1 and h_2 of $S^1 \times S^2$ by

$$h_1(x, y) = (k_1(x), y), \; h_2(x, y) = (x, k_2(y)) .$$

Kwun [5] considered PL involutions of $S^1 \times S^2$ and proved the following two theorems:

<u>Theorem</u>: Let h be a PL involution of $S^1 \times S^2$ with homogeneously two dimensional fixed point set F. If F is not connected, then h is PL equivalent to h_1.

Theorem: Let h be a PL involution of $S^1 \times S^2$ with 2-dimensional connected fixed point set F and orientable orbit space. Then h is PL equivalent to h_2.

Fremon [2] completed the work and proved that all possible fixed point sets of a PL involution of $S^1 \times S^2$ are $S^2 \cup S^2$, $S^1 \times S^1$, S^2 plus two points, and a Klein bottle. Hence by the Lemma 2.3 we obtain:

Lemma 2.4: Let N be the connected sum $P_3 \# P_3$ and h a PL involution of $P_3 \# P_3$. Then all possible 2-dimensional, non-orientable components of a fixed point set F of h are a Projective plane and a Klein bottle.

We shall first consider the case that a projective plane is fixed and show that, in this case, the fixed point set F is a disjoint union of two projective planes and h is uniquely determined.

Theorem 2.5: Let N be the connected sum $P_3 \# P_3$ and h a PL involution of N. If a projective plane is contained in the fixed point set F, then F is the disjoint union of two projective planes.

Proof: Suppose that a projective plane A is contained in F and assume that the simplicial triangulation of N is such that h is simplicial, the simplicial neighborhood U of h is an invariant regular neighborhood of A, and $h|_{U - A}$ is fixed point free. Then we have seen that U is homeomorphic to P_3 minus a 3-cell and $N - U$ is homeomorphic to P_3 minus the interior of a 3-cell in Theorem 2.2. Moreover, we have seen that $h|_{N-U}$ can be extended to an orientation reversing PL involution h' of P_3 obtained from $N - U$ by attaching a 3-cell. Let F' be the fixed point set of h'. Then by the parity theorem and the Lefschetz fixed point formula, the dimension of F' is either 0 or 2. But by work of Kwun [6], the dimension of F' cannot be 0, and F' is the disjoint union of a projective plane and a point p. By the way we extended h to h', the cone vertex must be an isolated fixed point. Hence p is the cone vertex, and p lies in the 3-cell attached to $N - U$ to obtain a P_3. Therefore $h|_{N-U}$ has a projective plane as the fixed point set. Using the same argument for $h|_U$ as above, we obtain that F is the disjoint union of two projective planes. This completes the proof.

<u>Theorem 2.6</u>: Let N be the connected sum $P_3 \# P_3$ and h a PL involution of N. If the fixed point set F of h contains a projective plane, then h is unique and h is of the form $h_1 \# h_2$, where h_i is the unique PL involution on P_3 with a projective plane and a point as the fixed point set.

<u>Proof</u>: By Theorem 2.5, F is the disjoint union of two projective planes. Let A and A' be two projective

planes whose union is F. We may assume that h is simplicial and simplicial neighborhood U and U' are invariant regular neighborhoods of A and A', respectively. Furthermore, assume that U is disjoint from U'. Then U and U' are isomorphic to P_3 minus a 3-cell, and by the unique decomposition theorem for 3-manifolds, $N - (U \cup U')$ is isomorphic to S^3 - two open 3-cells $= S^2 \times [0, 1]$ such that $h(S^2 \times i) = S^2 \times i$ for $i = 0, 1$, $h|_{N - (U \cup U')}$ is fixed point free, and $h(N - (U \cup U')) = N - (U \cup U')$. Work of Livesay [7] shows that if f is any fixed point free involution on $S^2 \times [0, 1]$ satisfying $f(S^2 \times i) = S^2 \times i$ for $i = 0, 1$, then there exists a homeomorphism $t: S^2 \times [0, 1] \to S^2 \times [0, 1]$ such that $tft^{-1} = g$, where g is the involution on $S^2 \times [0, 1]$ defined by $g(x, t) = (-x, t)$ for $x \in S^2$, $t \in [0, 1]$ and $-x$ denotes the antipode of x. That is, there is a unique involution g on $S^2 \times [0, 1]$ up to PL equivalences. Hence it suffices to analyse $h|_U$ and $h|_{U'}$.

We now analyse $h|_U$ as in Kwun [6]. Let O be the orbit space and let $f: \overline{U} \to O$ be the orbit map. Then O is a compact 3-manifold and the boundary components are $f(Bd(U))$ and $f(A)$. Let V be a regular neighborhood of $f(A)$ in O disjoint from $f(Bd(U))$. Let W be $f^{-1}(V)$. Then W is a neighborhood of A in U and f is 2-to-1 except on $f(A)$, since h is fixed point free on $\overline{U} - A$. We triangulate W in such a way that for each simplex s in V, there corresponds two copies of s in W. By collapsing two corresponding simplices as we collapse s, we can collapse W to A. Hence W is a regular neighborhood of A in U, which is disjoint from $Bd(U)$. Therefore $Cl(U - W)$ is homeomorphic to $S^2 \times [0, 1]$ on which h is fixed point free. By Livesay [8], the orbit space of $h|_{Cl(U - W)}$ is homeomorphic to $P_2 \times [0, 1]$ with $P_2 \times 1$ as $f(Bd(U))$. Since V is a collar of $f(A) = P_2$, V is homeomorphic to $P_2 \times [0, 1]$. Therefore O is homeomorphic to $P_2 \times [0, 1]$ such that $P_2 \times 0$ and $P_2 \times 1$ correspond to $f(A)$ and $f(Bd(U))$, respectively. We will construct a PL involution on U and show that for any two PL involutions h_1 and h_2 on U, there exists a PL homeomorphism $t\colon U \to U$ such that $h_1 = t^{-1}h_2t$. Consider a PL double covering $j\colon Bd(U) = S^2 \to A$. Then there exists only one non-trivial covering transformation g of j. Let $h|_{Bd(U)}$ be the covering transformation g. Then U is a mapping cylinder of j. Hence g can induce a PL involution on this mapping cylinder in the obvious way. Let h_1 and h_2 be any two such PL involutions on U and let g_1 and g_2 be orbit maps: $U \to P_2 \times [0, 1]$ corresponding to h_1 and h_2, respectively, with $P_2 \times 0$ corresponding to the fixed point set A. Since $g_i|_{U - A}$ is a universal covering, there exists a PL homeomorphism t from $U - A$ to $U - A$ such that $g_1 = g_2t$. Then t can be uniquely extended to a PL homeomorphism $t\colon U \to U$ such that $g_1 = g_2 t$. But $th_1 = h_2 t$ as t respects covering translation. This is true on $U - A$ and by continuity this is also true on U. Hence $h_1 = t^{-1}h_2 t$ and therefore there exists a unique PL involution of U up to PL equivalences.

The same argument applied to $h|_{U'}$ shows that there exists exactly one PL involution on U' up to PL equivalences and $h|_{U'}$ is exactly the same type of involution as $h|_U$. Therefore there exists a unique PL involution on N with a projective plane in the fixed point set.

Now $h = h_1 \# h_2$, where h_i is the unique involution on P_3 with a projective plane and a point as the fixed point set, since $h_1|_U = h|_U$, $h_2|_{U'} = h|_{U'}$, and $h_1|_{P_3 - U}$ $(h_2|_{P_3 - U'})$ is the cone over $h_1|_{Bd(U)}$ $(h_2|_{Bd(U')})$. This completes the proof.

From now on we shall consider the case that F contains a Klein bottle and shall show that F is the disjoint union of a Klein bottle and two points. To prove this we need following lemmas:

<u>Lemma 2.7</u>: Let N be the connected sum $P_3 \# P_3$ and let K be a Klein bottle contained in N and U a regular neighborhood of K in N. Then $\pi_1(N - U) = Z$.

Proof: Suppose that U is a regular neighborhood of K in $P_3 \# P_3$. Consider the double covering $p_1: N_1 \to N$ obtained from N by cutting along K. Then $p_1^{-1}(K)$ is connected and separates N_1 into two components. Since N_1 is orientable, $p_1^{-1}(K)$ is also orientable, otherwise $p_1^{-1}(K)$ cannot separate N_1. $p_1|_{p_1^{-1}(K)}: p_1^{-1}(K) \to K$ is 2-to-1. Hence $p_1^{-1}(K)$ must be homeomorphic to $S^1 \times S^1$. Now consider $p_1^{-1}(U)$. $p_1^{-1}(U)$ is a regular neighborhood of $p_1^{-1}(K)$, since each component of $N_1 - p_1^{-1}(K)$ maps homeomorphically onto $N - K$, and hence $p_1^{-1}(U)$ is a collar of $p_1^{-1}(K)$. Hence each component of the boundary of $p_1^{-1}(U)$ is homeomorphic to $S^1 \times S^1$ and p_1 maps $S^1 \times S^1$ homeomorphically onto the boundary of U. Therefore $Bd(U)$ is homeomorphic to $S^1 \times S^1$. By the Meyer-Vietoris sequence

$$H_2(N) \to H_1(S^1 \times S^1) \xrightarrow{f} H_1(\overline{U}) \oplus H_1(N - U) \xrightarrow{g} H_1(N) \to 0$$

we obtain

$$0 \to Z \oplus Z \xrightarrow{f} (Z + Z_2) \oplus H_1(N - U) \xrightarrow{g} Z_2 + Z_2 \to 0$$

where f is one-to-one and the kernel of g is $f(Z \oplus Z)$. Therefore $H_1(N - U)$ is a group of rank 1.

$\pi_1(N - U)$ is abelian. For, since $S^1 \times S^2$ covers N in 2-to-1 fashion, $S^1 \times S^2$ - a Klein bottle, which is homeomorphic to $R^2 \times S^1$, covers $N - U$ 2-to-1. Hence we obtain an exact sequence

$$0 \to Z \xrightarrow{f} \pi_1(N - U) \xrightarrow{g} Z_2 \to 0$$

Hence if we choose $a \in \pi_1(N - U)$ such that $g(a) \neq 0$ and a generator b in Z, then $\pi_1(N - U)$ is generated by a and b. Suppose $aba^{-1} = b^{-1}$. We abelianize it. Then $1 = aba^{-1}b^{-1} = (aba^{-1})b^{-1} = b^{-1}b^{-1} = (b^{-1})^2$. Hence $b^2 = 1$. And a is not of infinite order either, since $g(a^2) = g(a)g(a) = 1$ implies that $a^2 \in Z$, and therefore $a^2 = 1$ or $a^4 = 1$ in $H_1(N - U)$. Therefore $H_1(N - U)$ is finite. This contradicts the fact that the rank of $H_1(N - U)$ is 1. Hence $\pi_1(N - U)$ is abelian, which implies

$$\pi_1(N - U) = Z + \text{Torsion part.}$$

Consider the covering space $S^1 \times S^2$ of N. Since $S^1 \times S^2$ minus a Klein bottle is homeomorphic to $R^2 \times S^1$, we have a universal covering

$$R^2 \times R^1 \to R^2 \times S^1 \to N - U.$$

But no non-trivial finite group can act freely on a finite dimensional, contractible space. Hence $\pi_1(N - U) = Z$. This proves the lemma.

<u>Lemma 2.8</u>: Let N, K, and U be the same as in the Lemma 2.7. Then $N - U$ is irreducible and is homeomorphic to $D^2 \times S^1$.

Proof: Suppose $N - U$ is not irreducible. Then there exists a nice 2-sphere which does not bound a 3-cell. But every nice 2-sphere in $N - U$ is a bounding 2-sphere, and actually every bounding 2-sphere bounds a 3-cell. Let S be any nice 2-sphere in $N - U$, and consider the following diagram:

$$\begin{tikzcd} & R^2 \times S^1 \arrow[d, "p"] \arrow[r, phantom, "\subset"] & R^3 \arrow[r, phantom, "\subset"] & S^3 \\ S \arrow[ur, dashed, "\bar{i}"] \arrow[r, "i"] & N - U \end{tikzcd}$$

Since $\pi_1(S) = 0$, by the lifting theorem there exists $\bar{i}\colon S \to R^2 \times S^1$ such that $i = p\bar{i}$. S separates $N - U$ into two parts W and V, and

$$\pi_1(W \cup V) = \frac{\pi_1(W) * \pi_1(V)}{\pi_1(W \cap V)} = \frac{\pi_1(W) * \pi_1(V)}{\pi_1(S)} = \pi_1(W) * {}_1(V).$$

But by Lemma 2.7, $\pi_1(W \cup V) = \pi_1(N - U) = Z$. Hence $\pi_1(W) = 0$, which implies that $\bar{i}(S)$ consists of disjoint copies of S in $R^2 \times S^1$. Then it can be embedded in S^3. By the theorem of Alexander [1], it bounds a 3-cell and therefore S bounds a 3-cell. Hence $N - U$ is irreducible.

Since $N - U$ is an orientable, irreducible, compact 3-manifold with the fundamental group isomorphic to Z and the boundary is homeomorphic to $S^1 \times S^1$, $N - U$ is homeomorphic to $D^2 \times S^1$. This completes the proof.

<u>Lemma 2.9</u>: Consider $S^1 \times S^1$ as the boundary of $D^2 \times S^1$. Let $f: S^1 \times S^1 \to S^1 \times S^1$ be a homeomorphism such that $f_*: \pi_1(S^1 \times S^1) \to \pi_1(S^1 \times S^1)$ is presented by

$$\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}$$

using canonical generators for $\pi_1(S^1 \times S^1)$. Then f may be extended to a homeomorphism $\bar{f}: D^2 \times S^1 \to D^2 \times S^1$.

<u>Proof</u>: By [9], isotopy classes of homeomorphisms of $S^1 \times S^1$ are precisely isomorphism classes of $\pi_1(S^1 \times S^1, *)$ (We disregard base points as $\pi_1(S^1 \times S^1)$ is abelian.). Since extendability is an isotopy invariant, we may suppose that $f(e^{2\pi it}, e^{2\pi is}) = (e^{2\pi i(t + qs)}, e^{2\pi is})$. Define $\bar{f}(e^{2\pi it}, \rho e^{2\pi is}) = (e^{2\pi i(t + qs)}, \rho e^{2\pi is})$. This proves the lemma.

<u>Theorem 2.10</u>: Let N be the connected sum $P_3 \# P_3$ and let h be a PL involution of N. If a Klein bottle K is contained in the fixed point set F of h, then F is the disjoint union of a Klein bottle and two points.

<u>Proof</u>: Let U be an invariant regular neighborhood of K in N. Then by Lemma 2.8, $N - U$ is homeomorphic to $D^2 \times S^1$. Denote $h|_{D^2 \times S^1} = h'$. Then either the fixed point set $F_{h'}$ of h' is of dimension 0 or 2, or h' is fixed point free.

We shall show that, in the case that dimension of $F_{h'}$ is 0, the number of fixed points is 2 and shall rule out the case the dimension of $F_{h'}$ is 2 and the case h' is fixed point free.

Case I): $\dim F_{h'} = 0$. Suppose that h' fixes $x_1, x_2, \ldots, x_k$ in $D^2 \times S^1$ and no other point. We may assume that h' is simplicial with x_i as vertices and that closed stars of x_i are mutually disjoint. Let M be obtained from $D^2 \times S^1$ by removing open stars of x_i. Then $h'' = h|_M$ is a free involution on M reversing orientation of each boundary component of M. Then the Lefschetz number of h'' is $1 + 1 - k + 0 = 0$. Hence $k = 2$.

Case II): $\dim F_{h'} = 2$. Consider the double covering $p\colon S^1 \times S^2 \to P_3 \# P_3$. Since $F_{h'} \cap \mathrm{Bd}(S^1 \times D^2) = \emptyset$, $F_{h'}$ is contained in the interior of $S^1 \times D^2$, and hence $P^{-1}(F_{h'})$ is disjoint union of two 2-dimensional components and $p^{-1}(F_{h'})$ must be a fixed point set of a PL involution of $S^1 \times S^2$. By Fremon [2] $p^{-1}(F_{h'})$ is the disjoint union of two 2-spheres, which implies that $F_{h'}$ is a 2-sphere. Since $S^1 \times D^2$ is irreducible by Lemma 2.9, $F_{h'}$ bounds a 3-cell. But the other side of $F_{h'}$ cannot be a 3-cell, and therefore h' will not be able to interchange those two parts. Hence the dimension of $F_{h'}$ cannot be 2.

Now in order to rule out the case $F_{h'}$ is fixed point free, we need to see how the boundary $S^1 \times S^1$ of U is attached to the boundary $S^1 \times S^1$ of $S^1 \times D^2 = N - U$. Hence let $f: S^1 \times S^1 \to S^1 \times S^1$ be the attaching map and let a and b be the canonical generators of $\pi_1(S^1 \times S^1)$ of $S^1 \times S^1$ covering K and α and β the canonical generators of $\pi_1(S^1 \times S^1)$ of $S^1 \times S^1$ contained in $S^1 \times D^2$. Let $p: S^1 \times S^1 \to K$ be the covering projection and $i: S^1 \times S^1 \to S^1 \times D^2$ the inclusion. Suppose $f_\#(a) = p\alpha + q\beta$ and $f_\#(b) = r\alpha + s\beta$. We may assume that the determinant

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 1 .$$

Since the covering projection p takes a to a and b to b^2 with relation $bab^{-1}a = 1$, and the inclusion i takes α to c and β to 1, we have

$$Z_2 * Z_2 = \pi_1(P_3 \# P_3) = \{a, b, c \mid bab^{-1}a = 1, a = c^p, b^2 = c^r\}$$

with generators a, b, c and relations $bab^{-1}a = 1$, $a = c^p$, and $b^2 = c^r$. Since $bc^pb^{-1}c^p = 1$ and $b^{2p} = c^{rp}$, $c^{rp} = b(b^{2p})b^{-1} = b(c^p)^rb^{-1} = (bc^pb^{-1})^r = (c^{-p})^r = c^{-rp}$. Hence $c^{2rp} = 1$.

Case (i): The order of c is finite. Since $a = c^p$, the order of a is also finite. But actually the order of a is infinite. For, consider the subgroup $\{a\}$ generated by a. Since $bab^{-1} = a^{-1}$ and $a = c^p$ commutes with c, $\{a\}$ is a normal subgroup of $Z_2 * Z_2$. Therefore we obtain

$$0 \to \{a\} \xrightarrow{f} Z_2 * Z_2 \xrightarrow{g} Z_2 * Z_2/\{a\} \to 0$$

which is exact. Hence $Z_2 * Z_2/\{a\}$ is presented by

$$\{a, b, c \mid bab^{-1}a = 1,\ a = c^p,\ b^2 = c^r\ ,\ a = 1\}$$
$$= \{b, c \mid bb^{-1} = 1,\ c^p = 1,\ b^2 = c^r\}$$
$$= \{b, c \mid c^p = 1,\ b^2 = c^r\}\ .$$

Since $ps - rq = 1$, $c^{ps}c^{-rq} = c$. On the other hand, $c^{ps}c^{-rq} = (c^p)^s c^{-rq} = c^{-rq} = (c^r)^{-q} = (b^2)^{-q} = b^{-2q}$. Hence $b^{-2q} = c$. Therefore $Z_2 * Z_2/\{a\}$ is generated by $\bar{c}$, where $\bar{c}$ is the image of c under the projection. Since $Z_2 * Z_2 \xrightarrow{g} \bar{c} = Z_2 * Z_2/\{a\}$ can be factored through $Z_2 + Z_2$, we obtain

$$\begin{tikzcd}
Z_2 * Z_2 \arrow[rr, "g"] \arrow[dr] & & Z_2 * Z_2/\{a\} = \{\bar{c}\} \\
& Z_2 + Z_2 \arrow[ur, "g'"'] &
\end{tikzcd}$$

where g' is onto. Hence $\{\bar{c}\} = 0$ or Z_2. But $\{\bar{c}\}$ cannot be trivial, since $Z_2 * Z_2 \neq \{a\}$, which implies $\{\bar{c}\} = Z_2$ is finite. Consequently, $\{a\}$ must be infinite, which shows that the order of a is infinite. Therefore c cannot be of finite order.

Case (ii): The order of c is infinite. Since $c^{2rp} = 1$, $rp = 0$, which implies either $r = 0$ or $p = 0$. But p cannot be 0. For, if $p = 0$, then we would have

$$Z_2 * Z_2 = \{b, c \mid b^2 = c^r\}.$$

Hence we could define a homomorphism $\gamma\colon Z_2 * Z_2 \to Z$ as follows:

$$\gamma(b) = r \quad \text{and} \quad \gamma(c) = 2,$$

which is impossible. Therefore $r = 0$, and consequently we obtain that $p = s = \pm 1$. We may further assume that $p = s = 1$, and have that $f_\#$ is presented by

$$\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}.$$

Considering $S^1 \times S^1$ as the double covering space of K and t as the non-trivial covering transformation of the covering projection, $t_\#\colon \pi_1(S^1 \times S^1) \to \pi_1(S^1 \times S^1)$ is presented by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and we have the following commutative diagram

$$\begin{array}{ccc}
\pi_1(S^1 \times S^1) & \xrightarrow{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} & \pi_1(S^1 \times S^1) \\
{\scriptstyle \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}} \\
\pi_1(S^1 \times S^1) & \xrightarrow[h']{} & \pi_1(S^1 \times S^1)
\end{array}$$

where $h'\colon S^1 \times S^1 \to S^1 \times S^1$ is a PL involution of $S^1 \times S^1$ considered as the boundary of $S^1 \times D^2$. Consequently, $h'_{\#}$ is presented by

$$\begin{bmatrix} -1 & -2q \\ 0 & 1 \end{bmatrix}$$

and by Lemma 2.9, h' can be extended to a PL involution h' on $S^1 \times D^2$, and $h'_{\#}\colon \pi_1(S^1 \times D^2) \to \pi_1(S^1 \times D^2)$ sends 1 to -1, which implies that the Lefschetz number is not 0. Therefore h' cannot be fixed point free.

Hence F is the disjoint union of a Klein bottle and two points. This proves the theorem.

<u>Remark</u>: The uniqueness question for h in case F is the disjoint union of a Klein bottle and two points is not settled. If h is unique, then $h = h_1 \# h_2$.

For, let F_{h_i} be the disjoint union of a projective plane A_i and a point p_i which is the fixed point set of h_i. Taking the connected sum $P_3 \# P_3$ along invariant neighborhood of a_i, where $a_i \in A_i$, we obtain a PL involution $h_1 \# h_2$ whose fixed point set $F_{h_1 \# h_2}$ is the disjoint union of a Klein bottle and two points $(A_1 \# A_2 \cup \{p_1, p_2\})$.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] J. W. Alexander, On the subdivision of 3-space by a polyhedron, Proc. Nat. Acad. Sci. U.S.A. 10(1924), 6-8.

[2] R. L. Fremon, Finite cyclic group actions on $S^1 \times S^n$, Thesis, Michigan State University, 1969.

[3] M. Hall, Jr., The theory of groups, Macmillan, New York, 1959.

[4] K. W. Kwun, Non existence of orientation reversing involutions on some manifolds, Proc. Amer. Math. Soc. 23(1969), 725-726.

[5] ________, Piecewise linear involutions of $S^1 \times S^2$, Mich. Math. J. 16(1969), 93-96.

[6] ________, Scarcity of orientation reversing PL involutions of lens spaces, to appear.

[7] G. R. Livesay, Fixed point free involutions on the 3-sphere, Ann. of Math. 72(1960), 603-611.

[8] ________, Involutions with two fixed points on the three sphere, Ann. of Math. 78(1963), 582-593.

[9] W. Mangler, Die Klassen von topologischen Abbildungen einer geschlossenen Flache auf sich, Math. Z. 44 (1939), 541-554.

[10] J. Milnor, A unique decomposition theorem for 3-manifolds, Amer. J. of Math. 84(1962), 1-7.

[11] L. Neuwirth, In groups and imbeddings of n-compleces in (n + 1)-manifolds, Bull. Amer. Math. Soc. (1964), 737-738.

[12] E. H. Spanier, Algebraic topology, McGraw-Hill, New York, 1966.