FLEXIBLE LIE-ADMISSIBLE ALGEBRAS

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ABSTRACT

FLEXIBLE LIE-ADMISSIBLE ALGEBRAS

By

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The purpose of this thesis is a study of noncommutative, flexible nilalgebras. For an algebra A, the algebra A^- is defined as the same vector space as A, but with a multiplication given by $[x,y] = xy - yx$ for all x and y in A. An algebra A is said to be <u>Lie-admissible</u> if A^- is a Lie algebra.

Assume that A is a finite dimensional, flexible, Lie-admissible algebra over a field F of characteristic $\neq 2$. Let A_α denote the root space of A^- for a split Cartan subalgebra H of A^- corresponding to the root α. The main results center around the following theorems:

<u>Theorem 1</u>: Suppose that A^- has a split, abelian Cartan subalgebra H which is a nil A-subalgebra. If A_α is one-dimensional for every $\alpha \neq 0$ and the center of A^- is 0, then A is a Lie algebra isomorphic to A^-.

<u>Theorem 2</u>: Let F be of characteristic 0 and the solvable radical N of A^- be nilpotent. Then a Levi-factor S of A^- is an ideal of A if and only if S has a split Cartan subalgebra H being a nil A-subalgebra and $[N,H] = 0$.

Theorem 3: Suppose that A is power-associative and that A^- is reductive such that there exists a split Cartan subalgebra H of A^- with $h^3 = 0$ for all h in H. Then the center of A^- is an ideal of A.

Theorem 4: If A is simple and $x^3 = 0$ for all x in A, then A is a Lie algebra.

Theorem 5: Let F be of characteristic 0. If A is a central simple nilalgebra of dimension ≤ 4, then A is either commutative or a Lie algebra, and in the latter case A is the 3-dimensional simple Lie algebra.

Theorem 6: If A is a noncommutative nilalgebra of dimension ≤ 4 such that A^- is nilpotent, then A is nilpotent such that all products of any 4 elements in A are 0.

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INTRODUCTION

In 1948, A. A. Albert [1] first introduced flexible Lie-admissible algebras. He also defined Jordan-admissible algebras and proved that any finite dimensional, flexible algebra A such that A^+ is a simple Jordan algebra is either quasi-associative or a Jordan algebra. Albert proposed to investigate an analogous problem for flexible Lie-admissible algebras. In 1962, Laufer and Tomber [9] partially solved this problem by showing that if A is a finite dimensional, flexible, power-associative algebra over an algebraically closed field of characteristic 0 such that A^- is a simple Lie algebra, then A is a Lie algebra isomorphic to A^-. In Chapter I, we extend this result to the case of Rad $A^- \neq 0$ as well as generalize the result of Laufer and Tomber.

Although a Levi-factor S of A^- may be an ideal of A^-, S is not an ideal of A even in case A is a nilalgebra. In terms of a Cartan subalgebra of S, we give a necessary and sufficient condition that the Levi-factor S of A^- be an ideal of A when Rad A^- is nilpotent. This result provides some interesting applications to the case that A^- is reductive.

The class of flexible algebras includes all commutative and anticommutative algebras, in particular, Jordan and Lie algebras, and alternative algebras, as well as the generalized standard algebras of R. Schafer [13]. The generalized standard algebras generalize both alternative algebras and standard algebras of Albert [1]. Any standard algebra is flexible Lie-admissible. Schafer [13] proved that any generalized standard nilalgebra is nilpotent and hence there is no simple generalized standard nilalgebra. Simple Lie algebras give examples that being nil does not imply nilpotency for flexible Lie-admissible algebras. If a simple, flexible algebra A is not nil, then various structures are known, while very few results are known if A is nil. If A^- is a simple Lie algebra, then it is shown in Oehmke [10] that A is a simple nilalgebra, and in this case A is a Lie algebra. At the present time, the only known simple, flexible, Lie-admissible nilalgebras are Lie algebras. Since any commutative algebra is trivially Lie-admissible, these remarks suggest the conjecture that any simple, flexible, Lie-admissible nilalgebra is either a Lie algebra or commutative. In Chapter II, we treat in part this conjecture and deal with certain nilalgebras whose minus-algebras are nilpotent.

In Chapter III, by using the known structures of Lie algebras of dimension 4, we determine all 4-dimensional, flexible, Lie-admissible nilalgebras whose minus -algebras are not non-nilpotent, solvable Lie algebras. This enables us to show that any noncommutative, flexible nilalgebra A of dimension ≤ 4 such that A^- is a nilpotent Lie algebra is nilpotent.

CHAPTER I

GENERAL RESULTS IN CASE Rad $A^- \neq 0$

Flexible Lie-admissible algebras were first introduced by A. A. Albert in [1]. As an analog to the result that Albert has proved for a flexible algebra A such that A^+ is a simple Jordan algebra, in [9] Laufer and Tomber proved that if A is a finite dimensional, flexible, power-associative algebra over an algebraically closed field of characteristic 0 such that A^- is a simple Lie algebra, then A is a Lie algebra isomorphic to A^-. In this case it is shown that A is a simple nilalgebra. The proof mostly depends on the structure of the simple Lie algebra A^-. In this chapter we generalize the result of Laufer and Tomber in point of view of the structure of Lie algebra A^- and obtain some general results in case that A^- has non-zero radical.

We briefly introduce the standard terminologies, definitions, and notations in nonassociative algebras.

For an algebra A, the algebra A^- is defined as the same vector space as A, but with a multiplication given by $[x,y] = xy - yx$ where juxtaposition denotes multiplication in A. An algebra A is said to be <u>Lie-admissible</u> if A^- is a Lie algebra, that is, (1.1) $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$

and

$$[x,y] + [y,x] = 0$$

for all x, y, z in A. It is shown in [1] that an algebra A is Lie-admissible if and only if $R_{[x,y]} = [R_x - L_x, R_y - L_y]$ where R_a and L_a are right and left multiplications by a in the algebra A.

An algebra A is called <u>flexible</u> if it satisfies the flexible law

(1.2) $(xy)x = x(yx)$,

or equivalently

(1.3) $L_xR_x = R_xL_x$

for all x, y in A. The linearizations of (1.2) and (1.3) implies

(1.4) $(xy)z + (zy)x = x(yz) + z(yx)$

and

(1.5) $L_{xy} - L_yL_x = R_{yx} - R_yR_x$,

respectively. In [1] Albert has shown that an algebra is a flexible Lie-admissible algebra if and only if

(1.6) $R_{[x,y]} = [R_x, R_y - L_y]$.

If we denote the linear transformation $R_x - L_x$ by $D_x = D(x)$, then (1.6) may be written as $R_{xD(y)} = [R_x, D_y]$ or in a flexible Lie-admissible algebra D_y is a derivation.

An algebra A is said to be <u>power-associative</u> if the subalgebra generated by x is associative for every x in A. It is also shown in [1] that a flexible algebra A of characteristic $\neq 2,3,5$ is power-associative if and only if $x^2x^2 = x^3x$ for all x in A. An element x in a power-associative algebra A is called nilpotent in case there is a positive integer n such that $x^n = 0$. A power-associative algebra is called nil if every element is nil-potent. An algebra A is called simple in case A has no proper ideals and $A^2 \neq 0$.

In this chapter we will make extensive use of the standard terminologies and the known results of Lie algebras as found, for example, in Chevalley [5], Jacobson [8], and Seligman [14].

We first prove a sequence of lemmas which will be used throughout this thesis. In this chapter we assume that A denotes a finite dimensional, flexible, Lie-admissible al-gebra over a field F.

Lemma 1.1: Let H be a split Cartan subalgebra of A^- and A_α the root space of A^- for H corresponding to the root α. Then

$$A_\alpha A_\beta \subseteq \begin{cases} 0 & \text{if } \alpha + \beta \text{ is not a root;} \\ A_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root.} \end{cases}$$

In particular, H is a subalgebra of A.

Proof: Let D be a derivation of A such that the characteristic roots of D belong to the base field F. Let λ be a characteristic root of D and $A_\lambda = \{x \in A \mid x(D - \lambda I)^k = 0$ for some $k\}$. Then it is known that

$$A_\lambda A_\mu \subseteq \begin{cases} 0 & \text{if } \lambda + \mu \text{ is not a characteristic root;} \\ A_{\lambda+\mu} & \text{if } \lambda + \mu \text{ is a characteristic root.} \end{cases}$$

Since D_h is a derivation of A for every h in H and $A_\alpha = \bigcap_{h \in H} A_{\alpha(h)}$, $A_\alpha A_\beta \subseteq A_{\alpha(h)} A_{\beta(h)} \subseteq A_{\alpha(h) + \beta(h)} = A_{(\alpha+\beta)(h)}$ for all h in H and hence $A_\alpha A_\beta \subseteq \bigcap_{h \in H} A_{(\alpha+\beta)(h)} = A_{\alpha+\beta}$. Since H is the root space corresponding to the root 0, this in particular implies $HH \subseteq H$. This proves the lemma.

In the following lemmas we assume that the characteristic of F is not 2. A_α denotes the root space of A^- for a split Cartan subalgebra H corresponding to the root α and $F[x]$ the subalgebra of A generated by x in A.

Lemma 1.2: Let H be a split Cartan subalgebra of A^- and h an element of H such that D_y is a scalar on A_α, $\alpha \neq 0$, for all y in $F[h]$. If $h^3 = 0$, then $[A_\alpha, h^2] = 0$ for all $\alpha \neq 0$.

Proof: Let α be any non-zero root of A^- for H. Let $D_h = R_h - L_h = \lambda E$ and $D_{h^2} = \mu E$ on A_α for some λ, μ in F, where E is the identity map on A_α. Since $h^4 = 0$, (1.5) implies that $L_{h^2}{}^2 = R_{h^2}{}^2$ and hence

(1.7) $$\mu(2R_{h^2} - \mu E) = 0.$$

Since $h^3 = 0$, (1.5) implies $L_{h^2} L_h = R_{h^2} R_h$, hence $(R_{h^2} - \mu E)(R_h - \lambda E) = R_{h^2} R_h$, that is,

(1.8) $$-\lambda R_{h^2} - \mu R_h + \lambda\mu E = 0.$$

Now suppose that $\mu \neq 0$. From (1.7) we have $2R_{h^2} - \mu E = 0$ and combine this with (1.8) to obtain $\mu(\lambda E - 2R_h) = 0$. Since $\mu \neq 0$, this implies

(1.9) $$2R_h = \lambda E.$$

On the other hand, from $L_{h^2} - L_h^2 = R_{h^2} - R_h^2$ or $(R_{h^2} - \mu E) - (R_h - \lambda E)^2 = R_{h^2} - R_h^2$, we have $\mu E = \lambda(2R_h - \lambda E)$. Hence by (1.9), $\mu = 0$ and this is a contradiction. Therefore we have $\mu = 0$ or $D_{h^2} = 0$ on A_α for any non-zero root α. This completes the proof.

<u>Lemma 1.3</u>: Let H be the same as in Lemma 1.2. Suppose that H is power-associative and that h is an element of H such that both D_y and R_y are scalars on A_α, $\alpha \neq 0$, for all y in $F[h]$. If $h^4 = 0$, then $[A_\alpha, h^3] = 0$ for all $\alpha \neq 0$.

<u>Proof:</u> Let α be any non-zero root of A^- for H. Let $D_h = R_h - L_h = \lambda E$, $D_{h^2} = R_{h^2} - L_{h^2} = \mu E$ and $R_{h^2} = \nu E$ on A_α for some λ, μ, ν in F. Since $h^4 = 0$, (1.5) implies $L_{h^2}^2 = R_{h^2}^2$ and so $\mu(2R_{h^2} - \mu E) = 0$, that is, $\mu(2\nu - \mu) = 0$. Suppose $\mu \neq 0$, then $\mu = 2\nu$. Since $h^5 = 0$, equation (1.5) implies $L_{h^3}L_{h^2} = R_{h^3}R_{h^2}$. Hence, it follows that $L_{h^3}(R_{h^2} - \mu E) = R_{h^3}R_{h^2}$. Thus $(\nu - \mu)L_{h^3} = \nu R_{h^3}$, and since $\mu = 2\nu$, this implies $L_{h^3} = - R_{h^3}$. Hence $L_{h^3} - L_h L_{h^2} = R_{h^3} - R_h R_{h^2}$ gives $2R_{h^3} - \mu(R_h - \lambda E) - \lambda\nu R_h = 0$, since $L_{h^2} = (\nu - \mu)E$ and $L_h = R_h - \lambda E$.

Therefore

(1.10) $$4R_{h^3} - \mu(2R_h - \lambda E) = 0.$$

Using $h^4 = 0$ and (1.5), $L_{h^3}L_h = R_{h^3}R_h$ and hence $R_{h^3}(2R_h - \lambda E) = 0$. Thus by (1.10), this implies $4(R_{h^3})^2 = 0$, and since R_{h^3} is a scalar on A_α, this gives $R_{h^3} = 0$ and $L_{h^3} = 0$. Hence $D_{h^3} = 0$ on A_α. Now suppose $\mu = 0$. Then $R_{h^2} = L_{h^2} = \nu E$, and hence (1.5) applied to h^2 and h^3 implies $\nu D_{h^3} = 0$. Now (1.5) applied to h and h^2 gives $D_{h^3} = \nu D_h$. Therefore, since D_{h^3} is a scalar on A_α, $D_{h^3} = 0$ on A_α. This completes the proof.

<u>Lemma 1.4</u>: Let H be a split Cartan subalgebra of A^- such that D_h is a scalar on A_α, $\alpha \neq 0$, for all h in H. If H is a zero algebra, that is, $hh' = 0$ for all h, h' in H, then $R_h + L_h = 0$ for all h in H.

<u>Proof:</u> Let $D_h = R_h - L_h = \lambda E$ on A_α, $\alpha \neq 0$. Since $h^2 = 0$, $L_h{}^2 = R_h{}^2$ and hence

(1.11) $$\lambda(2R_h - \lambda E) = 0.$$

If $\lambda = 0$, $R_h = L_h$. Since α is a non-zero root of A^- for H, we may then choose $h' \in H$ such that $D_{h'} = \mu E \neq 0$ on A_α. By (1.5), $L_h L_{h'} = R_h R_{h'}$, and so $R_h D_{h'} = 0$. But since $D_{h'} = \mu E \neq 0$, this implies $R_h = 0$ and in particular $2R_h = \lambda E = 0$. If $\lambda \neq 0$, $2R_h = \lambda E$ by (1.11). Hence we obtain $2R_h = \lambda E$ on A_α, and therefore $R_h + L_h = 0$ on A. This proves the lemma.

<u>Theorem 1.5</u>: Suppose that A^- has a split, abelian Cartan subalgebra H which is a nil subalgebra of A and F is of characteristic $\neq 2$. If A_α is one-dimensional for any $\alpha \neq 0$ and the center of A^- is 0, then A is a Lie algebra isomorphic to A^-.

<u>Proof:</u> Since H is finite dimensional, there is a $t > 1$ such that $h^t = 0$ for all $h \in H$. We first show that H is a zero algebra. Suppose that $t \geq 3$ and let n be the least integer such that $3n \geq t$. For any element h of H, let $g = h^n$ and then $g^3 = 0$. Since A_α for $\alpha \neq 0$ is one-dimensional, it follows from Lemmas 1.1 and 1.2 that g^2 belongs to the center of A^-, and hence $g^2 = 0$ or $g^2 = h^{2n} = 0$. If $2n > 4$, let m be the least positive integer with $3m \geq 2n$ and then $m < n$. The above argument implies $h^{2m} = 0$. Hence by repeated applications of this, it follows that either $h^4 = 0$ or $h^2 = 0$. If $h^4 = 0$, $h^3 = 0$ by Lemma 1.3 and so $h^2 = 0$. Since H is abelian, it follows that H is a zero algebra. Therefore if $A_\alpha = Fx$ and $[x,h] = \alpha(h)x$, then $xh = -hx = \frac{1}{2}\alpha(h)x$ by Lemma 1.4. Let α, β be any non-zero roots. If $\alpha + \beta$ is not a root, then $A_\alpha A_\beta = A_\beta A_\alpha = 0$ by Lemma 1.1. Now suppose that $\alpha + \beta$ is a root. If $\alpha + \beta = 0$, choose $h \in H$ with $\alpha = \alpha(h) \neq 0$ and let $xh = \pi x$, so $hx = (\pi - \alpha)x$. Equation (1.4) applied to h, x, y implies that $(hx)y - h(xy) + (yx)h - y(xh) = 0$, and since xy and yx are in H, it follows that $yx = -\frac{\pi}{\alpha}[x,y]$. Therefore

if $\alpha + \beta$ is a root, we may set

(1.12) $xy - yx = \lambda z$, $xy = \mu z$, $yx = (\mu - \lambda)z$,

where $A_\beta = Fy$ and z is a non-zero element in $A_{\alpha + \beta}$. Choose $h \in H$ with $\beta(h) \neq 0$ and let $\alpha = \alpha(h)$, $\beta = \beta(h)$. Then

(1.13) $xh = - hx = \frac{1}{2}\alpha x$, $zh = - hz = \frac{1}{2}(\alpha + \beta)z$.

By (1.4), $(hx)y - h(xy) + (yx)h - y(xh) = 0$, and by (1.12) and (1.13) this implies that $\frac{1}{2}[-\alpha\mu + (\alpha + \beta)\mu + (\mu - \lambda)(\alpha + \beta) - \alpha(\mu - \lambda)]z = 0$. Since $z \neq 0$, this implies $\beta(2\mu - \lambda) = 0$ and so $\lambda = 2\mu$. Therefore $xy = - yx = \frac{1}{2}[x,y]$ and this holds for all x, y in A. This completes the proof.

In [9] Laufer and Tomber have proved that if A^- is semi-simple over an algebraically closed field of characteristic 0, then A is a direct sum of simple ideals A_i of A such that A_i^- are simple Lie algebras. In this case it follows from Oehmke [10] that if A is power-associative, then it is a nilalgebra, and therefore A^- satisfies the conditions in Theorem 1.5. Hence Theorem 1.5 generalizes the result of Laufer and Tomber. In Theorem 1.5 the conditions that H is abelian, $\dim A_\alpha = 1$ for $\alpha \neq 0$, and center $A^- = 0$ do not seem to be strong enough to imply that H is a nil subalgebra of A. The condition that the center of A^- is 0 is essential in the theorem. Examples for these two facts are given in the following:

<u>Example 1.1</u>: Let F be of characteristic $\neq 2$. Let $A(\alpha)$ be an algebra over F with basis $\{x, y, h\}$ such that the multiplications are given by

$$xh = x,\ yh = \frac{1}{2}(\alpha + 1)y,$$
$$hy = \frac{1}{2}(1 - \alpha)y,\ h^2 = h,$$

and all other products are 0, where $\alpha \neq 0,\ 1$ in F. Then $A(\alpha)$ is a flexible algebra. In fact, if $u = \lambda x + \mu y + \nu h$ is any element of $A(\alpha)$,

$$R_u = \begin{bmatrix} \nu & 0 & 0 \\ 0 & \frac{1}{2}(1 + \alpha)\nu & 0 \\ 0 & \frac{1}{2}(1 - \alpha)\mu & \nu \end{bmatrix},\ L_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(1 - \alpha)\nu & 0 \\ \lambda & \frac{1}{2}(1 + \alpha)\mu & \nu \end{bmatrix}$$

relative to $\{x, y, h\}$. A direct computation shows that $L_uR_u = R_uL_u$ and hence $A(\alpha)$ is flexible. Then $A(\alpha)^-$ is given by

$$[x, y] = 0,\ [x, h] = x,\ [y,h] = \alpha y,$$

and hence it follows that $A(\alpha)^-$ is a solvable Lie algebra. Therefore $A(\alpha)$ is flexible Lie-admissible. Since $\alpha \neq 0, 1$, $A(\alpha)^- = Fh + Fx + Fy$ is the Cartan decomposition of $A(\alpha)^-$ for Cartan subalgebra $H = Fh$, and Fx and Fy are the root spaces corresponding to roots 1 and α. The center of $A(\alpha)^-$ is 0, but H is not a nilalgebra. We notice that $A(\alpha)$ is isomorphic to $A(\beta)$ if and only if $\alpha = \beta$, since $A(\alpha)^-$ is isomorphic to $A(\beta)^-$ if and only if $\alpha = \beta$.

The above example tells us that the algebra A^- in Theorem 1.5 need not be semi-simple.

<u>Example 1.2</u>: Let F be of characteristic $\neq 2$. Let A be an algebra over F with basis $\{x, y, h, z\}$ such

that the multiplications are given by

$$xy = z + \frac{1}{2}h, \; yx = z - \frac{1}{2}h, \; xh = -hx = \frac{1}{2}x,$$
$$yh = -hy = -\frac{1}{2}y, \; h^2 = -z,$$

and all other products are 0. In Chapter III, Theorem 3.1, we will show that this algebra is flexible Lie-admissible. Then $H = Fh + Fz$ is an abelian Cartan subalgebra of A^-, and $A^- = H + Fx + Fy$ is the Cartan decomposition of A^- for H where Fx and Fy are the root spaces corresponding to roots 1 and -1, respectively. The multiplication table shows that H is a nilalgebra such that $u^3 = 0$ for all u in H. We note that the center of A^- is Fz. But A is not a Lie algebra, since $h^2 = -z \neq 0$.

A finite dimensional Lie algebra L over a field F of characteristic $\neq 2, 3$ is called <u>classical</u> if: (1) the center of L is 0; (2) $L = [L, L]$; (3) L has an abelian Cartan subalgebra H (called a classical Cartan subalgebra), relative to which: a) $L = \sum_{\alpha} L_\alpha$ where ad h is a scalar on L_α for any h in H and α; b) if $\alpha \neq 0$ is a root, $[L_\alpha, L_{-\alpha}]$ is one-dimensional; c) if α and β are roots and $\beta \neq 0$, then not all $\alpha + k\beta$ are roots. The family of classical Lie algebras includes the family of finite dimensional, split semi-simple Lie algebras over a field of characteristic 0. For any classical Lie algebra L, it is shown in Seligman [14, p. 30] that L_α for $\alpha \neq 0$ is one-dimensional. Therefore from Theorem 1.5 we obtain:

<u>Corollary 1.6</u>: Suppose that A^- is a classical Lie algebra such that A^- has a classical Cartan subalgebra which is a nil subalgebra of A. Then A is a Lie algebra isomorphic to A^-.

Let F be of characteristic 0 and S a Levi-factor of A^-. Then S is not in general an ideal of A although it is an ideal of A^-. In fact, let B be the algebra of 2×2 matrices over F and let

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $S = Fx + Fy + Fh$ is a Levi-factor of B^- which is an ideal of B^- but S is not a subalgebra of B. Here we notice that a Cartan subalgebra Fh of S is not a subalgebra of B. Next, let A be the algebra given in Example 1.2. Then $S = Fx + Fy + Fh$ is a Levi-factor of A^- which is an ideal of A^-, and Fh is a Cartan subalgebra of S. But S is not a subalgebra of A, since $h^2 = -z \notin S$. In this case, we notice that Fh is not a subalgebra of A. But h is nilpotent, that is, $h^3 = 0$, and A is not associative, since $(yh)h \neq yh^2 = 0$.

Now, we wish to give a necessary and sufficient condition that a Levi-factor of A^- is an ideal of A when the solvable radical of A^- is nilpotent.

<u>Theorem 1.7</u>: Suppose that A is a finite dimensional, flexible, Lie-admissible algebra over an algebraically closed field F of characteristic 0 such that the solvable radical N of A^- is nilpotent. Then a Levi-factor S of A^- is an ideal of A if and only if S has a Cartan subalgebra H that is a nil subalgebra of A and such that $[H, N] = 0$. In this case, N is a subalgebra of A, and furthermore if A is simple, then either A^- is nilpotent or A is a Lie algebra.

<u>Proof:</u> If the Levi-factor S is an ideal of A, then S has a Cartan subalgebra H satisfying the conditions since $[S, N] = 0$. Now suppose that the Levi-factor S of A^- has a Cartan subalgebra H satisfying the conditions. We show that $[S, N] = 0$ and hence S is an ideal of A^-. Let S_{α_0} be the root space of S for H corresponding to the root $\alpha_0 \neq 0$ and then $S_{\alpha_0} = Fx$. If h is an element of H such that $\alpha_0(h) \neq 0$, by (1.1), $\alpha_0(h)[N, x]$ $= [N, \alpha_0(h)x] \subseteq [N, [x, h]] \subseteq [[N, h], x] + [[x, N], h] = 0$. Hence it follows that $[N, x] = 0$, and so $[N, S] = 0$. It follows from Dixmier [6, p. 20] that there exists a Cartan subalgebra H' of A^- such that $H' = H + H' \cap N$ and $A_\alpha = A_\alpha \cap S + A_\alpha \cap N$ where A_α is the root space of A^- for H' corresponding to the root α. Since $[N, H] = 0$ and H and N are nilpotent, it follows that $H + N$ is nilpotent in A^-. Since H' is a maximal, nilpotent subalgebra of A^-, this implies that $H' = H + N$. Therefore,

for any non-zero root α, $A_\alpha \subseteq S$ and hence $S \supseteq H + \sum_{\alpha\neq 0} A_\alpha$, that is, $S = H + \sum_{\alpha\neq 0} A_\alpha$. Since $\alpha(z) = 0$ for all z in N and α, this implies that $S = H + \sum_{\alpha\neq 0} A_\alpha$ is the Cartan decomposition of S for H, and hence A_α is one-dimensional for all $\alpha \neq 0$. Since H is a commutative, nil subalgebra of A and the center of S is 0, from Lemmas 1.2 and 1.3 it follows that H is a zero algebra. Hence by Lemma 1.4, $L_h + R_h = 0$ on S for all h in H, and so

$$(1.14) \qquad xh = -hx = \tfrac{1}{2}[x, h] = \tfrac{1}{2}\alpha(h)x$$

for x in A_α. We first prove that S is a subalgebra of A. If α and β are non-zero roots, then $A_\alpha A_\beta \subseteq S$ if $\alpha + \beta$ is a non-zero root. Let x and y be non-zero elements of A_α and $A_{-\alpha}$, respectively. Then by Lemma 1.1, $xy \in H'$, and let $xy = h + z$ for h in H and z in N. Since α is a non-zero root, by (1.14) we may choose h' in H such that $yh' = -h'y = \lambda y$ with $\lambda \neq 0$ in F. By (1.4), $(xy)h' + (h'y)x = x(yh') + h'(yx)$ and hence $(h + z)h' - \lambda yx = \lambda xy + h'(h'' + z)$ since $yx = h'' + z$ for some h'' in H. Since H is a zero algebra and $[N, H] = 0$, this implies that $\lambda(xy + yx) = 0$ and $xy + yx = 0$ since $\lambda \neq 0$. Hence $h + h'' + 2z = 0$ and so $h + h'' = z = 0$, because $H + N$ is a vector space direct sum. Therefore $xy = -yx$ belongs to H, and hence this proves that S is a subalgebra of A. By Theorem 1.5, S is a Lie subalgebra of A. Now let α be any non-zero root and $\{x, y, h\}$ a basis of the 3-dimensional simple subalgebra of S

corresponding to α such that

(1.15) $$xh = x,\ yh = -y,\ xy = h.$$

Let z be any element of N and then $xz = zx = \lambda x$, $yz = zy = \mu y$ for λ, μ in F, since $[N, S] = 0$. Let $hz = zh = \nu h + u$ where $H' = Fh \oplus B$ (vector space direct sum) and $u \in B$. Then by (1.4), $(xy)z + (zy)x = x(yz) + z(yx)$ and by (1.15) this implies $2(\nu h + u) = 2\mu h$ and so $\nu = \mu$, $u = 0$. Similarly $(yx)z + (zx)y = y(xz) + z(xy)$ and (1.15) imply $\nu = \lambda$. Therefore S is an ideal of A and moreover we have

(1.16) $$xz = zx = \lambda x,\ yz = zy = \lambda y,\ hz = zh = \lambda h$$

for z in N and non-zero root α, where λ in F depends on z and α.

Since $[S, N] = 0$ and the center of S is 0, it follows that $N = \{x \in A \mid [x, S] = 0\}$. Hence by (1.6), $[xy, s] = x[y, s] + [x, s]y = 0$ for all x, y in N and s in S. Therefore N is a subalgebra of A. If A^- is not nilpotent, then $S \neq 0$. Hence if A is simple and A^- is not nilpotent, then $A = S$ and so A is a Lie algebra. This completes the proof of Theorem 1.7.

<u>Corollary 1.8</u>: Let A, N, S, and H be the same as in Theorem 1.7. Then for any non-zero root α, there exists a subalgebra B_α of A such that:

(1) the Lie algebra ${B_\alpha}^- = N + S^{(\alpha)}$ is a Levi-decomposition of ${B_\alpha}^-$ where $S^{(\alpha)}$ is the 3-dimensional simple Lie algebra, and

(2) the multiplications between N and $S^{(\alpha)}$ are given by $sz = zs = f_\alpha(z)s$ for all z in N and s in $S^{(\alpha)}$, where f_α is a linear function on N.

<u>Proof:</u> Since N and $S^{(\alpha)}$ are subalgebras of A and $NS^{(\alpha)} = S^{(\alpha)}N \subseteq S^{(\alpha)}$, $B_\alpha = N + S^{(\alpha)}$ is a subalgebra of A. Since N is also the radical of B_α^- and $S^{(\alpha)}$ is a simple Lie algebra, it follows that the Lie algebra $N + S^{(\alpha)}$ is a Levi-decomposition of B_α^-. It follows from (1.16) that f_α is a linear function on N.

<u>Corollary 1.9</u>: Let A and N be the same as in Theorem 1.7. If A is power-associative and N is a nil subalgebra of A, then the linear function f_α is 0 for any $\alpha \neq 0$, and hence N is an ideal of A.

For the proof we first prove the following lemma:

<u>Lemma 1.10</u>: Suppose that A is a finite dimensional, flexible, power-associative algebra over a field of characteristic 0. If x is a nilpotent element of A such that D_x is nilpotent, then both R_x and L_x are nilpotent.

<u>Proof:</u> Let A^+ be the algebra defined as the same vector space as A but with a multiplication given by $x \cdot y = \frac{1}{2}(xy + yx)$. Then A^+ is a commutative and power-associative algebra. If x is nilpotent in A, so is

x in A^+. Hence it follows from Gerstenhaber [7] that $T_x = \frac{1}{2}(R_x + L_x)$ is nilpotent. The flexible law $L_x R_x = R_x L_x$ implies that D_x commutes with T_x. Hence if D_x is nilpotent, so are $T_x + \frac{1}{2} D_x = R_x$ and $T_x - \frac{1}{2} D_x = L_x$. This proves the lemma.

Proof of Corollary 1.9: Let z be an element of N and α a non-zero root. Since D_z is nilpotent, so is R_z by Lemma 1.10. By Corollary 1.8, $f_\alpha(z)$ is a characteristic root of R_z, and $f_\alpha(z) = 0$. Hence N is an ideal of A.

A finite dimensional Lie algebra L over a field of characteristic 0 is called reductive if ad L is completely reducible. It is shown in Chevalley [5, p. 255] that the definition is equivalent to the fact that the quotient algebra L/Z of L by the center Z of L is semi-simple. Then the radical R of L coincides with Z. In fact, $(R + Z)/Z$ is a solvable ideal of a semi-simple algebra L/Z and hence $(R + Z)/Z = 0$ or $R = Z$. Let S be a Levi-factor of L and then $[L, L] = [S, S] = S$, since S is semi-simple. Hence if L is reductive, then L has unique Levi-factor $[L, L]$. Therefore, as an immediate consequence of Theorem 1.7, we obtain:

Corollary 1.11: Suppose that A^- is a reductive Lie algebra over a field of characteristic 0 such that $[A, A]$ has a split Cartan subalgebra being a nil subalgebra of A. Then $[A, A]$ is an ideal of A and so a Lie algebra. Moreover if A is simple then A is either commutative or a Lie algebra.

Theorem 1.12: Suppose that A is power-associative and A^- is reductive over a field of characteristic 0 such that there exists a split Cartan subalgebra H of A^- with $h^3 = 0$ for all h in H. Then the center Z of A^- is an ideal of A. Moreover if A is simple, then it is a Lie algebra.

Proof: Since A_α for $\alpha \neq 0$ is one-dimensional and H is abelian, it follows from Lemma 1.2 that h^2 belongs to Z for every h in H. Let h_1 and h_2 be any elements of H. Then $(h_1 + h_2)^2 = h_1^2 + 2h_1h_2 + h_2^2$ is in Z, and hence $h_1h_2 \in Z$, i.e., $H^2 \subseteq Z$. Since every element of Z is nilpotent, it follows from Lemma 1.10 that $ZA_\alpha = A_\alpha Z = 0$ for $\alpha \neq 0$. Since $Z \subseteq H$, this implies that Z is an ideal of A. If A is simple, then either $Z = 0$ or $Z = A$. If $Z = A$, then A is a commutative algebra with $x^3 = 0$ for all x in A. Since A is commutative, the linearization of $x^2x = 0$ implies $(xy)z + (yz)x + (zx)y = 0$, or equivalently

$$R_{xy} + R_xR_y + R_yR_x = 0.$$

Then $R_{x^2} = -2R_x^2$ and hence R_{x^2} commutes with R_x, that is, $(xy)x^2 = x(yx^2)$ for all x and y in A. A is thus a Jordan algebra. However, since there is no simple Jordan nilalgebra of finite dimension, we must have $Z = 0$.

Therefore it follows from Theorem 1.5 that A is a Lie algebra. This completes the proof.

An element x of A is called an <u>absolute zero divisor</u> of A if $xA = Ax = 0$.

<u>Corollary 1.13</u>: Suppose that A is power-associative and the radical R of A^- is one-dimensional. If there exists a split Cartan subalgebra H of A^- with $h^3 = 0$ for all h in H, then R is the set of absolute zero divisors of A.

<u>Proof:</u> We first show that A^- is reductive. Let S be a Levi-factor of A^- and $R = Fz$. Let H' be a split Cartan subalgebra of S. If $[H', R] \neq 0$, it follows that H' is also a Cartan subalgebra of A^-. Let S_α be the root space of S for H' corresponding to the root α and $A^- = R + \sum_\alpha S_\alpha$. Since $[R, H'] \neq 0$, z belongs to a non-zero root σ of A^- for H'. Since R is an ideal of A^-, $[z, S_\alpha]$ is contained in $R \cap S_{\alpha+\sigma}$ for $\alpha \neq 0$, and hence $[z, S_\alpha] = 0$ for $\alpha \neq 0$. Let $\alpha \neq 0$ and $h = [x, y] \neq 0$ for x in S_α and y in $S_{-\alpha}$. By (1.1), $[z,[x, y]] = [z, h] = \sigma(h)z = [y,[x, z]] + [x,[z, y]] = 0$ for all $\alpha \neq 0$, and hence $\sigma = 0$. This is a contradiction and therefore $[H', R] = 0$. Hence A^- is reductive. It follows from Theorem 1.12 that $RH \subseteq R$ and $RA_\alpha = A_\alpha R = 0$ for $\alpha \neq 0$. Since H is a commutative nilalgebra containing R and R is one-dimensional, by Lemma 1.10 this implies $RH = HR = 0$. This completes the proof.

We notice that the algebra A in Theorem 1.12 and Corollary 1.13 need not be a Lie algebra. The algebra given in Example 1.2 in fact satisfies the conditions above but is not a Lie algebra.

We now wish to give examples of algebras satisfying the conditions in Theorem 1.7, which are not Lie algebras.

Example 1.3: Let F be of characteristic $\neq 2$. Let A be the algebra given by

(1.17) $sz = zs = \alpha s$ for all s in S and $z^2 = \beta z$, $\alpha, \beta \in F$, where S is the 3-dimensional simple Lie algebra and z is a basic element of A. Let $\{x, y, h\}$ be a basis of S such that

$$xh = x, \quad yh = -y, \quad xy = h.$$

Let $u = \lambda x + \mu y + \nu h + \sigma z$ be any element of A. The multiplication table gives

$$R_u = \begin{bmatrix} \nu + \alpha\sigma & 0 & \mu & 0 \\ 0 & -\nu + \alpha\sigma & -\lambda & 0 \\ -\lambda & \mu & \alpha\sigma & 0 \\ \alpha\lambda & \alpha\mu & \alpha\nu & \beta\sigma \end{bmatrix}$$

and

$$L_u = \begin{bmatrix} -\nu + \alpha\sigma & 0 & -\mu & 0 \\ 0 & \nu + \alpha\sigma & \lambda & 0 \\ \lambda & -\mu & \alpha\sigma & 0 \\ \alpha\lambda & \alpha\mu & \alpha\nu & \beta\sigma \end{bmatrix}$$

relative to the basis $\{x, y, h, z\}$. Direct computations show $R_u L_u = L_u R_u$ and hence A is flexible. A^- is also a Lie algebra and $A^- = Fz + S^-$ where Fz is the center of A^-. Hence A is flexible Lie-admissible. If $\alpha = \beta = 0$, then A is a Lie algebra isomorphic to A^-. Conversely if A is a Lie algebra, then $\beta = 0$ from $z^2 = \beta z$. $(xy)z + (zx)y + (yz)x = 0$ implies $\alpha = 0$. Therefore A is a Lie algebra if and only if $\alpha = \beta = 0$. Let the algebra given in (1.17) denote by $A(\alpha, \beta)$. If $\beta \neq 0$, then $A(\alpha, 0)$ is not isomorphic to $A(\alpha, \beta)$ for every α, since $A(\alpha, \beta)$ has a non-zero idempotent $\beta^{-1}z$ but $A(\alpha, 0)$ does not. We note that $A(\alpha, 0)$ is power-associative if and only if $\alpha = 0$. If $\beta \neq 0$, then $A(\alpha\beta^{-1}, 1)$ is isomorphic to $A(\alpha, \beta)$. Let $A(\alpha) = A(\alpha, 1)$ and then z is a non-zero idempotent of $A(\alpha)$. If $(s + \lambda z)^2 = s + \lambda z$ for $\lambda \neq 0$ in F and $s \neq 0$ in S, then $\alpha = \frac{1}{2}$ and $\lambda = 1$. Conversely if $\alpha = \frac{1}{2}$, then $s + z$ is a non-zero idempotent of $A(\frac{1}{2})$ for every s in S. Hence z is only non-zero idempotent of $A(\alpha)$ if and only if $\alpha \neq \frac{1}{2}$. Suppose that $A(\alpha)$ is isomorphic to $A(\beta)$. If either $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$, then $\alpha = \beta = \frac{1}{2}$ from the above remark. Let $\alpha \neq \frac{1}{2}$ and $\beta \neq \frac{1}{2}$. Then by the remark z is mapped into itself under the isomorphism. 1 and β are only characteristic roots of R_z in $A(\beta)$, but α is also a characteristic root of R_z in $A(\beta)$ since $A(\alpha) \cong A(\beta)$. Similarly β is a characteristic root of R_z in $A(\alpha)$. Hence $\alpha = \beta$ and so we proved that $A(\alpha)$ is isomorphic to $A(\beta)$ if and only if $\alpha = \beta$. Therefore we obtain a family of non-Lie algebras satisfying the conditions in Theorem 1.7. $A(\alpha)$ is not in general power-associative. In fact, $u^3u = u^2u^2$ holds for every u in $A(\alpha)$ if and only if $2\alpha^3 - 3\alpha^2 + \alpha = 0$, that is, $A(\alpha)$ is power-associative if and only if $\alpha = 0$, $\alpha = \frac{1}{2}$, or $\alpha = 1$.

Correction

In the proof of Theorem 2 of the paper of Laufer and Tomber [9], the step (i), which proves the Cartan subalgebra H of A^- is a nil subalgebra of A, is not correct. Assuming that H is not a nilalgebra, there exists an idempotent $e \neq 0$ in H. Setting $ex - xe = \alpha x$, $ex = \beta x$ and $ey - ye = -\alpha y$, $ey = \beta' y$ where $\alpha \neq 0$ in F and $x \neq 0$ and $y \neq 0$ belong to the root $\alpha(H)$ and $-\alpha(H)$, respectively, they conclude $\beta + \beta' = 0$ by claiming that xy or yx belongs to the root $\beta + \beta'$ relative to e. But since R_e and L_e are not in general derivations of A, $\beta + \beta'$ may not be a root. In fact, (1.5) applied to e implies $R_e - R_e^2 = L_e - L_e^2$ and this applied to x, y in turn implies $\beta = \frac{1}{2}(1 + \alpha)$ and $\beta' = \frac{1}{2}(1 - \alpha)$. Thus $\beta + \beta' = 1$. However, if A is not nilalgebra, it follows from Oehmke [10] that A has an identity element. But since A^- is simple, this is impossible. Therefore A is nil and so is H.

CHAPTER II

SOME SIMPLE NILALGEBRAS

The class of flexible algebras includes all commutative and anticommutative algebras, in particular, Jordan and Lie algebras, and alternative algebras, as well as the noncommutative Jordan algebras introduced by Schafer [11]. All algebras we consider here are finite dimensional over a field of characteristic $\neq 2$. In [13] Schafer defined the generalized standard algebras which include all alternative algebras and the standard algebras of Albert [1]. Any commutative Jordan algebra is included in the class of generalized standard algebras. It is shown in Schafer [13] that any generalized standard algebra is a noncommutative Jordan algebra A, i.e., a flexible algebra with $(x^2y)x = x^2(yx)$ for all x, y in A. Hence the class of noncommutative Jordan algebras includes most of the well known nonassociative algebras. Any generalized standard algebra is power-associative. Any standard algebra is flexible Lie-admissible. However, there is no implication between the class of generalized standard algebras and the class of flexible Lie-admissible algebras. In fact, flexible Lie-admissibility does not imply power-associativity as shown by the algebras given in Example 1.3. Any nonassociative alternative algebra of characteristic $\neq 2, 3$ is not flexible Lie-admissible, since it is shown in Albert [1] that an alternative algebra of characteristic $\neq 2, 3$ is Lie-admissible if and only if it is associative.

It is also shown in [13] that any generalized standard nilalgebra is nilpotent, where an algebra A is called <u>nilpotent</u> in case there exists an integer t such that every product of t elements of A, no matter how associated, is 0. Hence there are no simple generalized standard nilalgebras. Simple Lie algebras give examples that the result is not valid for noncommutative Jordan algebras and flexible Lie-admissible algebras. The nilpotency of flexible nilalgebras is not known even in case of commutative, power-associative nilalgebras. The existence of simple, commutative nilalgebras has not been settled, but the relatively simple case of dimension ≤ 3 was settled by Gerstenhaber [7] in the following:

<u>Theorem</u>: Let A be a commutative power-associative nilalgebra over a field of characteristic 0. If A is of dimension ≤ 3, then it is nilpotent.

Hence there is no simple, commutative, power-associative nilalgebra of dimension ≤ 3.

Let A now be a noncommutative, flexible, simple nilalgebra. All simple, anticommutative algebras are such algebras. R. Block [2] has recently proved the theorem:

Theorem: Let A be a finite dimensional flexible algebra over a field of characteristic $\neq 2$. If A is simple and not anticommutative, then either A^+ is simple or A has an identity element 1.

Here the algebra A^+ is defined as the same vector space as A but with a multiplication given by $x \cdot y = \frac{1}{2}(xy + yx)$, and hence A^+ is a commutative algebra. If A is a power-associative nilalgebra, then so is A^+. Therefore, if A is a simple, flexible nilalgebra and not anticommutative, then A^+ is a commutative, simple nilalgebra by this theorem. Thus one might conjecture that a noncommutative, flexible, simple nilalgebra is anticommutative. At the present time, no proof is known.

A Jordan algebra is a commutative algebra A satisfying the Jordan identity $(xy)x^2 = x(yx^2)$ for all x, y in A. An algebra A is said to be Jordan-admissible in case A^+ is a Jordan algebra. It is shown in [1] that any noncommutative Jordan algebra is Jordan-admissible.

The above remarks lead one to seek a sufficient condition that a noncommutative, flexible, simple nilalgebra A be anticommutative. If, in particular, A is a simple noncommutative Jordan nilalgebra, then it follows from the above theorem that it is anticommutative, since A^+ is a Jordan nilalgebra and there is no simple Jordan nilalgebra. Laufer and Tomber have concluded that if A is power-associative and A^- is a simple Lie algebra over an algebraically closed field of characteristic 0, then A is anticommutative. In this case the simplicity of A^- implies that A is a simple nilalgebra. Since any commutative algebra is trivially Lie-admissible, this result suggests the conjecture that any simple, flexible, Lie-admissible nilalgebra is either commutative or anticommutative and in the latter case A is a Lie algebra. In Chapter I we have already discussed such simple algebras A under certain restrictions on the Lie algebras A^-. In most of the cases we assumed only that a Cartan subalgebra H of A^- is a nil subalgebra of A. The Cartan decomposition of A^- by H then worked as an analog to the idempotent decomposition in flexible, power-associative, non-nil algebras.

The principal purpose of this chapter is to answer in part the above questions. We will also deal with certain nilalgebras whose minus-algebras are nilpotent Lie algebras. The proofs are mainly based on the two theorems above, the structures of A^-, and the results in Chapter I.

We will assume throughout this chapter that A is a finite dimensional algebra over a field F and all nilalgebras are power-associative.

Theorem 2.1: Let A be a simple, flexible algebra over a field of characteristic $\neq 2$ such that $x^3 = 0$ for all x in A. Then A is anticommutative. Furthermore, if A is Lie-admissible, then it is a Lie algebra.

Proof: We first show that A is Jordan-admissible. The linearization of $x^2x = 0$ implies $(x\cdot y)\cdot z + (z\cdot x)\cdot y + (y\cdot z)\cdot x = 0$, or equivalently $T_{x\cdot y} + T_xT_y + T_yT_x = 0$ where $x\cdot y = \frac{1}{2}(xy + yx)$ and $T_x = \frac{1}{2}(R_x + L_x)$. Then $T_{x^2} = -2T_x{}^2$ and hence T_{x^2} commutes with T_x, that is, $(x\cdot y)\cdot x^2 = x\cdot(y\cdot x^2)$ for all x, y in A. A^+ is thus a Jordan algebra. Suppose that A is not anticommutative. Then since A is simple, it follows, from the result of Block quoted above, that A^+ is simple. Since A^+ is a nilalgebra and there is no simple Jordan nilalgebra, this is a contradiction. This completes the proof.

A simple, flexible nilalgebra may be both Jordan-admissible and Lie-admissible as shown by simple Lie algebras. Since it follows from the theorem of Block that any simple, flexible, Jordan-admissible nilalgebra is anticommutative, we obtain:

Theorem 2.2: A simple, flexible nilalgebra of characteristic $\neq 2$ is both Jordan-admissible and Lie-admissible if and only if it is a Lie algebra.

In general, the fact that $u^n = 0$, $n > 3$, for all u in A does not imply that A is Jordan-admissible, as shown by the followings:

Example 2.1: Let A be a 4-dimensional algebra of characteristic $\neq 2$ given by

$$x^2 = z,$$

$$xz = zx = zh = hz = z^2 = hx = y,$$

and all other products are 0, where $\{x, y, z, h\}$ is a basis of A. Then A is a flexible nilalgebra such that $u^4 = 0$ for all u in A, but $x^3 = y \neq 0$. A is also Lie-admissible, since $[A, [A, A]] = 0$. Let $T_x = \frac{1}{2}(R_x + L_x)$. Relative to the basis $\{x, y, z, h\}$,

$$T_x = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad T_{x^2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then $T_x T_{x^2} \neq T_{x^2} T_x$, since the (1, 2)-entry of $T_x T_{x^2}$ is 1 but that of $T_{x^2} T_x$ is 0. Hence A is not a Jordan-admissible algebra.

<u>Theorem 2.3</u>: Let A be an n-dimensional, flexible, Lie-admissible nilalgebra over a field F of characteristic 0 such that A^- is nilpotent. If A^- has an abelian ideal of dimension $n - 1$, then the center Z of A^- is an ideal of A.

For the proof we first prove:

<u>Lemma 2.4</u>: Let A be an n-dimensional, flexible, Lie-admissible nilalgebra over the field F. If A is not commutative and B is an abelian ideal of A^- of dimension $n - 1$, then B is an ideal of A.

Proof: Writing $A^- = B \oplus Fh$ as a vector space, $[A, A] = [h, B] \neq 0$ since A is not commutative. Let g be a non-zero element of $[h, B]$ and then $[h, z] = g$ for some z in B. Let x be any element of B and $xh = u + \alpha h$ for u in B and α in F. Since B is abelian, $[xh, z] = x[h, z] + [x, z]h$ implies $xg = gx = \alpha g$. Since $D_x^2 = 0$ and $g \neq 0$, $\alpha = 0$ by Lemma 1.10, and hence xh is in B. Likewise hx is in B and this proves the lemma.

Proof of Theorem 2.3: If A is commutative, then it is trivial. Let A be noncommutative. Since A^- is nilpotent, $n \geq 3$. Let B be an abelian ideal of A^- of dimension $n - 1$. Writing $A^- = B \oplus Fh$, $[B, h] \neq 0$ and D_h induces a nilpotent linear transformation on B, since A^- is a nilpotent Lie algebra. Then B can be expressed as a direct sum of subspaces $B = M_1 \oplus M_2 \oplus \dots \oplus M_r$ such that each M_i is a cyclic subspace of B relative to D_h, $n_1 \geq n_2 \geq \dots \geq n_r$ where n_i is the dimension of M_i, and n_1 is the nil index of D_h in B. Let $\{x_{i,1}, x_{i,2}, \dots , x_{i,n_i}\}$ be a basis of M_i with $[x_{i,k-1}, h] = x_{i,k}$ and $[x_{i,n_i}, h] = 0$, $k = 2, \dots , n_i$. Since $[B, B] = 0$ and $[B, h] \neq 0$, it follows that the center Z of A^- is the subspace generated by $\{x_{1,n_1}, x_{2,n_2}, \dots , x_{r,n_r}\}$. By Lemma 2.4, $hx_{i,n_i} = x_{i,n_i}h$ belongs to B. Then $[hx_{i,n_i}, h] = h[x_{i,n_i}, h] + [h, h]x_{i,n_i} = 0$ implies

(2.1) $\quad hx_{i,n_i} = x_{i,n_i}h \in Z, \; i = 1, 2, \ldots, r.$

Since $[B, h] \neq 0$, $n_1 \geq 2$, and let p be such that $n_1 \geq n_2 \geq \ldots \geq n_p \geq 2$ and $n_{p+1} = \ldots = n_r = 1$. Let x be any element of B. Since xh and hx belong to B by Lemma 2.4, if $i \leq p$ then $0 = [x_{i,n_i-1}, xh] = x[x_{i,n_i-1}, h] = xx_{i,n_i}$ and similarly $x_{i,n_i}x = 0$. Hence

(2.2) $\quad Bx_{i,n_i} = x_{i,n_i}B = 0, \; i = 1, \ldots, p.$

If $j > p$ then by (2.1), $0 = [x_{i,k}, x_{j,n_j}h] = x_{j,n_j}[x_{i,k}, h] = x_{j,n_j}x_{i,k+1}$ where $i = 1, 2, \ldots, p$, $1 \leq k \leq n_i - 1$. Hence

(2.3) $$x_{j,1}x_{i,k} = x_{i,k}x_{j,1} = 0, \quad 1 \leq i \leq p, \; j > p, \; 2 \leq k \leq n_i.$$

Let $i \leq p$ and then $[x_{j,1}x_{i,1}, h] = x_{j,1}[x_{i,1}, h] = x_{j,1}x_{i,2} = 0$ by (2.3) if $j > p$. The equations $[x_{j,1}x_{i,1}, h] = 0$ imply that $x_{j,1}x_{i,1} = x_{i,1}x_{j,1}$ belong to Z for $j > p$ and $1 \leq i \leq p$. Therefore, since Z is a subalgebra of A, by (2.1), (2.2), and (2.3) this implies that Z is an ideal of A. This proves Theorem 2.3.

Theorem 2.3 and Lemma 2.4 are not valid for non-nil algebras as shown by the following:

<u>Example 2.2</u>: Let A be a 4-dimensional algebra with multiplication given by

$$x^2 = 2x, \; xy = y + z,$$
$$yx = y, \; xz = zx = y^2 = z,$$
$$uh = hu = u \text{ for all } u \text{ in } A,$$

and all other products are 0, where $\{x, y, z, h\}$ is a basis of A. Then A is flexible and A^- is a nilpotent Lie algebra since $[A, [A, A]] = 0$. $Fy + Fz + Fh$ is an abelian ideal of A^- and $Fz + Fh$ is the center of A^-, but none of them is an ideal of A. In this case A is not associative since $x^2y \neq x(xy)$.

If A is a nilalgebra of dimension n over a field F then $x^{n+1} = 0$ for all x in A. If K is an extension field of F, let A_K be the scalar extension of A to an algebra over K. If F has an infinitely many elements, it follows from Braun and Koecher [4, p. 23] that A_K is also a nilalgebra over K. If A is a flexible Lie-admissible algebra over F of characteristic $\neq 2, 3$, then so is A_K for any extension K of F since (1.6) is a bilinear identity. An algebra A over F is called <u>central simple</u> in case A_K is simple over K for every extension K of F.

<u>Lemma 2.5</u>: Let A be a 4-dimensional, flexible nilalgebra over an algebraically closed field F of characteristic 0 such that A^- is a solvable Lie algebra. If A is not commutative, then it is not simple.

<u>Proof:</u> If A^- is nilpotent, it follows from Bourbaki [3, p. 120] that A^- has an abelian ideal of dimension 3. Hence by Theorem 2.3, A is not simple. Now suppose that A^- is not nilpotent and let H be a Cartan subalgebra of A^-. Let A_α denote the root space of A^- for H corresponding to root $\alpha \neq 0$. Then the dimension of H is 1, 2, or 3.

If the dimension of H is 3, then it follows from Lemma 1.1 that A_α for $\alpha \neq 0$ is an ideal of A. The proof continues by examining the two other possibilities for H.

Case I: dim $H = 1$. The possible Cartan decompositions of A^- for H are $A^- = H + A_\alpha$, $A^- = H + A_\alpha + A_\beta$, and $A^- = H + A_\alpha + A_\beta + A_\gamma$. In the first case A_α is an ideal of A. In the last case it follows from Theorem 1.5 that A is a solvable Lie algebra and not simple. In the remaining case we want to show that $A_\alpha + A_\beta$ is an ideal of A. If $\alpha + \beta \neq 0$, then by Lemma 1.1, $A_\alpha + A_\beta$ is an ideal of A. Now suppose $\alpha + \beta = 0$ and let $H = Fh$. We may then assume $A_\alpha = Fx + Fy$ and $A_\beta = Fz$. Setting $zh = \gamma z$, then $hz = (\gamma + \alpha)z$ since $zh - hz = -\alpha z$. Since $h^2 = 0$, (1.4) applied to z, h, h implies $\gamma^2 = (\gamma + \alpha)^2$ and $\alpha = -2\gamma$. It follows that

$$zh = -hz = -\tfrac{1}{2}\alpha z. \tag{2.4}$$

Since xz and zx belong to H and $h^2 = 0$, (1.4) applied to x, z, h and (2.4) imply $xz + zx = 0$, and similarly $yz + zy = 0$. If $[x, z] \neq 0$, the derived algebra $[A, A]$ of A^- contains h and z, and hence $[A, A]$ is not a nilpotent Lie algebra. Since the derived algebra of a solvable Lie algebra is nilpotent, this is a contradiction. Therefore we obtain $[x, z] = 0$ and similarly $[y, z] = 0$. Hence $A_\alpha A_\beta = A_\beta A_\alpha = 0$ and $A_\alpha + A_\beta$ is an ideal of A.

Case II: dim $H = 2$. H is a commutative nil subalgebra of A and $u^3 = 0$ for all u in H. The possible Cartan decompositions of A^- for H are $A^- = H + A_\alpha$ and $A^- = H + A_\alpha + A_\beta$. In the first case A_α is an ideal of A. In the second case both A_α and A_β are one-dimensional. Let Z in this case be the center of A^-. If $Z = 0$, then it follows from Theorem 1.5 that A is a Lie algebra and not simple. If $Z \neq 0$, then it follows from Lemmas 1.2 and 1.10 that $HH \subseteq Z$ and $ZA_\alpha = ZA_\beta = 0$. Since Z is contained in H, this implies that Z is an ideal of A.

Therefore A is not simple and this proves the lemma.

Theorem 2.6: Let A be a central simple, flexible, Lie-admissible nilalgebra over a field F of characteristic 0. If the dimension of A is not greater than 4, then A is either commutative or a Lie algebra, and in the latter case A is the 3-dimensional simple Lie algebra.

Proof: Let K be the algebraic closure of F. Then by the remarks above A_K is a simple algebra satisfying the conditions in the theorem. Hence without loss of generality we may assume that the field F is algebraically closed. If the dimension of A is not greater than 3, then by the result of Gerstenhaber the algebra A^+ is nilpotent. Hence it follows from the theorem of Block that A is anticommutative. Therefore A is in this case the 3-dimensional simple Lie algebra. Now suppose that A is a noncommutative algebra of dimension 4. If A^- is a solvable Lie algebra, from Lemma 2.5 it follows that A is not simple. If A^- is not solvable, then A^- is a reductive Lie algebra with one-dimensional center Z. In this case A^- has a Cartan subalgebra of dimension 2. Hence it follows from Theorem 1.12 that Z is an ideal of A. Consequently A must be commutative if the dimension of A is 4. This completes the proof.

<u>Remark on Theorem 2.6</u>: (1) The condition that A is central may be replaced by the fact that if A^- is a non-nilpotent Lie algebra of dimension 4, then it has a split Cartan subalgebra. (2) Although there is no simple, flexible nilalgebra of dimension 2, we may prove that any noncommutative, flexible, Lie-admissible nilalgebra of dimension 2 is a Lie algebra. Indeed, in this case A has a basis $\{x, y\}$ with $xy - yx = x$. The equation $[x, x^2] = [y, y^2] = 0$ imply $x^2 = y^2 = 0$. From $[x, xy] = x[x, y] = x^2 = 0$, we obtain $xy = \lambda x$ for some λ in F. Since $y^2 = 0$, (1.4) applied to x, y, y implies $(xy)y = y(yx)$ and this gives $\lambda^2 = (\lambda - 1)^2$ or $\lambda = \frac{1}{2}$. Hence A is a Lie algebra.

CHAPTER III

NILALGEBRAS OF DIMENSION LOWER THAN 5

Let L be a 4-dimensional Lie algebra over an algebraically closed field of characteristic 0. Then L is completely known unless it is non-nilpotent solvable. In this chapter, by using the known structures of Lie algebras of dimension 4, we determine all 4-dimensional, flexible, Lie-admissible nilalgebras whose minus-algebras are not non-nilpotent, solvable Lie algebras. We also prove that such algebras whose minus-algebras are non-abelian nilpotent are nilpotent.

We assume throughout this chapter that A denotes a finite dimensional, flexible, Lie-admissible algebra over a field F of characteristic 0 and that all nilalgebras are power-associative.

Theorem 3.1: Let F be algebraically closed. Then A is a 4-dimensional nilalgebra over F such that A^- is not solvable if and only if it is either a non-solvable Lie algebra or given by

$$xy = z + \frac{1}{2}h,\ yx = z - \frac{1}{2}h,\ xh = -hx = \frac{1}{2}x,$$
$$yh = -hy = -\frac{1}{2}y,\ h^2 = -z,$$

and all other products are 0, where $\{x, y, h, z\}$ is a basis of A. In this case A is a nilalgebra with $u^3 = 0$ for all u in A.

Proof: Suppose that A is a 4-dimensional nilalgebra such that A^- is not solvable. Then $A^- = Fz + S$ is a Levi-decomposition of A^- with the 3-dimensional simple Lie algebra S and the center Fz of A^-. Let $\{x, y, h\}$ be a basis of S with $[x, h] = x$, $[y, h] = -y$, $[x, y] = h$. Then $H = Fz + Fh$ is a Cartan subalgebra of A^- and thus a subalgebra of A. Hence $v^3 = 0$ for all v in H and it follows from Corollary 1.13 that z is an absolute zero divisor of A. By Lemma 1.2, $h^2 = \alpha z$ for α in F. Hence $0 = [x, h^2] = h[x, h] + [x, h]h = hx + xh$ and this with $[x, h] = x$ implies $xh = -hx = \frac{1}{2}x$, and similarly $hy = -yh = \frac{1}{2}y$. Since x and y belong to roots 1 and -1 of A^- for H, respectively, we obtain $xy = \beta z + \gamma h$ and $yx = \beta z + (\gamma - 1)h$. Then $(xy)h + (hy)x = x(yh) + h(yx)$ implies $\beta = -\alpha$ and $\gamma = \frac{1}{2}$. If $\alpha = 0$, then A is a Lie algebra isomorphic to A^-. If $\alpha \neq 0$, then we replace $-\alpha z$ by z to obtain the algebra given in the theorem. Conversely suppose that A is given by the multiplication table stated in the theorem. Let $u = \lambda x + \mu y + \nu h + \sigma z$ be any element of A. The matrix representations of R_u and L_u relative to the basis $\{x, y, h, z\}$ are

$$R_u = \begin{bmatrix} \frac{1}{2}\nu & 0 & \frac{1}{2}\mu & \mu \\ 0 & -\frac{1}{2}\nu & -\frac{1}{2}\lambda & \lambda \\ -\frac{1}{2}\lambda & \frac{1}{2}\mu & 0 & -\nu \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$L_u = \begin{bmatrix} -\frac{1}{2}\nu & 0 & -\frac{1}{2}\mu & \mu \\ 0 & \frac{1}{2}\nu & \frac{1}{2}\lambda & \lambda \\ \frac{1}{2}\lambda & -\frac{1}{2}\mu & 0 & -\nu \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then both R_uL_u and L_uR_u are equal to

$$\frac{1}{4}\begin{bmatrix} \lambda\mu-\nu^2 & -\mu^2 & -\mu\nu & 0 \\ -\lambda^2 & \lambda\mu-\nu^2 & -\lambda\nu & 0 \\ \lambda\nu & \mu\nu & 2\lambda\mu & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus A is flexible. We obtain $u^2 = (2\lambda\mu - \nu^2)z$ and hence $u^3 = 0$. From the multiplication table, it follows that A^- is isomorphic to the Lie algebra $Fz + S$ and thus A is Lie-admissible. This completes the proof.

The algebra in Example 1.2 is obtained from the algebra above.

<u>Theorem 3.2</u>: A is a 4-dimensional nilalgebra such that A^- is a nilpotent Lie algebra with one-dimensional center if and only if it is given by

$$x^2 = \alpha z,\ xh = -\frac{1}{2}y + \beta z,$$
$$hx = \frac{1}{2}y + \beta z,\ yh = -\ hy = -\frac{1}{2}z,$$
$$h^2 = \gamma z,\ \alpha,\ \beta,\ \gamma \text{ in } F,$$

and all other products are 0, where $\{x, y, z, h\}$ is a basis of A. In this case A is a nilalgebra with $u^4 = 0$ for all u in A and a Lie algebra if and only if $\alpha = \beta = \gamma = 0$.

Proof: Suppose that A is a nilalgebra such that A^- is a nilpotent Lie algebra with one-dimensional center. Then it follows from Bourbaki [3, p. 120] that A^- has a basis $\{x, y, z, h\}$ such that $[h, x] = y$, $[h, y] = z$ and all other products under $[\ ,\ ]$ are 0. Let $B = Fx + Fy + Fz$ and $Z = Fz$. Then B is an abelian ideal of A^- and Z is the center of A^-. Hence it follows from Theorem 2.3 and Lemma 2.4 that B and Z are ideals of A. By Lemma 1.10, z is an absolute zero divisor of A. Since xh is in B, $0 = [xh, x] = x[h, x] = xy$ and thus $xy = yx = 0$. $[h, yh] = [h, y]h = zh = 0$ implies $yh = \lambda z$ for λ in F. From $[xh, h] = [x, h]h = -yh$, we obtain $xh = \lambda y + \mu z$ for μ in F. We use $[h, x^2] = xy + yx = 0$ to obtain $x^2 = \alpha z$ for α in F. The equations $[h, h^2] = 0$ and $[h^2, y] = hz + zh = 0$ give $h^2 = \alpha z$. Therefore, since $hy - yh = z$ and $yh = \lambda z$, $0 = [h^2, x] = hy + yh = (2\lambda + 1)z$ and thus $\lambda = -\frac{1}{2}$. Hence A has the multiplication table given in the theorem. Conversely, suppose that A is the algebra given in the theorem. If $u = \lambda x + \mu y + \nu z + \sigma h$ is any element of A, then

$$R_u = \begin{bmatrix} 0 & -\frac{1}{2}\sigma & \alpha\lambda + \beta\sigma & 0 \\ 0 & 0 & -\frac{1}{2}\sigma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\lambda & \beta\lambda + \gamma\sigma + \frac{1}{2}\mu & 0 \end{bmatrix}$$

and

$$L_u = \begin{bmatrix} 0 & \frac{1}{2}\sigma & \alpha\lambda + \beta\nu & 0 \\ 0 & 0 & \frac{1}{2}\sigma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\lambda & \beta\lambda + \gamma\sigma - \frac{1}{2}\mu & 0 \end{bmatrix}$$

relative to $\{x, y, z, h\}$. A direct calculation shows $R_u L_u = L_u R_u$ and thus A is flexible. It follows from the multiplication table that A^- is a nilpotent Lie algebra with one-dimensional center Fz. Moreover we obtain $R_u{}^3 = L_u{}^3 = 0$ and $A^2 \subseteq Fy + Fz$ and so $A^2A^2 = 0$. Therefore $u^3u = u^2u^2 = 0$ for all u in A and thus completes the proof.

<u>Theorem 3.3</u>: A is a 4-dimensional nilalgebra such that A^- is a nilpotent Lie algebra with two-dimensional center if and only if it is given by

$$x^2 = \alpha y + \beta z,\ xz = zx = \gamma y,$$
$$xh = \delta y + \lambda z,\ hx = (\delta + 1)y + \lambda z,$$
$$z^2 = \mu y,\ zh = hz = \nu y,\ h^2 = \sigma y + \tau z$$

for $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \sigma, \tau$ in F with $\mu\beta^2 = \mu\lambda^2 = \mu\tau^2 = 0$, and all other products are 0, where $\{x, y, z, h\}$ is a basis of A. In this case A is a nilalgebra with $u^4 = 0$ for all u in A and a Lie algebra if and only if $\delta = -\frac{1}{2}$ and all other parameters are 0.

<u>Proof:</u> Suppose that A is a 4-dimensional nilalgebra such that A^- is nilpotent and has two-dimensional center. Then it also follows from Bourbaki [3] that A^- has a basis

$\{x, y, z, h\}$ such that $[h, x] = y$ and all products under $[\ ,\ ]$ are 0. Let $B = Fx + Fy + Fz$ and $Z = Fy + Fz$. Then B is an abelian ideal of A^- and Z is the center of A^-. Thus B and Z are ideals of A. Hence $0 = [h, xy] = [h, x]y = y^2$ and similarly $yz = zy = 0$. Since xy and x^2 are in Z and B, respectively, $[h, x^2] = 2xy$ implies that xy and x^2 are in Fy and $Fy + Fz$, respectively. Thus $xy = yx = 0$ by Lemma 1.10 and $x^2 = \alpha y + \beta z$ for α, β in F. It follows, from $[h^2, x] = h[h, x] + [h, x]h = 2yh$, that $yh = hy$ belongs to Fy. This implies that $yh = hy = 0$ and $h^2 = \sigma y + \tau z$ for σ, τ in F. Hence $[h, xh] = [h, x]h = yh = 0$ and this gives $xh = \delta y + \lambda z$ and $hx = (\delta + 1)y + \lambda z$ for δ, λ in F. Setting $zx = xz = \gamma y + \gamma' z$ for γ, γ' in F, then $(xz)x = \gamma' xz$ and thus $\gamma' = 0$ by Lemma 1.10. Similarly $hz = zh = \nu y$. Since Z is a nilalgebra of dimension 2, $z^3 = 0$ and this implies $z^2 = \mu y$. Since x belongs to B, $x^2x^2 = (\alpha y + \beta z)^2 = \beta^2 z^2 = \beta^2 \mu z = 0$. Since h belongs to the subalgebra $Fy + Fz + Fh$, $h^2h^2 = 0$ implies $\mu\tau^2 = 0$. Similarly, we obtain $\mu\lambda^2 = 0$ from $(xh)^2(xh)^2 = 0$. Therefore A is the algebra given in the theorem. A similar computation as in Theorem 3.2 shows that A is flexible and $R_u^3 = L_u^3 = 0$ for all u in A. From the multiplication table we obtain that $A^2 \subseteq Fy + Fz$ and $A^2A^2 \subseteq F \cdot \mu y$. If $\mu \neq 0$, then $\beta = \lambda = \tau = 0$ and thus $A^2A^2 = Fy^2 = 0$. Hence $u^3u = u^2u^2 = 0$ for all u in A. If $\delta = -\frac{1}{2}$ and all other parameters are 0, then A is a Lie algebra isomorphic to A^-. If A is Lie, then $\delta = -\frac{1}{2}$ from $hx + xh = 0$ and all other parameters are 0 and thus completes the proof.

Theorems 3.1, 3.2, and 3.3 determine all 4-dimensional nilalgebras whose minus-algebras are not non-nilpotent solvable. None of these algebras is associative, since $(hh)x \neq h(hx)$ in Theorems 3.1 and 3.2 and in Theorem 3.3 with $\mu = \nu = 0$ and $\gamma = \tau = 1$.

Theorem 3.4: If A is a noncommutative nilalgebra of dimension ≤ 4 such that A^- is nilpotent, then A is nilpotent such that all products of any four elements in A are 0.

Proof: If A^- is a nilpotent Lie algebra of dimension ≤ 2, A^- is abelian. Hence the dimension of A is in this case 3 or 4. If the dimension of A is 4, then A is either the algebra in Theorem 3.2 or in Theorem 3.3. In Theorem 3.2 we see that $A^2 \subseteq Fy + Fz$ and that AA^2 and A^2A are contained in Fz. Since z is an absolute zero divisor of A, it follows that $A(A^2A) = (A^2A)A = A(AA^2) = A^2A^2 = 0$. In Theorem 3.3 we also notice that $A^2 \subseteq Fy + Fz$ and that AA^2 and A^2A are contained in Fy. Since y is in this case an absolute zero divisor of A, we obtain the same result. Now suppose that the dimension of A is 3. Then A^- has a basis $\{x, y, z\}$ such that $[x, y] = z$, $[x, z] = [y, z] = 0$, and Fz is the center of A^-. By

Theorem 2.3, Fz is an ideal of A and hence z is an absolute zero divisor of A. Hence $[xy, x] = x[y, x] = -xz = 0$ and $[xy, y] = [x, y]y = zy = 0$, and these imply that xy and yx belong to Fz. From $[x^2, x] = [x^2, y] = 2xz = 0$, it follows that x^2 belongs to Fz, and similarly y^2 is in Fz. Therefore $A^2 \subseteq Fz$ and thus $A^2 A = AA^2 = 0$. This completes the proof.

Theorem 3.4 is not valid in case A^- is not nilpotent, as shown by the algebra in Theorem 3.1 and non-nilpotent, solvable Lie algebras.

At the present time, Theorem 3.4 for arbitrary dimension is not settled.

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