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BOCHNER PROPERTY IN BANACH SPACES

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ABSTRACT

BOCHNER PROPERTY IN BANACH SPACES

By

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In this thesis we study a class of Banach spaces in which some analogues of the following theorems hold.

(1) (Bochner's Theorem) A function φ defined on R^n is the characteristic functional of a probability measure on R^n iff φ is positive definite, $\varphi(0) = 1$ and it is continuous in the usual topology.

(2) A necessary and sufficient condition for the tightness of a family $\{\mu_\alpha : \alpha \in I\}$ of probability measures on R^n is the equicontinuity of the corresponding characteristic functionals $\{\hat{\mu}_\alpha\}$ at zero.

Let E be a real separable Banach space with metric approximation property, and let E' be its topological dual. D.H. Mustari has shown that embeddability of E into a space of random variables is a sufficient condition for the validity of Bochner's Theorem in E. A simpler proof of this result is given by first proving an inequality analogous to the finite dimensional one which connects the

behaviour at zero of a characteristic functional with the probability of compact sets. A result of Lindenstrauss and Pelczynski is used to show that the embeddability condition is also necessary if E is further assumed to be of stable type p. For such spaces we also give an explicit form of the topology on E' that is associated with the Bochner Theorem. For this we use some results of deAcosta and V. Mandrekar.

It is known that the validity of Bochner's Theorem implies that the space is of cotype 2. Results of Maurey and Rosenthal are exploited to show that the converse is not true. This also gives cotype 2 spaces which are not embeddable.

Finally, in Section IV, we consider (2). We denote by $\tilde{\tau}_2$ the topology on E' generated by characteristic functionals of Gaussian probability measures. Sazanov's result shows that for real separable Hilbert spaces, the equicontinuity in $\tilde{\tau}_2$ of characteristic functionals of a family of probability measures implies the tightness of the family. It is shown that the validity of this result, in fact, characterizes the cotype 2 spaces.

TO MY PARENTS

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SECTION 0

INTRODUCTION

Let E be a real separable Banach space with (topological) dual E' and duality denoted by $\langle\ ,\ \rangle$. Let φ be a function from E' into the set of complex numbers C, with $\varphi(0) = 1$ and which is positive definite in the sense that for each integer n, $\sum_{i,j} c_i \bar{c}_j \varphi(y_i - y_j) \geq 0$ for $c_i \in C$ and $y_i \in E'$ $(i = 1,2,\ldots,n)$. The solutions to the following problems have played a useful role in the case where E is one-dimensional.

Problem I. Give necessary and sufficient conditions on φ so that φ is a characteristic functional (or Fourier transform) of a probability measure μ on the Borel subsets $\mathcal{B}(E)$ of E; i.e., $\varphi(y) = \int_E \exp[i \langle y,x\rangle]\mu(dx)$ $(= \hat{\mu}(y)$, say).

Problem II. Let $\{\hat{\mu}_\alpha : \alpha \in I\}$ be the characteristic functionals (ch.f. for short) of a family $\{\mu_\alpha\}$ of probability measures on $\mathcal{B}(E)$. Give necessary and sufficient conditions on $\{\hat{\mu}_\alpha : \alpha \in I\}$ in order that the family is tight. (For the definitions of "tight" and other concepts used, the reader is referred to Section I.)

Bochner's work ([17], p.207 and p.193) provides complete answers to the above problems in the case where E is finite-dimensional. Namely, Problem I has a solution iff φ is continuous at zero and Problem II has a solution iff $\{\hat{\mu}_\alpha : \alpha \in I\}$ is an equicontinuous family.

It is known that the exact analogue of the above solutions fails in case E is an infinite-dimensional Hilbert space ([18],[27]). In this case, however, Sazanov [29] and Gross [10] gave a complete solution to Problem I; namely, Problem I has a solution iff φ is continuous in the topology S, generated by the non-negative trace class operators. In ([25],[29]) it was shown that sufficiency part of Problem II is valid if $\{\hat{\mu}_\alpha : \alpha \in I\}$ is a family of functions equicontinuous at zero in the S-topology, but the necessary part is not, in the case where E is an infinite dimensional Hilbert space.

In this work, we study a class of Banach spaces in which some analogues of the above solutions hold. To facilitate this we introduce

<u>Bochner Property I</u>: E has Bochner Property I if there exists a topology on E' such that continuity in it of φ solves Problem I for every φ.

<u>Bochner Property II</u>: E has Bochner Property II if there exists a topology τ on E' such that $\{\hat{\mu}_\alpha : \alpha \in I\}$ is equicontinuous at zero in τ implies $\{\mu_\alpha : \alpha \in I\}$ is tight.

From the previous comments we get that the class of Banach spaces with Bochner Property I and Bochner Property II includes all separable Hilbert spaces. It is known that a separable Hilbert space is embeddable in $L^0(\Omega,P)$ [30], and has approximation property (in fact, has a basis). In ([22], Theorem 1) Mustari has shown that

(0.1) All embeddable (in $L^0(\Omega,P)$) Banach spaces with approximation property have Bochner Property I

and

(0.2) All spaces with Bochner Property I are of cotype 2.

He raises the question of whether all cotype 2 spaces have Bochner Property I. We show that this is not true by means of examples. These are of general interest as they provide examples of spaces of cotype 2 which are not embeddable. We present them in Section 2.

We observe that the main tool of Mustari's proof of (0.1) is an inequality first proved by Lévy for which the above two hypotheses of embeddability and metric approximation property seem to be tailor-made. In order to bring this point across we give, in Section 3, a simple proof of Mustari's result, based on what we now term as a Lévy Inequality.

This proof shows that one can characterize subclasses of Banach spaces having Bochner Property I with respect to the smallest topologies making ch.f. of certain probability measures continuous. We give such a characterization of Banach spaces embeddable in L^p using topologies associated with ch.f. of symmetric stable probability measures.

In the last section we study the class of Banach spaces having Bochner Property II with respect to topologies generated by ch.f. of stable measures. For the case of stable measures of index p, where $1 \leq p < 2$ we only get a partial characterization. However for $p = 2$, i.e., in the case of Gaussian measures we get that a Banach space has Bochner Property II iff it is of cotype 2. This also shows that Bochner Property I is different from Bochner Property II, since by [22] we know that a Banach space has Bochner Property I with respect to the Gaussian topology iff it is isomorphic to a Hilbert space.

We conclude by giving some final remarks and open problems. For a review of other methods used in solution of Bochner Problems I and II see [18].

SECTION I

PRELIMINARIES AND NOTATION

This section contains a description of the tools used in later sections. We also fix the notation used throughout.

A 'Banach space' E will mean a real, separable, complete normed linear space with norm $\| \|_E$. We will denote its topological dual by E'.

A measure space is a pair (Ω,μ), where μ is a countably additive, non-negative measure on some σ-algebra of subsets of Ω. It is a probability space if $\mu(\Omega) = 1$. $L^\circ(\Omega,\mu)$ is the equivalence classes of real valued measurable functions where functions, equal μ a.e., are identified. (Throughout this work we consider $L^\circ(\Omega,\mu)$ with μ a probability measure and the topology to be that of convergence in probability). For $1 \leq p < \infty$, $L^p(\Omega,\mu)$ is the Banach space with norm $\|\cdot\|_p$ consisting of all $f \in L^\circ(\Omega,\mu)$ such that

$$\|f\|_p^p = \int |f|^p \, d\mu < \infty.$$

In case $\Omega = [0,1]$ and μ is the Lebesgue measure we denote this space by L^p and for the case $\Omega = Z^+$ (natural numbers) and μ the counting measure, by ℓ^p. We denote by $\ell^p(E)$ the Banach space $\{(x_i) \in E^\infty | \sum_{i=1}^{\infty} \|x_i\|_E^p < \infty\}$ with the norm

$$\|(x_i)\| = (\sum_{i=1}^{\infty} \|x_i\|_E^p)^{1/p}, \quad 1 \leq p < \infty.$$

<u>Definition 1.1</u>. A space E is said to be embeddable in a linear metric space F, if there exists a linear topological isomorphism of E into F.

<u>Lemma 1.2</u>. A separable (closed) subspace G of an infinite dimensional $L^p(\Omega,\mu)$, for an arbitrary measure μ, can be embedded in L^p, $1 \leq p < \infty$.

<u>Proof</u>: By ([6], Lemma 5, p.168 and [11], Theorem C, p.173) we obtain that G is embeddable in $L^p \oplus \ell^p$. However ℓ^p embeds in L^p ([16], p.133). Thus $L^p \oplus \ell^p$ embeds in L^p, giving the result.

<u>Definition 1.3</u> [20]. Let (Ω,μ) be a probability space. Then E is said to be strongly embeddable in $L^p(\Omega,\mu)$, $0 < p < \infty$, if E is embeddable in $L^p(\Omega,\mu)$ and the L^p-convergence and the L^0-convergence coincide on the image of E.

<u>Definition 1.4</u>. We say that a separable Banach space E has the metric approximation property if there exists a sequence $\{\pi_n\}$ of operators of finite rank such that $\|\pi_n x - x\|_E \to 0$ for each $x \in E$.

Definition 1.5. A Banach space E is said to be of Rademacher type p if for every sequence $\{x_i\}_{i=1}^{\infty} \subset E$ with $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$ we have $\sum x_i \varepsilon_i$ converges a.e. where ε_i are independent identically distributed (i.i.d. for short) symmetric Bernoulli random variables, i.e., $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$.

Definition 1.6. A real valued random variable with ch.f. $\exp(-|t|^p)$ for t real, is called a symmetric stable random variable of index p $(1 \leq p < 2)$. For $p = 2$ it is called a standard Gaussian random variable.

Definition 1.7. A Banach space E is said to be of stable type p if for every sequence $\{x_j\}_{j=1}^{\infty} \subset E$ with $\sum \|x_j\|^p < \infty$ we have $\sum x_j \eta_j$ converges a.e., where η_j are i.i.d. symmetric stable random variables of index p, $1 \leq p \leq 2$.

Remark 1.8. Spaces of Rademacher type 2 may be equivalently defined over the Gaussian system, i.e., E is of Rademacher type 2 if for every sequence $\{x_i\} \subset E$ with $\sum \|x_i\|^2 < \infty$ we have $\sum x_i \gamma_i$ converges a.e., where γ_i are i.i.d. standard Gaussian random variables. Thus Rademacher type 2 is equivalent to stable type 2 and we refer this case as of type 2. For $p \neq 2$ we have the following Lemma ([21], p.79).

Lemma 1.9. (i) E is of stable type p implies E is of Rademacher type p.

(ii) E is of Rademacher type p implies E is of stable type q for all $q < p$.

Definition 1.10. E is said to be of cotype 2 (Rademacher) if for every sequence $\{x_i\} \subset E$ satisfying $\sum x_i \varepsilon_i$ converges a.e. we have $\sum \|x_i\|^2$ is finite. Or equivalently ([21], Thms. 1.1,1.2 and Cor. 1.3) the space is of cotype 2 if a.e. convergence of $\sum x_i \gamma_i$ implies $\sum \|x_i\|^2$ is finite, where γ_i are i.i.d. standard Gaussian random variables.

The following lemma shows that an analogous definition of stable cotype p, $1 \le p < 2$ does not restrict the class of Banach spaces.

Lemma 1.11. If η_j are i.i.d. symmetric stable random variables of index p, $1 \le p < 2$, then the a.e. convergence of $\sum x_j \eta_j$ implies $\sum \|x_j\|^p$ is finite, for $\{x_j\}_{j=1}^{\infty}$ is any Banach space E.

Proof: First note that $\sum \|x_j\|^p$ converges iff $\sum P(|\eta_j| > \frac{1}{\|x_j\|})$ converges, ([7], p.544). But $\sum P(|\eta_j| > \frac{1}{\|x_j\|}) = \sum_j P(\|\eta_j x_j\| > 1)$ which is finite by Borel-Cautelli Lemma, since $\sum \eta_j x_j$ converges a.e. implies $\|\eta_j x_j\| \to 0$ a.e.

We also need some results on the weak convergence of measures on linear spaces ([4]). We start by defining cylinder measures ([3]).

Let G and F be two Hausdorff topological vector spaces in separating duality, with the duality denoted by $\langle\ ,\ \rangle$. Let M be a finite dimensional subspace of F, $M^{\perp}$ the annahilator $\{x \mid x \in G, \langle x,x'\rangle = 0$ for all $x' \in M\}$ and let $G_M = G/M^{\perp}$. Then G_M is finite dimensional.

If $M_1 \subseteq M_2$ are finite dimensional subspaces of F, we denote by ρ_{M_1,M_2} the projection of G_{M_2} onto G_{M_1}. $\mathcal{B}(G_M)$ will denote the σ-field of Borel subsets of G_M and ρ_M the canonical map of G into G_M. We note that $G = \rho_m^{-1}(G_M)$ and call $\bigcup_M \rho_M^{-1}(\mathcal{B}(G_M)) = \mathcal{C}$ the algebra of cylinder sets.

<u>Definition 1.12</u>. A cylinder measure μ on G is a family of probability measures $\{\mu_M : M$ a finite-dimensional ($\sigma(G,F)$ closed) subspace of $F\}$ such that μ_M is a probability measure on $\mathcal{B}(G_M)$ and if $M_1 \subseteq M_2$ then $\mu_{M_1} = \mu_{M_2} \circ \rho_{M_1,M_2}^{-1}$.

Given a cylinder measure we can define a finitely additive measure μ on $\bigcup_M \rho_M^{-1}(\mathcal{B}(G_M))$ by $\mu(\rho_M^{-1}(A)) = \mu_M(A)$ for $A \in \mathcal{B}(G_M)$. Conversely given any finitely additive measure μ on $\bigcup_M \rho_M^{-1}(\mathcal{B}(G_M))$, with $\mu(G) = 1$ and μ restricted to $\rho_M^{-1}(\mathcal{B}(G_M))$ countably additive, then the family $\{\mu_M = \mu \circ \rho_M^{-1}\}$ gives a cylinder measure. In particular, any countably additive measure μ on $\bigcup_M \rho_M^{-1}(\mathcal{B}(G_M))$ gives a cylinder measure.

<u>Definition 1.13</u>. Given a cylinder measure μ on G, then for each $x' \in F$, $\mu \circ x'^{-1}$ is a probability measure. We define the ch.f. of μ on F as $\hat{\mu}(x') = \int_R e^{is} \mu \circ x'^{-1}(ds)$.

<u>Proposition 1.14</u> ([3], p.19). The ch.f. defines a one-one onto map between cylinder measures on G and comple-valued positive definite functions φ on F with restriction to each finite dimensional subspace continuous and $\varphi(0) = 1$.

<u>Definition 1.15</u>. Given a topological vector space F, a linear continuous map Ψ of F into $L^\circ(\Omega,\mu)$ (with (Ω,μ) a probability space) is called a random linear functional.

<u>Remark</u>. Without loss of generality $\Psi(x)$ is assumed to be symmetric and of the form $U\tilde{\Psi}(x)$ where $\tilde{\Psi}$ is an embedding and U is uniform on $[-1,1]$ and is independent of $\{\tilde{\Psi}(x),\ x \in E\}$.

If G and F are in separating duality and Ψ is a random linear functional on F, then $\hat{P}_\Psi(x) = \int_\Omega e^{i\Psi(x)} \mu(dx)$ defines a continuous complex-valued positive definite function on F and hence a cylinder measure on G ([3], Exposé no. 2). If further Ψ is an embedding and F is a Banach space then we get, given $\varepsilon > 0$ there exists an $h(\varepsilon) > 0$ such that

$$\|x\|_F > \varepsilon \quad \text{implies} \quad |1 - \hat{P}_\Psi(x)| > h(\varepsilon).$$

Remark 1.16. Let E be a separable Banach space with E' its topological dual and $\langle \, , \, \rangle$ denoting the duality. Then $\mathcal{B}(E)$ is the σ-algebra generated by $\mathcal{C}$ (the algebra of cylinder sets), and hence by Proposition 1.14 we get that the ch.f. determines μ uniquely.

A separable Banach space E is a complete separable metric space. In order to study convergence of measures on E, we need some general concepts and results on the weak convergence of measures on the Borel subsets $\mathcal{B}(S)$ of a metric space S [4].

Definition 1.17. A sequence of probability measures $\{\mu_n\}$ on $\mathcal{B}(S)$ is said to converge weakly to the probability measure μ on $\mathcal{B}(S)$ if $\int_S f d\mu_n \to \int_S f d\mu$ for every bounded continuous real valued function defined on S.

If μ_n converges weakly to μ, we write $\mu_n \Rightarrow \mu$.

Definition 1.18. A family $\{\mu_\alpha : \alpha \in I\}$ of probability measures on $(S, \mathcal{B}(S))$ is said to be relatively compact if every sequence of elements $\{\mu_{\alpha_n}\}_{n=1}^{\infty}$ contains a weakly convergent subsequence.

Definition 1.19. A probability measure μ on $(S, \mathcal{B}(S))$ is said to be tight if for each $\epsilon > 0$, there exists a compact set K such that $\mu(K) > 1 - \epsilon$.

Note that on a separable complete metric space each probability measure is tight ([4], p.10).

Definition 1.20. A family $\{\mu_\alpha : \alpha \in I\}$ of probability measures is said to be tight, if for each $\epsilon > 0$, there exists a compact set K, such that $\mu_\alpha(K) > 1 - \epsilon$ for all $\alpha \in A$.

The next proposition is due to Proherov, for a proof see ([4], p.37).

Proposition 1.21. Let S be a separable and complete metric space. Then a family $\{\mu_\alpha : \alpha \in I\}$ of probability measures on $(S, \mathcal{B}(S))$ is tight iff the family is relatively compact.

We conclude this section by giving the following known facts from the theory of probability measures on Banach spaces.

Theorem 1.22 ([12], Theorem 3.1 and 4.1 and [32]). Let $\{X_k\}_{k=1}^{\infty}$ be independent E-valued random variables with $S_n = \sum_1^n X_k$ and μ_n the distribution of S_n. Then the following are equivalent.

(a) S_n converges a.e.

(b) S_n converges in probability.

(c) μ_n converges weakly.

If further $\{X_k\}$ are symmetrically distributed then (a), (b), (c) are equivalent to

(d) $\{\mu_n\}$ is tight.

(e) There exists an E-valued random variable S such that $\langle y, S_n\rangle \to \langle y, S\rangle$ for every $y \in E'$.

(f) There exists a probability measure μ on E such that $\varphi_{\mu_n}(y) \to \varphi_\mu(y)$ for all $y \in E'$.

Definition 1.23. A probability measure μ on a separable Banach space E is said to be symmetric stable of order p $(1 \leq p \leq 2)$, if for each $y \in E'$, $\mu \circ y^{-1}$ is a symmetric stable measure of order p on the real line R. For $p = 2$ it is called (symmetric) Gaussian measure on E.

Theorem 1.24 ([2],[19], for a proof see [19]). Let E be a Banach space and $p \in (0,2]$, then the following are equivalent.

(1) E is of stable type p.

(2) Every function of the form $\varphi(y) = \exp[-\int_\Gamma |\langle y,u\rangle|^p \lambda(du)]$ for a finite measure λ on Γ, the boundary of the unit ball of E, is the ch.f. of a (necessarily stable) probability measure on E.

(3) Every function of the form $\varphi(y) = \exp[-\int_E |\langle y,u\rangle|^p \nu(du)]$, where $\int_E \|x\|^p \nu(dx)$ is finite, is the ch.f. of a (necessarily stable) probability measure on E.

Definition 1.25. A probability measure μ on $\mathcal{B}(E)$ is said to be infinitely divisible if for each integer n, there exists a probability measure μ_n on E, so that $\mu = \mu_n^{*n}$ (here $*$ denotes convolution of measures).

Theorem 1.26 ([19]). For a real separable Banach space the following are equivalent:

(1) E is of Rademacher type p.

(2) The positive definite function given by

$$\varphi(y) = \exp[\int (e^{i\langle y,x\rangle} - 1 - i \langle y,x\rangle/1 + \|x\|^p) F(dx)]$$

where

(a) F is a σ-finite measure on E with $F(\{0\}) = 0$ and F finite outside neighborhood of zero.

(b) $\int_{\|x\|\le 1} \|x\|^p \, F(dx) < \infty$, $(0 < p \le 2)$

is a ch.f. of a non-Gaussian infinitely divisible measure.

The above theorems use the concept of "flat concentration" due to deAcosta ([1], p.279).

Definition 1.27. A family $\{\mu_\alpha : \alpha \in I\}$ of probability measures on $(E,\mathcal{B}(E))$ is flatly concentrated if for every $\varepsilon > 0$ and $\delta > 0$, there exists a finite dimensional subspace M of E, such that

$$\mu_\alpha\{x \in E | \inf\{\|x - z\| : z \in M\} \le \varepsilon\} \ge 1 - \delta$$

for all $\alpha \in I$.

SECTION II

EXAMPLES OF SPACES OF COTYPE 2 WHICH ARE NOT EMBEDDABLE IN $L^\circ(\Omega,\mu)$

We need the following results.

<u>Theorem 2.1</u> ([20], Theorem 98). Let (Ω,μ) be a probability space. If E embeds in $L^\circ(\Omega,\mu)$ then the following are equivalent for $0 < p \leq 2$.

(a) E is of stable type p.

(b) E can be strongly embedded in $L^p(\Omega,\mu)$.

<u>Theorem 2.2</u> ([28], p.775). If $\ell^p(E)$ embeds in L^1, then E embeds in L^q for all $q < p$.

<u>Lemma 2.3</u> ([21], p.55). If E is of cotype 2, then so is $\ell^p(E)$ for $1 \leq p \leq 2$.

<u>Lemma 2.4</u> ([21], Proposition 2.2, Corollary 2.2). If E is of stable type q it is of stable type r for all $r \leq q$ and so is $\ell^p(E)$ for all p, $q < p$.

<u>Example 2.5</u>. Choose a real separable Banach space E of cotype 2, stable type q for $q > 1$, and not

embeddable in L^{q_0} for some q_0 $(1 \leq q_0 < 2)$. Then for p, with $\max(q,q_0) < p \leq 2$, $\ell^p(E)$ is of cotype 2, but not embeddable in $L^\circ(\Omega,\mu)$.

<u>Proof</u>: Suppose $\ell^p(E)$ is embeddable in $L^\circ(\Omega,\mu)$, then by the assumptions on E, $\ell^p(E)$ is of stable type 1, thus by Theorem 2.1 it embeds in $L^1(\Omega,\mu)$. But E is separable, giving $\ell^p(E)$ is separable, thus its image is a separable subspace of $L^1(\Omega,\mu)$. By Lemma 1.2 we can assume that $\ell^p(E)$ is embeddable in L^1. Using Theorem 2.2, we get that E embeds in L^r for all $r < p$. This contradicts the choice of E. Thus $\ell^p(E)$ cannot embed in $L^\circ(\Omega,\mu)$, but by Lemma 2.3, is of cotype 2.

The spaces ℓ^q, $(1 < q < 2)$, satisfy the assumptions on E. Therefore for any $1 < q < p \leq 2$, $\ell^p(\ell^q)$ give examples of the required spaces.

SECTION III

BOCHNER PROPERTY I

We first prove an analogue of Lévy Inequality ([5], Proposition 8.29, p.171) for a real separable Banach space E, embeddable in $L^0(\Omega,P)$ and having metric approximation property.

We denote the embedding by Ψ, and by P_Ψ the cylinder measure induced on E' with ch.f. $\hat{P}_\Psi$. Without loss of generality we can assume that Ψ maps E into the set of symmetric real valued random variables, i.e., $\hat{P}_\Psi$ is real valued.

Let the sequence of finite dimensional operators associated with the metric approximation property of E be $\{\pi_n, n \in N\}$. Denote by π_n' the transpose of π_n.

With the above notation we get,

<u>Lemma 3.1</u> (Lévy Inequality). Any probability measure μ on E satisfies the following inequality, given $\varepsilon > 0$, there exists an $h(\varepsilon) > 0$ such that

$$\mu\{x \mid \|(\pi_m - \pi_k)x\| > \varepsilon\} \leq \frac{1}{h(\varepsilon)} \int_{E'} [1 - \hat{\mu}((\pi_m - \pi_k)'y)]\, P_\Psi(dy).$$

Proof: From embeddability we get that given $\epsilon > 0$, there exists $h(\epsilon) > 0$, such that whenever $\|x\| > \epsilon$, $1 - \hat{P}_\psi(x) > h(\epsilon)$ for all $x \in E$. Using this and Chebychev's inequality

$$\mu\{x \mid \|(\pi_m - \pi_k)x\| > \epsilon\} \leq \mu\{x \mid (1 - \hat{P}_\psi((\pi_m - \pi_k)x)) > h(\epsilon)\}$$

$$\leq \frac{1}{h(\epsilon)} \int_E [1 - \hat{P}_\psi((\pi_m - \pi_k)x)]\mu(dx)$$

$$= \frac{1}{h(\epsilon)} \int_E [\int_{E'} (1 - e^{i\langle y,x\rangle}) P_\psi^{(\pi_m - \pi_k)'}(dy)]\mu(dx)$$

where $P_\psi^{(\pi_m-\pi_k)'}$ denotes the probability measure on E' with the finite dimensional support $(\pi_m - \pi_k)'E'$.

Since the measures involved above are probability measures and the function is jointly measurable, by Fubini Theorem we get

$$\mu\{x \mid \|(\pi_m - \pi_k)x\| > \epsilon\} \leq \frac{1}{h(\epsilon)} \int_{E'} [1 - \hat{\mu}((\pi_m - \pi_k)'y)] P_\psi(dy).$$

Note that $\hat{P}_\psi$ being real valued all integrals above are real. This completes the proof.

Let $\varkappa$ denote the set of certain positive definite functions θ on E' such that,

(3.2) (i) θ is real valued.

(ii) $\theta(0) = 1$.

(iii) θ is continuous in the norm topology on E'.

(iv) If θ_1 and θ_2 belong to $\varkappa$ then the product $\theta_1 \cdot \theta_2$ is in $\varkappa$.

Let

$$U_{\theta,\varepsilon} = \{y \in E' \mid 1 - \theta(y) < \varepsilon\}.$$

Then the system $\{U_{\theta,\varepsilon} : \theta \in \varkappa,\ \varepsilon > 0\}$ forms a sub-basis of neighborhoods at zero and generates a topology with respect to which exactly all functions in $\varkappa$ are continuous.
We will denote this topology by $\tau_{\varkappa}$. The following gives an example of such a family: For μ_1 and μ_2, two probability measures on E, we denote their convolution by the probability measure

$$\mu_1 * \mu_2(A) = \int_E \mu_1(A - x)\mu_2(dx),$$

and let $\varkappa_0 = \{\hat{\mu} \mid \mu$ belongs to a subclass of symmetric probability measures on E, which is closed under convolution$\}$.
Then $\varkappa_0$ satisfies conditions (3.2).

<u>Lemma 3.3</u>. If E has metric approximation property and is embeddable in $L^\circ(\Omega, P)$ with embedding Ψ, then the continuity in $\tau_{\varkappa_0}$ of a positive definite function φ, with $\varphi(0) = 1$, implies

$$\lim_k \sup_n \lim_m \int_{E'} \operatorname{Re}[1 - \varphi(\pi_n'(\pi_m - \pi_k)'y)]P_\Psi(dy) = 0.$$

<u>Proof</u>: Since φ is continuous in $\tau_{\varkappa_0}$, given $\varepsilon > 0$, there exists a $\delta > 0$ and a symmetric probability measure ν with $\hat{\nu} \in \varkappa_0$ such that

$$\operatorname{Re}(1 - \varphi(y)) < \varepsilon \quad \text{whenever} \quad 1 - \hat{\nu}(y) < \delta.$$

Using the fact that $\mathrm{Re}(1-\varphi(y)) \leq 2$ for all $y \in E'$ we get,

$$\mathrm{Re}(1-\varphi(y)) \leq \frac{2}{\delta}\,(1-\hat{\nu}(y)) + \varepsilon \quad \text{for all} \quad y \in E'.$$

Thus

$$\int_{E'} \mathrm{Re}[1-\varphi(\pi_n'(\pi_m-\pi_k)'y)]\,P_\Psi(dy)$$

$$\leq \frac{2}{\delta}\int_{E'}[1-\hat{\nu}(\pi_n'(\pi_m-\pi_k)'y)]\,P_\Psi(dy) + \varepsilon$$

$$= \frac{2}{\delta}\int_{E'}\Big(\int_E (1-e^{i\langle \pi_n'(\pi_m-\pi_k)'y,x\rangle})\,\nu(dx)\Big)P_\Psi(dy) + \varepsilon$$

$$= \frac{2}{\delta}\int_E (1-\hat{P}_\Psi((\pi_m-\pi_k)\pi_n x))\,\nu(dx) + \varepsilon.$$

But $\|\pi_m x - x\| \to 0$ for all $x \in E$ and $\hat{P}_\Psi$ is continuous, thus

$$\limsup_{k}\ \sup_{n}\ \lim_{m} \int_E [1-\hat{P}_\Psi((\pi_m-\pi_k)\pi_n x)]\,\nu(dx) = 0,$$

giving

$$\lim_{k}\ \sup_{n}\ \lim_{m} \int_{E'} \mathrm{Re}[1-\varphi(\pi_n'(\pi_m-\pi_k)'y)]\,P_\Psi(dy) \leq \varepsilon,$$

but ε is arbitrary and thus we get the result.

Lemma 3.4. If a function on E' is continuous in $\tau_{\varkappa_0}$, then it is sequentially weak-star continuous.

Proof: It is enough to show that the ch.f. of any probability measure is sequentially weak-star continuous. This can be shown by using Lebesgue Dominated Convergence Theorem.

Using the above lemmas we get the following result which includes a result of Mustari ([22], Theorem 1(C)).

Theorem 3.5. If E is embeddable in $L^0(\Omega,P)$ for a probability space (Ω,P), and has the metric approximation property, then every positive definite function φ, with $\varphi(0) = 1$, and which is continuous in τ_{κ_0}, is a ch.f. of a probability measure on E.

Proof: We use the notations as defined above.

By Lemma (3.4) φ is sequentially weak-star continuous. Hence its restriction to every finite dimensional subspace of E' is continuous. Thus there exists a cylinder measure μ associated with φ (1.14). Let $\mu_n = \mu \circ \pi_n^{-1}$, then for each n, μ_n is a Borel probability measure on E having finite dimensional support $\pi_n(E)$ and $\hat{\mu}_n(y) = \varphi(\pi_n'(y))$. Using (3.1) we get the Lévy inequality; given $\varepsilon > 0$, there exists $h(\varepsilon) > 0$ such that

$$\mu_n\{x \mid \|(\pi_m - \pi_k)x\| > \varepsilon\} \le \frac{1}{h(\varepsilon)} \int_{E'} [1 - \varphi(\pi_n'(\pi_m - \pi_k)'y] P_\Psi(dy).$$

By Lemma 3.3 we have

$$\lim_k \sup_n \lim_m \int_{E'} [1 - \varphi(\pi_n'(\pi_m - \pi_k)'y] P_\Psi(dy) = 0.$$

Therefore

$$\lim_k \sup_n \mu_n\{x \mid \|(I - \pi_k)x\| > \varepsilon\}$$

$$= \lim_k \sup_n \lim \mu_n\{x \mid \|(\pi_m - \pi_k)x\| > \varepsilon\} = 0.$$

Thus for $\varepsilon > 0$, $\varepsilon' > 0$ there exists a finite-dimensional subspace $\pi_{n_0}(E)$ of E such that $\sup_n \mu_n\{x \mid \|(I - \pi_{n_0})x\| > \varepsilon\} < \varepsilon'$, giving $\{\mu_n\}_{n \in N}$ is flatly concentrated in the sense of (1.27).

Now as φ is sequentially weak-star continuous and $\langle \pi_n' y, x \rangle \to \langle y, x \rangle$ for each $x \in E$, we get that $(\hat{\mu}_n(y) =)\varphi(\pi_n' y)$ converges pointwise to $\varphi(y)$. It also follows that restriction of φ to each one-dimensional subspace of E' is continuous.

Thus by ([1], Theorem 2.4, p.280) we get the existence of a probability measure μ, such that $\hat{\mu}(y) = \varphi(y)$.

<u>Corollary 3.6</u>. If E has metric approximation property and embeds in $L^0(\Omega, P)$, and the topology τ_{κ_0} is such that the ch.f. of every probability measure is continuous in it, then E has Bochner Property I with respect to it.

<u>Theorem 3.7</u>. If E has metric approximation property and embeds in $L^0(\Omega, P)$, then E has Bochner Property I.

<u>Proof</u>: In Theorem 3.5 we take the set κ_0 to consist of ch.f. of all symmetric probability measures on E. Then the continuity of ch.f. in τ_{κ_0} of every probability measure μ follows from the inequality $|1 - \hat{\mu}(y)|^2 \leq 2(1 - \mathrm{Re}\,\hat{\mu}(y))$ and the fact that $\mathrm{Re}\,\hat{\mu}(y)$ is the ch.f. of the symmetric probability measure ν given by $\nu(A) = \frac{1}{2}(\mu(A) + \mu(-A))$.

We next show that for embeddable spaces with metric approximation property and of Rademacher or of stable type p, we can get topologies of the form τ_κ where κ can be described explicitly.

Corollary 3.8. Let E be of Rademacher type p, embeddable in $L^0(\Omega, P)$ and have the metric approximation property. Let F be a symmetric measure on E satisfying

(i) F is σ-finite on E with $F\{0\} = 0$ and finite outside every neighborhood of zero.

(ii) $\int_{\|x\| \leq 1} \|x\|^p F(dx)$ is finite.

Let

$$\varkappa_p = \{\psi : E' \to C \mid \psi(y) = \exp \int (\cos \langle y,x\rangle - 1) F(dx), \text{ for all } F \text{ satisfying (i) and (ii)}\}.$$

Then if a positive definite function φ, with $\varphi(0) = 1$ is continuous in $\tau_{\varkappa_p}$, it is a ch.f. of a probability measure on E.

Proof: We note that $\varkappa_p$ is the set of ch.f.'s of symmetric non-Gaussian infinitely divisible measures on E by Theorem 1.26, and satisfies conditions (3.2). The result now follows from Theorem 3.5.

Next we consider topologies associated with symmetric stable measures. We will denote by τ_p (for $p \in [1,2]$) the topology $\tau_\varkappa$ obtained by taking

$$\varkappa = \{\psi \mid \psi(y) = \exp[-\int_E |\langle y,x\rangle|^p \nu(dx), \text{ as } \nu \text{ varies through the set of measures for which } \int_E \|x\|^p \nu(dx) < \infty\}.$$

Let $\tilde{\tau}_p$ be the topology $\tau_\varkappa$ obtained by taking $\varkappa$ to be the set of ch.f.'s of all symmetric stable measures of index p on E. Then in general, $\tilde{\tau}_p$ is weaker than τ_p [33], but in the case where E is of stable type p, τ_p coincides with $\tilde{\tau}_p$ by Theorem 1.24 and E' is a topological vector space under τ_p ([30], p.125).

For the sake of abbreviation we say that E has B(I,p) if it has Bochner Property I with respect to the topology τ_p. We note that (Theorem 1.24) if E has B(I,p) then E is of stable type p.

<u>Corollary 3.9</u>. If E is of stable type p, embeddable in $L^0(\Omega,P)$ and has metric approximation property, then E has B(I,p), $1 \leq p \leq 2$.

<u>Proof</u>: Since E is of stable type p, τ_p topology coincides with $\tilde{\tau}_p$. Therefore by Corollary 3.6 it is enough to show that every ch.f. is continuous in τ_p.

Suppose φ is the ch.f. of a probability measure μ on E, i.e.,

$$\varphi(y) = \int_E e^{i\langle y,x\rangle}\,\mu(dx).$$

Then

$$|1-\varphi(y)|^2 \leq 2\,\mathrm{Re}(1-\varphi(y)) = 2\int_E (1-\cos\langle y,x\rangle)\mu(dx).$$

Since μ is a probability measure on E, given $\varepsilon > 0$, there exists a compact set $K \subset E$ such that $\mu(K) > 1-\varepsilon/2$, (1.19). Therefore

$$\begin{aligned} 2\int_E (1-\cos\langle y,x\rangle)\mu(dx) &\leq 2\int_K (1-\cos\langle y,x\rangle)\mu(dx) + \varepsilon \\ &\leq 4\int_K |\langle y,x\rangle|^p\,\mu(dx) + \varepsilon, \\ &\qquad 1 \leq p \leq 2. \end{aligned}$$

Let ν be the measure on E such that for all $A \in \mathcal{B}(E)$, $\nu(A) = \mu(K \cap A)$. Thus

$$\int_E \|x\|^p \nu(dx) = \int_K \|x\|^p \nu(dx) \leq \sup_{x \in k} \|x\|^p \cdot \nu(K) < \infty,$$

and

$$|1 - \varphi(y)|^2 \leq 4\int_E |\langle y,x\rangle|^p \nu(dx) + \varepsilon.$$

Hence φ is continuous in τ_p.

It is well known that φ is positive definite and $\varphi(0) = 1$.

Remark 3.10. The fact that τ_p gives a necessary topology is true without any assumptions on E.

Corollary 3.11. If E is of Rademacher type p, $1 < p < 2$, has metric approximation property and is embeddable in $L^0(\Omega,P)$, then E has Bochner Property I with respect to $\tau_{\varkappa_p}$.

Proof: By Corollary 3.8 it is enough to show that ch.f. of any probability measure μ is continuous in $\tau_{\varkappa_p}$.

Since E is Rademacher p, it is stable q for all $q < p$, hence E has $B(I,q)$ by (3.9). Thus given $\varepsilon > 0$, there exists a symmetric stable measure ν of index q and a $\delta > 0$, such that $\mathrm{Re}(1 - \hat{\mu}(y)) < \varepsilon$ whenever $1 - \hat{\nu}(y) < \delta$.

We show that $\hat{\nu}$ belongs to $\varkappa_p$. It is known [33] that $\hat{\nu}(y) = \exp[-\int_\Gamma |\langle y,x\rangle|^q \lambda(dx)]$ where λ is a finite measure on the boundary Γ of the unit ball of E. Since

$\int_0^\infty (\cos ts - 1) \frac{c_q}{s^{1+q}} ds = -|t|^q$ for a constant c_q $(0 < c_q < \infty)$ ([5], p.205), we get

$$\hat{\nu}(y) = \exp[\int_\Gamma \int_0^\infty (\cos \langle y,x \rangle s - 1) \frac{c_q}{s^{1+q}} ds \, \lambda(dx)].$$

Identify $E = \Gamma \times [0,\infty)$ and define measure on $\mathcal{B}(E)$ as in ([19], p.323) by

$$F_1(A) = \int_\Gamma \int_0^\infty 1_A(x,s) c_q \frac{ds}{s^{1+q}} \lambda(dx)$$

and

$$F_2(A) = \int_\Gamma \int_{-\infty}^0 1_A(x,-s) \frac{ds}{|s|^{1+q}} \lambda(dx).$$

Let $F = \frac{1}{2}(F_1 + F_2)$, then F is symmetric, finite outside every neighborhood of zero, and

$$\int_{\|x\| \leq 1} \|x\|^p F(dx) = \int_\Gamma \int_0^1 \|u\|^p s^p \frac{ds}{s^{1+q}} \lambda(du)$$

$$= \lambda(\Gamma) \int_0^1 s^{p-q-1} ds,$$

which is finite since $q < p$.

Finally since F is symmetric we get

$$\hat{\nu}(y) = \exp \int_E (\cos \langle y,x \rangle - 1) F(dx),$$

giving $\hat{\nu} \in \varkappa_p$. Thus $\hat{\mu}$ is continuous in $\tau_{\varkappa_p}$.

To prove the converse of Corollaries 3.9 and 3.11 (without metric approximation property) we need a result of [15].

<u>Definition 3.12</u> ([15], p.282). Let X,Y be Banach spaces, $B(X,Y)$ the set of bounded linear operators from X into Y. Define

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\| \mid T \in B(X,Y)\}.$$

(If X and Y are not isomorphic then $d(X,Y) = \infty$).

Theorem 3.13[1] ([15], p.313). Let X be a Banach space, for $p \geq 1$, $\lambda \leq \infty$ there exists a measure ν and a subspace Y of $L^p(\nu)$, such that $d(X,Y) \leq \lambda$ iff whenever for all $y \in X'$

$$\sum_{i=1}^{n} |\langle y,u_i\rangle|^p \geq \sum_{j=1}^{m} |y,v_j\rangle|^p, \quad \{u_i\}_{i=1}^{n}, \{v_j\}_{j=1}^{m} \in X$$

then

$$\lambda^p \sum_{i=1}^{n} \|u_i\|^p \geq \sum_{j=1}^{m} \|v_j\|^p.$$

Theorem 3.14. If E has $B(I,p)$, $1 \leq p \leq 2$, then E is embeddable in $L^p(\Omega,\mu)$ for some measure μ.

Proof: Let $1 \leq p < 2$. Suppose E has $B(I,p)$ but is not embeddable in a $L^p(\Omega,\mu)$. Then for every subspace Y of $L^p(\Omega,\mu)$ and any integer $k \geq 1$, $d(E,Y) > 2^k$. Hence by Theorem 3.13, for each K, there exists finite sequences $\{u_i^k\}_{i=1}^{n_k}$ and $\{v_j^k\}_{j=1}^{m_k}$ in E, such that for all $y \in E'$

$$(3.15) \quad \begin{cases} \sum_{i=1}^{n_k} |\langle y,u_i\rangle|^p \geq \sum_{j=1}^{m_k} |\langle y,v_j\rangle|^p, \quad \text{but} \\ 2^{kp} \sum_{i=1}^{n_k} \|u_i^k\|^p < \sum_{j=1}^{m_k} \|v_j^k\|^p. \end{cases}$$

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[1] I thank Professor A. Weron for bringing this result to my attention in this context.

Without loss of generality we can assume that $\sum_{j=1}^{m_k} \|v_j^k\|^p = 1$. (Since if $\sum_{j=1}^{m_k} \|v_j^k\|^p = M_k$, we can replace v_j^k by $w_j^k = \frac{v_j^k}{M_k^{1/p}}$ and u_i^k by $x_i^k = \frac{u_i^k}{M_k^{1/p}}$. Then (3.15) holds for the finite sequences $\{x_i^k\}_{i=1}^{n_k}$ and $\{w_j^k\}_{j=1}^{m_k}$ with $\sum_{j=1}^{m_k} \|w_j^k\|^p = 1$).

We note that $\sum_{k=1}^{\infty} \sum_{i=1}^{n_k} \|u_i^k\|^p < \sum_{k=1}^{\infty} \frac{1}{2^{kp}} < \infty$, and $\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \|v_j^k\|^p = \infty$. Thus we have two sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ such that for all $y \in E'$,

$$(3.16)\quad \begin{cases} \sum_{i=1}^{\infty} |\langle y, u_i\rangle|^p \geq \sum_{j=1}^{\infty} |\langle y, v_j\rangle|^p \\ \sum_{i=1}^{\infty} \|u_i\|^p < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \|v_j\|^p = \infty. \end{cases}$$

We have already noted that since E has $B(I,p)$ it is of stable type p. Thus by definition of stable p, $\sum \|u_i\|^p$ is finite implies that $\sum u_i \eta_i$ converges a.e., for $\{\eta_i\}$ i.i.d. symmetric stable random variables of index p. Thus there exists a probability measure μ on E such that $\hat{\mu}(y) = e^{-\sum_{i=1} |\langle y, u_i\rangle|^p}$ for each $y \in E'$. Moreover μ is symmetric p-stable measure on E. Using this and the first inequality in (3.16) we can show that the positive definite function $\varphi(y) = e^{-\sum_{i=1}^{\infty} |\langle y, v_i\rangle|^p}$ is continuous in τ_p. Also $\varphi(0) = 1$. Thus by the hypothesis, there exists a probability measure ν on E, such that

$$\hat{\nu}(y) = \varphi(y) = e^{-\sum_{i=1}^{\infty} |\langle y, v_i\rangle|^p}.$$

By Theorem 1.22 we get, $\sum \nu_i \eta_i$ converges a.e. for $\{\eta_i\}$ i.i.d. symmetric p-stable random variables. Therefore $\sum \|v_i\|^p < \infty$ by Lemma 1.11, which contradicts 3.16. This gives the result.

For $p = 2$: τ_2 is generated by symmetric bilinear forms, hence E is isomorphic to a Hilbert space ([23]), thus embeds in L^2.

Remark 3.17. Using Lemma 1.2 we obtain that E is embeddable in L^p if it has $B(I,p)$. However ([30]) we know that L^p, $1 \leq p \leq 2$ is embeddable in $L^\circ(\Omega,P)$ for some probability space (Ω,P). Thus we obtain the following result.

Theorem 3.18. Let E have metric approximation property. Then E is of stable type p and embeddable in $L^\circ(\Omega,P)$ for some probability space (Ω,P) iff it has $B(I,p)$.

Lemma 3.19. If E is of stable type p and has Bochner Property I, then it has $B(I,p)$.

Proof: Suppose E has Bochner Property I with respect to some topology τ. As already noted for a space of stable type p, the τ_p topology coincides with the topology $\tilde{\tau}_p$ generated by ch.f.'s of symmetric stable measures, but these are continuous in τ. Thus a positive definite function which is continuous in τ_p is continuous in τ, and therefore is the ch.f. of a probability measure.

In general a ch.f. of a probability measure is necessarily continuous in τ_p (ch. 3.10). Thus E has B(I,p).

Remark 3.20. Theorem 3.18 and the examples in Section II with the above Lemma, show that cotype 2 is not a sufficient condition for a space to have Bochner Property I. Though, as we have already noted that it is a necessary condition ([19],[22]).

Remark 3.21. Combining a result of Maurey ([20], Theorem 98) with Theorem 3.14 and Remark 3.17 we get for real separable Banach spaces with metric approximation property the following are equivalent:

(1) E has B(I,p).

(2) E is strongly embeddable in $L^p(\Omega,\mu)$ for some probability space (Ω,μ).

SECTION IV

BOCHNER PROPERTY II

We recall our definition of Bochner Property II. A Banach space E is said to have Bochner Property II, if there exists a topology τ on E', such that given a family $\{u_\alpha : \alpha \in I\}$ of probability measures on E, the equicontinuity in τ of their ch.f. is a sufficient condition for tightness.

Let $\varkappa$ denote the set consisting of ch.f.'s of all probability measures on E and $\tau_\varkappa$ the corresponding topology as defined in Section III.

<u>Theorem 4.1</u>. If E has Bochner Property II with respect to $\tau_\varkappa$ then E is of cotype 2.

Proof is as in the necessity part of our next theorem.

Spaces having Bochner Property I are also of cotype 2, however the following theorem shows that Property I is not equivalent to Property II in view of [23].

We first make a few definitions.

Definitions 4.2 ([24], p.154). An S-operator on a Hilbert space H is a linear, symmetric, non-negative, compact operator having finite trace.

Given an E-valued symmetric Gaussian random variable X defined on some probability space (Ω, P). Let $\mu = P \circ X^{-1}$. Then by Fernique's Theorem [8], we know that $\int_E \|x\|^2 \mu(dx)$ is finite. Define an operator A from E' into E by

$$Ay = \int_E \langle y,x \rangle \, x \mu(dx)$$

where the integral is in the sense of Bochner. Then A is called the covariance operator of X and the ch.f. of μ is $\exp(-\frac{1}{2}\langle Ay,y \rangle)$. Thus $\tilde{\tau}_2$ coincides with the topology for which a basis of neighborhoods of zero is given by the system of sets $\{y \in E' | \langle Ay,y \rangle < 1\}$, where A runs through the set of Gaussian covariance operators. We note that if E is a separable Hilbert space then $\tilde{\tau}_2$ is the same as the S-topology and A is an S-operator [24].

Theorem 4.3. E has Bochner Property II with respect to the $\tilde{\tau}_2$ topology iff E is of cotype 2.

Proof: Suppose E is of cotype 2, and $\{\mu_\alpha\}_{\alpha \in I}$ a family of probability measures such that their ch.f. $\hat{\mu}_\alpha$ are equicontinuous at 0 in $\tilde{\tau}_2$. Then given $\epsilon > 0$, there exists a Gaussian covariance operator A_ϵ, such that

$$(4.4) \qquad \langle A_\epsilon y,y \rangle \le 1 \text{ implies } 1 - \operatorname{Re} \hat{\mu}_\alpha(y) \le |1 - \hat{\mu}_\alpha(y)| < \epsilon, \quad \forall \alpha \in I.$$

Let $\{X_k\}_{k\in Z^+}$ be independent, E-valued symmetric Gaussian random variables with X_k having covariance $\langle A_{1/k}y,y\rangle$. Observe that $\mathcal{E}\|X_k\| < \infty$ [8], where $\mathcal{E}$ denotes expectation. Define

$$X = \sum 2^{-n} \sigma_n X_n \quad \text{where} \quad \sigma_n = \mathcal{E}\|X_n\|^{-1},$$

$$\text{(here } \sigma_n X_n = 0 \text{ if } \mathcal{E}\|X_n\| = 0).$$

Then X is a symmetric Gaussian E-valued random variable with covariance $\langle Ay,y\rangle = \sum 2^{-2n} \sigma_n^2 \langle A_{1/n}y,y\rangle$. Further for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$(4.5) \qquad |1 - \hat{\mu}_\alpha(y)| < \varepsilon, \quad \text{whenever} \quad \langle Ay,y\rangle < \delta.$$

Let λ be the Gaussian measure corresponding to X (i.e., the distribution of X). Now, since E is of cotype 2, there exists a Hilbert space H, a continuous linear operator U from H into E and a symmetric Gaussian measure λ_1 on H such that $\lambda = \lambda_1 \circ U^{-1}$ ([9], Theorem 4). Without loss of generality we can (and we do!) assume that U is one-one. Let T be the covariance operatoryof λ_1, then T is an S-operator and $\hat{\lambda}_1(h) = e^{-1/2(Th,h)}$. Therefore $\hat{\lambda}(y) = e^{-1/2\langle UTU^*y,y\rangle}$, where U^* denotes the adjoint. By Remark 1.16, $A = UTU^*$.

Define $\hat{\nu}_\alpha$ on $U^*(E')$ by $\hat{\nu}_\alpha(U^*y) = \hat{\mu}_\alpha(y)$. If $U^*y_1 = U^*y_2$ then $UTU^*y_1 = UTU^*y_2$ i.e., $Ay_1 = Ay_2$. Hence by (4.5) and positive-definiteness of $\hat{\mu}_\alpha$ we get $\hat{\mu}_\alpha(y_1) = \hat{\mu}_\alpha(y_2)$. Hence $\hat{\nu}_\alpha$ is well defined on the range of U^* and

$$|1 - \hat{\nu}_\alpha(U^*y)| < \varepsilon \quad \text{if} \quad (TU^*y,U^*y)_H < \delta.$$

But the range of U^* is dense in H and $\hat{\nu}_\alpha$ is uniformly continuous on the range of U^* giving

(4.6) $\quad |1-\hat{\nu}_\alpha(h)| < \epsilon$ whenever $(Th,h)_H < \delta$.

In other words, $\{\hat{\nu}_\alpha : \alpha \in I\}$ is equicontinuous in the S-topology in the sense of ([24], p.155). By ([29]) $\{\nu_\alpha : \alpha \in I\}$ is tight. Since U is continuous, $\{\nu_\alpha \circ U^{-1} : \alpha \in I\}$ is tight on E ([4], p.30). But $\nu_\alpha \circ U^{-1} = \mu_\alpha$ by Remark 1.16, completing the sufficiency part.

To prove the necessity part suppose that E has Bochner Property II with respect to $\tilde{\tau}_2$ but is not of cotype 2. Then there exists a sequence $\{x_k\}_{k=1}^\infty$ such that for γ_k independent standard Gaussian random variables, $\sum x_k\gamma_k$ converges a.e., but $\sum \|x_k\|^2 = \infty$. Let $a_k = (\sum_{i=1}^{k} \|x_i\|^2)^{1/2}$, then $a_k^2 \to \infty$ and $\sum_{k=1}^{n} \frac{\|x_k\|^2}{a_k^2} \to \infty$.

Define E-valued independent symmetric random variables $\{Y_k\}_{k=1}^\infty$ by

$$\begin{cases} P(Y_k = \frac{a_k x_k}{\|x_k\|}) = P(Y_k = \frac{-a_k x_k}{\|x_k\|}) = \frac{1}{4}\frac{\|x_k\|^2}{a_k^2} \\ \text{and} \quad P(Y_k = 0) = 1 - \frac{1}{2}\frac{\|x_k\|^2}{a_k^2} . \end{cases}$$

Since $\sum_k P\{\|Y_k\| > \epsilon\} = \frac{1}{2}\sum_k \frac{\|x_k\|^2}{a_k^2} = \infty$, by Borel-Cantelli Lemma we get that $Y_k \not\to 0$ a.e., and hence $\sum Y_k$ diverges a.e.,

Let φ_k denote the ch.f. of the Gaussian E-valued random variable $\sum_{n=1}^{k} x_n\gamma_n$ and φ the ch.f. of the Gaussian E-valued random variable $\sum_{n=1}^{\infty} x_n\gamma_n$. Then $\varphi_k(y)$ and $\varphi(y)$ are continuous in $\tilde{\tau}_2$. Moreover

(4.7) $$1-\varphi_k(y) \le 1-\varphi(y) \quad \text{for all } k.$$

$$1-\varphi(y) < \varepsilon \quad \text{implies} \quad \frac{1}{2}\sum_{k=1}^{\infty} \langle y,x_k\rangle^2 < -\log(1-\varepsilon)$$

which tends to '0' as $\varepsilon \to 0$. Thus we can take ε small such that $1-\varphi(y) < \varepsilon$ implies $|\langle y,x_k\rangle| < 1$ for all k, then

(4.8) $$e^{-1/2\langle y,x_k\rangle^2} \le 1 - \frac{1}{4}\langle y,x_k\rangle^2.$$

Note that

$$1 - \cos\langle y, \frac{a_k}{\|x_k\|} x_k\rangle \le \frac{a_k^2}{\|x_k\|^2} \frac{\langle y,x_k\rangle^2}{2}$$

therefore

$$\frac{\|x_k\|^2}{a_k^2} - \frac{\|x_k\|^2}{a_k^2} \cos\langle y, \frac{a_k x_k}{\|x_k\|}\rangle \le \frac{\langle y,x_k\rangle^2}{2},$$

and

$$1 - \frac{1}{4}\langle y,x_k\rangle^2 \le 1 - \frac{1}{2}\frac{\|x_k\|^2}{a_k^2}\left(1-\cos\langle y, \frac{a_k x_k}{\|x_k\|}\rangle\right),$$

for all k.

Thus if $1-\varphi(y) < \varepsilon$ for small ε then using (4.8) we get

(4.9) $$\varphi_k(y) \le \prod_{n=1}^{k}\left(1-\frac{1}{4}\langle y,x_n\rangle^2\right)$$

$$\le \prod_{n=1}^{n}\left[1-\frac{1}{2}\frac{\|x_n\|^2}{a_n^2}\left(1-\cos\langle y, \frac{a_n x_n}{\|x_n\|}\rangle\right)\right].$$

Let ν_k denote the distribution of $\sum_{n=1}^{k} Y_n$, then from (4.9), $\varphi_k(y) \le \hat{\nu}_k(y)$ for all k, therefore

$$1-\hat{\nu}_k(y) \le 1-\varphi_k(y) \le 1-\varphi(y) \quad \text{for all } k.$$

Thus the family $\{\hat{\nu}_k\}_{k=1}^{\infty}$ is equicontinuous in $\tilde{\tau}_2$. Thus by hypothesis the family $\{\nu_k\}_{k=1}^{\infty}$ is tight.

$\{Y_k\}$ are symmetric, hence by Theorem 1.22 we get that ΣY_k converges a.e., contradicting ΣY_k diverges. Thus E is of cotype 2. This completes the proof.

Let τ_p be the topology as before. We say that a Banach space has $B(II,p)$ if it has Bochner Property II with respect to τ_p. We get that if a Banach space has $B(II,2)$ then it is of type 2 (cf. Theorem 4.10), hence $\tilde{\tau}_2 = \tau_2$ (1.24), and in view of Theorem 4.3 it is isomorphic to a Hilbert space [14]. Now we consider $B(II,p)$, for $1 \leq p < 2$. However, the following theorem is valid for $p = 2$.

<u>Theorem 4.10</u>. If E has $B(II,p)$ then E is of stable type p and embeddable in $L^p(\mu)$.

<u>Proof</u>: We will first show that E is stable type p. Suppose $\{x_i\}_{i=1}^{\infty}$ is a sequence in E with $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$, let η_i be i.i.d. symmetric stable random variables of order p. Let μ_n be the distribution of $\sum_{i=1}^{n} x_i\eta_i$. Then $\hat{\mu}_n(y) = e^{-\sum_{i=1}^{n}|\langle y,x\rangle|^p}$.

Let $\nu = \frac{1}{2} \sum_{i=1}^{\infty} (\delta_{x_i} + \delta_{-x_i})$. Then

$$\int_E \|x\|^p \, \nu(dx) = \Sigma \|x_i\|^p \text{ is finite,}$$

and

$$1 - \hat{\mu}_n(y) \leq 1 - e^{-\int_E |\langle y,x\rangle|^p \nu(dx)} \qquad \text{for all } n,$$

giving $\{\hat{\mu}_n\}$ is equicontinuous in τ_p. By hypothesis $\{\mu_n\}$ is tight. Thus by Theorem 1.22 $\Sigma x_i \eta_i$ converges a.e., implying E is of stable type p. This proof does not depend on the fact $(1 \leq p < 2)$.

Suppose E is not embeddable in $L^p(\mu)$. Proceeding in the same way as in the proof of Theorem 3.11, we can choose $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ in E such that

$$\Sigma \|u_i\|^p < \infty \quad \text{and} \quad \Sigma \|v_i\|^p = \infty \quad \text{with}$$
$$(4.11) \qquad \sum_{i=1}^{\infty} |\langle y, u_i\rangle|^p \geq \sum_{i=1}^{\infty} |\langle y, v_i\rangle|^p \quad \text{for all } y \text{ in } E'.$$

Now since E is of stable type p, we have $\Sigma u_i \eta_i$ converges a.e.. Thus there exists a symmetric stable measure μ on E, such that $\hat{\mu}(y) = e^{-\Sigma_{i=1}^{\infty} |\langle y, u_i\rangle|^p}$ for each $y \in E'$.

Let ν_n be the distribution of $\sum_{i=1}^{n} v_i \eta_i$. Then $\hat{\nu}_n(y) = e^{-\Sigma_{i=1}^{n} |\langle y, v_i\rangle|^p}$. Therefore for all n,

$$1 - \hat{\nu}_n(y) \leq 1 - e^{-\Sigma_{i=1}^{\infty} |\langle y, v_i\rangle|^p} \leq 1 - \hat{\mu}(y),$$

giving $\hat{\nu}_n(y)$ equicontinuous in τ_p. Thus $\{\nu_n\}$ is tight. Again using Theorem 1.22 and symmetry of η_i we get that $\Sigma v_i \eta_i$ converges a.e. Hence $\Sigma \|v_i\|^p < \infty$ by Lemma 1.11, which contradicts (4.11) giving the result.

<u>Theorem 4.12</u>. If E has metric approximation property, is of stable type p and embeddable in $L^0(\Omega, P)$ then E has B(II,p).

<u>Proof</u>: Let $\{\mu_\alpha\}_{\alpha \in I}$ be a family of probability measures such that $\{\hat{\mu}_\alpha\}$ are equicontinuous in τ_p. Then from Lévy Inequality (Lemma 3.1) we get for $\varepsilon > 0$, there exists $h(\varepsilon) > 0$ such that

$$(4.13) \qquad \mu_\alpha\{x \mid \|(\pi_m - \pi_k)x\| > \varepsilon\} \leq \frac{1}{h(\varepsilon)} \int_{E'} (1 - \hat{\mu}_\alpha((\pi_m - \pi_k)'y))P_\Psi(dy)$$

where Ψ denotes the embedding into the space of symmetric real random variables and $\{\pi_m\}$ the sequence corresponding to the metric approximation property.

Since E is of stable type p, topology $\tau_p \equiv \tilde{\tau}_p$ (Theorem 1.24). Then using an argument as in the proof of Lemma 3.3 we get that equicontinuity of $\{\hat{\mu}_\alpha\}$ in τ_p implies

$$\lim_k \sup_\alpha \lim_m \int_{E'} [1 - \hat{\mu}_\alpha((\pi_m - \pi_k)'y)]P_\Psi(dy) = 0.$$

Combining with (4.13) we get

$$\lim_k \sup_\alpha \mu_\alpha\{x \mid \|(I - \pi_k)x\| > \varepsilon\} = 0$$

Thus $\{\mu_\alpha\}_{\alpha \in I}$ is flatly concentrated. We can show that $\{\mu_\alpha \circ y^{-1}\}_{\alpha \in I}$ is tight for each $y \in E'$ using equicontinuity and ([17], p.193). Hence by ([1], Theorem 2.3, p.279) $\{\mu_\alpha : \alpha \in I\}$ is a tight family, giving E has $B(II,p)$.

<u>Remark 4.14</u>. For a separable Banach space E having the metric approximation property the following are equivalent:

(i) E has $B(I,p)$

(ii) E has $B(II,p)$

(iii) E is strongly embeddable in $L^p(\Omega,\mu)$ for some probability space (Ω,μ).

Remarks 4.15. (1) Examples of Section II show that cotype 2 does not imply B(II,p) for any p.

(2) We observe that Theorem 4.3 is a generalization of a theorem of Kuelbs ([13], Theorem 6.3) and an extension of the result for Hilbert spaces [29].

(3) Theorem 4.3 does not have assumption of the approximation property. In this context, however, the support of measures involved does have this property. This fact allows us to çircumvent the problem.

SECTION V

FINAL REMARKS

(I) Is the support of a stable measure on a stable type p space, $L^{p'}$, $(\frac{1}{p} + \frac{1}{p'} = 1)$? Under condition B(I,p) it is known[1] that the support is $L^{p'}$. Solving problem (I) will enable us to remove the assumption of approximation property. In fact (I) can equivalently be stated as, 'Does embeddability alone imply B(I,p) for stable type p spaces?'

If the Lindenstrauss-Pelczynski result (Theorem 3.13) can be generalized to L^p $(0 < p < 1)$, it would lead to the solution of:

(II) Is the converse of Theorem 3.7 true? (At least for spaces with metric approximation property).

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[1] Personal communication by Professors V. Mandrekar and A. Weron.

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