ESSENTIAL FIXED POINTS AND ALMOST CONTINUOUS FUNCTIONS

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ABSTRACT

ESSENTIAL FIXED POINTS AND ALMOST CONTINUOUS FUNCTIONS

by Somashekhar Amrith Naimpally

The concept of an essential fixed point was first introduced by M. K. Fort, Jr. for single-valued continuous self-mappings on a compact metric space with the fixed point property. Schmidt extended the theory to a compact Hausdorff space with the fixed point property. Jiang Jia-he has extended the theory to upper semi-continuous multi-valued self-mappings on a compact metric space. In the first chapter of this thesis the theory of essential fixed points is further extended to upper semi-continuous multi-valued self-mappings on a compact Hausdorff space.

Almost continuous functions were introduced by Stallings. In Chapter II two new topologies on the function spaces of almost continuous functions are introduced. Then the theory of essential fixed points is extended to single-valued almost continuous self-mappings, first on a compact metric space and then on a compact Hausdorff space both having the fixed point property.

In Chapter III consideration is restricted to real-valued almost continuous functions defined on a closed

interval. The chief results are: (1) almost continuity is equivalent to the connected graph property; (2) an almost continuous function is continuous if and only if the inverse image of each point is closed. Finally the relationship between almost continuous functions and other non-continuous functions such as neighborly functions and locally recurrent functions is investigated.

In the last chapter Borsuk's concept of a retract is extended to that of an almost retract. It is shown that whereas almost retracts inherit the fixed point property from the original space, they do not always inherit such properties as local connectedness.

ESSENTIAL FIXED POINTS AND ALMOST CONTINUOUS FUNCTIONS

By

Somashekhar Amrith Naimpally

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DEDICATION

To my mother and my wife for their patience and encouragement.

CONTENTS

CHAPTER I

ESSENTIAL FIXED POINTS OF MULTIVALUED MAPPINGS ON A UNIFORM SPACE

Essential fixed points were first introduced by M. K. Fort, Jr. in [4] for single valued continuous self-mappings of a compact metric space with the fixed point property. The central result in Fort's paper was that continuous self-mappings on such a space can be approximated arbitrarily closely by those which have all fixed points essential. Schmidt [13] extended the theory of essential fixed points to continuous self-mappings on a compact Hausdorff space with the fixed point property. Jiang Jia-he [7] considered upper semi-continuous multivalued self-mappings on a compact metric space which need not have the fixed point property. In this chapter we extend the theory to the case of essential fixed points of upper semi-continuous multivalued mappings on a compact Hausdorff space. We use the techniques of Schmidt and Jiang Jia-he.

We use the standard terminology of Hocking and Young [6] and Kelley [9].

Let (X, τ) be a compact Hausdorff space with topology τ. (X need not have the fixed point property.) Let $B(X)$

be the family of all open symmetric neighborhoods of the diagonal in the product space $X \times X$. Then $B(X)$ is a base for the uniformity $\mathcal{U}$ of (X, τ). Let $C(X)$ be the space of all nonempty compact subsets of X. For each U in $\mathcal{U}$ let

$$M(U) = \left\{ (K, K') \text{ in } C(X) \times C(X) \;\middle|\; K \subset U(K'),\quad K' \subset U(K) \right\}.$$

$\left\{ M(U) \right\}$, with U in $\mathcal{U}$, is a base for the uniformity $\mathcal{P}$ of $C(X)$. $C(X)$ is a compact Hausdorff space with uniformity $\mathcal{P}$.

Let $f : Y \longrightarrow C(X)$ be a mapping of Y into $C(X)$, (f is a multi-valued mapping on Y to X). Let $\mathcal{W}$ be a uniformity for Y.

<u>Definition 1.1</u>

f is <u>upper semi-continuous</u> at y in Y if and only if for each U in $B(X)$, there is a W in $\mathcal{W}$ such that for all (y, y') in W, $f(y') \subset U[f(y)]$.

<u>Definition 1.2</u>

f is <u>lower semi-continuous</u> at y in Y if and only if for each U in $B(X)$ there is a W in $\mathcal{W}$ such that for all (y, y') in W, $f(y) \subset U[f(y')]$. f is continuous if and only if it is upper semi-continuous and lower semi-continuous. Let D be a directed set and n be in D. It is well known that f is upper semi-continuous at y in Y if and only if $y_n \longrightarrow y$, $x_n \longrightarrow x$, n in D, x_n in $f(y_n)$, implies x is in $f(y)$. Let $S(Y, C(X))$ be the space of all upper semi-continuous mappings on Y to $C(X)$. For each P in $\mathcal{P}$ let

$W(P) = \{(f,g) \mid f,g \text{ in } S(Y,C(X)),\ (f(y), g(y)) \text{ in } P \text{ for all } y \text{ in } Y\}$.

$\{W(P)\}$, with P in $\mathcal{P}$ is a base for the uniformity $\mathcal{N}$ of $S(Y, C(X))$.

Theorem 1.1 $S(Y,C(X))$ is complete.

Proof. This is proved by showing that $S(Y,C(X))$ is closed in the space of all functions on Y to $C(X)$. Let f be a limit point of $S(Y,C(X))$. For each W $[M(U)]$ there is a g in $S(Y,C(X))$ such that $(f(y),g(y))$ belongs to $M(V)$ for all y in Y where we may assume that $VoVoV \subset U$. This implies that $g(y) \subset V\ [f(y)]$ and $f(y) \subset V\ [g(y)]$. Now there exists a W in $\mathcal{W}$ such that for all (y,y') in W, $g(y') \subset V[g(y)]$. Now

$$\begin{aligned} f(y') &\subset V[g(y')] \\ &\subset VoV\ [g(y)] \\ &\subset VoVoV\ [f(y)] \\ &\subset U\ [f(y)]. \end{aligned}$$

This shows that f is an element of $S(Y,C(X))$.

Q.E.D.

Definition 1.3

Let f be in $S(X,C(X))$ and x be in X. Then x is called a fixed point of f if and only if x is in $f(x)$.

Let $\tilde{S} \subset S(X,C(X))$ be the subspace of all f in $S(X,C(X))$ which have at least one fixed point.

Theorem 1.2 $\tilde{S}$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy net in $\tilde{S}$. Since $S(X,C(X))$ is complete, there is an f in $S(X,C(X))$ such that

$f_n \longrightarrow f$. Since each f_n is in $\tilde{S}$, there exist x_n in X such that each x_n belongs to $f_n(x_n)$. Since X is a compact space there exists a convergent subnet of $\{x_n\}$ and we can then consider the corresponding subnet of $\{f_n\}$. So there is no loss of generality in assuming that the net $\{x_n\}$ itself is convergent. So let $x_n \longrightarrow x$. Since $f_n \longrightarrow f$ there exists a subnet $\{x_{n_m}\}$ of $\{x_n\}$ such that for any U in $\mathcal{U}$, we can choose $\tilde{x}_{n_m}$ in $f(x_{n_m})$ such that $(\tilde{x}_{n_m}, x_{n_m})$ belongs to U. Then $\tilde{x}_{n_m} \longrightarrow x$ and since f is upper semi-continuous at x it follows that x belongs to f(x). This shows that f is an element of $\tilde{S}$, i.e. $\tilde{S}$ is complete.

Q.E.D.

<u>Definition 1.4</u> For f in $\tilde{S}$ let <u>F(f)</u> = the set of all fixed points of f. Then F is a function on $\tilde{S}$ to C(X).

<u>Theorem 1.3</u> F is in $S(\tilde{S}, C(X))$.

<u>Proof</u>. Clearly for f in $\tilde{S}$, F(F) is in C(X). Let $f_n \longrightarrow f$, $x_n \longrightarrow x$ where x_n is in $F(f_n)$. Choose $\tilde{x}_{n_m}$ in $f(x_{n_m})$ such that $(\tilde{x}_{n_m}, x_{n_m})$ is in U for given U in $\mathcal{U}$, $\tilde{x}_{n_m} \longrightarrow x$. Since f is upper semi-continuous at x, x is in F(f).

Q.E.D.

<u>Definition 1.5</u> x in F(f) is essential if and only if for each U in B(X), there is an N in $\mathcal{N}$ such that whenever (f,g) is in N, x is in U [F(g)].

<u>Definition 1.6</u> f in $\tilde{S}$ is an <u>essential fixed point map</u> if and only if all fixed points of f are essential.

<u>Theorem 1.4</u> f in $\tilde{S}$ is an essential fixed point map if and only if $F : \tilde{S} \longrightarrow C(X)$ is continuous at f.

<u>Proof</u>. Let f in $\tilde{S}$ be an essential fixed point map. For U in $B(X)$ let V be in $B(X)$ such that $V \circ V \subset U$. Since $F(f)$ is compact there is a finite subset $\{x_1, x_2, \ldots x_n\}$ of $F(f)$ such that $F(f) \subset \bigcup_{i=1}^{n} V[x_i]$. Since x_i in $F(f)$ is essential for each $i=1, 2, \ldots n$, there is an N in $\mathcal{N}$ such that for (f,g) in N, g in $\tilde{S}$,

$$x_i \text{ is in } V[F(g)],\ i = 1, 2, \ldots n.$$

Therefore, $F(f) \subset \bigcup_{i=1}^{n} V[x_i] \subset \bigcup_{i=1}^{n} V \circ V [F(g)] \subset \bigcup_{i=1}^{n} U [F(g)]$ which shows that F is lower semi-continuous at f. But F is upper semi-continuous by theorem 1.3. It follows that F is continuous at f.

Next let F be continuous at f. Let U be in $B(X)$ and x be in $F(f)$. Since F is lower semi-continuous, there is an N in $\mathcal{N}$ such that for g in $\tilde{S}$, (f,g) in N implies $F(f) \subset V[F(g)]$, i.e. x is in $V[F(g)]$, which means that x is essential for f. This shows that f is an essential fixed point map.

Q.E.D.

<u>Theorem 1.5</u> If f in $\tilde{S}$ has a single fixed point then this fixed point is essential.

<u>Proof</u>. Let $\{p\} = F(f)$. For any U in $B(X)$ there is an N in $\mathcal{N}$ such that whenever g is in $\tilde{S}$, (f,g) is in N, $F(g) \subset V[F(f)] = V[p]$ which means that p is in $V[F(g)]$. Therefore, p is essential.

Q.E.D.

<u>Theorem 1.6</u> Let f be in $\tilde{S}$ and U, V be in $B(X)$ such that $V \circ V \subset U$. Then there is an N in $\mathcal{N}$ such that if g is in $\tilde{S}$ and (f, g) is in N then $V[F(g)] \subset U[F(f)]$.

<u>Proof</u>. Since F is upper semi-continuous at f, there is an N in $\mathcal{N}$ such that if g is in $\tilde{S}$, (f,g) is in N then $F(g) \subset V[F(f)]$.

If p is in $V[F(g)]$, then p is in $V[q]$ for some q in $F(g)$ and q is in $V[r]$ for some r in $F(f)$. Therefore, p is in $V \circ V[r] \subset U[r]$, that is $V[F(g)] \subset U[F(f)]$.

Q.E.D.

CHAPTER II

ESSENTIAL FIXED POINTS OF ALMOST CONTINUOUS FUNCTIONS

In this chapter we extend the theory of essential fixed points to almost continuous self-mappings on compact metric and compact Hausdorff spaces respectively. Stallings first introduced almost continuous functions in [14] and proved that an almost continuous self-mapping of a Hausdorff space with the fixed point property has a fixed point. In order to consider the theory of essential fixed points for almost continuous functions, we introduce two new topologies on function spaces of these functions. We shall show that these topologies agree with the usual topologies for function spaces of continuous functions. We then prove that the theorems of Fort [4] and Schmidt [13] hold when the functions under consideration are almost continuous.

Let X and Y be topological spaces. Let $f:X \longrightarrow Y$ be a function. Let the graph of f be denoted by

$$\Gamma(f) = \left\{(x,f(x)) \;\middle|\; x \text{ in } X\right\} \subset X \times Y.$$

Let $X \times Y$ be assigned the usual product topology. The following definition is due to Stallings [14].

<u>Definition 2.1</u> $f : X \longrightarrow Y$ is <u>almost continuous</u> if and only if for each open set N in $X \times Y$ containing $\Gamma(f)$, there is a continuous function $g : X \longrightarrow Y$ such that $\Gamma(g) \subset N$.

<u>Definition 2.2</u> X has <u>the fixed point property</u> if and only if every continuous function f on X to X has a fixed point, i.e. there is a p in X such that $f(p) = p$.

The following proposition due to Stallings [14] shows the existence of fixed points for almost continuous functions.

<u>Proposition 2.1</u> Let X be a Hausdorff space with the fixed point property. Then every almost continuous function f on X to X has a fixed point.

We not investigate the "essential" character of the fixed points of almost continuous functions. The main result is that if X has the fixed point property and f is an almost continuous function on X to X then f can be approximated, in a certain sense, by an essential fixed point map.

We first study metric spaces (up to Theorem 2.4) and then uniform spaces.

So let X and Y be compact metric spaces with metrics d , d' respectively and let $X \times Y$ be assigned the product metric $D((x_1 , y_1), (x_2 , y_2)) = d(x_1 , x_2) + d'(y_1 , y_2)$. Let H be the Hausdorff metric (see [9] page 131) on the hyperspace of all non-empty closed subsets of $X \times Y$.

We introduce a metric ρ in Y^X. This was used by Kuratowski [10] in discussing continuous functions defined on X as well as on subsets of X.

<u>Definition 2.3</u> For f and g in Y^X we define $\underline{\rho(f,g)}$ $= H(\overline{\Pi(f)}, \overline{\Pi(g)})$.

Clearly (Y^X, ρ) is a pseudometric space. We make it into a metric space by agreeing that $f \equiv g$ if and only if $\overline{\Pi(f)} = \overline{\Pi(g)}$ for f, g in Y^X and then passing to the quotient space with respect to this relation.

Let S(X,Y) be the set of all continuous functions on X to Y and for f, g in S(X,Y) define $\rho_1(f,g) = \sup_{p \in X} d'(f(p),g(p))$ where d' is the metric for Y. It is easily seen that ρ_1 is a metric on S(X,Y). The next theorem relates the metrics ρ and ρ_1.

<u>Theorem 2.1</u> The metrics ρ and ρ_1 are equivalent for S(X,Y).

<u>Proof</u>. Let $\varepsilon > 0$ be arbitrary and let $U(f, \varepsilon) = \{ g \mid f,g \text{ in } S(X,Y) \text{ such that } \rho(f,g) < \varepsilon \}$, $U_1(f, \varepsilon) = \{ g \mid f,g \text{ in } S(X,Y) \text{ such that } \rho_1(f,g) < \varepsilon \}$. It is easily seen that $U_1(f, \varepsilon) \subset U(f, \varepsilon)$. We now show that there is a $\delta > 0$ such that $U(f, \delta) \subset U_1(f, \varepsilon)$. Since f is continuous on a compact set X, f is uniformly continuous. Therefore, there exists a $\delta > 0$ which we can, without any loss of generality, assume to be $< \varepsilon/3$, such that for all x, y in X $d(x,y) < \delta$ implies $d'(f(x),f(y)) < \varepsilon$.

Now let g be in $U(f, \delta)$ and let p be an arbitrary point in X. Then there is a q in X such that $d(p,g) + d'(g(p), f(q)) < \delta$. Now $d'(f(p), g(p)) \leq d(p,q) + d'(g(p),f(q)) + d(p,q) + d'(f(p),f(q)) < 2\delta + \varepsilon/3 < \varepsilon$.

This means g is in $U_1(f, \varepsilon)$. Therefore, ρ and ρ_1 are equivalent for $S(X,Y)$.

Q.E.D.

Let $(\mathcal{A}, \rho)$ be the space of all almost continuous functions on X to X where X is a compact metric space with the fixed point property. Let $C(X^2)$ denote the space of all non-empty closed subsets of $X^2 = X \times X$ with the Hausdorff metric H. Let $\Delta = \{(p,p) \mid \text{for } p \text{ in } X\} \subset X^2$ be the diagonal in X^2. We modify the definition of essential fixed points to suit their study in relation to almost continuous functions.

<u>Definition 2.4</u> For f in $\mathcal{A}$ we define <u>F(f)</u> $= \overline{\Pi(f)} \cap \Delta$. Clearly F is a function on $\mathcal{A}$ to $C(X^2)$.

<u>Definition 2.5</u> p in F(f) is an <u>essential point of f</u> in $\mathcal{A}$ if and only if corresponding to each open set V in X^2 containing p, there exists an open set W in $\mathcal{A}$ containing f such that for all g in W, $F(g) \cap V \neq \phi$.

<u>Theorem 2.2</u> $F : \mathcal{A} \longrightarrow C(X^2)$ is upper semi-continuous.

<u>Proof</u>. Let $\varepsilon > 0$ be arbitrary. Let δ be the Hausdorff distance between Δ and $\overline{\Pi(f)} - U(F(f), \varepsilon)$ if $\overline{\Pi(f)} \not\subset U(F(f), \varepsilon)$ and let $\delta = 1$ otherwise. If g is

in $\mathcal{A}$ such that $\rho(f,g) < \delta$ then $F(g) \subset U(F(f), \varepsilon)$. Therefore, F is upper semi-continuous.

Q.E.D.

Theorem 2.3 If F(f) is a single point p then it is an essential point for f.

Proof. By Theorem 2.2 for $\varepsilon > 0$ there exists a $\delta > 0$ such that for all g in $\mathcal{A}$ such that $\rho(f,g) < \delta$, $F(g) \subset U(p, \varepsilon)$. That is $F(g) \cap U(p, \varepsilon) \neq \phi$. Therefore, p is essential for f.

Q.E.D.

Definition 2.6 f in $\mathcal{A}$ is an essential fixed point map if and only if all points of F(f) are essential for f.

Theorem 2.4 F is lower semi-continuous at f in $\mathcal{A}$ if and only if f is an essential fixed point map.

Proof. Let F be lower semi-continuous at f in $\mathcal{A}$. Then for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all g in $\mathcal{A}$, $\rho(f,g) < \delta$ implies $F(f) \subset U(F(g), \varepsilon)$. Let p be in F(f). Then p is in $U(F(g), \varepsilon)$, implies $F(g) \cap U(p, \varepsilon) \neq \phi$. Therefore, p is essential for f. Thus f is an essential fixed point map.

Next let f be an essential fixed point map. Then every point in F(f) is essential for f. Let $\varepsilon > 0$. For each p in F(f), there is a $\delta p > 0$ such that for all g in $\mathcal{A}$, $\rho(f,g) < \delta p$ implies $F(g) \cap U(p, \varepsilon/2) \neq \phi$. Since F(f) is compact a finite number of $U(p, \varepsilon/2)$ cover F(f). Let δ be the minimum of all the corresponding

δ_p's. Now if $\rho(f,g)<\delta$ then each p in F(f) is in U(F(g), ε). Therefore, F(f) $\subset$ U(F(g), ε). This implies F is lower semi-continuous at f.

Q.E.D.

Theorem 2.5 For f in $\mathcal{A}$ and $\varepsilon > 0$, there is an essential fixed point map g in $\mathcal{A}$ such that $\Gamma(g) \subset U(\overline{\Gamma(f)}, \varepsilon)$.

Proof. Since f is in $\mathcal{A}$, there is a continuous function h on X to X such that $\Gamma(h) \subset U(\overline{\Gamma(f)}, \varepsilon)$. Since $\Gamma(h)$ is compact, there is a $\delta > 0$ such that $U(\Gamma(h), \delta) \subset U(\overline{\Gamma(f)}, \varepsilon)$. By Fort's theorem [4], there is a continuous function g on X to X, which is an essential fixed point map and $\rho_1(g,h) < \delta$. Then clearly $\Gamma(g) \subset U(\Gamma(h), \delta) \subset U(\overline{\Gamma(f)}, \varepsilon)$.

Q.E.D.

It should be noted that we have actually proved that g can be chosen to be continuous.

Next we consider the essential points of almost continuous functions on X to X where X is a compact Hausdorff space with the fixed point property.

Let $\mathcal{A}$ be the space of all almost continuous functions on X to X and let S be the subspace of $\mathcal{A}$ containing only continuous functions.

We introduce a topology τ for $\mathcal{A}$ by defining the Kuratowski closure operator for a subset B of $\mathcal{A}$ (see [9], p. 43).

<u>Definition 2.7</u> <u>f is in B'</u> (the set of limit points of B) if and only if for each open set $V \subset X^2$ containing $\sqcap(f)$, there is a $g(\neq f)$ in B such that $\sqcap(g) \subset V$.

Let $B^c = B \cup B'$. We shall now prove that the operator 'c' satisfies the conditions (a) to (d) of the Kauratowski closure operator ([9], page 43).

(a) Clearly $\phi^c = \phi$.

(b) For each $B \subset \mathcal{A}$, obviously $B \subset B^c$.

(c) We want to show that $B^{cc} = B^c$ for each $B \subset \mathcal{A}$. Clearly $B^c \subset B^{cc}$. Suppose f is in B^{cc} but not in B^c. Then for each open set V containing $\sqcap(f)$ there is a g in B^c different from f such that $\sqcap(g) \subset V$. If g is in B' then clearly there is an h in B different from f such that $\sqcap(h) \subset V$. Therefore, f is in B'. If g is in B then f is in B'. In both cases f is in B^c, a contradiction, therefore, $B^{cc} = B^c$.

(d) Let B_1 and B_2 be subsets of $\mathcal{A}$. We wish to show that $B_1^c \cup B_2^c = (B_1 \cup B_2)^c$.

Let f be in $B_1^c \cup B_2^c$. Then either f is in B_1^c or f is in B_2^c. Suppose f is in B_1^c. Then f is in B_1 or B_1'. If f is in B_1, then f is in $B_1 \cup B_2$. If f is in B_1' then f is in $(B_1 \cup B_2)^c$.

Therefore, $B_1^c \cup B_2^c \subset (B_1 \cup B_2)^c$. Next let f be in $(B_1 \cup B_2)^c$. Then f is in $B_1 \cup B_2$ or in $(B_1 \cup B_2)'$. If f is in $B_1 \cup B_2$, then f is in B_1 or B_2, i.e. f is in

B_1^c or B_2^c, i.e. f is in $B_1^c \cup B_2^c$. If f is in $(B_1 \cup B_2)'$ then clearly f is B_1' or B_2'. Therefore, 'c' is the Kuratowski closure operator.

It is easy to see that the topology τ of $\mathcal{A}$ induced by 'c' is the same as the one generated by the basis consisting of sets of the form $\mathcal{A}_V = \{ f \mid f \text{ in } \mathcal{A} \text{ such that } \Gamma(f) \subset V \}$ for V an open set in X^2.

<u>Theorem 2.6</u> If X is a compact metric space then $(\mathcal{A}, \rho)$ is equivalent to $(\mathcal{A}, \tau)$.

<u>Proof</u>. Let f be an element of $\mathcal{A}$ and let V be an open subset of X^2 containing $\overline{\Gamma(f)}$. Since $\overline{\Gamma(f)}$ is compact there is a positive δ such that $U[\overline{\Gamma(f)}, \delta] \subset V$. Now if g belongs to $U(f, \delta)$ then $\rho(\overline{\Gamma(f)}, \overline{\Gamma(g)}) < \delta$. This implies that $\overline{\Gamma(g)} \subset V$, i.e. $U(f, \delta) \subset \mathcal{A}_V$.

Next corresponding to an arbitrary positive ε, choose a positive $\delta < \varepsilon/2$. Let $V = U[\overline{\Gamma(f)}, \delta]$. If g is an element of $\mathcal{A}_V$ then clearly $\overline{\Gamma(g)} \subset U[\overline{\Gamma(f)}, \delta]$ and $\overline{\Gamma(f)} \subset U[\overline{\Gamma(g)}, 2\delta]$. This implies that $\rho(\overline{\Gamma(f)}, \overline{\Gamma(g)}) < 2\delta < \varepsilon$, i.e. $\mathcal{A}_V \subset U(f, \delta)$.

Q.E.D.

We shall now prove that the topology generated by the basis $\mathcal{A}_V$ (for V in X^2) for S is equivalent to the topology of uniform convergence. We use the notation of Kelley([9],p. 226). Let $\mathcal{U}$ be the uniformity for X. The family of sets $W(U) = \{ (f,g) \mid (f(x), g(x)) \in U$ for U in $\mathcal{U}$ form a basis for the topology of uniform convergence of S.

<u>Theorem 2.7</u> The subspace topology induced on S by $\mathcal{T}$ is equivalent to the usual topology of uniform convergence on S.

<u>Proof</u>. Let V be an open set in X^2 and let f be in S such that $\Gamma(f) \subset V$. Since f is continuous and X is compact, $\Gamma(f)$ is compact. Therefore, there is a U in $\mathcal{U}$ such that $\bigcup_{x \text{ in } X} x \times U[f(x)] \subset V$. If g is in W(U) [f], then (f(x),g(x)) belongs to U for all x in X. Therefore, $\Gamma(g) \subset V$. This means W(U) [f] $\subset \mathcal{A}_V$. Next consider W(U)[f] where U is in $\mathcal{U}$ and f is in S. Let U_2 be in $\mathcal{U}$ such that $U_2 \circ U_2 \subset U$. Since f is continuous on the compact set X, it is uniformly continuous on X. Therefore, corresponding to U_2 there is a U_1 in $\mathcal{U}$ (we may suppose U_1, U_2 to be symmetric) such that for all p, q in X, (p,q) in U_1 implies (f(p) , f(q)) is in U_2. Now consider the open set $V = \bigcup_{x \text{ in } X} U_1[x] \times U_2 [f(x)]$ in X^2, and let g be in S such that $\Gamma(g) \subset V$. Let p be any element in X. Then there is a q in X such that (p,g(p)) is in $U_1[q] \times U_2 [f(q)]$, i.e. p is in $U_1[q]$ and g(p) is in $U_2 [f(q)]$. Therefore, f(q) is in $U_2 [f(p)]$. This means g(p) is in $U_2 \circ U_2 [f(p)]$, i.e. g(p) is in U[f(p)], i.e. g is in W(U) [f]. Therefore, $\mathcal{A}_V \subset$ W(U) [f]. This proves the equivalence.

Q.E.D.

Definition 2.8 For f in $\mathcal{A}$ let F(f) = $\Gamma(f) \cap \Delta \subset X^2$.
The next theorem generalizes theorem 2.2.

Theorem 2.8 Corresponding to each open set V containing F(f), f in $\mathcal{A}$, there is an open set W containing $\Gamma(f)$ such that for all g in $\mathcal{A}$, $\Gamma(g) \subset$ W implies F(g) $\subset$ V.

Proof. X is compact Hausdorff implies X^2 is compact Hausdorff (see [6], Exercise 2-7, p. 40). Therefore, X^2 is normal (see [6], p. 41). Also Δ is closed in X^2 ([6],p.39). Therefore, to each p in $\Gamma(f) - \Delta$, there exists an open set $W_p \subset X^2$ such that $W_p \cap \Delta = \phi$. Let $W = V \cup \bigcup_{p \text{ in } \Gamma(f) - \Delta} W_p$. W is an open set containing $\Gamma(f)$ and clearly if g is in $\mathcal{A}$ such that $\Gamma(g) \subset$ W then F(g) $\subset$ V.

Q.E.D.

Definition 2.9 p in F(f), for f in $\mathcal{A}$, is called an essential point for f, if and only if for each open set V containing p, there is an open set W containing $\Gamma(f)$ such that if g is in $\mathcal{A}$ such that $\Gamma(g) \subset$ W then F(g) $\cap$ V $\neq \phi$.

The following theorem generalizes theorem 2.3.

Theorem 2.9 For f in $\mathcal{A}$ if F(f) consists of a single point then it is essential for f.

Proof. Let F(f) = $\{p\}$. By theorem 2.8 corresponding to each open set V containing p, there is an open set W containing $\Gamma(f)$ such that for all g in $\mathcal{A}$, $\Gamma(g) \subset$ W implies F(g) $\subset$ V, i.e. F(g) $\cap$ V $\neq$ 0. Therefore, p is an essential point for f.

Q.E.D.

The next two theorems generalize theorem 2.4.

<u>Theorem 2.10</u> Let f be in $\mathcal{A}$ and let for U in $\mathcal{U}$ there be a V in $\mathcal{U}$ such that for all g in $\mathcal{A}$, $\Gamma(g) \subset V[\Gamma(f)]$ implies $F(f) \subset U[F(g)]$. Then f is an essential fixed point map.

<u>Proof</u>. Let p be in F(f) and let U be in $\mathcal{U}$. Then there is a V in $\mathcal{U}$ such that for all g in $\mathcal{A}$, $\Gamma(g) \subset V[\Gamma(f)]$ implies $F(f) \subset U[F(g)]$. This means that there is a q in F(g) such that p is in U [q], i.e. q is in U [p]. Therefore, p is essential. This proves that f is an essential fixed point map.

Q.E.D.

<u>Theorem 2.11</u> Let f in $\mathcal{A}$ be an essential fixed point map and let F(f) be compact. Then corresponding to each U in $\mathcal{U}$, there is a V in $\mathcal{U}$ such that for all g in $\mathcal{A}$ such that $\Gamma(g) \subset V[\Gamma(f)]$, $F(g) \subset U[F(f)]$ and $F(f) \subset U[F(g)]$.

<u>Proof</u>. Corresponding to U in $\mathcal{U}$ let U' be in $\mathcal{U}$ such that $U' \circ U' \subset U$. Since f is an essential fixed point map, corresponding to each p in F(f), there is a Vp in $\mathcal{U}$ such that for all g in $\mathcal{A}$, $\Gamma(g) \subset V_p[\Gamma(f)]$, implies $F(g) \cap U'[p] \neq \phi$. Since F(f) is compact, there is a finite set $\{p_1, p_2 \ldots, p_m\} \subset F(f)$ such that $F(f) \subset \bigcup_{k=1}^{m} U'[p_k]$. Let $V = \bigcap_{k=1}^{m} V_{p_k}$. If g is in $\mathcal{A}$ such that $\Gamma(g) \subset V[\Gamma(f)]$, then $F(g) \subset \bigcup_{k=1}^{m} U'[p_k] \subset U[F(f)]$. To each p_k corresponds a q_k in F(g), such

that p_k is in $U'[q_k]$. Therefore, $F(f) \subset \bigcup_{k=1}^{m} U'[p_k] \subset \bigcup_{k=1}^{m} U' \circ U'[q_k] \subset \bigcup_{k=1}^{m} U[q_k] \subset U[F(g)]$.

Q.E.D.

The following theorem shows that any f in $\mathcal{A}$ can be approximated by a g in $\mathcal{A}$ which is an essential fixed point map. This generalizes Schmidt's theorem [13] to almost continuous functions.

Theorem 2.12 Let f be in $\mathcal{A}$. Corresponding to each open set W containing $\Gamma(f)$, there is an essential fixed point map g in $\mathcal{A}$ such that $\Gamma(g) \subset W$.

Proof. Since f is almost continuous, there is a continuous function h in $\mathcal{A}$ such that $\Gamma(h) \subset W$. Since h is continuous it follows from Schmidt's theorem [13] that there is an essential fixed point map g (in fact continuous) such that $\Gamma(g) \subset W$.

Q.E.D.

CHAPTER III

PROPERTIES OF ALMOST CONTINUOUS FUNCTIONS

As it was mentioned earlier, almost continuous functions were first defined by Stallings in [14]. Stallings proved a few properties of almost continuous functions and also raised a few questions which remained unanswered. Inspired by an interesting article in the American Mathematical Monthly by Marcus [12[, we shall investigate the properties of real valued almost continuous functions defined on a closed interval [a, b]. We shall find an equivalent condition for almost continuity and answer a question of Stallings in a special case. Next a necessary and sufficient condition for an almost continuous function to be continuous will be given. Finally we shall investigate the relationships between almost continuous functions and other non-continuous functions such as neighborly functions (see Bledsoe [1])and locally recurrent functions (see Marcus [12]). The following proposition is due to Stallings [14].

Proposition 3.1 (Stallings) If X , Y are topological spaces such that $X \times Y$ is completely normal and X is connected then $f : X \longrightarrow Y$ is almost continuous implies $\Gamma(f)$ is connected.

It follows that the graph $\Gamma(f)$ of a real valued almost continuous function f on [a,b] is connected. The question naturally arises as to whether this property characterizes an almost continuous function. The following theorem answers this question in the affirmative.

Theorem 3.1 If f is a real valued function defined on [a,b] such that the graph $\Gamma(f)$ is connected then f is almost continuous.

Proof. Let U be any open set in $[a,b] \times R^1$ containing $\Gamma(f)$. It is sufficient to show that there is a set of points $\{p_i\}$ $i = 1, 2, \ldots, n$ such that $a = p_1 < p_2 < \ldots < p_n = b$ and the straight line segments joining $(p_i, f(p_i))$, $(p_{i+1}, f(p_{i+1}))$ lie within U for $i = 1, 2, \ldots, n - 1$. For if such a set exists then the resulting polygonal arc in U joining (a, f (a)) and (B, f(b)) is obviously the graph of a single-valued continuous function, since each vertical straight line meets the arc in at most one point.

Let X be the subset of $\Gamma(f)$ such that for each (x, f(x)) in X, there is a set of points $\{p_i\}$ $i = 1, 2, \ldots, m$ such that the segments joining $(p_i, f(p_i))$, $(p_{i+1}, f(p_{i+1}))$ $i = 1, 2, \ldots, m - 1$ lie within U and $a = p_1 < p_2 < \ldots < p_m = x$. X is not empty for if S_a is any open disc within U containing (a, f(a)) there is a point (p, f(p)) of $\Gamma(f)$ $(p \neq a)$ such that (p, f(p)) is in S_a. This shows that (p, f(p)) belongs to X. (The existence of the point (p, f(p)) within S_a follows from the fact that since $\Gamma(f)$ is connected no point of $\Gamma(f)$ is an isolated point.)

Let c be the supremum of all x in [a,b] such that (x, f(x)) belongs to X. If c = b we are done. So let us assume that c < b. Let $W_1 = \{(x, y) \mid a \leq x < c\}$, $W_2 = \{(x, y) \mid c < x \leq b\}$. W_1, W_2 are disjoint open subsets of $[a,b] \times R^1$ and $(a, f(a)) \in W_1$, $(b, f(b)) \in W_2$. If (c, f(c)) is not a limit point of X then there exists an open disc S_c within U containing (c, f(c)) and not containing any other point of X. In this case W_1 and $S_c \cup W_2$ separate $\Gamma(f)$, a contradiction. This shows that (c, f(c)) is a limit point of X and there is a point x in [a,b] such that x < c and $(x, f(x)) \in X \cap S_c$. Since the segment joining (x, f(x)), (c, f(c)) lies within U, it follows that (c, f(c)) is an element of X. In this case S_c cannot contain any point (x, f(x)) with $c < x \leq b$ for this will contradict the construction of c. Then $W_1 \cup S_c$ and W_2 separate $\Gamma(f)$, again a contradiction. Thus c = b.

Q.E.D.

The above theorem provides us with the following equivalent statements.

<u>Theorem 3.2</u> Let f be a real-valued function on a closed interval. The following statements are equivalent:

(a) f is almost continuous.

(b) $\Gamma(f)$ is connected.

<u>Definition 3.1</u> A real-valued function f defined on [a,b] is said to be <u>Darboux continuous</u> if and only if for every [c,d] $\subset$ [a,b], f(x) takes every value between f(c) and f(d) in the open interval (c,d).

The following proposition is due to Stallings [14].

<u>Proposition 3.2</u> If $f : X \longrightarrow Y$ is almost continuous and $C \subset X$ is closed then $f|C : C \longrightarrow Y$ is almost continuous.

<u>Theorem 3.3</u> A real-valued almost continuous function f defined on [a,b] is Darboux continuous.

<u>Proof</u>. Let f be almost continuous on [a,b] and let $[c,d] \subset [a,b]$. By proposition 3.2 f is almost continuous on [c,d] and by proposition 3.1 the graph of f restricted to [c,d] is connected. This shows that the image of [c,d] under f is connected and therefore, f is Darboux continuous.

Q.E.D.

It might be conjectured that the converse of theorem 3.3 is true. But the following counter example shows that this is not the case. It is a modification of an example constructed by Halperin [5, p. 117].

<u>Example 3.1</u> Let $a_1, a_2, \ldots, a_\alpha, \ldots$ $(\alpha < \Omega)$ be a Hamel basis for the real numbers. Every real number x can be expressed uniquely as $x = \sum_\alpha r_\alpha a_\alpha$ with rational coefficients r_α of which only a finite number differ from zero. Such a basis can be rearranged into a two-fold sequence

$b_1, b_2, \ldots, b_\alpha, \ldots (\alpha < \Omega)$; $c_1, c_2, \ldots, c_\alpha, \ldots (\alpha < \Omega)$. Now we define the function h(x) by setting $h(x) = \sum_\alpha r_\alpha a_\alpha$ for $x = \sum_\alpha r_\alpha b_\alpha + \sum_\beta s_\beta c_\beta$. For any interval (c,d) and any $u = \sum_\alpha r_\alpha a_\alpha$ we shall have $h(x) = u$ if $x = \sum_\alpha r_\alpha b_\alpha + \sum_\beta s_\beta c_\beta$ for any choice of s_β; in particular if only one $s_\beta \neq o$, say s_{β_0}, and it is chosen so that $c - \sum_\alpha r_\alpha b_\alpha < s_{\beta_0} c_{\beta_0} < d - \sum_\alpha r_\alpha b_\alpha$. Hence h(x) will equal u for uncountably many x in (c,d).

For each x in [o, 1] for which h(x) = x, we define f(x) = x + 1 and for all other x in [o, 1] we define f(x) = h(x). Then for every real u and every $(c,d) \subset [o, 1]$, f(x) will equal u for uncountably many x in (c,d). This shows that f is Darboux continuous in [o, 1]. But the diagonal Δ in $[o, 1] \times R^1$ is closed and does not contain any points of $\Gamma(f)$. Consequently the disjoint open sets U and V of $[o, 1] \times R^1$ defined by $U = \{(x,y) \mid y > x\}$, $V = \{(x,y) \mid y < x\}$ each contain points of $\Gamma(f)$ and together contain the whole of $\Gamma(f)$. This shows that $\Gamma(f)$ is not connected and so by theorem 3.2 f is not almost continuous.

The function f constructed above is unbounded, in fact $\Gamma(f)$ is dense in $[a,b] \times R^1$. The question naturally arises as to whether there is a counter example with a bounded Darboux continuous function. Such a function can be easily constructed by modifying an example given by

Lebesque [11]. Let every x be written as a non-terminating decimal I . $a_1\, a_2 \dots a_n \dots$. If the decimal . $a_1\, a_3 \dots a_{2n-1} \dots$ is not periodic, set $g(x) = 0$; if it is periodic and the first period commences with a_{2n-1}, set $g(x) = .a_{2n}\, a_{2n+2}\, a_{2n+4} \dots$. The function g takes on every value between 0 and 1 inclusive in every interval, no matter how small and $0 \leq g(x) \leq 1$ for all x. Now if $g(x) = x$ for any x in [o, 1] we change it to another value different from x but still lying between 0 and 1. In this case g takes on every value between 0 and 1 countably many times in every non-degenerate subinterval of [o, 1] and so g is Darboux continuous. But g is not almost continuous as its graph $\Gamma(g)$ is not connected.

The function f constructed above is Darboux continuous but $\Gamma(f)$ is not connected. The question arises as to whether there is an example of a Darboux continuous function whose graph is totally disconnected. The answer is in the affirmative and this can be done by modifying the function f such that $\Gamma(f)$ has no points on the countable collection of straight lines $\{y = \gamma x \mid \text{where } \gamma \text{ is rational}\}$. Then given any two points P, Q of $\Gamma(f)$, there exists a straight line $y = \gamma x$ whose complement contains $\Gamma(f)$ and such that P and Q lie in the disjoint open components of that complement.

We now give a few examples of almost continuous functions to serve as illustrations.

<u>Example 3.2</u> Let $f(x) = \sin \frac{1}{x}$ $(x \neq 0)$

$= 0$ $(x = 0)$

where $-1 \leq x \leq 1$.

Clearly $\Gamma(f)$ is connected in $[-1, 1] \times R^1$. Therefore, by theorem 3.2 f is almost continuous but clearly f is not continuous.

<u>Example 3.3</u> We now give an example of an unbounded almost continuous function.

$$f(x) = \frac{1}{x} \sin \frac{1}{x} \quad (x \neq 0)$$
$$= 0 \quad (x = 0)$$

where $-1 \leq x \leq 1$.

<u>Example 3.4</u> Stallings [14] asked the following question if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are almost continuous then under what conditions is $gf : X \longrightarrow Z$ almost continuous? Clearly if we restrict ourselves to real-valued almost continuous functions and $X = [a, b]$ then by theorem 3.2, gf is almost continuous if and only if $\Gamma(gf)$ is connected. As a special case the following function is almost continuous.

$$h(x) = \sin \left[\left(\sin \frac{1}{x}\right)^{-1} \right] \quad \left(x \neq \frac{1}{n\pi}\right)$$
$$= 0 \quad \left(x = \frac{1}{n\pi} \right).$$
$$-1 \leq x \leq 1.$$

Here $h = f^2$ where f is the function of example 3.2.

Jones ([8] p. 117) has constructed an example of a real-valued function f whose graph is connected but which is discontinuous everywhere. By theorem 3.2 it follows that there exist almost continuous functions which are not Riemann integrable.

Suppose $\{f_n\}$ is a sequence of almost continuous functions and $f_n \longrightarrow f$, then the question arises under what kind of convergence will f be almost continuous. The answer is obvious from the topology used in the function space $\mathcal{A}$ namely : for each open set U containing $\Gamma(f)$ there is a positive integer m such that $\Gamma(f_m) \subset U$.

Next we give a necessary and sufficient condition for an almost continuous function to be continuous.

<u>Theorem 3.4</u> A real-valued almost continuous function f defined on [a, b] is continuous if and only if for every real number x, the set $f^{-1}(x)$ is closed in [a, b].

<u>Proof</u>. If f is continuous then $f^{-1}(x)$ is closed for each real x.

On the other hand if f is almost continuous then f is Darboux continuous by theorem 3.3. Let c be any point in [a,b] and let ε be any arbitrary positive number. The sets $U = f^{-1}[f(c) + \varepsilon]$ and $V = f^{-1}[f(c) - \varepsilon]$ are disjoint and closed in [a, b] and c does not belong to either set. Therefore, there is a positive δ such that the open interval $(c - \delta, c + \delta)$ is disjoint from both U and V. Also f(x) does not equal $f(c) + \varepsilon$ or $f(c) - \varepsilon$ in $(c - \delta, c + \delta)$. This shows that for all x in $(c - \delta, c + \delta)$ f(x) lies between $f(c) - \varepsilon$ and $f(c) + \varepsilon$, for if not $f(x)$ must equal $f(c) - \varepsilon$ or $f(c) + \varepsilon$ since f is Darboux continuous. This means that f is continuous.

Q.E.D.

Finally we discuss the relationships between almost continuous functions and other non-continuous functions.

Definition 3.2 A real-valued function f of a real variable is neighborly at a real x if and only if for every positive ε, there exists an open interval I such that for all y in I, $|x-y| + |f(x) - f(y)| < \varepsilon$ (Bledsoe [1]). Clearly if I contains x then f is continuous. We say that f is neighborly if f is neighborly at all real x. Example 3.1 shows that there exist non-continuous neighborly functions.

Defintion 3.3 A real-valued function f of a real variable is locally recurrent at x if and only if every deleted neighborhood of x, $N(x)$, contains an element y such that $f(y) = f(x)$ (Bush [3]).

Definition 3.4 A real-valued function f of a real variable is almost locally recurrent at x if and only if there is a sequence $\{x_n\}$ such that $x_n \longrightarrow x$ and $f(x_n) \longrightarrow f(x)$.

If in definition 3.4, $x_n > x (x_n < x)$ for all n then we say that f is almost locally recurrent from the right (left) at x.

Obviously a locally recurrent function is almost locally recurrent but the converse is not true (see example 3.6).

The following theorems are easy consequences of the definitions and so we omit the proofs.

<u>Theorem 3.5</u> If a real-valued function f of a real variable is neighborly at x then it is almost locally recurrent at x.

<u>Theorem 3.6</u> If a real-valued function f on [a,b] is almost continuous then it is almost locally recurrent for all x in [a,b].

<u>Example 3.5</u> The Dirichlet function f(x) = 0 when x is rational, f(x) = 1 when x is irrational, shows that even a function which is locally recurrent everywhere need not be neighborly or almost continuous.

<u>Example 3.6</u> Let $f(x) = x \sin \frac{1}{1-x}$, $(x \neq 1)$

$$= 1, \ (x = 1)$$

$$\text{where} \quad 0 \leq x \leq 1.$$

Here f is almost continuous, neighborly and almost locally recurrent but f is not locally recurrent at 1.

<u>Example 3.7</u> Let $f(x) = 0$ for $0 \leq x \leq 1$

$$f(x) = 1 \text{ for } 1 < x \leq 2$$

Clearly f is neighborly at all x in [0, 2] but since the graph of f is not connected, f is not almost continuous. So a neighborly function need not be almost continuous. Bledsoe [1] has shown that the points of discontinuity of a neighborly function form a set of the first category.

On the other hand, as remarked earlier, there exist almost continuous functions which are discontinuous everywhere. This means that an almost continuous function need not be neighborly.

Now we prove a theorem which gives sufficient conditions for an almost continuous function to be neighborly.

<u>Theorem 3.7</u> If f is a real-valued almost continuous function on [a,b] and f has only one point of discontinuity then f is neighborly.

<u>Proof</u>. Let p in [a,b] be the point of discontinuity of f. At all other points f is continuous and so f is neighborly. By theorem 3.6 f is almost locally recurrent at p, therefore, corresponding to any positive ε there is a q in [a,b] different from p such that $|p - q| < \varepsilon/4$ and $|f(p)-f(q)| < \varepsilon/4$. Since f is continuous at q, there is a positive δ (less than $\varepsilon/4$) such that for all y in [a,b] such that $|y - q| < \delta$, $|f(y) - f(q)| < \varepsilon/4$. Therefore, for all y in the open interval $(q - \delta, q + \delta)$, $|p - y| + |f(p) - f(y)| \leq |p - q| + |q - y| + |f(p) - f(q)| + |f(q) - f(y)| < \varepsilon$. This shows that f is neighborly at p.

Q.E.D.

<u>Corollary 3.8</u> It is easy to see that the conclusion of theorem 3.7 holds even if f has an infinite number of discontinuities which have a finite number of limit points.

Corollary 3.9   An almost continuous function cannot have a removable discontinuity.

Next we prove a partial converse of theorem 3.6.

Theorem 3.10   Let f be a real-valued function continuous at all points except at p in the interior of [a,b] and let f be almost locally recurrent from the right and from the left at p. Then f is almost continuous.

Proof. The conclusion easily follows from the fact that the graph of f is connected but we give a direct proof below. Let V be any open set containing $\Gamma(f)$. There exists a positive ε such that the open disc $S((p,f(p)), \varepsilon)$ lies in V. From the given conditions we know that there are x_1, x_2 in [a,b] such that $x_1 < p < x_2$ and $(x_1\ f(x_1))$ and $(x_2, f(x_2))$ lie in $S((p, f(p)), \varepsilon)$. Let g be a function on [a,b] as follows.

$$g(x) = f(x) \text{ for } x \text{ in } [a,b] - (x_1, x_2)$$
$$= \frac{(x-x_1)\ f(x_2) + (x_2-x)\ f(x_1)}{x_2-x_1} \quad \text{for } x \text{ in } [x_1,x_2].$$

Clearly g is continuous and $\Gamma(g) \subset V$ which shows that f is almost continuous.

Q.E.D.

CHAPTER IV

ALMOST RETRACTS

Following Borsuk [2], we shall study almost retracts which are "retracts" under almost continuous functions. We shall show in this chapter that almost retracts inherit the fixed point property as do retracts but that they do not always inherit some other properties such as local connectedness. We construct a non-locally connected set with the fixed point property.

The following propositions are due to Stallings [14].

<u>Proposition 4.1</u> If $f : X \longrightarrow Y$ is almost continuous and $g : Y \longrightarrow Z$ is continuous then $gf : X \longrightarrow Z$ is almost continuous.

<u>Proposition 4.2</u> Let X be a compact Hausdorff space, Y a Hausdorff space and Z any topological space. If $f : X \longrightarrow Y$ is continuous and $g : Y \longrightarrow Z$ is almost continuous then $gf : X \longrightarrow Z$ is almost continuous.

We now give a counter example to show that theorem 3 page 91 of Kelley [9] cannot be extended to almost continuous functions. This answers negatively a natural conjecture.

Example 4.1 Define $f : [o, 1] \longrightarrow [-1, 1] \times [-1, 1]$ by setting $f(o) = (o, o)$

$$f(x) = (\sin \frac{1}{x}, \cos \frac{1}{x}) \text{ for } x \neq 0.$$

If $\Gamma(f)$ denotes the graph of f, then $\Gamma(f) = \{(x, f(x)) \mid x \in [o,1]\} \subset [o, 1] \times [-1, 1] \times [-1, 1]$. Since (o, o, o) is a point on $\Gamma(f)$ and for $x \neq o$ the distance between (o, o, o) and $(x, \sin \frac{1}{x}, \cos \frac{1}{x})$ is greater than one, the closed sphere S of radius $\frac{1}{2}$ with center at (o, o, o) does not include any other point of $\Gamma(f)$. This shows that $\Gamma(f)$ is not connected and by proposition 3.1, f is not almost continuous. This example shows that a function $f : Y \longrightarrow \prod_{a \in A} X_a$ is not necessarily almost continuous when $P_a of$ is almost continuous for each projection P_a.

Stallings [14] remarked that the composition of two almost continuous functions need not be almost continuous but did not give an example. By employing a technique similar to that used in example 4.1 we give below an example.

Example 4.2 Let $Y = [-1, 1]$ and Y^n denote the product of Y with itself n times. Define $f : Y^2 \longrightarrow Y^2$ by

$$f(o, y) = (o, y)$$
$$f(x, y) = (\sin \frac{1}{x}, y) \text{ for } x \neq 0.$$

Let U be any open set in Y^4 containing $\Gamma(f)$. There is a positive real number r such that the sphere S_r (with center at (o, o, o, o) and radius r) lies within U. There is a positive integer n such that $\frac{1}{n\pi} < r$. Let F be a function defined by

$$F(x, y) = (o, y) \text{ for } |x| \leq \frac{1}{n\pi}$$
$$= f(x,y) \text{ for } |x| > \frac{1}{n\pi}.$$

Then F is a continuous function and $\Gamma(F)$ lies within U which shows that f is almost continuous. Next let g be a function on Y^2 into itself defined by $g(x, y) = (x, \cos \frac{1}{y})$. Then g is almost continuous. Now consider the composition of f and g namely fog on Y^2 to Y^2 given by fog (x, y) $= (\sin \frac{1}{x}, \cos \frac{1}{y})$. If fog were almost continuous, then its restriction to Δ the diagonal in Y^2 would be almost continuous by proposition 3.2. But by using arguments similar to those used in example 4.1 we see that fog : $\Delta \longrightarrow Y^2$ defined by fog $(x, x) = (\sin \frac{1}{x}, \cos \frac{1}{x})$ is not almost continuous. This example shows that the composition of two almost continuous functions is not necessarily almost continuous.

<u>Definition 4.1</u> Let Y be a subset of a topological space X. Y is called <u>an almost retract</u> of X if and only if there is an almost continuous function f on X onto Y such that for all y in Y, $f(y) = y$.

The following theorem states an equivalent condition (Cf. Borsuk [2] page 154). The proof is omitted as it is simple.

<u>Theorem 4.1</u> $Y \subset X$ is an almost retract of X if and only if there is an almost continuous function f on X onto Y such that for each x in X, $f(f(x)) = f(x)$.

<u>Theorem 4.2</u> If X is a Hausdorff space with the fixed point property and Y is an almost retract of X then Y has the fixed point property.

<u>Proof</u>. Let f be an almost continuous function on X onto Y such that for all y in Y, $f(y) = y$. Let g be a continuous function on Y to Y. By proposition 4.1, $gf : X \longrightarrow X$ is almost continuous and by proposition 2.1, gf has a fixed point p in X, i.e. $gf(p) = p$. Clearly p is in Y and so $f(p) = p$. This implies $g(p) = p$ which means that g has a fixed point in Y. Therefore, Y has the fixed point property. Q.E.D.

<u>Example 4.3</u> Let X by the square $[-1, 1] \times [-1, 1]$; and let $Y = \left\{ (x, f(x)) \;\middle|\; f(x) = \sin \frac{1}{x} \ (x \neq 0) \text{ and } f(0) = 0 \text{ for } x \text{ in } [-1, 1] \right\}$. Let Δ be the diagonal of X. The function g on X onto Δ defined by $g(x, y) = (x, x)$ is continuous. The function h on X into X defined by $h(x, y) = (x, f(y))$ is almost continuous. Also since Δ is closed in X the function $h \mid \Delta : \Delta \longrightarrow Y$ is also almost continuous by proposition 3.2. By proposition 4.2 the function hg on X onto Y is almost continuous and for each p in Y, $hg(p) = p$. This shows that Y has the fixed point property. However, Y is not locally connected and is not closed. In case of retracts the property of being closed and locally connected is preserved (see Borsuk [2] page 155).

A retract of a retract of a set X is a retract of X but as the composition of two almost continuous functions need not be almost continuous (example 4.2), we cannot extend this result to almost retracts. However, propositions 4.1, 4.2 provide us with the following theorems the proofs of which we omit.

Theorem 4.3 If Y is an almost retract of X and Z is a retract of Y then Z is an almost retract of X.

Theorem 4.4 If Y is a retract of a compact Hausdorff space X and Z is an almost retract of Y then Z is an almost retract of X.

Definition 4.2 Let X and Y be any topological spaces and let X_1 be a subset of X. Let f be a continuous function on X_1 to Y. We say that f admits an almost continuous extension to X if and only if there is an almost continuous function F on X to Y such that for all x in X, $F(x) = f(x)$.

Theorem 4.5 X_1 is an almost retract of X if and only if every continuous function f on X_1 to Y (Y arbitrary) admits an almost continuous extension to X.

Proof. Let X_1 be an almost retract of X, and let $r : X \longrightarrow X_1$ be the almost retraction.

Then $F = fr$ is an almost continuous function (proposition 4.1) on X to Y and for all x in X_1, $F(x) = fr(x) = f(x)$, i.e. f admits an almost continuous extension to X. On the other hand if $f : X_1 \longrightarrow Y$ admits an almost continuous extension $F : X \longrightarrow Y$, then choosing $Y = X_1$ and f the identity map on X_1 we find that F is an almost retraction of X onto X_1.

Q.E.D.

Theorem 4.6 If B, a closed subset of a compact metric space A, is homeomorphic to an almost retract R of the Hilbert cube I^{ω} then B is an almost retract of A.

Proof. The homeomorphism h on B onto R admits a continuous extension H on A to I (see Borsuk [2] page 158). By theorem 4.5 there is an almost continuous extension of $h^{-1} : R \longrightarrow B$ say $f : I^{\omega} \longrightarrow B$. By proposition 4.2, fH is an almost continuous function on A onto B and fH is the identity function on B. This means that B is an almost retract of A.

Q.E.D.

Definition 4.3 R is called a metric absolute almost retract if and only if X is any metric space and R' is a closed subset of X that is homemorphic to R, then R' is an almost retract of X.

The following theorem is obvious.

Theorem 4.7 The property of being a metric absolute almost retract is invariant under a homeomorphism.

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