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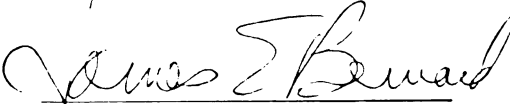
AN EFFICIENT METHOD TO PREDICT
THE EFFECT OF DESIGN MODIFICATIONS
ON THE DYNAMICS OF STRUCTURES

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AN EFFICIENT METHOD TO PREDICT
THE EFFECT OF DESIGN MODIFICATIONS
ON THE DYNAMICS OF STRUCTURES

By

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ABSTRACT

AN EFFICIENT METHOD TO PREDICT
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Static and dynamic analyses are important tools for the improvement of structural systems. Consider an analysis where a number of different changes may be synthesized to improve the dynamic and static behavior of a system. Such an analysis usually depends on numerical optimization schemes for systems with a large number of degrees of freedom. These analyses become computationally burdensome for redesign study.

This thesis presents an efficient method to predict the effect of design modifications on mechanical and structural systems with a large number of degrees of freedom. The method minimizes a cost function which includes natural frequencies, size of design change and the static deflections. It uses a finite element preprocessor to find derivatives of mass and stiffness matrices and computationally efficient techniques to find eigenvalue and eigenvector derivatives.

DEDICATION

To mom and dad, Muriel and Andrew TenEyck.

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I particularly wish to express my gratitude to my major professor and good friend Dr. James Bernard, for his guidance and constant encouragement as a teacher and a friend on this work and throughout my graduate studies.

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CHAPTER I

INTRODUCTION

Numerical optimization methods for specifying dynamic characteristics of structures require eigenvalues and eigenvectors of the modified system for each iteration. This can be a computationally burdensome process for systems with a large number of degrees of freedom.

This thesis presents an efficient method to determine the effect of design modifications on the eigenvalues and eigenvectors of a system, and uses this method to expedite design optimization. The next chapter will review the literature in the areas of 1) numerical optimization with dynamic constraints and 2) derivatives of natural frequencies and mode shapes with respect to design changes. Chapter 3 presents an efficient method to determine the effect of a single design modification. The method improves the dynamic characteristics of structure by removing natural frequencies from bands where excitations are likely to occur. It illustrates the method with examples. Chapter 4 examines design modification as a function of many possible changes and introduces static deflection to the optimization. It illustrates the power and the flexibility of the method with examples. Chapter 5 presents conclusions and some remarks on the extension of this work.

CHAPTER II

LITERATURE REVIEW

Dynamic response of structures in the absence of damping is typically described by a matrix equation of the form [1],

$$[M]\ddot{x} + [K]x = F \quad (2.1)$$

where $[M]$ and $[K]$ are, respectively, the mass and stiffness matrices of the structure. Their order n corresponds to the degrees of freedom of the system. The vectors $\ddot{x}$, x and F represent the acceleration, displacement and external loads respectively.

The mathematical model can be used to compute the natural frequencies and the mode shapes of the system. Should these indicate that the system is unsatisfactory, there is motivation to change the system.

This thesis develops and illustrates methods which are useful for finding changes which improve the frequency spectrum of the system. This review will discuss those areas of the literature which underpin the methods developed here. The review begins with the literature concerning optimization of the frequency spectra which result from models of the form 2.1. Since optimization, particularly the techniques to be developed here, makes use of derivatives of natural frequencies and mode shapes with respect to design changes, a review of the literature in that area will also be included.

2.1 Design Optimization

Design optimization entails a search for the best set of design variables satisfying an objective such as minimum weight or following rules concerning the frequency spectrum. Such an optimization may be thought of in either of two categories. The first category relies on variational calculus to support a search for a function or functions that will find minimum or maximum of some performance function subject to prescribed constraints. These methods work with a continuous model that is described by continuous functions of spatial coordinates. The optimum design is found through the solution of a system of equations expressing the conditions for optimality.

Variational methods are well suited for fundamental optimization studies. On the other hand, numerical methods employ iterative techniques to seek a near-optimal design [2,3]. An initial guess is used as starting point for a systematic search for better designs. The search is terminated when criteria are satisfied that ensure the current design is sufficiently close to the true optimum.

Frequency constraints are among the dynamic restrictions that have been considered. Much of the research in dynamic constraints optimization applies equality or inequality constraints on the fundamental frequency and specified frequency range of the structure. A good text on numerical optimization is Applied Optimal Design by Haug and Arora [4].

The computational methods with equality constraints allow less flexibility of design than those with inequality constraints. Equality constraints may be unnecessarily restrictive since an alternative and

more convenient design might be acceptable. But whether the problem deals with equality or inequality constraints, the eigenvalue problem consumes a major portion of the total computation effort. The process may prohibitively be slow for systems with a large number of degrees of freedom. Feng, Arora and Haug have presented a modal method which reduces the size of the problem and thus limits the needed computer resources [5].

Starkey proposed a new method to modify an existing design to improve the dynamic characteristics [6]. The approach was to select design changes that improve the dynamic characteristics of structure by removing natural frequencies from undesirable bands, for example, ranges of frequency where excitation is likely to occur. This procedure seeks to minimize a penalty function which becomes smaller as the design improves. Starkey suggested a penalty function $P(w,e)$ of the form

$$P(w,e) = H(F(w), S(e)) \quad (2.2)$$

where w is n vector of natural frequencies, e is m vector that represents design variables, $F(w)$ is a function which is large when natural frequencies are in undesirable ranges, $S(e)$ is a function which becomes large for undesirable size of variables and function H combines the effect of the two functions to make up the penalty function. The best designs are selected from local minima of the function $P(w,e)$.

Starkey used linear approximation via first order Taylor series expansion to predict the effects of various design changes on natural frequencies in the computation of the search for a minimum penalty function [6]. He suggested that, for small systems, the eigenvalue problem might be solved at every step of the optimization. But for the system with many

degrees of freedom, the solution of the eigenvalue problem is computationally burdensome since it requires $O(n^3)$ calculations where n is the number of degrees of freedom. On the other hand, the linear approximation of the modified natural frequencies has a limited range of validity.

It is the goal of this thesis to proceed along a path laid out by Starkey, i.e., to minimize a penalty function which is dependent on several variables. But we proposed to attack large systems and large changes. Thus, we seek, among other things, an improved method of approximating eigenvalues as a function of design changes. The next section discusses the literature in this area.

2.2 Derivatives of Eigenvalues and Eigenvectors

The eigenvalue problem associated with equation 2.1 may be written as

$$[M][U][\lambda] = [K][U] \quad (2.3)$$

where $[U]$ is an $n \times n$ matrix whose columns are n eigenvectors of the structure and $[\lambda]$ is an $n \times n$ diagonal matrix of the corresponding eigenvalues.

The determination of derivatives of eigenvalues with respect to a system parameter goes back to the nineteenth century work of Jacobi [7]. Wittrick presented an application of Jacobi's work to systems with an infinite number of degrees of freedom [8]. Fox and Kapoor showed the first derivative of an eigenvalue for discrete systems with symmetric mass and stiffness matrices [9]. They also derived two

methods for determining an eigenvector derivative. Rogers has generalized the second approach in reference 9 to include non-symmetric matrices [10]. Rudisill and Bhatia used second derivatives of eigenvalues to find the role of change of the flutter velocity of an aircraft structure with respect to structural parameters [11]. This work was extended for general matrices by Rudisill [12]. Nelson proposed an alternate procedure for calculating eigenvector derivatives of an arbitrary n th order symmetric or non-symmetric systems [13]. This method offered a significant computational improvement for sparse systems. Whitesell presented a method to calculate the eigenvector derivatives using only $O(n^2)$ calculations independent of the sparsity of the constituent matrices [14,15].

Since Whitesell's work is a key element of the work presented in this thesis, it will be useful to present some details. Consider mass and stiffness matrices which are functions of only one variable. Equation 2.3 can be written as follows

$$[M(e)][U(e)][\lambda(e)] = [K(e)][U(e)] \quad (2.4)$$

where $[M(e)]$ and $[K(e)]$ are real symmetric on $e \in \mathbb{R}$, furthermore $[M(0)]$ is positive definite and $[\lambda(0)]$ are non-repeated. Assume that equation 2.4 can be expanded with power series:

$$\left(\sum_{i=0}^{\infty} e^i ([K]^{(i)} - \sum_{k=0}^i \lambda_m^{(i-k)} [M]^{(k)}) \right) \left(\sum_{i=0}^{\infty} e^i \underline{U_m^{(i)}} \right) = 0 \quad (2.5)$$

where a superscript in parentheses denotes power series coefficient, a superscript without parentheses denotes the i th power of that variable, and subscript m specifies the m th eigenvalue and m th eigenvector. Also in this analysis, (e) is dropped from each term for convenience. Collecting terms in e in equation 2.5 yields

$$\sum_{i=0}^{\infty} e^i \sum_{J=0}^i ([K]^{(i-J)} - \sum_{k=0}^{i-J} \lambda_m^{(i-J-k)} [M]^{(k)}) \underline{U}_m^{(J)} = 0 \quad (2.6)$$

Each coefficient of e^i , $i \geq 1$, must vanish

$$\sum_{J=0}^i ([K]^{(i-J)} - \sum_{k=0}^{i-J} \lambda_m^{(i-J-k)} [M]^{(k)}) \underline{U}_m^{(J)} = 0 \quad (2.7)$$

Subtracting the $i = j$ term

$$\begin{aligned} ([K] - \lambda_m [M]) \underline{U}_m^{(i)} &= - \sum_{J=0}^{i-1} ([K]^{(i-J)} \\ &- \sum_{k=0}^{i-J} \lambda_m^{(i-J-k)} [M]^{(k)}) \underline{U}_m^{(J)} \end{aligned} \quad (2.8)$$

Multiplying both sides of equation 2.8 by $\underline{U}_m^T$, where T denotes the transpose of the vector $\underline{U}_m$, will result in zero on the left hand side of the equation since $\underline{U}_m^T ([K] - \lambda_m [M])$ is an eigenvalue problem. Therefore

$$0 = \underline{U}_m^T \left(\sum_{J=0}^{i-1} ([K]^{(i-J)} - \sum_{k=0}^{i-J} \lambda_m^{(i-J-k)} [M]^{(k)}) \underline{U}_m^{(J)} \right) \quad (2.9)$$

If the $j=0$ term is separated from equation 2.9 and the $k=0$ term is carried to the left hand side,

$$\begin{aligned} \lambda_m^{(i)} \underline{U}_m^T [M] \underline{U}_m &= \underline{U}_m^T ([K]^{(i)} - \sum_{k=1}^i \lambda_m^{(i-k)} [M]^{(k)}) \underline{U}_m \\ &+ \underline{U}_m^T \left(\sum_{J=1}^{i-1} ([K]^{(i-J)} - \sum_{k=0}^{i-J} \lambda_m^{(i-J-k)} [M]^{(k)}) \underline{U}_m^{(J)} \right) \end{aligned} \quad (2.10)$$

Therefore equation 2.10 is the formulation of the power series coefficients for the m th eigenvalue where $m=i, \dots, n$. The derivatives of eigenvalues with respect to a design change e can be calculated with a simple substitution

$$\frac{d^i \lambda_m}{de^i} = i! \lambda_m^{(i)} \quad (2.11)$$

Equation 2.8 can be written in the following form to calculate derivatives of the m th eigenvector [16],

$$\begin{aligned} \underline{U}_m^{(i)} &= -([K] - \lambda_m [M])^I \sum_{J=0}^{i-1} ([K]^{(i-J)} \\ &- \sum_{k=0}^{i-J} \lambda_m^{(i-J-k)} [M]^{(k)}) \underline{U}_m^{(J)} + C_i \underline{U}_m \end{aligned} \quad (2.12)$$

where the superscript I denotes a generalized inverse of the singular matrix $([K] - \lambda_m [M])$ such that

$$\underline{U}_m^T [M] ([K] - \lambda_m [M])^I = 0 \quad \text{and} \quad ([K] - \lambda_m [M])^I [M] \underline{U}_m = 0 \quad (2.13)$$

Whitesell [14] showed that the generalized inverse matrix in equation 2.12 can be computed as follows

$$([K] - \lambda_m [M])^I = ([I] - [G][M]) ([K] - \tilde{\lambda}_m [M])^{-1} ([I] - [M][G]) \quad (2.14)$$

$$\text{where } [G] = \frac{\underline{U}_m \underline{U}_m^T}{\underline{U}_m^T [M] \underline{U}_m}, \quad I = \text{a unit matrix}$$

and

$$\tilde{\lambda}_m = \lambda_m + \epsilon$$

The term C_i in the equation 2.12 can also be written as

$$C_i = - \sum_{J=0}^{i-1} \left(\sum_{k=0}^{i-J} \underline{U}_m^{(i-J-k)T} [M]^{(k)} \right) \underline{U}_m^{(J)} \quad (2.15)$$

Thus the derivatives of eigenvectors with respect to a single design change e , can be calculated with the following substitution.

$$\frac{d^i \underline{U}_m}{de^i} = i! \underline{U}_m^{(i)} \quad (2.16)$$

In the solution for the derivative of eigenvalues and eigenvectors, the only term that is candidate for $O(n^3)$ calculations is the inverse in equation 2.12. But if the eigenvalue problem for λ and U is solved with the inverse iteration method [17], the matrix $([K] - \tilde{\lambda}_m [M])^{-1}$ is calculated in the process. Thus the calculations of the derivatives of the eigenvalue and the eigenvectors of the system with size n are $O(n^2)$.

The next chapter illustrates the use of such a Taylor series in support of the optimization procedure offered by Starkey. At this stage, we will limit ourselves to one design change to facilitate the use of equations 2.10 and 2.12. Later chapters will present a technique to remove this restriction.

CHAPTER III

DESIGN OPTIMIZATION WITH ONE DESIGN VARIABLE

The traditional optimization methods for specifying dynamic characteristics of structures have important limitations. In particular, the methods which search for particular changes for natural frequencies and/or mode shapes as a function of the size of a given change are unduly restrictive since the designer must select an exact value for frequency modification out of a large variety of acceptable modifications.

A less restrictive method that removes unwanted natural frequencies from specified ranges was presented by Starkey [6]. The strength of this procedure lies in the fact that the designer needs only to specify what is not wanted, for example, a natural frequency occurring in a certain frequency band. References [18] and [19], which illustrate the effectiveness of the procedure, deal with problems in which a linear approximation relating natural frequency to design change was adequate.

In this thesis, we will be concerned with economical methods for the redesign of large structure via changes which may themselves be large. The implications of the size of the system and the size of the changes are these: We wish to deduce changes which will improve the system with as few eigenvalue solutions as possible. Thus a major step in this work is to improve upon the linear approximations used in [18] and [19]. Of course, to be effective, whatever approximations are used must be more economical than re-solving the eigenvalue problem.

This chapter presents an efficient method to determine the effect of a single design modification. The proposed method is intended to remove unwanted natural frequencies from the specified frequency ranges in structures that have many degrees of freedom. The chapter includes some examples which illustrate the procedure. Subsequent chapters will deal with design modification as a function of many possible changes.

3.1 Cost Function

The optimal design is the best choice of the feasible designs. The method presented in this chapter considers 'best' to be the design which minimizes cost function $C(w,e)$, which decreases as the design improves.

The form of $C(w,e)$ to be used here was developed by Starkey [6],

$$C(w,e) = F(F(w), S(e)) \quad (3.1)$$

where w is an $n \times 1$ vector of natural frequencies of the modified system, n is the size of the system, $F(w)$ is a function that is large when natural frequencies are in the undesirable range, e is an $m \times 1$ vector of design variables, $S(e)$ is a function that becomes large when design variables begin to exceed prescribed limits. Figure 3.1 shows the characteristics of the cost function.

The frequency content function, $F(w)$, is largest near the center of the undesirable frequency band since the center of the frequency range is the most critical frequency. Its magnitude drops as natural frequencies move away from the center of the function toward the edges of the band.

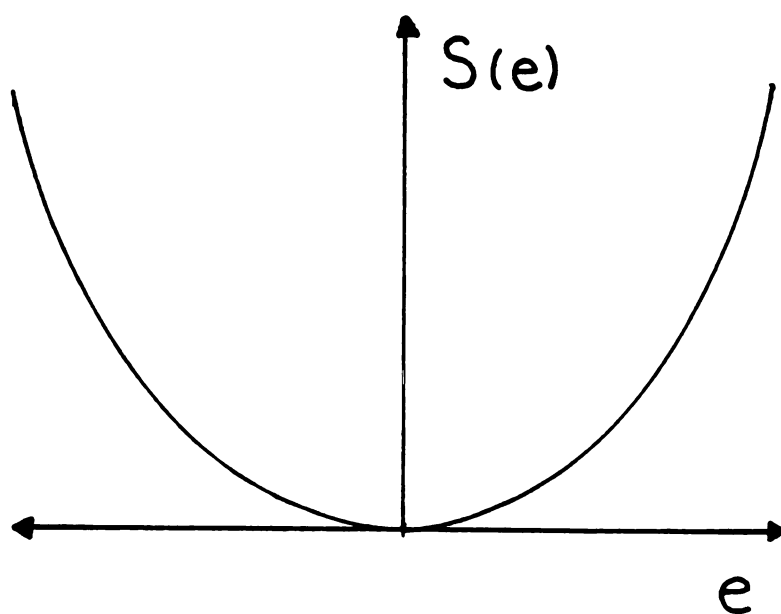
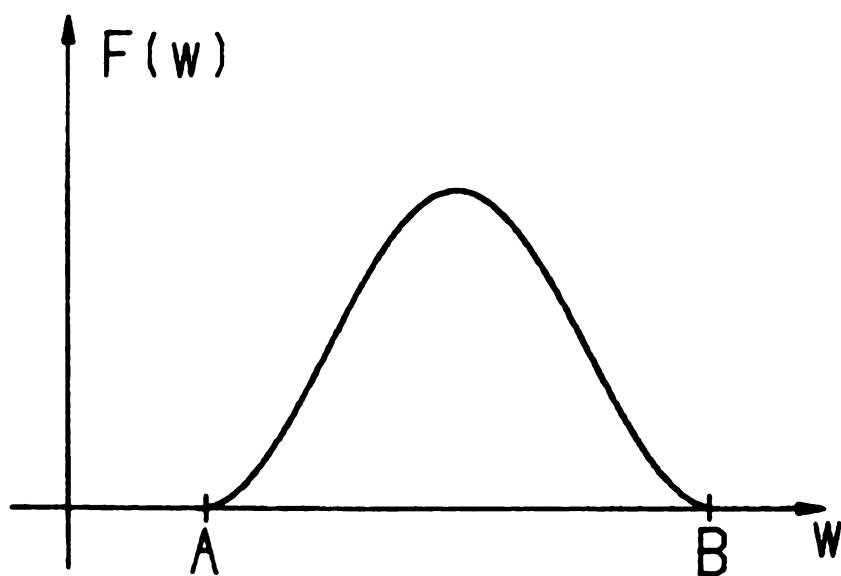


Figure 3.1 Frequency content and size-of-change functions.

The size of change function, $S(e)$, becomes larger as the size of the change increases. The total cost function is

$$C(w,e) = A S(e) + \sum_{i=1}^k B_i F_i(w) \quad (3.2)$$

The parameter B_i weighs the relative importance of each critical frequency band and A is a factor that weighs the importance of the size of change.

To minimize C , it is necessary to relate the changes e to the natural frequencies w . This can be done by re-evaluating the eigenvalue problem as a function of the changes or by developing functions which approximate the natural frequencies as a function of the change. For example, one might use a Taylor series:

$$w_{JP}^2 = w_{J0}^2 + \frac{d w_J^2}{de} e + \frac{1}{2!} \frac{d^2 w_J^2}{de^2} e^2 + \frac{1}{3!} \frac{d^3 w_J^2}{de^3} e^3 + \dots \quad (3.3)$$

The total cost function is then

$$C(e) = A S(e) + \sum_{i=1}^k B_i F_i(w(e)) \quad (3.4)$$

and C can be minimized as a function of e .

Starkey minimized $C(e)$ via the linear terms of the Taylor series. Since we expect the w, e relationship to be fundamentally nonlinear, it is clear that the range of usefulness of the linear series is limited. In this chapter, we will show how the higher order terms can be used to advantage.

3.2 Prediction of Eigenvalues and Eigenvectors

Numerical optimization is an iterative process which searches for the best design. It requires eigenvalues and perhaps eigenvectors at every iteration. But since each eigensolution requires $O(n^3)$ calculations, recalculation of eigenvalues and eigenvectors computationally burdensome.

In the method presented in this chapter, eigenvalues (square of natural frequencies) and eigenvectors (mode shapes) of the modified structure are approximated with a Taylor series including higher order terms. The approximation is based in large part on theory developed by Whitesell [14], which is summarized in some detail in Chapter 2. The derivatives of eigenvalues have the form

$$\begin{aligned} \lambda_k^{(i)} = & \frac{1}{\underline{U}_k^T [\underline{M}] \underline{U}_k} \left[\underline{U}_k^T ([\underline{K}]^{(i)} - \sum_{k=1}^i \lambda^{(i-k)} [\underline{M}]^{(k)}) \underline{U}_k \right. \\ & \left. + \underline{U}_k^T \left(\sum_{J=0}^{i-1} ([\underline{K}]^{(i-J)} - \sum_{k=0}^{i-J} \lambda^{(i-J-k)} [\underline{M}]^{(k)}) \underline{U}_k^{(J)} \right) \right] \end{aligned} \quad (3.5)$$

$$\text{where; } \lambda_k^{(i)} = \frac{1}{i!} \frac{d^i \lambda_k}{de^i}, \quad [\underline{k}]^{(i)} = \frac{1}{i!} \frac{d^i [\underline{K}]}{de^i},$$

$$[\underline{M}]^{(i)} = \frac{1}{i!} \frac{d^i [\underline{M}]}{de^i}, \quad \underline{U}_k^{(J)} = \frac{1}{J!} \frac{d^J \underline{U}_k}{de^J}$$

and derivatives of eigenvectors have the form

$$\underline{U}_k^{(i)} = -([K] - \lambda_k[M])^{-1} \sum_{J=0}^{i-1} ([K])^{(i-J)} - \sum_{k=0}^{i-J} \lambda_k^{(i-J-k)} [M]^{(k)} \underline{U}_k^{(J)} + C_i \underline{U}_k \quad (3.6)$$

where;

$$C_i = - \sum_{J=0}^{i-1} \left(\sum_{k=0}^{i-J} \underline{U}_k^{(i-J-k)} [M]^{(k)} \right) \underline{U}_k^{(J)}$$

$$([K] - \lambda_k[M])^{-1} = ([I] - [G][M])([K] - \tilde{\lambda}_k[M])^{-1}([I] - [M][G])$$

$$[G] = \frac{\underline{U}_k \underline{U}_k^T}{\underline{U}_k^T [M] \underline{U}_k} \quad \tilde{\lambda}_k = \lambda_k + \epsilon$$

Equations 3.5 and 3.6 require derivatives of mass and stiffness matrices with respect to the design variable. It is unwieldy to construct mass and stiffness matrices with analytical functions in their entries which are appropriate for the modeling of a wide variety of potential design changes. Therefore, it is difficult to obtain derivatives of these matrices via differentiation with respect to a design variable. However, these derivatives can be approximated with finite difference methods. For example, the first derivative of a mass matrix, $[M(e)]$, and a stiffness matrix, $[K(e)]$, with respect to a design variable e , can be approximated with the central finite difference method as:

$$\frac{d[M(e)]}{de} = \frac{[M(e')] - [M(-e')]}{2e'} \quad (3.7)$$

$$\frac{d[K(e)]}{de} = \frac{[K(e')] - [K(-e')]}{2e'} \quad (3.8)$$

$[K(e')]$, $[M(e')]$ and $[K(-e')]$, $[K(-e')]$ are mass and stiffness matrices that are perturbed by e' and $-e'$ respectively. These matrices can be constructed with a finite element preprocessor. Higher order derivatives of mass and stiffness matrices can also be calculated via finite differences. The central difference formulations for these derivatives is given in Appendix A.

To illustrate the power of the procedures presented here, it will be instructive to consider an example.

3.3 Example

Consider the horizontal beam of Figure 3.2. The legs are twice as thick as the horizontal part of the beam. The design variable is the thickness h of the horizontal beam. The objective is to remove any natural frequency from the frequency range of 400 Hz - 580 Hz.

3.3.1 Initial Design

The initial design is modeled with the six beam finite elements (four horizontal, two vertical) via the ANSYS finite element preprocessor. Each beam element has two nodes with three degrees of freedom (two translation and one rotation) at each node [20]. Figure 3.3 shows the elements and nodes of the beam. Table 3.1 lists the fifteen natural frequencies that correspond to the fifteen degrees of freedom. These were obtained by solving

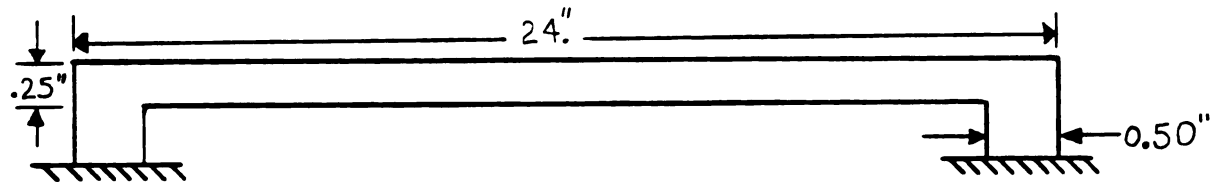


Figure 3.2 Beam with legs fixed.

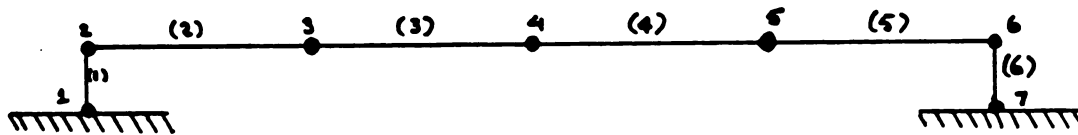


Figure 3.3 Finite element representation of the beam.
Numbers in parenthesis represent the element number.

Table 3.1 Natural frequencies of the beam at $h=.25$ inches

Beam Frequencies (Hz)

1.	88
2.	245
3.	490
4.	918
5.	1520
6.	2323
7.	2782
8.	5197
9.	7721
10.	10260
11.	14054
12.	19394
13.	20694
14.	44783
15.	44921

$$([K] - w_1^2[M])\underline{U}_1 = 0 \quad (3.9)$$

where $[K]$ and $[M]$ are the stiffness and mass matrices respectively, w_1 is the 1th natural frequency of the system and $\underline{U}_1$ is the associated mode shape.

3.3.2 The Design Change

The only admissible design change is assumed to be the thickness h of the horizontal part of the beam. The derivatives of the eigenvalues and eigenvectors with respect to h are calculated using equations 3.5 and 3.6. The derivatives of the mass and stiffness matrices were obtained via the ANSYS preprocessor and the central difference method as discussed in section 3.2.

3.3.3 The Cost Function

The penalty function $C(e)$ can be minimized with routinely available software [21]. In this example, there is one undesirable frequency band which is in the band 400 Hz - 580 Hz. A frequency content penalty that meets our needs is

$$F(w) = \sum_{j=1}^m (1 - \cos(2\pi(w_j - p)/(u-p))) \quad (3.10)$$

where p and u are lower and upper bounds of the frequency band respectively, m is the number of modes in the finite element model and w_j is the j th modified natural frequency.

We assume that the size of change of the penalty function is quadratic in e

$$S(e) = e^2 \quad (3.11)$$

and the linear combination of equations 3.10 and 3.11 with scaling factors A and B at 1 and 100, respectively, becomes the total cost function as indicated in equation 3.2

3.3.4 The Procedure

An optimal design for this example is sought according to the following procedure.

1. Construct the stiffness and mass matrices for the original design via a finite element preprocessor.
2. Extract the dynamic characteristics of the system, that is, the natural frequencies and mode shapes.
3. Calculate mass and stiffness matrix derivatives as shown in Appendix A.
4. Calculate eigenvalue and eigenvector derivatives using equations 3.5 and 3.6.
5. Input the scaling parameters A and B in equation 3.2.
6. Minimize the total cost function $C(e)$ as it is defined by equation 3.4.
7. Verify the results by solving eigenvalue problem for the modified model with the proposed design change.

3.3.5 Results

The algorithm used for this example approximates the modified frequency with 9 terms in the Taylor series. Figures 3.4 through 3.8 illustrate the approximation. The figures show clearly that the first five frequencies of the example do not have linear relationships with the design variable.

The results of the minimization are shown in Table 3.2. The optimal thickness of the beam found was to be .056 inches less than the original .25 inch design, and the corresponding frequencies are out of the unwanted frequency band. Note that the approximated frequencies for the new thickness agree with the eigensolution of the new design, indicating that the series has satisfactorily replaced the re-solution of the eigenvalue problem.

3.4 The Convergence of the Series

The eigenvalues and eigenvectors of a modified system are approximated with the Taylor series. This approximation is accurate only within the radius of convergence of the series. For example, consider the following 2x2 matrix

$$A = \begin{bmatrix} 1 & Z \\ Z & 2 \end{bmatrix} \quad (3.12)$$

for this case, the eigenvalues are

$$\lambda_1 = \frac{3}{2} + \frac{1}{2} \sqrt{4Z^2 + 1} \quad (3.13)$$

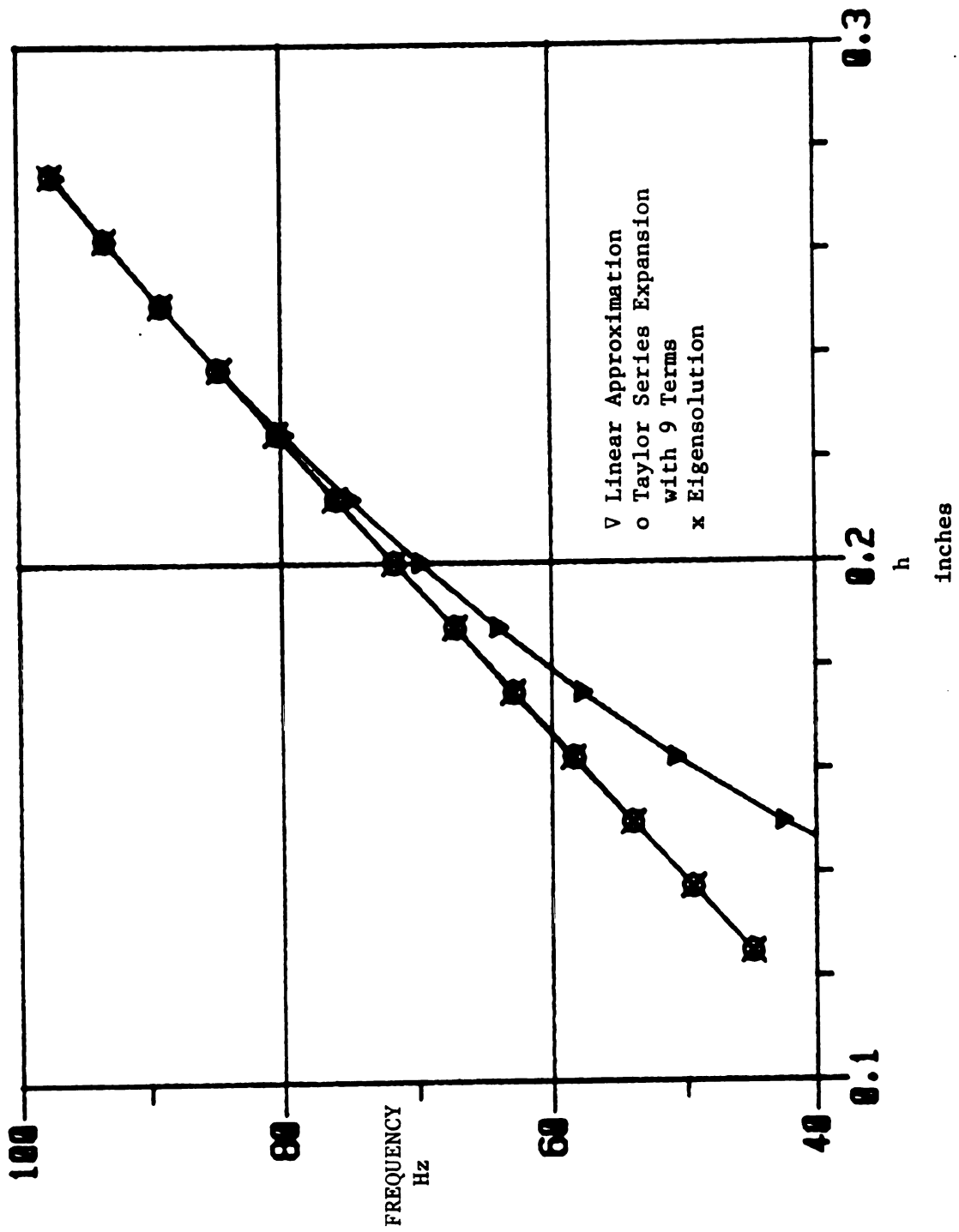


Figure 3.4 Comparison Between The Linear Approximation And The Higher Order Taylor Series Of the First Frequency Of The Beam.

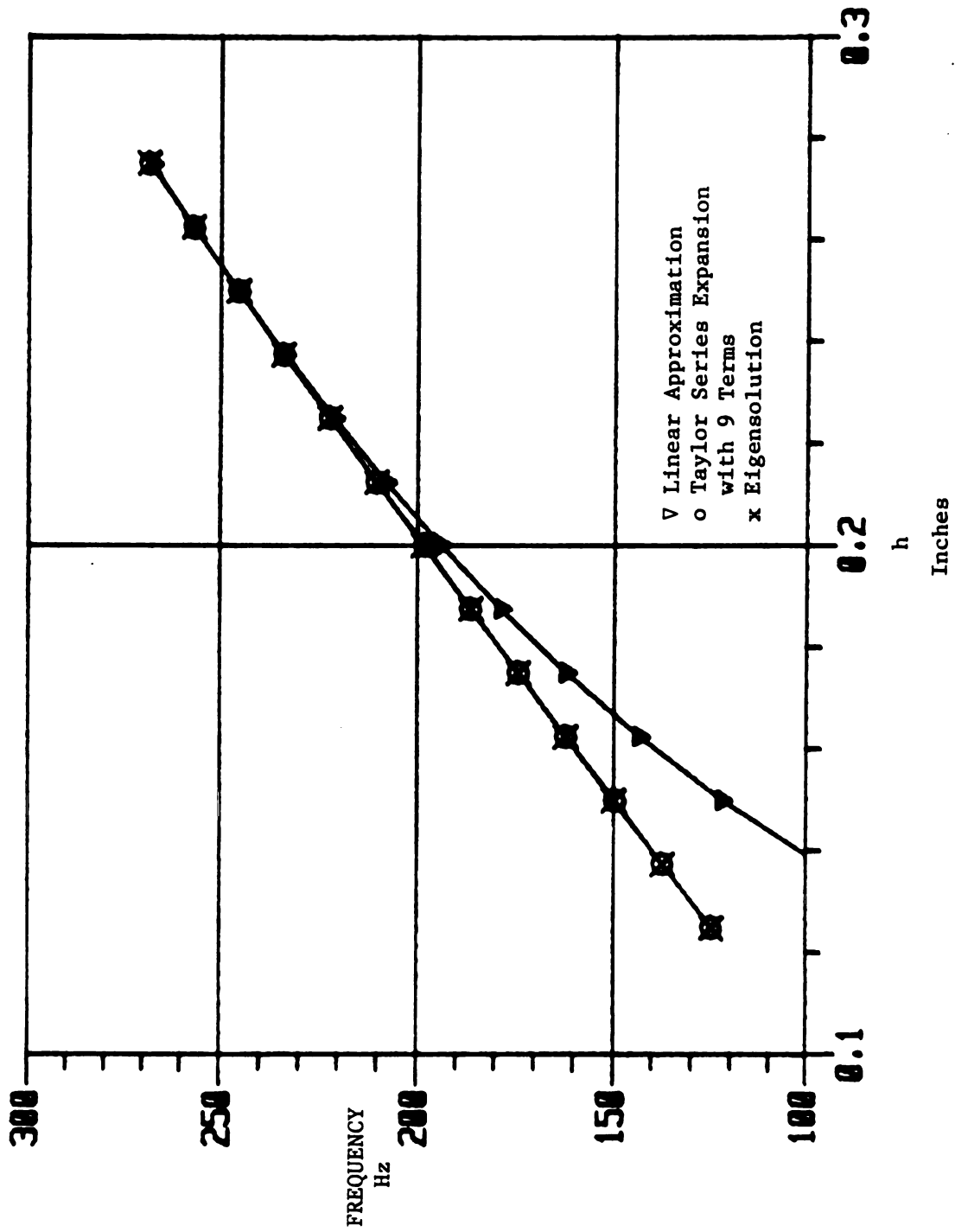


Figure 3.5 Comparison Between The Linear Approximation And The Higher Order Taylor Series Of The Second Frequency Of The Beam.

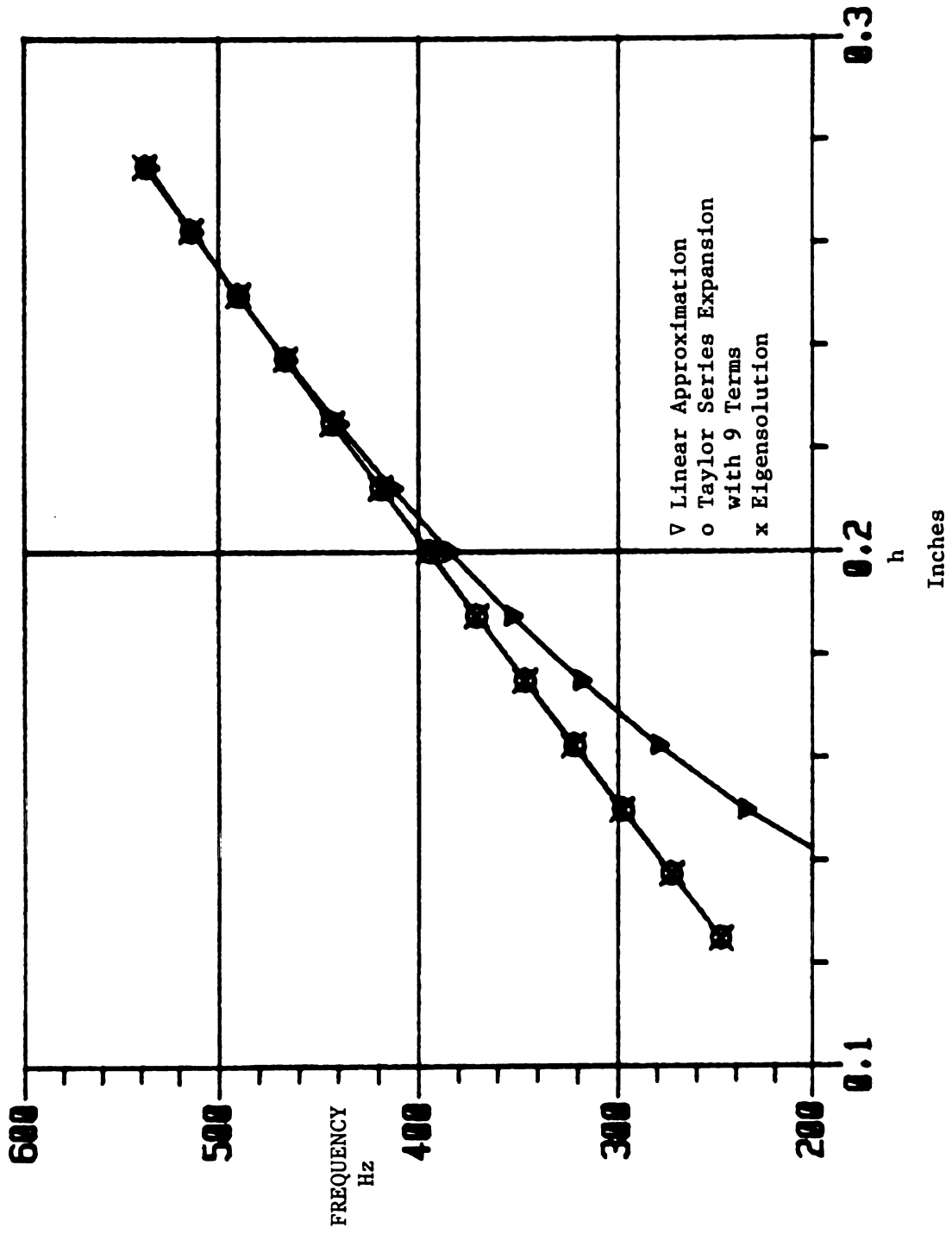


Figure 3.6 Comparison Between The Linear Approximation And The Higher Order Taylor Series Of the Third Frequency Of The Beam.

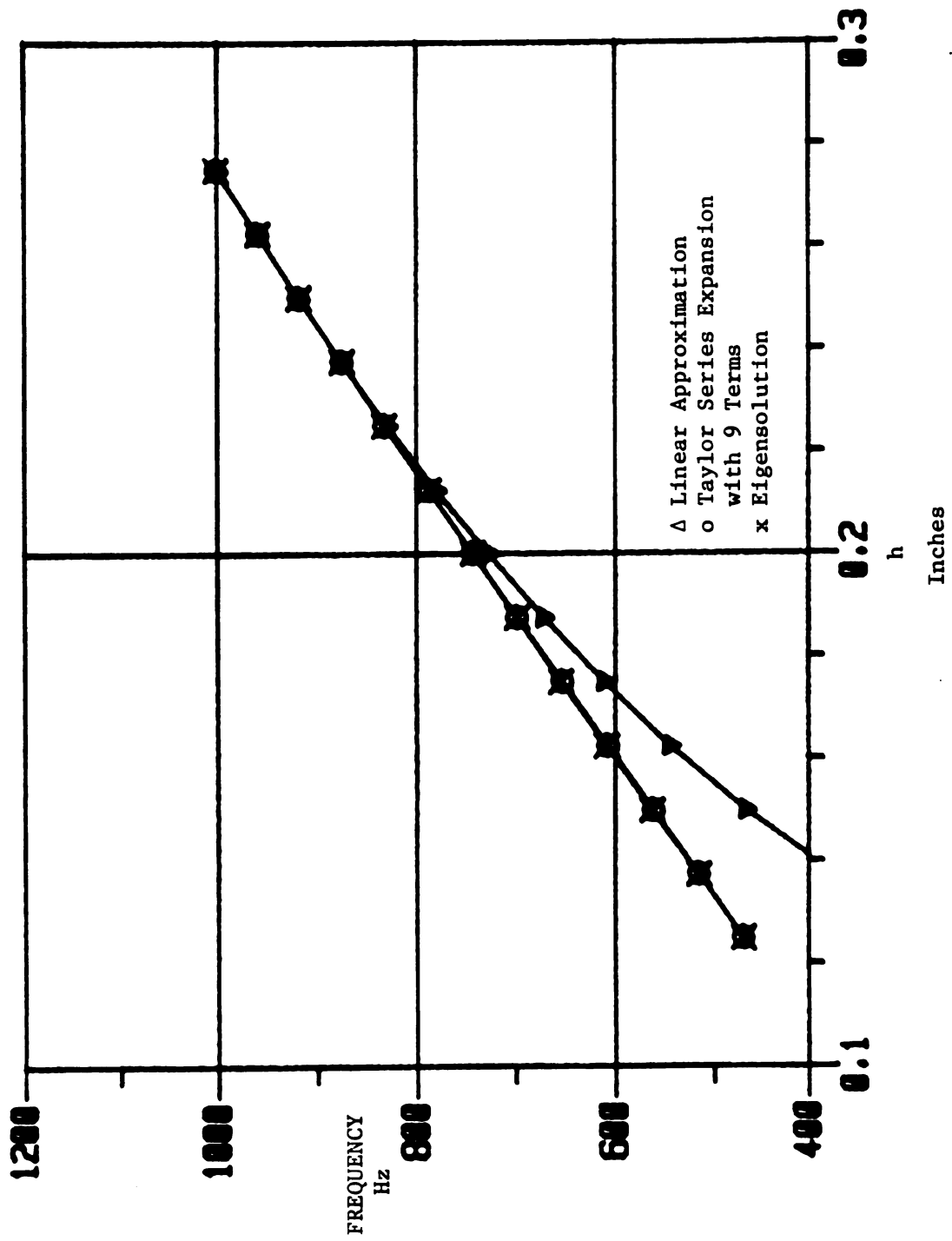


Figure 3.7 Comparison Between The Linear Approximation And The Higher Order Taylor Series Of the Fourth Frequency Of The Beam.

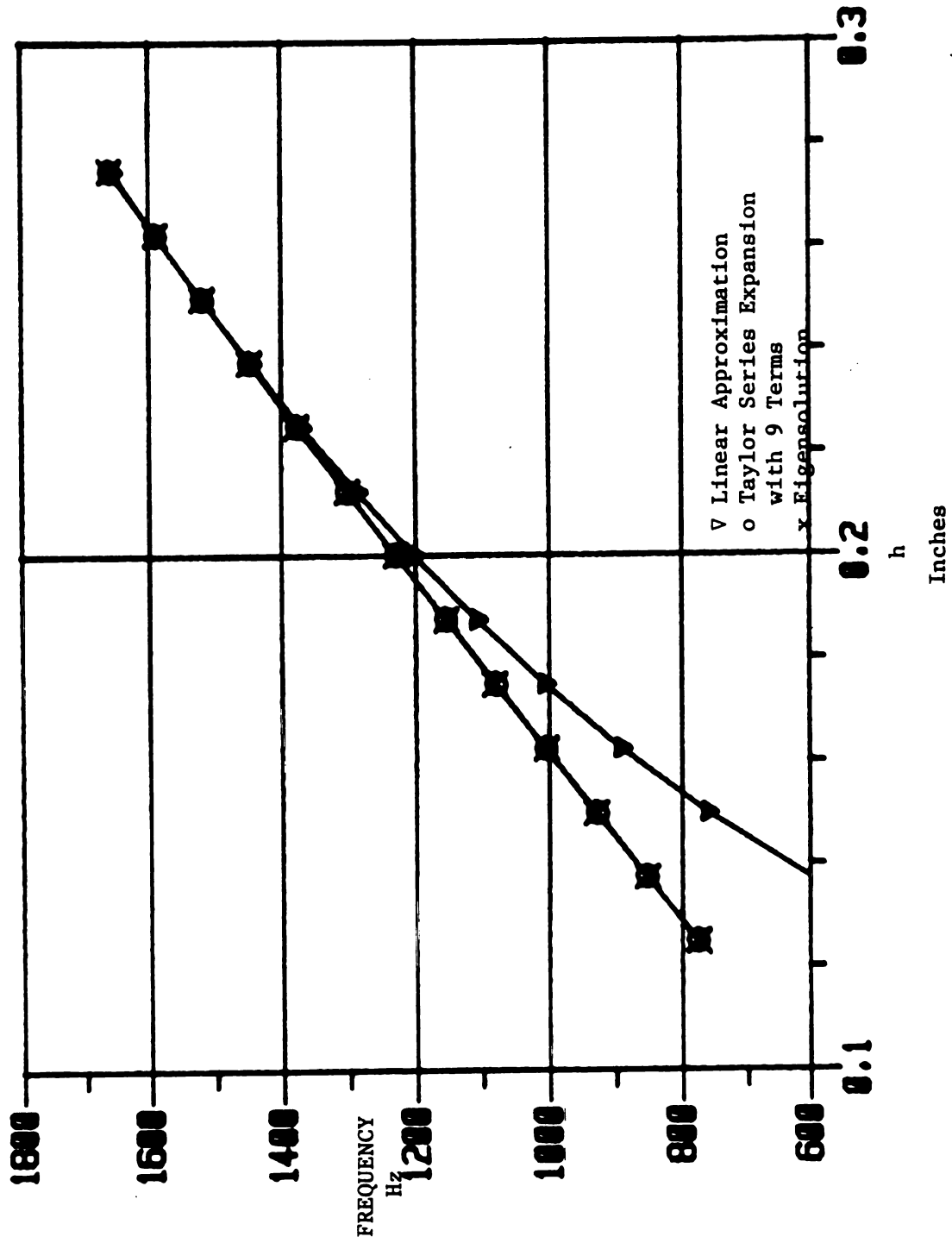


Figure 3.8 Comparison Between The Linear Approximation And The Higher Order Taylor Series Of the Fifth Frequency Of The Beam.

Table 3.2 Comparison of the predicted natural frequencies and the frequencies from the eigenvalue solver at $h=.194$ inches

	PREDICTED FREQUENCIES (Hz)	ANSYS EIGENVALUE SOLVER (Hz)
1.	69.5	69.5
2.	192.6	192.6
3.	383.1	383.2
4.	721.6	721.8
5.	1193.1	1193.5

$$\lambda_2 = \frac{3}{2} - \frac{1}{2} \sqrt{4z^2 + 1} \quad (3.14)$$

where z is variable. The eigenvalues are complex and the matrix A has a double root at

$$z = \pm \frac{1}{2} i \quad \text{where} \quad i = \sqrt{-1} \quad (3.15)$$

The magnitude of this z value is the radius of convergence of the Taylor series about $z=0$. This simple example illustrates that the radius of convergence of the series may depend on complex values of the design parameter. Figures 3.9 and 3.10 are plots of λ_2 and λ_2 for real values of z .

Of course, in a large complicated problem, this radius is not at all obvious and may, in fact, have to be dealt with via rather ad-hoc methods. In section 3.3, for example, the series had a large enough radius of convergence to cover frequencies of the problem. But, in the same example, if the unwanted frequency range is set to 2000 Hz - 3000 Hz, then the frequency penalty includes sixth and seventh frequencies. Figures 3.11 and 3.12 indicate that the radius of convergence of the Taylor series for these two frequencies is smaller than for the first five frequencies. Therefore, the usefulness of the Taylor series approximation of these frequencies will be limited.

To illustrate the consequences of this limitation, consider Table 3.3, which presents the results of an attempt to drive frequencies out of the range of 2000 to 3000 Hz. The predicted frequencies in Table 3.3 show that the frequencies are removed from range. However, when these

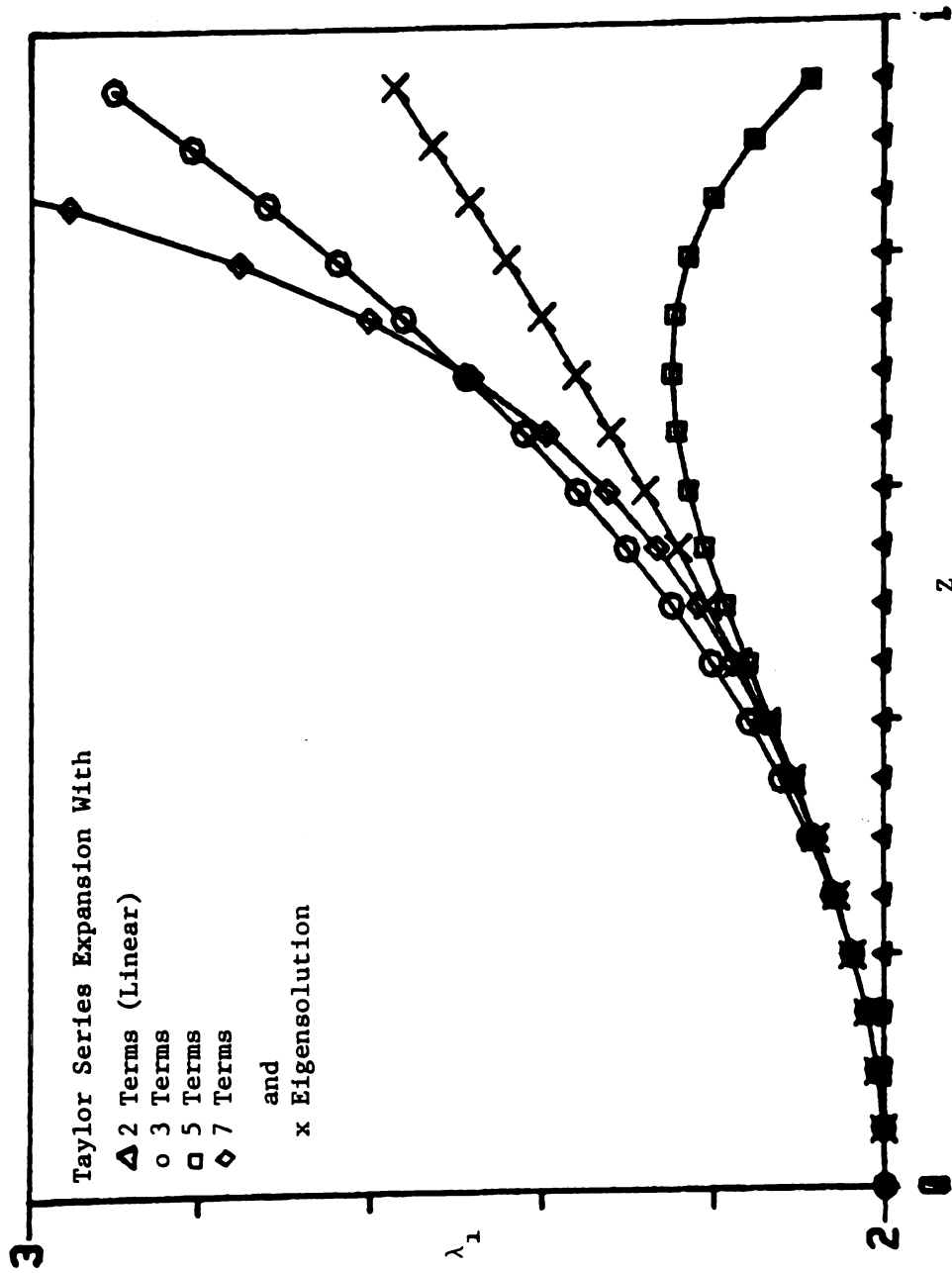
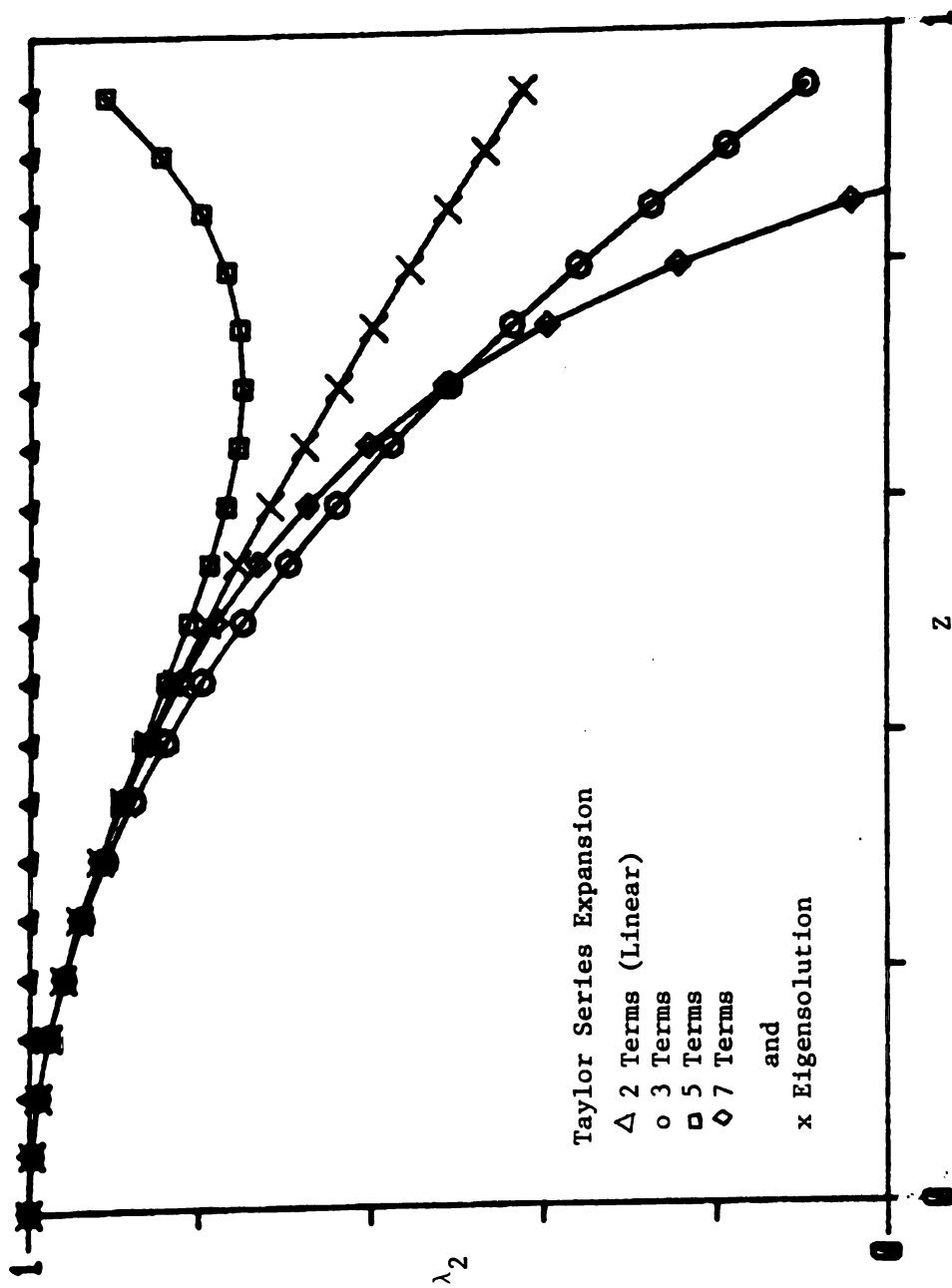


Figure 3.9 Predictions Of The First Eigenvalue At Different Z Values.

Figure 3.10 Predictions Of The Second Eigenvalue at Different Z Values.

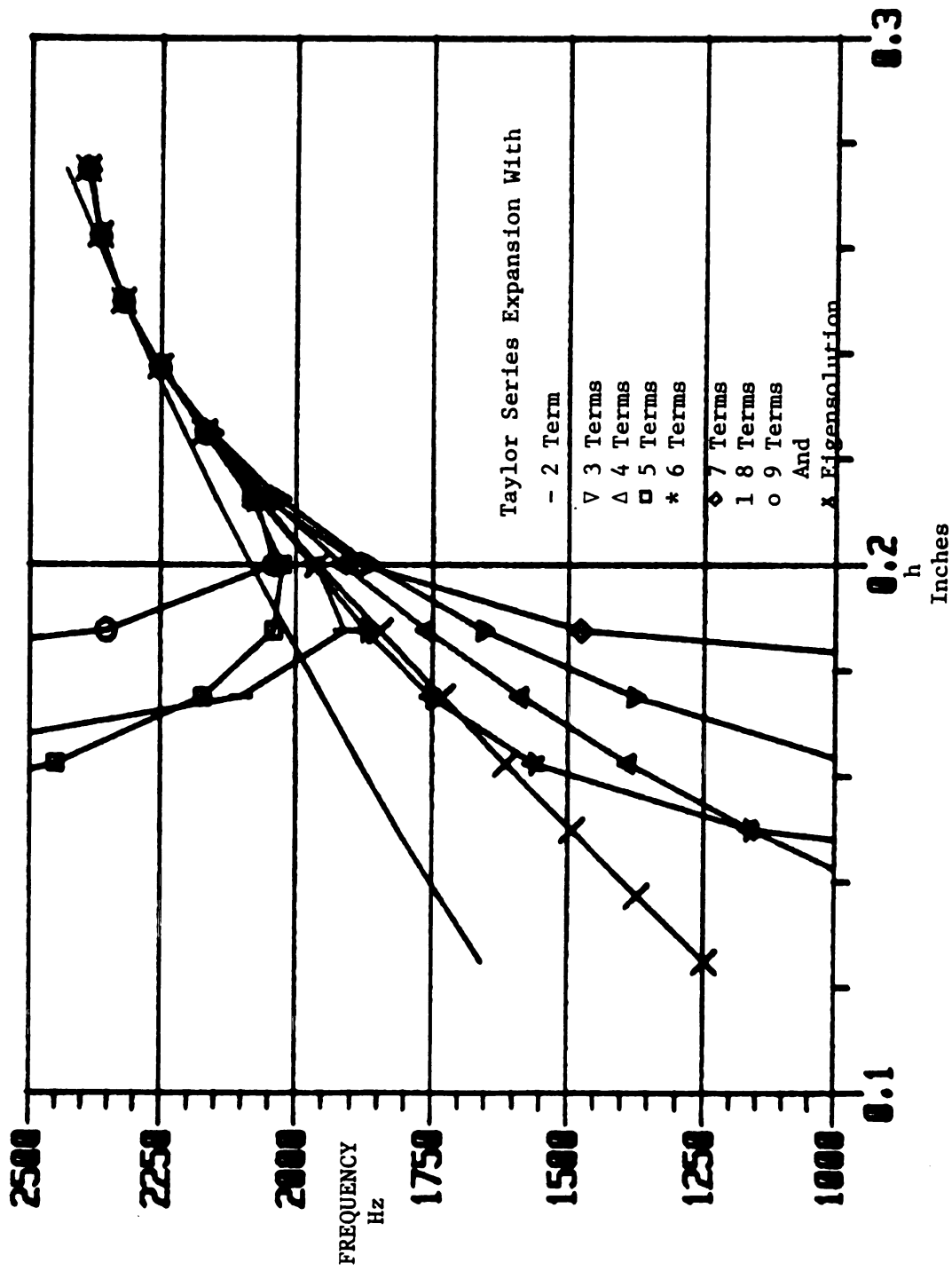


Figure 3.11 Predictions Of The 6th Frequency Of The Beam At Different Thicknesses.

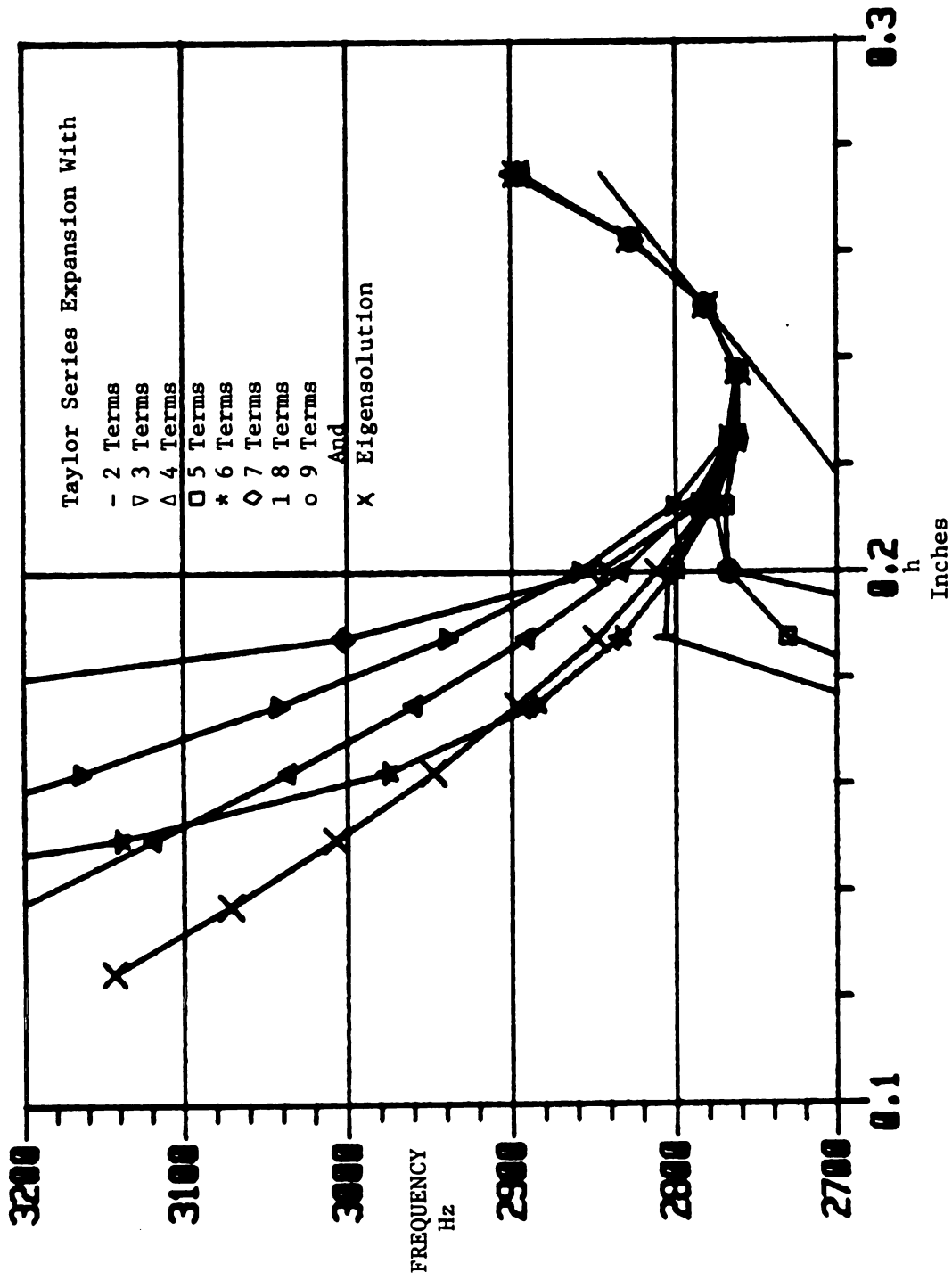


Figure 3.12 Predictions Of The 7th Frequency Of The Beam At Different Thicknesses.

Table 3.3 Comparison of the predicted frequencies
and the frequencies from the eigenvalue solver
at $h=.1732$ inches for the frequency band of 2000 Hz-3000 Hz

PREDICTED FREQUENCIES (Hz)		ANSYS EIGENVALUE SOLVER (Hz)
1.	63.8	63.8
2.	176.9	177.0
3.	351.6	351.9
4.	663.1	663.7
5.	1096.3	1097.2
*6.	1905.7	*1759.2
*7.	3172.3	*2885.7
8.	5537.2	5539.2
9.	8065.3	8067.6
10.	10780.0	10782.9
11.	14771.0	14776.2
12.	20875.0	20882.6
13.	21777.0	21784.9
14.	47545.0	47555.0
15.	47669.0	47678.7

results are compared with the eigensolution at the indicated thickness, it is obvious that the approximations of the sixth and seventh frequencies are inaccurate at the proposed thickness, and that, in fact, the seventh frequency has not been moved from the unwanted frequency range.

The next section presents a method to deal with the series convergence problem.

3.5 Restart

Restart is a re-evaluation procedure used when resolution of the eigenvalue problem indicates that the Taylor series did not converge for a frequency of interest. In short, the fact that the eigenvalue problem has been redone to check the accuracy of the series allows new derivatives to be calculated to expand about the new operating point.

The example from Section 3.4, with an objective of removing frequencies from the 2000 Hz - 3000 Hz range, yielded a design which, according to the approximation, drove all frequencies out of the undesirable range. But re-evaluation of the eigenvalue problem at the new design point indicated that the series did not faithfully represent the eigenvalues at the indicated minimum. Figures 3.13 and 3.14 verify that the next step in the optimization, which is an expansion about the new design point based on the eigenvalue check solution, results in locally accurate solutions. With this new starting point the optimizer found a design change that met the objective. In this case, re-resolution indicated that the series remained in an accurate range. (See Table 3.4).

The steps of the algorithm that are explained in this chapter can be summarized as follows:

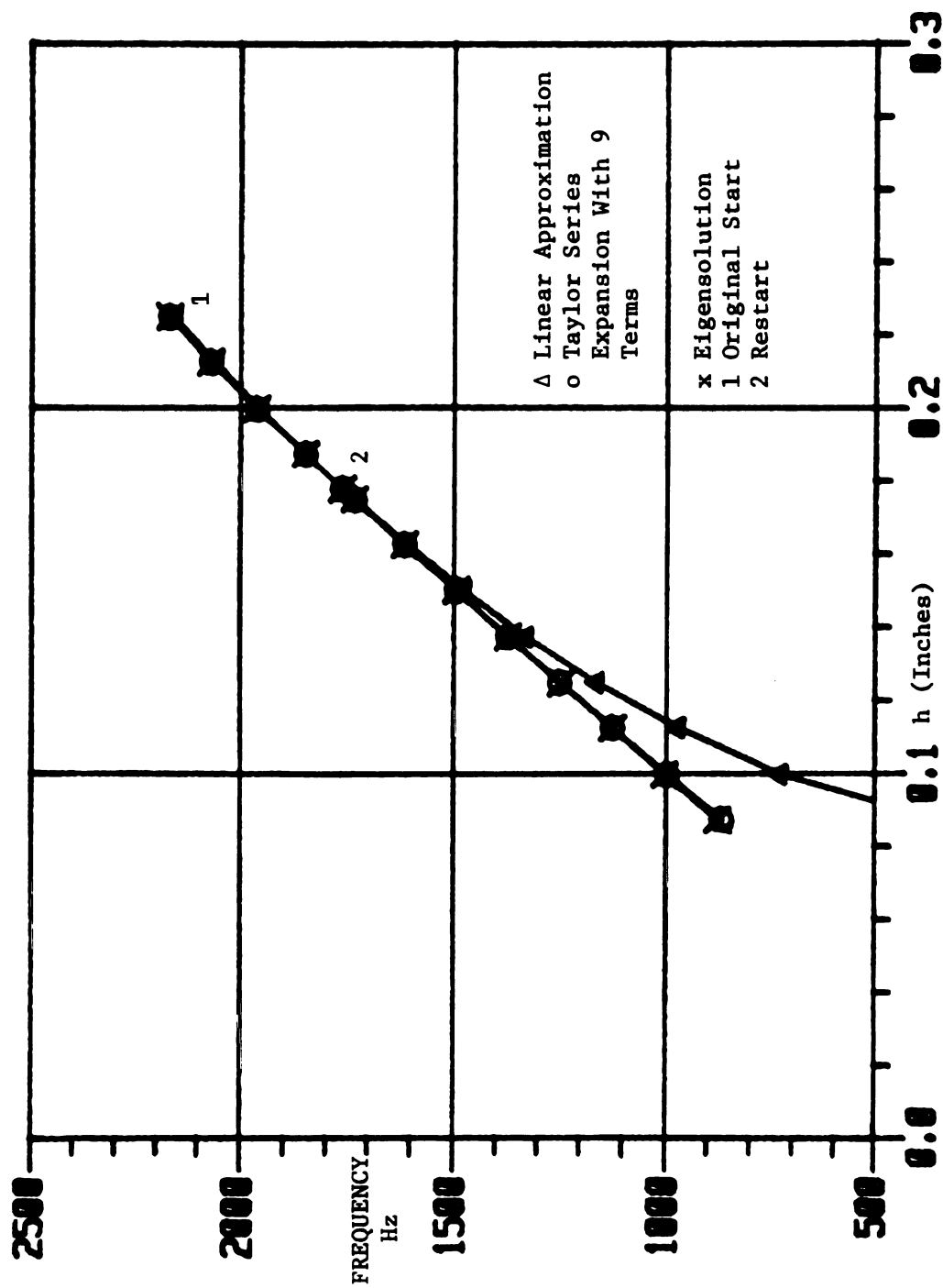


Figure 3.13 The Effect Of The Restart On The Predictions For The Sixth Frequency.

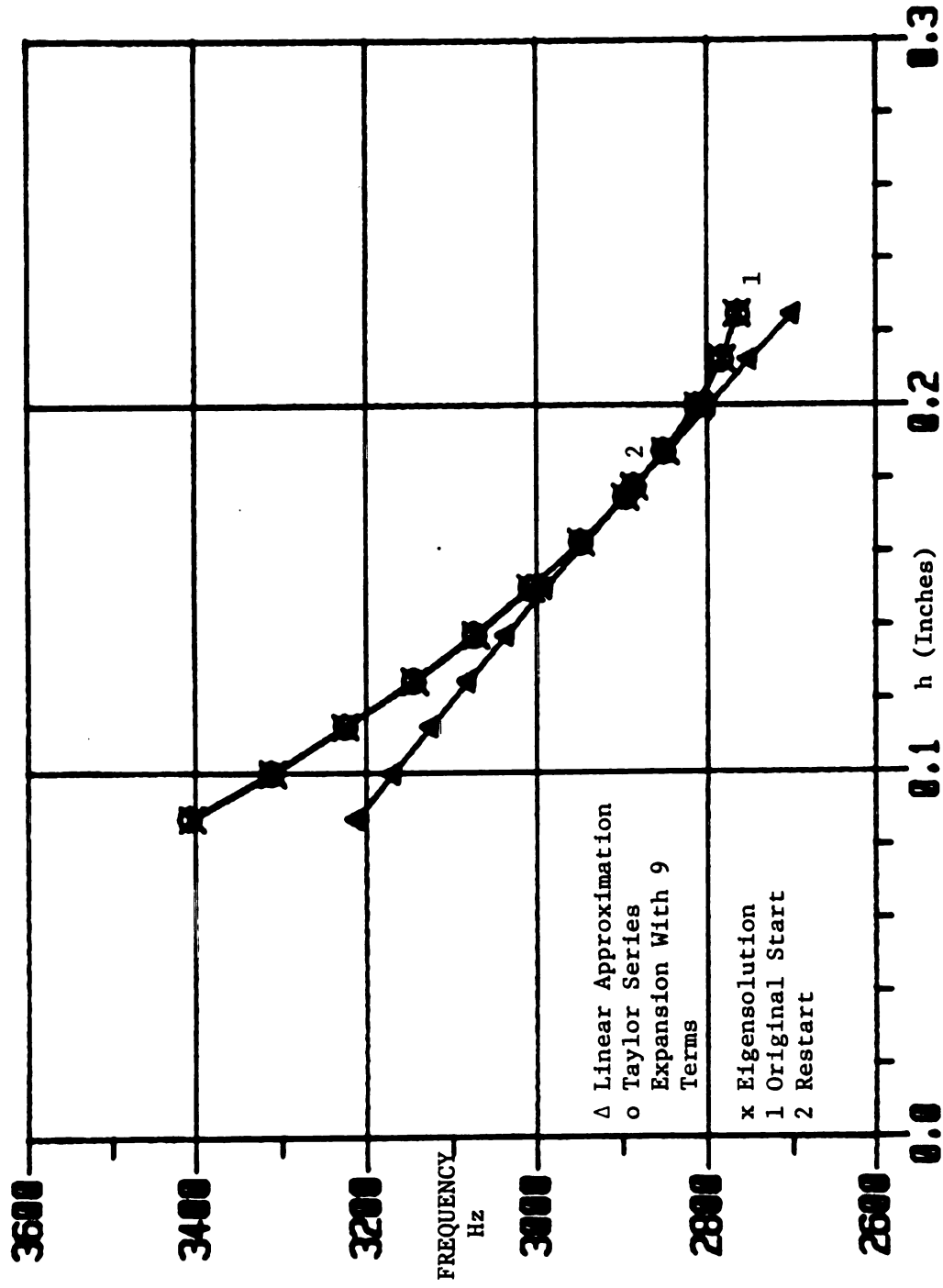


Figure 3.14 The Effect Of The Restart On The Predictions For The Seventh Frequency.

Table 3.4 Restart Results
h=.1351 inches

PREDICTED FREQUENCIES (Hz)		ANSYS EIGENVALUE SOLVER (Hz)
1.	50.2	50.3
2.	139.6	139.6
3.	277.1	277.2
4.	523.6	523.7
5.	865.5	865.8
6.	1392.6	1393.0
7.	3061.9	3061.3
8.	5856.7	5855.4
9.	8389.1	8387.8
10.	11124.0	11122.5
11.	15179.0	15178.6
12.	22101.0	22097.1
13.	22788.0	22785.4
14.	49235.0	49228.8
15.	49349.0	49342.8

- Generate the Finite Element Model of the original design
- Obtain the eigensolution of the model
- Assess the results
- Set up the cost function
- Calculate derivatives of mass and stiffness matrices via the finite element preprocessor and finite difference methods
- Calculate derivatives for eigenvalues and eigenvectors for the natural frequencies of interest
- Execute the minimization algorithm to find a new design
- Solve the eigenvalue problem at the new design point
- Compare the solution of the eigenvalue problem with the series approximation
- Restart if necessary.

This chapter dealt with design optimization using one design variable and a cost penalty which is a simple function of the size of the design change. The next chapter deals with the multi-variate design optimization and introduces static deflection to the cost function.

CHAPTER IV

DESIGN OPTIMIZATION WITH MULTIPLE DESIGN VARIABLES

This chapter presents an efficient method for predicting the effects of design modifications which may be dependent on several variables. The particular problem of interest here is to find an improved design, as indicated by eigenvalues and eigenvectors and by static deflection. We assume that the finite element analysis finds the eigenvalues and eigenvectors of the initial design, and that this analysis indicates a need to improve the system.

Additional eigenvalue analyses for optimization purposes are computationally burdensome process for systems with a large number of degrees of freedom. Thus the procedure presented here seeks to avoid resolution of the eigenvalue problem. To avoid unnecessary resolutions, the eigenvalues and eigenvectors of the modified system are approximated using the Taylor series expansion about the original design point. Since the series will require only $O(n^2)$ calculations, as opposed to $O(n^3)$ for an eigenvalue problem, the optimization procedure requires only $O(n^2)$ calculations to find the improved design.

Initially, the design analysis for several variables will be developed without static deflection as part of the design criteria. Then in section 4.5 static deflection function will be added to the cost function. The following sections explain these steps in detail. The power of the method will be illustrated with examples.

4.1 The Cost Function

As in previous chapters, the cost function is

$$C(w, e) = F(F(w), S(e)) \quad (4.1)$$

where w is an $n \times 1$ vector of natural frequencies of the modified system, $F(w)$ is a function that is large when natural frequencies are in an undesirable range, e is an $m \times 1$ vector of design variables, $S(e)$ is a function that becomes large when design variables begin to exceed prescribed limits. The methods given in chapter 3 allow efficient minimization of such penalties as a function of one change. This chapter presents an efficient minimization procedure which includes many changes.

4.2 Calculation of Optimum Design Changes

The calculation of optimum changes require a search for the minimum specified cost function. The key issue in the total cost function is determining eigenvalues and eigenvectors of a modified system for the frequency content function. This can be done in two steps, 1) a linear approximation of the eigenvalues of interest, a procedure which can handle multiple design variables while using only $O(n^2)$ calculations, and 2) nonlinear approximation which improves the results of the linear calculations. In the first step, the design variables are determined based on the assumption that the rates of change of eigenvalues are constant. For the second step, the ratios of the design variables with respect to each other are kept constant and a new variable that

scales the magnitude of the variables is determined. In this case, the Taylor series, including higher order terms, is used to estimate the eigenvalues of interest. The next two sections explain the steps in detail.

4.2.1 Linear Approximation

The penalty function can be computed as a function of the modified eigenvalues. As a first step, the approximations of the eigenvalues are based on a linear expansion, therefore, the predictions of the natural frequencies of a modified system are

$$w_1^2 = w_{10}^2 + \sum_{j=1}^m \frac{\partial w_1^2}{\partial e_j} e_j \quad (4.2)$$

where w_1 and w_{10} are the modified and the original i th frequencies, respectively, and $\frac{\partial w_1^2}{\partial e_j}$ is the rate of change of the i th frequency with respect to the j th design variable. For distinct eigenvalues, the derivative can be computed as follows [9],

$$\frac{\partial w_1^2}{\partial e_j} = \underline{U}_1^T \left(\frac{\partial [K]}{\partial e_j} - w_1^2 \frac{\partial [M]}{\partial e_j} \right) \underline{U}_1 \quad (4.3)$$

$$\text{where } \underline{U}_1^T [M] \underline{U}_1 = 1$$

This procedure has $O(n^2)$ calculations, where n is the number of degrees of freedom. The first derivatives of stiffness and mass matrices can be approximated via the central difference method as explained in chapter 3. Therefore, the final solution becomes a combination of the design variables that best removes the unwanted natural frequencies under the constraint of the size-of-change penalties.

4.2.2 Nonlinear Approximation

If design variables have a linear relationship with the natural frequencies of the structure, the linear approximation will give the correct solution. But since the natural frequencies are, in fact, nonlinear functions of the design variables, the linear results have a limited range of validity. The use of higher order terms can lead to accuracy over a much wider range of changes. The method that is presented here uses the Taylor series including the higher order terms to improve the linear results.

An initial set of design variables $e_k'^*$ is found using linear approximations of the natural frequencies. This is followed by a nonlinear optimization using the methods of chapter 3. The single design variable for the nonlinear optimization is $\bar{e}$, where

$$e_J = \bar{e} e_J'^* \quad (4.4)$$

The natural frequencies of the system are now a function of the scale variable $\bar{e}$. Since $\bar{e}$ scales the change along the path indicated by the linear analysis, this may be viewed as a steepest descent procedure. Equations 3.5 and 3.6 yield the higher order terms in the Taylor series for approximation of the modified natural frequencies and mode shape vectors with respect to $\bar{e}$. In particular,

$$w_1^2 = w_{10}^2 + \frac{dw_1^2}{d\bar{e}} \bar{e} + \frac{1}{2!} \frac{d^2 w_1^2}{d\bar{e}^2} \bar{e}^2 + \frac{1}{3!} \frac{d^3 w_1^2}{d\bar{e}^3} \bar{e}^3 + \dots \quad (4.5)$$

The solution is a set of design variables $\bar{e} e_J'^*$ which minimize the total cost function under the constraint that the design variables retain the ratio to each other that was indicated by the linear

calculations. Thus there remain two questions, namely 1) does the Taylor series yield an adequate representation of the penalty along the path indicated by the e_J^* , and 2) did the e_J^* indicate a path that led to a minimum of $C(w,e)$.

To answer the first question, it is necessary to find the eigensolution of the system modified by the changes $\bar{e}e_J^*$, a procedure which requires $O(n^3)$ calculations. If the eigenvalues and eigenvectors at $\bar{e}e_J^*$ do not closely match the values predicted by the series expansion, then it is apparent that the series representation is inadequate. In this case it is necessary to use the restart procedure. If the eigensolution at $\bar{e}e_J^*$ matches the predicted eigenvalues, it remains necessary to verify that the $\bar{e}e_J^*$ indicate a minimum penalty. This also invalidates the restart procedure.

4.3 Restart

This procedure entails finding the optimum design variables by minimizing the cost function based on linear approximations of the eigenvalues. Then, these variables are scaled according to the minimization procedure of the total cost function with the estimation of the eigenvalues via the Taylor series including the higher order terms. This process assumes constant ratios of the design variables and improves the optimization of the variables by scaling. But this enhancement may be inaccurate if the direction that is assumed constant is incorrect. In this case, the optimization requires a new ratios of e_J^* .

Thus for the initial step of the restart the eigenvalues become

$$w_k^2 = w_J^2(\bar{e}e_J^*) + \sum_{J=1}^m \frac{\partial w_k^2}{\partial e_J} \Delta e_J \quad (4.6)$$

If the linear optimization about $\bar{e}e_k^*$ returns the solution $\Delta e_k \approx 0$, this indicates that $\bar{e}e_k^*$ minimizes $C(w, e)$. Consider, on the other hand, the case wherein one or more of the Δe_k are non-zero, say they are Δe_k^* . Then ratio of Δe_k must be established, and nonlinear optimization is again used. During the nonlinear approximation analysis, the new design variable size function becomes

$$e_J = \bar{e}^* e_J^* + \bar{e} \Delta e_J^* \quad (4.7)$$

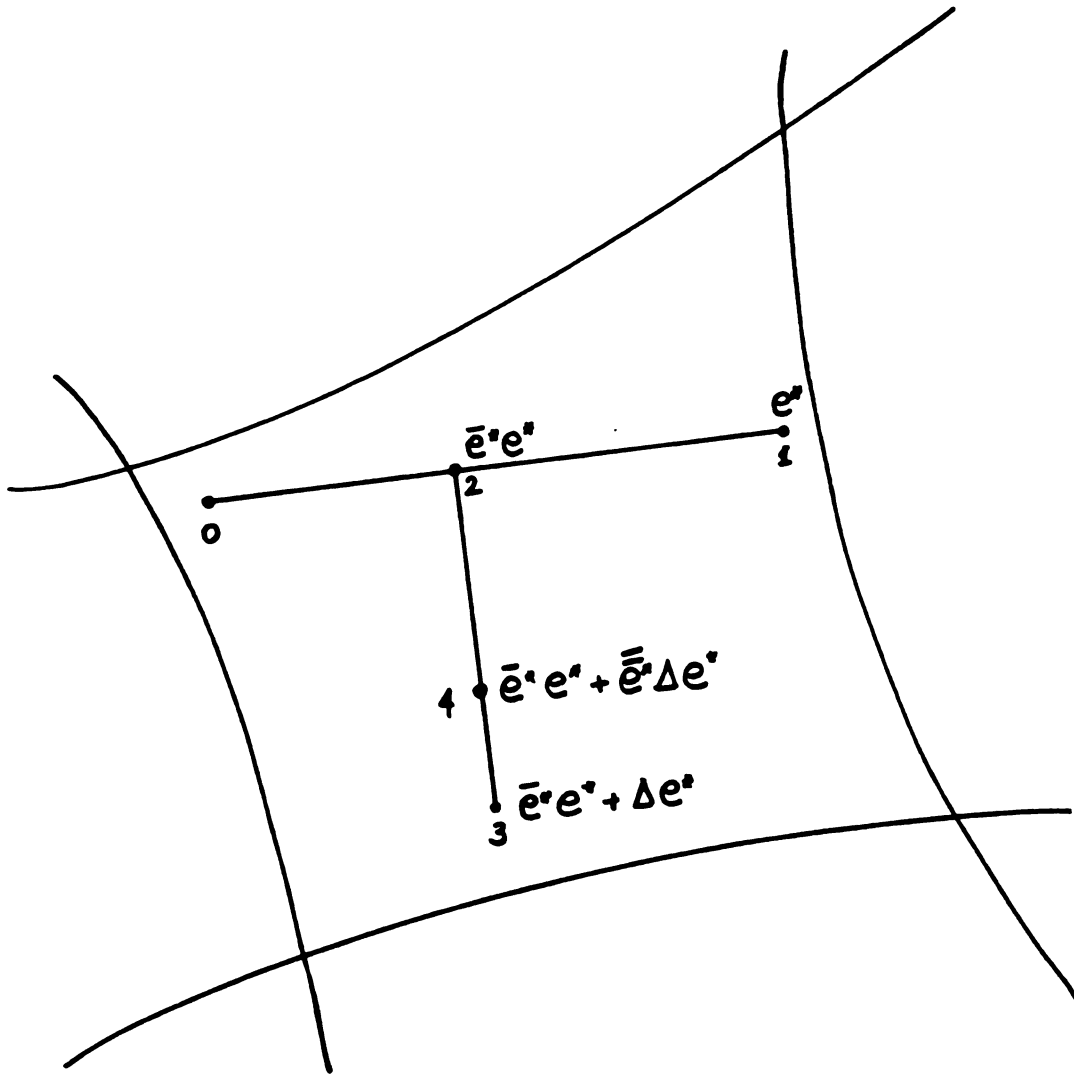
where $\bar{e}$ is now the single nonlinear design variable.

Figure 4.1 presents a pictorial explanation of the restart procedure in the case of an incorrect direction for the search of an optimum result with one variable. Again, this can be viewed as optimization via steepest descents.

The next section illustrates the power and the flexibility of the analysis with an example problem.

4.4 Example

In this section, the method is applied to the design of a fixed-fixed beam (see Figure 4.2). The design variable is the height of the



- 0 Starting point
- 1 Linear approximation
- 2 Nonlinear approximation
- 3 Linear approximation
- 4 Restart with nonlinear approximation

Figure 4.1 Schematic Representation Of The Restart.

top of the beam. The objective is to remove natural frequencies from the frequency band of 450 Hz - 550 Hz. The size of design change is included in the cost function.

4.4.1 Initial Design

The fixed-fixed beam is modeled with ten beam finite elements using ANSYS finite element preprocessor. The beam elements each have two nodes and each node has three degrees of freedom (two translation and one rotation). Figure 4.3 shows the element and the nodes of the beam. The eigensolution from ANSYS finite element processor found the frequencies which correspond to twenty-seven degrees of freedom (Table 4.1).

To facilitate the redesign of the shape of the top of the beam, we write the height of the beam, h , as a function of four cubic parametric equations. Figure 4.4 shows the four parametric equations that model each half of the fixed-fixed beam [22]. The redesigned beam will remain symmetric about the mid point.

The parameter t ranges from 0 to 1 on each of the plots. The first two plots, B_1 and B_2 , have zero slope at each end point, and have non-zero coordinate only at $t=0$ for B_1 and $t=1$ for B_2 . B_3 and B_4 , on the other hand, have zero coordinate at each end and non-zero slope only at $t=0$ for B_3 and $t=1$ for B_4 .

The design variable h , which is the top surface of the beam, may be written for each half of the beam as

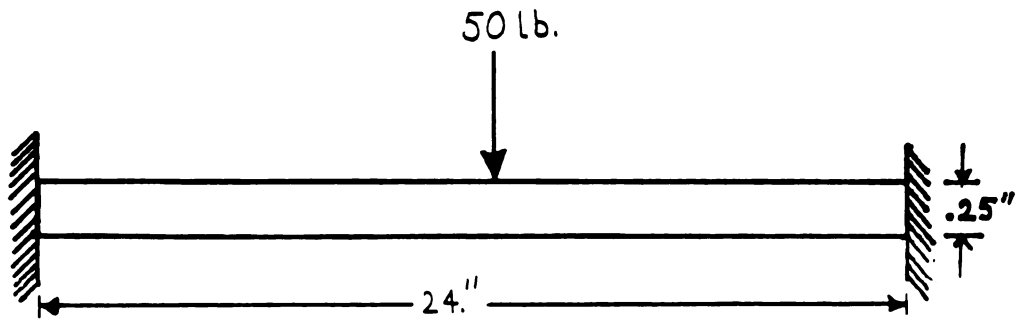


Figure 4.2 Fixed-Fixed Beam.

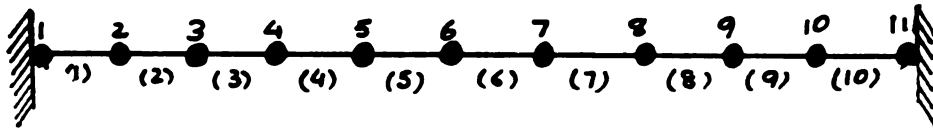


Figure 4.3 Finite Element Representation Of The Fixed-Fixed Beam.

Table 4.1 Natural frequencies of the fixed-fixed
beam at $h=.25$ inches

FREQUENCY (Hz)

1.	90
2.	248
3.	487
4.	806
5.	1208
6.	1695
7.	2271
8.	2937
9.	3651
10.	4226
11.	4867
12.	5099
13.	7172
14.	8557
15.	8683
16.	10462
17.	12529
18.	13098
19.	14842
20.	17195

Table 4.1 Continued

FREQUENCY (Hz)

21. 17953

22. 19096

23. 23205

24. 28874

25. 34798

26. 40446

27. 44757

$$h = h_0 + e_1 B_1 + e_2 B_2 + e_3 B_3 + e_4 B_4$$

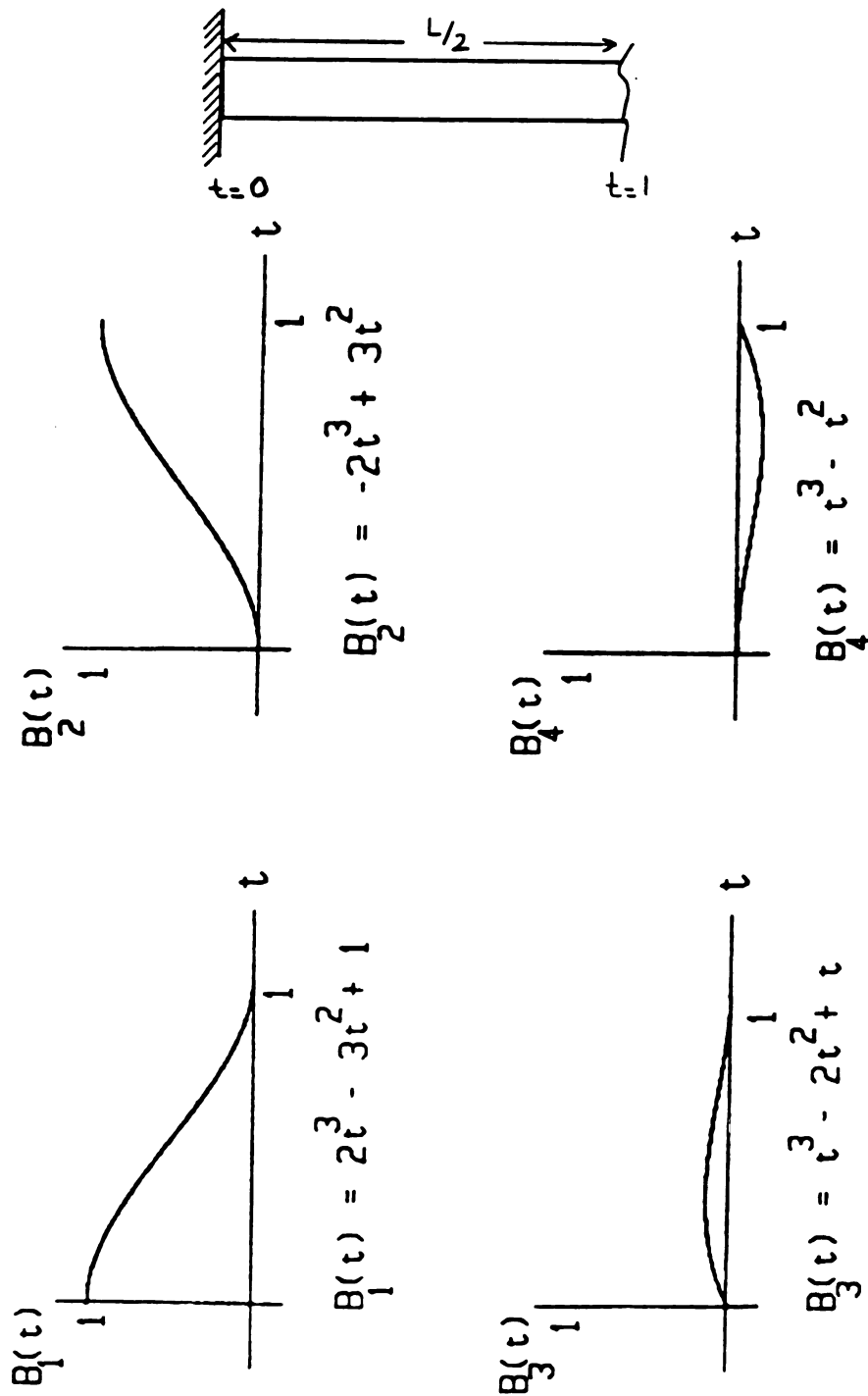


Figure 4.4 Cubic Polynomials That Represent Each Half Of The Fixed-Fixed Beam.

$$\begin{aligned}
h = h_0 + \frac{e_1}{20}(2t^3 - 3t^2 + 1) + \frac{e_2}{20}(-2t^3 + 3t^2) \\
+ \frac{e_3}{3}(t^3 - 2t^2 + t) + e_4(t^3 - t^2)
\end{aligned} \tag{4.8}$$

The variables e_1 , e_2 , e_3 and e_4 are the design variables which determine the top surface of the beam. The design parameters are scaled such that $e_1=1$ corresponds to a 20% change in height at the end of the beam, and $e_2=1$ corresponds to a 20% change in height at the mid point of the beam. The slope scaling is such that $e_3=1$ corresponds to a maximum of 15% change in height at an interior point fairly near the ends of the beam ($t = \frac{1}{3}$). To ensure slope continuity at the mid point, e_4 is set to zero. Thus for the left hand side of the beam, h may be written as

$$\begin{aligned}
h = h_0 + \left(\frac{e_1}{20} - \frac{e_2}{10} + \frac{1}{3} e_3\right)t^3 + \left(-\frac{3}{20} e_1 + \frac{3}{20} e_2 - \frac{2}{3} e_3\right)t^2 \\
+ \left(\frac{1}{3} e_3\right)t + \frac{e_1}{20}
\end{aligned} \tag{4.9}$$

and for the nonlinear approximation, h may be written

$$\begin{aligned}
h = h_0 + \bar{e} \left[\left(\frac{e_1^*}{10} - \frac{e_2^*}{10} + \frac{1}{3} e_3^*\right)t^3 + \left(-\frac{3}{20} e_1^* - \frac{2}{3} e_3^*\right)t^2 \right. \\
\left. + \left(\frac{1}{3} e_3^*\right)t + \frac{e_1^*}{20} \right]
\end{aligned} \tag{4.10}$$

where e_i^* s are the values of the design which result from the linear approximation.

4.4.2 The Design Change

The approximation of the natural frequencies and mode shapes of the modified design requires the first derivative of mass and stiffness matrices with respect to the e_i and the high order derivatives of the mass and the stiffness matrices with respect to $\bar{e}$. They are computed via the central finite difference method as explained in section 3.2 using ANSYS finite element preprocessor.

4.4.3 The Cost Function

The cost function is separated into two functions. First, the frequency content function is represented by the following mathematical form,

$$F(w) = \sum_{j=1}^m (1 - \cos(2\pi(w_j - p)/(u - p))) \quad (4.11)$$

where p and u are lower and upper bounds of the frequency band, respectively, m is the number of modes in the finite element model and w_i is the i th natural frequency. The w_i are found by the linear approximation at each design change. The design change size functions are quadratic functions of e_i where $i=1,2,3$. That is,

$$S(e) = \sum_{i=1}^3 e_i^2 \quad (4.12)$$

Figure 4.5 summarizes the cost functions for this example.

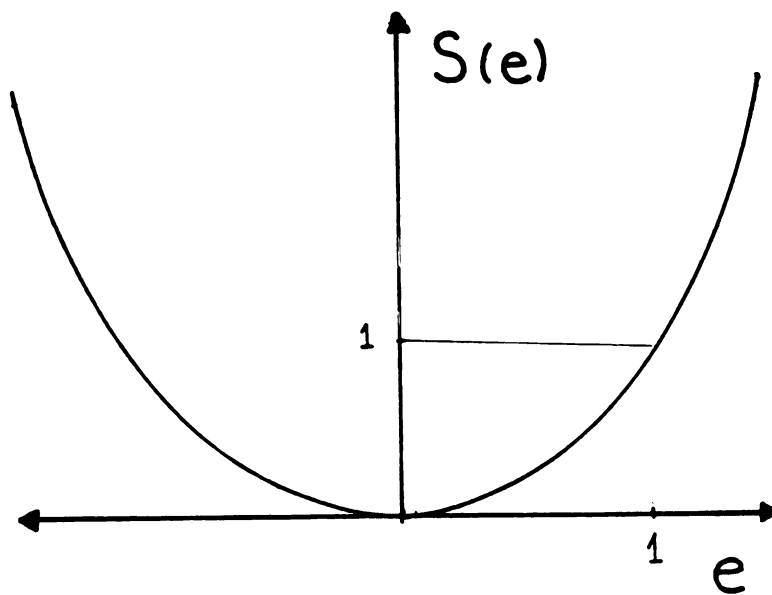
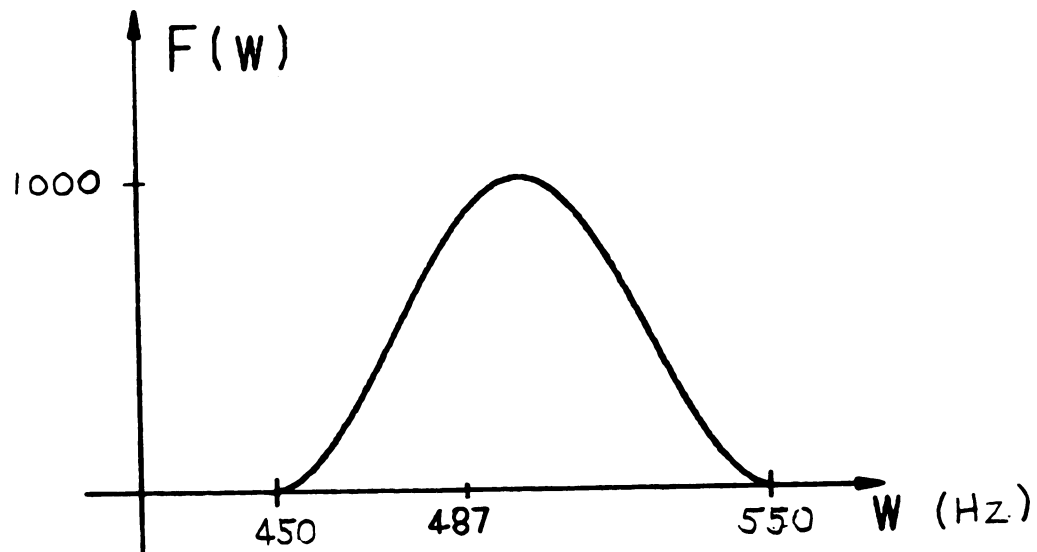


Figure 4.5 Frequency Content And Size-Of-Change Functions For The Example Problem.

4.4.4 The Procedure

The procedure for this example is as follows:

1. Construct the Finite element model
2. Extract the natural frequencies and mode shapes of the initial design
3. Calculate the first derivative of mass and stiffness matrices with respect to each design variable e_j at the initial design via the ANSYS finite element preprocessor
4. Calculate the first derivative of the natural frequencies with respect to each design variable e_j
5. Minimize the total cost function in equation 4.1 using the linear approximation for the frequencies. This yields e_j^*
6. Formulate the new design variable $\bar{e}$ by defining the design change in terms of e_j^* and $\bar{e}$ according to equation 4.4
7. Calculate the higher derivatives of mass and stiffness matrices with respect to $\bar{e}$ via the central finite difference method using finite element preprocessor
8. Calculate the higher derivatives of the natural frequencies and the eigenvectors with respect to $\bar{e}$, according to equations 3.5 and 3.6
9. Minimize the total cost function to find $\bar{e}$
10. Resolve the eigenvalue problem at $\bar{e}e_j^*$
11. Repeat steps 1-5 with the modified design
12. If the eigensolution matches the predicted results from step 10 and if $\Delta e_k = 0$ at the completion of step 11, then the procedure is completed

13. If $\Delta e_k \neq 0$, use the restart procedure

14. Do a restart until $\Delta e_k = 0$, which indicates the minimum solution for the given cost function.

4.4.5 The Results

Table 4.2 shows the results obtained by linear approximation. It also shows the natural frequencies that are computed with the suggested design changes via eigenvalue analysis. The table indicates that the predicted natural frequencies deviate considerably from the correct solutions. In fact, the proposed design failed to remove the natural frequency from the specified frequency range of 450 Hz - 550 Hz.

Table 4.3 shows results after the nonlinear optimization. It scales down the change considerably from the linear-based analysis. This proposed design does not have any frequencies in the specified frequency range of 450 Hz - 550 Hz. The table indicates that the eigenvalue analysis with this proposed design matches the predicted natural frequencies.

Further linear approximation with the modified design as an operating point results $\Delta e_k = 0$. This means that the path that is taken with the linear approximation lead to the minimum cost function. Figure 4.6 shows the shape of the beam with the new changes.

The analysis and figure 4.6 indicate that the design change has thinned the beam. Thus, while the design objective with regard to frequency spectrum has been met, the designer might well be concerned with the static strength of the new design.

Table 4.2 Comparison of the predicted
 natural frequencies via linear approximation
 and the natural frequencies via the
 eigenvalue solver at
 $e_1^* = -.2623$ $e_2^* = -1.9237$ $e_3^* = -1.3012$

MODE	FREQUENCY VIA (Hz) LINEAR APPROXIMATION	FREQUENCY VIA (Hz) EIGENVALUE SOLVER
1.	63	66
2.	151	170
3.	279	324
4.	441	522
5.	643	770
6.	890	1068
7.	1188	1419
8.	1570	1848
9.	1896	2277
10.	2519	3043
11.	3138	3755
12.	3706	4457
13.	4394	4541
14.	4431	5478
15.	5273	6567
16.	6224	7845
17.	7208	9066
18.	7620	9221

Table 4.2 Continued

MODE	FREQUENCY VIA (Hz) LINEAR APPROXIMATION	FREQUENCY VIA (Hz) EIGENVALUE SOLVER
19.	8770	10276
20.	8886	11461
21.	13266	13347
22.	18043	18075
23.	23205	23205
24.	28769	28732
25.	34584	34483
26.	40049	39839
27.	44629	44581

Table 4.3 Comparison of the predicted
 natural frequencies via nonlinear approximation
 and the natural frequencies via the eigenvalue solver at
 $\bar{e}^* = .2799$ Design Change $= \bar{e}^* f(e_1^*, e_2^*, e_3^*)$

MODE	FREQUENCY VIA Hz NONLINEAR APPROXIMATION	FREQUENCY VIA Hz EIGENVALUE SOLVER
1.	83	83
2.	226	226
3.	441	441
4.	727	727
5.	1086	1086
6.	1522	1522
7.	2038	2038
8.	2641	2641
9.	3275	3275
10.	4278	4278
11.	4366	4366
12.	5306	5306
13.	6436	6437
14.	7784	7784
15.	8660	8660
16.	9369	9369
17.	11208	11209
18.	13150	13150
19.	13253	13254

Table 4.3 Continued

MODE	FREQUENCY VIA Hz NONLINEAR APPROXIMATION	FREQUENCY VIA Hz EIGENVALUE SOLVER
20.	15255	15256
21.	16975	16977
22.	17980	17980
23.	23205	23205
24.	28842	28842
25.	34732	34732
26.	40323	40323
27.	44718	44718

----- New Design

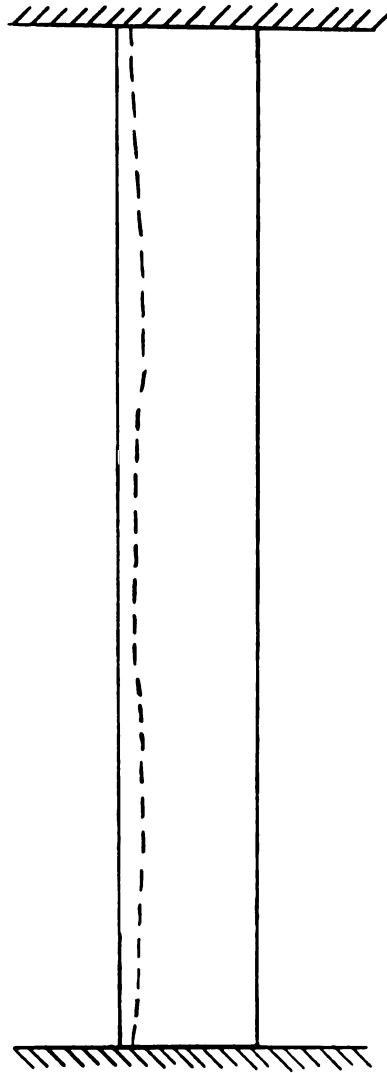


Figure 4.6 Shape Of The Fixed-Fixed Beam After The Nonlinear Approximation.

One way to codify the loss in load carrying capacity is to calculate the deflection of the new and the original designs under a static load. For example, calculations would indicate that under centerpoint loading the new design deflects about 30% more at the center than the original design.

To deal with these concerns, the next section considers static deflection as a design criterion. It seeks for an improved design indicated by static deflection as well as eigenvalues and eigenvectors.

4.5 Static Deflection Function

The changes which produce a desirable frequency spectrum may so weaken the structure that it will not withstand expected static loads. The static deflection penalty to be presented here is meant to prevent the formulation of such a structure.

The static deflection cost function, $D(d_s)$, reflects the relative desirability of all values that static deflections can accrue. That is, reasonable values of static deflection should be associated with a small cost and less desirable ones should have larger cost.

Since it is an goal here to limit the number of calculations, we will use the solution to the vibration problem to determine the static deflections. Then the economical series approximation for eigenvalues and eigenvectors can be used to determine expected changes in static deflection as a function of changes in the system.

The determination of the static deflections of a modified system is given by the following method. The system is defined by

$$[M]\ddot{\underline{x}} + [K]\underline{x} = \underline{F} \quad (4.13)$$

where

$[M]$ = mass matrix (nxn)

$[K]$ = stiffness matrix (nxn)

$\underline{F}$ = force vector of static loads (nx1)

$\ddot{\underline{x}}$ = acceleration vector (nx1)

$\underline{x}$ = displacement vector (nx1)

If the eigenvalues are distinct and non-zero, equation 4.13 can be uncoupled by the following transformation

$$\underline{x} = [U]\underline{y} \quad (4.14)$$

where $\underline{y}$ is an nx1 generalized coordinate vector and U is the nxn modal matrix with columns which are eigenvectors. Assume the eigenvectors are normalized such that $[U]^T[M][U] = [I]$, and premultiply by U^T . Equation 4.13 becomes

$$[I]\ddot{\underline{y}} + [\lambda]\underline{y} = [U]^T \underline{F} \quad (4.15)$$

where $[\lambda]$ is a diagonal nxn matrix of eigenvalues.

In the static case, that is, $\ddot{y}=0$, equation 4.15 becomes

$$[\lambda]\underline{y} = [U]^T \underline{F} \quad (4.16)$$

or

$$\underline{y} = [\lambda]^{-1}[U]^T \underline{F} \quad (4.17)$$

where $[\lambda]^{-1}$ is the inverse of the diagonal matrix $[\lambda]$.

The generalized coordinate vector y can be obtained from equation 4.17 for the modified system. The entries of $[\lambda]^{-1}$ and $[U]$ are the reciprocal of the modified eigenvalues and the modified eigenvectors for a given design change. The static deflection can then be found from 4.14.

4.6 Total Cost Function

A total cost function can be written as a linear combination of the size, frequency content and static deflection functions:

$$C(w,e,d_s) = \sum_{J=1}^k A_J F_J(w) + \sum_{J=1}^r B_J S_J(e) + \sum_{J=1}^m E_J D_J(d_s) \quad (4.18)$$

where k is the number of unwanted frequency ranges, r is number of design variables, m is the number of sections of the structure where static deflections are of concern, d_s is the static deflection of a point on the structure under a set of prescribed static loads and $D(d_s)$ is a function that becomes large for undesirable static deflections of the system. Coefficients of each function weight the relative importance of each frequency, size and static deflection.

As indicated in chapter 3, the frequency content function is a function of design variables,

$$w_i = f(e_J) \quad J = 1, 2, \dots, m \quad (4.19)$$

where w_i is modified ith natural frequency and the m e_J are design variables. The static deflection is also a function of design variables e_J as indicated in equations 4.17 and 4.14. Therefore, the total cost function becomes a function of the design variables.

$$\begin{aligned} C(e_J) = & \sum_{J=1}^k A_J F_J(w(e_J)) + \sum_{J=1}^r B_J S_J(e_J) \\ & + \sum_{J=1}^m E_J(D_J(e_J)) \end{aligned} \quad (4.20)$$

4.7 Example with Static Deflection Function

In this section, the beam example from section 4.4 is considered to illustrate an application of the static deflection penalty. In this case, the static deflection penalty is included in the cost function. In particular, a linear static deflection function for the mid point of the beam under mid-point loading is used. The linear equation is

$$D(ds) = Z \frac{d - d_s^*}{d_s^*} \quad (4.21)$$

where Z is a scale factor, d_s^* is the static deflection of the middle

node of the original design under a give external load at that node, and d is the static deflection of the middle node for the various designs. The total cost function is the linear combination of the three functions of equation 4.20. For this example, the scaling factors A , B and Z are set to 1000, 1 and 125, respectively.

4.7.1 Results

The results of this example show the behavior similar to the last example. The linear approximation eigenvalue results deviate from the eigensolution of the modified design (Table 4.4). However, the nonlinear approximation procedure with constant ratios of design variables yields design that does not have an frequencies in the range of 450 Hz - 550 Hz and the predicted natural frequencies agree with the eigensolution of the design (Table 4.5). Figure 4.7 shows the modified beam after the nonlinear approximations. A further linear approximation with the modified design as an operating point yields $\Delta e_k = 0$, which indicates that the given set of design changes for this example lead to the minimum cost function.

Although this example is similar to the previous one, the analysis synthesized a remarkably different design modification. In this case, this modified beam has mid-point a static deflection only about 13% higher than the original design, whereas, the previous example lead to a static deflection about 30% more than the original design. Figure 4.8 shows the final design for each case. Clearly, the particular deflection chosen here lead to increased thickness in key areas of the beam.

Table 4.4 Comparison of the predicted natural frequencies via linear approximation including the static deflection and the natural freequencies via the eigenvalue solver at

$$e_1^* = 2.1646 \quad e_2^* = -2.3705 \quad e_3^* = -3.2625$$

MODE	FREQUENCY VIA (Hz) LINEAR APPROXIMATION	FREQUENCY VIA (Hz) EIGENVALUE SOLVER
1.	82.	71
2.	176	177
3.	300	326
4.	442	504
5.	612	733
6.	685	1055
7.	758	1294
8.	823	1659
9.	1088	1984
10.	1494	2640
11.	1686	3384
12.	2209	4164
13.	2915	4829
14.	3213	5078
15.	3690	5983
16.	4196	7053
17.	4663	8897
18.	4705	9842

Table 4.4 Continued

MODE	FREQUENCY VIA (Hz) LINEAR APPROXIMATION	FREQUENCY VIA (Hz) EIGENVALUE SOLVER
19.	4834	10002
20.	9361	10153
21.	13517	13662
22.	18178	18180
23.	23205	23205
24.	28610	23611
25.	34255	34087
26.	39442	38895
27.	44377	44281

Table 4.5 Comparison of the predicted natural frequencies via nonlinear approximation including the static deflection and the natural frequencies via the eigenvalue solver at $\bar{e}^* = .2964$
Design Change = $\bar{e}^* \times f(e_1^*, e_2^*, e_3^*)$

MODE	FREQUENCY VIA (Hz) NONLINEAR APPROXIMATION	FREQUENCY VIA (Hz) EIGENVALUE SOLVER
1.	87	87
2.	230	230
3.	442	442
4.	721	721
5.	1072	1072
6.	1496	1496
7.	1989	1989
8.	2596	2596
9.	3208	3209
10.	4278	4279
11.	4383	4383
12.	5232	5233
13.	6329	6331
14.	7639	7640
15.	8831	8831
16.	9175	9176
17.	10957	10959
18.	12920	12921

Table 4.5 Continued

MODE	FREQUENCY VIA (Hz)	
	NONLINEAR APPROXIMATION	EIGENVALUE SOLVER
19.	13233	13233
20.	14680	14681
21.	16262	16263
22.	18022	18022
23.	23205	23205
24.	28793	28794
25.	34626	34627
26.	40120	40121
27.	44637	44638

----- New Design

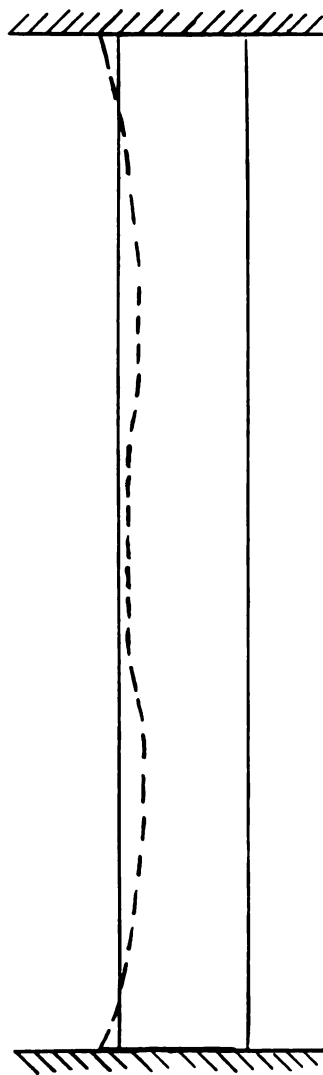


Figure 4.7 Shape Of The Beam After The Nonlinear Approximation Including Static Deflections In The Cost Function.

---- Without Static
 Deflection
 -.-.- With Static
 Deflection

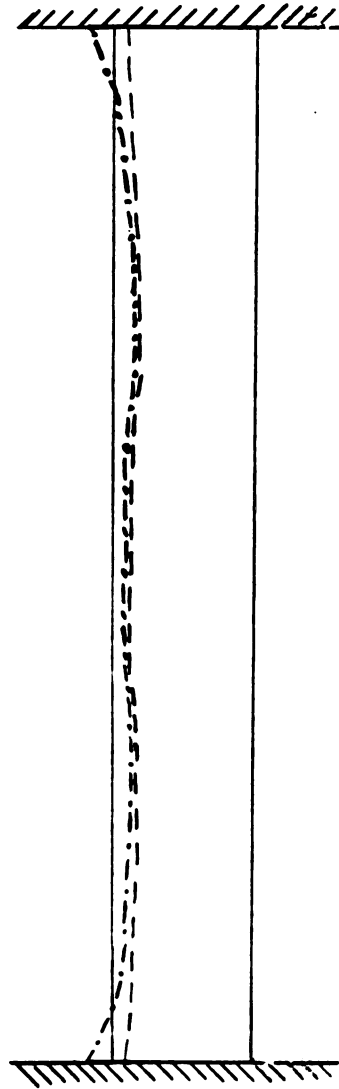


Figure 4.8 Shape Of The Beam With Nonlinear Approximation
 Results With And Without Static Deflection
 Functions In The Analysis.

CHAPTER V

CONCLUSIONS

Design modifications to improve the frequency spectrum of large systems often requires many solutions of large eigenvalue problems. Thus, this has been computationally inefficient. This thesis develops a more efficient approach to the problem.

The procedure developed here uses a finite element preprocessor and series approximation to develop an approximation for eigenvalues and eigenvectors as a function of design changes. Thus, the iterations which lead to the optimal redesign take only $O(n^2)$ calculations. Examples indicate that the approximations are useful over a wide range of design changes.

Future work should include applications of this method to systems with damping and the investigation of the possibility for use of a more sophisticated choice of direction of the change vector in restart applications.

APPENDICES

APPENDIX A

The central finite difference formulations for the various derivatives involve values of the function on both sides of the x value at which the derivative of the function is desired. By utilizing the appropriate Taylor series expansions, one can obtain expressions for derivatives as follows:

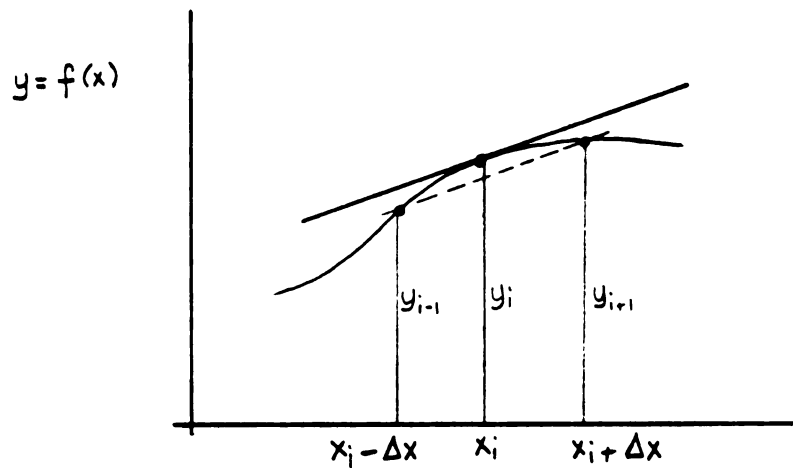


Figure A.1 Approximation of the derivative at x_i

The Taylor series for a function $y=f(x)$ at $(x_1+\Delta x)$ expanded about x_1 is

$$y(x_1 + \Delta x) = y_1 + y_1'(\Delta x) + \frac{y_1''(\Delta x)^2}{2!} + \frac{y_1'''(\Delta x)^3}{3!} + \dots \quad (\text{A.1})$$

where y_1 is the ordinate corresponding to x_1 and $(x_1+\Delta x)$ is in the region of convergence. The function at $(x_1-\Delta x)$ is similarly given by

$$y(x_1 - \Delta x) = y_1 - y_1'(\Delta x) + \frac{y_1''(\Delta x)^2}{2!} - \frac{y_1'''(\Delta x)^3}{3!} + \dots \quad (\text{A.2})$$

Using only the first 3 terms of each expansion, we can obtain an expression for y_1' by subtracting equation (A.2) from (A.1), yielding

$$y_1' = \frac{y(x_1 + \Delta x) - y(x_1 - \Delta x)}{2(\Delta x)} \quad (\text{A.3})$$

Equation (A.3) is called the first central-difference approximation of y' at x_1 . Graphically, the approximation represents the slope of the dashed line in Figure A.1. The actual derivative is represented by the solid line drawn tangent to the curve at x_1 . If we add equations (A.1) and (A.2), the second derivative can be approximated as

$$y_1'' = \frac{y(x_1 + \Delta x) - 2y_1(\Delta x) + y_1(x_1 - \Delta x)}{(\Delta x)^2} \quad (\text{A.4})$$

The four terms on the right hand side of each equations (A.1) and (A.2) can be used to obtain an expression for the third derivative. Subtracting equation (A.2) from equation (A.1) yields

$$y_1(x_1 + \Delta x) - y_1(x_1 - \Delta x) = 2y_1'(\Delta x) + \frac{2y_1'''(\Delta x)^3}{3!} \quad (\text{A.5})$$

If we expand the Taylor series about x_1 to obtain expressions for $y=f(x)$ at $(x_1+2\Delta x)$ and $(x_1-2\Delta x)$, respectively, we obtain

$$y(x_1 + 2\Delta x) = y_1 + y_1'(2\Delta x) + \frac{y_1''(2\Delta x)^2}{2!} + \frac{y_1'''(2\Delta x)^3}{3!} \quad (\text{A.6})$$

$$y(x_1 - 2\Delta x) = y_1 - y_1'(2\Delta x) + \frac{y_1''(2\Delta x)^2}{2!} - \frac{y_1'''(2\Delta x)^3}{3!} \quad (\text{A.7})$$

Subtracting equation A.7 from equation A.6, and using just the 4 terms of each expansion shown, gives

$$y_1(x_1 + 2\Delta x) - y_1(x_1 - 2\Delta x) = 4y_1'(\Delta x) + \frac{8}{3} y_1'''(\Delta x)^3 \quad (\text{A.8})$$

The simultaneous solution of equations (A.5) and (A.8) yields

$$y_1''' = \frac{y_1(x_1 + 2\Delta x) - 2y_1(x_1 + \Delta x) + 2y_1(x_1 - \Delta x) - y_1(x_1 - 2\Delta x)}{2(\Delta x)^3} \quad (\text{A.9})$$

Equation (A.9) gives the central difference difference expression for the third derivative of y at x_1 .

Successively higher derivatives can be obtained by this method, but, since they require the solution of increasingly larger number of simultaneous equations, the process becomes quite tedious. Derivations for the higher derivatives are accomplished with much greater facility and far less labor by using difference, averaging and derivative

operators []. Central-difference expressions, however, are more accurate than either forward or backward difference expressions. The following is a summary of the differentiation formulas which may be obtained from Taylor series expansions.

$$\begin{aligned}
 y'_i &= \frac{y_{i+1} - y_{i-1}}{2(\Delta x)} \\
 y''_i &= \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \\
 y'''_i &= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2(\Delta x)^3} \\
 y''''_i &= \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{(\Delta x)^4}
 \end{aligned} \tag{A.10}$$

where

$$\begin{aligned}
 y_{i+1} &= y_i(x_i + \Delta x) \\
 y_{i+2} &= y_i(x_i + 2\Delta x) \\
 y_{i-1} &= y_i(x_i - \Delta x) \\
 y_{i-2} &= y_i(x_i - 2\Delta x)
 \end{aligned}$$

Similarly the derivatives of mass and stiffness matrices may be formulated via central difference method. The following is a summary of the derivatives of mass matrix [M] and stiffness matrix [K] with respect to a design variable e.

$$[M]' = \frac{[M]_{i+1} - [M]_{i-1}}{2(\Delta e)}$$

$$[M]'' = \frac{[M]_{i+1} - 2[M]_i + [M]_{i-1}}{(\Delta e)^2}$$

$$[M]''' = \frac{[M]_{i+2} - 2[M]_{i+1} + 2[M]_{i-1} - [M]_{i-2}}{2(\Delta e)^3}$$

$$[M]'''' = \frac{[M]_{i+2} - 4[M]_{i+1} + 6[M]_i - 4[M]_{i-1} + [M]_{i-2}}{(\Delta e)^4}$$

where

$$[M]_{i+1} = M_i(e_i + \Delta e)$$

$$[M]_{i+2} = M_i(e_i + 2\Delta e)$$

$$[M]_{i-1} = M_i(e_i - \Delta e)$$

$$[M]_{i-2} = M_i(e_i - 2\Delta e)$$

$$[K]' = \frac{[K]_{i+1} - [K]_{i-1}}{2(\Delta e)}$$

$$[K]'' = \frac{[K]_{i+1} - 2[K]_i + [K]_{i-1}}{(\Delta e)^2}$$

$$[K]''' = \frac{[K]_{i+2} - 2[K]_{i+1} + 2[K]_{i-1} - [K]_{i-2}}{2(\Delta e)^3}$$

$$[K]'''' = \frac{[K]_{i+2} - 4[K]_{i+1} + 6[K]_i - 4[K]_{i-1} + [K]_{i-2}}{(\Delta e)^4}$$

where

$$[K]_{i+1} = K_i(e_i + \Delta e)$$

$$[K]_{i+2} = K_i(e_i + 2\Delta e)$$

$$[K]_{i-1} = K_i(e_i - \Delta e)$$

$$[K]_{i-2} = K_i(e_i - 2\Delta e)$$

APPENDIX B

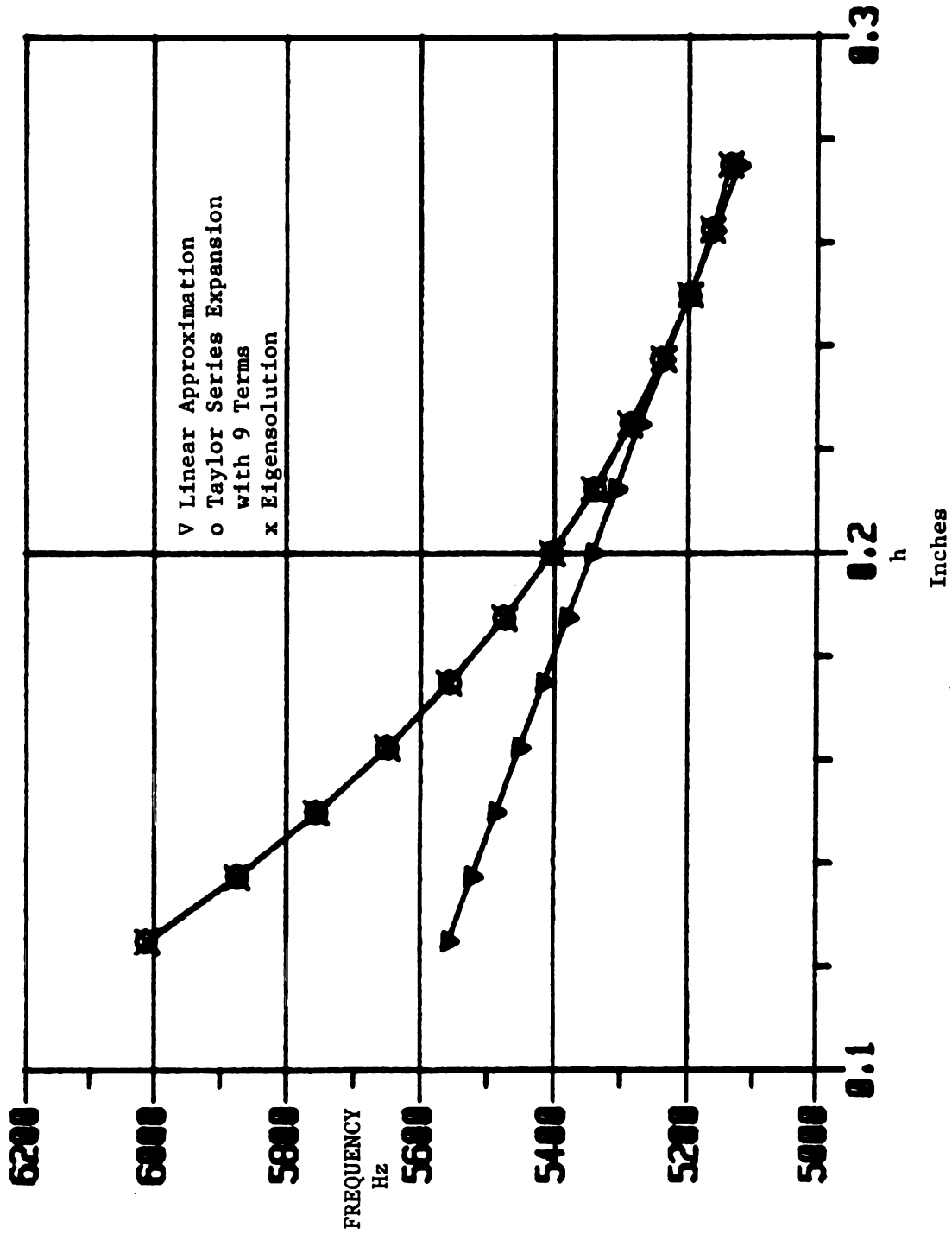


Figure B.1 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Eighth Frequency Of The Beam.

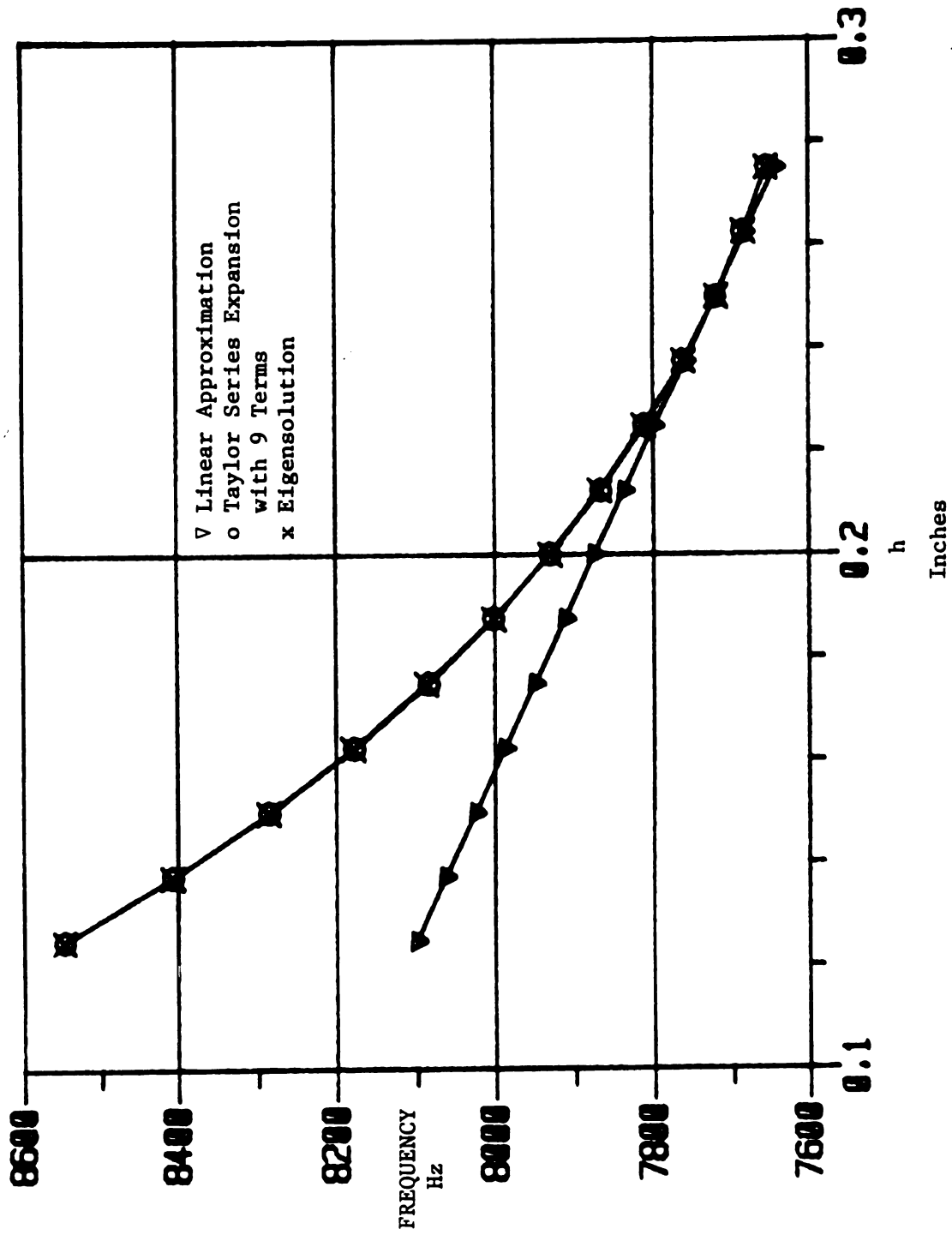


Figure B.2 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Ninth Frequency Of The Beam.

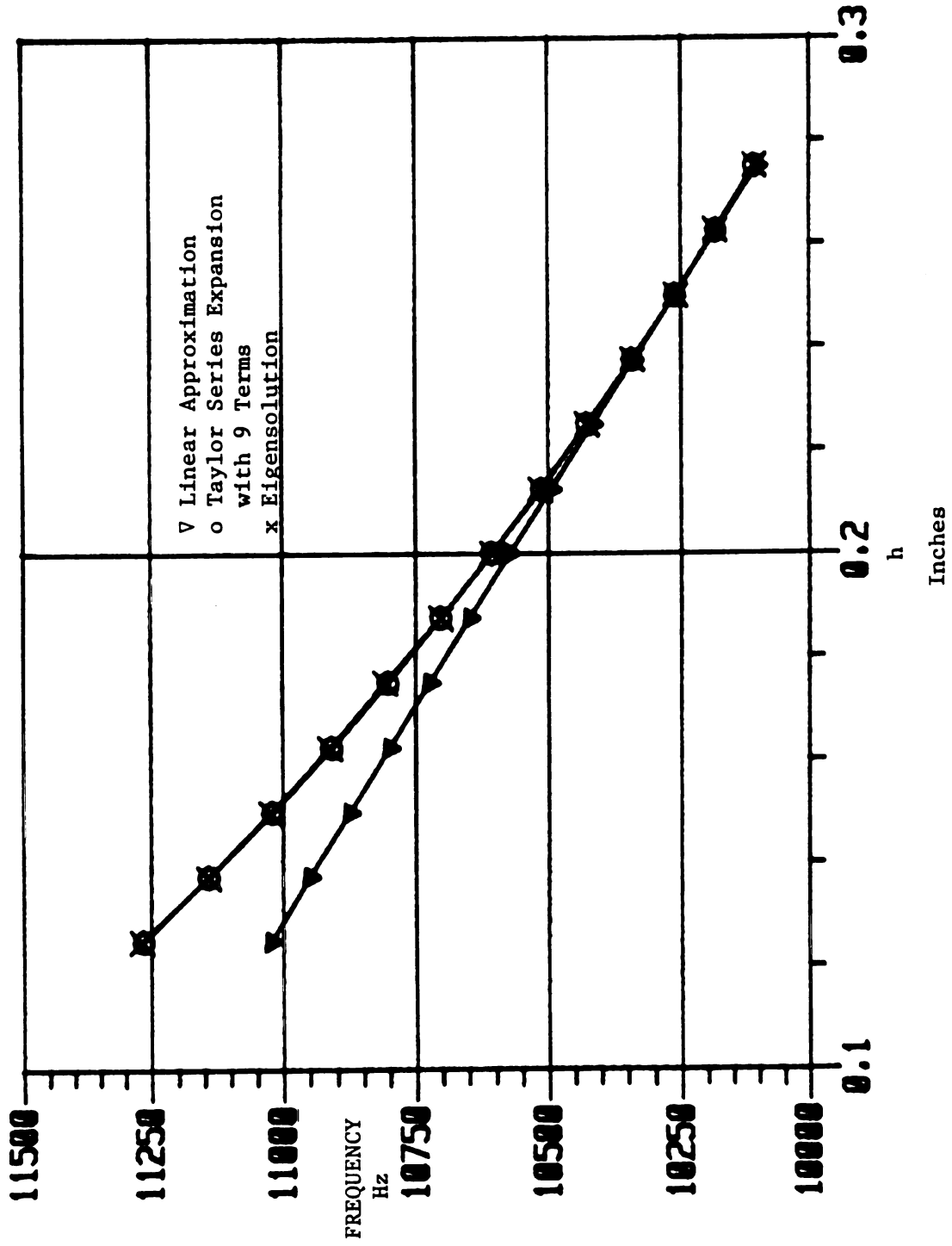


Figure B.3 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Tenth Frequency Of The Beam.

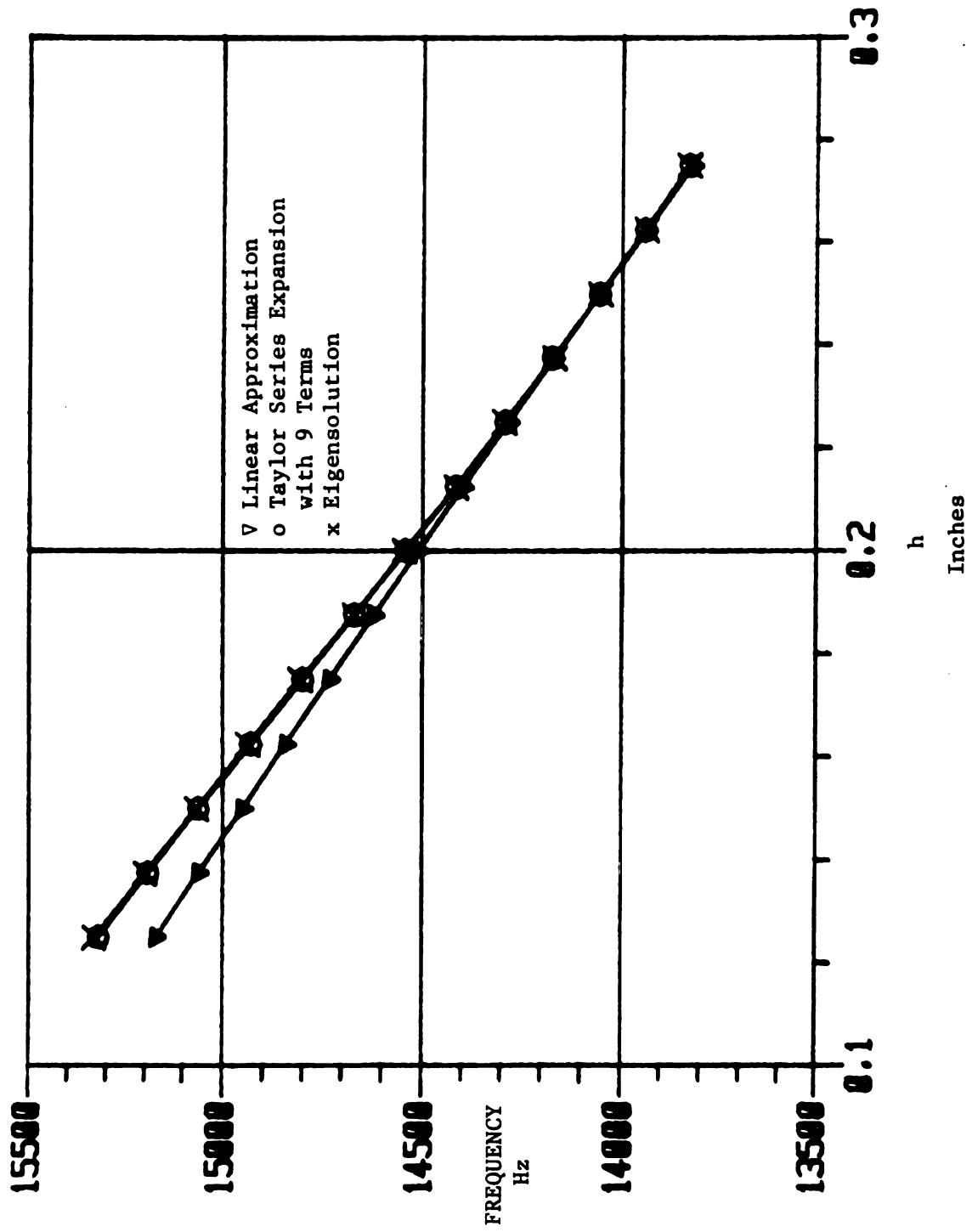


Figure B.4 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Eleventh Frequency Of The Beam.

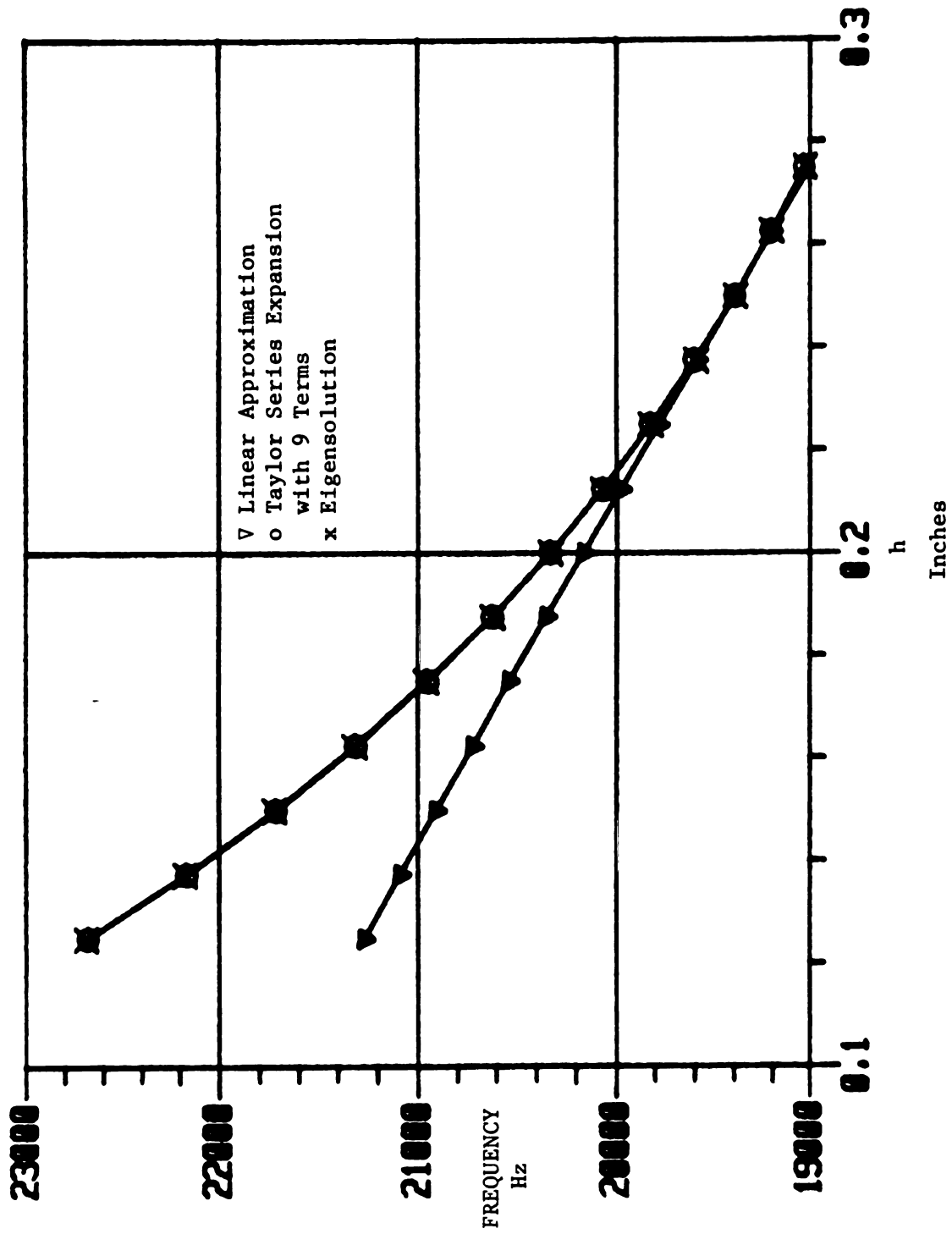


Figure B.5 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Twelfth Frequency Of The Beam.

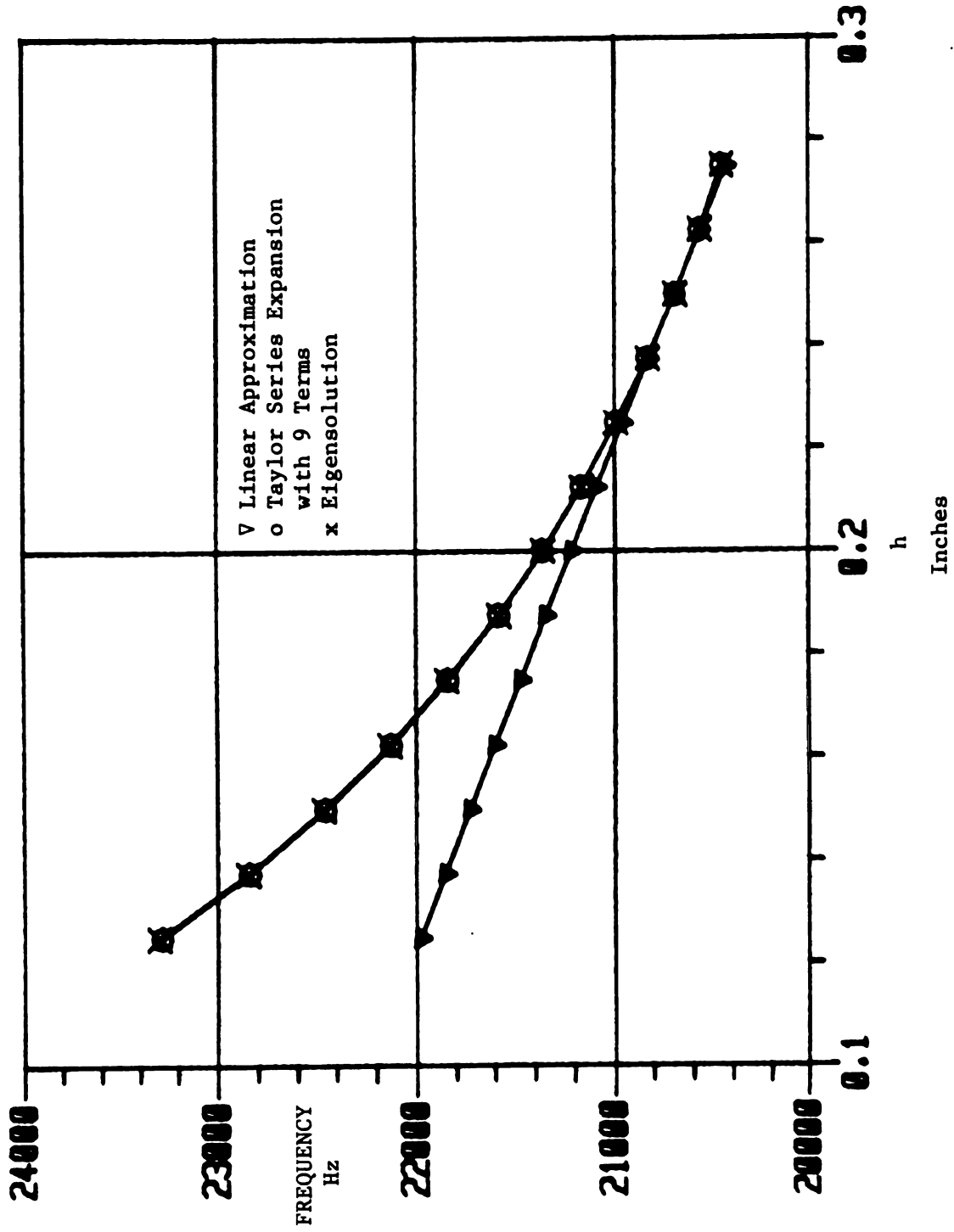


Figure B.6 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Thirteenth Frequency Of The Beam.

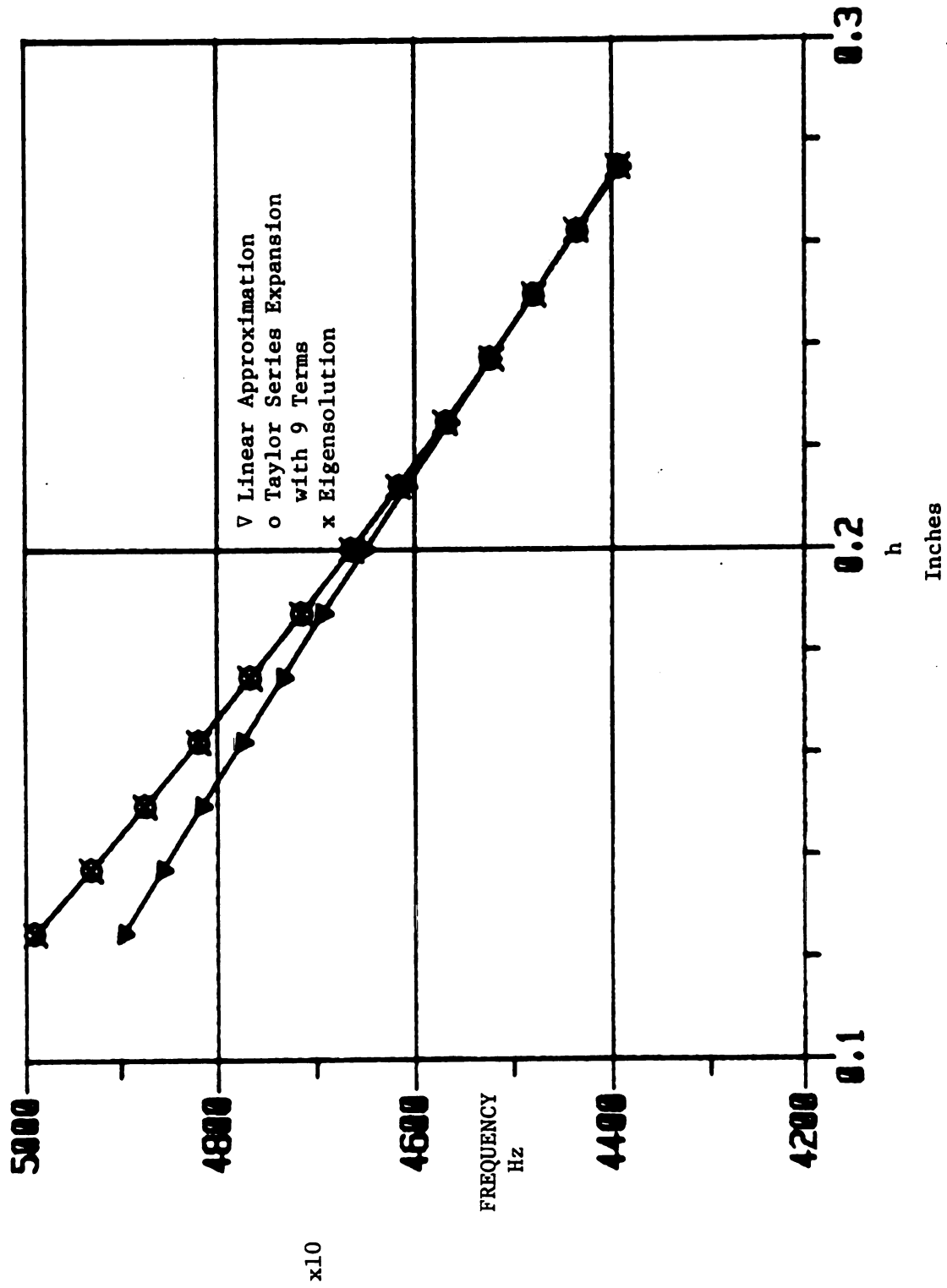


Figure B.7 Comparison Between The Linear Approximation And Higher Order Taylor Series Of The Fourteenth Frequency Of The Beam.

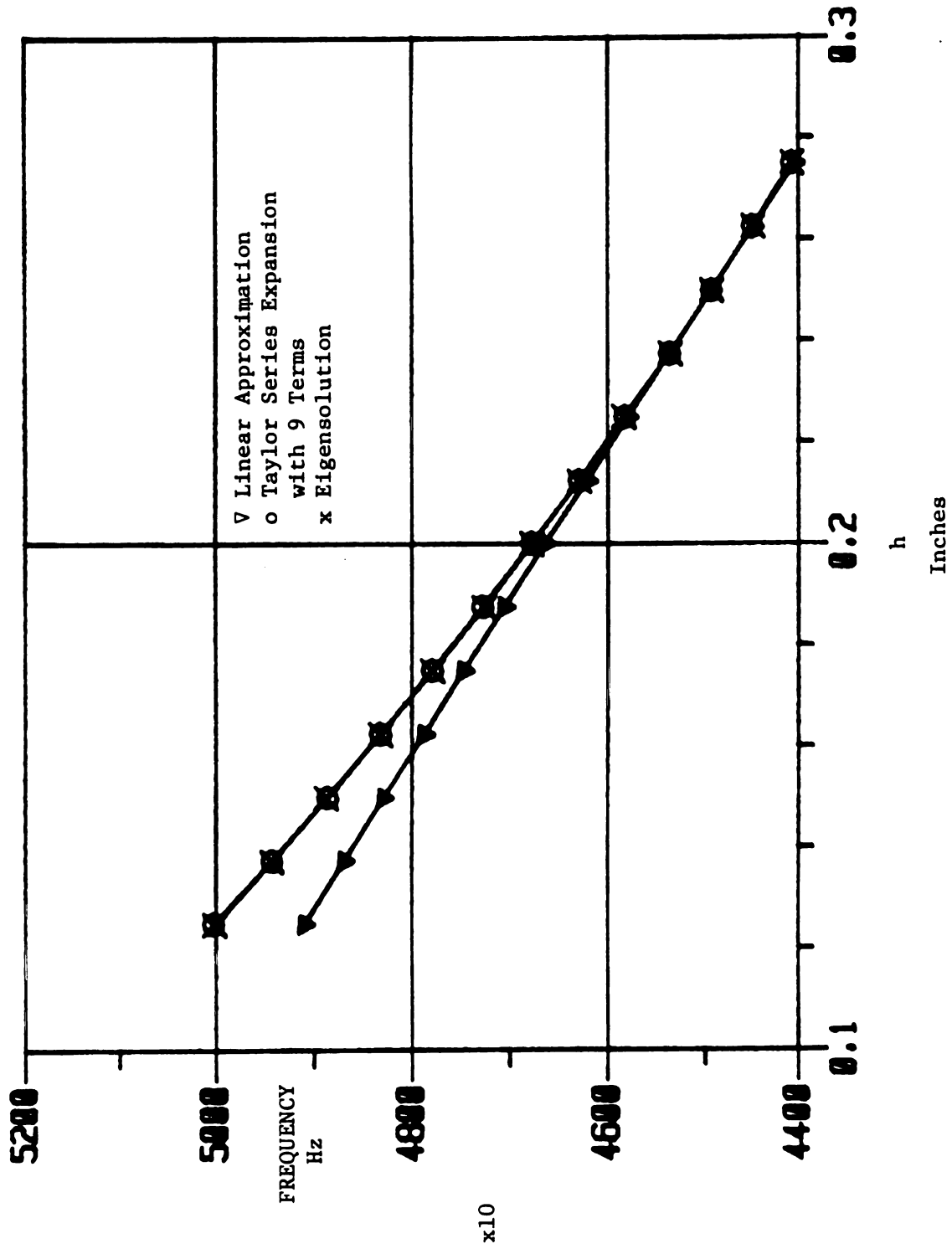


Figure B.8 Comparison Between The Linear-Approximation And Higher Order Taylor Series Of The Fifteenth Frequency Of The Beam.

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