This is to certify that the

thesis entitled

A SYSTEMATIC STUDY OF ADJUSTMENT TIME IN

MODELS OF ECONOMIC GROWTH

presented by

Myung Jai Rhee

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Economics

William Y.C. Koo
Major professor

Date September 20, 1977

A SYSTEMATIC STUDY OF ADJUSTMENT TIME IN MODELS OF ECONOMIC GROWTH

By

Myung Jai Rhee

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

1977

3108070

ABSTRACT

A SYSTEMATIC STUDY OF ADJUSTMENT TIME IN MODELS OF ECONOMIC GROWTH

By

Myung Jai Rhee

This dissertation investigates the time length of the adjustment process to a steady state path for several models of economic growth. Most modern growth models attempt to explain actual economic growth in terms of steady state solutions of a model economy. The practical relevance of steady state solutions as an approximation of reality, however, depends upon the adjustment time required for the model economy to reach a reasonably close vicinity of steady state path after any initial disturbance occurs.

The investigation begins with a discussion of the concept of adjustment time. A proportional concept of adjustment time is adopted and, as a result, the question of "indicator" variable representing the behavior of the entire system is raised and discussed. It is demonstrated analytically and numerically that the proportional adjustment time of a model economy may vary significantly with the choice of indicator variable, since the time path of an

indicator variable is generally different from the time paths of other indicator variables.

The investigation of adjustment time is extended to growth models with different saving functions such as a Classical saving function, an Ando-Modigliani saving function, and a Kaldor saving function. This investigation is further extended to monetary growth models. The analysis indicates that change in the specification of savings behavior within a model or the introduction of money to the model can have a considerable effect on the adjustment time. The magnitude and direction of the effect on the adjustment time, however, are different for different specifications of savings functions and of models.

In conducting this investigation, relationships among time paths and adjustment times of different indicator variables are explicitly derived and are related to the so-called fundamental equation of a growth model. Thus, the method adopted in this study organizes previous investigations in a more consistent frame of reference and eliminates some difficulties associated with different analytic procedures for different indicator variables and different specifications of growth models.

It is concluded that the adjustment time of an economic growth model depends on various factors such as the indicator variable chosen as a representative of a system, the values of parameters, and the specification of the model. Based on the estimates of this study, the adjustment time ranges from 14 to 172 years for a 90 percent adjustment towards its steady state path. Therefore, it may not be possible to give a proper judgment on the practical relevance of steady state solutions as an approximation to actual economic growth until we investigate this problem of adjustment time for a variety of models, especially for growth models which reflect more properly the actual economy. If the adjustment time proves to be too long, it will be necessary to investigate the dynamic path of the growth model in full, including both the adjustment path and steady state path in the analysis of economic growth.

ACKNOWLEDGMENTS

I would like to express my sincere appreciation to Professor Anthony Koo, chairman of my committee, for his thoughtful help, guidance, and encouragement during the preparation of this dissertation. I am also very thankful to Professor Norman Obst, who turned my interest to growth theory and gave me valuable comments and suggestions on various aspects of this study. My gratitude is also expressed to Professor Robert Gustafson and Professor Mark Ladenson for their professional advice.

I am deeply indebted to the Korean Agricultural Sector Study, which provided financial support during my years of graduate study, and particularly to Dr. Ho-tak Kim and Dr. Jong-tack Yoo for their friendly help in relation to the study. I would like to add a special thanks to Mr. Robert King for his contribution to the readability of this thesis.

Finally, I thank my wife Hejae and sons Sang-hyun and Sung-hyun, who provided a great deal of help in the form of love, patience and encouragement.

TABLE OF CONTENTS

		Page
LIST OF TABLES		iv
LIST OF FIGURES		v
Chapter		
I.	INTRODUCTION	1
II.	ADJUSTMENT TIME IN ECONOMIC GROWTH MODELS	10
	2.1 Concept of Adjustment Time	10
	2.2 Indicator Variable of an Economic System	21
III.	NEOCLASSICAL ONE-SECTOR MODEL OF GROWTH	29
	3.1 Neoclassical Growth Model	30
	3.2 Adjustment Path to Steady State	37
	3.3 Adjustment Time to Steady State	48
IV.	EXTENSION OF THE MODEL: SAVING BEHAVIOR	64
	4.1 Classical Saving Function	65
	4.2 Ando-Modigliani Saving Function	74
	4.3 Kaldor Saving Function	78
V.	EXTENSION OF THE MODEL: INCLUSION OF MONEY	88
	5.1 Money as a Consumption Good	89
	5.2 Money as a Production Good	100
	5.3 Keynes-Wicksell Approach	106
VI.	CONCLUSIONS	113
REFERENCES		120

LIST OF TABLES

Table | Page

3.3.1 Adjustment Time (in Years) of Neoclassical Economy with Cobb-Douglas Production Function 57

3.3.2 Adjustment Time (in Years) of Neoclassical Economy with C.E.S. Production Function. . 62

4.3.1 Adjustment Time (in Years) of Neoclassical Economy with Kaldor Saving Function . . . 84

LIST OF FIGURES

Figure		Page
2.1.1	Adjustment of Neoclassical Economy	16
3.1.1	Equilibrium and Stability of Neoclassical Economy	35
3.2.1	Adjustment of Neoclassical Economy with Cobb-Douglas Production Technology	46
3.2.2	Time Paths of Capital-Output Ratio (a), Growth Rate of Output (b), and Capital-Effective Labor Ratio (c)	47
4.1.1	Equilibrium and Stability of Neoclassical Economy with Classical Saving Function . . .	67
5.1.1	Equilibrium in Monetary Growth Model of Tobin .	101
5.2.1	Equilibrium in Monetary Growth Model of Levhari-Patinkin	107
5.3.1	Equilibrium in Monetary Growth Model of Stein .	112

CHAPTER I

INTRODUCTION

Growth economics deals in general with the evolution of an economy over time, which possibly fluctuates in the short run but exhibits a trend in the long run. Modern economic growth theory has focused primarily in the investigation of the long run trend, especially in advanced industrial economies. Relatively little attention has been given to the process of adjustment to a steady state, by which the long run trend is to be explained. In this dissertation the time length of this adjustment process to the steady state is investigated.

Economic growth has been a long standing concern of economists. Both the classical and neoclassical economists of eighteenth and nineteenth centuries showed a deep interest in the problem of economic growth. Their method of analysis, however, is quite different from that of modern growth theory. As Hicks (20, p. 29) states, "The causes of economic progress were one of their main concerns. What is true is that they had a very special approach to dynamic problems: their method of treating them was by the tools of static theory. That was a more inadequate treatment." In contrast to these, modern growth theory is characterized by its explicit and formal use of dynamic tools such as differential equations in the explanation of the economic growth process.

Modern growth theory has a relatively short history. The publication of Harrod (19) in 1939 and a similar but independently developed paper by Domar (13) published in 1946 are usually claimed to be the beginning of modern theory of economic growth. Extensive investigations have been undertaken since then and the theory has developed to a considerable degree in less than four decades.

Despite the seminal roles of Harrod and Domar in reawakening interest in the problem of economic growth, modern growth theory is, perhaps, best represented by the neoclassical approach, which can be said to stem from papers by Solow (36) and Swan (40). The neoclassical approach is characterized by the proposition that an economy moves towards a path of steady state growth, if there exists a stable equilibrium, along which the growth rate of the economy is equal to constant exogeneous growth rate of the labor force and thus is entirely independent of the proportion of income saved and invested. Based on this property of convergence to a steady state, most growth models attempt to explain the growth process of a real economy in terms of steady state solutions of a model economy. Solow's statement (37, p. 4) may well represent the characteristic feature of modern growth theories: "Most of the modern theory of economic growth is devoted to analyzing the properties of steady states and to finding out whether an economy not initially in a steady state will evolve into one if it proceeds under specified rules of the game. It is worth looking at some figures if the steady state picture does actually give a fair shorthand summary of the facts of life in advanced industrial economies."

It is important, however, to notice that this steady state path is a special type of path a model economy follows, which can only be attained theoretically as time goes to infinity. That is, it takes an infinite amount of time for the model economy to complete the whole process of adjustment to the steady state path from any initial (non-steady state) position. More precisely speaking, the steady state can never be attained in a real sense and thus the explanation of reality in terms of steady state solutions may be meaningless. Steady state solutions as an approximation of reality may only be justified on the ground that it takes a relatively short time for the economy to complete "almost all" of the adjustment process. Therefore, the practical usefulness of the steady state solutions as an approximation of reality depends crucially upon how long the economy takes to reach a reasonably close vicinity of the steady state path from any initial disturbance. If the adjustment process is too long, steady state solutions are of very limited value as an explanation g

t

g

c

s,

di

co

ju

fo

ste

of the growth process of a real economy. If a model economy adjusts very slowly, as Hahn and Matthews (17, p. 32) have commented in their survey article, "in order to ascertain the real-life implications of any given model, it is necessary to investigate its equilibrium dynamics in full. This is naturally much more difficult than investigating the properties of the steady state solution."

The steady state is a very useful analytic concept, whether it exists or not in reality, for analyzing the long run trend of economic growth. Along the steady state path of growth, the growth rates of some variables are constant over time. This means that each period is essentially identical to that of the previous period except in scale. Therefore, a static mode of analysis may be applied to the essentially dynamic problem of growth. According to Stieglitz and Uzawa (39, p. 7), "Steady states are to growth theory what perfect competition and monopoly are to the theory of firm. One can learn a great deal about the growth processes from studying these admittedly special cases . . .," and "if the economy converges with reasonable speed to the steady state, then the steady state becomes directly empirically relevant. How quickly the economy converges is a moot question." In order to add practical justification to this convenient analytic jargon, therefore, a short adjustment time is required. As a first step to this end, intensive investigation on the adjustment time for a variety of models is needed. However, relatively little work has been done on this problem of adjustment.

The main contributions in this area have been made by Atkinson (5), Conlisk (10), Furuno (15), Ramanathan (28, 29), K. Sato (32), and R. Sato (33, 34). They measure adjustment time of a model economy in terms of the time required for a certain indicator variable of a model economy to cover a specified proportion of total initial displacement from the steady state path. That is, they use the concept of "proportional" adjustment time.

R. Sato (33, 34) investigates, for the first time in literature, the adjustment time for a neoclassical one-sector growth model with a Cobb-Douglas production function. He looks at the adjustment time of the output-capital ratio ($1/v$), which converges to a constant value at the steady state. Assuming that the economy was at a steady state before being disturbed and that the initial impact was given by changing saving rate, he measures the required time for the output-capital ratio to cover a certain proportion of the initial displacement and shows that it may take a hundred years for the neoclassical economy to cover 90 percent of the initial disturbance from its steady state path. According to Ramanathan's numerical calculations (29), for the neoclassical model with C.E.S. production function, the adjustment time of the growth rate of output ($\hat{Q}$) ranges from 22 to over 150 years depending on values of parameters.

While K. Sato (32) shows that the adjustment process of the capital-output ratio (v) rather than the output-capital ratio (1/v) takes from 25 to 37.5 years in covering 90 percent of the initial displacement for the one-sector vintage model and argues that a realistic modification of the neoclassical model reduces the adjustment time by as much as three quarters of R. Sato's, Furuno's estimate (15), using the output-capital ratio (1/v) as an indicator, ranges from 3 to 7 centuries in covering the same proportion of adjustment for the neoclassical model with Pasinetti saving function.

In addition to these, Conlisk (10) estimates the adjustment time of the growth rate of output ($\hat{Q}$) for his modified version of the neoclassical model and Ramanathan (28, 29) examines two sector growth models. Atkinson (5) gives numerical estimates for the neoclassical model with nonneutral technical progress, Goodwin's model of cyclical growth, and the Shell and Stiglitz's one-sector model with two types of capital, and concludes (5, p. 151) that "examination of the time scale provides useful information about their (growth model's) behavior. This is not altogether surprising, since some kind of time dimension is implicit or explicit in our thinking about any real economy; and we should expect the time scale of these models to be important for understanding their relationship to the real world."

Even though some research has been done by the above invdstigators on this problem of adjustment time, several problems still remain uninvestigated. The primary objectives of this dissertation are to investigate some of these remaining problems, as outlined below, and to organize previous findings in a more consistent frame of reference.

The concept of "adjustment time" or "speed" has not been systematically considered, even though previous writers have freely used this term according to their conveniences. The magnitude of adjustment time may depend not only on the model specifications and values of the model parameters but also on the definition of adjustment time itself. Thus, the concept of adjustment time should be made clarified before attempting to measure the adjustment time of any particular model economy. Moreover, previous writers have bypassed another important point, that is, the multi-dimensional nature of a system. Any economic system cannot, strictly speaking, be represented by one variable, but should be represented by a combination of all the economic variables constituting the system. One may look at the behavior of a system by examining the behavior of a particular variable, depending on the nature or purpose of the analysis. However, the adjustment time of the variables within a model may not be the same even if they are interrelated functionally. Therefore, the adjustment time of any model economy depends on which variable one chooses as an indicator of the economic

a
g
a
e
mo
ad
pr

system. These problems of the concept of adjustment time and of the choice of an indicator variable are the main subjects of Chapter II.

In order to get more realistic estimates of the adjustment time of an economy, the analysis may need to be conducted for more complex and realistic models. The basic concepts and methods of the analysis may, however, be more conveniently pursued for a simple model. In Chapter III, the analysis of adjustment time for a simple neoclassical growth model is conducted based on the discussions of Chapter II. Analytic results and numerical calculations are given.

The adjustment time responds very sensitively to changes in model specifications. For example, R. Sato has shown that it may take, for reasonable values of parameters, as much as 50 to 190 years for a simple neoclassical economy to cover 90 percent of the initial disturbance and has argued that the neoclassical model may, as a result, lose its "basic foundations." Meanwhile, according to other investigators such as K. Sato and Conlisk, the adjustment time is much shorter for more complex models of growth. They have shown "a somewhat more optimistic view" about the relevance of steady state solutions as an explanation of reality. In fact, one might expect that the more flexible and general is the model economy, the shorter adjustment time may be. Therefore, in order to make a proper evaluation of the relevance of steady state solutions as an approximation of reality it may be necessary to investigate the question for more realistic models.

Simple growth models have adopted the simplest form of saving function; a constant rate of saving to output. The analytical simplicity of this type of saving function is gained, however, only by sacrificing realistic elements of real life saving behavior. Many economists have analyzed the effects of different saving behavior on the steady state path of growth. The effects of different saving behaviors on the adjustment time, however, have not been investigated in the literature except by Furuno (15). In Chapter IV, the effects of different saving functions on the adjustment time are analyzed.

Another area which has been neglected is the consideration of monetary elements in investigation of adjustment time. Although some theorizing has been done on the role of money in the context of the modern growth models, an investigation of its effects on the adjustment time has not yet appeared in the literature. This is the subject of Chapter V. The conclusions of this dissertation are given in Chapter VI.

c

b

n

m

v

t

wi

th

by

of

giv

sys

are

grow

of a

CHAPTER II

ADJUSTMENT TIME IN ECONOMIC GROWTH MODELS

While several writers have investigated the problem of adjustment time for different growth models, two basically conceptual problems, as I understand them, are not yet well discussed. The first is the concept of adjustment time and speed; the second is the choice of indicator variable of a system. Since the measurement of adjustment time is based on these basic concepts, this chapter deals with these problems before attempts are made to measure the adjustment time in the later chapters.

2.1 Concept of Adjustment Time

A model of a growing economy is usually represented by a set of differential equations. An explicit solution of this system of differential equations therefore will give the best information about the behavior of the economic system over a period of time. Because explicit solutions are rarely possible or are too complex for most of economic growth models, however, we are often content with the study of a limiting case (i.e., steady state case), which can often be analyzed without an explicit solution. The study of this special situation, of course, gives much information about the growth process of the economy.

However, in order for this limiting case to be relevant as an approximation of reality, the model economy should be expected to exist in this special situation, the steady state path. This means that the time-length of the adjustment process towards a steady state path from any initial position is required to be very short. The shorter the adjustment time, the better the steady state situation can serve as a representation of the growth process of an economy. If the adjustment time is very long, the practical usefulness of a steady state solution is much reduced, since the functioning of the economy is not likely to be approximated by a steady state path, especially when the model parameters are often changing over time. This is why the problem of adjustment time is so important in relation to the relevance of steady state solutions as an approximation of reality.

The adjustment time (or speed) in growth theory refers to the length of time required for an economy not in a steady state to adjust towards its steady state path. Even if an economy is initially on a steady state path, it will no longer be in that steady state once a disturbance (such as change of saving rate) has occurred in the economy. The disturbed economy adjusts towards a new steady state path which is determined by new values of the parameters.

Theoretically, it takes an infinite amount of time for the economy to complete the whole process of adjustment to the new steady state from any (non-steady state) initial position. It does not make sense, therefore, to ask how long an economy takes to complete its entire adjustment towards a new steady state path. It takes forever. Instead, we ask how long it takes for the economy to complete "almost all" of the adjustment process towards its steady state path. More precisely, we ask how much time is required to reach a certain prescribed vicinity or neighborhood of the steady state path.

In order to measure properly the adjustment time, we need a variable which converges to a constant value in a steady state. When we attempt to measure the adjustment time using variables which do not converge to a constant value but which continue to increase, the concept of adjustment time loses much of its practical meaning. The values of these variables are infinite in a steady state and thus adjustment time required to cover even "almost all" of the adjustment process will be infinite. For this reason, all previous analysis of the adjustment times of growth models have used variables which converge to constant values in a steady state.

An important point in relation to the concept of adjustment time is how we technically define the adjustment time itself. A definition of adjustment time should properly represent the convergence time required to cover "almost all" of the movement towards a steady state path. The problem, then, is how to specify the level of "almost all" of the adjustment. The predeternined level of allowance may be a certain specified constant value, it may be a specific percentage of the initial disturbance from the steady state value, or it may be based on some other measure.

Let us consider, for example, a simple neoclassical growth model[1] with an aggregate production function $Q(t) = F(K(t),L(t))$, a rate of capital accumulation $\dot{K} = s\,Q$, and a proportional growth rate of labor supply $\hat{L} = n$.

Where $Q(t)$ = national output at time t.

$F(K,L)$ = an aggregate production function with two factors of production, which is homogeneous of degree one.

$K(t)$ = amount of capital stock an economy holds at time t.

$L(t)$ = amount of labor force an economy supplies at time t.

$\dot{K} = dK/dt$.

$\hat{L} = (dL/dt)/L$.

s = the saving rate of an economy as a whole.

n = the proportional growth rate of labor force.

---

[1]Refer to Chapter III of this dissertation for a more complete discussion of the neoclassical growth model.

This model economy can be discussed in terms of the following differential equation, called the "fundamental equation" of the neoclassical growth model. We can obtain the fundamental equation by differentiating the capital-labor ratio (k=K/L) lagarithmically.

(2.1.1)
$$\hat{k} = \hat{K} - \hat{L}$$
$$= \frac{sQ}{K} - n$$
$$= \frac{sf(k)}{k} - n$$
or,
$$\dot{k} = sf(k) - nk$$

where $\dot{k} = dk/dt$, $\hat{k} = \dot{k}/k$, and $f(k)$ $(=f(K/L,1))$ represents the intensive form of the aggregate production function which is homogeneous of degree one.

By solving this differential equation (2.1.1) we can find the time path of k and of other variables. Steady state solutions, however, can be found without explicitly solving the equation. In a steady state, k converges to a constant value (k^*) and thus the steady state solution is obtained by setting $\dot{k}$ to zero.

(2.1.2)
$$\dot{k}^* = s\ f\ (k^*) - nk^* = 0$$
or,
$$s\ f\ (k^*) = nk^*$$

Now we suppose that the neoclassical economy in a steady state is disturbed by a change of saving rate from s_o to s_1 (refer to Figure 2.1.1). As the saving rate increases ($s_o \rightarrow s_1$), the capital-labor ratio (k) begins to increase from its initial value (k_o) and eventually reaches its steady state value (k*) as time goes to infinity. At the same time, other variables such as the capital-output ratio (v), the output per unit of labor (q), and the growth rate of output ($\hat{Q}$) also change simultaneously and move from their initial values (v_o, q_o, and $\hat{Q}_o$) to their steady state values (v*, q*, and $\hat{Q}^*$) respectively as time goes to infinity.

The basic concern of this study is how long it takes for a variable (v,$\hat{Q}$,k, or q) to adjust from its initial position (v_o, $\hat{Q}_o$, k_o, or q_o) to a certain predetermined value of allowance (v_Δ, $\hat{Q}_\Delta$, k_Δ, or q_Δ), since the adjustment time to its steady state value (v*, $\hat{Q}^*$, k*, or q*) is infinite and thus meaningless. The adjustment time required for a variable, x, to reach the specific value, x_Δ, will be given by (41, p. 437)

$$t_x(\varepsilon) = \int_{x_o}^{x_\Delta} \frac{dt}{dx}\, dx \tag{2.1.3}$$

The adjustment time of k, for example, is given by

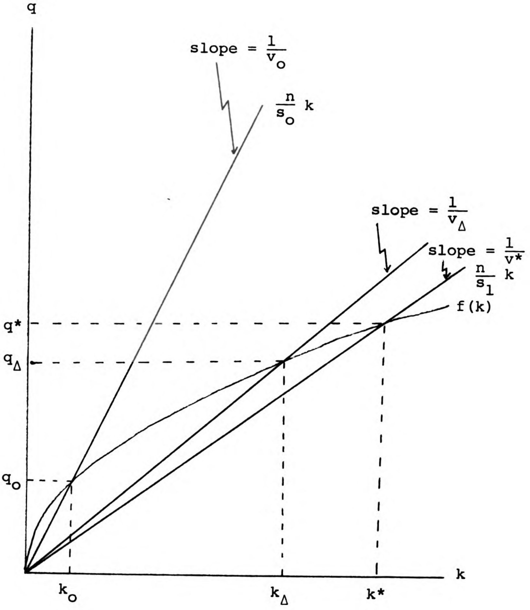

Figure 2.1.1

Adjustment of Neoclassical Economy

$$t_k(\varepsilon) = \int_{k_o}^{k_\Delta} \frac{dt}{dk}\, dk \tag{2.1.4}$$

$$= \int_{k_o}^{k_\Delta} \frac{dk}{sf(k) - nk}$$

since $dt/dk = 1/(sf(k)-nk)$ from equation (2.1.1).

The practical problem which remains is how to technically specify the predetermined level of x, that is x_Δ. The choice of x_Δ may in principle depend on purpose and on analytic convenience, but the following proportional concept has been adopted in previous investigations in this area.

$$x_\Delta = x_o + \varepsilon_x(x^* - x_o) \tag{2.1.5}$$

or,

$$\varepsilon_x = \frac{x_\Delta - x_o}{x^* - x_o}$$

where ε is a predetermined proportion of adjustment.

Now equation (2.1.3) can be written as follows if the proportional concept is adopted.

$$t_x = \int_{x_o}^{x_o + \varepsilon (x^* - x_o)} \frac{dt}{dx}\, dx \tag{2.1.6}$$

This definition of the adjustment time, called proportional adjustment time, however, causes two problems. Our basic concern is to find the adjustment time required to reach a reasonaly close vicinity of the steady state. If we adopt this proportional concept, the level of the reasonaly close vicinity may depend on the size of the initial displacement. They may be far below the steady state value when the initial displacement is large even for a high value of ε. This means that we may have to change the adjustment proportion (ε) according to the size of initial displacement in order to test properly the relevance of steady state solutions as an approximation to reality.

In addition to this difficulty, the proportional concept causes a more serious problem. The proportional adjustment time may not be the same for different variables within a system. In Figure 2.1.1, the values of v_Δ, Q_Δ, k_Δ, and q_Δ are the functionally related values of the variables. Therefore, the adjustment time required for a variable to reach the predetermined level will be the same whichever variable we may choose as an indicator. However, the proportion of adjustment (ε_v, $\varepsilon_{\hat{Q}}$, ε_k, and ε_q) is in general not the same for each variable. In other words, the proportional adjustment time may be different for different variables even for the same proportion of adjustment ($\varepsilon_v = \varepsilon_{\hat{Q}} = \varepsilon_q = \varepsilon_k$).

This difference in adjustment time, of course, is the combined result of the difference in the shape of the adjustment paths, and the definition of proportional adjustment time. Therefore, the adjustment time of an economy will depend on which variable we choose as an indicator of the system as well as on the model specification and parameters. Thus, this definition of adjustment time raises the problem of choosing an indicator variable of a system. Despite these difficulties, this proportional concept of adjustment time will be adopted in this dissertation simply because no better substitute is available.

Another conceptual problem, which should be clarified, concerns the terminology of adjustment "time" and "speed." The average adjustment speed (AV) corresponding to the proportional adjustment time (equation (2.1.6)) may be defined as

$$AV = \frac{x_\Delta - x_o}{t_x} \tag{2.1.7}$$

From this equation, we know that the adjustment speed is inversely related to the adjustment time (t_x). For a given magnitude of adjustment ($x_\Delta - x_o$), thus, high adjustment time means low adjustment speed and vice versa. On this account, the terms adjustment time and speed are often interchangeably used in literature.

It seems, however, that adjustment time is a better terminology than adjustment speed, because the conventional concept of "speed" (i.e., distance per unit time) may cause some confusion in relation to growth models. For example, the proportional adjustment time of the capital-output ratio (v) for the neoclassical model with a Cobb-Douglas production function is constant for a given level of adjustment proportion (ε_v) even though the magnitude of the required adjustment ($x_\Delta - x_o$) may differ.[2] This means that the adjustment speed is high for longer distances and low for shorter distances even though the adjustment time is the same for a given level of adjustment proportion (ε). In this situation it might erroneously be said that "the variable (v) adjusts at the same 'speed' regardless of the size of initial displacement." This is obviously contradictory to the conventional concept of speed.

In comparing the adjustment time of two different variables, we face another problem, if the terminology of adjustment speed is used. The "unit" problem of variables arises. Average speed of adjustment is defined as average distance covered by unit time interval. However, the distance and thus speed depend on the units adopted. But it is not an easy problem to find a completely agreeable common unit for various variables. In these respects, adjustment time seems to be a better terminology than adjustment speed.

---

[2]Constancy of proportional adjustment time for given adjustment proportion (ε) comes from the linearity of the time path of v. See equations (3.2.9) and (3.3.5).

2.2 Indicator Variable of an Economic System

An economic system is composed of various economic variables. Thus, strictly speaking, the behavior of a system should be represented by a simultaneous description of the behavior of all the variables contained in the system. Even though we may, for practical convenience, look at the behavior of a system through changes in the value of a particular variable, it should be made clear that the behavior of variables within a system may be different even though they are functionally related. As discussed in the previous section, the proportional adjustment time is different for different variables for a given level of adjustment. Previous investigators of the problem of adjustment time have not given attention to this problem. Different writers have adopted different indicator variables without providing any convincing justification for their choice.[3]

There is no a priori criterion by which we can choose an indicator variable. The choice of an indicator variable depends essentially on the purpose and operational convenience of the particular analysis. However, it may be useful to examine the relevance of some variables as an

---

[3]For example, R. Sato (33) uses $1/v$ and K. Sato (32) adopts v while Conlisk (10) and Ramanathan (28, 29) adopt $\hat{Q}$.

indicator variable. For explanatory convenience, system variables are classified into three types; basic variables, rate variables, and ratio variables.

Consider the previous neoclassical growth model. In this economic system there are three "basic" variables; output (Q), capital (K), and labor (L). In referring to the model, however, we often use the proportional rate of change of these basic variables, which will be called "rate" variables ($\hat{Q}$, $\hat{K}$, and $\hat{L}$). In addition to these rate variables, we often use "ratio" variables, which are defined as the ratio of any two basic variables ($k = K/L$, $q = Q/L$, and $v = K/Q$).

Basic Variables (Q, K, and L)

Basic variables, dominant in static analysis, are not popular in the analysis of dynamic models. When we use these variables in representing the growth process, as indicated in the above discussion, we face some mathematical difficulties in the analysis of steady state properties and of adjustment time since these variables are changing over time even in the steady state. This is the main reason why these variables are rarely used in the analysis of the growth process in modern growth models which are primarily concerned with the steady state properties. However, these variables may have more real-life implications when we are especially interested in the adjustment process rather than in the steady state path.

Rate Variables ($\hat{Q}$, $\hat{K}$, and $\hat{L}$)

The rate variable, "a concept which has been little used in economic theory," has become "an extremely useful instrument of economic analysis" since growth economics has begun to receive the attention of the modern economists (14, p. 147). These rate variables converge to certain constant values in a steady state if the economy has a stable equilibrium. Therefore, a static mode of analysis can be applied to the dynamic problem of growth when we are mainly interested in the path of steady state growth. This may be one of the reasons why rate variables are so popular in modern growth models. The property of convergence to constant value in a steady state is a very useful device also for the measurement of adjustment time. If these variables were changing in the steady state, it would be very difficult to measure the adjustment time in the manner defined above.

For a given growth rate of labor ($\hat{L} = n$), a combined behavior of both $\hat{K}$ and $\hat{Q}$ represents the behavior of the system. For practical convenience, however, we may choose either of them as a indicator of the system. The choice is a matter of purpose and convenience. It seems to me, however, that the capital stock is not in itself an interesting variable but is sometimes adopted because of its analytic convenience. National income, on the other hand, is one of the most important economic variables. Thus, it may be natural to investigate the growth process of an economic system by means of national income. In fact, Conlist (10) and Ramanathan (28, 29) adopt this variable ($\hat{Q}$) for the measurement of adjustment time of their model economies.

Ratio Variables (k, q, and v)

The ratio variables are defined as the ratio of two basic variables. Thus, we may think of a variety of ratio variables. It is interesting to note that the inverse of any ratio variables, in contrast to the other types of variables, may have economic significance. Many economic growth models are analysed using one or a combination of these ratio variables. This may be due to the fact that most rate variables are functionally related to one of these ratio variables and most ratio variables converge to constant values at steady state. If we employ ratio variables rather than rate variables, growth models can often be more easily analysed. This is one of the important advantages associated with their use.

The choice of any particular ratio variable as an indicator variable may need to be decided on the basis of analytic convenience on the one hand and the economic meaning in relation with rate variables on the other hand.

1. Capital-Labor Ratio (k)

This variable has played a dominant role in the Solow growth model (36) and in recent text books analysis. When the aggregate production function is homogeneous of degree one, the average product of labor ($q = Q/L$) depends only on capital per unit of labor ($k = K/L$). There exist a number of similarities between this intensive form of the production function ($q = f(k)$) and conventional short-run production function ($Q = F(L)$). Thus, we may utilize many useful results of microeconomic analysis by choosing k as an indicator variable. In addition, the time path of other variables can easily be found through simple relationships when the time path of k is known. In short, the choice of k as an indicator variable gives much analytic convenience in the analysis of the steady state properties. However, the analytic convenience of this variable is not so powerful when we are interested in the process of adjustment towards the steady state mainly because the growth (differential) equation expressed in terms of k is very complex and thus it is difficult to derive analytic solutions. This might be one of the reasons why previous investigators do not use this variable as an indicator variable in the analysis of adjustment time.

2. Output-Labor Ratio (q)

Because of its real-life implications we are often interested in this variable. However, the growth process expressed in terms of q is less convenient than in k. Moreover, the real-life implications of this variable are much reduced when L represents not the simple labor but effective-labor. We are in a practical sense interested in output per worker but not in output per unit of effective labor. Output per worker in this situation is changing even in a steady state and thus is inconvenient for the analysis of steady state and of adjustment time.

3. Output-Capital Ratio ($1/v$) and Capital-Output Ratio (v)

Most economic growth models are consistent with the Kaldor "stylized fact" of a constant capital-output (or output-capital) ratio in a steady state. While the Harrod-Domar growth model assumes a constant capital-output (or output-capital) ratio over time by employing a fixed coefficient production function, the Solow neoclassical model allows v (or $1/v$) to vary over time assuming factor substitutability within the aggregate production function. However, v (or $1/v$) finally reaches a constant value in a steady state for the neoclassical model. Along with this historical background, Swan (40) has described the growth process in terms of this variable rather than k .

We may use either v or $1/v$ when we simply talk about the constancy of the ratio of the two basic variables. However, we may have to distinguish these two variables when we use them in relation to growth process because the shape of their time paths are not identical. The growth rate of total output ($\hat{Q}$) has a linear relationship with $1/v$ if relative factor shares are constant over time. This relationship can be derived by differentiating the production function ($Q = F(K,L)$) logarithmically and by utilizing the assumptions of the neoclassical model.

$$\hat{Q} = \eta_K \hat{K} + \eta_L \hat{L} \tag{2.2.1}$$

$$= \eta_K \frac{I}{K} + \eta_L n$$

$$= \frac{\eta_K s}{v} + \eta_L n$$

where $\eta_K (= \frac{\partial Q}{\partial K} \frac{K}{Q})$ represents the elasticity of output w.r.t. K and thus capital share of output in a competitive economy.

$\eta_L (= \frac{\partial Q}{\partial L} \frac{L}{Q})$ represents the elasticity of output w.r.t. L and thus labor share of output.

When η_K and η_L are constant in this equation, it is evident that $\hat{Q}$ has a linear relationship with $1/v$.

In fact, R. Sato measures the adjustment time of a simple neoclassical economy with a Cobb-Douglas production function in terms of this variable ($1/v$). The resulting estimates, therefore, are consistent with those of Conlisk and Ramanathan using $\hat{Q}$, when the production function is of the Cobb-Doublas type for which η_K and η_L are constant.

However, the capital-output ratio (v) has one interesting analytic convenience when the relative factor shares are constant in the simple neoclassical economy. The growth process of v is represented by a "linear" differential equation, which is very convenient in the analysis of the growth process. From the definition of v, we know that $Q = K/v$. By differentiating it logarithmically,

$$\hat{Q} = \hat{K} - \hat{v} \tag{2.2.2}$$

$$= \frac{s}{v} - \frac{\dot{v}}{v}$$

Substituting equation (2.2.1) into equation (2.2.2), we obtain the following differential equation.

$$\dot{v} = - \eta_L \, nv + (1 - \eta_K) s \tag{2.2.3}$$

This equation becomes a linear differential equation when η_L and η_K are constant.

CHAPTER III

NEOCLASSICAL ONE-SECTOR MODEL OF GROWTH

The neoclassical one-sector model of economic growth is the basis for most recent models of economic growth. One advantage of the neoclassical model is that it can be set out simply and clearly as a system of equations, and it is easily adapted to accommodate different assumptions. The neoclassical model stems from papers by Solow (36) and Swan (40), which were published in 1956, although most of the characteristics of the neoclassical approach were included in a paper by Tobin (42) published in the previous year.

The growth model used in this chapter is similar to that of Solow though some modifications such as inclusion of capital depreciation and of technical progress have been added. For this model, a complete analytic solution and numerical calculations are given for the adjustment time assuming a Cobb-Douglas technology. As an extension, numerical analysis is also done assuming a C.E.S. production function.

3.1 Neoclassical Growth Model

Specification of Model

A simple neoclassical one-sector model of growth may be represented as follows (36 and 44, pp. 32-63):

1. Production Function

There is one commodity (Q) in an economy, the production of which utilizes the full capacity of homogeneous capital (K) and the full supply of homogeneous effective labor (E), and is represented by the linearly homogeneous production function.

(3.1.1) $$Q\ (t) = F\ [K\ (t),\ E\ (t)]$$

Because it is homogeneous of degree one, the production function can be represented in its intensive form as follows.

(3.1.2) $$q = F\ (\frac{K}{E},\ 1) = f\ (k)$$

where q and k represent output per unit of effective labor (Q/E) and capital per unit of effective labor (K/E) respectively.

The production function is assumed to satisfy the following so-called Inada conditions (21 and 44, pp. 37-40).

(3.1.3) $$\text{(i)}\ \frac{df}{dk} = f'(k) > 0$$

$$\text{(ii)}\ \frac{d^2f}{dk^2} = f''\ (k) < 0$$

(iii) $\lim_{k \to o} f'(k) = \infty$

(iv) $\lim_{k \to \infty} f'(k) = o$

(v) $f(o) = 0$

(vi) $f(\infty) = \infty$

2. Capital Formation

A constant fraction (s) of total output is saved and automatically added to the stock of capital which is subject to depreciation at a constant rate of δ.

(3.1.4) $$\dot{K} = sQ - \delta K$$

or,

$$\hat{K} = \frac{sQ}{K} - \delta$$

where $\dot{K} = dK/dt$ and $\hat{K} = \dot{K}/K$.

3. Labor Force Growth

The effective labor force (E) grows at a constant proportional rate $(n + \lambda)$ independent of any economic variable in the system, where n is the growth rate of natural labor force (L) and λ is the improvement rate of labor effectiveness, that is the rate of labor augmenting technical progress.

(3.1.5) $$\dot{E} = (n + \lambda) E$$

or,

$$\hat{E} = n + \lambda$$

where $\dot{E} = dE/dt$ and $\hat{E} = \dot{E}/E$.

This neoclassical model is closed since it has three unknowns (Q, K, and E) and three equations (3.1.1), (3.1.4), and (3.1.5). Therefore, for given initial values of three variables (Q_o, K_o, and E_o), the time path of the model economy can be found by solving the above system of three equations. Solow reduces this system, however, to a single differential equation in terms of the capital-effective labor ratio (k), which may not, itself, be of ultimate interest but which helps to describe the growth process of the system in a simple and convenient way (36, pp. 77-80).

Since k = K/E, we can obtain the following equations by differentiating it logarithmically and by substituting equation (3.1.4) and (3.1.5) into the result.

$$(3.1.6) \qquad \hat{k} = \hat{K} - \hat{E}$$

$$= \frac{sQ - \delta K}{K} - (n + \lambda)$$

$$= \frac{sf\ (k)}{k} - (n + \lambda + \delta)$$

or,

$$\dot{k} = s\ f\ (k) - (n + \lambda + \delta)k$$

where $\dot{k} = dk/dt$ and $\hat{k} = \dot{k}/k$.

This fundamental equation of neoclassical economic growth, governing the path to be followed by the capital-effective labor ratio (k), can be interpreted as the rate of change of the capital-effective labor ratio ($\dot{k}$) being determined by the difference between the amount of saving (and investment) per unit of effective labor ($sf(k)$) and the amount required to keep the capital-effective labor ratio constant ($(n + \lambda + \delta)k$) as the effective labor force grows.

By solving the fundamental equation, we can derive the explicit time path of the capital-effective labor ratio (k) and of the time pattern of other variables such as total output (Q) and capital (K), which are functionally related to k. Such a growth model leads, in a mathematical sense, to the study of a differential equation, and in this study there is no perfect substitute for an explicit solution. Since analytic solutions are, however, rare for non-linear differential equations and the immediate concern of the neoclassical model is with the asymptotic behavior of the model, researchers have concentrated on the investigation of steady state properties, which can be analysed without explicitly solving the differential equation.

Steady State Properties

A path of steady state growth is the time path along which the proportional rates of growth of all the relevant variables remain constant over time. Along a path of steady state growth, then, output (Q), effective labor (E), and capital stock (K) are all growing exponentially while the capital-effective labor ratio (k), the growth rate of output ($\hat{Q}$) and the capital-output ratio (v) remain constant.

The neoclassical economy, under the conditions specified above, has a unique and stable steady state path in the sense that whatever the initial values of all the variables, the economy converges to a steady state path uniquely determined by the parameters of the model (44, pp. 35-40). The existence and stability of the steady state path is shown graphically in Figure 3.1.1. The intensive production function ($q = f(k)$) is strictly concave and has an infinite slope at $k = o$ and a zero slope at $k = \infty$ from the Inada conditions of equation (3.1.3). Therefore, the straight line $q = (n + \lambda + \delta)k/s$ with a positive slope $(n + \lambda + \delta)/s$, meets $f(k)$ somewhere between $k = o$ and $k = \infty$. This means that there exists a finite value of k, which satisfies the fundamental equation (3.1.6). To the left of k^* in Figure 3.1.1, output per unit of effective labor (q) exceeds the output level needed to maintain the k ratio constant. There is enough investment both to equip each new E unit of effective labor with the existing K/E ratio and to increase the K/E ratio by $\dot{k}$. In this case, $\dot{k} > o$ and k increases. To the right of k^*, there is not enough output per unit of effective labor to outfit each E unit of effective labor at existing K/E ratio, so k decreased at a rate of $\dot{k}$. At $k = k^*$, the level of q ($=q^*$) is the amount which yields just enough investment to maintain k at a constant level as E grows. Therefore, $\dot{k} = o$ at $k = k^*$ and the system reaches a stable equilibrium.

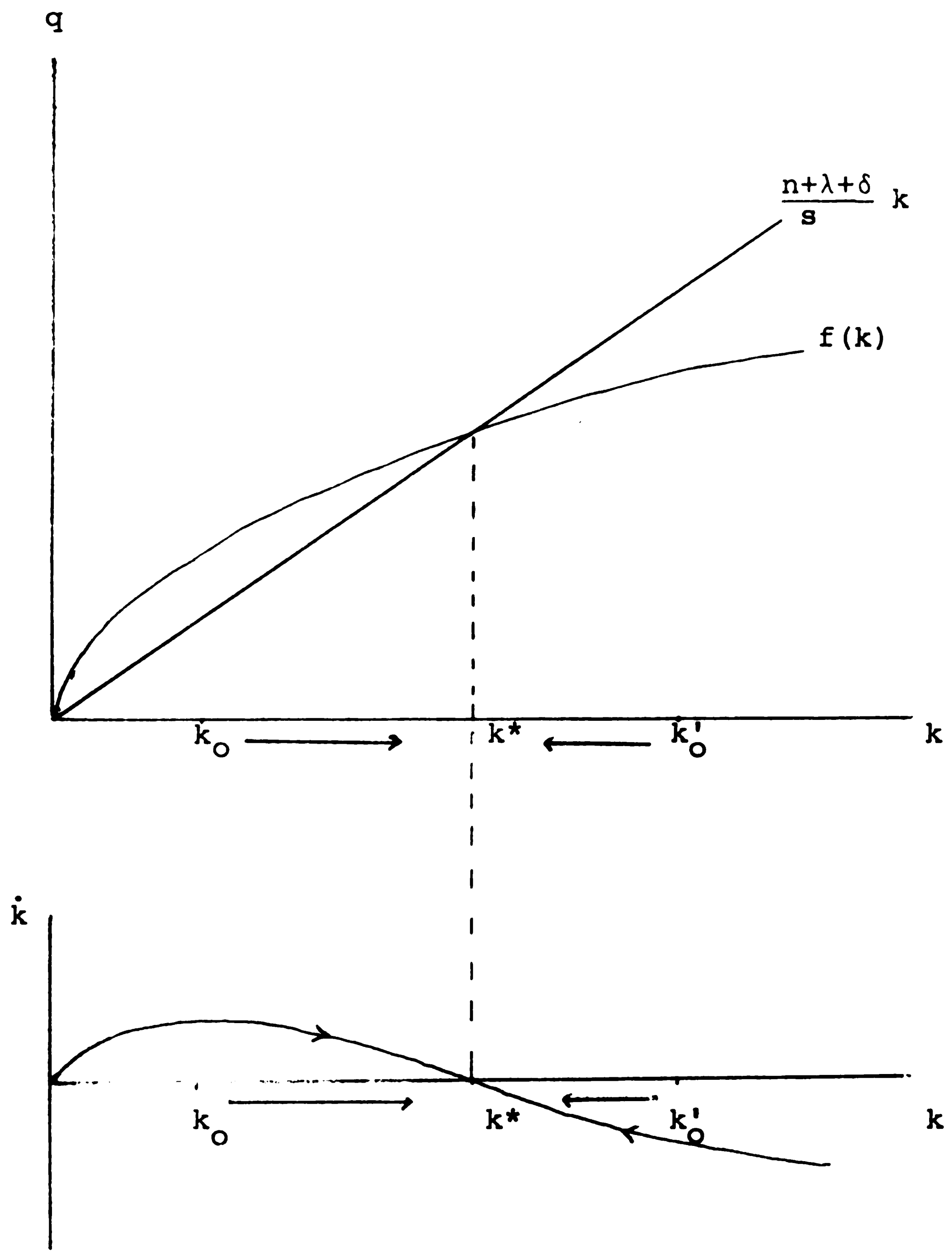

Figure 3.1.1

Equilibrium and Stability of Neoclassical Economy

Since the capital per unit of effective labor (k) is constant along the steady state path of growth, the steady state value of k (i.e., k*) is algebraically obtained by setting $\dot{k} = o$ in equation (3.1.6).

$$\dot{k}^* = s\ f\ (k^*) - (n + \lambda + \delta)k^* = o \tag{3.1.7}$$

or,

$$s\ f\ (k^*) = (n + \lambda + \delta)k^*$$

Steady state values of the capital-output ratio (v^*) and of the growth rate of output ($\hat{Q}^*$) can be easily obtained from the constancy of k in the steady state.

$$v^* = \frac{k^*}{f(k^*)} \tag{3.1.8}$$

Since k^* is constant, v^* is constant.

National output (Q) is the amount of output per unit of effective labor ($q = f(k)$) multiplied by the amount of effective labor (E), that is $Q = f(k)E$. By differentiating it logarithmically,

$$\begin{aligned} \hat{Q} &= \hat{f}(k) + \hat{E} \\ &= \hat{f}(k) + (n + \lambda) \end{aligned} \tag{3.1.9}$$

Since f(k) is constant in the steady state, $\hat{Q}$ is also constant. That is,

$$\hat{Q}^* = n + \lambda \tag{3.1.10}$$

In conclusion, the neoclassical economy converges in the long run to the steady state path, along which the growth rate of output ($\hat{Q}$) is constant and equal to that of the effective labor force ($n + \lambda$) and thus is entirely independent of the proportion of income saved. Several ratio variables such as the capital-output ratio (v) and the capital-effective labor ratio (k) also are constant in the steady state. The magnitudes of these ratio variables, however, depend on the propensity to save (s) as well as on the other parameters.

3.2 Adjustment Path to Steady State

As explained in the previous section, the neoclassical economy converges in the long run to a steady state path of growth. The adjustment path of the economy to steady state is, however, rarely discussed in the literature, partly because the neoclassical approach is primarily concerned with the long run characteristics of economic growth and partly because the assumption of instantaneous market adjustment is far from reality and thus the model may not be relevant for the explanation of the short run phenomena of economic growth.

In order to measure the adjustment time of the economy, however, we need explicit information on the adjustment path. Since the proportional adjustment time may be different for different variables, we need to deal with several variables which can serve as an indicator variable of the system. This section deals in particular with the adjustment path of the capital-effective labor ratio (k) because of its dominant role in recent economic models, the capital-output ratio (v) because of its analytic simplicity, and finally the growth rate of output ($\hat{Q}$) because of its economic importance. The adjustment time of other variables, if of interest, can be analysed by exactly the same method as used for the above variables.

The growth path of the capital-effective labor ratio (k) was given in equation (3.1.6). Rewriting the equation,

$$\dot{k} = s\ f\ (k) - (n + \lambda + \delta)k \tag{3.2.1}$$

The growth path of the capital-output ratio ($v = k/f(k)$) is derived by differentiating it logarithmically.

$$\begin{aligned} \hat{v} &= \hat{k} - \hat{f}(k) \\ &= \hat{k} - \eta_k\ \hat{k} \\ &= (1 - \eta_k)\ \hat{k} \end{aligned} \tag{3.2.2}$$

where η_k represents the elasticity of output per unit of effective labor (q) w.r.t. the capital per unit of effective labor (k). Substitution of equation (3.2.1) into equation (3.2.2) yields

$$\hat{v} = -\ (1 - \eta_k)\ (n + \lambda + \delta) + (1 - \eta_k)\ \frac{s}{v} \tag{3.2.3}$$

or,

$$\dot{v} = - (1 - \eta_k)\ (n + \lambda + \delta)v + (1 - \eta_k)\ s$$

or,

$$\dot{v} = - \eta_E\ (n + \lambda + \delta)v + \eta_E\ s$$

since $1 - \eta_k = \eta_E$ when the production function is homogeneous of degree one.[1] Where η_E represents the output elasticity w.r.t. effective labor (E) and η_K represents the output elasticity w.r.t. capital (K).

Since the differential equation representing the time path of v is often easy to manipulate, it is convenient to express time paths of other variables in terms of v. The time path of the growth rate of output ($\hat{Q}$) can be expressed in terms of v as follows. By differentiating the aggregate production function Q = f (K,L),

$$\hat{Q} = \eta_K\ \hat{K} + \eta_E\ \hat{E} \tag{3.2.4}$$

---

1. $\eta_K = \frac{\partial Q}{\partial K}\frac{K}{Q} = \frac{\partial (f\ (k)\ E)}{\partial K}\ \frac{K}{Q}$

$= f'(k)\ \frac{\partial k}{\partial K}\ \frac{E\ K}{Q} = f'\ (k)\ (\frac{E}{E^2})\ \frac{E\ K}{Q}$

$= f'\ (k)\ \frac{k}{f} = \eta_k$

Therefore, $\eta_E = 1 - \eta_K = 1 - \eta_k$ for a linearly homogeneous production function (refer to Allen (2), pp. 41-48).

By substituting equation (3.1.4) for $\hat{K}$ and equation (3.1.5) for $\hat{E}$ into equation (3.2.4),

$$(3.2.5) \qquad \hat{Q} = \eta_K \frac{sQ - \delta K}{K} + \eta_E (n + \lambda)$$
$$= \frac{s \ \eta_K}{v} + [(n + \lambda + \delta) \ \eta_E - \delta)]$$

Using this equation, the time path of $\hat{Q}$ can be found if the time path of v is given. It is interesting to notice that $\hat{Q}$ has a linear relationships with $1/v$ when the output elasticies w.r.t. factors (η_K and η_E) are constant over time.

The growth path of k can also be expressed in terms of v. Since $v(=k(f(k))$ is a function of k, we can express k in terms of v. Let $g(k) = f(k)/k$, then $g(k) = v$.

$$(3.2.6) \qquad k = g^{-1} (v)$$

where g^{-1} is an inverse function of g.

We can find the time path of k using this equation (3.2.6) if the time path of v is given. Therefore, the time path of all the variables in question (v, k, and $\hat{Q}$) can be found by solving the differential equation (3.2.3) and by substituting the solution in equations (3.2.5) and (3.2.6), which are algebraic equations.

<u>Cobb-Douglas Production Function Case</u>

The time path of the capital-effective labor ratio (v) given in equation (3.2.3) results in a linear differential equation when the output elasticities of the production function are constant over time. Since both output elasticities w.r.t. capital (η_K) and w.r.t. effective labor (η_E) are constant for a Cobb-Douglas (C-D) function, which satisfies all the Inana conditions of equation (3.1.3), we adopt a C - D function to derive analytic solutions for the time path of v and thus of k and $\hat{Q}$.

A Cobb-Douglas function is written as

$$Q = A\ K^{\alpha}\ E^{1-\alpha} \tag{3.2.7}$$

where A and α are constant parameters of the function.

The output elasticities w.r.t. factors are given by

$$\eta_K = \frac{\partial Q}{\partial K}\frac{K}{Q} = \alpha \tag{3.2.8}$$

$$\eta_E = \frac{\partial Q}{\partial L}\frac{L}{Q} = 1-\alpha$$

Substituting these output elasticities into differential equation (3.2.3), we obtain the following linear differential equation.

$$\dot{v} = -(1-\alpha)\ (n + \lambda + \delta)v + (1-\alpha)s \tag{3.2.9}$$

The solution of this linear differential equation is given by (7, pp. 7-12)

$$(3.2.10)\quad v(t) = v^* + (v_o - v^*)\, e^{-(1-\alpha)(n + \lambda + \delta)t}$$

or,

$$v(t) = B_1 + B_2 e^{-\theta t}$$

where v_o = the initial value of v at $t = o$

v^* = the steady state value of v at $t = \infty$

$\theta = (1 - \alpha)(n + \lambda + \delta)$

$$B_1 = v^* = \frac{s}{n + \lambda + \delta} = \frac{(1-\alpha)\, s}{\theta}$$

$$B_2 = v_o - v^*$$

The time path of $\hat{Q}$ is expressed as follows for the C-D function using equation (3.2.5).

$$(3.2.11)\qquad \hat{Q} = \frac{\alpha s}{v} + [(1 - \alpha)(n + \lambda + \delta) - \delta]$$

Substitution of the expression for v (equation (3.2.10)) into equation (3.2.11) yields

$$(3.2.12)\qquad \hat{Q} = \frac{\alpha s}{B_1 + B_2 e^{-\theta t}} + (\theta - \delta)$$

$$\hat{Q} = [\hat{Q}^* - \alpha(\hat{Q}^* + \delta)] + \alpha(\hat{Q}^* + \delta) / \left[1 - \frac{(\hat{Q}_o - \hat{Q}^*) e^{-\theta t}}{(\hat{Q}_o - \hat{Q}^*) + \alpha(\hat{Q}^* + \delta)}\right]$$

---

[2]This is the expression Conlisk (10, p. 551) adopts for his simple model.

where $\hat{Q}_o$ = the initial value of $\hat{Q}$ at $t = o$

$$= \frac{\alpha s}{v_o} + (\theta - \delta)$$

$$= \frac{\alpha s}{B_1 + B_2} + (\theta - \delta)$$

$\hat{Q}^*$ = the steady state value of $\hat{Q}$ at $t = \infty$

$$= \frac{\alpha s}{v^*} + (\theta - \delta)$$

$$= \frac{\alpha s}{B_1} + (\theta - \delta)$$

$$= n + \lambda$$

The time path of k is obtained as follows.

(3.2.13)
$$v = \frac{k}{f(k)}$$

$$= \frac{k}{A\,k^{\alpha}}$$

$$= A^{-1}\,k^{1-\alpha}$$

Therefore,

(3.2.14)
$$k = (Av)^{\frac{1}{1-\alpha}}$$

$$= [\,A\,(B_1 + B_2 e^{-\theta t})\,]^{\frac{1}{1-\alpha}}$$

where the following relationships hold.

k_o = the initial value of k at $t = o$

$$= (Av_o)^{\frac{1}{1-\alpha}}$$

$$= [\,A\,(B_1 + B_2)\,]^{\frac{1}{1-\alpha}}$$

k^* = the steady state value of k at $t = \infty$

$$= (Av^*)^{\frac{1}{1-\alpha}}$$

$$= (AB_1)^{\frac{1}{1-\alpha}}$$

The shapes of the time paths of the variables (v, $\hat{Q}$, and k) may be determined using first and second order derivatives of their time paths. By taking first and second order derivatives of the time path of v (equation (3.2.10)) w.r.t. time t,

$$\dot{v} = -\theta B_2 e^{-\theta t} \tag{3.2.15}$$

and,

$$\ddot{v} = \theta^2 B_2 e^{-\theta t}$$

By differentiating the time path of $\hat{Q}$ (equation (3.2.11)),

$$\dot{\hat{Q}} = -\alpha s v^{-2} \dot{v} \tag{3.2.16}$$

and,

$$\ddot{\hat{Q}} = \alpha s v^{-2} (2v^{-1} \dot{v}^2 - \ddot{v})$$

By differentiating the time path of k (equation (3.2.14)),

$$\dot{k} = \frac{A^{\frac{1}{1-\alpha}}}{1-\alpha} v^{\frac{\alpha}{1-\alpha}} \dot{v} \tag{3.2.17}$$

and,

$$\ddot{k} = \frac{A^{\frac{1}{1-\alpha}} v^{\frac{\alpha}{1-\alpha}}}{(1-\alpha)^2} [\alpha v^{-1} \dot{v}^2 + (1 - \alpha)\ddot{v}]$$

Suppose that the neoclassical economy is disturbed by a change of saving rate from s_o to s_1 where $s_1 > s_o$. Then, it is evident from Figure 3.2.1, which shows the adjustment of the neoclassical economy, that the initial value of the capital-output ratio (v_o) is smaller than the steady state value of the variable (v^*). Thus, $B_2(=v_o - v^*)$ is negative. In this case, therefore, $\dot{v} > o$ and $\ddot{v} < o$ since B_2 is negative. This means that the time path of v is strictly concave from the origin. It follows from equation (3.2.16) that that $\dot{\hat{Q}} < o$ and $\ddot{\hat{Q}} > o$, which means that the time path of $\hat{Q}$ is strictly convex from origin. It also follows that $\dot{k} > o$ from equation (3.2.17), but the sign of $\ddot{k}$ is not clear. It depends on the level of v and the value of α. However, we know that $\ddot{k}$ has a positive value as time goes to infinity because v goes to zero as time goes to infinity. Thus, the time path of k has a positive slope ($\dot{k}$) during all time intervals and a negative rate of change of the slope ($\ddot{k}$) at large values of t. Based on this information the time paths of the variables can be plotted out as in Figure 3.2.2.

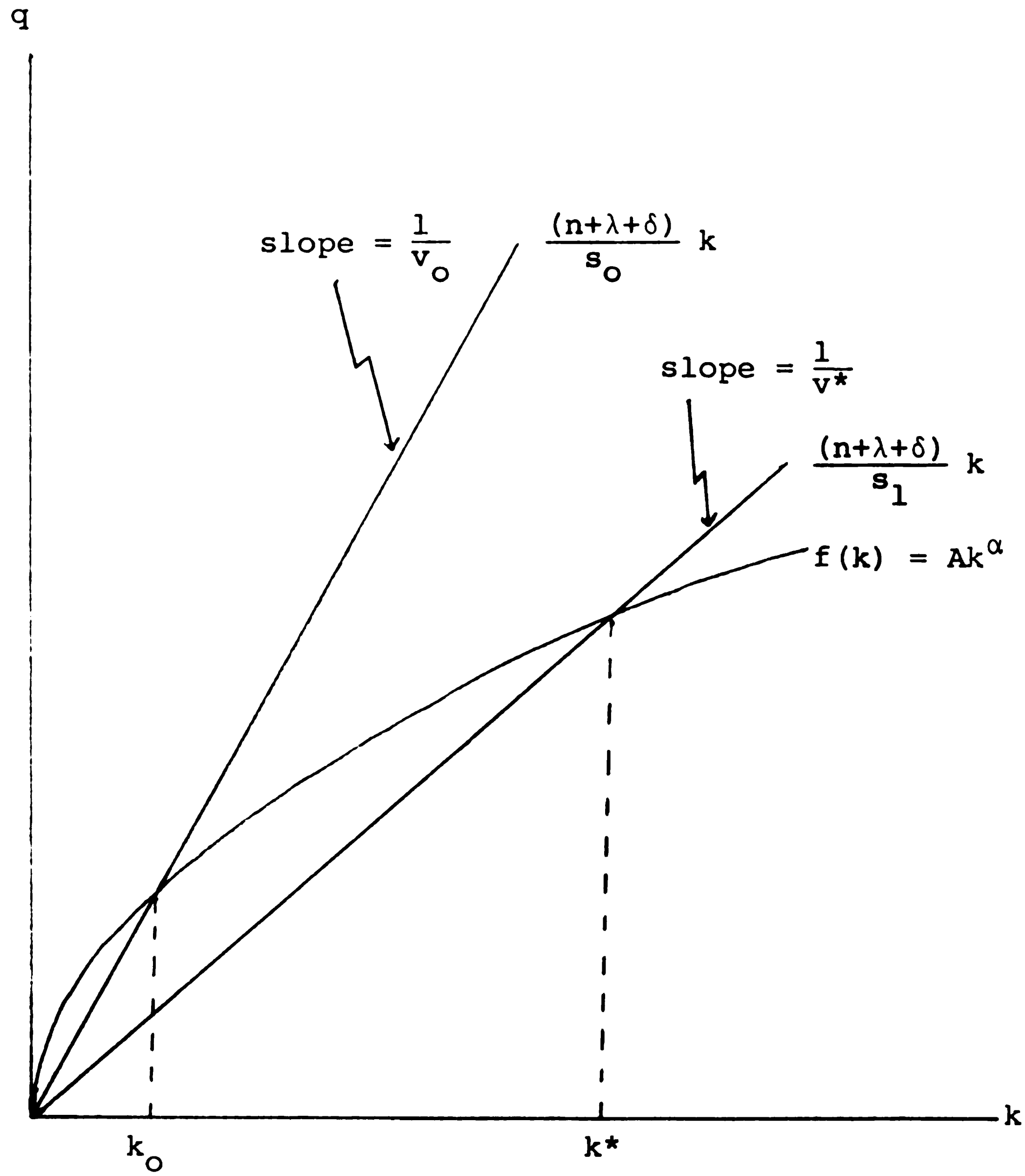

Figure 3.2.1

Adjustment of Neoclassical Economy with Cobb-Douglas Production Technology

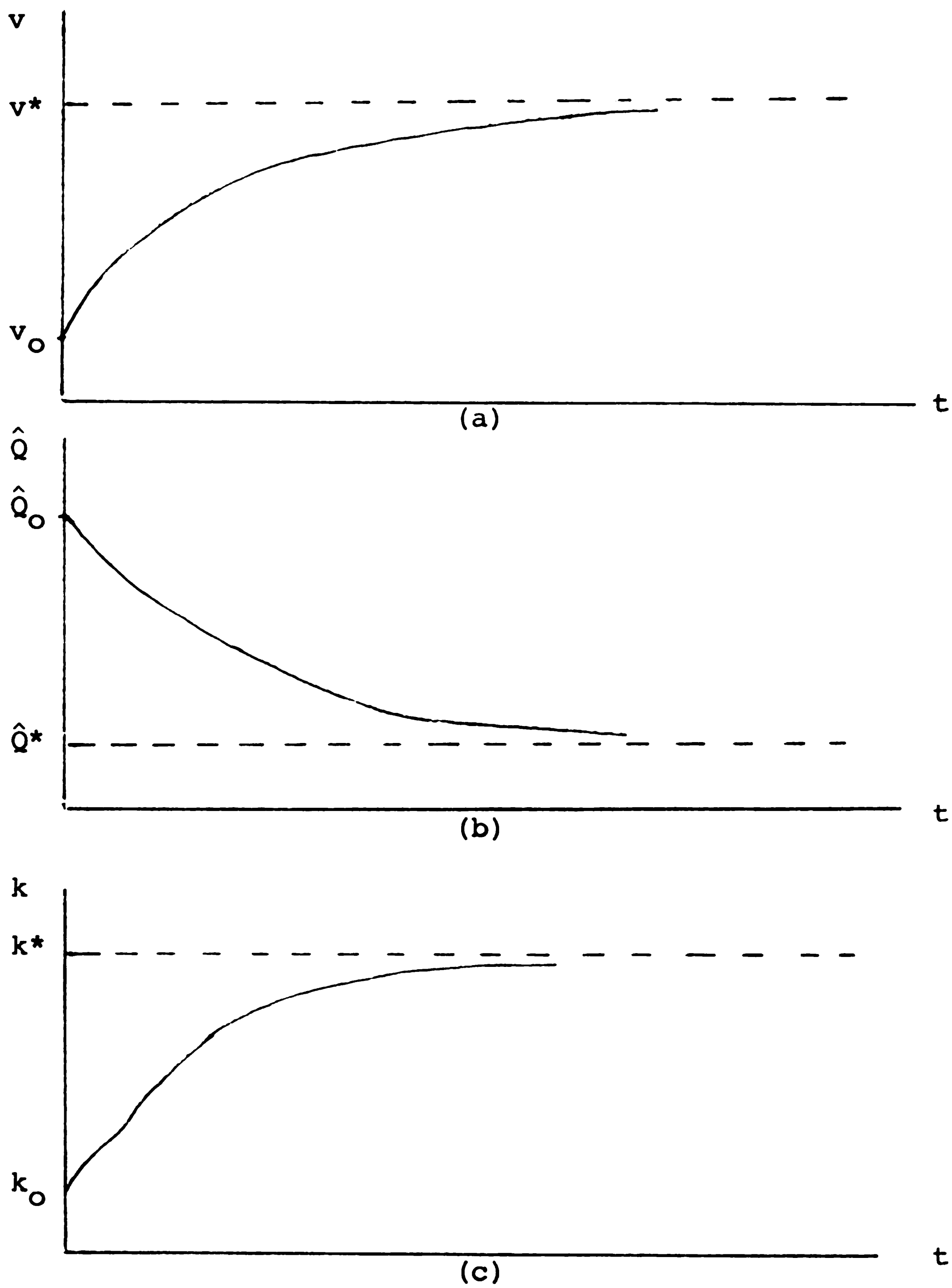

Figure 3.2.2

Time Paths of Capital-Output Ratio (a), Growth Rate of Output (b), and Capital-Effective Labor Ratio (c)

3.3 Adjustment Time to Steady State

The variables in question (v, $\hat{Q}$, and k) of the neo-classical model reach their steady state values as time goes to infinity. We are interested here in the time required for an indicator variable of the economy (v, $\hat{Q}$, or k) to cover a certain proportion of an initial displacement from its steady state value, which has been defined as the proportional adjustment time of the variable.

As shown in Chapter II, the proportional adjustment time is different for different variables because the shape of time path of a variable is in general different from that of other variables even for a given level of proportion of adjustment. The proportional adjustment time of a variable (t_x) for a given level of adjustment proportion (ε) has been given in equation (2.1.6). Rewriting the equation,

$$(3.3.1) \qquad t_x\ (\varepsilon) = \int_{x_o}^{x_o + \varepsilon\ (x^* - x_o)} \frac{dt}{dx}\, dx$$

where x_o and x^* represent initial and steady state values of x respectively, and $\varepsilon = (x\ (t) - x_o)\ /\ (x^* - x_o)$.

Solving this equation (3.3.1) the following equation, which determines the time required for the variable (x) to complete 100ε percent of the initial displacement towards its steady state value (x^*) from its initial value (x_o), can be derived.

$$t_x(\varepsilon) = H\ (x^*,\ x_o,\ \varepsilon) \tag{3.3.2}$$

However, it is difficult to find analytic solutions for most of economic growth models if the production function does not take a specific form.

Cobb-Douglas Production Function Case

Since we can easily derive an explicit formula for the adjustment time corresponding to equation (3.3.2) for the neoclassical model with a Cobb-Douglas production function, we examine the C - D case first.

1. Adjustment Time of v

The expression for the adjustment time of the capital-output ratio (v) can be derived in the following manner. From the definition of ε_v,

$$\varepsilon_v = \frac{v\ (t) - v_o}{v^* - v_o} \tag{3.3.3}$$

By substituting the expression for $v(t)$, v_o, and v^* from equation (3.2.10) into equation (3.3.3),

$$\begin{aligned} \varepsilon_v &= \frac{(B_1 + B_2\ e^{-\theta t}) - (B_1 + B_2)}{B_1 - (B_1 + B_2)} \\ &= 1 - e^{-\theta t} \end{aligned} \tag{3.3.4}$$

Taking logarithm for both sides, we obtain the following formula for the adjustment time of v (t_v).

(3.3.5) $$t_v = \frac{1}{\theta} \log \left(\frac{1}{1 - \varepsilon}\right)$$ [3]

It is important to notice that the adjustment time of v is independent of the size and cause of the initial impact. This property comes, of course, from the linearity of the time path of v. This equation (3.3.5) shows that the adjustment time has an inverse relationship with $\theta [= (1-\alpha)(n + \lambda + \delta)]$. It means that the larger are the growth rate of the effective labor force $(n + \lambda)$, the rate of capital depreciation (δ), and the relative share of labor $(1 - \alpha)$, the shorter is the adjustment time for a given proportion of the adjustment (ε). More precisely,

(3.3.6) (i) $$\frac{\partial t_v}{\partial \theta} = \frac{-1}{\theta^2} \log \left(\frac{1}{1 - \varepsilon}\right) < o$$

(ii) $$\frac{\partial t_v}{\partial n} = \frac{\partial t_v}{\partial \theta} \frac{\theta}{n} = - \left(\frac{1-\alpha}{\theta^2}\right) \log \left(\frac{1}{1-\varepsilon}\right) < o$$

(iii) $$\frac{\partial t_v}{\partial \lambda} = \frac{\partial t_v}{\partial \theta} \frac{\partial \theta}{\partial \lambda} = - \left(\frac{1-\alpha}{\theta^2}\right) \log \left(\frac{1}{1-\varepsilon}\right) < o$$

(iv) $$\frac{\partial t_v}{\partial \delta} = \frac{\partial t_v}{\partial \theta} \frac{\partial \theta}{\partial \delta} = - \left(\frac{1-\alpha}{\theta^2}\right) \log \left(\frac{1}{1-\varepsilon}\right) < o$$

(v) $$\frac{\partial t_v}{\partial (1-\alpha)} = \frac{\partial t_v}{\partial \theta} \frac{\partial \theta}{\partial (1-\alpha)} = - \frac{(n+\lambda+\delta)}{\theta^2} \log \left(\frac{1}{1-\varepsilon}\right) < o$$

(vi) $$\frac{\partial t_v}{\partial \varepsilon} = \frac{1}{\theta(1-\varepsilon)} > o$$

[3] This is the formula K. Sato (32, p. 265) adopts to measure the adjustment time of the neoclassical vintage model.

2. Adjustment Time of $\hat{Q}$

The expression for the adjustment time of the growth rate of output ($\hat{Q}$) is derived in the same manner as in the capital output ratio (v). From the definition of $\varepsilon_{\hat{Q}}$,

(3.3.7)
$$\varepsilon_{\hat{Q}} = \frac{\hat{Q}(t) - \hat{Q}_o}{\hat{Q}^* - \hat{Q}_o}$$

By substituting the expressions for $\hat{Q}(t)$, $\hat{Q}_o$, and $\hat{Q}^*$ from the equation (3.2.11) into equation (3.3.7),

$$\varepsilon_{\hat{Q}} = \frac{[\frac{\alpha s}{v(t)} + (\theta - \delta)] - [\frac{\alpha s}{v_o} + (\theta - \delta)]}{[\frac{\alpha s}{v^*} + (\theta - \delta)] - [\frac{\alpha s}{v_o} + (\theta - \delta)]}$$

By arranging this expression,

(3.3.8)
$$\varepsilon_{\hat{Q}} = \frac{\frac{1}{v(t)} - \frac{1}{v_o}}{\frac{1}{v^*} - \frac{1}{v_o}}$$

$$= \frac{v(t) - v_o}{v^* - v_o} \; \frac{v^*}{v(t)}$$

$$= (1 - e^{-\theta t}) \; (\frac{B_1}{B_1 + B_2 e^{-\theta t}})$$

Taking logarithm on the both sides, we obtain the following formula for the adjustment time of $\hat{Q}$.

(3.3.9)
$$t_{\hat{Q}} = \frac{1}{\theta} \log [\frac{B_1 + \varepsilon B_2}{(1-\varepsilon) B_1}]$$

or,

$$t_{\hat{Q}} = \frac{1}{\Theta} \log \left[\frac{\dfrac{\hat{Q}_o - \hat{Q}^*}{\hat{Q}^* + \delta} + \dfrac{\alpha}{1 - \varepsilon}}{\dfrac{\hat{Q}_o - \hat{Q}^*}{\hat{Q}^* + \delta} + \alpha}\right]$$

It is evident from this equation (3.3.8) that the adjustment time of the growth rate of output ($\hat{Q}$) is the same as that of the output-capital ratio (1/v). This is so because of the linear relationship between them when the production function is of the Cobb-Douglas form for the neoclassical model. Therefore, R. Sato's estimates using 1/v (33, 34) are the same for $\hat{Q}$.

It is evident from equation (3.3.9) that $t_{\hat{Q}}$ has an inverse relation with $\theta (= (1-\alpha)(n+\lambda+\delta))$ and a positive relationship with ε. Mathematically,

(3.3.10)

$$\text{(i)} \quad \frac{\partial t_{\hat{Q}}}{\partial \theta} = \frac{-1}{\theta^2} \log \left[\frac{B_1 + \varepsilon B_2}{(1 - \varepsilon)\, B_1}\right] < o$$

$$\text{(ii)} \quad \frac{\partial t_{\hat{Q}}}{\partial \varepsilon} = \frac{1}{\theta} \frac{(B_1 + B_2)}{(1 - \varepsilon)\,(B_1 + \varepsilon B_2)} > o$$

It is interesting to note that $t_{\hat{Q}}$ is independent of saving rate (s) from the second expression of equation (3.3.9) if the initial position of the economy is specified in terms of $\hat{Q}_o$ rather than in terms of initial values of other variables. Therefore, R. Sato's conclusion (33, p. 21) that "(1) the greater the initial saving ratio, the longer the adjustment period. (2) The higher the new saving ratio, the shorter the adjustment period" may be valid only if the initial position of the economy is not expressed in terms of initial value of the growth rate of output $(\hat{Q}_o)$.

3. Adjustment Time of k

A similar expression of the adjustment time of the capital per unit of effective labor (k) is also found by the same method. From the definition of ε_k,

$$\varepsilon_k = \frac{k(t) - k_o}{k^* - k_o} \tag{3.3.11}$$

Manipulating this equation in the same manner as in v and $\hat{Q}$, we obtain the adjustment time of k.

$$t_k = \frac{1}{\theta} \log\left[\frac{B_2}{\left[\varepsilon B_1^{\frac{1}{1-\alpha}} + (1-\varepsilon)(B_1 + B_2)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} - B_1}\right] \tag{3.3.12}$$

or,

$$t_k = \frac{1}{\theta} \log \frac{(k_o^{\,1-\alpha} - k^{*\,1-\alpha})/A}{[\varepsilon k^* + (1-\varepsilon)k_o]^{1-\alpha} - k^{*1-\alpha}\ /A}$$

This equation (3.3.12) also shows that t_k has an inverse relationship with θ and a positive relationship with ε.

4. Comparison of Adjustment Time

Since the adjustment time formulas have been derived for the several variables in question it may be useful to compare the time required for these variables to complete the same proportion of their adjustment; that is, $\varepsilon_v = \varepsilon_Q = \varepsilon_k$. Let us rewrite the formulas for convenient comparison.

$$t_v = \frac{1}{\theta} \log \left(\frac{1}{1-\varepsilon}\right)$$

$$t_{\hat{Q}} = \frac{1}{\theta} \log \left[\frac{B_1 + \varepsilon B_2}{(1-\varepsilon)\, B_1}\right]$$

$$t_k = \frac{1}{\theta} \log \left[\frac{B_2 / B_1}{\left\{\varepsilon + (1-\varepsilon)\left(1 + \frac{B_2}{B_1}\right)^{\frac{1}{1-\alpha}}\right\}^{1-\alpha} - 1}\right]$$

The difference of adjustment time between $t_{\hat{Q}}$ and t_v is given by

$$t_{\hat{Q}} - t_v = \frac{1}{\theta} \log \left(1 + \frac{\varepsilon B_2}{B_1}\right) \tag{3.3.13}$$

Since $B_1 = v^*$ and $B_2 = v_o - v^*$ from equation (3.2.10), the equation (3.3.13) can alternatively be expressed in the following way.

$$t_{\hat{Q}} - t_v = \frac{1}{\theta} \log \left[1 + \frac{\varepsilon (v_o - v^*)}{v^*}\right] \tag{3.3.14}$$

Therefore,

if $B_2 < o$ (i.e. $v_o < v^*$), $t_v > t_{\hat{Q}}$

and,

if $B_2 > o$ (i.e. $v_o > v^*$), $t_v < t_{\hat{Q}}$

This means that the relative magnitude of adjustment time depends on the direction of the adjustment. When K. Sato (using v) criticized R. Sato (using $\frac{1}{v}$ and thus $\hat{Q}$), arguing that the adjustment time of a neoclassical economy could be reduced by a realistic modification of the model, he examined the case of upwards adjustment ($v_o < v^*$). Thus, K. Sato's argument could have been more effective if he had made estimates of adjustment time for the case of downwards adjustment ($v_o > v^*$).

The difference between $t_{\hat{Q}}$ and t_v decreases as $(v_o - v^*)$ becomes smaller. In other words, the difference of adjustment time becomes smaller as the size of initial disturbance becomes smaller. This is an expected result because the shapes of time paths of both v and $\hat{Q}$ become more similar and can be linearized as the size of displacement becomes smaller. The proportional adjustment time, as discussed in Chapter II, is same for the variables which have the same shapes of time paths.

The difference between $t_{\hat{Q}}$ and t_v becomes larger as the proportion of adjustment (ε) becomes bigger. This is due to the fact that as ε becomes larger the size of adjustment becomes larger and thus the difference in shapes of the time paths of $\hat{Q}$ and v become more pronounced.

Analytic comparisons of t_k and t_v, and t_k and $t_{\hat{Q}}$ are difficult because of the complexity of the time formula of k. Numerical illustrations will give some ideas for the relative magnitude among t_v, $t_{\hat{Q}}$, and t_k.

5. Numerical Calculation

It has been shown that the proportional adjustment time of an economy depends both on which variable is chosen as an indicator of the system and on the values of parameters of the economic system. Table 3.3.1 shows the proportional adjustment time of the variables discussed (v, $\hat{Q}$, and k) for a neoclassical model with a Cobb-Douglas production function having the values of parameters adopted by R. Sato (33). We know that this economy adjusts to a steady state path where $\hat{Q}^* = n+\lambda = 0.035$ as time goes to infinity.

The adjustment times of the variables are, however, quite different from each other, even for a given level of adjustment proportion. When an initial growth rate of output is 0.025 (i.e., $\hat{Q}_o = 0.025$), for example, the capital-output ratio (v) takes 101.2 years, the growth rate of output ($\hat{Q}$) takes 172.0 years, and the capital per unit of effective labor (k) takes 81.4 years for a 90 percent adjustment towards their steady state values. It is also interesting to note that the adjustment time of $\hat{Q}$ becomes shorter and that of k becomes larger as $\hat{Q}_o$ increases for a given level of $\hat{Q}^*$.

The adjustment times of $\hat{Q}$ and k are different for different levels of initial values even for the same level of adjustment proportion (ε), while the adjustment time of v is the same for a given level of adjustment proportion regardless of the size of initial disturbance because of

Table 3.3.1

Adjustment Time (in Years) of Neoclassical Economy With Cobb-Douglas Production Function

Values of Parameters: $n=0.015$, $\lambda=0.02$, $\delta=0.0$, $\alpha=0.35$, $s=0.1254$

Adjustment Proportion (ε)	$\hat{Q}_o=0.025$			$\hat{Q}_o=0.0359$			$\hat{Q}_o=0.05$		
	t_v	$t_{\hat{Q}}$	t_k	t_v	$t_{\hat{Q}}$	t_k	t_v	$t_{\hat{Q}}$	t_k
0.1	4.6	20.8	3.4	4.6	4.3	4.7	4.6	2.1	5.8
0.2	9.8	37.8	7.3	9.8	9.2	10.0	9.8	4.7	12.0
0.3	15.7	52.9	11.7	15.7	14.7	15.9	15.7	7.7	18.8
0.4	22.5	67.4	16.8	22.5	21.2	22.8	22.5	11.5	26.4
0.5	30.5	81.9	23.0	30.5	28.9	30.9	30.5	16.3	35.2
0.6	40.3	97.4	30.7	40.3	38.4	40.0	40.3	22.7	45.7
0.7	53.0	115.1	40.7	53.0	50.7	53.5	53.0	31.5	59.0
0.8	70.5	137.4	55.3	70.5	68.2	71.4	70.5	45.2	77.5
0.9	101.2	172.0	81.4	101.2	98.3	102.0	101.2	71.2	108.5

the linearity of its time path. As the initial (non-steady state) position is specified closer to the steady state path (Q_o = 0.0359), however, the adjustment times become closer to each other, since difference in the shapes of time paths is not so influential when the initial disturbance is small.

While for a case where $\hat{Q}^* < \hat{Q}_o$ (that is, $v^* > v_o$) i.e., cases where $\hat{Q}_o = 0.0359$ and $\hat{Q}_o = 0.05$, the adjustment time of $\hat{Q}$ ($t_{\hat{Q}}$) is shorter than adjustment time of $v(t_v)$. The adjustment time of $\hat{Q}$ is larger than adjustment time of v when $\hat{Q}^* > \hat{Q}_o$ (that is, $v^* < vo$) i.e., case where $\hat{Q}_o = 0.025$, which has been discussed above.

Later investigators of adjustment time have modified values of the parameters adopted by R. Sato. K. Sato, especially, argues that the estimates of the adjustment time by R. Sato are not realistic because R. Sato does not consider capital depreciation (i.e., $\delta = 0.0$). In fact, the adjustment time of the variables (v, $\hat{Q}$, and k) becomes shorter as $(1-\alpha)(n+\lambda+\delta)$ increases. Estimates of the adjustment time for more conventional parameter values for a C-D case are given in the discussion of the C.E.S. production function case with a unitary elasticity of substitution.

C.E.S. Production Function Case

Cobb-Douglas production function has a unitary elasticity of substitution. Therefore, the effect of elasticity of substitution (σ) on the adjustment time remains to be investigated. We examine here the role of the elasticity of substitution in determining the adjustment time by adopting a C.E.S. production function for the neolcassical one-sector model.[4] The Cobb-Douglas function is treated here as a special case where $\sigma = 1$ for convenient comparison. Since it is difficult to find analytic solutions, however, we concentrate on numerical analysis.

A C.E.S. production function is written as

$$Q = A\,[\alpha K^{-\rho} + (1-\alpha)E^{-\rho}]^{-\frac{1}{\rho}} \tag{3.3.15}$$

or,

$$q = A\,[\alpha k^{-\rho} + (1-\alpha)]^{-\frac{1}{\rho}}$$

where A, α, and ρ are constant parameters of the function.

The elasticity of substitution between factors (σ) and output elasticities w.r.t. factors (η_K and η_E) are given as follows (2, pp. 52-55).

$$\sigma = \frac{1}{1 + \rho} \tag{3.3.16}$$

---

[4]The C.E.S. production function does not satisfy the Inada condition, which means that a stable steady state path may not exist for certain values of parameters. Despite this fact, it is here assumed that a steady state path exists for the one-sector model. In fact, it is numerically verified that there exists a steady state path and it is locally stable for the values of initial condition and parameters adopted in this analysis. See Burmeister and Dobell (9, pp. 30-36), and Wan (44, pp. 37-40) for details.

and,

$$(3.3.17) \qquad \eta_K = \frac{\partial Q}{\partial K}\frac{K}{Q} = \alpha v^{-\rho} = \alpha v^{1-\frac{1}{\sigma}}$$

$$\eta_E = \frac{\partial Q}{\partial E}\frac{E}{Q} = 1 - \alpha v^{-\rho} = 1 - \alpha v^{1 - \frac{1}{\sigma}}$$

The time path of the capital-output ratio (v) given in equation (3.2.3) is modified by substituting equations (3.3.16) and (3.3.17) into equation (3.2.3).

$$(3.3.18) \qquad \dot{v} = - \eta_E (n + \lambda + \delta) v + \eta_E s$$

$$= - (1-\alpha v^{-\rho}) (n + \lambda + \delta) v + (1 - \alpha v^{-\rho}) s$$

$$= \alpha(n + \lambda + \delta) v^{-\rho+1} - \alpha s v^{-\rho} - (n+\lambda+\delta)v + s$$

$$= (n+\lambda+\delta)v^{2 - \frac{1}{\sigma}} - \alpha s v^{1 - \frac{1}{\sigma}} - (n+\lambda+\delta)v + s$$

The time paths of the growth rate of output ($\hat{Q}$) and the capital per unit of effective labor (k) are expressed in terms of v as follows from equations (3.2.5) and (3.3.17), and equations (3.2.6) and (3.3.15).

$$(3.3.19) \qquad \hat{Q} = \frac{s\,\eta_K}{v} + [(n + \lambda + \delta)\,\eta_E - \delta]$$

$$= (n + \lambda) - \alpha(n + \lambda + \delta)^{1- \frac{1}{\sigma}} + \alpha s^{- \frac{1}{\sigma}}$$

and,

$$(3.3.20) \qquad v = \frac{k}{f(k)}$$

$$= \frac{k}{A[\alpha k^{-\rho} + (1-\alpha)]^{-\frac{1}{\rho}}}$$

By expressing this equation (3.3.20) explicitly for k,

(3.3.21) $$k = \left[\frac{(Av)^{\rho} - \alpha}{1-\alpha}\right]^{\frac{1}{\rho}}$$

$$= \left[\frac{(Av)^{\frac{1}{\sigma} - 1} - \alpha}{1 - \alpha}\right]^{\frac{\sigma}{1-\sigma}}$$

Therefore, the time paths of all the variables in question (v, $\hat{Q}$, and k) can be found by solving the differential equation (3.3.18) and by substituting the solution into equations (3.3.19) and (3.3.21). Since analytic solutions for both the time path and for the adjustment time are difficult to find, however, we have obtained numerical solutions to the differential equation (3.3.19) using a fourth-order Runge-Kutta method (16, pp. 388-399).

Numerical estimates for 90 percent adjustment are given in Table 3.3.2. The adjustment time for the C.E.S. function with $\sigma = 1$, that is for the Cobb-Douglas function, is about one-third of that for the previous R. Sato case (refer to Table 3.3.1). This is simply because $(1-\alpha)(n+\lambda+\delta)$, which has an inverse relationship with adjustment time from equation (3.3.5), (3.3.9) and (3.3.12), increases in this case.

While the adjustment time of the variables is not affected by changes in saving rate for the C-D function case when initial condition is specified in terms of $\hat{Q}_o$,

Table 3.3.2

Adjustment Time (in Years) of Neoclassical Economy with C.E.S. Production Function

Values of Parameters and Initial Position: $\delta=0.05$, $\alpha=0.25$, $\varepsilon=0.9$, $\hat{Q}_o=0.05$

Parameters			Adjustment Time		
Elasticity of Substitution (σ)	Growth Rate of Effective Labor ($n + \lambda$)	Saving Rate (s)	t_v	$t_{\hat{Q}}$	t_k
0.5	0.042	0.10	34.6	28.1	34.6
		0.15	31.4	23.7	31.4
	0.052	0.10	30.9	33.2	30.9
		0.15	28.0	31.2	28.0
1.0	0.042	0.10	33.4	29.5	34.0
		0.15	33.4	29.5	34.0
	0.052	0.10	30.1	31.1	30.0
		0.15	30.1	31.1	30.0
2.0	0.042	0.10	34.3	32.0	35.3
		0.15	37.1	35.1	38.4
	0.052	0.10	30.7	31.7	30.5
		0.15	33.7	34.1	33.4

it is affected by changes in the saving rate for the C.E.S. function with a non-unitary elasticity of substitution.

The magnitude of elasticity of substitution affects the adjustment time of the variables under investigation, but its effect is not large according to our estimates, which are consistent with Ramanathan's estimates (29).

CHAPTER IV

EXTENSION OF THE MODEL: SAVING BEHAVIOR

To this point, the analysis of the adjustment time has been restricted to the neoclassical growth model with a simple saving function, a constant rate of saving to output. The analytic simplicity of this type of saving function is gained, however, only by sacrificing realistic elements of real-life saving behavior and thus the estimated adjustment time may lose much of its practical value.

Consumption (or saving) behavior has been extensively investigated in relation to static macroeconomics by Dusenberry, Friedman, Modigliani and Brumberg, and others (8, pp. 167-197). Kaldor, Pasinetti, and Samuelson and Modigliani, among others (44, pp. 184-214), have attempted to integrate these earlier works into the subsequent development of modern growth theory. As is characteristic of modern growth theory, however, analytic emphasis has been given to the effect of different forms of saving behavior on the steady state path rather than to their effect on the general growth process including both the adjustment process and the steady state path.

More specifically, the effect on the adjustment time of alternative forms of saving behavior has not been investigated in literature except by Furuno (15) who examined a neoclassical model with Pasinetti saving function by numerical methods. Since different forms of saving behavior have different effects on the adjustment time of an economy, further investigations for a variety of saving functions are needed. For this reason three types of saving functions--a Classical saving function, an Ando-Modigliani saving function, and a Kaldor saving function--are adopted for use in the neoclassical model discussed in Chapter III. All other specifications of the neoclassical growth model discussed in Chapter III are maintained in this chapter.

4.1 Classical Saving Function

The Classical saving function makes the saving ratio (s) a function of the profit rate (r). If the reason for saving and investing is to increase future consumption possibilities one might expect that as the rate of return on investment (r) falls with a rising capital-effective labor ratio (K/E), the rate of saving should also fall, since the future consumption payoff is reduced.

A Classical saving function, under this assumption, can be written as (8, pp. 402-06).

$$s = s\ (r) \tag{4.1.1}$$

where s represents an overall saving rate of the economy and $s'(r)$ $(=ds/dr)$ is positive.

Since the rate of return on capital (r) is equal to marginal product of capital $(f'(k))$ in a competitive economy, $r = f'(k)$. From the assumption on the production function of the neoclassical model (equation (3.1.3)), $f'(k)$ decreases as the capital per unit of effective labor (k) increases, that is $f''(k) < o$. Therefore, the overall saving rate of the economy $(s(r))$ is decreasing as k is increasing because $s'(r) > o$. That is,

$$\frac{ds(r)}{dk} < o \tag{4.1.2}$$

Combining this Classical saving function with the neoclassical growth model discussed in Chapter III, we can derive its fundamental equation. By substituting the saving function $(s = s(r))$ into the fundamental equation of the neoclassical economy (equation (3.1.6)),

$$\dot{k} = s(r)\ f(k) - (n + \lambda + \delta)k \tag{4.1.3}$$

This is the fundamental equation of the neoclassical economy with the Classical saving function.

The Classical saving function ensures the existence of a stable equilibrium in the neoclassical one-sector growth model if the production function satisfies the Inada conditions of equation (3.1.3), as Figure 4.1.1 shows. To the left of k^*, $\dot{k} > o$ because $s(r)\ f(k) > (n + \lambda + \delta)k$ and thus k is increasing towards k^*, while to the right of k^*,

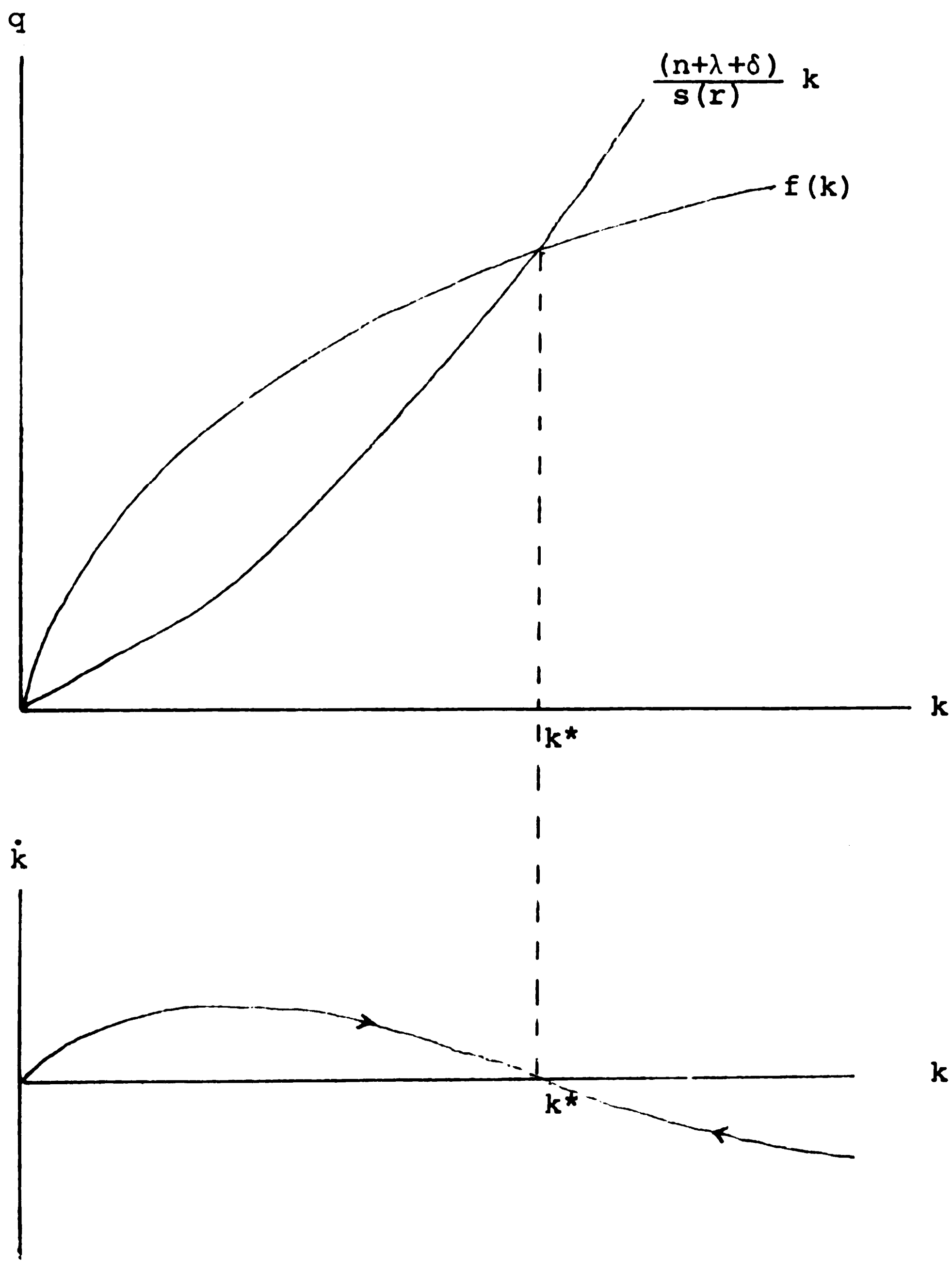

Figure 4.1.1

Equilibrium and Stability of Neoclassical Economy with Classical Saving Function

$\dot{k}<o$ because $s(r)\ f(k)<(n + \lambda + \delta)k$, and thus k is decreasing towards k*. From this fact, it is clear that k* is a stable equilibrium (8, pp. 403-403).

The steady state solution of this economy can be found from the equation (4.1.3) by setting $\dot{k} = o$.

$$s(f'(k^*))\ f(k^*) - (n + \lambda + \delta)k^* = o \tag{4.1.4}$$

where parameters $(n + \lambda)$ and δ represent the growth rate of effective labor (E) and the depreciation rate of capital (K).

Steady state values of the capital-output ratio (v) and the growth rate of output ($\hat{Q}$) are easily obtained as follows.

$$\begin{aligned} \hat{Q}^* &= (f\,\hat{}(k^*)\ E) \\ &= \hat{E} \\ &= n + \lambda \\ &= \text{constant} \end{aligned} \tag{4.1.6}$$

where the variables with star (*) represent steady state values of the variables.

Although the steady state results obtained from the model with the classical saving function are similar to those for the model with the simple constant rate saving function, it should be recognized that the level of k* and v* may be, in general, different for different saving functions. These differences in the level of k^* and v^* are partly related to the difference in adjustment time between two models.

The problem of adjustment time can be more easily analysed using the time path of the capital-output ratio (v) rather than that of the capital-effective labor ratio (k). The time path of v has been derived for the neoclassical model with constant rate of saving in Chapter III, which is equation (3.2.3). We can obtain the time path of v for the neoclassical model with the Classical saving funtion by simply substituting the Classical saving function (equation (4.1.1)) into the equation (3.2.3).

$$\text{(4.1.7)} \qquad \dot{v} = -\eta_E\ (n + \lambda + \delta)v + \eta_E\ s(r)$$

where η_E represents the output elasticity w.r.t. the effective labor.

Cobb-Douglas Production Function Case

When the production function takes a form of Cobb-Douglas function, the time path of v can be simplified as follows.

$$\text{(4.1.8)} \qquad \dot{v} = -\ (1 - \alpha)\ (n + \lambda + \delta)v + (1-\alpha)\ s(r)$$

where $\eta_E = (1-\alpha)$ for a Cobb-Douglas function which takes the form of $Q = AK^{\alpha}\ L^{1-\alpha}$.

When the saving rate (s) depends on the profit rate (r), however, it is difficult to find analytic solutions for the adjustment path and adjustment time even for a Cobb-Douglas production function. In order to simplify the problem we assume, for heuristic purposes, the following particular type of Classical saving function, which is convenient for finding analytic solutions.

(4.1.9) $$s = s_1 - s_2 v^{1}$$

where s_1 and s_2 represent positive constants.

This particular saving function satisfies the property of the Classical saving function that the saving rate is an increasing function of r and thus a decreasing function of v, because r has a negative relationship with v. By adopting this particular form of the Classical saving function, the time path of v can be written as follows for the Cobb-Douglas production function case. By substitution equation (4.1.9) into equation (4.1.8),

(4.1.10) $$\dot{v} = -(1-\alpha)(n + \lambda + \delta)v + (1-\alpha)(s_1 - s_2 v)$$

$$= -(1-\alpha)(n + \lambda + \delta + s_2)v + (1-\alpha)s_1$$

---

[1] This specific form of saving function is adopted only for analytic simplicity. It can be justified, however, if the Classical saving function takes the form of $s = s_1 - a/r$. Since $r = \alpha/v$ for a Cobb-Douglas function, $s = s_1 - av/\alpha = s_1 - s_2 v$ where $s_2 = a/\alpha$.

The solution of this linear differential equation is given by

$$(4.1.11) \qquad v\ (t) = v^* + (v_o - v^*)\ e^{-\ (1-\alpha)\ (n+\lambda+\delta+s_2)\ t}$$

or,

$$v\ (t) = B_1 + B_2\ e^{-(1-\alpha)\ (n+\lambda+\delta+s_2)\ t}$$

where v_o = the initial value of v at $t = o$

v^* = the steady state value of v at $t = \infty$

$$= \frac{s_1}{n + \lambda + \delta + s_2}$$

$B_1 = v^*$

$B_2 = (v_o - v^*)$

Since this time path of v (equations (4.1.10) or (4.1.11)) is the same as that for the constant rate of saving case (equation (3.2.9) or (3.2.10)) except for a minor modification of the coefficient of the equation, the proportional adjustment time of v for this Classical saving function case is easily obtained using equation (3.3.5), which represents the adjustment time for the neo-classical model with constant rate of saving. We can now derive the following equation which represents the adjustment time required for v to cover 100ε percent of initial disturbance towards its steady state path for this Classical saving function case.

$$(4.1.12) \qquad t_v = \frac{1}{(1-\alpha)\ (n+\lambda+\delta+s_2)} \log \frac{1}{(1-\varepsilon)}$$

The time path and adjustment time of the growth rate of output ($\hat{Q}$) can be easily obtained using equations (3.2.12) and (3.3.9), which represent adjustment path and time of $\hat{Q}$ for the constant rate of saving case. The growth path of $\hat{Q}$ for this Classical saving function is derived by using equation (3.2.12).

$$(4.1.13) \qquad \hat{Q} = \frac{\alpha[s_1 - s_2 v]}{v} + [(1-\alpha)\ (n+\lambda+\delta) - \delta]$$

$$= \frac{\alpha s_1}{v} + [(1-\alpha)\ (n+\lambda+\delta+s_2) - s_2 - \delta]$$

Since time path of v is given in equation (4.1.11), the time path of $\hat{Q}$ is obtained by substituting equation (4.1.11) into equation (4.1.13).

The proportional adjustment time of $\hat{Q}$ for this Classical saving function model is derived using equation (3.3.9).

$$(4.1.14) \qquad t_{\hat{Q}} = \frac{1}{(1-\alpha)\ (n+\lambda+\delta+s_2)} \log \left[\frac{B_1 + \varepsilon B_2}{(1-\varepsilon) B_1}\right]$$

The adjustment path and adjustment time of the capital-effective labor ratio (k) can also be easily obtained using equations (3.2.14) and (3.3.12), which represent their relationships for the constant rate of saving case. The growth rate of k for this Classical saving function is derived using equation (3.2.4).

$$k = (Av)^{\frac{1}{1-\alpha}} \tag{4.1.15}$$

Since the time path of v is given in equation (4.1.11), the time path of k is found by substituting equation (4.1.11) into equation (4.1.15).

The proportional adjustment time of k for this Classical saving function model is found using equation (3.3.12), which represents the adjustment time of k for the constant rate of saving model.

$$t_k = \frac{1}{(1-\alpha)(n+\lambda+\delta+s_2)} \log \left[\frac{B_2}{\left[\varepsilon B_1^{\frac{1}{1-\alpha}} + (1-\varepsilon)(B_1+B_2)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} - B_1}\right] \tag{4.1.16}$$

We can determine the effect of this Classical saving function on the adjustment time by comparing the adjustment time formulas for the two cases--the Classical saving function and the constant rate of saving. Each of the adjustment formulas for the Classical saving function model (equations (4.1.12), (4.1.14), and (4.1.16)) has one additional positive constant (s_2) in its denominator, when compared to the adjustment formulas for the constant rate of saving model (equations (3.3.5), (3.3.9), and (3.3.12)). This means that the adjustment time for the Classical saving function model is, other factors being equal, shorter than the adjustment time for the constant rate of saving model. The difference in the adjustment time of the capital-output ratio (v), for example, is given by

$$(4.1.17) \qquad t_v - t_v\ (r) = \frac{s_2}{H} \log[\frac{1}{1-\varepsilon}] > o$$

where t_v and $t_v(r)$ represent the adjustment time of v for the constant rate of saving model and that of the Classical saving function model respectively, and $H = (1-\alpha)(n+\lambda+\delta)(n+\lambda+\delta+s_2)$.

As this equation (4.1.17) indicates, the difference in adjustment time is larger as the value of s_2 increases. This means that as saving rate responds more sensitively to the profit rate, the adjustment time becomes shorter.

4.2 Ando-Modigliani Saving Function

According to Ando and Modigliani, consumption (C) depends on both labor income (W) and net wealth, which is given by capital stock (K) in our simple one-sector economy. Thus, the Ando-Modigliani (A-M) consumption function can be expressed in the following way (3 and 8, p. 409).

$$(4.2.1) \qquad C = c_1W + c_2K$$

where c_1 and c_2 are marginal propensities to consume out of wage income and wealth, respectively, which satisfy the condition that $o<c_2<c_1<1$.

This consumption function can be converted into a saving function in the following manner. Since the economy has been assumed to be full employment equilibrium condition,

(4.2.2) $$S = Q - C$$
$$= Q - c_1W - c_2K$$

Since the economy's total output (Q) is divided into wage income (W) and profit income (P) in the two factors (labor and capital) economy (that is, $Q = W + P$), and profit income is the profit rate on capital (r) multiplied by the amount of capital (K) (that is, $P = rK$),

(4.2.3) $$S = (1 - c_1)\ Q - (c_2 - c_1r)\ K$$

Thus the overall saving rate (s) is given by

(4.2.4) $$s = \frac{S}{Q}$$
$$= (1 - c_1) - (c_2 - c_1\ r)v$$

where v represents the capital-output ratio and it is assumed that $(c_2 - c_1\ r) > o$.

The overall saving rate, as a result, depends on both the profit rate (r) and the capital-output ratio (v). Substituting this overall saving rate into the fundamental equation of the neoclassical economy (equation (3.1.6)), a modified fundamental equation for the A-M saving function can be derived.

(4.2.5) $$\dot{k} = [(1 - c_1) - (c_2 - c_1\ r\ v]\ f\ (k) - (n+\lambda+\delta)k$$

A stable equilibrium exists if $(n+\lambda+\delta)/s$ does not increase as k increases. In general it is assumed for the life-cycle behavior that the saving rate (s) falls as k rises and, thus as v rises. Indeed, this will be the case if $(c_2 - c_1 r) > o$. Therefore, the one sector model with the A-M consumption function has, if we accept the assumption $(c_2 - c_1 r) > o$, a stable equilibrium (8, pp. 410-12).

We derive the time path of the capital-output ratio (v) using the same approach as has been used in the previous analysis, since the investigation of adjustment time can be more easily handled with this equation. Equation (3.2.3) can be used as a starting point in deriving the time path of v. By substituting equation (4.2.4) into equation (3.2.3),

$$
\begin{aligned}
(4.2.6) \quad \dot{v} &= -\eta_E \,(n + \lambda + \delta)v + \eta_E \, s \\
&= -\eta_E \,(n + \lambda + \delta)v + \eta_E \,[(1-c_1) - (c_2-c_1 \, r)v] \\
&= -\eta_E \,[n + \lambda + \delta + (c_2-c_1 \, r)] \, v + \eta_E \,(1 - c_1)
\end{aligned}
$$

Since this differential equation is difficult to solve analytically we concentrate, as before, on the Cobb-Douglas production function case.

Cobb-Douglas Production Function Case

Since the rate of profit on capital (r) is equal to marginal product of capital in competitive economy, r is expressed as follows for the Cobb-Douglas production function.

$$(4.2.7) \qquad r = \frac{\partial Q}{\partial K} = \frac{\alpha}{v}$$

where α is the output elasticity w.r.t. capital.

By substituting this equation into equation (4.2.4),

$$(4.2.8) \qquad s = (1 - c_1) - (c_2 - c_1 r)v = c_3 - c_2 v$$

where $c_3 = 1 - c_1 (1 - \alpha)$.

The resulting saving rate (s) associated with the A-M consumption function takes exactly the same form as the particular type of Classical saving function (equation (4.1.9)) discussed in the previous section 4.1. This means that the effect of the A-M saving function on the adjustment time is the same as that of the particular type of Classical saving function used in the analysis above, even if the magnitude of the effect (that is the reduction of adjustment time) may be different depending on the values of coefficients (s_1 and s_2; c_3 and c_2) of two saving functions.

The adjustment path of the capital-output ratio (v), for example, is obtained by substituting equation (4.2.8) into the equation (4.2.6).

$$(4.2.9) \qquad \dot{v} = -(1 - \alpha)(n + \lambda + \delta + c_2)v + (1 - \alpha)c_3$$

By solving this linear differential equation and by manipulating the solution in the same manner as in section 3.3, we obtain the following formulas for the adjustment time of v.

$$(4.2.10) \quad t_v = \frac{1}{(1 - \alpha)\ (n + \lambda + \delta + c_2)} \log \left(\frac{1}{1 - \varepsilon}\right)$$

Comparison of the adjustment time of v for two cases--a Classical saving function model (equation (4.1.12)) and the Ando-Modigliani saving function model (equation (4.2.10))--indicates that two equations are the same except that c_2 is substituted for s_2. We know from equation (4.2.10) that as the marginal propensity to consume out of wealth (c_2) goes up, the adjustment time is reduced, while the adjustment time is not affected by a change in the marginal propensity to consume out of wage income (c_1) because the equation does not contain c_1.

4.3 Kaldor Saving Function

The idea that it may be fruitful to distinguish between the propensities to save of capitalists and workers, or between the propensities to save out of different kind of income, has a long history in economics (22, p. 143). In recent years, Kaldor (23) has suggested a saving function, originally proposed as a "Keysian alternative theory of distribution," which has become one of the distinguishing characteristics of the Cambridge School.

According to Kaldor, an aggregate saving (S) consists of the saving from wage income (W) and the saving from profit income (P) (8, pp. 406-08), which leads to

$$S = s_w W + s_p P \tag{4.3.1}$$

where it is assumed that the marginal propensity to save out of wage income (s_w) is less than that of profit income (s_p). Since $Q = W + P$ in a two factor economy,

$$S = s_w Q + (s_p - s_w) P \tag{4.3.2}$$

Therefore, the overall saving rate is given as

$$\begin{aligned} s &= \frac{S}{Q} \\ &= s_w + (s_p - s_w) \frac{P}{Q} \\ &= s_w + (s_p - s_w) r v \end{aligned} \tag{4.3.3}$$

Substituting the overall saving rate into the fundamental equation of the neoclassical economy (equation (3.1.6)), we obtain a modified fundamental equation for the Kaldor saving function model.

$$\dot{k} = [s_w + (s_p - s_w) r v] f(k) - (n+\lambda+\delta)k \tag{4.3.4}$$

It is not difficult to prove that the neoclassical one-sector model with the Kaldor saving function has a stable equilibrium (8, pp. 407-08).

In order to investigate the effect of the Kaldor saving function on the adjustment time, we derive the time path of the capital-output ratio (v) by substituting equation (4.3.3) into equation (3.2.3).

$$\text{(4.3.5)} \quad \dot{v} = -\eta_E \ (n + \lambda + \delta)v + \eta_E \ [s_w + (s_p - s_w) \ r \ v]$$

$$= -\eta_E \ [n + \lambda + \delta - (s_p - s_w) \ r] \ v + \eta_E \ s_w$$

In order to determine the adjustment time, it is necessary to specify a production function. The Cobb-Douglas production function, which has been adopted in the previous chapters, is not relevant for the Kaldor saving function since it is reduced to a constant saving rate when the relative factor shares are constant. Recognizing $r = \alpha/v$ for a Cobb-Douglas function,

$$\text{(4.3.6)} \quad s = s_w + (s_p - s_w) \ \frac{\alpha}{v} \ v$$

$$= s_w + \alpha \ (s_p - s_w)$$

$$= \text{constant}$$

<u>C.E.S. Production Function Case</u>

Alternatively, we can assume that the aggregate production function is of the C.E.S. type in order to determine the effect of this specification of the saving function on the adjustment time. When we adopt a C.E.S.

production function, the time path of v takes the following form.[2]

(4.3.7) $$\dot{v} = -\eta_E (n + \lambda + \delta)v + \eta_E s$$
$$= \alpha (n+\lambda+\delta)v^{2-\frac{1}{\sigma}} - \alpha s v^{1-\frac{1}{\sigma}} - (n+\lambda+\delta)v + s$$

where the following C.E.S. function is assumed:

1. The production function takes the form of

$$Q = A [\alpha K^{-\rho} + (1 - \alpha)E^{-\rho}]^{-\frac{1}{\rho}}$$

or,

$$q = A [\alpha k^{-\rho} + (1 - \alpha)]^{-\frac{1}{\rho}}$$

2. The elasticity of substitution between factors is

$$\sigma = \frac{1}{1 + \rho}$$

3. The profit rate is

$$r = \frac{\partial Q}{\partial K} = \alpha v^{-\frac{1}{\sigma}}$$

4. The output elasticities w.r.t. factors are

$$\eta_K = \frac{\partial Q}{\partial K}\frac{K}{Q} = \alpha v^{1-\frac{1}{\sigma}}$$

---

[2] C.E.S. production function does not satisfy the Inada condition, which means that a stable steady state path may not exist for certain values of parameters. Despite this fact, it is here assumed that a steady state path exists. It is numerically verified in fact that there exists a stable equilibrium for the parameters adopted in this investigation. See Burmeister and Dobell (9, pp. 30-36), and Wan (44, pp. 37-40).

$$\eta_E = 1 - \eta_K = 1 - \alpha v^{1 - \frac{1}{\sigma}}$$

The Kaldor saving function takes the following form when the production function is of the C.E.S. type. From Equation (4.3.3),

$$(4.3.8) \qquad \begin{aligned} s &= s_w + (s_p - s_w)\, r\, v \\ &= s_w + (s_p - s_w)\, \alpha v^{-\frac{1}{\sigma}}\, v \\ &= s_w + (s_p - s_w)\, \alpha v^{1 - \frac{1}{\sigma}} \end{aligned}$$

By substituting this overall saving rate into equation (4.3.7), we can get the time path of v for the neoclassical model with the C.E.S. function and the Kaldor saving function.

$$(4.3.9) \qquad \begin{aligned} \dot{v} = &-\alpha^2 (s_p - s_w) v^{2 - \frac{2}{\sigma}} + \alpha(n + \lambda + \delta) v^{2 - \frac{1}{\sigma}} \\ &+ \alpha(s_p - 2s_w)\, v^{1 - \frac{1}{\sigma}} - (n + \lambda + \delta) v + s_w \end{aligned}$$

The time path of the growth rate of output $(\hat{Q})$ is derived starting with equation (3.2.5).

$$(4.3.10) \qquad \hat{Q} = \frac{\eta_K\, s}{v} + [\eta_E\, (n + \lambda + \delta) - \delta]$$

By substituting the expressions for η_K and η_E into the above equation,

$$(4.3.11) \qquad \hat{Q} = (n + \lambda) - \alpha(n + \lambda + \delta) v^{1 - \frac{1}{\sigma}} + \alpha s v^{-\frac{1}{\sigma}}$$

By substituting the saving rate (s) of Kaldor saving function into the equation,

$$(4.3.12) \qquad \hat{Q} = \alpha^2 (s_p - s_w)\, v^{1 - \frac{2}{\sigma}} - \alpha(n + \lambda + \delta) v^{1 - \frac{1}{\sigma}} + \alpha s_w v^{-\frac{1}{\sigma}} + (n + \lambda)$$

The time path of the capital-effective labor ratio (k) is derived from the definition of v.

$$(4.3.13) \qquad v = \frac{k}{f(k)} = \frac{k}{A[\alpha k^{-\rho} + (1 - \alpha)]^{-\frac{1}{\rho}}}$$

By solving this equation explicitly for k ,

$$(4.3.14) \qquad k = \left[\frac{(Av)^{\frac{1}{\sigma} - 1} - \alpha}{1 - \alpha}\right]^{\frac{\sigma}{1-\sigma}}$$

Since the differential equation (4.3.9) is difficult to solve analytically, a numerical method--a fourth-order Runge-Kutta procedure (16)--is adopted to measure the adjustment time of the variables. Estimates of the adjustment time required for an indicator variable to cover 90 percent of its initial displacement are given in Table 4.3.1, where the case with $\sigma = 1$ represents the Cobb-Douglas case and the case with $s_w = s_p$ represents the constant rate of saving case.

The results indicate that an increase in the growth rate of effective labor $(n+\lambda)$, in general, reduces the

Table 4.3.1

Adjustment Time (in Years) of Neoclassical Economy with Kaldor Saving Function

Values of Parameters and Initial Position:
$\alpha=0.25$, $\delta=0.05$, $\varepsilon=.90$, $\hat{Q}_o=0.05$

Parameters				Adjustment Time		
σ	$n+\lambda$	s_w	s_p	t_v	$t_{\hat{Q}}$	t_k
0.5	0.42	0.15	0.15	31.4	23.7	31.4
		0.1	0.3	25.2	19.8	25.2
		0.0	0.6	16.5	13.9	16.5
	0.52	0.15	0.15	28.0	31.2	28.0
		0.1	0.3	22.9	24.9	22.9
		0.0	0.6	15.4	16.2	15.4
1.0	0.42	0.15	0.15	33.4	29.5	34.0
	0.52	0.15	0.15	30.1	31.1	30.0
2.0	0.42	0.15	0.15	37.1	35.1	38.4
		0.1	0.3	47.5	44.7	49.5
		0.0	0.6	84.9	81.1	91.5
	0.52	0.15	0.15	33.7	34.1	33.4
		0.1	0.3	42.3	42.9	41.8
		0.0	0.6	72.2	73.5	70.8

adjustment time regardless of the elasticity of substitution (σ). They also indicate that as the elasticity of substitution increases, other things being equal, the adjustment time increases. The effect of the saving rate on the adjustment time is, on the other hand, quite dependent upon the elasticity of substitution. The adjustment time decreases as $(s_p - s_w)$ increases when $\sigma < 1$, while the adjustment time increases as $(s_p - s_w)$ increases when $\sigma > 1$. The saving rate does not affect the adjustment time when $\sigma = 1$. In conclusion, the adjustment time for the Kaldor saving function model compared to that for the constant rate of saving model, depends upon the elasticity of substitution. The Kaldor saving function raises the adjustment time when $\sigma > 1$, does not affect the adjustment time when $\sigma = 1$, and reduces the adjustment time when $\sigma < 1$.

It is interesting to note that the saving rate of the Kaldor saving function (equation (4.3.3)) is very similar to the saving rate of the Ando-Modigliani saving function (equation (4.2.4)). Both are expressed as a function of the profit rate (r) and the capital-output ratio (v). In fact they may be treated as a category of saving functions for the economy where the physical capital (K) is the only asset and thus the only source of profit income (P). However, they are not identical because the profit income (P) in the Kaldor saving function is not identical to the capital stock (K) in the A-M function, since the profit rate (r) changes as K changes. The difference between the two saving functions has been demonstrated for the Cobb-Douglas production function. While the Kaldor saving rate is constant, the A-M saving rate is a decreasing function of k and thus v .

Despite this difference, some information on the effect of the Ando-Modigliani saving function on the adjustment time for the model with the C.E.S. production function can be inferred from the estimates of the effect of the Kaldor saving function on the adjustment time. For the A-M saving function, the time path of the capital-output ratio (v) can be derived by combining equation (4.2.4) and (4.3.7).

$$(4.3.15)\qquad \dot{v} = -\alpha^2 c_1 v^{2 - \frac{2}{\sigma}} + \alpha(n+\lambda+\delta+c_2)\, v^{2 - \frac{1}{\sigma}} + \alpha(2c_1 - 1)\, v^{1 - \frac{1}{\sigma}} - (n+\lambda+\delta+c_2)v + (1 - c_1)$$

Comparison of two time paths of v--the Kaldor case (equation (4.3.9)) and the A-M case (equation (4.3.15))--indicates that they are in fact the same form of differential equation with different coefficients. Recognizing this fact, we may expect that the A-M saving function affects the adjustment time in a similar to the Kaldor saving function. It should be noticed, however, that the time path of v for the A-M case has one more element in the coefficients of the terms $v^{2 - \frac{1}{\sigma}}$ and v , which may be analytically regarded as the increase in the growth rate of effective labor $(n+\lambda)$. Therefore, we may say that the A-M saving function will have an effect on the adjustment time analogous to the combined effect of the Kaldor saving function and an increase of $(n+\lambda)$. Though the net effect will depend on the magnitude and direction of both effects, we may in general say that the A-M function decreases the adjustment time, compared to that for the constant rate of saving model, when $\sigma \leq 1$ and raises the adjustment time when $\sigma > 1$.

CHAPTER V

EXTENSION OF THE MODEL: INCLUSION OF MONEY

Thus far, our investigation of adjustment time has been based on growth models in which only flows of real goods and stocks of a real capital good have been considered. That is, the models examined so far have been exclusively "real" models without financial assets.

In fact, money is one of the characteristic features of modern economy. The role of money in the capitalist economy has been extensively investigated, especially in the context of "static" macroeconomics. Some theorizing has been attempted in the context of modern growth models. However, the role of money in the modern economy is complex, and existing theory is not satisfactory, especially from the point of view of modern growth models. "The only consensus economists can reach today about monetary growth models is that much work remains to be done" (44, p. 264).

The effect of the inclusion of money in growth models on the adjustment time has not been explicitly analyzed in the literature. In order to illustrate this type of analysis, we will consider three types of monetary growth models.[1] One model focuses on the role of money as a consumption good, while another considers money as a production good. Both of these growth models can be regarded as "neoclassical" in the sense that they assume the presence of money but avoid the problem of capacity utilization and unemployment by continuing to assume automatic equality of planned saving to desired investment. The third model called the "Keynes-Wicksell" type model, assumes that there are independent savings and investment functions. Therefore, in this type of model neither the full utilization of capital nor the full employment of labor is assumed.

In order to simplify the analysis, we concentrate on the adjustment time of the capital-output ratio (v). The analysis for the other variables can be also done, as in the previous chapters, if of interest. It is also assumed for simplicity that no technical progress and no depreciation of capital exists. The extension of the model to include these factors is a simple algebraic matter.

5.1 Money as a Consumption Good

According to portfolio balance theory, the existence of money (M) affects the consumption behavior of consumers, since the public's total wealth consists of not only

---

[1] Numerical illustration is not given in this chapter since it is difficult to estimate values of parameters, within a limited time, defined in the functional relationships of the monetary growth models.

physical capital (K) but also of real balances (M/P) in the two asset world. A monetary growth model based on this idea has been formulated by Tobin (42, 43). Monetary growth models of this type continue, as in the basic neo-classical model discussed in the previous chapters, to avoid the problems of capacity utilization and unemployment by assuming that there are no independently determined investment devices but rather that all saving plans are realized.

Following Stein's representation (38) of Tobin's model (43), let the specifications of the model be given in the following way:

1. Production Function

The aggregate production function is of the neo-classical type satisfying all the Inada conditions.

(5.1.1) $$Q = F(K, L)$$

From the property of linear homogeneity, output per worker (q) is given by

(5.1.2) $$q = f(k)$$

where $k = K/L$.

2. Labor Force Growth

The labor force (L) grows at a constant proportional rate (n).

$$L = L_o \, e^{nt} \tag{5.1.3}$$

3. Price Change

The price level (P) changes at a rate of π which brings equilibrium in the money market.

$$P = P_o \, e^{\pi t} \tag{5.1.4}$$

4. Expected Rate of Inflation

The expected rate of inflation (π_e) is always assumed to be equal to the equilibrium value of the rate of price change (π^*).

$$\pi_e = \pi^* \tag{5.1.5}$$

5. Money Supply

The money supply (M^s) increases at a constant proportional rate (u), which is controlled by the government.[2]

$$M^s = M_o \, e^{ut} \tag{5.1.6}$$

The supply of real balances per worker (m^s) is given by

$$\begin{aligned} m^s &= \frac{M^s}{P\,L} \\ &= \frac{M_o}{P_o L_o} \, e^{(u - n - \pi)t} \end{aligned} \tag{5.1.7}$$

---

[2] All money is assumed to be of the outside type which constitutes debt of the government and has a yield rate specified by the government.

6. Demand for Money

The demand for real balances per worker (m^D) is positively related to the capital per worker (k) and negatively related to the expected rate of inflation (π_e).

(5.1.8) $$m^D = \frac{M}{P\ L}$$

$$= R\ (k,\ \pi_e)$$

where R_1 $(=\partial R/\partial k) > o$ and R_2 $(=\partial R/\partial \pi_e) < o$

7. Consumption Function

Consumption per worker (C/L) is a function of the disposable income per worker, which constitues the sum of output per worker (f(k)) and the increament of real balances per worker $[(\dot{\frac{M}{P}})/L]$.

(5.1.9) $$\frac{C}{L} = g\ [f\ (k) + \frac{(\dot{\frac{M}{P}})}{L}]$$

where g' > o.

By utilizing these assumptions, we can derive a modified version of the fundamental equation for the neo-classical economy with money. To make the analysis more simple, consumption per worker is assumed to be a linear function of disposable income, which is the assumption Tobin adopts in his 1965 article.

(5.1.10) $$\frac{C}{L} = g\ [f\ (k) + \frac{(\frac{\dot{M}}{P})}{L}]$$

$$= (1 - s_d)\ [f\ (k) + \frac{(\frac{\dot{M}}{P})}{L}]$$

where $(1 - s_d)$ is the marginal propensity to consume out of disposable income, which is assumed to be positive. By converting this equation into a saving function,

(5.1.11) $$S = W - L\ \frac{C}{L}$$

$$= Q - L\ (1 - s_d)\ [f\ (k) + \frac{(\frac{\dot{M}}{P})}{L}]$$

$$= s_d\ Q - (1 - s_d)\ (\frac{\dot{M}}{P})$$

Thus, the overall saving rate (s) is given by

(5.1.12) $$s = \frac{S}{Q}$$

$$= s_d - (1 - s_d)\ \frac{(\frac{\dot{M}}{P})}{Q}$$

By differentiating equation (5.1.7) logarithmically, we obtain the following equation.

(5.1.13) $$(\frac{\dot{M}}{P})/L = (u - \pi)\ m^S$$

Substituting equation (5.1.13) into equation (5.1.12), we obtain the overall saving rate in this monetary economy.

(5.1.14) $$s = s_d - (1 - s_d)\ \frac{(u - \pi)\ m^S}{f\ (k)}$$

It is important to note that this overall saving rate is not yet completely specified in the sense that the value and sign of $(u - \pi)$ is undetermined because π is not specified in this neoclassical model. We may assume, however, that $(u - \pi)$ is positive since an increase in real balances will raise consumption and thus reduce saving. Accepting this interpretation, we transform this saving rate into a simplified form, which is essentially the same modification performed by Stein (38, p. 88).

(5.1.15) $$s = s_d - s_m m^s$$

where s_m is positive constant which satisfies that $s_d > s_m m$.

By utilizing the equilibrium condition of the money market we can show that

(5.1.16) $$s = s_d - s_m R(k, \pi_e)$$

This equation implies that the overall saving rate is negatively related to real balances and thus negatively related to k and positively associated with π_e since $R_1 > o$ and $R_2 < o$. That is,

(5.1.17) $$s = s[R(k, \pi_e)]$$

where s' (=ds/dr) $< o$, $\partial s/\partial k < o$, and $\partial s/\partial \pi_e > o$.

The fundamental equation of this monetary growth model is derived as follows. Since $k = K/L$, we obtain the following equation differentiating it logarithmically.

$$(5.1.8) \qquad \hat{k} = \hat{K} - \hat{L}$$

$$= \frac{s\ Q}{K} - n$$

$$= \frac{s\ f(k)}{k} - n$$

or,

$$\dot{k} = s\ f\ (k) - n\ k$$

By substituting equation (5.1.17) into equation (5.1.18), we arrive at the fundamental equation of this monetary growth model.[3]

$$(5.1.19) \qquad \dot{k} = s\ [R\ (k,\ \pi_e)]\ f\ (k) - n\ k$$

We derive the time path of the capital-output ratio (v) since the problem of adjustment time is most easily examined in terms of the time path of v . Since v = k/f(k) we obtain the time path of v by differentiating it logarithmically.

$$(5.1.20) \qquad \hat{v} = \hat{k} - \hat{f}\,(k)$$

$$= \hat{k} - \eta_k\ \hat{k}$$

$$= (1 - \eta_k)\ \hat{k}$$

---

[3]This neoclassical economy has a stable equilibrium when $\pi_e = \pi^*$. See Stein (38, pp. 96-97) and Burmeister and Dobell (9, pp. 166-80).

Substituting equation (5.1.19) into equation (5.1.20),

$$(5.1.21)\quad \hat{v} = (1 - \eta_k)\ [\frac{s\ [R\ (k,\ \pi_e)]\ f\ (k)}{k} - n]$$

$$= \frac{(1 - \eta_k)\ s\ [R\ (k,\ \pi_e)]}{v} - (1 - \eta_k)\ n$$

or,

$$\dot{v} = -\ n\ (1 - \eta_k)\ v + (1 - \eta_k)\ s\ [R\ (k,\ \pi_e)]$$

where $\eta_k = (\partial q/\partial k)\ (k/q)$.

Comparing the time path of v (equation (5.1.21)) for this monetary economy with the time path of v (equation (3.2.3)) for the barter economy discussed in Chapter III, we see that the only difference is in the overall saving rate. While saving rate of the barter economy is constant, the saving rate of the monetary economy is a function of real balances per worker (m), which depends both on capital per worker (k) and on the expected rate of inflation (π_e). Since the saving rate is the only difference, we can determine the adjustment time of this monetary economy by specifying the saving rate in more detail.

When the economy is in a steady state, real balances per worker are constant. Therefore, we know from equation (5.1.7) that

(5.1.22) $$m^* = (u - n - \pi^*) = o$$

or,

$$\pi^* = u - n$$

Since $\pi_e = \pi^*$ from assumption (5.1.5), we obtain

(5.1.23) $$\pi_e = u - n$$

Then the saving rate is represented by

(5.1.24) $$s = s\ [R\ (k,\ u - n)]$$

where $\partial s/\partial k < o$ and $\partial s/\partial (u-n) > o$.

Since (u-n) is an exogeneous constant determined by the government, the saving rate is changed only by a change of k for a given value of (u-n). In other words, the saving rate (s) is decreasing as k is increasing for a given level of (u-n). The analytic behavior of this saving rate is quite similar to that of the Classical saving function (equation (4.1.1)) discussed in Chapter IV, even though the underlying economic reasons are very different. While the saving rate is positively associated with the profit rate on capital and thus is negatively associated with the level of capital per worker (k) for the Classical saving function, the saving rate for this monetary model is negatively related to the demand for real balances and thus negatively related to capital per worker (k).

Since we have already examined the effect of the Classical saving function on the adjustment time, we may use the results given in Chapter IV in determining the adjustment time of this monetary economy. As it is the case for the Classical saving function model, it is difficult to derive any analytic result if the saving function does not take a particular form. If we assume that the saving rate is a linear function of the capital-output ratio (v) as is assumed for the Classical saving function, we know that the adjustment time of the monetary economy can be easily found for a model with a Cobb-Douglas production function.

Suppose that the saving rate is a linear function of the capital-output ratio (v), which may be one possible form of the saving rate because the saving rate is a decreasing function of k, and k has a positive relationship with v.

(5.1.25) $$s = s_1 - s_2 \, (u - n) \, v$$

where s_1 and s_2 are positive constant for given value of $(u - n)$ where $\partial s_2/\partial(u-n) < o$ and thus $\partial s/\partial(u-n) > o$.

Assuming that the production function is of the Cobb-Douglas type, we can obtain the following linear differential equation by substituting equation (5.1.25) into equation (5.1.21).

$$(5.1.26)\quad \dot{v} = -(1-\alpha)n\,v + (1-\alpha)\,[s_1 - s_2\,[u-n]\,v]$$

$$= (1-\alpha)\,[n + s_2\,(u-n)]\,v + (1-\alpha)\,s_1$$

where α is the elasticity of output w.r.t. capital.

Recalling the adjustment time formula (equation (4.1.12)) of the Classical saving function model, we can obtain the following adjustment time formula for this monetary economy with a Cobb-Douglas function.

$$(5.1.27)\qquad t_v = \frac{1}{(1-\alpha)\;(n+s_2)}\log\left[\frac{1}{1-\varepsilon}\right]$$

where n is the growth rate of labor and ε represents the proportion of adjustment covered by time t_v.

Comparing this adjustment time to that of the neo-classical barter economy discussed in Chapter III, it is evident that the introduction of money reduces the adjustment time since the equation (5.1.27) has an additional positive constant (s_2) in the denominator. It is worthwhile, however, to note that the above result is reached partly due to the assumption that the expected rate of inflation (π_e) is equal to the equilibrium value of inflation (π^*). If we specify the price expectation differently, the effect on the adjustment time of this monetary growth model may be different.

Suppose that the government increases the expansion rate of the money supply from u_o to u_1. Then, the expected rate of inflation (π_e) will increase from $(u_o - n)$

to $(u_1 - n)$ since $\pi_e = u - n$. This means that the demand for real balances are reduced for a given capital intensity and thus the saving rate increases since it has a negative relationship with real balances. Therefore, the economy will move to a higher level of capital intensity and a higher level of output per worker (refer to Figure 5.1.1). However, it is important to note that the saving rate gradually decreases from this initial rate as the economy moves to a higher k, since the demand for real balances will increase as k increases. Equation (5.1.27) represents the adjustment time required for the capital-output ratio (v) to cover 100ε percent of the total displacement $(v^* - v_o)$ towards its steady state value (v^*) from its initial value (v_o).

5.2 Money as a Production Good

Among the criticisms of Tobin by Levhari and Patinkin (25), Tobin's negligence of money as a means of production was one major point. Money as an explicit medium of exchange raises, according to them, the productivity of the economy by permitting a more efficient means of distribution and hence a greater rate of production with given aggregate inputs of capital and labor. For this reason, they argue that aggregate output may be, at least, a non-decreasing function of real balances, regardless of whether money is of the inside or of the outside type. The model discussed in this section still

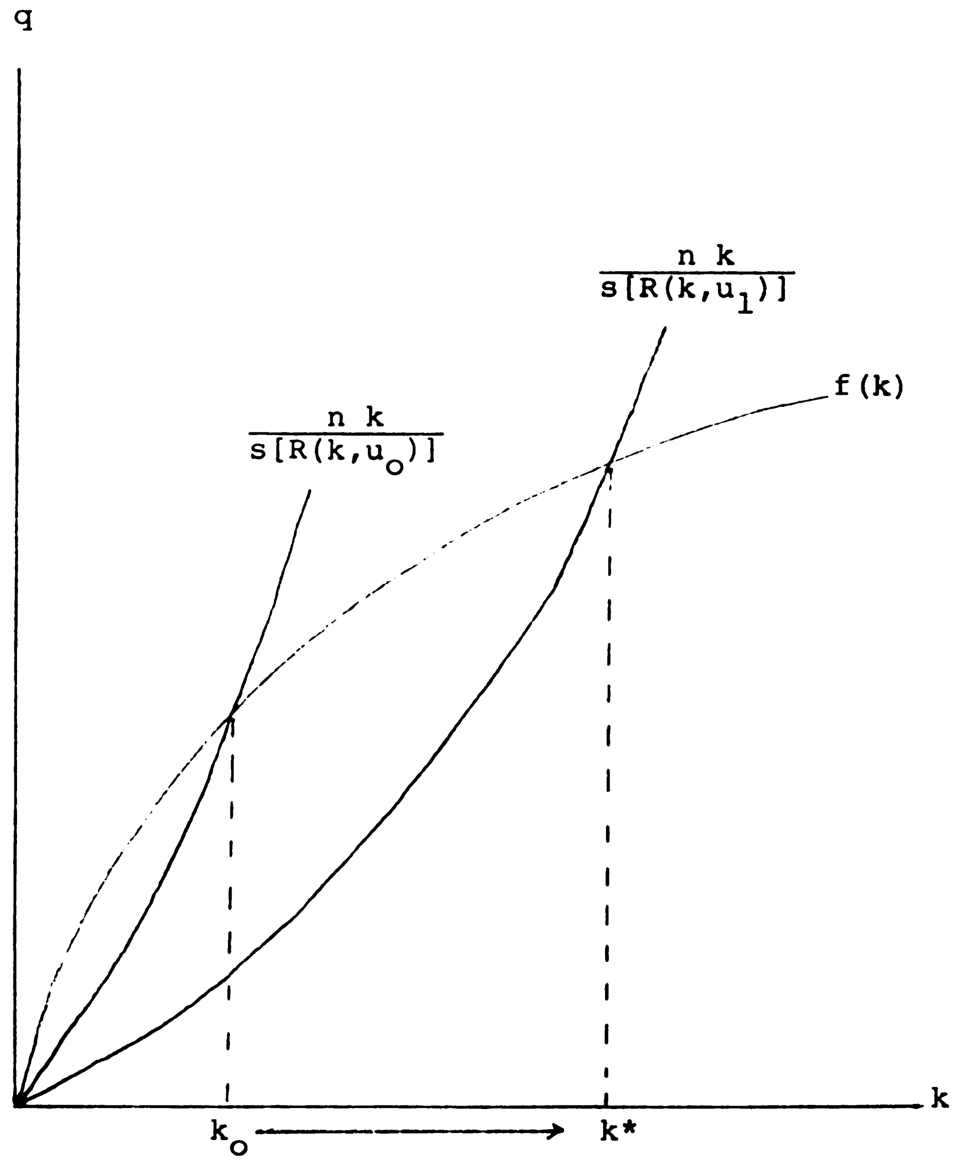

Figure 5.1.1

Equilibrium in Monetary Growth Model of Tobin assumes, following the neoclassical approach, that the rate of capital is identically equal to planned savings and all markets are always in the equilibrium situation regardless of the rate of price change.

Let all the assumptions of the previous monetary growth model in section 5.1 be maintained except that concerning the production function and the saving function. The saving rate is assumed constant as in the barter model of the neoclassical economy. The production function of Levhari and Patinkin's economy can be described as follows.

$$q = f\ (k,\ m) \tag{5.2.1}$$

where $f_1(=\partial f/\partial k) > o$ and $f_2(=\partial f/\partial m) \geqq o$. When the economy already has a fully developed set of financial institutions, the increase in real balances does not affect the productivity of capital per labor (k) and thus $f_2 = o$.

The fundamental equation of this monetary model is derived as follows by substituting equation (5.2.1) into equation (5.1.18),

$$\dot{k} = s\ f\ (k,\ m) - n\ k \tag{5.2.2}$$

In order to find the proportional adjustment time of the capital-output ratio (v) for this model, we first derive the time path of v. From the definition of v,

$$v = \frac{k}{f(k,m)} \tag{5.2.3}$$

By differentiating it logarithmically.

$$(5.2.4) \qquad \hat{v} = \hat{k} - f\ (\hat{k}, m)$$

$$= \hat{k} - \eta_k \hat{k} - \eta_m \hat{m}$$

$$= \hat{k}\ (1 - \eta_k) - \eta_m \hat{m}$$

where $\eta_k = (\partial q/\partial k)\ (k/q)$ and $\eta_m = (\partial q/\partial m)\ (m/q)$.

Substituting equation (5.2.2) into equation (5.2.4), we obtain the time path of v in terms of the following differential equation.

$$(5.2.5) \qquad \hat{v} = (1 - \eta_k)\ [\frac{s\ f\ (k, m)}{k} - n] - \eta_m \hat{m}$$

$$= (1 - \eta_k)\ \frac{s}{v} - [(1 - \eta_k)\ n + \eta_m \hat{m}]$$

or,

$$\dot{v} = -\ [(1 - \eta_k)\ n + \eta_m \hat{m}]\ v + (1 - \eta_k)\ s$$

In order to make the analysis simple, we assume that production function is the Cobb-Douglas type with three factors of production.

$$(5.2.6) \qquad Q = K^{\alpha}\ L^{1-\alpha}\ (\frac{M}{P})^{1-\alpha-\beta}$$

where α and β are constant parameters.

By converting this production function into the intensive form,

$$(5.2.7) \qquad q = \frac{Q}{L}$$

$$= k^{\alpha}\, m^{\,1-\alpha-\beta}$$

where $k = K/L$ and $m = M/(PL)$.

The output elasticities w.r.t. factors are given by

$$(5.2.8) \qquad \eta_k = \frac{\partial q}{\partial k}\,\frac{k}{q} = \alpha$$

$$\eta_m = \frac{\partial q}{\partial m}\,\frac{m}{q} = 1 - \alpha - \beta$$

By substituting these elasticities into equation (5.2.5), we can obtain the following equation.

$$(5.2.9) \qquad \dot{v} = -\,[(1 - \alpha)\, n + (1 - \alpha - \beta)\, \hat{m}]\, v + (1 - \alpha)\, s$$

The time path and the adjustment time of this monetary economy will vary greatly depending on the value of $\hat{m}$, for given values of the other parameters, if $(1-\alpha-\beta) \neq o$. Let us specify $\hat{m}$ by utilizing the assumptions of this model. From the assumptions on the demand for real balances (equation (5.1.7)) and on the expected rate of inflation (equation (5.1.21)),

$$(5.2.10) \qquad m = R\,(k,\, \pi_e) = R\,(k,\, u - n)$$

By differentiating it logarithmically, we derive the proportional rate of change of m.

$$(5.2.11) \qquad \hat{m} = \S_k\, \hat{k} + \S_{(u\,-\,n)}\, (u \hat{-} n)$$

Since $(u - n)$ is assumed an exogeneous constant, $(u \hat{-} n) = o$. Therefore,

$$(5.2.12) \qquad \hat{m} = §_k \hat{k}$$

where $§_k$ $(=(\partial m/\partial k)(k/m))$ is the elasticity of demand for real balances per worker with respect to k, which is assumed to be positive.

By substituting equation (5.2.2) into equation (5.2.12),

$$(5.2.13) \qquad \hat{m} = §_k \left[\frac{s\ f\ (k,\ m)}{k} - n\right]$$

$$= \frac{s\ §_k}{v} - n\ §_k$$

Substituting this equation (5.2.13) into equation (5.2.9), we derive the time path of v for the monetary growth model.

$$(5.2.14) \qquad \dot{v} = - [(1-\alpha)n + (1-\alpha-\beta)\left(\frac{s\ §_k}{v}\right) - n\ §_k)]v + (1-\alpha)s$$

$$= -n[(1-\alpha) - (1-\alpha-\beta)\ §_k]v + s[(1-\alpha) - (1-\alpha-\beta)\ §_k]$$

When $§_k$ is constant, this equation (5.2.14) is a familiar linear differential equation. The adjustment time of v, then, is given by

$$(5.2.15) \qquad t_v = \frac{1}{(1-\alpha)n - (1-\alpha-\beta)n\ §_k} \log \left[\frac{1}{1-\varepsilon}\right]$$

Comparing this equation (5.2.15) with the equation (3.3.5) for the adjustment time of the neoclassical barter economy, this formula has an additional negative constant in the denominator since $(1-\alpha-\beta) > o$ and $§_k > o$. This means that the existence of money increases the adjustment time for this monetary economy compared to that for the barter economy. The existence of money, in this model, makes the economy converge more slowly towards the steady state path. If $(1-\alpha-\beta) = o$, this monetary model becomes identical to the barter economy and thus there will be no change in the adjustment time and steady state solutions.

Suppose that the government raises the expansion rate of money supply from u_o to u_1. The demand for real balances per worker are reduced because the expected rate of inflation ($\pi_e = u-n$) rises from ($u_o - n$) to ($u_1 - n$). As real balances per worker (m) fall, the output per worker ($q = f(k, m)$) goes down. Therefore, the economy moves to a lower level of capital intensity and of output per worker (refer to Figure 5.2.1). The proportional adjustment time towards a new steady state is given by equation(5.2.15).

5.3 Keynes-Wicksell Approach

The analytic simplicity of the neoclassical models examined in the previous sections is based on assumptions which sacrifice reality with respect to disequilibrium phenomena. These are probably of considerable importance in a monetary economy. Several writers such as Rose (30), Stein (38), and Nagatani (27), called proponents of the Keynes-Wicksell approach, have attempted to build models which include such disequilibrium phenomena. The

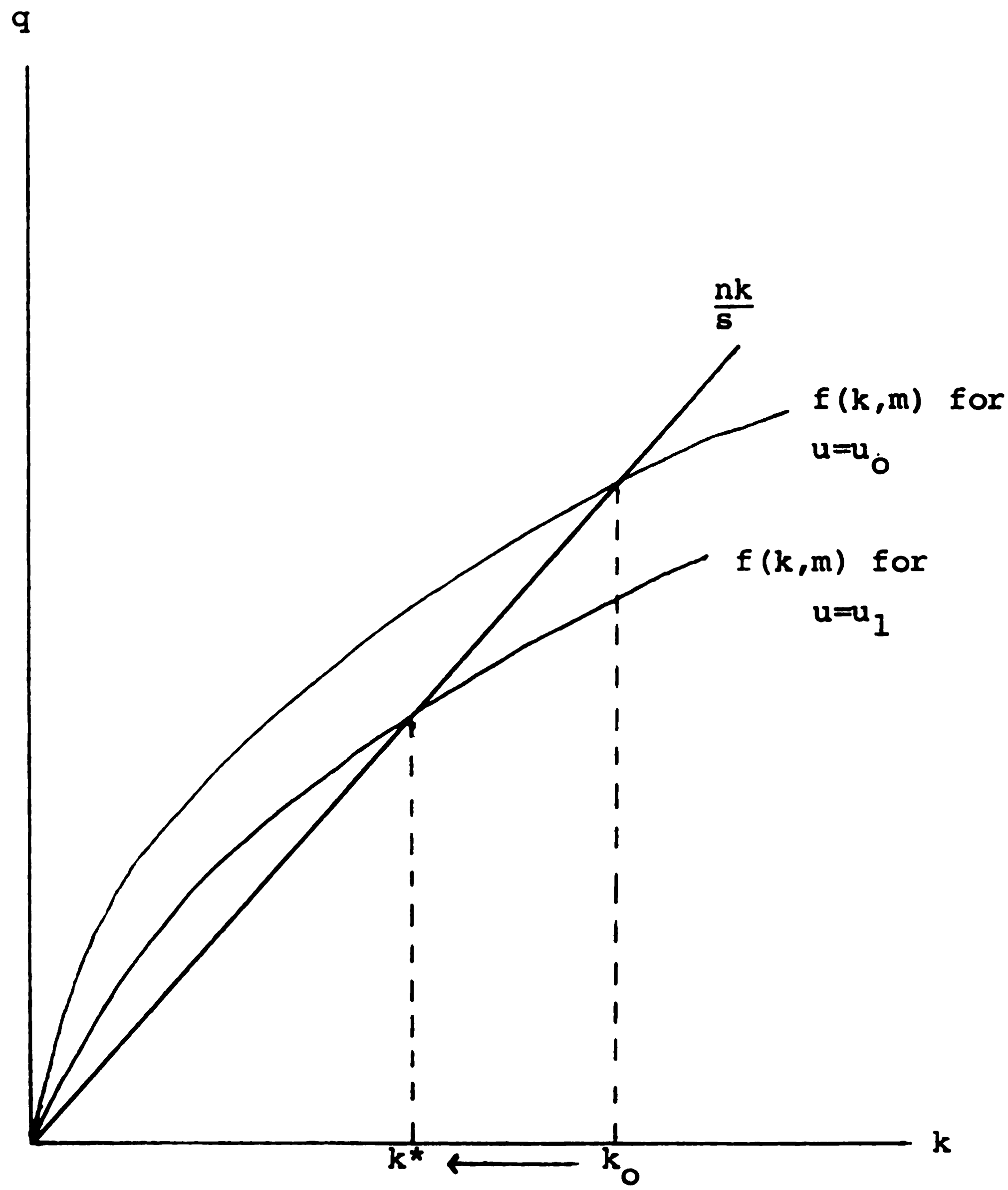

Figure 5.2.1

Equilibrium in Monetary Growth Model of Levhari-Patinkin characteristic features of this approach are (38, p. 97); "(a) prices are changing if, and only if, the goods markets is not in equilibrium and (b) there are independent savings and investment functions."

In this section, we will examine the adjustment time for Stein monetary growth model (38). We assume, following Stein, that the production function is of the neoclassical type and that the specifications of the money and labor markets are the same as those in the previous neoclassical monetary growth models. In other words, all the assumptions of the neoclassical monetary growth model in section 5.1 continue to hold except the equations for capital formation and price change. In addition, the economy is assumed to be in a period of continued inflation, as is assumed by Stein.

(1) The price (P) is changing if and only if there exists disequilibrium in the goods market. The rate of price change (π) is assumed to be proportional to excess aggregate demand deflated by the stock of capital.

$$(5.3.1) \qquad \pi = \lambda\left(\frac{I^d}{K} - \frac{S^d}{K}\right)$$

where I^d and S^d represent the desired level of investment and planned level of saving respectively, and λ is assumed to be a positive constant.

(2) During inflationary periods ($\pi > o$), the desired level of investment (I^d) exceeds the planned level of saving (S^d). Actual saving (S) and investment (I) are

assumed to be less than desired investment but more than planned saving.

$$(5.3.2) \qquad \frac{I}{K} = \frac{S}{K} = \frac{a\ I^d}{K} + (1 - a)\ \frac{S^d}{K}$$

where the coefficient a is institutionally determined such that $o < a < 1$ during the period of excess aggregate demand.

We derive the fundamental equation of Stein model using the above assumptions. Rearranging assumption (5.3.1),

$$(5.3.3) \qquad \frac{I^d}{K} = \frac{\pi}{\lambda} + \frac{S^d}{K}$$

By substituting this equation (5.3.3) into equation (5.3.2),

$$(5.3.4) \qquad \hat{K} = \frac{I}{K} = a\ \left(\frac{\pi}{\lambda} + \frac{S^d}{K}\right) + (1 - a)\ \frac{S^d}{K}$$

$$= \frac{a\pi}{\lambda} + \frac{S^d}{K}$$

Since $S^d = s^d\ Q$,

$$(5.3.5) \qquad \hat{K} = \frac{a\pi}{\lambda} + \frac{s^d\ f(k)}{k}$$

where s^d is the constant rate of the desired saving to output.

Since $\hat{k} = \hat{K} - n$, we derive the fundamental equation of this monetary growth model in the following way.

(5.3.6) $$\hat{k} = \frac{s^d \; f\ (k)}{k} - (n - \frac{a\pi}{\lambda})$$

or,

$$\dot{k} = s^d \; f\ (k) - (n - \frac{a\pi}{\lambda})\ k$$

Let us now consider the adjustment time of this model. The time path of the capital-output ratio (v) is derived as before. Since $v = k/f(k)$, we obtain the following equation by differentiating it logarithmically.

(5.3.7) $$\hat{v} = \hat{k} - \eta_k \hat{k}$$

$$= \hat{k}\ (1 - \eta_k)$$

Substituting equation (5.3.6) into this equation (5.3.7), we obtain the time path of v .

(5.3.8) $$\dot{v} = -\ (1 - \eta_k)\ (n - \frac{a\pi}{\lambda})\ v + (1 - \eta_k)\ s^d$$

Assuming that π is constant, the time path of v is expressed as follows for a Cobb-Douglas production function since $\eta_k = \alpha$.

(5.3.9) $$\dot{v} = -\ (1 - \alpha)\ (n - \frac{a\pi}{\lambda})\ v + (1 - \alpha)\ s^d$$

This is a familiar linear differential equation. When the time path of v is expressed by a linear differential equation, the proportional adjustment time of v is given by

$$(5.3.10) \qquad t_v = \frac{1}{(1-\alpha)\ (n- \frac{a\pi}{\lambda})} \log \left(\frac{1}{1-\varepsilon}\right)$$

Since this equation contains an additional negative constant $(- \frac{a\pi}{\lambda})$ when compared to the time formula for the barter economy, we know that the adjustment time of this model is larger than that of the barter economy if other things are equal.

Therefore, the existence of money raises the adjustment time for the Stein model during the periods of continuous inflation. It is worthwhile to note that this result comes from his rather controversial assumption (equation (5.3.1)) concerning the actual formation of capital.

Suppose that Stein economy enjoys initially a stable price level (π = o) and then goes into an inflationary period, the rate of inflation (π) being constant. In this inflationary period actual saving is larger than planned saving and thus the economy will move to a higher steady state situation which is represented in Figure 5.3.1. Equation (5.3.10) represents the adjustment time (t_v) required for the capital-output ratio (v) to cover 100ε percent of the initial displacement ($v^* - v_o$) towards its steady state value (v^*) from its initial value (v_o) for the Stein monetary growth model.

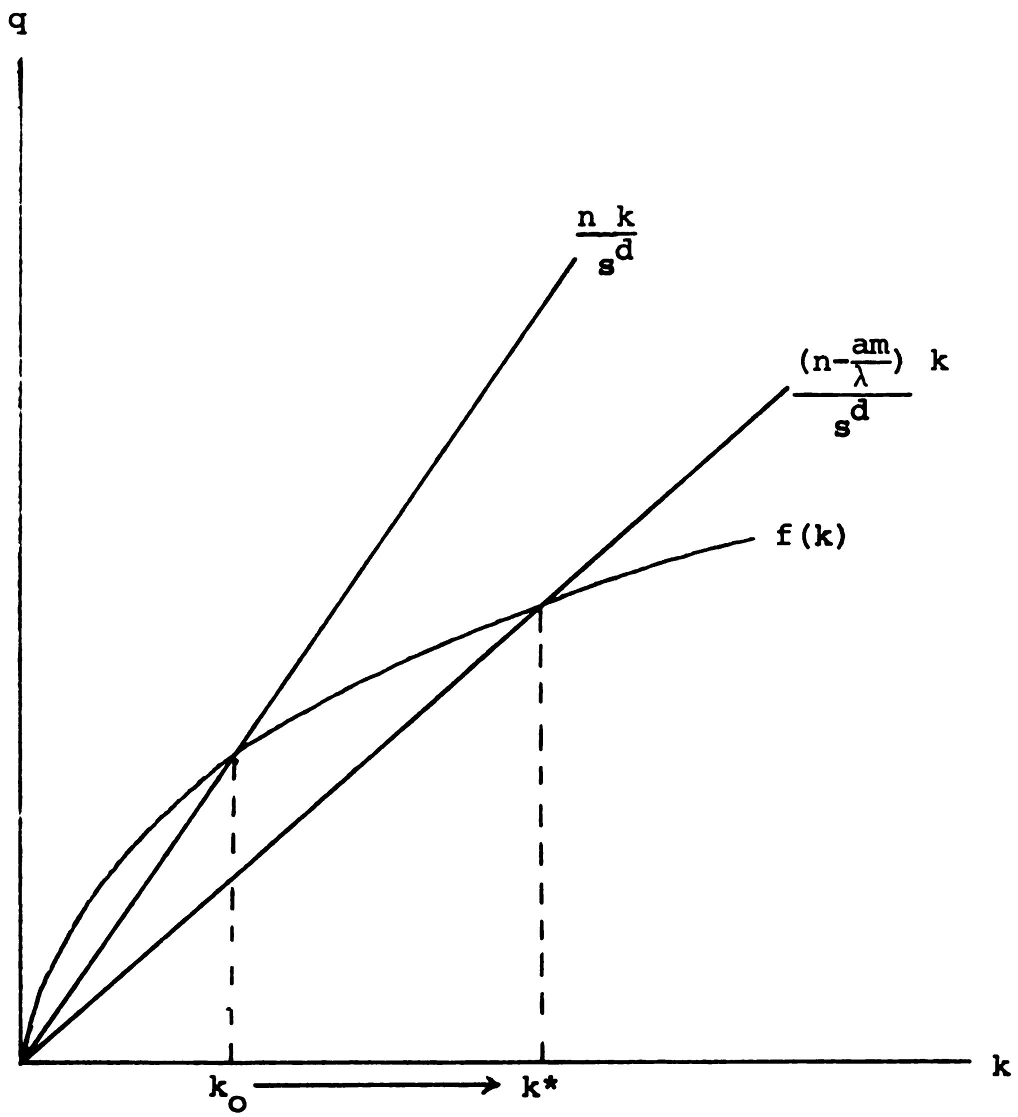

Figure 5.3.1

Equilibrium in Monetary Growth Model of Stein

CHAPTER VI

CONCLUSIONS

In this study the adjustment time towards a steady state path--the time path which explains the long run trend of growth in the economy--has been investigated for several models of economic growth. Adjustment time is of considerable importance because the relevance of a steady state solution as an approximation to reality depends crucially on the length of time the model economy takes to adjust to its steady state path after a disturbance occurs.

Initially two related conceptual problems--the first concerning the measurement of adjustment time and the second the choice of an indicator variable--are raised and discussed. A proportional measure of adjustment time is adopted for use in the study despite some drawbacks associated with it. In particular its use leads to problems regarding the choice of an indicator variable which represents the adjustment process of the entire system, since the time paths of adjustment towards a steady state are generally different for different variables within a system.

Several possible indicator variables have been examined, but ultimately the choice of an indicator variable depends on the nature and purpose of the analysis. It has been demonstrated analytically and numerically for several neoclassical growth models that the adjustment times of indicator variables, specifically the capital-output ratio (v), the growth rate of output ($\hat{Q}$), and the capital-effective labor ratio (k) are quite different from each other, especially when the economy is initially far from its steady state paths. It is also noted that for a neoclassical model with a Cobb-Douglas production function the adjustment time required to cover a given proportion of total displacement is constant for the capital-output ratio regardless of the size of the initial displacement. On the other hand the proportional adjustment time of the growth rate of output decreases and that of the capital-effective labor ratio increases as the initial value of the growth rate of output increases. It is also shown that the adjustment time of the output-capital ratio (1/v) is the same as that of the growth rate of output ($\hat{Q}$) for this particular neoclassical model with a Cobb-Douglas technology.

Second, it has been shown that the adjustment time responds very sensitively to changes in values of the parameters of a system. It has been demonstrated that the growth rate of labor forces, the rate of technical progress, the rate of capital depreciation, and the labor share of total output are very influential on the adjustment time. Their inverse relationship with the adjustment time has been analytically derived for a neoclassical model with a Cobb-Douglas technology and a numerical illustration has been given adopting a C.E.S. production function. It has also been shown numerically that changes in the elasticity of substitution between factors of production affects the adjustment time. As the value of the elasticity of substitution becomes larger, the adjustment time generally increases.

It has been shown, in fact, that R. Sato's argument that a neoclassical economy takes 100 years for 90 percent adjustment towards its steady state (which, if true, reduces much of the practical value of neoclassical models) can easily be attacked by slightly raising the values of any of the above parameters. Criticisms of R. Sato by later investigators such as K. Sato and Conlisk are partly based on such modifications of parameter values.

Third, it has been demonstrated that the adjustment time depends sensitively on the specification of savings behavior used in a growth model. The constant savings rate of the standard neoclassical model is, in general, not an influential factor, and its level has no impact on adjustment time when the production function of the model is of a Cobb-Douglas type and the initial position of the economy is expressed in terms of the growth rate of output. It has been shown, however, that the use of either a

Classical, an Ando-Modigliani, or Kaldor saving function in a growth model has a significant influence on the adjustment time of the economy. The magnitude and direction of the effect depend on the elasticity of substitution of the production function.

It has been demonstrated that the use of either a Classical or an Ando-Modigliani saving function reduces significantly the adjustment time for a neoclassical model with a Cobb-Douglas production function, while the introduction of a Kaldor saving function has no effect because it is reduced to a constant rate of saving in the Cobb-Douglas case. The more sensitive the savings rate is to the adjustment in the economy, the more the adjustment time is reduced.

Numerical calculations have been given for a Kaldor saving function, adopting a C.E.S. production function. When the elasticity of substitution is less than unity, the Kaldor saving function leads to an adjustment time shorter than that for a model with a constant rate of saving. Results of the analysis also indicate that as the marginal propensity to save out of profit increases the adjustment time becomes shorter. When the elasticity of substitution is greater than unity the savings behavior represented by the Kaldor saving function implies a larger adjustment time, and in this case adjustment time increases as the marginal propensity to save out of profit increases. It has been also mentioned that similar results may hold for an Ando-Modigliani saving function because of the similarity of two saving functions.

Fourth, three different types of monetary growth models have been investigated, partly to demonstrate that the adjustment time depends on the specification of a model as well as on values of its parameters, and partly to demonstrate that the existence of money affects adjustment time. It has been shown that the adjustment time is reduced, under certain assumptions, for a monetary growth model which includes money as a consumption good and thus the existence of money affects the overall saving rate of economy.

The adjustment time increases, on the other hand, for a monetary growth model in which money is considered to be a production good. In this case the existence of money affects production in the economy, even when levels of labor and capital are constant. The adjustment time also increases for the third model called a Keynes-Wicksell type model, which allows, in contrast to the above models, independent saving and investment functions and, as a result, both actual saving and investment are affected by the existence of money.

Finally, in addition to the above results, relationships among time paths and adjustment times of different indicator variables are explicitly derived. Using these relationships, previous findings in this area of adjustment time by several researchers choosing different indicator variables as a representative of a model are organized in a more consistent frame of reference. In conducting this investigation, method of analysis is also more systemized by adopting the analytic procedure by which time paths of indicator variable are directly derived from the so-called fundamental equation of a growth model. Therefore, the method adopted here eliminates some difficulties associated with different analytic procedures for different indicator variables and different specifications of growth models.

In conclusion, the adjustment time of an economic growth model depends on various factors such as the indicator variable chosen as a representative of the model, the specification of the model, and the values of the model parameters. Based on the estimates given in this dissertation for example, it ranges from 14 to 172 years for a 90 percent adjustment towards the steady state path of growth. Therefore, it may not be possible to give a proper judgment on the practical relevance of steady state solutions as an approximation to actual economic growth until we investigate this problem of adjustment time for a variety of models, especially for growth models which reflect more properly the actual economy. As more realistic models of economic growth are developed, the analysis of their adjustment time should accompany the investigation of their steady state properties until the validity of steady state solutions as an approximation of reality is established. If the adjustment time proves to be too long, it will be necessary to investigate the dynamic path of the model economy in full, including both the adjustment path and the steady state path in the analysis of the economic growth process.

REFERENCES

REFERENCES

1. Allen, R. G. E. Mathematical Analysis for Economists. New York: St. Martin's Press, 1967.

2. Allen, R. G. D. Macro-Economic Theory: A Mathematical Treatment. London: Macmillan, 1968.

3. Ando, A. and F. Modigliani. "The 'Life Cycle' Hypothesis of Saving:Aggregate Implication and Test." American Economic Review, March 1963, pp. 55-84.

4. Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. H. Solow. "Capital Labor Substitution and Economic Efficiency." Review of Economics and Statistics, August 1961, pp. 225-50.

5. Atkinson, A. B. "The Timescale of Economic Models: How Long is the Long-Rune?" Review of Economic Studies, April 1969, pp. 137-52.

6. Blaug, M. Economic Theory in Retrospect. 2nd edition. Illinois: Irwin, Homewood, 1968.

7. Boyce, W. E. and R. C. DiPrima. Introduction to Differential Equations. New York: John Wiley and Sons, 1970.

8. Branson, W. H. Macroeconomic Theory and Policy. New York: Harper and Row, 1970.

9. Burmeister, E. and A. R. Dobell. Mathematical Theories of Economic Growth. New York: Collier-Macmillan, 1970.

10. Conlisk, J. "Unemployment in a Neoclassical Growth Model: The Effect on Speed of Adjustment." Economic Journal, September 1966, pp. 550-66.

11. Conlisk, T. "The Analysis and Testing of the Asymtotic Behavior of Aggregate Growth Models." Doctoral Dissertation, Stanford University, Stanford, 1965.

12. Denison, E. F. Why Growth Rates Differ: Post-War Experience in Nine Western Countries. Washington: Brookings Institute, 1967.

13. Domar, E. D. "Capital Expansion, Rate of Growth and Employment." Econometrica, April, 1946, pp. 137-47.

14. Domar, E. D. Essays in the Theory of Economic Growth. New York: Oxford University Press, 1957.

15. Furuno, Y. "Convergence Time in the Samuelson-Modigliani Model." Review of Economic Studies, April 1970, pp. 221-32.

16. Haggerty, G. B. Elementary Numerical Analysis with Programming. Boston: Allyn and Bacon, 1972.

17. Hahn, F. H. and R. C. O. Matthews. "The Theory of Economic Growth: A Survey." In Surveys in Economic Theory. Vol. 2. American Economic Association/Royal Economic Society, eds. London: Macmillan, 1965.

18. Hamberg, D. Models of Economic Growth. New York: Harper and Row, 1971.

19. Harrod, R. F. "An Essay in Dynamic Theory." Economic Journal, March 1939, pp. 14-33.

20. Hicks, J. R. Capital and Growth. Oxford: Oxford University Press, 1965.

21. Inada, K. "On a Two-Sector Model of Economic Growth: Comments and a Generalization." Review of Economic Studies, June 1963, pp. 119-27.

22. Jones, H. G. An Introduction to Modern Theories of Economic Growth. New York: McGraw-Hill, 1976.

23. Kaldor, N. "Alternative Theories of Distribution." in Readings in the Modern Theory of Economic Growth. J. Stiglitz and H. Uzawa, eds. Cambridge: M.I.T. Press, 1969.

24. Kaldor, N. "Marginal Productivity and the Macro-Economic Theories of Distribution." Review of Economic Studies, October 1966, pp. 309-19.

25. Levhari, D. and D. Patinkin. "The Role of Money in a Monetary Economy." American Economic Review, September 1968, pp. 713-53.

26. Meade, J. E. A Neo-Classical Theory of Economic Growth. London: Allen and Unwin, 1961.

27. Nagatani, K. "A Monetary Growth Model with Variable Employment." Journal of Money, Credit, Banking, May 1969, pp. 188-206.

28. Ramanathan, R. "Adjustment Time in the Two-Sector Growth Model with the Fixed Coefficients." Economic Journal, December 1973, pp. 1236-44.

29. Ramanathan, R. "The Elasticity of Substitution and the Speed of Convergence in Growth Models." Economic Journal, September 1975, pp. 612-13.

30. Rose, H. "Unemployment in a Theory of Growth." International Economic Review, September 1966, pp. 260-82.

31. Samuelson, P. A. "Parable and Realism in Capital Theory: The Surrogate Production Function." Review of Economic Studies, June 1962, pp. 193-206.

32. Sato, K. "On the Adjustment Time in Neoclassical Growth Models." Review of Economic Studies, July 1966, pp. 263-68.

33. Sato, R. "Fiscal Policy in a Neoclassical Growth Model: An Analysis of the Time Required for Equibrating Adjustment." Review of Economic Studies, February 1963, pp. 13-23.

34. Sato, R. "The Harrod-Domar Model versus the Neo-classical Growth Models." Economic Journal, June 1964, pp. 380-87.

35. Sen, A. K., ed. Growth Economics. Harmondsworth: Penguin, 1970.

36. Solow, R. M. "A Contribution to the Theory of Economic Growth." Quarterly Journal of Economics, February 1956, pp. 65-94.

37. Solow, R. M. Growth Theory: An Exposition. Oxford: Oxford University Press, 1970.

38. Stein, J. L. "Monetary Growth Theory in Perspective." American Economic Review, March 1970, pp. 85-106.

39. Stiglitz, J. E. and H. Uzawa, eds. Readings in the Modern Theory of Economic Growth. Cambridge: M.I.T. Press, 1969.

40. Swan, T. W. "Economic Growth and Capital Accumulation." Economic Record, November 1956, pp. 334-61.

41. Takayama, A. Mathematical Economics. Hinsdale: Dryden Press, 1974.

42. Tobin, J. "A Dynamic Aggregative Model." Journal of Political Economy, April 1955, pp. 103-15.

43. Tobin, J. "Money and Economic Growth." Econometrica, October 1965, pp. 671-84.

44. Wan, H. Y. Economic Growth. New York: Harcourt Brace Jovanovich, 1971.