SOME SUBGROUPS OF THE AUTOMORPHISM GROUP OF A FINITE ABELIAN GROUP

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

WILLIAM L. HIGHTOWER

1970

This is to certify that the

thesis entitled

Some Subgroups of the Automorphism Group

of a Finite Abelian Group

presented by

William L. Hightower

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

J. E. Adney

Major professor

Date August 6, 1970

ABSTRACT

SOME SUBGROUPS OF THE AUTOMORPHISM GROUP
OF A FINITE ABELIAN GROUP

By

William L. Hightower

A main objective of this thesis is to determine the structure of certain subgroups of the automorphism group of a finite abelian p-group. This is done by analyzing their action on certain series of subgroups of the abelian p-group.

For $s: G=G_0 \geq G_1 \geq \cdots \geq G_t=1$ a chain of subgroups of the group G, define

$$S_0(s) = \{\alpha \in Aut(G) \mid G_i^{\alpha}=G_i \text{ for } i=0,1,\ldots,t\} \text{ and}$$

$$S_{i+1}(s)=\{\alpha \in S_i(s) \mid g^{-1}g^{\alpha} \in G_{i+1} \text{ for all } g \in G_i\},$$

$$i=0,1,\ldots,t-1.$$

The chain $S_0(s) \geq S_1(s) \geq \cdots \geq S_t(s)$ is known as the stability series of s and $S_t(s)$ as the stability group of s. P. Hall has shown $S_t(s)$ to be nilpotent and it is well known that, for s a composition series in a solvable group, $S_0(s)$ is solvable.

The following two theorems from Chapter II on stability series are of some interest in their own right:

If G is a finite abelian p-group and if $s: G=G_0>G_1>\cdots>G_t=1$ is a composition series, there exist elements $a_i \in G_i$ and subgroups $H_i \leq G_{i+1}$ such that $G_i=\langle a_i\rangle x H_i$ and $G_{i+1}=\langle a_i^p\rangle x H_i$, $i=0,1,\ldots,t-1$. If, for some i with $0\leq i<t$, $\langle a_i\rangle$ is not a direct factor of G or if there is an element $h \in \Phi(G)$ such that $a_i h \in G_{i+1}$ then $S_i(s)=S_{i+1}(s)$.

If the finite group G is supersolvable and $s: g=G_0>G_1>\cdots>G_t=1$ is a chief series, then $S_0(s)$ is supersolvable.

The following notation is used throughout Chapter III:

G is a finite abelian p-group and $G=H_1 x\cdots x H_t$ where H_i is homocyclic of exponent p^{n_i} and $[H_i:\Phi(H_i)]=p^{d_i}$ with $n_i>n_{i+1}$ for $i=1,\ldots,t$, taking $n_{t+1}=0$. Define $f_0=0$ and $f_i=\sum_{j=1}^{i} d_j$ for $i=1,\ldots,t$ and $d=f_t=\sum_{i=1}^{t} d_i=[G:\Phi(G)]$. Let $H_i=\langle a_{f_{i-1}+1}\rangle x\cdots x\langle a_{f_i}\rangle$ where $\operatorname{ord}(a_{f_{i-1}+1})=\cdots=\operatorname{ord}(a_{f_i})$ for $i=1,\ldots,t$. Then $G=\langle a_1\rangle x\cdots x\langle a_d\rangle$ and $\operatorname{ord}(a_1)\geq\cdots\geq\operatorname{ord}(a_d)>0$. For $q\geq 0$, $0<r\leq d$ and $m = qd+r$, define $G_m=\mathcal{U}_{q+1}(\langle a_1\rangle x\cdots x\langle a_r\rangle)\ x\ \mathcal{U}_q(\langle a_{r+1}\rangle x\cdots x\langle a_d\rangle)$ and $G_0=G$. Let s_1 denote the series $G_0\geq G_1\geq\cdots$. For $q\geq 0$, $0<r\leq t$ and $m = qt+r$, define $M_m = \mathcal{U}_{q+1}(H_1 x\cdots x H_r)\ x\ \mathcal{U}_q(H_{r+1} x\cdots x H_t)$ and $M_0=G$. Let s_2 denote the chain $M_0\geq M_1\geq\cdots$.

Using this notation and that for stability series given above, the following theorems are established in Chapter III:

(1) $S_{n_1 d}(s_1)$ is a p-Sylow subgroup of Aut(G) with order p^n where $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i(d_i+1)/2)$.

(2) $N_{Aut(G)}(S_{n_1 d}(s_1)) = S_o(s_1)$ which is supersolvable and has order $(p-1)^d p^n$ with n as in (1).

(3) If H_1 is cyclic, $C_{Aut(G)}(S_{n_1 d}(s_1)) = \{\alpha \in Aut(G) \mid a_i^{\alpha} = a_i^m$ for $(m,p) = 1$ and $1 \leq i \leq d\}$ which has order $(p-1)p^{(n_1-1)}$.

(4) If H_1 is not cyclic, $C_{Aut(G)}(S_{n_1 d}(s_1)) = \{\alpha \in Aut(G) \mid a_1^{\alpha} = a_1^m h$; $a_i^{\alpha} = a_i^m$ if $1 < i \leq d$, with $(m,p)=1$ and $h \in \Omega_1(\langle a_{d_1} \rangle)\}$ which has order $(p-1)p^{n_1}$.

(5) The intersection, W, of the p-Sylow subgroups of Aut(G) is equal to $S_{n_1 t}(s_2)$ which has order p^n where

$n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$.

(6) If G is not elementary abelian, $C_{Aut(G)}(W) = \{\alpha \in Aut(G) \mid a_i^{\alpha} = a_i^m h_i$ for $1 \leq i \leq d_1$ and $a_i^{\alpha} = a_i^m$ for $d_1 < i \leq d$ where $(m,p)=1$ and $h_i \in \Omega_1(H_1)$ for $1 \leq i \leq d_1\}$ which has order $(p-1)p^{(n_1+d_1^2-2)}$.

Also in Chapter III, the Fitting subgroup of $S_o(s_1)$, the Fitting subgroup of Aut(G) and the maximal normal solvable subgroup of Aut(G) are characterized by showing how their elements act upon the elements a_i. In each case the order of the subgroup is obtained, giving criteria for the nilpotence or solvability of Aut(G).

It is then shown that Aut(G) is supersolvable if, and only if, G is elementary abelian of order 4 or $ord(a_1) > ord(a_2) > \cdots > ord(a_t)$, i.e., the invariants of the abelian p-group G are all distinct.

The final section of Chapter III contains the following two theorems on the existence of normal Hall subgroups of the automorphism group of a finite abelian p-group:

If Aut(G) is not supersolvable, then Aut(G) has no proper non-trivial normal Hall subgroup.

If G is not elementary abelian of order 4 and if Aut(G) is supersolvable but not nilpotent, with $ord(Aut(G)) = \lambda\mu$ where $(\lambda,\mu) = 1$, $\mu \neq 1 \neq \lambda$ and p divides λ, then Aut(G) has a normal subgroup of order λ and does not have a normal subgroup of order μ.

Chapter IV consists of the extension of some of the results of Chapter III to the case where G is a finite abelian group and not necessarily of prime power order.

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By

William L. Hightower

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1970

ACKNOWLEDGMENTS

I would like to express my deep feeling of gratitude toward my major professor, Dr. J. E. Adney, for his patience and encouragement in the preparation of this thesis. His cheerful willingness to give of his time in spite of a busy schedule has been especially appreciated. For his guidance and direction in the research leading to this thesis I am greatly indebted.

TABLE OF CONTENTS

INTRODUCTION

All groups referred to herein are finite.

An abelian group G is the direct product of its Sylow subgroups and its automorphism group is the direct product of the automorphism groups of the Sylow subgroups of G. Hence many results concerning automorphism groups of abelian p-groups can be extended, in some fashion, to automorphism groups of abelian groups in general.

A major objective of this thesis is to determine properties of the p-Sylow subgroup of the automorphism group of an abelian p-group.

In section 1.3 the order of the automorphism group of an abelian p-group is quoted from P. Hall, [4], p. 65. Also in this section are a number of conditions related to a cyclic subgroup of an abelian p-group G being a direct factor of G.

Section 1.4 deals with the definition and elementary properties of the stability series of a chain of subgroups of a group.

Section 1.5 contains some elementary properties of homocyclic abelian p-groups and the result that if G is a homocyclic abelian p-group, then any automorphism of $\mathcal{V}_k(G)$ or of $G/\mathcal{V}_k(G)$ can be extended to an automorphism of G.

Section 2.1 gives conditions for the coincidence of consecutive terms of the stability series of a composition series of an abelian p-group. In the following section the supersolvability of the group of automorphisms which fix a chief series of a supersolvable group is established.

Throughout Chapter III G denotes an abelian p-group. In the first section two series, s_1 and s_2, of subgroups of G are defined. s_1 contains a composition series and is a refinement of s_2 which is characteristic. The stability group of s_1 is shown to be a p-Sylow subgroup of Aut(G). Its normalizer is shown to be the group of automorphisms which fix s_1. The centralizer of this p-Sylow subgroup of Aut(G) is also characterized as is Z(Aut(G)).

The intersection, W, of the p-Sylow subgroups of Aut(G) is proven to be the stability group of s_2. C(W) is characterized, showing in the process that it is supersolvable if G is not elementary abelian.

Characterizations of the Fitting subgroup of the group of automorphisms which fix s_1 and of the Fitting subgroup of Aut(G) show these to be the product of Z(Aut(G)) with the stability group of s_1 and s_2 respectively, excepting the latter when G is elementary abelian of order 4 or 9. Necessary and sufficient conditions for Aut(G) to be nilpotent are obtained.

The maximal normal solvable subgroup of Aut(G) is characterized and in the process is shown to be supersolvable if

$p>3$. Necessary and sufficient conditions for Aut(G) to be solvable are obtained.

Section 3.6 contains the result that Aut(G) is supersolvable if, and only if, all of the invariants of G are distinct or G is elementary abelian of order 4.

In the final section of Chapter III it is shown that if Aut(G) is not supersolvable, then Aut(G) has no proper non-trivial normal Hall subgroup and if Aut(G) is supersolvable but not nilpotent and if G is not elementary abelian of order 4, then Aut(G) has a normal Hall subgroup of each possible order divisible by p but no non-trivial ones of orders not divisible by p.

In Chapter IV some of the results of Chapter III are extended to Aut(G) where G is any abelian group. In particular, necessary and sufficient conditions for Aut(G) to be solvable, supersolvable or nilpotent, respectively, are given as well as necessary and sufficient conditions for Aut(G) to have a normal Hall subgroup of a given order.

The reader is asked to consult the index of notation for identification of symbolic notations of groups, sets and relations.

CHAPTER I

BASIC DEFINITIONS AND PROPERTIES

This chapter contains definitions and properties which are used in the chapters which follow.

1.1 Abelian groups.

If G is an abelian group, it is the direct product of its Sylow subgroups and Aut(G) is the direct product of the automorphism groups of the Sylow subgroups of G (cf. M. Hall [3], p. 85). Hence an analysis of Aut(G) can be undertaken by analyzing the automorphism groups of abelian p-groups.

1.2 p-groups.

Throughout this section G is a p-group. $\Phi(G)$, the Frattini subgroup of G, is the intersection of the maximal subgroups of G. The first part of the following theorem is the Burnside Basis Theorem (cf. M. Hall [3], pp. 176-178).

Theorem 1.1: The factor group $G/\Phi(G)$ is elementary abelian. If its order is p^r then every set of elements $z_1,\ldots,z_s$ which generates G contains a subset of r elements $x_1,\ldots,x_r$ which generate G. In the mapping $G\to G/\Phi(G)$, the elements $x_1,\ldots,x_r$ are mapped onto a basis of $G/\Phi(G)$. Conversely any set of r elements of G which, in $G\to G/\Phi(G)$ is

mapped onto a set of generators of $G/\Phi(G)$, will generate G. The order of Aut(G), where $ord(G)=p^n$, divides $p^{r(n-r)}\prod_{i=1}^{r}(p^r-p^{i-1})$ and the order of the group of automorphisms which fix $G/\Phi(G)$ elementwise, is a divisor of $p^{r(n-r)}$.

Definition 1.2: Where the order of $G/\Phi(G)$ is p^r and the set of r elements $x_1,\ldots,x_r$ generates G, $x_1,\ldots,x_r$ is called a minimal basis of G.

For the following theorem G need not be a p-group.

Theorem 1.3: If $G=HxK$, $\Phi(G)=\Phi(H)x\Phi(K)$, (cf. Scott, [10], Theorem 7.3.23, p. 164).

Lemma 1.4: If H is a subgroup of G and K is a subgroup of H, with $[H:K]=p$, then $[H\Phi:K\Phi]=p$ or 1. If $[H\Phi:K\Phi]=p$, then $[H\cap\Phi:K\cap\Phi]=1$ and if $[H\Phi:K\Phi]=1$, then $[H\cap\Phi:K\cap\Phi]=p$.

Proof: $[H\Phi:K\Phi]=(|H|\cdot|K\cap\Phi|)/(|K|\cdot|H\cap\Phi|)=p|K\cap\Phi|/|H\cap\Phi|$. Since $K\subset H$, $K\cap\Phi\subseteq H\cap\Phi$ and hence $|K\cap\Phi|/|H\cap\Phi|\leq 1$. Thus $[H\Phi:K\Phi]\leq p$ and since $[H\Phi:K\Phi]$ is a power of p, $[H\Phi:K\Phi]=p$ or 1. Also, $[H\cap\Phi:K\cap\Phi]=p/[H\Phi:K\Phi]$. Hence, if $[H\Phi:K\Phi]=p$, then $[H\cap\Phi:K\cap\Phi]=1$ and if $[H\Phi:K\Phi]=1$, then $[H\cap\Phi:K\cap\Phi]=p$.

Lemma 1.5: If H is a subgroup of G and K is a subgroup of H, with $[H:K]=p$, and if $H\Phi=K\Phi$, then $H=K(H\cap\Phi)$.

Proof: $(H\cap\Phi)K\subseteq H$. If $H\cap\Phi\subseteq K$, then $H\cap\Phi\subseteq K\cap\Phi$ and hence $K\cap\Phi=H\cap\Phi$ contrary to Lemma 1.4. Therefore, $H\cap\Phi\not\subseteq K$ and $K\subset(H\cap\Phi)K\subseteq H$. Since p is a prime, $(H\cap\Phi)K=H$.

Definition 1.6: If A is a group of automorphisms of G and H is a subgroup of G then $[H,A]=\langle\{h^{-1}h^{\alpha} \mid h\in H \text{ and } \alpha\in A\}\rangle$. For any positive integer n, $[H,A^{n+1}]=[[H,A^{n}],A]$.

Theorem 1.7: If A is a p'-group of automorphisms of G, then $[G,A^2]=[G,A]$ (cf. Gorenstein, [2], Theorem 5.3.6, p. 181).

Theorem 1.8: If A is a p'-group of automorphisms of the group G, then $[G,A^n]=[G,A]$ for each positive integer n.

Proof: The conclusion holds for n=1 since $[G,A]=[G,A]$. If $[G,A^k]=[G,A]$, then $[G,A^{k+1}]=[[G,A^k],A]=[[G,A],A]=[G,A^2]=[G,A]$ by Theorem 1.7.

1.3 Abelian p-groups.

Throughout this section G is an abelian p-group.

Theorem 1.9: There exist elements of G $a_i\neq 1$, i=1,...,d, such that $G=\langle a_1\rangle x\ldots x\langle a_d\rangle$. The number d is invariant as are the numbers $\text{ord}(a_i)$, i=1,...,d, except for their arrangement. (cf. M. Hall, [3], Theorems 3.3.1 and 3.3.2, pp. 40 and 41).

Definition 1.10: If each $a_i\neq 1$, for $1\leq i\leq d$, and if $G=\langle a_1\rangle x\ldots x\langle a_d\rangle$, then $a_1,\ldots,a_d$ is a direct basis of G.

From the definition and Theorem 1.1, if $a_1,\ldots,a_d$ is a direct basis of G and if $[G:\Phi(G)]=p^r$, then d=r and $a_1,\ldots,a_d$ is a minimal basis of G. In general, however, it is not the case that every minimal basis of G is a direct basis.

Example 1.11: Let $G=\langle a\rangle x\langle b\rangle$ where $ord(a)=p^2$ and $ord(b)=p$. Then a, ab is a minimal basis of G but not a direct basis since $ord(a)=ord(ab)=p^2$ and the order of a direct product is equal to the product of the orders of the direct factors.

Definition 1.12: An abelian p-group is homocyclic if it has a direct basis each of whose elements has the same order.

Theorem 1.13: If $G=H_1x\ldots xH_t$ where H_i is homocyclic of exponent p^{n_i} and $[H_i:\Phi(H_i)]=p^{d_i}$ with $n_i>n_{i+1}$ for $1\leq i\leq t$ and $n_{t+1}=0$ and if $f_i=\sum_{j=1}^{i}d_j$ for $1\leq i\leq t$, then $ord(Aut(G))=mp^n$ where $n=\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i(d_i+1)/2)$ and $m=\prod_{i=1}^{t}(\prod_{j=1}^{d_i}(p^j-1))$ (cf. P. Hall, [4], p. 65).

Definition 1.14: For a p-group H and a non-negative integer i define $\Omega_i(H)=\langle\{h\in H\mid h^{p^i}=1\}\rangle$ and $\mathcal{V}_i(H)=\langle\{h^{p^i}\mid h\in H\}\rangle$.

$\Omega_i(H)$ and $\mathcal{V}_i(H)$ are characteristic subgroups of H. If G is an abelian p-group, then $\Omega_i(G)=\{g\in G\mid g^{p^i}=1\}$ and $\mathcal{V}_i(G)=\{g^{p^i}\mid g\in G\}$.

Theorem 1.15: If $G=H_1x\ldots xH_t$ is an abelian p-group, then $\Omega_m(G)=\Omega_m(H_1)x\ldots x\Omega_m(H_t)$ and $\mathcal{V}_m(G)=\mathcal{V}_m(H_1)x\ldots x\mathcal{V}_m(H_t)$, for each non-negative integer m.

Proof: If $g=\prod_{i=1}^{t}h_i\in\Omega_m(G)$ where $h_i\in H_i$, $1\leq i\leq t$, then $1=g^{p^m}=\prod_{i=1}^{t}h_i^{p^m}$ and hence $h_i^{p^m}=1$ for all i, $1\leq i\leq t$, whence $g\in\Omega_m(H_1)x\ldots x\Omega_m(H_t)$.

If $g=\prod_{i=1}^{t} h_i \in \Omega_m(H_1)x\ldots x\Omega_m(H_t)$ where $h_i \in \Omega_m(H_i)$ for $1\leq i\leq t$, then ${h_i}^{p^m}=1$ for $i=1,\ldots,t$. Hence $g^{p^m}=\prod_{i=1}^{t} {h_i}^{p^m}=1$ and $g \in \Omega_m(G)$.

Therefore, $\Omega_m(G)=\Omega_m(H_1)x\ldots x\Omega_m(H_t)$.

If $g \in \mathcal{V}_m(G)$, then there is an element $x \in G$ such that $g=x^{p^m}$. $x=\prod_{i=1}^{t} h_i$ for some $h_i \in H_i$, $1\leq i\leq t$. Hence $g=\prod_{i=1}^{t} {h_i}^{p^m}$ $\in \mathcal{V}_m(H_1)x\ldots x\mathcal{V}_m(H_t)$.

If $g=\prod_{i=1}^{t} h_i \in \mathcal{V}_m(H_1)x\ldots x\mathcal{V}_m(H_t)$ where $h_i \in \mathcal{V}_m(H_i)$ for $1\leq i\leq t$, then there are elements $x_i \in H_i$ such that $h_i={x_i}^{p^m}$ for $1\leq i\leq t$. Hence $g=(\prod_{i=1}^{t} x_i)^{p^m} \in \mathcal{V}_m(G)$.

Therefore, $\mathcal{V}_m(G)=\mathcal{V}_m(H_1)x\ldots x\mathcal{V}_m(H_t)$.

Theorem 1.16: If G is an abelian p-group and $i\leq j$ are non-negative integers, then $\mathcal{V}_i(G/\mathcal{V}_j(G))=\mathcal{V}_i(G)/\mathcal{V}_j(G)$.

Proof: Let $x\mathcal{V}_j(G) \in \mathcal{V}_i(G)/\mathcal{V}_j(G)$. Then there is an element $g \in G$ such that $g^{p^i}=x$. Hence $x\mathcal{V}_j(G)=(g\mathcal{V}_j(G))^{p^i} \in$ $\mathcal{V}_i(G/\mathcal{V}_j(G))$.

Let $x\mathcal{V}_j(G) \in \mathcal{V}_i(G/\mathcal{V}_j(G))$. Then there is an element $g \in G$ such that $(g\mathcal{V}_j(G))^{p^i}=x\mathcal{V}_j(G)$. Hence $x \in g^{p^i}\mathcal{V}_j(G)$ and there is an element $h \in G$ such that $x=g^{p^i}h^{p^j}=$ $(gh^{p^{(j-i)}})^{p^i} \in \mathcal{V}_i(G)$. Thus $x\mathcal{V}_j(G) \in \mathcal{V}_i(G)/\mathcal{V}_j(G)$.

Therefore, $\mathcal{V}_i(G/\mathcal{V}_j(G))=\mathcal{V}_i(G)/\mathcal{V}_j(G)$.

Theorem 1.17: If G is of exponent p^n and if r and s are integers with $0 \leq r \leq s \leq n$, then $\mathcal{V}_{n-r}(G) \leq \Omega_s(G)$. (cf. P. Hall, [5], p. 79).

Theorem 1.18: $\mathcal{V}_1(G) = \Phi(G)$, since G is an abelian p-group. (cf. Huppert, [7], Satz 3.14, p. 272).

The following two theorems give the relationship between the Frattini index of G and that of a maximal subgroup.

Theorem 1.19: If H is a maximal subgroup of G, then $[G:\Phi(G)] \geq [H:\Phi(H)]$.

Proof: Let $[G:\Phi(G)] = p^d$. Since $\Phi(G) \leq H$, $H/\Phi(G)$ is elementary abelian of order p^{d-1}. Let $a_1, \ldots, a_{d-1}$ be representatives in H of a minimal basis of $H/\Phi(G)$. Let $a_d \in G-H$. Since $H \supseteq \Phi(G)$, $a_d \notin H\Phi(G)$ and $a_1\Phi(G), \ldots, a_d\Phi(G)$ is a minimal basis of $G/\Phi(G)$. Hence $a_1, \ldots, a_d$ is a minimal basis of G.

$\langle a_1, \ldots, a_{d-1}, a_d^p \rangle \leq H$ since $a_d^p \in H$. Let $x \in H$. Then $x = a_1^{\alpha_1} \cdots a_{d-1}^{\alpha_{d-1}} m$ where $m \in \Phi(G)$, since $a_1\Phi(G), \ldots, a_{d-1}\Phi(G)$ is a minimal basis for $H/\Phi(G)$. By Theorem 1.18 $m \in \mathcal{V}_1(G) = \langle a_1^p, \ldots, a_d^p \rangle \leq \langle a_1, \ldots, a_{d-1}, a_d^p \rangle$. Thus $x \in \langle a_1, \ldots, a_{d-1}, a_d^p \rangle$ and $H \leq \langle a_1, \ldots, a_{d-1}, a_d^p \rangle$. Therefore, $H = \langle a_1, \ldots, a_{d-1}, a_d^p \rangle$. By Theorem 1.1 $[H:\Phi(H)] \leq p^d$.

Theorem 1.20: If H is a maximal subgroup of G, where $[G:\Phi(G)] = p^d$, and if $\Phi(G) = \Phi(H)$, then $[H:\Phi(H)] = p^{d-1}$. If $\Phi(H) \subset \Phi(G)$, then $[H:\Phi(H)] = p^d$.

Proof: If $\Phi(H)=\Phi(G)$, then $[H:\Phi(H)]=[H:\Phi(G)]=[G:\Phi(G)]/[G:H]=p^d/p=p^{d-1}$.

If $\Phi(H)\subset\Phi(G)$, then $[H:\Phi(H)]>[H:\Phi(G)]=p^{d-1}$. By Theorem 1.19, $[H:\Phi(H)]\leq p^d$ and since $[H:\Phi(H)]$ is a power of p, $[H:\Phi(H)]=p^d$.

It is not generally the case that for an arbitrary subgroup of G one can choose a direct basis for G in such a way that the subgroup is the direct product of subgroups of the cyclic groups generated by the elements of the direct basis. However, as indicated by the next theorem, this can be done when the subgroup is maximal.

Example: Let $G=\langle a\rangle x\langle b\rangle$ where $ord(a)=p^3$ and $ord(b)=p$. Let $H=\langle a^p b\rangle$. If H is the direct product of subgroups of the cyclic groups generated by the elements of some direct basis, then, since H is cyclic, H is a subgroup of the cyclic group generated by some element, say c, of a direct basis. By Theorem 1.9, $ord(c)=p$ or p^3. Since $ord(H)=p^2$, $ord(c)=p^3$. Hence $H=\langle c^p\rangle$. But $c^p\in\Phi(G)$ and $a^p b$ is not an element of the maximal subgroup $\langle a\rangle$ and hence $a^p b\notin\Phi(G)$. Hence H is not a subgroup of the cyclic group generated by an element of a direct basis of G.

Theorem 1.21: If H is a maximal subgroup of G, where $[G:\Phi(G)]=p^d$, then there is a direct basis $a_1,\ldots,a_d$ of G such that $H=\langle a_1^p\rangle x\langle a_2\rangle x\cdots x\langle a_d\rangle$.

Proof: Let $g_1,\ldots,g_d$ be a direct basis of G with $\text{ord}(g_i)\leq\text{ord}(g_{i+1})$ for $i=1,\ldots,d-1$. Let $j=\min\{i|g_i\notin H\}$. $\{i|g_i\notin H\}\neq\emptyset$ since $H\subset G$. Since $[G:H]=p$, there is an integer n_i, with $1\leq n_i\leq p$ and $h_i\in H$ such that $g_j^{n_i}h_i=g_{j+i}$ for $i=1,\ldots,d-j$.

Since $\text{ord}(g_j)\leq\text{ord}(g_{j+i})$ and $\langle g_j\rangle\cap\langle g_{j+i}\rangle=1$, $\text{ord}(h_i)=\text{ord}(g_j^{-n_i}g_{j+i})=\text{ord}(g_{j+i})$ for $1\leq i\leq d-j$. Hence $\text{ord}(G)=\prod_{i=1}^{d}\text{ord}(g_i)=(\prod_{i=1}^{j}\text{ord}(g_i))(\prod_{i=1}^{d-j}\text{ord}(h_i))$.

Let $x\in G$. Then $x=\prod_{i=1}^{d}g_i^{m_i}=[\prod_{i=1}^{j-1}g_i^{m_i}][\prod_{i=1}^{d-j}(g_j^{-n_i}g_{j+i})^{m_{j+i}}]$

$[g_j^{m_j+\sum_{i=1}^{d-j}n_im_{j+i}}]=[\prod_{i=1}^{j-1}g_i^{m_i}][g_j^{(m_j+\sum_{i=1}^{d-j}n_im_{j+i})}][\prod_{i=1}^{d-j}h_i^{m_{j+i}}]$

$\in\langle g_1,\ldots,g_j,h_1,\ldots,h_{d-j}\rangle$. Hence $G=\langle g_1,\ldots,g_j,h_1,\ldots,h_{d-j}\rangle$. Since also $\text{ord}(G)=[\prod_{i=1}^{j}\text{ord}(g_i)][\prod_{i=1}^{d-j}\text{ord}(h_i)]$, $g_1,\ldots,g_j,h_1,\ldots,h_{d-j}$ is a direct basis of G.

By definition of j, $g_i\in H$ for $i<j$ and $h_i\in H$ for $i=1,\ldots,d-j$. Since $[G:H]=p$, $K=\langle g_1\rangle\times\cdots\times\langle g_{j-1}\rangle\times\langle g_j^p\rangle\times\langle h_1\rangle\times\cdots\times\langle h_{d-j}\rangle\subseteq H$. Since $\text{ord}(K)=\text{ord}(H)$, $K=H$. The result now follows upon relabeling.

Corollary 1.22: If H is a maximal subgroup of G, where $[G:\Phi(G)]=p^d$, then there is an element $g\in G$ and a subgroup K of H such that $G=K\times\langle g\rangle$ and $H=K\times\langle g^p\rangle$.

Proof: By Theorem 1.21 there is a direct basis $a_1,\ldots,a_d$ of G such that $H=\langle a_1^p\rangle\times\langle a_2\rangle\times\cdots\times\langle a_d\rangle$. Let $g=a_1$ and $K=\langle a_2\rangle\times\cdots\times\langle a_d\rangle$.

The next series of theorems deals with conditions relating to a subgroup being a direct factor.

Lemma 1.23: If $G=Kx\langle g\rangle$ and if $g\in\Phi(G)$, then $g=1$. (cf. M. Hall, [3], Theorem 10.4.1, p. 156).

Theorem 1.24: If H is a maximal subgroup of G, then $\Phi(G)=\Phi(H)$ if, and only if, H is a direct factor of G.

Proof: Suppose H is a direct factor of G. Then, since $[G:H]=p$, $G=HxK$ where K is cyclic of order p. Hence $\Phi(K)=1$ and by Theorem 1.3, $\Phi(G)=\Phi(H)x\Phi(K)=\Phi(H)$.

Suppose $\Phi(G)=\Phi(H)$ and $[G:\Phi(G)]=p^d$. By Theorem 1.21, there is a direct basis $a_1,\ldots,a_d$ of G such that $\langle a_1^p\rangle x\langle a_2\rangle x\cdots x\langle a_d\rangle=H$. $a_1^p\in\Phi(G)=\Phi(H)$. By Lemma 1.23, $a_1^p=1$. Hence $H=\langle a_2\rangle x\cdots x\langle a_d\rangle$ which is a direct factor of G.

Theorem 1.25: If H is a subgroup of G, with $G=\langle g\rangle xH$ and if $h\in\Phi(G)$, then $\text{ord}(gh)\geq\text{ord}(h)$.

Proof: By Theorems 1.3 and 1.18, since $h\in\Phi(G)$, there is an element $k\in\Phi(H)$ and an integer n such that $h=g^{pn}k$. Let $m=\text{ord}(gh)$. Then $1=[g(g^{pn}k)]^m=g^{(pn+1)m}k^m$. Hence $g^{(pn+1)m}=1$ and $k^m=1$. Let $\text{ord}(g)=p^i$. Since $(p^i,pn+1)=1$, p^i divides m and hence $g^m=1$. Thus, $g^{pnm}=1$ and $k^m=1$. Hence $(g^{pn}k)^m=h^m=1$. Therefore, ord(h) divides m and $\text{ord}(h)\leq m=\text{ord}(gh)$.

Theorem 1.26: If H is a subgroup of G, with $\langle g\rangle xH=G$ and $h\in\Phi(G)$, then $\langle gh\rangle$ is a direct factor of G if, and only if, $ord(h)\leq ord(g)$.

Proof: By Theorems 1.3 and 1.18, since $h\in\Phi(G)$, there is an element $k\in\Phi(H)$ and an integer n such that $h=g^{pn}k$. Let $ord(g)=p^i$. Since $(pn+1,p^i)=1$, there is an integer j such that $j(pn+1)\equiv 1 \bmod p^i$. Let m be an integer and $y\in H$. Then $g^m y=[g(g^{pn}k)]^{mj}k^{-mj}y=(gh)^{mj}k^{-mj}y\in\langle gh\rangle H$. Therefore, $G=\langle gh\rangle H$.

Suppose $ord(h)\leq ord(g)$. Then since G is an abelian p-group, $ord(gh)\leq ord(g)$ and $ord(\langle gh\rangle \cap H)=ord(\langle gh\rangle)\ ord\ (H)/ord(G)\leq ord(\langle g\rangle)\ ord(H)/ord(G)=ord(\langle g\rangle \cap H)=1$. Hence $ord(\langle gh\rangle \cap H)=1$ and $G=\langle gh\rangle xH$.

Suppose $\langle gh\rangle$ is a direct factor of G. Then by Theorem 1.25 $ord(g)=ord(ghh^{-1})\geq ord(h^{-1})=ord(h)$.

Theorem 1.27: If $g\in G$, then either $g\in\Phi(G)$ or there is an element $h\in\Phi(G)$ such that $\langle gh\rangle$ is a direct factor of G.

Proof: Let $a_1,\ldots,a_d$ be a direct basis of G with $ord(a_i)\leq ord(a_{i+1})$ for $i=1,\ldots,d-1$. Then $g=\prod_{i=1}^{d} a_i^{n_i}$. If $g\notin\Phi(G)$ then there is an i, $1\leq i\leq d$, such that p does not divide n_i. Let $j=\max\{i \mid p \text{ does not divide } n_i\}$. Let $A=\{i \mid p \text{ does not divide } n_i\}$ and $B=\{i \mid p \text{ divides } n_i\}$. Let $a=\prod_{i\in A} a_i^{n_i}$ and $b=\prod_{i\in B} a_i^{n_i}$. Then $g=ab$ and by definition of B, $b\in\Phi(G)$.

Claim that $a_1,\ldots,a_{j-1},a,a_{j+1},\ldots,a_d$ is a direct basis of G.

By definition of j, $\text{ord}(a)=\text{ord}(a_j)$. Hence $\text{ord}(G)=\prod_{i=1}^{d}\text{ord}(a_i)=\text{ord}(a)\prod_{i\neq j}\text{ord}(a_i)$.

Let $x\in G$. Then $x=\prod_{i=1}^{d}a_i^{m_i}$. Since p does not divide n_j, there is an integer k such that $kn_j\equiv m_j \bmod \text{ord}(a_j)$. Then $x=\prod_{i=1}^{d}a_i^{m_i}=[\prod_{i\in B}a_i^{m_i}][\prod_{i\in A-\{j\}}a_i^{(m_i-kn_i)}][\prod_{i\in A-\{j\}}a_i^{kn_i}]a_j^{kn_j}\in \langle a_1,\ldots,a_{j-1},a,a_{j+1},\ldots,a_d\rangle$ since $\prod_{i\in A}a_i^{kn_i}=a^k$. Hence $G=\langle a_1,\ldots,a_{j-1},a,a_{j+1},\ldots,a_d\rangle$ and since $\text{ord}(G)=\text{ord}(a)\prod_{i\neq j}\text{ord}(a_i)$, $a_1,\ldots,a_{j-1},a,a_{j+1},\ldots,a_d$ is a direct basis for G.

Thus $\langle a\rangle$ is a direct factor of G. Take $h=b^{-1}\in\Phi(G)$. Then $gh=a$ and $\langle gh\rangle$ is a direct factor of G.

Corollary 1.28: If $g\in G$ and $\langle g\rangle$ is not a direct factor of G, then $g\in\Phi(G)$ or there is a $k\in\Phi(G)$ and $h\in G$ such that $\langle h\rangle$ is a direct factor of G, $\text{ord}(k)>\text{ord}(h)$ and $g=hk$.

Proof: If $g\notin\Phi(G)$ then by Theorem 1.27 there is an element $f\in\Phi(G)$ such that $\langle gf\rangle$ is a direct factor of G. Since $\langle gff^{-1}\rangle$ is not a direct factor of G, $\text{ord}(f^{-1})>\text{ord}(gf)$ by Theorem 1.26. Take $k=f^{-1}$ and $h=gf$.

The following theorem gives an aid in the construction of automorphisms of G.

Theorem 1.29: If $a_1,\ldots,a_d$ and $b_1,\ldots,b_d$ are direct bases of G with $\text{ord}(a_i)=\text{ord}(b_i)$ for $i=1,\ldots,d$, then the map α defined by $(\prod_{i=1}^{d}a_i^{n_i})^{\alpha}=\prod_{i=1}^{d}b_i^{n_i}$ is an automorphism of G.

Proof: $(\prod_{i=1}^{d} a_i^{n_i})^{\alpha}(\prod_{i=1}^{d} a_i^{m_i})^{\alpha}=(\prod_{i=1}^{d} b_i^{n_i})(\prod_{i=1}^{d} b_i^{m_i})=$ $\prod_{i=1}^{d} b_i^{n_i+m_i}=(\prod_{i=1}^{d} a_i^{n_i+m_i})^{\alpha}=[(\prod_{i=1}^{d} a_i^{n_i})(\prod_{i=1}^{d} a_i^{m_i})]^{\alpha}$. α is defined on G since $a_1,\ldots,a_d$ is a direct basis for G. $\prod_{i=1}^{d} a_i^{n_i}=1$ if, and only if, $a_i^{n_i}=1$, for $i=1,\ldots,d$, if, and only if, $b_i^{n_i}=1$, for $i=1,\ldots,d$ (since $\text{ord}(a_i)=\text{ord}(b_i)$) if, and only if, $\prod_{i=1}^{d} b_i^{n_i}=1$. Hence α is well defined and one to one. α is onto since $b_1,\ldots,b_d$ is a direct basis of G.

1.4 Stability Series.

Definition 1.30: Let $s:G=G_0\geq G_1\geq\cdots\geq G_{t-1}\geq G_t=1$ be a chain of subgroups for an arbitrary group G. Define $S_0(s)=\{\theta\in\text{Aut}(G)\mid G_i^{\theta}=G_i \text{ for } i=0,1,\ldots,t\}$ and $S_k(s)=\{\theta\in S_{k-1}(s)\mid g^{-1}g^{\theta}\in G_k, \forall g\in G_{k-1}\}$ for $1\leq k\leq t$. The series $S_0(s)\geq S_1(s)\geq\cdots\geq S_t(s)$ is called the stability series of s.

Each $S_i(s)$, $0\leq i\leq t$, is a subgroup of $\text{Aut}(G)$ and $S_i(s)\trianglelefteq S_0(s)$, $0\leq i\leq t$ (cf. Polimeni, [8], Theorem 1.1, p. 4). $S_t(s)$, the stability group of s, has been shown by P. Hall [6] to be nilpotent. It is well known that if s is a composition series in a solvable group, then $S_0(s)$ is solvable.

Theorem 1.31: Let $s_1:G=G_0\geq\cdots\geq G_t=1$ and $s_2:G=H_0\geq\cdots\geq H_t=1$ be two chains of subgroups of G such that $s_1^{\theta}=s_2$ for some $\theta\in\text{Aut}(G)$. Then $\theta^{-1}S_k(s_1)\theta=S_k(s_2)$ for $k=0,1,\ldots,t$ (cf. Polimeni, [8], Theorem 1.2, p. 7).

Theorem 1.32: If $s:G=G_0 \geq G_1 \geq \cdots \geq G_{t-1} \geq G_t=1$ is a chain of subgroups of an arbitrary group G, and if p is a prime dividing ord $(S_t(s))$, then p divides ord(G). (Cf. Feit-Thompson, [1], Lemma 8.1, p. 795).

Theorem 1.33: If $s:G=G_0 > G_1 > \cdots > G_{t-1} > G_t=1$ is a composition series in the p-group G, then $[S_i(s):S_{i+1}(s)]$ divides p-1 for $0 \leq i \leq t-1$ and $S_t(s)$ is the p-Sylow subgroup of $S_0(s)$.

Proof: Define $\phi_i:S_i(s) \to Aut(G_i/G_{i+1})$, $0 \leq i \leq t-1$ by $(gG_{i+1})^{\alpha^{\phi_i}}=g^{\alpha}G_{i+1}$ for $\alpha \in S_i(s)$, $g \in G_i$. $\alpha^{\phi_i} \in Aut(G_i/G_{i+1})$ since α fixes G_{i+1}. A computation shows that $(\alpha\beta)^{\phi_i}=\alpha^{\phi_i}\beta^{\phi_i}$ so ϕ_i is a homomorphism. $Ker(\phi_i)=S_{i+1}(s)$ so $S_i(s)/S_{i+1}(s)$ is isomorphic to some subgroup of $Aut(G_i/G_{i+1})$. Since s is a composition series, $ord(G_i/G_{i+1})=p$ and hence $ord(Aut(G_i/G_{i+1}))=p-1$. Therefore, $ord(S_i(s)/S_{i+1}(s))$ divides p-1.

Let $\alpha \in S_0(s)$ and $ord(\alpha)=p^n$. If $\alpha \in S_i(s)$, then $ord(\alpha^{\phi_i})$ divides p^n and $ord(\alpha^{\phi_i})$ divides p-1. Hence $ord(\alpha^{\phi_i})=1$ and $\alpha \in S_{i+1}(s)$. Therefore, $\alpha \in S_t(s)$. Thus $S_t(s)$, in view of Theorem 1.32, is the p-Sylow subgroup of $S_0(s)$.

Theorem 1.34: If $s:G=G_0 \geq G_1 \geq \cdots \geq G_{t-1} \geq G_t=1$ is a chain of subgroups of the p-group G and if a p'-element $\alpha \in S_k(s)$ for some k, $0 \leq k \leq t$, then $g^{-1}g^{\alpha} \in G_k$ for all $g \in G$.

Proof: If α is a p'-element of $S_k(s)$, then $\langle\alpha\rangle$ is a p'-subgroup of $S_k(s)$. Hence $g^{-1}g^{\alpha} \in [G,\langle\alpha\rangle]=[G,\langle\alpha\rangle^k]$ (by Theorem 1.8) $\subseteq [G, S_k(s)^k] \leq G_k$ by definition of $S_k(s)$.

Theorem 1.35: If $s:G=G_0>G_1>\cdots>G_{t-1}>G_t=1$ is a composition series in the p-group G and if for some i, $0\leq i\leq t$, $\alpha\in S_i(s)$, then α^{p^n} is a p'-element of Aut(G), where $p^n=\mathrm{ord}(S_t(s))$.

Proof: By Theorem 1.33, $S_t(s)$ is the p-Sylow subgroup of $S_0(s)$ and hence of $S_i(s)$. Therefore there is a positive integer m such that $\mathrm{ord}(S_i(s))=p^n m$ and $(p,m)=1$. Hence $(\alpha^{p^n})^m=\alpha^{p^n m}=1$ and $\mathrm{ord}(\alpha^{p^n})$ divides m. Therefore, α^{p^n} is a p'-element.

1.5 Homocyclic abelian p-groups.

This section contains the result that if the abelian p-group G is homocyclic, then the automorphisms of $\mathcal{U}_i(G)$ and $G/\mathcal{U}_i(G)$ can be extended to automorphisms of G.

Throughout this section G is a homocyclic abelian p-group.

Theorem 1.36: If $\exp(G)=p^n$ and if r and s are integers with $0\leq r\leq s\leq n$, then $\Omega_{n-s}(G)\leq\mathcal{U}_r(G)$.

Proof: Let $a_1,\ldots,a_d$ be a direct basis of G. Then $\mathrm{ord}(a_i)=p^n$ for $i=1,\ldots,d$. Let $x\in\Omega_{n-s}(G)$. Then $x=\coprod_{i=1}^{d} a_i^{m_i p^{n_i}}$ with $(p,m_i)=1$ and $\coprod_{i=1}^{d} a_i^{m_i p^{(n_i+n-s)}}=x^{p^{(n-s)}}=1$. Hence $a_i^{m_i p^{(n_i+n-s)}}=1$ for all i, $1\leq i\leq d$. Therefore, p^n divides $m_i p^{(n_i+n-s)}$ and $n\leq n_i+n-s$. Thus $n_i\geq s\geq r$ and hence $x=\coprod_{i=1}^{d} a_i^{m_i p^{n_i}}\in\mathcal{U}_r(G)$. Therefore, $\Omega_{n-s}(G)\leq\mathcal{U}_r(G)$.

Theorem 1.37: If $b_1,\ldots,b_d$ is a minimal basis of G, it is also a direct basis of G.

Proof: As remarked following Definition 1.10 a direct basis of G has the same number of elements as a minimal basis. Let $a_1,\ldots,a_d$ be a direct basis of G with $ord(a_i)=p^n$ for $i=1,2,\ldots,d$. Then $ord(G)=p^{nd}$ and $exp(G)=p^n$. Hence $ord(b_i)\leq p^n$ for $i=1,\ldots,d$. Since $G=\langle b_1,\ldots,b_d\rangle$, $ord(G)\leq \prod_{i=1}^{d} ord(b_i)\leq p^{nd}=ord(G)$. Therefore, $ord(G)=\prod_{i=1}^{d} ord(b_i)$. Since $ord(G)=\prod_{i=1}^{d} ord(b_i)$ and $G=\langle b_1,\ldots,b_d\rangle$, $b_1,\ldots,b_d$ is a direct basis of G.

Lemma 1.38: If $exp(G)=p^n$, $n>1$ and $[G:\Phi(G)]=p^d$, then $\mathcal{V}_1(G)$ is homocyclic of exponent p^{n-1} and $[\mathcal{V}_1(G):\Phi(\mathcal{V}_1(G))]=p^d$.

Proof: Let $a_1,\ldots,a_d$ be a direct basis of G. By Theorem 1.18, $\Phi(G)=\mathcal{V}_1(G)=\langle a_1^p,\ldots,a_d^p\rangle$. Since $ord(a_i)=p^n$, $ord(a_i^p)=p^{n-1}$ for $1\leq i\leq d$. $ord(\mathcal{V}_1(G))=ord(\Phi(G))=p^{nd}/p^d=p^{d(n-1)}=\prod_{i=1}^{d} ord(a_i^p)$. Since $\mathcal{V}_1(G)=\langle a_1^p,\ldots,a_d^p\rangle$ and $ord(\mathcal{V}_1(G))=\prod_{i=1}^{d} ord(a_i^p)$ and $a_i^p\neq 1$ for $1\leq i\leq d$, $a_1^p,\ldots,a_d^p$ is a direct basis of $\mathcal{V}_1(G)$. Hence $\mathcal{V}_1(G)$ is homocyclic of exponent p^{n-1} with $[\mathcal{V}_1(G):\Phi(\mathcal{V}_1(G))]=p^d$.

Theorem 1.39: If $exp(G)=p^n$ and $[G:\Phi(G)]=p^d$, then $\mathcal{V}_i(G)$ is homocyclic of exponent p^{n-i} with $[\mathcal{V}_i(G):\Phi(\mathcal{V}_i(G))]=p^d$ for $0\leq i<n$.

Proof: The conclusion holds for i=0 since $\mathcal{V}_0(G)=G$.

Suppose $\mathcal{V}_k(G)$ is homocyclic of exponent p^{n-k} with $[\mathcal{V}_k(G):\Phi(\mathcal{V}_k(G))]=p^d$ for some k, $0\le k<n-1$. Then $n-k>1$ and by Lemma 1.38 $\mathcal{V}_1(\mathcal{V}_k(G))=\mathcal{V}_{k+1}(G)$ is homocyclic of exponent p^{n-k-1} with $[\mathcal{V}_{k+1}(G):\Phi(\mathcal{V}_{k+1}(G))]=p^d$.

Lemma 1.40: If $\exp(G)=p^n$, $n>1$ and $[G:\Phi(G)]=p^d$ and if $x_1,\ldots,x_d$ is a direct basis for $\mathcal{V}_1(G)$ with $a_i^p=x_i$ for $1\le i\le d$, then $a_1,\ldots,a_d$ is a direct basis of G.

Proof: By Lemma 1.38, $\mathcal{V}_1$ is homocyclic of exponent p^{n-1}. Hence $\operatorname{ord}(x_i)=p^{n-1}$ for $1\le i\le d$. Thus $\operatorname{ord}(a_i)=p^n$ for $1\le i\le d$. If $\prod_{i=1}^{d} a_i^{n_i}=1$, then $(\prod_{i=1}^{d} a_i^{n_i})^p=\prod_{i=1}^{d} x_i^{n_i}=1$. Hence $x_i^{n_i}=1$ for all i, $1\le i\le d$. Therefore $n_i=p^{n-1}m_i$, $1\le i\le d$. Since $n>1$, $1=\prod_{i=1}^{d} a_i^{n_i}=\prod_{i=1}^{d} a_i^{p^{(n-1)}m_i}=\prod_{i=1}^{d} x_i^{p^{(n-2)}m_i}$. Hence $x_i^{p^{(n-2)}m_i}=1$ for all i, $1\le i\le d$ and $p^{(n-1)}$ divides $p^{n-2}m_i$. Thus p divides m_i and p^n divides n_i for all i, $1\le i\le d$. Hence $a_i^{n_i}=1$ for all i, $1\le i\le d$. Therefore, $\langle a_1,\ldots,a_d\rangle=\langle a_1\rangle\times\cdots\times\langle a_d\rangle$. Also $\operatorname{ord}(\langle a_1,\ldots,a_d\rangle)=\prod_{i=1}^{d}\operatorname{ord}(a_i)=p^{nd}=\operatorname{ord}(G)$. Because $n>1$, $\operatorname{ord}(a_i)=p^n>1$ and $a_1,\ldots,a_d$ is a direct basis of G.

Theorem 1.41: If $\exp(G)=p^n$ and $[G:\Phi(G)]=p^d$ and if $x_1,\ldots,x_d$ is a direct basis for $\mathcal{V}_j(G)$ with $a_i^{p^j}=x_i$ for $1\le i\le d$, then $a_1,\ldots,a_d$ is a direct basis for G, $0\le j<n$.

Proof: For $j=0$, $a_i=x_i$ and $a_1,\ldots,a_d$ is a direct basis of G.

Suppose that if $y_1,\ldots,y_d$ is a direct basis of $\mathcal{V}_k(G)$ with $b_i^{p^k}=y_i$ for $1\le i\le d$, then $b_1,\ldots,b_d$ is a direct basis of G, for some k, $o\le k<n-1$.

Let $x_1,\ldots,x_d$ be a direct basis of $\mathcal{V}_{k+1}(G)$ with $a_i^{p^{k+1}}=x_i$ for $1\le i\le d$. By Lemma 1.40 and Theorem 1.39 $a_1^{p^k},\ldots,a_d^{p^k}$ is a direct basis of $\mathcal{V}_k(G)$ and hence by the induction hypothesis $a_1,\ldots,a_d$ is a direct basis of G.

Theorem 1.42: If $\exp(G)=p^n$ and $[G:\Phi(G)]=p^d$, then, for $1\le j\le n$, $G/\mathcal{V}_j(G)$ is homocyclic of exponent p^j and $[G/\mathcal{V}_j(G):\Phi(G/\mathcal{V}_j(G))]=p^d$.

Proof: Let $a_1,\ldots,a_d$ be a direct basis of G. If $a_i^{p^{m_i}}\in\mathcal{V}_j(G)$, then $a_i^{p^{(m_i+n-j)}}=1$ since, by Theorem 1.39 $\mathcal{V}_j(G)$ is homocyclic of exponent p^{n-j}. Hence $m_i\ge j$ since $\operatorname{ord}(a_i)=p^n$. Therefore, $\operatorname{ord}(a_i\mathcal{V}_j(G))=p^j$ for $1\le i\le d$. By Theorem 1.39 $\operatorname{ord}(\mathcal{V}_j(G))=p^{d(n-j)}$. Hence $[G:\mathcal{V}_j(G)]=p^{dj}$. Since $\langle a_1\mathcal{V}_j(G),\ldots,a_d\mathcal{V}_j(G)\rangle=G/\mathcal{V}_j(G)$ and $\operatorname{ord}(G/\mathcal{V}_j(G))=p^{dj}=\prod_{i=1}^{d}\operatorname{ord}(a_i\mathcal{V}_j(G))$ and $a_i\mathcal{V}_j(G)\neq\mathcal{V}_j(G)$ for $1\le i\le d$, $a_1\mathcal{V}_j(G),\ldots,a_d\mathcal{V}_j(G)$ is a direct basis of $G/\mathcal{V}_j(G)$ and hence $G/\mathcal{V}_j(G)$ is homocyclic of exponent p^j with $[G/\mathcal{V}_j(G):\Phi(G/\mathcal{V}_j(G))]=p^d$.

Lemma 1.43: If $\exp(G)=p^n$ and $[G:\Phi(G)]=p^d$ and if $a_1\mathcal{V}_1(G),\ldots,a_d\mathcal{V}_1(G)$ is a direct basis of $G/\mathcal{V}_1(G)$, then $a_1,\ldots,a_d$ is a direct basis of G.

Proof: By Theorem 1.18, $\mathcal{V}_1(G)=\Phi(G)$. Hence, by Theorem 1.1, $a_1,\ldots,a_d$ is a minimal basis of G and, by Theorem 1.37, $a_1,\ldots,a_d$ is a direct basis of G.

Theorem 1.44: If $\exp(G)=p^n$ and $[G:\Phi(G)]=p^d$ and if $a_1\mathcal{V}_j(G),\ldots,a_d\mathcal{V}_j(G)$ is a direct basis of $G/\mathcal{V}_j(G)$, then $a_1,\ldots,a_d$ is a direct basis of G, $1\leq j\leq n$.

Proof: By Theorem 1.42 $G/\mathcal{V}_j(G)$ is homocyclic of exponent p^j. By Theorem 1.39 $\mathcal{V}_{j-1}(G/\mathcal{V}_j(G))$ is homocyclic of exponent p with $[\mathcal{V}_{j-1}(G/\mathcal{V}_j(G)):\Phi(\mathcal{V}_{j-1}(G/\mathcal{V}_j(G)))]=p^d$. Since also $\mathcal{V}_{j-1}(G/\mathcal{V}_j(G))=\langle a_1^{p^{j-1}}\mathcal{V}_j(G),\ldots,a_d^{p^{j-1}}\mathcal{V}_j(G)\rangle$, $a_1^{p^{j-1}}\mathcal{V}_j(G),\ldots,a_1^{p^{j-1}}\mathcal{V}_j(G)$ is minimal basis and hence, by Theorem 1.37, a direct basis of $\mathcal{V}_{j-1}(G/\mathcal{V}_j(G))$. By Theorem 1.16, $\mathcal{V}_{j-1}(G/\mathcal{V}_j(G))=\mathcal{V}_{j-1}(G)/\mathcal{V}_j(G)$. By Theorem 1.39, $\mathcal{V}_{j-1}(G)$ is homocyclic and since $\mathcal{V}_1(\mathcal{V}_{j-1}(G))=\mathcal{V}_j(G)$, $a_1^{p^{j-1}},\ldots,a_d^{p^{j-1}}$ is a direct basis for $\mathcal{V}_{j-1}(G)$, by Lemma 1.43. Hence, by Theorem 1.41, $a_1,\ldots,a_d$ is a direct basis of G.

Theorem 1.45: If $\exp(G)=p^n$ and $[G:\Phi(G)]=p^d$ and if, for some j with $1\leq j<n$, $\beta\in\mathrm{Aut}(\mathcal{V}_j(G))$, then there is an $\alpha\in\mathrm{Aut}(G)$ such that $\alpha|_{\mathcal{V}_j(G)}=\beta$.

Proof: Let $x_1,\ldots,x_d$ be a direct basis of $\mathcal{V}_j(G)$ and $y_i=x_i^{\beta}$ for $1\leq i\leq d$. Then $y_1,\ldots,y_d$ is a direct basis for $\mathcal{V}_j(G)$. There exist $a_i,b_i\in G$ such that $a_i^{p^j}=x_i$ and $b_i^{p^j}=y_i$ for $1\leq i\leq d$. By Theorem 1.41, $a_1,\ldots,a_d$ and $b_1,\ldots,b_d$ are direct bases of G. Define α by $\alpha:a_i\to b_i$ for $1\leq i\leq d$. By Theorem 1.29, $\alpha\in Aut(G)$. $x_i^{\alpha}=(a_i^{p^j})^{\alpha}=(a_i^{\alpha})^{p^j}=b_i^{p^j}=y_i=x_i^{\beta}$, for $1\leq i\leq d$. Hence $\alpha|_{\mathcal{V}_j(G)}=\beta$.

Theorem 1.46: If $\exp(G)=p^n$ and $[G:\Phi(G)]=p^d$ and if, for some j with $1\leq j\leq n$, $\beta\in Aut(G/\mathcal{V}_j(G))$, then there is an $\alpha\in Aut(G)$ such that α induces β on $G/\mathcal{V}_j(G)$.

Proof: Let $a_1\mathcal{V}_j,\ldots,a_d\mathcal{V}_j$ be a direct basis of $G/\mathcal{V}_j(G)$ and $(a_i\mathcal{V}_j(G))^{\beta}=(b_i\mathcal{V}_j(G))$ for $1\leq i\leq d$. Then $b_1\mathcal{V}_j(G),\ldots,b_d\mathcal{V}_j(G)$ is a direct basis of $G/\mathcal{V}_j(G)$. By Theorem 1.44, $a_1,\ldots,a_d$ and $b_1,\ldots,b_d$ are direct bases of G. Define α by $\alpha:a_i\to b_i$ for $1\leq i\leq d$. $\alpha\in Aut(G)$ by Theorem 1.29. $(a_i\mathcal{V}_j(G))^{\alpha}=a_i^{\alpha}\mathcal{V}_j(G)=b_i\mathcal{V}_j(G)=(a_i\mathcal{V}_j)^{\beta}$. Hence α induces β on $G/\mathcal{V}_j(G)$.

CHAPTER II

STABILITY SERIES

For basic definitions see section 4 of Chapter I.

2.1 Some sufficient conditions for $S_i(s)=S_{i+1}(s)$.

This section gives some sufficient conditions for $S_i(s)=S_{i+1}(s)$ when G is an abelian p-group and s is a composition series.

Throughout this section G is an abelian p-group and $s: G=G_0>G_1>\cdots>G_{t-1}>G_t=1$ is a composition series. For $0\leq i<t$, $a_i \in G_i$ and H_i is a subgroup of G_{i+1} (the existence of which is guaranteed by Corollary 1.22) such that $G_i=\langle a_i\rangle xH_i$ and $G_{i+1}=\langle a_i^p\rangle xH_i$.

Theorem 2.1: If, for some i with $0\leq i<t$, $a_i \in \Phi(G)$, then $S_i(s)=S_{i+1}(s)$.

Proof: Suppose $\alpha \in S_i(s)$. Let $\mathrm{ord}(S_t(s))=p^n$ (cf. Theorem 1.33). By Theorem 1.35, α^{p^n} is a p'-element of Aut(G). Hence, by Theorem 1.34, $g^{-1}g^{\alpha^{p^n}} \in G_i$, for all $g \in G$. If $a_i \in \Phi(G)$, then, by Theorem 1.18, there exists $g_i \in G$ such that $g_i^p=a_i$. Since $g_i^{-1}g_i^{\alpha^{p^n}} \in G_i$, $(g_i^{-1}g_i^{\alpha^{p^n}})^p=a_i^{-1}a_i^{\alpha^{p^n}} \in G_{i+1}$ and hence

$\alpha^{p^n} \in S_{i+1}(s)$. By Theorem 1.33, $\text{ord}(S_i(s)/S_{i+1}(s))$ divides $p-1$. Since $(p^n, p-1)=1$, $\alpha \in S_{i+1}(s)$. Hence $S_i(s)=S_{i+1}(s)$.

Theorem 2.2: If, for some i with $0 \leq i < t$, $\langle a_i \rangle$ is not a direct factor of G, then $S_i(s)=S_{i+1}(s)$.

Proof: Suppose $\langle a_i \rangle$ is not a direct factor of G and let $\alpha \in S_i(s)$ with $\text{ord}(S_t(s))=p^n$. Then, by Theorem 1.35, α^{p^n} is a p'-element of $S_i(s)$.

If $a_i \in \Phi(G)$, then, by Theorem 2.1, $S_i(s)=S_{i+1}(s)$.

Suppose $a_i \notin \Phi(G)$. Then, by Corollary 1.28, there exist $b \in \Phi(G)$ and $g \in G$ such that $\langle g \rangle$ is a direct factor of G, $\text{ord}(b)>\text{ord}(g)$ and $a_i=gb$. By Theorem 1.34, $g^{-1}g^{\alpha^{p^n}} \in G_i = \langle a_i \rangle x H_i$. Hence $g^{-1}g^{\alpha^{p^n}} = a_i^m h$ for some $h \in H_i$ and m an integer.

If $g^{-1}g^{\alpha^{p^n}} \notin G_{i+1} = \langle a_i^p \rangle x H_i$, then $(p,m)=1$ and $\text{ord}(a_i^m h) \geq \text{ord}(a_i^m) = \text{ord}(a_i)$. By Theorem 1.25, $\text{ord}(a_i) = \text{ord}(gb) \geq \text{ord}(b) > \text{ord}(g)$. Hence $\text{ord}(g^{-1}g^{\alpha^{p^n}}) = \text{ord}(a_i^m h) > \text{ord}(g)$. But $\text{ord}(g^{-1}g^{\alpha^{p^n}}) \leq \text{ord}(g)$.

Therefore, $g^{-1}g^{\alpha^{p^n}} \in G_{i+1}$. Since $b \in \Phi(G)$, by Theorem 1.18, there exists $x \in G$ such that $x^p=b$. Hence, by Theorem 1.34, $x^{-1}x^{\alpha^{p^n}} \in G_i$ and $(x^{-1}x^{\alpha^{p^n}})^p = b^{-1}b^{\alpha^{p^n}} \in G_{i+1}$. Thus, $a_i^{-1}a_i^{\alpha^{p^n}} = g^{-1}g^{\alpha^{p^n}} b^{-1}b^{\alpha^{p^n}} \in G_{i+1}$. Therefore, $\alpha^{p^n} \in S_{i+1}(s)$. By Theorem 1.33, $\text{ord}(S_i(s)/S_{i+1}(s))$ divides $p-1$. Since $(p^n, p-1)=1, \alpha \in S_{i+1}(s)$. Hence $S_i(s)=S_{i+1}(s)$.

Theorem 2.3: If, for some i with $o \leq i < t$, there is an element $h \in \Phi(G)$ such that $a_i h \in G_{i+1}$, then $S_i(s) = S_{i+1}(s)$.

Proof: Suppose there is an element $h \in \Phi(G)$ such that $a_i h \in G_{i+1}$. Let $\alpha \in S_i(s)$ and $ord(S_t(s)) = p^n$. Since $a_i h \in G_{i+1}$ and $\alpha \in S_i(s) \trianglelefteq S_o(s)$, $a_i^{-1} a_i^{\alpha^{p^n}} h^{-1} h^{\alpha^{p^n}} \in G_{i+1}$. Since $h \in \Phi(G)$, by Theorem 1.18, there exists $g \in G$ such that $g^p = h$. By Theorem 1.35, α^{p^n} is a p'-element of $S_i(s)$ and by Theorem 1.34 $g^{-1} g^{\alpha^{p^n}} \in G_i$. Since $[G_i : G_{i+1}] = p$, $h^{-1} h^{\alpha^{p^n}} = (g^{-1} g^{\alpha^{p^n}})^p \in G_{i+1}$. Since $a_i^{-1} a_i^{\alpha^{p^n}} h^{-1} h^{\alpha^{p^n}} \in G_{i+1}$ and $h^{-1} h^{\alpha^{p^n}} \in G_{i+1}$, $a_i^{-1} a_i^{\alpha^{p^n}} \in G_{i+1}$. Since $G_i = \langle a_i \rangle x H_i$ and $G_{i+1} = \langle a_i^p \rangle x H_i$, $\alpha^{p^n} \in S_{i+1}(s)$. By Theorem 1.33, $ord(S_i(s)/S_{i+1}(s))$ divides $p-1$. Since $(p^n, p-1) = 1$, $\alpha \in S_{i+1}(s)$. Hence $S_i(s) = S_{i+1}(s)$.

2.2 A sufficient condition for $S_o(s)$ to be supersolvable.

In this section it is shown that $S_o(s)$ is supersolvable when s is a chief series in a supersolvable group.

Theorem 2.4: If H is a subgroup of the group G and K is a subgroup of H with [G:H] and ord(K) both prime (not necessarily the same) and if $C = \{\alpha \in Aut(G) \mid h^\alpha = h \; \forall h \in H$ and $g^{-1} g^\alpha \in K \; \forall g \in G\}$, then C is a subgroup of Aut(G) and either ord(C) = ord(K) or ord(C) = 1.

Proof: $1 \in C$. If $\alpha, \beta \in C$, then $h^{\alpha\beta} = h^\beta = h, \forall h \in H$ and $g^{-1} g^{\alpha\beta} = (g^{-1} g^\alpha)((g^{-1})^\alpha (g^\alpha)^\beta) \in K, \forall g \in G$. Hence $\alpha\beta \in C$ and since Aut(G) is finite, C is a subgroup of Aut(G).

Since [G:H] is prime there is an element $g_1 \in G-H$ such that $\forall g \in G$, $\exists h \in H$ and m an integer such that $g=g_1^m h$. Hence if $\alpha,\beta \in C$ and $g_1^\alpha = g_1^\beta$, then $\alpha=\beta$. Thus if $\alpha \in C$ and $\alpha \neq 1$, then $g_1^\alpha = g_1 k$ with $k \neq 1$ and $k \in K$. Since ord(K) is prime, $\langle k \rangle = K$. Hence if $\beta \in C$, $\exists$ an integer m, $1 \leq m \leq ord(K)$ such that $g_1^\beta = g_1 k^m = g_1^{\alpha^m}$. Since C is a group, $\alpha^m \in C$ and, by the above, $\alpha^m = \beta$. Hence $C=\langle \alpha \rangle$ and $ord(C)=ord(\alpha)=ord(k)=ord(K)$.

Theorem 2.5: If G is solvable and $s: G=G_0 > G_1 > \cdots > G_{t-1} > G_t = 1$ is a composition series, then $(S_0(s))' \leq S_t(s)$.

Proof: Since G is solvable and G_i/G_{i+1} is a composition factor, G_i/G_{i+1} is cyclic of prime order, say p_i. Suppose $G_i/G_{i+1} = \langle g_i G_{i+1} \rangle$. Let $\alpha,\beta \in S_0(s)$ and $g_i^\alpha = g_i^{m_1} a_1$, $g_i^{\alpha^{-1}} = g_i^{m_2} a_2$, $g_i^\beta = g_i^{m_3} a_3$, $g_i^{\beta^{-1}} = g_i^{m_4} a_4$ with $a_1,a_2,a_3,a_4 \in G_{i+1}$. $g_i = g_i^{\alpha\alpha^{-1}} = (g_i^{m_1} a_1)^\alpha = (g_i^{m_2} a_2)^{m_1} a_1^{\alpha^{-1}} = g_i^{m_1 m_2} a$ with $a \in G_{i+1}$ since $G_{i+1} \triangleleft G_i$ and $\alpha^{-1} \in S_0(s)$. Hence $g_i^{m_1 m_2^{-1}} \in G_{i+1}$ which gives $m_1 m_2 \equiv 1 \bmod p_i$. Similarly $m_3 m_4 \equiv 1 \bmod p_i$. $g_i^{\alpha^{-1}\beta^{-1}\alpha\beta} = (g_i^{m_2} a_2)^{\beta^{-1}\alpha\beta} = [(g_i^{m_4} a_4)^{m_2} a_2^{\beta^{-1}}]^{\alpha\beta} = ([(g_i^{m_1} a_1)^{m_4} a_4^\alpha]^{m_2} a_2^{\beta^{-1}\alpha})^\beta = (((g_i^{m_3} a_3)^{m_1} a_1^\beta)^{m_4} a_4^{\alpha\beta})^{m_2} a_2^{\beta^{-1}\alpha\beta} = g_i^{m_1 m_2 m_3 m_4} a_5$ with $a_5 \in G_{i+1}$ since $G_{i+1} \triangle G_i$. Since $m_1 m_2 m_3 m_4 \equiv 1 \bmod p_i$, $g_i^{-1} g_i^{\alpha^{-1}\beta^{-1}\alpha\beta} = g_i^{m_1 m_2 m_3 m_4 - 1} a_5 \in G_{i+1}$. Since $G_i/G_{i+1} = \langle g_i G_{i+1} \rangle$ and $G_{i+1}^{\alpha^{-1}\beta^{-1}\alpha\beta} = G_{i+1}$ and $g_i^{-1} g_i^{\alpha^{-1}\beta^{-1}\alpha\beta} \in G_{i+1}$,

$g^{-1}g^{\alpha^{-1}\beta^{-1}\alpha\beta} \in G_{i+1}$, $\forall g \in G_i$. Since this is the case for $i=0,1,\ldots,t-1$, $\alpha^{-1}\beta^{-1}\alpha\beta \in S_t(s)$ and $(S_0(s))' \subseteq S_t(s)$.

Theorem 2.6: If G is supersolvable and $s: G=G_0 > G_1 > \cdots > G_{t-1} > G_t=1$ is a chief series, then $S_0(s)$ is supersolvable.

Proof: Since G is supersolvable and s is a chief series, $[G_i:G_{i+1}]$ is prime for $i=0,1,\ldots,t-1$. Hence, by Theorem 2.5, $(S_0(s))' \subseteq S_t(s)$. For $k \geq t$ let $G_k=1$. Define $B_{i,j}$, for i a non-negative integer and $0 \leq j \leq t-1$ by $B_{i,j} = \{\alpha \in S_0(s) \mid$ for $0 \leq k \leq j$, $g^{-1}g^{\alpha} \in G_{k+i+1}$, $\forall g \in G_k$ and for $0 \leq k \leq t-1$ $g^{-1}g^{\alpha} \in G_{k+i}$, $\forall g \in G_k\}$. Then $B_{0,t-1}=S_t(s)$ and if $i_1 > i_2$ or if $i_1=i_2$ and $j_1 > j_2$, $B_{i_1,j_1} \subseteq B_{i_2,j_2}$. For $0 \leq j < t-1$, $B_{i,j+1} = \{\alpha \in B_{i,j} \mid g^{-1}g^{\alpha} \in G_{j+i+2}, \forall g \in G_{j+1}\}$ and for $i \geq 0$, $B_{i+1,0} = \{\alpha \in B_{i,t-1} \mid g^{-1}g^{\alpha} \in G_{i+2}, \forall g \in G_0\}$. Each $B_{i,j}$ is a group since $1 \in B_{i,j}$ and $g^{-1}g^{\alpha\beta} = g^{-1}g^{\alpha}[(g^{-1})^{\alpha}(g^{\alpha})^{\beta}]$. Let $\alpha \in S_0(s)$ and $\beta \in B_{i,j}$. Let $g \in G_k$. Then $g^{-1}g^{\alpha^{-1}\beta\alpha} = [(g^{-1})^{\alpha^{-1}}(g^{\alpha^{-1}})^{\beta}]^{\alpha} \in G_{k+i+1}$ if $1 \leq k \leq j$ and $g^{-1}g^{\alpha^{-1}\beta\alpha} \in G_{k+i}$ if $1 \leq k \leq t-1$. Hence $\alpha^{-1}\beta\alpha \in B_{i,j}$ and $B_{i,j} \triangleleft S_0(s)$.

For $0 \leq j < t-2$ and $i \geq 1$, the elements of $B_{i,j}$ induce automorphisms on G_{j+1}/G_{j+i+2}. Let ϕ be the map from $B_{i,j}$ to $\mathrm{Aut}(G_{j+1}/G_{j+i+2})$ where, for $\alpha \in B_{i,j}$, α^{ϕ} is the automorphism induced on G_{j+1}/G_{j+i+2} by α. ϕ is a homomorphism. From the definition of $B_{i,j}$, the elements of $B_{i,j}^{\phi}$ fix G_{j+2}/G_{j+i+2} elementwise and induce the identity automorphism on $(G_{j+1}/G_{j+i+2})/(G_{j+i+1}/G_{j+i+2})$. Since $[G_{j+1}/G_{j+i+2}: G_{j+2}/G_{j+i+2}]$ and $\mathrm{ord}(G_{j+i+1}/G_{j+i+2})$ are both prime, $\mathrm{ord}(B_{i,j}^{\phi}) =$

$[G_{j+i+1}:G_{j+i+2}]$ or $\text{ord}(B_{i,j}{}^{\phi})=1$, by Theorem 2.4. By the above, $\ker\phi=B_{i,j+1}$. Hence either $[B_{i,j}:B_{i,j+1}]$ is prime or $[B_{i,j}:B_{i,j+1}]=1$.

For $i\geq 1$, $B_{i,t-2}=B_{i,t-1}$. By an argument similar to that above, for $i\geq 0$ $[B_{i,t-1}:B_{i+1,0}]$ is equal to either $[G_{i+1}:G_{i+2}]$ or 1.

Thus for $i\geq 1$ and $1\leq j<t-1$, $[B_{i,j}:B_{i,j+1}]$ is prime or $B_{i,j}=B_{i,j+1}$ and, for $i\geq 0$, $[B_{i,t-1},B_{i+1,0}]$ is prime or $B_{i,t-1}=B_{i+1,0}$. Also $B_{t-1,0}=1$. Hence from the $B_{i,j}$ a series of subgroups from $S_t(s)$ to 1 can be extracted, each member of which is normal in $S_0(s)$ with factors of prime order. Since also $S_0(s)/S_t(s)$ is abelian, $S_0(s)$ is supersolvable.

Corollary 2.7: If the solvable group G has a characteristic composition series, then Aut(G) is supersolvable.

CHAPTER III

THE AUTOMORPHISM GROUP OF AN ABELIAN P-GROUP

The notation developed in the first section of this chapter will be used throughout the chapter.

3.1 Notation and Basic Properties.

(3.1.1) G is an abelian p-group. $G=H_1x\cdots xH_t$ where H_i is homocyclic of exponent p^{n_i} and $[H_i:\Phi(H_i)] = p^{d_i}$ for $i=1,\ldots,t$. $n_i > n_{i+1}$ for $i=1,\ldots,t$ with $n_{t+1} = 0$.

(3.1.2) $f_o = 0$ and $f_i = \sum_{j=1}^{i} d_j$ for $i=1,\ldots,t$.

$$d=f_t = \sum_{i=1}^{t} d_i \; ; \; [G:\Phi(G)]=p^d.$$

(3.1.3) $a_{f_{i-1}+1},\ldots,a_{f_i}$ is a direct basis of H_i for $i=1,\ldots,t$. Hence $a_1,\ldots,a_d$ is a direct basis of G and ord $(a_i) \geq$ ord (a_{i+1}) for $i=1,\ldots,d-1$.

(3.1.4) $K_i = \langle a_1\rangle x\cdots x\langle a_{i-1}\rangle x\langle a_{i+1}\rangle x\cdots x\langle a_d\rangle$ for $i=1,\ldots,d$.

(3.1.5) For $q \geq o$, $o<r\leq d$ and $m = qd+r$, $G_m = \mathcal{U}_{q+1}(\langle a_1\rangle x\cdots x\langle a_r\rangle)\; x\mathcal{U}_q\;(\langle a_{r+1}\rangle x\cdots x\langle a_d\rangle)$ and $G_o = G$.

From this definition and Theorem 1.15 it follows that $G_m = \mathcal{U}_q(G_r)$ and for $i \geq 0$, $[G_i : G_{i+1}] \leq p$ and $G_{n_1 d} = 1$.

(3.1.6) s_1 denotes the chain $G_o \geq G_1 \geq \cdots$.

(3.1.7) For $q \geq o$, $o < r \leq t$ and $m = qt+r$, $M_m = \mathcal{U}_{q+1}(H_1 x \cdots x H_r) \times \mathcal{U}_q(H_{r+1} x \cdots x H_t)$ and $M_o = G$.

From this definition and Theorem 1.15 it follows that $M_m = \mathcal{U}_q(M_r)$ and $M_{n_1 t} = 1$.

(3.1.8) s_2 denotes the chain $M_o \geq M_1 \geq \cdots$.

From the above definitions it follows directly that s_1 is a refinement of s_2.

Theorem 3.1: s_2 is a characteristic series.

Proof: It suffices to show that, for $o \leq q$ and $o < r \leq t$, $\mathcal{U}_{q+1}(H_1 x \cdots x H_r) \times \mathcal{U}_q(H_{r+1} x \cdots x H_t) = \mathcal{U}_{q+1}(G) \cdot [\Omega_{n_{r+1}-q}(G) \cap \mathcal{U}_q(G)]$ where $\Omega_k(G) = 1$ if $k \leq 0$.

Let $x \in \mathcal{U}_{q+1}(H_1 x \cdots x H_r)$ and $y \in \mathcal{U}_q(H_{r+1} x \cdots x H_t)$. Then there are elements g and h with $g \in H_1 x \cdots x H_r$ and $h \in H_{r+1} x \cdots x H_t$ such that $x = g^{p^{q+1}}$ and $y = h^{p^q}$. Since $\exp(H_{r+1} x \cdots x H_t) = p^{n_{r+1}}$, $h^{p^{n_{r+1}}} = 1$. Hence $y^{p^{n_{r+1}-q}} = h^{p^{n_{r+1}}} = 1$ and $y \in \Omega_{n_{r+1}-q}(G)$, if $n_{r+1} \geq q$ and $y = 1$ if $n_{r+1} < q$. Also $x \in \mathcal{V}_{q+1}(G)$ and $y \in \mathcal{V}_q(G)$. Thus $xy \in \mathcal{V}_{q+1}(G) \cdot$

$[\Omega_{n_{r+1}-q}(G) \cap \mathcal{V}_q(G)]$. Therefore, $\mathcal{V}_{q+1}(H_1x\cdots xH_r) \times \mathcal{V}_q(H_{r+1}x\cdots xH_t) \leq \mathcal{V}_{q+1}(G)[\Omega_{n_{r+1}-q}(G) \cap \mathcal{V}_q(G)]$.

Let $x \in \mathcal{V}_{q+1}(G)$ and $y \in \mathcal{V}_q(G) \cap \Omega_{n_{r+1}-q}(G)$. Then there are elements h_1, h_2, g_1, g_2 with h_1, $g_1 \in H_1x\cdots xH_r$ and h_2, $g_2 \in H_{r+1}x\cdots xH_t$ such that $x = {g_1}^{p^{q+1}} {g_2}^{p^{q+1}}$ and $y = {h_1}^{p^q} {h_2}^{p^q}$ and $y^{p^{n_{r+1}-q}} = {h_1}^{p^{n_{r+1}}} {h_2}^{p^{n_{r+1}}} = 1$ or $y = 1$. By Theorems 1.15 and 1.36, and since $n_r > n_{r+1} \geq 0$, $\Omega_{n_{r+1}}(H_1x\cdots xH_r) \subseteq \mathcal{V}_1(H_1x\cdots xH_r)$. Therefore, since ${h_1}^{p^{n_{r+1}}} = 1$, there is an element $h_3 \in H_1x\cdots xH_r$ such that $h_1 = {h_3}^p$. Hence $xy = (g_1h_3)^{p^{q+1}}(g_2{}^ph_2)^{p^q} \in \mathcal{V}_{q+1}(H_1x\cdots xH_r) \times \mathcal{V}_q(H_{r+1}x\cdots xH_t)$. Thus $\mathcal{V}_{q+1}(G)[\Omega_{n_{r+1}-q}(G) \cap \mathcal{V}_q(G)] \leq \mathcal{V}_{q+1}(H_1x\cdots xH_r) \times \mathcal{V}_q(H_{r+1}x\cdots xH_t)$ and the conclusion follows.

The next two theorems provide an aid in the construction of automorphisms.

Theorem 3.2: If when $1 \leq j \leq t$ and $f_{j-1} < i \leq f_j$, $g_i \in G_i \cap \mathcal{V}_{n_j}(G)$ and $0 < m_i < p$, then the map α, defined by $\alpha: a_i \to a_i{}^{m_i} g_i$ for $i = 1,\ldots,d$, is an automorphism of G.

Proof: By Theorem 1.29, the map β, defined by $\beta: a_i \to a_i{}^{m_i}$, $0 < m_i < p$, $i = 1,\ldots,d$, is an automorphism of G.

Since $G_d = \mathcal{U}_1(G) = \Phi(G)$, $G = \langle a_1,\ldots,a_{d-1},a_dg_d\rangle$. $\text{ord}(a_dg_d) \leq \text{ord}(a_d)$ since $g_d \in \Omega_{n_t}(G)$. Hence $a_1,\ldots,a_{d-1}$, a_dg_d is a direct basis of G and $\text{ord}(a_dg_d) = \text{ord}(a_d)$.

Suppose $a_1,\ldots,a_i,a_{i+1}g_{i+1},\ldots,a_dg_d$ is a direct basis of G and $\text{ord}(a_{i+j}g_{i+j}) = \text{ord}(a_{i+j})$ for $1\leq j\leq d-i$.

Then $g_i = (\pi_{j=1}^{i} a_j^{r_j})(\pi_{j=i+1}^{d} a_j^{r_j} g_j^{r_j})$. Since $g_i \in G_i =$

$\mathcal{U}_1(\langle a_1\rangle \times\cdots\times\langle a_i\rangle) \times (\langle a_{i+1}\rangle\times\cdots\times\langle a_d\rangle) =$

$\mathcal{U}_1(\langle a_1\rangle \times\cdots\times\langle a_i\rangle) \times (\langle a_{i+1}\, g_{i+1}\rangle\times\cdots\times\langle a_dg_d\rangle)$, p divides

r_i. Hence $g_i^{r_i+1} = (\pi_{j=1}^{i-1} (a_j^{r_j}))(\pi_{j=1}^{d} a_j^{r_j} g_j^{r_j}) \in$

$\langle a_1,\ldots,a_{i-1}, a_ig_i,\ldots, a_dg_d\rangle$ and p does not divide r_i+1.

Thus g_i and hence $a_i \in \langle a_1,\ldots,a_{i-1},a_ig_i,\ldots, a_dg_d\rangle$.

Therefore $G = \langle a_1,\ldots, a_{i-1}, a_ig_i,\ldots, a_dg_d\rangle$. Since $\text{ord}(a_ig_i) \leq \text{ord}(a_i)$ and $\text{ord}(a_{i+j}g_{i+j}) = \text{ord}(a_{i+j})$ for $1\leq j\leq d-i$, $a_1,\ldots,a_{i-1}, a_ig_i,\ldots, a_dg_d$ is a direct basis of G and $\text{ord}(a_ig_i) = \text{ord}(a_i)$.

It follows that $a_1g_1,\ldots, a_dg_d$ is a direct basis of G and $\text{ord}(a_ig_i) = \text{ord}(a_i)$ for $1\leq i\leq d$. By Theorem 1.29, the map γ, defined by $\gamma: a_i \to a_ig_i$, is an automorphism of G.

The conclusion follows since the G_i are the same whether obtained from $a_1,\ldots,a_d$ or $a_1^{m_1},\ldots,a_d^{m_d}$ for $o<m_i<p$.

Theorem 3.3: If when $1 \leq j \leq t$ and $f_{j-1} < i \leq f_j$, $g_i \in G_i \cap \Omega_{n_j}(G)$ and $o < m_i < p$, then the map α, defined by $\alpha: a_i \to a_i^{m_i} g_i$ for $i=1,\ldots,d$, is an element of $S_o(s_1)$ and if all $m_i \equiv 1 \bmod p$, then $\alpha \in S_{n_1 d}(s_1)$.

Proof: By Theorem 3.2, $\alpha \in \mathrm{Aut}(G)$. Since $G_{d-1} = \langle a_d, G_d \rangle = \langle a_d, \Phi(G) \rangle$, α fixes G_{d-1}. Suppose α fixes G_i for some i with $1 \leq i \leq d$. Since $G_{i-1} = \langle a_i, G_i \rangle$, α fixes G_{i-1} as well. Hence α fixes G_i for $i=0,1,\ldots,d$. Since if $i = qd+r$ where $o < r \leq d$, $G_i = \mathcal{U}_q(G_r)$ and α fixes G_r, therefore α fixes the characteristic subgroup $G_i = \mathcal{U}_q(G_r)$ of G_r for $i = o,1,\ldots$. Hence $\alpha \in S_o(s_1)$.

Suppose each $m_i \equiv 1 \bmod p$. Since $G_{i-1} = \langle a_i, G_i \rangle$ for $1 \leq i \leq d$ and $G_i^{\alpha} = G_i$ and $a_i^p \in \Phi(G) = G_d \leq G_i$, $g^{-1}g^{\alpha} \in G_i$ for each $g \in G_{i-1}$ when $1 \leq i \leq d$. Since if $i=qd+r$ where $o < r \leq d$, $G_i = \mathcal{U}_q(G_r)$ and $G_d = \mathcal{U}_1(G_o)$, $g^{-1}g^{\alpha} \in G_i$ for each $g \in G_{i-1}$ and for $i=1,2,\ldots$. Hence $\alpha \in S_{n_1 d}(s_1)$.

Lemma 3.4: For $1 \leq j \leq t$, $\mathrm{Aut}(M_{j-1}/M_j)$ is a homomorphic image of $\mathrm{Aut}(G)$.

Proof: By Theorem 1.46, any automorphism of $H_j/\mathcal{U}_1(H_j)$ can be induced by an automorphism of H_j. Hence any automorphism of M_{j-1}/M_j can be induced by an automorphism of G.

3.2 The p-Sylow Subgroups of Aut(G).

This section contains characterizations of a p-Sylow subgroup of Aut(G), its normalizer and centralizer in Aut(G) and, as a by-product of these, a characterization of Z(Aut(G)) and the fact that the normalizer in Aut(G) of a p-Sylow subgroup of Aut(G) is supersolvable.

Theorem 3.5: If $\alpha \in Aut(G)$ and $\alpha \notin S_o(s_1)$, then there is an element $\beta \in S_{n_1 d}(s_1)$ such that $\alpha\beta\alpha^{-1} \notin S_o(s_1)$.

Proof: Suppose $\alpha \in Aut(G)$ and $\alpha \notin S_o(s_1)$. Let $m = qd+r$, where $o<r\leq d$, be the smallest integer such that $G_m{}^{\alpha} \neq G_m$. Since $G_m = \mathcal{V}_q(G_r)$, $G_r{}^{\alpha} \neq G_r$ and $r = m$ by the choice of m. $G_r = \langle a_1^p \rangle x \cdots x \langle a_r^p \rangle x \langle a_{r+1} \rangle x \cdots x \langle a_d \rangle$. For $1\leq i\leq r$, $a_i^p \in \mathcal{V}_1(G) \subseteq G_r{}^{\alpha}$. Hence there is an integer u, $r<u\leq d$, such that $a_u \notin G_r{}^{\alpha}$.

$G_r{}^{\alpha} \subseteq G_{r-1}^{\alpha} = G_{r-1} = \langle a_r, G_r \rangle$ and $a_r^p \in G_r$. Hence $G_r{}^{\alpha} \subseteq G_{r-1} = G_r \dot{\cup} a_r G_r \dot{\cup} \cdots \dot{\cup} a_r^{p-1} G_r$. If $a_r G_r \cap G_r{}^{\alpha} = \emptyset$, then $a_r^i G_r \cap G_r{}^{\alpha} = \emptyset$ for all i with $o<i<p$ which would imply $G_r{}^{\alpha} \subseteq G_r$ which contradicts the assumption that $G_r{}^{\alpha} \neq G_r$. Hence $a_r G_r \cap G_r{}^{\alpha} \neq \emptyset$. Suppose $a_r^{\omega} h \in G_r{}^{\alpha}$ where p does not divide ω and $h \in G_r \cap K_r$ (cf. 3.1.4 in section 3.1).

Define the map β by $a_i{}^{\beta} = a_i$ if $i \neq r$ and $a_r{}^{\beta} = a_r a_u$. Since $r<u$, $a_u \in G_r \cap \Omega_{n_j}(G)$ where $f_{j-1}<r\leq f_j$. Hence, by Theorem 3.3, $\beta \in S_{n_1 d}(s_1)$.

Since $a_r{}^{\omega}h \in G_r{}^{\alpha}$ and $(a_r{}^{\omega}h)^{-1}(a_r{}^{\omega}h)^{\beta} = a_u{}^{\omega} \notin G_r{}^{\alpha}$, $s_1{}^{\alpha\beta} \neq s_1{}^{\alpha}$. Hence $s_1{}^{\alpha\beta\alpha^{-1}} \neq s_1{}^{\alpha\alpha^{-1}} = s_1$. Thus $\alpha\beta\alpha^{-1} \notin S_o(s_1)$.

Corollary 3.6: $N_{Aut(G)}(S_{n_1d}(s_1)) = S_o(s_1)$.

Corollary 3.7: $S_{n_1d}(s_1)$ is a p-Sylow subgroup of Aut(G).

Proof: By Theorem 1.33, $S_{n_1d}(s_1)$ is the p-Sylow subgroup of $S_o(s_1)$. Hence if $S_{n_1d}(s_1)$ were not a p-Sylow subgroup of Aut(G), there would be an element of Aut(G), outside of $S_o(s_1)$, which would normalize $S_{n_1d}(s_1)$. This would contradict Theorem 3.5.

Corollary 3.8: The normalizer in Aut(G) of a p-Sylow subgroup of Aut(G) is supersolvable.

Proof: By Corollaries 3.6 and 3.7, the normalizer of a p-Sylow subgroup of Aut(G) is conjugate to $S_o(s_1)$. By Theorem 2.6, $S_o(s_1)$ is supersolvable.

Definition 3.9: G is multicyclic if each H_i is cyclic, for $1 \leq i \leq t$.

Theorem 3.10: Aut(G) has a normal p-Sylow subgroup if, and only if, G is multicyclic.

Proof: If G is multicyclic, then $s_1 = s_2$. By Theorem 3.1, s_2 is characteristic. Hence $S_o(s_1)$ is Aut(G). Hence, by Corollary 3.6, $N_{Aut(G)}(S_{n_1 d}(s_1)) = Aut(G)$. Thus Aut(G) has a normal p-Sylow subgroup $S_{n_1 d}(s_1)$ (cf. Corollary 3.7).

If H_j is not cyclic, then $a_{f_{j-1}+1} \neq a_{f_j}$ and $ord(a_{f_{j-1}+1}) = ord(a_{f_j})$. Define the map β by $a_i^{\beta} = a_i$ if $i \neq f_{j-1}+1$ and $i \neq f_j$ and $a_{f_{j-1}+1}^{\beta} = a_{f_j}$ and $a_{f_j}^{\beta} = a_{f_{j-1}+1}$. By Theorem 1.29, $\beta \in Aut(G)$. Since $a_{f_{j-1}+1} \notin G_{f_{j-1}+1}$ and $a_{f_j} \in G_{f_{j-1}+1}$, $\beta \notin S_o(s_1)$ Hence $N_{Aut(G)}(S_{n_1 d}(s_1)) \neq Aut(G)$. Therefore, Aut(G) does not have a normal p-Sylow subgroup.

Lemma 3.11: Let α and β be the maps defined by $\alpha: a_i \to a_i^{m_i} g_i$ and $\beta: a_i \to a_i^{r_i} h_i$ where $o < m_i, r_i < p$ and $g_i, h_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, for $i=1,\ldots,d$. Then $\alpha = \beta$ if, and only if, $m_i = r_i$ and $g_i = h_i$ for $i = 1,\ldots,d$.

Proof: By Theorem 3.2, $\alpha, \beta \in Aut(G)$. If $m_i = r_i$ and $g_i = h_i$ for $i = 1,\ldots,d$, then $a_i^{\alpha} = a_i^{\beta}$ for $i = 1,\ldots,d$. Hence $\alpha = \beta$.

If $\alpha = \beta$, then $a_i^{\alpha} = a_i^{\beta}$ and $a_i^{m_i} g_i = a_i^{r_i} h_i$ for $i = 1,\ldots,d$.

Hence $a_i^{m_i-r_i} \in G_i$ for $i = 1,\ldots,d$. Thus p divides m_i-r_i for $i = 1,\ldots,d$. Since $0<m_i, r_i<p$, $-p<m_i-r_i<p$ and therefore $m_i = r_i$ for $i = 1,\ldots,d$. Since $a_i^{m_i} g_i = a_i^{r_i} h_i$ and $m_i = r_i$, $g_i = h_i$ for $i = 1,\ldots,d$.

Lemma 3.12: $\prod_{i=1}^{d} \operatorname{ord}(\Omega_{n_{j(i)}}(G)) = p^n$ where $j(i)$ is related to i by $f_{j(i)-1}<i\leq f_{j(i)}$ and $n = \sum_{k=1}^{t} f_k^2(n_k-n_{k+1})$.

Proof: $\prod_{i=1}^{f_1} \operatorname{ord}(\Omega_{n_1}(H_1)) = [\operatorname{ord}(H_1)]^{f_1} = p^{f_1^2 n_1}$.

Suppose, for some k with $1\leq k<t$, that

$\prod_{i=1}^{f_k} \operatorname{ord}(\Omega_{n_j}(H_1\times\cdots\times H_k)) = p^r$ where $r = (\sum_{j=1}^{k-1} f_j^2(n_j-n_{j+1}))$ $+ f_k^2 n_k$. Then $\prod_{i=1}^{f_{k+1}} \operatorname{ord}(\Omega_{n_j}(H_1\times\cdots\times H_{k+1})) =$

$\prod_{i=1}^{f_k} [\operatorname{ord}(\Omega_{n_j}(H_1\times\cdots\times H_k))\ \operatorname{ord}(H_{k+1})])\cdot(\prod_{i=f_k+1}^{f_{k+1}} \operatorname{ord}(\Omega_{n_{k+1}}$ $(H_1\times\cdots\times H_{k+1}))) = p^n$ where $n = r+f_k(f_{k+1}-f_k)n_{k+1} +$

$(f_{k+1}-f_k)f_{k+1}\, n_{k+1} = (\sum_{j=1}^{k-1} f_j^2(n_j-n_{j+1})) + f_k^2(n_k-n_{k+1}) +$

$f_{k+1}^2\, n_{k+1} = (\sum_{j=1}^{k} f_j^2(n_j-n_{j+1})) + f_{k+1}^2 n_{k+1}$.

Hence $\prod_{i=1}^{d} \text{ord}(\Omega_{n_j}(G)) = \prod_{i=1}^{f_t} \text{ord}(\Omega_{n_j}(G)) = p^n$ where

$n = (\sum_{j=1}^{t-1} f_j^2(n_j-n_{j+1})) + f_t^2 n_t = \sum_{j=1}^{t} f_j^2(n_j-n_{j+1})$ since n_{t+1} $= 0$.

Theorem 3.13: Let $P = \{\alpha | a_i^{\alpha} = a_i g_i$, where $g_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, for $i=1,\ldots,d\}$. Then $P = S_{n_1 d}(s_1)$ and $\text{ord}(S_{n_1 d}(s_1)) = p^n$ where $n = \sum_{j=1}^{t} [f_j^2(n_j-n_{j+1}) - d_j(d_j+1)/2]$.

Proof: By Theorem 3.3, $P \leq S_{n_1 d}(s_1)$. If $\alpha \in S_{n_1 d}(s_1)$, then $a_i^{\alpha} = a_i(a_i^{-1}a_i^{\alpha})$ and $a_i^{-1}a_i^{\alpha} \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, for $i=1,\ldots,d$. Hence $\alpha \in P$ and $S_{n_1 d}(s_1) \leq P$. Therefore $S_{n_1 d}(s_1) = P$.

By Lemma 3.11, $\text{ord}(P) = \prod_{i=1}^{d} \text{ord}(G_i \cap \Omega_{n_j}(G))$ where j is related to i by $f_{j-1} < i \leq f_j$. Using Lemma 3.12 and the definition of G_i, $\text{ord}(P) = p^n$ where

$n = \sum_{j=1}^{t} [f_j^2(n_j-n_{j+1}) - d_j(d_j+1)/2]$.

Theorem 3.14: Let $A = \{\alpha | a_i^{\alpha} = a_i^{m_i} g_i$, where $g_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, and $0 < m_i < p$, for $i=1,\ldots,d\}$.

Then $S_o(s_1) = A$ and $\text{ord}(S_o(s_1)) = (p-1)^d p^n$ where

$$n = \sum_{j=1}^{t} [f_j^2(n_j - n_{j+1}) - d_j(d_j+1)/2].$$

Proof: By Theorem 3.3, $A \leq S_o(s_1)$. Let $\alpha \in S_o(s_1)$. Then $a_i^{\alpha} \in G_{i-1}$ for $i=1,\ldots,d$. Since $G_{i-1} = \langle a_i, G_i \rangle$ and $a_i^p \in G_i$ for $1 \leq i \leq d$, $a_i^{\alpha} = a_i^{m_i} g_i$ where $g_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, and $o < m_i \leq p$, for $i=1,\ldots,d$, since $\text{ord}(a_i^{\alpha} a_i^{-m_i}) \leq \text{ord}(a_i) = p^{n_j}$. If, for some i, $1 \leq i \leq d$, $m_i = p$, then $a_i^{\alpha} \in G_i$ and hence $G_{i-1}^{\alpha} \leq G_i$ contrary to the fact that $G_i \subset G_{i-1}$. Thus $o < m_i < p$ for $i=1,\ldots,d$ and $\alpha \in A$ and $S_o(s_1) \leq A$. Therefore $S_o(s_1) = A$.

By Lemma 3.11, $\text{ord}(A) = (p-1)^d \prod_{i=1}^{d} \text{ord}(G_i \cap \Omega_{n_j}(G))$ where j is related to i by $f_{j-1} < i \leq f_j$. As in the proof of Theorem 3.13, $\text{ord}(A) = (p-1)^d p^n$ where $n = \sum_{j=1}^{t} [f_j^2(n_j - n_{j+1}) - d_j(d_{j+1})/2]$.

Corollary 3.15: The number of p-Sylow subgroups of Aut(G) is $\prod_{j=1}^{t} [\prod_{i=1}^{d_j - 1} (\sum_{k=o}^{i} p^k)]$.

Proof: By Corollaries 3.6 and 3.7, the normalizer in Aut(G) of a p-Sylow subgroup of Aut(G) is conjugate to

$S_o(s_1)$. By Theorems 3.14 and 1.13 the index of $S_o(s_1)$ in Aut(G) is $\prod_{j=1}^{t} [\prod_{i=1}^{d_j-1} (\sum_{k=o}^{i} p^k)]$.

Theorem 3.16: Let $C_1 = \{\alpha | a_i^{\alpha} = a_i^{m}$ for $(m,p)=1$ and $1 \leq i \leq d\}$ and let $C_2 = \{\alpha | a_1^{\alpha} = a_1^{m} h$; $a_i^{\alpha} = a_i^{m}$ if $i>1$, with $(m,p) = 1$ and $h \in \Omega_1(\langle a_{d_1} \rangle)\}$. If H_1 is cyclic, then $C_{Aut(G)}(S_{n_1 d}(s_1)) = C_1$ and $ord(C_{Aut(G)}(S_{n_1 d}(s_1))) = (p-1)p^{(n_1-1)}$. If H_1 is not cyclic, then $C_{Aut(G)}(S_{n_1 d}(s_1)) = C_2$ and $ord(C_{Aut(G)}(S_{n_1 d}(s_1)) = (p-1)p^{n_1}$.

Proof: Let $\alpha \in C_{Aut(G)}(S_{n_1 d}(s_1))$. By Corollary 3.6, $N_{Aut(G)}(S_{n_1 d}(s_1)) = S_o(s_1)$, so $\alpha \in S_o(s_1)$. Hence $G_1^{\alpha} = G_1$. Since $G = \langle a_1, G_1 \rangle$ and $G^{\alpha} \neq G_1$, there is an integer m with $(p,m)=1$ and an element $h \in K_1$ (cf. 3.1.4 in section 3.1) such that $a_1^{\alpha} = a_1^{m} h$.

Let k be an integer with $1<k \leq d$. Define the map β by $a_1^{\beta} = a_1 a_k$ and $a_i^{\beta} = a_i$ for $1<i \leq d$. By Theorem 3.3, $\beta \in S_{n_1 d}(s_1)$. Hence $a_1^{m} a_k^{m} h = (a_1^{m} h)^{\beta} = a_1^{\alpha\beta} = a_1^{\beta\alpha} = (a_1 a_k)^{\alpha} = a_1^{m} h\, a_k^{\alpha}$. Therefore $a_k^{\alpha} = a_k^{m}$. Thus, for each i with $1<i \leq d$, $a_i^{\alpha} = a_i^{m}$.

If there is an integer r and an integer $k \neq d_1$ with $1<k\leq d$ and an element $h_1 \in K_1 \cap K_k$ such that $a_k^r \neq 1$ and $h = a_k^r h_1$, define the map γ by $a_i^\gamma = a_i$ if $i \neq k$ and $a_k^\gamma = a_k a_{d_1}^{p^{n_1-n_j}}$ where $f_{j-1}<k\leq f_j$. If $n_1=n_j$, then $k<d_1$ and $a_{d_1} \in G_k \cap \Omega_{n_j}(G)$. If $n_1>n_j$, then $a_{d_1}^{p^{n_1-n_j}} \in \mathcal{V}_1(G) \cap \Omega_{n_j}(G) = G_d \cap \Omega_{n_j}(G) \leq G_k \cap \Omega_{n_j}(G)$. Hence, by Theorem 3.3, $\gamma \in S_{n_1 d}(s_1)$. $a_1^{\alpha\gamma} = (a_1^m h)^\gamma = a_1^m a_k^r a_{d_1}^{rp^{n_1-n_j}} h_1$ and $a_1^{\gamma\alpha} = a_1^\alpha = a_1^m a_k^r h_1$. $a_{d_1}^{rp^{n_1-n_j}} \neq 1$ since p^{n_j} does not divide r. Hence $a_1^{\alpha\gamma} \neq a_1^{\gamma\alpha}$ contrary to the assumption that $\alpha \in C_{Aut(G)}(S_{n_1 d}(s_1))$. Therefore $h \in \langle a_{d_1} \rangle$.

If H_1 is cyclic, then $d_1 = 1$ and $h \in \langle a_{d_1} \rangle \cap K_1 = 1$. Thus $\alpha \in C_1$ and $C_{Aut(G)}(S_{n_1 d}(s_1)) \leq C_1$.

If H_1 is not cyclic, then $d_1 > 1$. Define the map δ by $a_i^\delta = a_i$ if $i \neq d_1$ and $a_{d_1}^\delta = a_{d_1} a_1^p$. Since $a_1^p \in \mathcal{V}_1(G) = G_d \leq G_{d_1}$, $\delta \in S_{n_1 d}(s_1)$ by Theorem 3.3. If $h = a_{d_1}^{r_1}$, then $a_1^{\alpha\delta} = (a_1^m h)^\delta = a_1^m a_{d_1}^{r_1} a_1^{r_1 p}$ and $a_1^{\delta\alpha} = a_1^\alpha = a_1^m a_{d_1}^{r_1}$. Since $a_1^{\alpha\delta} = a_1^{\delta\alpha}$, $a_1^{r_1 p} = 1$ which implies that p^{n_1-1} divides r_1. Hence $h = a_{d_1}^{r_1} \in \Omega_{n_1-1}(\langle a_{d_1} \rangle) \leq \Omega_1(\langle a_{d_1} \rangle)$, by Theorem 1.17. Thus $\alpha \in C_2$ and $C_{Aut(G)}(S_{n_1 d}(s_1)) \leq C_2$.

$C_1 \leq z(\mathrm{Aut}(G)) \leq C_{\mathrm{Aut}(G)}(S_{n_1 d}(s_1))$. Hence if H_1 is cyclic, $C_{\mathrm{Aut}(G)}(S_{n_1 d}(s_1)) = C_1$.

Suppose H_1 is not cyclic and $\alpha \in C_2$ with $a_1^{\alpha} = a_1^{m}h$, $a_i^{\alpha} = a_i^{m}$ for $1<i\leq d$ where $(m,p)=1$ and $h \in \Omega_1(\langle a_{d_1} \rangle)$. By Theorem 3.3, $\alpha \in S_o(s_1)$. If $\beta \in S_{n_1 d}(s_1)$, then β fixes $G_{(n_1-1)d+d_1-1} = \mathcal{U}_{n_1}(\langle a_1 \rangle \times \cdots \times \langle a_{d_1-1} \rangle) \times \mathcal{U}_{n_1-1}(\langle a_{d_1} \rangle \times \cdots \times \langle a_d \rangle) = \mathcal{U}_{n_1-1}(\langle a_{d_1} \rangle)$ elementwise since $G_{(n_1-1)d+d_1} = \mathcal{U}_{n_1}(\langle a_1 \rangle \times \cdots \times \langle a_{d_1} \rangle) \times \mathcal{U}_{n_1-1}(\langle a_{d_1+1} \rangle \times \cdots \times \langle a_d \rangle) = 1$. Hence $h^{\beta}=h$, since $\Omega_1(\langle a_{d_1} \rangle) \leq \mathcal{U}_{n_1-1}(\langle a_{d_1} \rangle)$ by Theorem 1.36. Thus $a_1^{\alpha\beta} = (a_1^{m}h)^{\beta} = (a_1^{\beta})^{m}h$. Let $a_1^{\beta} = a_1^{rp+1}h_1$ where $h_1 \in K_1$. Then $a_1^{\beta\alpha} = a_1^{m(rp+1)}h^{(rp+1)}h_1^{m} = (a_1^{\beta})^{m} h^{rp+1} = (a_1^{\beta})^{m} h$ since $h \in \Omega_1(\langle a_{d_1} \rangle)$. Therefore $a_1^{\alpha\beta} = a_1^{\beta\alpha}$.

For $i>1$, let $a_i^{\beta} = a_1^{p^{r_i}}h_i$ where $h_i \in K_1$. This can always be done since $a_i \in G_{i-1} \leq G_1 = \langle a_1^{p} \rangle \times K_1$ and $G_{i-1}^{\beta} = G_{i-1}$. $a_i^{\beta\alpha} = (a_1^{pr_i}h_i)^{\alpha} = a_1^{mpr_i}h^{pr_i}h_i^{m} = (a_i^{\beta})^{m} h^{pr_i} = (a_i^{\beta})^{m}$ since $h \in \Omega_1(\langle a_{d_1} \rangle)$. Since $a_i^{\alpha\beta} = (a_i^{m})^{\beta} = (a_i^{\beta})^{m}$, $a_i^{\alpha\beta} = a_i^{\beta\alpha}$.

Hence $a_i^{\alpha\beta}=a_i^{\beta\alpha}$ for $1\leq i\leq d$ and $\alpha\beta=\beta\alpha$. Therefore $\alpha\in C_{(Aut(G)}(S_{n_1}d(s_1))$ and $C_2 \subseteq C_{Aut(G)}(S_{n_1}d(s_1))$. Hence $C_2 = C_{Aut(G)}(S_{n_1}d(s_1))$.

Suppose $\alpha:a_i\to a_i^m$ and $\beta:a_i\to a_i^n$ for $(n,p)=1=(m,p)$ and $i=1,\ldots,d$. If $m\equiv n \bmod p^{n_1}$, then $\alpha=\beta$. If $\alpha=\beta$, then $a_1^m=a_1^n$ and hence $m\equiv n \bmod p^{n_1}$. Thus $ord(C_1) = (p-1)p^{n_1-1}$. From this and Lemma 3.11, $ord\ (C_2) = (p-1)p^{n_1}$.

Corollary 3.17: If H_1 is cyclic, $Z(S_{n_1}d(s_1)) = \{\alpha \mid a_i^{\alpha}=a_i^m$ where $m\equiv 1 \bmod p$, $i=1,\ldots,d\}$ and $ord(Z(S_{n_1}d(s_1))) = p^{n_1-1}$. If H_1 is not cyclic, then $Z(S_{n_1}d(s_1)) = \{\alpha \mid a_1^{\alpha}=a_1^m h$, $a_i^{\alpha}=a_i^m$, $m\equiv 1 \bmod p$, $h\in\Omega_1(\langle a_{d_1}\rangle)$, $i=2,3,\ldots,d\}$ and $ord(Z(S_{n_1}d(s_1))) = p^{n_1}$.

Corollary 3.18: Taking C_1 as in Theorem 3.16, $Z(Aut(G)) = C_1$ and $ord(Z(Aut(G))) = (p-1)p^{n_1-1}$.

Proof: $C_1 \leq Z(Aut(G)) \leq C_{Aut(G)}(S_{n_1}d(s_1))$. From Theorem 3.16, if H_1 is cyclic $C_1 = Z(Aut(G)) \leq C_2$.

Suppose H_1 is not cyclic and let $\alpha \in Z(\mathrm{Aut}(G)) \subseteq C_2$, as in Theorem 3.16. Then $a_1^{\alpha}=a_1^{m}h$, $a_i^{\alpha}=a_i^{m}$ for $1<i\leq d$, $(m,p) = 1$ and $h \in \Omega_1(\langle a_{d_1} \rangle)$. Define the map β by $a_1^{\beta}=a_{d_1}$, $a_{d_1}^{\beta}=a_1$ and $a_i^{\beta}=a_i$ if $i \neq 1$ and $i \neq d_1$. By Theorem 1.29, $\beta \in \mathrm{Aut}(G)$. Since, by Theorem 1.36, $\Omega_1(\langle a_{d_1} \rangle) \subseteq \mathcal{U}_{n_1-1}(\langle a_{d_1} \rangle)$, let $h = a_{d_1}^{rp^{(n_1-1)}}$. Then $a_1^{\alpha\beta}=(a_1^{m}a_{d_1}^{rp^{(n_1-1)}})^{\beta} = a_1^{rp^{(n_1-1)}}a_{d_1}^{m}$ and $a_1^{\beta\alpha}=a_{d_1}^{\alpha}=a_{d_1}^{m}$. Since $\alpha \in Z(\mathrm{Aut}(G))$, $a_1^{\alpha\beta}=a_1^{\beta\alpha}$ and $a_1^{rp^{(n_1-1)}} = 1$. Hence p divides r and $h = 1$. Therefore if $\alpha \in Z(\mathrm{Aut}(G))$, $\alpha \in C_2$ and $h=1$. Thus $\alpha \in C_1$ and $Z(\mathrm{Aut}(G)) \subseteq C_1$.

Therefore, in either case, $Z(\mathrm{Aut}(G)) = C_1$. $\mathrm{ord}(C_1) = (p-1)p^{n_1-1}$ as in the proof of Theorem 3.16.

3.3 The Intersection of the p-Sylow Subgroups of Aut(G).

Throughout the rest of this chapter W is the intersection of the p-Sylow subgroups of Aut(G). W and $C_{\mathrm{Aut}(G)}(W)$ both characteristic subgroups of Aut(G), are characterized in this section.

Theorem 3.19: Let $T = \{\alpha | a_i^{\alpha} = a_i g_i$ where $g_i \in M_j \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j, i=1,\ldots,d\}$. Then $S_{n_1 t}(s_2) = T$ and $\text{ord}(S_{n_1 t}(s_2)) = p^n$ where $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$.

Proof: Let $\alpha \in T$. Since $g_i \in M_j \cap \Omega_{n_j}(G) \leq G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, $\alpha \in S_{n_1 d}(s_1)$ by Theorem 3.3. By theorem 3.1, each M_j is characteristic. For $1 \leq j \leq t$, $M_{j-1} = \langle a_{f_{j-1}+1}, \ldots, a_{f_j}, M_j \rangle$. Hence for each $g \in M_{j-1}, g^{-1}g^{\alpha} \in M_j$ for $1 \leq j \leq t$. If $j = qt+r$ where $o < r \leq t$, $M_j = \mathcal{V}_q(M_r)$. Also $M_t = \mathcal{V}_1(M_o)$. Hence $g^{-1}g^{\alpha} \in M_j$ for each $g \in M_{j-1}$ and for $j=1,2,\ldots$. Thus $\alpha \in S_{n_1 t}(s_2)$ and $T \leq S_{n_1 t}(s_2)$.

Let $\alpha \in S_{n_1 t}(s_2)$. If $f_{j-1} < i \leq f_j$, then $a_i \in M_{j-1}$. Therefore $a_i^{-1} a_i^{\alpha} \in M_j \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, and $a_i^{\alpha} = a_i(a_i^{-1} a_i^{\alpha})$ for i-1.,,,.d. Hence $\alpha \in T$ and $S_{n_1 t}(s_2) \leq T$. Thus $S_{n_1 t}(s_2) = T$.

By Lemma 3.11, $\text{ord}(T) = \prod_{i=1}^{t} [\text{ord}(M_i \cap \Omega_{n_i}(G)]^{d_i} =$ $\prod_{i=1}^{t} [\text{ord}(\Omega_{n_i}(G))/p^{d_i}]^{d_i} = p^n$ where $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$ (cf. Lemma 3.12).

Theorem 3.20: $W = S_{n_1 t}(s_2)$.

Proof: Let Q be a p-Sylow subgroup of Aut(G). By Corollary 3.7, there is an element $\beta \in Aut(G)$ such that $(S_{n_1 d}(s_1))^\beta = Q$. By Theorem 1.31, $(S_{n_1 d}(s_1))^\beta = S_{n_1 d}(s_1{}^\beta)$. Since s_1 is a refinement of s_2 and s_2 is characteristic, by Theorem 3.1, $s_1{}^\beta$ is a refinement of s_2. Hence $S_{n_1 t}(s_2) \subseteq S_{n_1 d}(s_1{}^\beta) = Q$. Therefore $S_{n_1 t}(s_2) \subseteq W$.

Let $\alpha \in W$. Then by Corollary 3.7, $\alpha \in S_{n_1 d}(s_1)$ and hence $a_i{}^\alpha = a_i g_i$ where $g_i \in G_i$ for $1 \leq i \leq d$. Suppose for some k, $1 \leq k \leq d$, $g_k \notin M_j$ when $f_{j-1} < k \leq f_j$. $M_{j-1} \supseteq G_{k-1} \supseteq G_k \supseteq M_j$ and since $g_k \in G_k$, $G_k \supset M_j$. Hence for some u, with $k < u \leq f_j$, $g_k = a_u{}^r h_k$ where $(r,p)=1$ and $h_k \in K_u$. Define the map β by $a_i{}^\beta = a_i$ if $i \neq k$ and $i \neq u$ and $a_k{}^\beta = a_u$ and $a_u{}^\beta = a_k$. Since $f_{j-1} < k < u \leq f_j$, $ord(a_k) = ord(a_u)$. Thus, by Theorem 1.29, $\beta \in Aut(G)$. $G_k{}^\beta = \mathcal{V}_1(\langle a_1 \rangle x \cdots x \langle a_{k-1} \rangle x \langle a_u \rangle)$ $x \langle a_k \rangle x \cdots x \langle a_{u-1} \rangle x \langle a_{u+1} \rangle x \cdots x \langle a_d \rangle$. Since $a_u{}^r \notin \langle a_u{}^p \rangle$ and $h_k \in K_u$, $g_k \notin G_k{}^\beta$. Hence $a_k{}^{-1} a_k{}^\alpha \notin G_k{}^\beta$ whereas $a_k \in G_{k-1} = G_{k-1}{}^\beta$. Therefore $\alpha \notin S_{n_1 d}(s_1{}^\beta)$ contradicting the fact that $W \subseteq (S_{n_1 d}(s_1))^\beta = S_{n_1 d}(s_1{}^\beta)$, by Theorem 1.31.

Hence, for $i = 1,\ldots, d$, $g_i \in M_j$ when $f_{j-1}<i\leq f_j$ and $\alpha\in S_{n_1 t}(s_2)$, by Theorem 3.19, since also $g_i\in\Omega_{n_j}(G)$ when $f_{j-1}<i\leq f_j$. Therefore $S_{n_1 t}(s_2) \geq W$ and it follows that $W = S_{n_1 t}(s_2)$.

Theorem 3.21: Let $V = \{\alpha \mid a_i{}^{\alpha}=a_i{}^{m_i}h_i$ for $1\leq i\leq d_1$ and $a_i{}^{\alpha}=a_i{}^{m_1}$ for $d_1<i\leq d$ where $(m_1,p) = 1$ and $m_i\equiv m_1 \bmod p^{n_1-1}$ for $2\leq i\leq d_1$ and $h_i\in\Omega_1(H_1)\cap K_i$ for $1\leq i\leq d_1$.$\}$. Then $C_{Aut(G)}(W)=V$ and $ord(C_{Aut(G)}(W))=(p-1)p^{(n_1+d_1{}^2-2)}$, if G is not elementary abelian.

Proof: Let $\alpha\in C_{Aut(G)}(W)$. Then $a_i{}^{\alpha}=a_i{}^{m_i}h_i$ where $h_i\in K_i$.

Let k be an integer with $1\leq k\leq d_1$. Define the map β by $a_k{}^{\beta} = a_k a_d{}^{p^{(n_t-1)}}$ and $a_i{}^{\beta} = a_i$ for $i \neq k$ and $1\leq i\leq d$. If $d>d_1$, then $a_d\in M_1$. If $d=d_1$, then, since G is not elementary abelian, $n_t-1\geq 1$ and $a_d{}^{p^{(n_t-1)}}\in M_1$. Thus, by Theorems 3.19 and 3.20, $\beta\in W$. Hence $a_k{}^{m_k}h_k a_d{}^{m_k p^{(n_t-1)}} = (a_k{}^{m_k}h_k)^{\beta} =$

$a_k{}^{\alpha\beta}=a_k{}^{\beta\alpha} = (a_k a_d{}^{p^{(n_t-1)}})^{\alpha} = a_k{}^{m_k}h_k(a_d{}^{p^{(n_t-1)}})^{\alpha}$. Therefore

$a_d{}^{m_1 p^{(n_t-1)}} = (a_d{}^{p^{(n_t-1)}})^{\alpha}$. Since $a_d{}^{p^{(n_t-1)}} \neq 1$,

$(a_d^{p^{(n_t-1)}})^\alpha \neq 1$ and hence $a_d^{m_k p^{(n_t-1)}} \neq 1$. Therefore, $(p,m_k) = 1$. Thus $(p,m_i) = 1$ for $i=1,\ldots,d_1$ and, in particular $(p,m_1) = 1$.

Let k be an integer with $1 \leq k \leq d_1$. Define the map γ by $a_1^\gamma = a_1 a_k^p$ and $a_i^\gamma = a_i$ for $2 \leq i \leq d$. By Theorems 3.19 and 3.20, $\gamma \in W$. Hence $a_1^{m_1} h_1 a_k^{m_k p} h_k^p = (a_1 a_k^p)^\alpha = a_1^{\gamma\alpha} = a_1^{\alpha\gamma} = (a_1^{m_1} h_1)^\gamma = a_1^{m_1} a_k^{m_1 p} h_1$. Thus $h_k^p = 1$ and $a_k^{m_k p} = a_k^{m_1 p}$. Therefore, for $1 \leq i \leq d_1$, $m_i \equiv 1 \bmod p^{n_1-1}$ and $h_i \in \Omega_1(G) \cap K_i$.

Suppose for some k with $1 \leq k \leq d_1$, $h_k \notin H_1$. Then there is an integer u with $d_1 < u \leq d$ and $f_{j-1} < u \leq f_j$ and $j > 1$ and there is an integer r with $(p,r)=1$ and an element $g_k \in K_k \cap K_u$ such that $h_k = a_u^{rp^{(n_j-1)}} g_k$ $(h_k \in \Omega_1(G) \cap K_k = \Omega_1(\langle a_u \rangle x K_u) x K_k = \Omega_1(\langle a_u \rangle) x \Omega_1(K_u \cap K_k) \subseteq \mho^{n_{j-1}}(\langle a_u \rangle) x\ \Omega_1(K_u \cap K_k)$, by Theorems 1.15 and 1.36). Define the map δ by $a_u^\delta = a_u a_k^{p^{(n_1-n_j)}}$ and $a_i^\delta = a_i$ for $i \neq u$ and $1 \leq i \leq d$. Since $j > 1$, $n_1 > n_j$ and $a_k^{p^{(n_1-n_j)}} \in M_t \subseteq M_j$. Hence, by Theorems 3.19 and 3.20,

$\delta \in W$. Therefore, $a_k^{m_k} g_k\, a_u^{rp^{(n_j-1)}}\, a_k^{rp^{(n_1-1)}} =$

$(a_k^{m_k}\, a_u^{rp^{(n_j-1)}}\, g_k)^{\delta} = a_k^{\alpha\delta} = a_k^{\delta\alpha} = a_k^{\alpha} = a_k^{m_k} g_k\, a_u^{rp^{(n_j-1)}}$.

Hence $a_k^{rp^{(n_1-1)}} = 1$ contradicting the fact that $(r,p) = 1$. Thus, for $1 \leq i \leq d_1$, $h_i \in \Omega_1(H_1) \cap K_i$.

If $d_1 < d$, let k be an integer with $d_1 < k \leq d$. Define the map β by $a_1^{\beta} = a_1 a_k$ and $a_i^{\beta} = a_i$ for $2 \leq i \leq d$. Since $d_1 < k$, $a_k \in M_1$ and, by Theorems 3.19 and 3.20, $\beta \in W$. Thus, $a_1^{m_1}\, a_k^{m_1} h_1 = (a_1^{m_1} h_1)^{\beta} = a_1^{\alpha\beta} =$

$a_1^{\beta\alpha} = (a_1 a_k)^{\alpha} = a_1^{m_1} h_1\, a_k^{m_k} h_k$. Hence $a_k^{m_1} = a_k^{m_k}$ and $h_k = 1$. Therefore, for $d_1 < i \leq d$, $a_i^{\alpha} = a_i^{m_1}$.

Thus, $C_{Aut(G)}(W) \subseteq V$.

Let $\alpha \in V$. Since G is not elementary abelian, $\alpha \in S_o(s_1)$ by Theorem 3.3. Let $\beta \in W$. By Theorem 3.20, $\beta \in S_{n_1 t}(s_2)$ and, by Theorem 3.19, $a_i^{\beta} = a_i g_i$ where $g_i \in M_j$ when $f_{j-1} < i \leq f_j$. $M_{(n_1-1)t} = \mathcal{U}_{(n_1-1)}(G) = \Omega_1(H_1)$ by Theorems 1.15, 1.17 and 1.36. Since also $M_{(n_1-1)t+1} = 1$, β fixes $\Omega_1(H_1)$ elementwise. Hence, for $1 \leq i \leq d_1$,

$a_i^{\alpha\beta}=(a_i^{m_i}h_i)^{\beta}=a_i^{m_i}g_i^{m_i}h_i$ and $a_i^{\beta\alpha}=(a_ig_i)^{\alpha}=a_i^{m_i}h_ig_i^{\alpha}$.

For $g\in M_1$, $g=(\prod_{i=1}^{d_1}a_i^{pr_i})(\prod_{i=d_1+1}^{d}a_i^{r_i})$. Hence $g^{\alpha}=(\prod_{i=1}^{d_1}a_i^{r_im_ip})(\prod_{i=d_1+1}^{d}a_i^{m_1r_i})$. For $1\le i\le d_1$, $m_i\equiv m_1 \bmod p^{n_1-1}$. Hence $pm_i\equiv pm_1 \bmod p^{n_1}$ and $a_i^{m_ip}=a_i^{m_1p}$. Thus $g^{\alpha}=g^{m_1}$. Therefore, for $1\le i\le d$, $g_i^{\alpha}=g_i^{m_1}$. For $1\le i\le d_1$, $g_i^{m_1-m_i}=1$ since $m_i\equiv m_1 \bmod p^{n_1-1}$ and $\exp(M_1)=p^{n_1-1}$. Hence $g_i^{m_i}=g_i^{m_1}=g_i^{\alpha}$ and $a_i^{\alpha\beta}=a_i^{\beta\alpha}$ for $1\le i\le d$.

For $d_1<i\le d$, $a_i^{\alpha\beta}=(a_i^{m_1})^{\beta}=a_i^{m_1}g_i^{m_1}$ and $a_i^{\beta\alpha}=(a_ig_i)^{\alpha}=a_i^{m_1}g_i^{m_1}$ since $g_i\in M_1$. Thus, for $1\le i\le d$, $a_i^{\alpha\beta}=a_i^{\beta\alpha}$. Therefore, $\alpha\beta=\beta\alpha$ and $\alpha\in C_{Aut(G)}(W)$. Hence $V\le C_{Aut(G)}(W)$ and it follows that $V=C_{Aut(G)}(W)$.

Let $\alpha,\beta\in V$ and $a_i^{\alpha}=a_i^{m_i}h_i$, $a_i^{\beta}=a_i^{r_i}g_i$ for $1\le i\le d_1$, with $(m_1,p)=1=(r_1,p)$, $m_i\equiv m_1 \bmod p^{n_1-1}$ and $r_i\equiv r_1 \bmod p^{n_1-1}$ and $h_i,g_i\in\Omega_1(H_1)\cap K_i$ and $a_i^{\alpha}=a_i^{m_1}$, $a_i^{\beta}=a_i^{r_1}$ for $d_1<i\le d$.

If $\alpha=\beta$, then $a_i^{m_i}h_i=a_i^{r_i}g_i$ for $1\le i\le d_1$. Hence $h_i=g_i$ and $m_i\equiv r_i \bmod p^{n_1}$ for $1\le i\le d_1$.

Suppose $h_i=g_i$ and $m_i\equiv r_i \bmod p^{n_1}$ for $1\le i\le d$. Then $a_i^{m_i}=a_i^{r_i}$ for $1\le i\le d_1$ and $a_i^{m_1}=a_i^{r_1}$ for $d_1<i\le d$. Hence $a_i^{\alpha}=a_i^{\beta}$ for $1\le i\le d$. Therefore $\alpha=\beta$.

Thus $\operatorname{ord}(C_{Aut(G)}(W))=\operatorname{ord}(V)=(p-1)p^{(n_1-1)}p^{(d_1-1)}p^{(d_1-1)d_1}=(p-1)p^{(n_1+d_1^2-2)}$.

Corollary 3.22: If G is not elementary abelian, then $Z(W)=\{\alpha \mid a_i^{\alpha}=a_i^{m_1}h_i$ for $1\leq i\leq d_1$, $a_i^{\alpha}=a_i^{m_1}$ for $d_1<i\leq d$, where $m_1\equiv 1 \bmod p$, $m_i\equiv m_1 \bmod p^{n_1-1}$ and $h_i\in\Omega_1(H_1)\cap K_i$ for $1\leq i\leq d_1\}$ and ord $(Z(W))=p^{(n_1+d_1^2-2)}$.

Corollary 3.23: If G is not elementary abelian, then $C_{Aut(G)}(W)$ is supersolvable.

Proof: By Theorems 3.14 and 3.21, and since G is not elementary abelian, $C_{Aut(G)}(W)\subseteq S_0(s_1)$ which, by Theorem 2.5, is supersolvable.

Theorem 3.24: If G is elementary abelian, then $C_{Aut(G)}(W)=$ Aut(G) and $Z(W)=W=1$.

Proof: By Theorems 3.19 and 3.20, and since $M_1=1$, $W=1$.

3.4 The Fitting Subgroup.

This section starts with a characterization of the Fitting subgroup of the normalizer of a p-Sylow subgroup of Aut(G) and ends with a characterization of the Fitting subgroup of Aut(G).

Definition 3.25: The Fitting subgroup of a group H, denoted Fit(H), is the maximal normal nilpotent subgroup of H. (cf. Gorenstein, [2], p. 218).

Theorem 3.26: Let $F=\{\alpha \mid a_i{}^{\alpha}=a_i{}^{m}g_i$ for $1\leq i\leq d$ where $o<m<p$ and $g_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1}<i\leq f_j\}$. Then $\mathrm{Fit}(S_0(s_1))=F$ and $\mathrm{ord}\,[\mathrm{Fit}(S_0(s_1))]=(p-1)p^n$ where $n=\sum_{i=1}^{t}(f_i{}^2[n_i-n_{i+1}]-d_i(d_i+1)/2)$.

Proof: By Theorem 1.33, $S_{n_1 d}(s_1)$ is the (normal) p-Sylow subgroup of $S_0(s_1)$. Hence $S_{n_1 d}(s_1)\subseteq \mathrm{Fit}(S_0(s_1))$.

Let $\alpha \in \mathrm{Fit}(S_0(s_1))$. Then $\alpha \in S_0(s_1)$. By Theroem 3.14, $a_i{}^{\alpha}=a_i{}^{m_i}g_i$, for $1\leq i\leq d$, and $o<m_i<p$ and $g_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1}<i\leq f_j$.

If $\alpha \notin F$, then for some k, $1<k\leq d$ and $f_{j-1}<k\leq f_j$, $m_1\neq m_k$. Define the map β by $a_i{}^{\beta}=a_i(g_i{}^{-1})^{\alpha^{-1}}$ for $1\leq i\leq d$. By Theorem 3.3, $\beta \in S_{n_1 d}(s_1)$. Hence $\gamma=\beta\alpha \in \mathrm{Fit}(S_0(s_1))$. $a_i{}^{\gamma}=a_i{}^{m_i}$ for $1\leq i\leq d$. Define the map δ by $a_1{}^{\delta}=a_1a_k{}^{p^{(n_j-1)}}$ and $a_i{}^{\delta}=a_i$ if $2\leq i\leq d$. Since $k>1$, $a_k \in G_1 \cap \Omega_{n_1}(G)$ and, by Theorem 3.3, $\delta \in S_{n_1 d}(s_1)$. δ fixes G_1 elementwise. Hence $a_1{}^{\delta^{-1}}=a_1a_k{}^{-p^{(n_j-1)}}$ and $a_1{}^{\delta\gamma\delta^{-1}\gamma^{-1}}=(a_1a_k{}^{p^{(n_j-1)}})^{\gamma\delta^{-1}\gamma^{-1}}=(a_1{}^{m_1}a_k{}^{m_k p^{(n_j-1)}})^{\delta^{-1}\gamma^{-1}}=(a_1{}^{m_1}a_k{}^{[m_k-m_1]p^{(n_j-1)}})^{\gamma^{-1}}=a_1a_k{}^{u_k[m_k-m_1]p^{(n_j-1)}}$ where $a_k{}^{\gamma^{-1}}=a_k{}^{u_k}$ and $(p,u_k)=1$ and $m_k u_k\equiv 1 \bmod p^{n_j}$. $a_i{}^{\delta\gamma\delta^{-1}\delta^{-1}}=(a_i{}^{m_i})^{\delta^{-1}\gamma^{-1}}=(a_i{}^{m_i})^{\gamma^{-1}}=a_i$ for $2\leq i\leq d$. Since p does not divide u_k and $o<m_1,m_k<p$ and $m_1\neq m_k$, p does not divide $u_k[m_k-m_1]$. Hence there is an integer m such that $mu_k[m_k-m_1]\equiv 1 \bmod p$. Thus $(\delta\gamma\delta^{-1}\gamma^{-1})^m=\delta\neq 1$ contradicting the fact that $\mathrm{Fit}(S_0(s_1))$ is nilpotent.

Therefore $\alpha \in F$ and $\mathrm{Fit}(S_0(s_1)) \subseteq F$.

By Theorem 3.3, $F \trianglelefteq S_0(s_1)$. $1 \in F$ and if $\alpha, \beta \in F$ with $a_i^{\alpha} = a_i^{m} g_i$ and $a_i^{\beta} = a_i^{r} h_i$ for $1 \leq i \leq d$ where $0 < m, r < p$ and g_i, $h_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$, then $a_i^{\alpha\beta} = a_i^{mr} h_i^{m} g_i^{\beta}$ for $1 \leq i \leq d$. Since $0 < m, r < p$ there are integers λ, μ such that $\lambda \geq 0$, $0 < \mu < p$ and $mr = \lambda p + \mu$. Hence $a_i^{\alpha\beta} = a_i^{\mu} a_i^{\lambda p} h_i^{m} g_i^{\beta}$. Since $\beta \in S_0(s_1)$ and $a_i^{p} \in G_i$ and $\Omega_{n_j}(G)$ is characteristic, $a_i^{\lambda p} h_i^{m} g_i^{\beta} \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$. Therefore $\alpha\beta \in F$ and F is a group. From Theorem 2.6, $F' \leq S_{n_1 d}(s_1)$.

Let $\alpha \in F$. For $1 \leq i \leq d$ and $g \in G_{i-1}$, $g^{\alpha} = g^{m} h$ with $h \in G_i$. Since if $i = qd + r$ with $0 < r \leq d$ and $q \geq 0$, $G_i = \mathcal{V}_q(G_r)$ and $G_d = \mathcal{V}_1(G)$, for $g \in G_{i-1}$, $g^{\alpha} = g^{m} h$ where $h \in G_i$ for $i = 1, 2, \ldots$.

Let $B_k = \{\alpha \in S_0(s_1) \mid g^{-1} g^{\alpha} \in G_{i+k}$ for each $g \in G_i$ and $i = 0, 1, \ldots\}$. Then $B_1 = S_{n_1 d}(s_1)$ and $B_{n_1 d} = 1$. Let $\beta \in B_k$ and $a_i^{\alpha^{-1}} = a_i^{u} h_i$ where $0 < u < p$ and $h_i \in G_i \cap \Omega_{n_j}(G)$ when $f_{j-1} < i \leq f_j$ and $mu \equiv 1 \bmod p$. Let $g \in G_i$ and $g^{\alpha} = g^{m} x_1$ where $x_1 \in G_{i+1}$. Let $g^{\beta} = g x_2$ where $x_2 \in G_{i+k}$ and $x_1^{\beta} = x_1 x_3$ where $x_3 \in G_{i+k+1}$ and $x_2^{\alpha^{-1}} = x_2^{u} x_4$ where $x_4 \in G_{i+k+1}$. Then $g^{\alpha\beta\alpha^{-1}\beta^{-1}} = (g^{m} x_1)^{\beta\alpha^{-1}\beta^{-1}} = (g^{m} x_2^{m} x_1 x_3)^{\alpha^{-1}\beta^{-1}} = (g x_2^{um} x_4^{m} x_3 \alpha^{-1})^{\beta^{-1}} = g(x_2^{um-1} x_4^{m} x_3 \alpha^{-1})^{\beta^{-1}}$. Since $(x_2^{um-1} x_4^{m} x_3 \alpha^{-1})^{\beta^{-1}} \in G_{i+k+1}$, $g^{-1} g^{\alpha\beta\alpha^{-1}\beta^{-1}} \in G_{i+k+1}$ and hence $\alpha\beta\alpha^{-1}\beta^{-1} \in B_{k+1}$. Hence $[F, B_k] \leq B_{k+1}$ and since $[F, F] \leq S_{n_1 d}(s_1) = B_1$, $[F, F^{(n)}] \leq B_n$. Therefore F is nilpotent.

Since $(S_0(s_1))' \leq S_{n_1 d}(s_1) \leq \mathrm{Fit}(S_0(s_1)) \leq F \leq S_0(s_1)$, $F \trianglelefteq S_0(s_1)$.

Therefore $F = \mathrm{Fit}(S_0(s_1))$.

From Lemmas 3.11 and 3.12, $\mathrm{ord}(F)=(p-1)\prod_{i=1}^{d}\mathrm{ord}((G_i)\cap\Omega_{n_j}(G))$ (where j is related to i by $f_{j-1}<i\leq f_j$)$=(p-1)p^n$ where $n=\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i(d_i+1)/2)$.

The next series of theorems is aimed at finding the Fitting subgroup of GL(n,p), i.e. of the automorphism group of an elementary abelian p-group of order p^n (cf. Rotman, [9], p. 155).

Theorem 3.27: The second center of GL(2,p) is equal to Z(GL(2,p)).

Proof: $\{Z(GL(2,p))=\{rE|o<r<p\}$ where E is the 2x2 identity matrix (cf. Rotman, [9], proof of Theorem 8.18, p. 159). Suppose the matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is an element of the second center of GL(2,p). Since $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\in GL(2,p)$, there is an integer r, $o<r<p$ such that $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=r\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. Hence $\begin{pmatrix}a & b\\ a+c & b+d\end{pmatrix}=r\begin{pmatrix}a+b & b\\ c+d & d\end{pmatrix}$. Since $b=rb$, either $r=1$ or $b=0$. If $b=0$, then since $a=r(a+b)$ and $\begin{pmatrix}0 & 0\\ c & d\end{pmatrix}\notin GL(2,p)$, $r=1$. If $r=1$, then since $a=r(a+b)$, $b=0$. Hence $r=1$ and $b=0$. Since $a+c=r(c+d)$, $a=d$. Since $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\in GL(2,p)$, there is an integer u, $o<u<p$ such that $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=u\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Hence $\begin{pmatrix}a+c & b+d\\ c & d\end{pmatrix}=u\begin{pmatrix}a & a+b\\ c & c+d\end{pmatrix}$. Since $b=0$ and $a=d$, $\begin{pmatrix}a+c & a\\ c & a\end{pmatrix}=u\begin{pmatrix}a & a\\ c & a+c\end{pmatrix}$. Since $a=ua$ and $a\neq 0$ because $\begin{pmatrix}0 & 0\\ c & d\end{pmatrix}\notin GL(2,p)$, $u=1$. Since $a+c=ua$, $c=0$. Hence $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in Z(GL(2,p))$.

Lemma 3.28: If B is an nxn matrix over any field with $n\geq 3$ and entry b_{ij} in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column and if for some u and v with $1\leq u,v\leq n$, $b_{ij}=0$ when $i\neq u$ and $j\neq v$, then $\det(B)=0$.

Proof: Expanding by the $v^{\underline{th}}$ column each cofactor is zero since the associated matrix contains a row of zeroes.

Theorem 3.29: The second center of GL(n,p) is equal to Z(GL(n,p)).

Proof: Let E be the nxn identity matrix and let E_{ij} be the matrix with entry 1 in the $i^{\underline{th}}$ row, $j^{\underline{th}}$ column and zeroes elsewhere.

If n=1, Z(GL(n,p))=GL(n,p) and the result follows and if n=2, the statement is just Theorem 3.27.

Suppose n>2. $Z(GL(n,p))=\{rE \mid o<r<p\}$ (cf. Rotman, [9], proof of Theorem 8.18, p. 159). Let C be an element of the second center of GL(n,p) with entry c_{ij} in the $i^{\underline{th}}$ row, $j^{\underline{th}}$ column. Let u and v be any two integ with $1 \leq u, v \leq n$ and $u \neq v$. Since n>2 let k be an integer with $1 \leq k \leq n$ and $k \neq u$ and $k \neq v$. Since $\det(E+E_{ku}) \neq 0$, $E+E_{ku} \in GL(n,p)$ and hence there is an integer r, $o<r<p$, such that $(E+E_{ku})C=rC(E+E_{ku})$. Hence $c_{ij}=rc_{ij}$ if $i \neq k$ and $j \neq u$ and $c_{ku}+c_{uu}=r(c_{ku}+c_{kk})$ and $c_{kj}+c_{uj}=rc_{kj}$ if $j \neq u$. By Lemma 3.28, and since $\det(C) \neq 0$ and $c_{ij}=rc_{ij}$ if $i \neq k$ and $j \neq u$, r=1. Hence $c_{uu}=c_{kk}$ and $c_{uj}=0$ if $j \neq u$. In particular $c_{uv}=0$. Since u,v,k were arbitrary integers between 1 and n inclusive, $C \in Z(GL(n,p))$ and the conclusion follows.

Lemma 3.30: If B is a solvable normal subgroup of GL(n,p) such that $Z(GL(n,p)) \leq B$ and if $n \neq 2$ or $p>3$, then $B=Z(GL(n,p))$.

Proof: If n=1, $GL(n,p)=Z(GL(n,p))\subseteq B\subseteq GL(n,p)$.

If n>2 or p>3, then PSL(n,p) is simple and not of prime order (cf. Rotman, [9], pp. 161-169). $Z(SL(n,p))\subseteq Z(GL(n,p))$ (cf. Rotman, [9], Theorem 8.18, p. 159). Hence $B\cap SL(n,p)=Z(SL(n,p))$. $[GL(n,p), B]\subseteq B\cap SL(n,p)=Z(SL(n,p))\subseteq Z(GL(n,p))$. Hence B is contained in the second center of GL(n,p) which, by Theorem 3.29, is equal to Z(GL(n,p)). Hence B=Z(GL(n,p)).

Theorem 3.31: If $n\neq 2$ or p>3, then Fit(GL(n,p))= Z(GL(n,p)).

Proof: The result follows from Lemma 3.30, since Fit(GL(n,p)) is a normal solvable subgroup of GL(n,p) and $Z(GL(n,p))\subseteq Fit(GL(n,p))$.

Theorem 3.32: Let $F_2=\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}\}$. Then $Fit(GL(2,2))=F_2$.

Proof: GL(2,2) is particularly well known since it is isomorphic to the symmetric group on 3 letters. It is of order 6, is not nilpotent and has normal 3-Sylow subgroup F_2.

Theorem 3.33: Let $F_3=\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&2\end{pmatrix}, \begin{pmatrix}2&2\\2&1\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}, \begin{pmatrix}2&1\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\2&0\end{pmatrix}, \begin{pmatrix}0&2\\1&0\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}\}$. Then $Fit(GL(2,3))=F_3$.

Proof: ord(GL(2,3))=48. F_3 is the subgroup generated by all elements of order 4. Hence F_3 is normal. Since $ord(F_3)=8$, F_3 is nilpotent. Thus $F_3\subseteq Fit(GL(2,3))$. F_3 is contained

in each 2-Sylow subgroup of GL(2,3). Therefore each 2-Sylow subgroup of GL(2,3) has at most 8 elements of order 8. If ord(Fit(GL(2,3)) were divisible by 16, then GL(2,3) would have at most 8 elements of order 8. Since $\left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&2\\2&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2&1\\2&2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2&2\\1&2\end{smallmatrix}\right)$,$\left(\begin{smallmatrix}2&2\\2&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ are all of order 8, ord(Fit(GL(2,3))) is not divisible by 16.

If ord(Fit(GL(2,3))) were divisible by 3, then GL(2,3) would have exactly 2 elements of order 3. Since $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ are all of order 3, ord(Fit(GL(2,3))) is not divisible by 3.

Therefore ord(Fit(GL(2,3)))=8 and Fit(GL(2,3))=F_3.

The following theorems deal with the homomorphic image of Fit(Aut(G)) in Aut(M_{j-1}/M_j).

Theorem 3.34: If $\alpha \in$ Fit(Aut(G)), then α induces an element of Fit(Aut(M_{j-1}/M_j)) on M_{j-1}/M_j for each j, $1 \leq j \leq t$.

Proof: By Lemma 3.4, Aut(M_{j-1}/M_j) is a homomorphic image of Aut(G). Hence the image of Fit(Aut(G)) is normal in Aut(M_{j-1}/M_j) as well as being nilpotent.

Lemma 3.35: If, for some j with $1 \leq j \leq t$, $H_j=\langle a_k \rangle x \langle a_{k+1} \rangle$ and if $\alpha \in$ Fit(Aut(G)) with $a_i^{\alpha}=h_i g_i$, $h_i \in H_j$, $g_i \in K_k \cap K_{k+1}$ for $1 \leq i < k$ and $k+1 < i \leq d$ and $a_k^{\alpha}=a_k^{\lambda_1} a_{k+1}^{\mu_1} g_k$ and $a_{k+1}^{\alpha}=a_k^{\lambda_2} a_{k+1}^{\mu_2} g_{k+1}$ where $\left(\begin{smallmatrix}\lambda_1&\mu_1\\\lambda_2&\mu_2\end{smallmatrix}\right) \in GL(2,p)$ and $g_k, g_{k+1} \in M_j$, then there is an element $\gamma \in$ Fit(Aut(G)) such that $a_i^{\gamma} \in K_k \cap K_{k+1}$ for $1 \leq i < k$ and $k+1 < i \leq d$ and $a_k^{\gamma}=a_k^{\lambda_1} a_{k+1}^{\mu_1}$ and $a_{k+1}^{\gamma}=a_k^{\lambda_2} a_{k+1}^{\mu_2}$.

Proof: Let i be an integer with $1\leq i\leq d$, $f_{\nu-1}<i\leq f_\nu$ and $\nu\neq j$. If $1\leq i<k$, then $n_\nu>n_j$ and $h_i\in\Omega_{n_\nu}(G)\cap M_\nu$. If $k+1<i\leq d$, then $n_j>n_\nu$ and since $h_i\in\Omega_{n_\nu}(G)$, $h_i\in\Omega_{n_j-1}(H_j)\leq\mathcal{V}_1(H_j)\leq M_\nu$. Hence, in either case, $h_i\in\Omega_{n_\nu}(G)\cap M_\nu$. Define the map β by $a_i^\beta=a_i(h_i^{-1})^{\alpha^{-1}}$ if $1\leq i<k$ or $k+1<i\leq d$ and $a_k^\beta=a_k(g_k^{-1})^{\alpha^{-1}}$ and $a_{k+1}^\beta=a_{k+1}(g_{k+1}^{-1})^{\alpha^{-1}}$. Since $(h_i^{-1})^{\alpha^{-1}}\in\Omega_{n_\nu}(G)\cap M_\nu$ when $f_{\nu-1}<i\leq f_\nu$ for $1\leq i<k$ or $k+1<i\leq d$ and since $(g_k^{-1})^{\alpha^{-1}}$, $(g_{k+1}^{-1})^{\alpha^{-1}}\in\Omega_{n_j}(G)\cap M_j$, $\beta\in W\leq Fit(Aut(G))$, by Theorems 3.19 and 3.20. Hence $\gamma=\beta\alpha\in Fit(Aut(G))$ is the required automorphism.

Theorem 3.36: If G is an abelian 2-group, not elementary abelian of order 4, and if $\alpha\in Fit(Aut(G))$, then α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j for each j, $1\leq j\leq t$.

Proof: Let $\alpha\in Fit(Aut(G))$. Then, by Theorem 3.34, α induces an element of $Fit(Aut(M_{j-1}/M_j)$ on M_{j-1}/M_j for $1\leq j\leq t$. Let j be an integer with $1\leq j\leq t$.

If $d_j\neq 2$, then, by Theorem 3.31, $Fit(Aut(M_{j-1}/M_j))=Z(Aut(M_{j-1}/M_j))$ since M_{j-1}/M_j is elementary abelian of order 2^{d_j}.

Suppose $d_j=2$ and $H_j=\langle a_k\rangle x\langle a_{k+1}\rangle$. Suppose $(a_kM_j)^\alpha=a_k^{\lambda_1}a_{k+1}^{\mu_1}M_j$, $(a_{k+1}M_j)^\alpha=a_k^{\lambda_2}a_{k+1}^{\mu_2}M_j$, $(a_kM_j)^{\alpha^{-1}}=a_k^{\lambda_3}a_{k+1}^{\mu_3}M_j$ and $a_k^{\lambda_4}a_{k+1}^{\mu_4}M_j=(a_{k+1}M_j)^{\alpha^{-1}}$ where $0\leq\lambda_i,\mu_i<2$ for $1\leq i\leq 4$. Then $\begin{pmatrix}\lambda_1&\mu_1\\ \lambda_2&\mu_2\end{pmatrix}$, $\begin{pmatrix}\lambda_3&\mu_3\\ \lambda_4&\mu_4\end{pmatrix}\in GL(2,2)$ and $\begin{pmatrix}\lambda_1&\mu_1\\ \lambda_2&\mu_2\end{pmatrix}^{-1}=\begin{pmatrix}\lambda_3&\mu_3\\ \lambda_4&\mu_4\end{pmatrix}$. By Lemma 3.35

there is an element $\gamma \in Fit(Aut(G))$ such that $a_i^{\gamma} \in K_k \cap K_{k+1}$ for $1 \leq i < k$ and for $k+1 \leq i \leq d$ and $a_k^{\gamma} = a_k^{\lambda_1} a_{k+1}^{\mu_1}$ and $a_{k+1}^{\gamma} = a_k^{\lambda_2} a_{k+1}^{\mu_2}$. By Theorems 3.34 and 3.32, $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} \in F_2$.

Suppose $j<t$ or $n_j>1$. If $j<t$, then $n_j>1$. If $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, define the map δ by $a_i^{\delta} = a_i$ for $1 \leq i < k$ and for $k+1 < i \leq d$ and $a_k^{\delta} = a_k a_{k+1}^{2^{(n_j-1)}}$ and $a_{k+1}^{\delta} = a_k^{2^{(n_j-1)}} a_{k+1}$. By Theorems 3.19 and 3.20, $\delta \in W \leq Fit(Aut(G))$. $(\gamma\delta\gamma^{-1}\delta^{-1})[\gamma(\gamma\delta\gamma^{-1}\delta^{-1})\gamma^{-1}(\gamma\delta\gamma^{-1}\delta^{-1})^{-1}] = \delta$ contradicting the fact that $Fit(Aut(G))$ is nilpotent. If $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $\begin{pmatrix} \lambda_3 & \mu_3 \\ \lambda_4 & \mu_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which leads to a contradiction as above. Hence, from Theorem 3.32, $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in Z(GL(2,2))$.

Suppose $j=t$ and $n_t=1$. Since G is not elementary abelian of order 4, $1<d-1=k$. From the part of the proof already given, γ induces an element of $Z(Aut(M_0/M_1))$ on M_0/M_1. Hence $(a_1^{2^{n_1-1}})^{\gamma} = a_1^{2^{n_1-1}}$. If $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, define the map δ by by $a_i^{\delta} = a_i$ if $1 \leq i < k$ and $a_k^{\delta} = a_1^{2^{(n_1-1)}} a_k$ and $a_{k+1}^{\delta} = a_{k+1}$. By Theorems 3.19 and 3.20, $\delta \in W \leq Fit(Aut(G))$. $(\gamma\delta\gamma^{-1}\delta^{-1}) \cdot [\gamma(\gamma\delta\gamma^{-1}\delta^{-1})\gamma^{-1}(\gamma\delta\gamma^{-1}\delta^{-1})^{-1}] = \delta$ contradicting the nilpotence of $Fit(Aut(G))$. If $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $\begin{pmatrix} \lambda_3 & \mu_3 \\ \lambda_4 & \mu_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which leads to a contradiction as above. Hence, from Theorem 3.32, $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in Z(GL(2,2))$.

Thus α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j for each j, $1\leq j\leq t$.

Theorem 3.37: If G is an abelian 3-group, not elementary abelian of ord 9, and if $\alpha\in Fit(Aut(G))$, then α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j for each j, $1\leq j\leq t$.

Proof: Let $\alpha\in Fit(Aut(G))$. Then, by Theorem 3.34, α induces an element of $Fit(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j, for $1\leq j\leq t$. Let j be an integer with $1\leq j\leq t$.

If $d_j\neq 2$, then, by Theorem 3.31, $Fit(Aut(M_{j-1}/M_j))=Z(Aut(M_{j-1}/M_j))$ since M_{j-1}/M_j is elementary abelian of order 2^{d_j}.

Suppose $d_j=2$ and $H_j=\langle a_k\rangle x\langle a_{k+1}\rangle$. Suppose $(a_kM_j)^\alpha=a_k^{\lambda_1}a_{k+1}^{\mu_1}M_j$, $(a_{k+1}M_j)^\alpha=a_k^{\lambda_2}a_{k+1}^{\mu_2}M_j$, $(a_kM_j)^{\alpha^{-1}}=a_k^{\lambda_3}a_{k+1}^{\mu_3}M_j$ and $(a_{k+1}M_j)^{\alpha^{-1}}=a_k^{\lambda_4}a_{k+1}^{\mu_4}M_j$ where $0\leq\lambda_i,\mu_i<3$ for $1\leq i\leq 4$. Then $\begin{pmatrix}\lambda_1&\mu_1\\ \lambda_2&\mu_2\end{pmatrix}$, $\begin{pmatrix}\lambda_3&\mu_3\\ \lambda_4&\mu_4\end{pmatrix}\in GL(2,3)$ and $\begin{pmatrix}\lambda_1&\mu_1\\ \lambda_2&\mu_2\end{pmatrix}^{-1}=\begin{pmatrix}\lambda_3&\mu_3\\ \lambda_4&\mu_4\end{pmatrix}$. By Lemma 3.35 there is an element $\gamma\in Fit(Aut(G))$ such that $a_i^\gamma\in K_k\cap K_{k+1}$ for $1\leq i<k$ and for $k+1<i\leq d$ and $a_k^\gamma=a_k^{\lambda_1}a_{k+1}^{\mu_1}$ and $a_{k+1}^\gamma=a_k^{\lambda_2}a_{k+1}^{\mu_2}$. By Theorems 3.34 and 3.33, $\begin{pmatrix}\lambda_1&\mu_1\\ \lambda_2&\mu_2\end{pmatrix}\in F_3$.

Suppose $j<t$ or $n_j>1$. If $j<t$, then $n_j>1$. If $\begin{pmatrix}\lambda_1&\mu_1\\ \lambda_2&\mu_2\end{pmatrix}=\begin{pmatrix}1&1\\1&2\end{pmatrix}$, $\begin{pmatrix}1&2\\2&2\end{pmatrix}$ or $\begin{pmatrix}0&1\\2&0\end{pmatrix}$, define the map δ by $a_i^\delta=a_i$ for $1\leq i<k$ and for $k+1<i\leq d$ and $a_k^\delta=a_k{}^{1+2\cdot 3^{n_j-1}}$ for case 1, $a_k{}^{1+3^{n_j-1}}$ in

case 2, and $a_k a_{k+1}^{3^{n_j-1}}$ in case 3 and $a_{k+1}^{\delta}=a_k^{2\cdot 3^{n_j-1}} a_{k+1}^{1+3^{n_j-1}}$ in case 1, $a_k^{2\cdot 3^{n_j-1}} a_{k+1}^{1+2\cdot 3^{n_j-1}}$ in case 2 and $a_k^{3^{n_j-1}} a_{k+1}$ in case 3. In each case, by Theorems 3.19 and 3.20, $\delta \in W \leq Fit(Aut(G))$. Also in each case $\gamma\delta\gamma^{-1}\gamma^{-1}=\delta$ contradicting the nilpotence of Fit(Aut(G)).

If $\begin{pmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{pmatrix} = \begin{pmatrix}2 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}$, then $\begin{pmatrix}\lambda_3 & \mu_3\\ \lambda_4 & \mu_4\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$ which leads to a contradiction as above.

Hence, from Theorem 3.33, $\begin{pmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{pmatrix} \in \{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\} = Z(GL(2,3))$.

Suppose $j=t$ and $n_t=1$. Since G is not elementary abelian of order 9, $1<d-1=k$. From the part of the proof already given, γ induces an element of $Z(Aut(M_0/M_1))$ on M_0/M_1. Hence $(a_1^{3^{n_1-1}})^{\gamma}=a_1^{3^{n_1-1}}$ or $a_1^{2\cdot 3^{n_1-1}}$. If $\begin{pmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$ define the map δ by $a_i^{\delta}=a_i$ for $1\leq i<k$ and $a_k=a_k a_1^{3^{(n_1-1)}}$ and $a_{k+1}^{\delta}=a_{k+1}a_1^{3^{(n_1-1)}}$. In each case, by Theorems 3.19 and 3.20, $\delta \in W \leq Fit(Aut(G))$. Also in each case, $(\gamma\delta\gamma^{-1}\gamma^{-1})^{-1}\cdot[\gamma(\gamma\delta\gamma^{-1}\delta^{-1})\gamma^{-1}(\gamma\delta\gamma^{-1}\delta^{-1})^{-1}]=\delta$ contradicting the nilpotence of Fit(Aut(G)). If $\begin{pmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{pmatrix} = \begin{pmatrix}2 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}$, then $\begin{pmatrix}\lambda_3 & \mu_3\\ \lambda_4 & \mu_4\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$ which leads to a contradiction as above. Hence, from Theorem 3.33, $\begin{pmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{pmatrix} \in \{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\} = Z(GL(2,3))$.

Thus α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j for each j, $1\leq j\leq t$.

Theorem 3.38: Let $U=\{\alpha \mid a_i^{\alpha}=a_i^{m}g_i$ where $o<m<p$ and $g_i \in M_j \cap \Omega_{n_j}(G)$ when $f_{j-1}<i\leq f_j$ for $1\leq i\leq d\}$. If G is an abelian p-group, not elementary abelian of order 4 or 9, then $Fit(Aut(G))=U$ and $ord(Fit(Aut(G)))=(p-1)p^n$ where $n=\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i^2)$.

Proof: When $f_{j-1}<i\leq f_j$, $G_i \supseteq M_j$. Hence, by Theorem 3.26, $U\subseteq Fit(S_0(s_1))$. Thus U is nilpotent.

If $\alpha \in U$ and $a_i^{\alpha}=a_i^{m}g_i$, then, for any $g \in M_{j-1}$, $g^{\alpha}=g^{m}h$ where $h \in M_j$, $1\leq j\leq t$. Hence if $\alpha \in U$ and $\beta \in Aut(G)$, then $a_i^{\beta^{-1}\alpha\beta}=[(a_i^{\beta^{-1}})^{m}h_i]^{\beta}=a_i^{m}h_i^{\beta}$ where $h_i \in M_j$ when $f_{j-1}<i\leq f_j$, since M_{j-1} is characteristic. Since $h_i^{\beta} \in M_j$ when $f_{j-1}<i\leq f_j$ and since $ord(h_i^{\beta})=ord(a_i^{-m}a_i^{\beta^{-1}\alpha\beta})\leq ord(a_i)$, $\beta^{-1}\alpha\beta \in U$ and hence $U \triangleleft Aut(G)$. Therefore $U \subseteq Fit(Aut(G))$.

Let $\alpha \in Fit(Aut(G))$. By Theorems 3.31, 3.34, 3.36 and 3.37, α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j for each j, $1\leq j\leq t$. Therefore $a_i^{\alpha}=a_i^{m_j}g_i$ where $o<m_j<p$ and $g_i \in M_j \cap \Omega_{n_j}(G)$ when $f_{j-1}<i\leq f_j$ for $1\leq i\leq d$. Hence, by Theorem 3.3, $\alpha \in S_0(s_1)$. Thus $\alpha \in S_0(s_1) \cap Fit(Aut(G)) \subseteq Fit(S_0(s_1))$. Therefore, by Theorem 3.26, $m_1=m_j$ for $j=1,2,\ldots,t$ and $\alpha \in U$. Thus $Fit(Aut(G)) \subseteq U$ and it follows that $Fit(Aut(G))=U$.

From Lemmas 3.11 and 3.12, $ord(U)=(p-1)_j\prod_{1}^{t}[ord(M_j \cap \Omega_{n_j}(G))]^{n_j}=(p-1)p^n$ where $n=\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i^2)$.

Corollary 3.39: For $p>2$, Aut(G) is nilpotent if, and only if, G is cyclic and, for $p=2$, Aut(G) is nilpotent if, and only if, G is multicyclic.

3.5 Sol(Aut(G)).

Definition 3.40: Sol(H) denotes the maximal normal solvable subgroup of the group H.

Theorem 3.41: If $\alpha \in Sol(Aut(G))$, then α induces an element of $Sol(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j for each j, $1 \leq j \leq t$.

Proof: By Lemma 3.4, $Aut(M_{j-1}/M_j)$ is a homomorphic image of Aut(G). Hence the image of Sol(Aut(G)) is normal in $Aut(M_{j-1}/M_j)$ as well as being solvable.

Theorem 3.42: If $n \neq 2$ or $p>3$, then $Sol(GL(n,p)) = Z(GL(n,p))$.

Proof: The result follows from Lemma 3.30 since Sol(GL(n,p)) is a normal solvable subgroup of GL(n,p) and $Z(GL(n,p)) \subseteq Sol(GL(n,p))$.

Theorem 3.43: Let $R_1 = \{\alpha \mid a_i^{\alpha} = a_i^{m_j} g_i$ where $0<m_j<p$ and $g_i \in \Omega_{n_j}(G) \cap M_j$ when $f_{j-1} < i \leq f_j$ for $1 \leq i \leq d\}$. If $p>3$, then $Sol(Aut(G)) = R_1$ and $ord(Sol(Aut(G)) = (p-1)^t p^n$ where $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$.

Proof: By Theorem 3.3, $R_1 \subseteq S_0(s_1)$. Therefore, by Theorem 2.6, R_1 is in fact supersolvable. If $\alpha \in R_1$, α induces an element of $Z(\mathrm{Aut}(M_{j-1}/M_j))$ on M_{j-1}/M_j for $1 \leq j \leq t$. For $\beta \in \mathrm{Aut}(G)$, $\beta^{-1}\alpha\beta$ also induces an element of $Z(\mathrm{Aut}(M_{j-1}/M_j))$ on M_{j-1}/M_j for $1 \leq j \leq t$. Hence $\beta^{-1}\alpha\beta \in R_1$ and $R_1 \triangleleft \mathrm{Aut}(G)$. Therefore $R_1 \subseteq \mathrm{Sol}(\mathrm{Aut}(G))$.

Let $\alpha \in \mathrm{Sol}(\mathrm{Aut}(G))$. By Theorems 3.41 and 3.42, α induces an element of $Z(\mathrm{Aut}(M_{j-1}/M_j))$ on M_{j-1}/M_j for $1 \leq j \leq t$. Hence $\alpha \in R_1$ and $\mathrm{Sol}(\mathrm{Aut}(G)) \subseteq R_1$. If follows that $\mathrm{Sol}(\mathrm{Aut}(G)) = R_1$.

From Lemmas 3.11 and 3.12, $\mathrm{ord}(R_1) = (p-1)^t \prod_{i=1}^{t} [\mathrm{ord}(\Omega_{n_i}(G) \cap M_i)]^{d_i} = (p-1)^t p^n$ where $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$.

Corollary 3.44: For $p > 3$, $\mathrm{Aut}(G)$ is solvable if, and only if, G is multicyclic.

Theorem 3.45: Let $R_2 = \{\alpha \mid a_i^{\alpha} = a_i^{m_j} g_i$ where $0 < m_j < p$ and $g_i \in \Omega_{n_j}(G) \cap M_j$ when $f_{j-1} < i \leq f_j$ and $d_j \neq 2$ and $a_i^{\alpha} = a_i^{\lambda_i} a_{i+1}^{\mu_i} g_i$ and $a_{i+1}^{\alpha} = a_i^{\lambda_{i+1}} a_{i+1}^{\mu_{i+1}} g_{i+1}$ where $\begin{pmatrix} \lambda_i & \mu_i \\ \lambda_{i+1} & \mu_{i+1} \end{pmatrix} \in GL(2,p)$ and $g_i, g_{i+1} \in \Omega_{n_j}(G) \cap M_j$ when $H_j = \langle a_i \rangle x \langle a_{i+1} \rangle\}$. If $p \leq 3$, then $\mathrm{Sol}(\mathrm{Aut}(G)) = R_2$ and $\mathrm{ord}(\mathrm{Sol}(\mathrm{Aut}(G))) = (p-1)^t[(p+1)p(p-1)]^r \cdot p^n$ where $r = \mathrm{ord}(\{j \mid d_j = 2\})$ and $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$.

Proof: By Theorem 1.29, $R_2 \subseteq \mathrm{Aut}(G)$. $R_2^{(p)} \subseteq W$ by Theorems 3.19 and 3.20 and the fact that $[GL(2,p)]^{(p)} = 1$ for $p \leq 3$. Since W is nilpotent, R_2 is solvable. Let $\alpha \in R_2$ and

$\beta \in Aut(G)$. When α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j, so does $\beta^{-1}\alpha\beta$. Hence $\beta^{-1}\alpha\beta \in R_2$ and $R_2 \triangleleft Aut(G)$. Therefore $R_2 \leq Sol(Aut(G))$.

Let $\alpha \in Sol(Aut(G))$. Then, by Theorems 3.41 and 3.42, α induces an element of $Z(Aut(M_{j-1}/M_j))$ on M_{j-1}/M_j whenever $d_j \neq 2$. Hence $\alpha \in R_2$ and $Sol(Aut(G)) \leq R_2$. It follows that $Sol(Aut(G)) = R_2$.

For $p \leq 3$, $ord(GL(2,p)) = (p+1)p(p-1)^2$. Hence $ord(R_2) = (p-1)^t[(p+1)p(p-1)]^r \prod_{j=1}^{t} [ord(\Omega_{n_j}(G) \cap M_j)]^{n_j} = (p-1)^t[(p+1)p(p-1)]^r \cdot p^n$ where $r = ord(\{j \mid d_j = 2\})$ and $n = \sum_{i=1}^{t} (f_i^2[n_i - n_{i+1}] - d_i^2)$.

Definition 3.46: $\omega(G) = max\{d_j \mid 1 \leq j \leq t\}$.

Corollary 3.47: For $p \leq 3$, Aut(G) is solvable if, and only if, $\omega(G) \leq 2$.

3.6 A Necessary and Sufficient Condition for Aut(G) to be Supersolvable.

This section contains the result that Aut(G) is supersolvable if, and only if, G is multicyclic or elementary abelian of order 4.

Theorem 3.48: If $p > 2$ and if Aut(G) is supersolvable, then G is multicyclic.

Proof: Suppose $p > 3$. If Aut(G) is supersolvable, then Aut(G) is solvable and the conclusion follows from Corollary 3.44.

Let p=3. Suppose for some j, $1 \leq j \leq t$, H_j is not cyclic. Then there is an integer k such that $f_{j-1} < k < k+1 \leq f_j$. Let $H = \langle a_k \rangle x \langle a_{k+1} \rangle$ and $K = K_k \cap K_{k+1}$. Then G=HxK. By Theorem 1.39, $\mathcal{V}_{n_j-1}(H)$ is an elementary abelian p-group of order p^2. Hence $\mathrm{Aut}(\mathcal{V}_{n_j-1}(H)) \cong GL(2,3)$. Since $\mathrm{ord}(GL(2,3)) = 2^4 \cdot 3$, if GL(2,3) were supersolvable, it would contain a normal 3-Sylow subgroup (cf. M. Hall, [3], Corollary 10.5.2, p. 159) and hence exactly two elements of order 3. But $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ are all elements of GL(2,3) each of whose order is 3. Hence $\mathrm{Aut}(\mathcal{V}_{n_j-1}(H))$ is not supersolvable. By Theorem 1.45, $\mathrm{Aut}(\mathcal{V}_{n_j-1}(H))$ is isomorphic to a factor group of Aut(H). Hence Aut(H) is not supersolvable. Since $\mathrm{Aut}(H) \leq \mathrm{Aut}(H) x \mathrm{Aut}(K) \leq \mathrm{Aut}(G)$, Aut(G) is not supersolvable.

Lemma 3.49: If $G = \langle a \rangle x \langle b \rangle$ where $\mathrm{ord}(a) = \mathrm{ord}(b) = 4$, then Aut(G) is not supersolvable.

Proof: $\mathrm{ord}(\mathrm{Aut}(G)) = 2^5 \cdot 3$. Since $\langle ab, ab^2 \rangle = G$, by Theorems 1.37 and 1.29, the map α defined by $a^\alpha = ab$ and $b^\alpha = ab^2$ is an automorphism of G. Similarly for β and γ defined by $a^\beta = ab^3$, $b^\beta = a^3b^2$, $a^\gamma = a^3b^3$ and $b^\gamma = a$, β and $\gamma \in \mathrm{Aut}(G)$. If Aut(G) were supersolvable, then Aut(G) would have a normal 3-Sylow subgroup and hence exactly 2 elements of order 3. But α, β and γ are all elements of Aut(G) each having order 3. Hence Aut(G) is not supersolvable.

Lemma 3.50: If $G = \langle a \rangle x \langle b \rangle x \langle c \rangle$ where $\mathrm{ord}(b) = \mathrm{ord}(c) = 2$ and $\mathrm{ord}(a) = 2^n$ with $n > 1$, then Aut(G) is not supersolvable.

Proof: $\text{ord}(\text{Aut}(G))=2^{n+4}\cdot 3$. abc, bc, b is a direct basis of G with ord(a)=ord(abc), ord(b)=ord(bc) and ord(c)=ord(b). Hence, by Theorem 1.29, the map α defined by $a^{\alpha}=abc$, $b^{\alpha}=bc$ and $c^{\alpha}=b$ is an automorphism of G. $a^{\alpha^2}=ab$, $b^{\alpha^2}=c$ and $c^{\alpha^2}=bc$. $\alpha^3=1$. Similarly the map β defined by $a^{\beta}=a$, $b^{\beta}=bc$, $c^{\beta}=b$ is an automorphism of G with $\beta^3=1$. If Aut(G) were supersolvable, then Aut(G) would have a normal 3-Sylow subgroup of order 3 and hence exactly 2 elements of order 3. But α,α^2 and β are all elements of Aut(G) having order 3. Hence Aut(G) is not supersolvable.

Theorem 3.51: If p=2 and if Aut(G) is supersolvable, then G is multicyclic or G is elementary abelian of order 4.

Proof: Suppose G is not elementary abelian of order 4 and for some j, $1\leq j\leq t$, H_j is not cyclic. Let $k+1=f_j$, $H=\langle a_k\rangle x\langle a_{k+1}\rangle$ and $K=K_k\cap K_{k+1}$. Then G=HxK.

If $n_j\geq 2$, then, by Theorem 1.39 and Lemma 3.49, $\text{Aut}(\mathcal{U}_{n_j-2}(H))$ is not supersolvable. By Theorem 1.45, $\text{Aut}(\mathcal{U}_{n_j-2}(H))$ is isomorphic to a factor group of Aut(H). Hence Aut(H) is not supersolvable. Since $\text{Aut}(H)\leq \text{Aut}(H)x\,\text{Aut}(K)\leq \text{Aut}(G)$, Aut(G) is not supersolvable.

If $n_j=1$ and $d_j=2$, then j>1 since G is not elementary abelian of order 4. Let $D_1=\langle a_{f_{j-1}}\rangle x\langle a_k\rangle x\langle a_{k+1}\rangle$ and $D_2=K_k\cap K_{k+1}\cap K_{f_{j-1}}$. Then $G=D_1xD_2$. By Lemma 3.50, since $n_{j-1}>1$, $\text{Aut}(D_1)$ is not supersolvable. Since $\text{Aut}(D_1)\leq \text{Aut}(D_1)x\,\text{Aut}(D_2)\leq \text{Aut}(G)$, Aut(G) is not supersolvable.

If $d_j>2$, then, by Corollary 3.47, Aut(G) is not solvable and hence not supersolvable.

Theorem 3.52: Aut(G) is supersolvable if, and only if, G is multicyclic or elementary abelian of order 4.

Proof: If Aut(G) is supersolvable, then, by Theorems 3.48 and 3.51, G is multicyclic or elementary abelian of order 4.

If G is elementary abelian of order 4, then Aut(G) is isomorphic to the symmetric group on three letters which is supersolvable.

If G is multicyclic, then, by Theorem 3.10, Aut(G) has a normal p-Sylow subgroup and hence, by Corollary 3.8, Aut(G) is supersolvable.

3.7 Normal Hall Subgroups of Aut(G).

This section contains the results that if Aut(G) is not supersolvable, Aut(G) has no proper non-trivial normal Hall subgroups and if Aut(G) is supersolvable but not nilpotent, then Aut(G) has a normal Hall subgroup for each possible order which is divisible by p but no others.

Theorem 3.53: If Aut(G) is not supersolvable, then Aut(G) has no proper non-trivial normal solvable Hall subgroups.

Proof: Suppose B is a normal solvable Hall subgroup of Aut(G). Let k be an integer with $1 \leq k \leq t$ such that $d_k \geq d_j$ for each j, $1 \leq j \leq t$. By Theorem 3.52, H_k is not cyclic. Hence $(p-1)^2$ divides ord(Aut(H_k)) and $(p-1)^{t+1}$ divides ord(Aut(G)).

Suppose p>3. By Theorem 3.43, ord(B) divides $(p-1)^t \cdot p^n$ where $n=\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i^2)$. The exponent of the highest power of p which divides ord(Aut(G)) is $\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i(d_i+1)/2)$. Since $d_k>1$, $d_k(d_k+1)/2<d_k^2$. Since $d_i(d_i+1)/2 \leq d_i^2$ for $1 \leq i \leq t$ and $d_k(d_k+1)/2<d_k^2$, $n<\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i(d_i+1)/2)$. Therefore p does not divide ord(B). Hence if q is a prime dividing ord(B), q divides (p-1). If q^r divides p-1 and q^{r+1} does not, then q^{rt} divides ord(B) and q^{rt+1} does not. But $q^{r(t+1)}$ divides ord(Aut(G)). Hence there is no prime dividing ord(B) and B=1.

Suppose p=3. Then, by Theorem 3.45, when a prime q divides ord(B), q=2 or q=3. Since 4 divides ord(Aut(H_k)), 4 divides ord(Aut(G)). By Theorem 3.38, 4 does not divide the order of Fit(Aut(G)) unless G is elementary abelian of order 9 in which case Theorem 3.33 gives B=1 or B=Aut(G). Hence Aut(G) does not have a normal 2-Sylow subgroup. By Theorem 3.10, Aut(G) does not have a normal 3-Sylow subgroup. Therefore, either B=1 or both 2 and 3 divide ord(B).

Suppose p=2. Then, by Theorem 3.45, when a prime q divides ord(B), then q=2 or q=3. By Theorem 3.38 and the fact that Aut(G) is supersolvable if ord(G)=4, Aut(G) does not have a normal 3-Sylow subgroup and, by Theorem 3.10,

Aut(G) does not have a normal 2-Sylow subgroup. Hence either B=1 or both 2 and 3 divide ord(B).

Suppose $p\leq 3$. From the above paragraphs, either B=1 or both 2 and 3 divide ord(B).

If $d_k=2$, then 2 and 3 are the only primes dividing ord(Aut(G)). Hence if both 2 and 3 divide ord(B), B=Aut(G).

If $d_k>2$, then $d_k(d_k+1)/2<d_k^2$. Let $r=\text{ord}(\{j \mid d_j=2\})$. Since $(2(2+1)/2)+1=2^2$ and $d_i(d_i+1)/2\leq d_i^2$ for $1\leq i\leq t$ and $d_k\neq 2$, $r+\sum_{i=1}^{t} d_i(d_i+1)/2<\sum_{i=1}^{t} d_i^2$. By Theorem 3.45, the exponent of the highest power of p which divides ord(B) must be $\leq n+r$ where $n=\sum_{i=1}^{t}[f_i^2(n_i-n_{i+1})-d_i^2]$. The exponent of the highest power of p which divides ord(Aut(G)) is $\sum_{i=1}^{t}[f_i^2(n_i-n_{i+1})-d_i(d_i+1)/2]$. From the above inequality, $r+\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i^2)<\sum_{i=1}^{t}(f_i^2[n_i-n_{i+1}]-d_i(d_i+1)/2)$. Thus, since B is a Hall subgroup, p does not divide ord(B). Hence B=1.

Theorem 3.54: If G is not elementary abelian of order 4, then Aut(G) has no non-trivial normal subgroup whose order is relatively prime to (p-1)p.

Proof: Let $B \triangleleft \text{Aut}(G)$ with $(\text{ord}(B), (p-1)p)=1$. Then B has odd order and by Feit-Thompson, [1], is solvable. Hence, by Theorems 3.43 and 3.45, B=1 for $p\geq 3$ and for p=2 ord(B) is some power of 3, in which case Theorem 3.38 gives B=1.

Theorem 3.55: If Aut(G) is not supersolvable, then Aut(G) has no proper normal Hall subgroup whose order is divisible by p.

Proof: Suppose B is a normal Hall subgroup of Aut(G) with p dividing ord(B). Let k be an integer with $1 \leq k \leq t$ such that $d_k \geq d_j$ for each j, $i \leq j \leq t$. By Theorem 3.52, $d_k > 1$. Hence $B \cap \mathrm{Aut}(H_k)$ is a normal Hall subgroup of $\mathrm{Aut}(H_k)$ with p dividing $\mathrm{ord}(B \cap \mathrm{Aut}(H_k))$. By the maximality of d_k, if q is a prime which divides ord(Aut(G)) and does not divide ord(B), then q divides $\mathrm{ord}(\mathrm{Aut}(H_k))$ and does not divide $\mathrm{ord}(B \cap \mathrm{Aut}(H_k))$. By Lemma 3.4, $\mathrm{Aut}(H_k/\Phi(H_k))$ is a homomorphic image of $\mathrm{Aut}(H_k)$. The order of the kernel of the homomorphism is $p^{(n_k-1)d_k^2}$. Since the highest power of p dividing $\mathrm{ord}[\mathrm{Aut}(H_k)]$ is $p^{n_k d_k^2 - d_k(d_k+1)/2}$ and $d_k > 1$, the image D of $B \cap \mathrm{Aut}(H_k)$ is a normal Hall subgroup of $\mathrm{Aut}(H_k/\Phi(H_k))$ with p dividing ord(D). Also if q is a prime which divides $\mathrm{ord}(\mathrm{Aut}(H_k))$ but not $\mathrm{ord}(B \cap \mathrm{Aut}(H_k))$, q divides $\mathrm{ord}[\mathrm{Aut}(H_k/\Phi(H_k))]$ but not ord(D).

The elements of order a power of p generate $SL(d_k,p)$ which is of index p-1 in $GL(d_k,p)$ (cf. Rotman, [9], Lemma 8.14 and Theorem 8.17, pp. 156 and 158). Since $d_k > 1$, p-1 divides $\mathrm{ord}(SL(d_k,p))$. Hence the only normal Hall subgroup of $GL(d_k,p)$ whose order is divisible by p is $GL(d_k,p)$.

Therefore $D = \mathrm{Aut}(H_k/\Phi(H_k))$, $B \cap \mathrm{Aut}(H_k) = \mathrm{Aut}(H_k)$ and $B = \mathrm{Aut}(G)$.

Theorem 3.56: If Aut(G) is not supersolvable, then Aut(G) has no proper non-trivial normal Hall subgroup.

Proof: Let B be a normal Hall subgroup of Aut(G). Let k be an integer with $1 \leq k \leq t$ such that $d_k \geq d_j$ for each j, $1 \leq j \leq t$. Then, by Theorem 3.52, $d_k > 1$.

By Theorem 3.53, if B is of odd order, B=Aut(G) or B=1.

If $d_k = 2$ and $p \leq 3$, then the only primes dividing ord(Aut(G)) are 2 and 3. Hence Aut(G) is solvable (cf. M. Hall, [3], Theorem 9.3.2, p. 143) and, by Theorem 3.53, B=1 or B=Aut(G).

Suppose $d_k > 2$ or $p > 3$ and suppose 2 divides ord(B). By the maximality of d_k, a prime divides ord(Aut(G)) if, and only if, it divides ord(Aut(H_k)). By Lemma 3.4, Aut($H_k/\Phi(H_k)$) is a homomorphic image of Aut(H_k). The kernel of the homomorphism has order $p^{(n_k-1)d_k^2}$ and $p^{n_k d_k^2 - d_k(d_k+1)/2}$ is the highest power of p which divides ord(Aut(H_k)) and $d_k > 1$. Hence a prime divides ord(Aut(H_k)) if, and only if, it divides ord(Aut($H_k/\Phi(H_k)$)). Also a prime divides ord(B) if, and only if, it divides the order of the image of $B \cap \mathrm{Aut}(H_k)$ in Aut($H_k/\Phi(H_k)$).

Suppose D is a normal Hall subgroup of $GL(d_k,p)$ with ord(D) divisible by 2. $D \cap SL(d_k,p)$ is a normal Hall subgroup of $SL(d_k,p)$ whose order is divisible by 2. Since ord[Z($SL(d_k,p)$)] divides p-1 and ord($SL(d_k,p)$) is divisible by (p-1)p(p+1), $D \cap SL(d_k,p) \not\subseteq Z(SL(d_k,p))$. Therefore, since $PSL(d_k,p)$ is simple, $[D \cap SL(d_k,p)][Z(SL(d_k,p))] = SL(d_k,p)$. Hence $[SL(d_k,p) : D \cap SL(d_k,p)] = [(D \cap SL(d_k,p))(Z(SL(d_k,p))) : D \cap SL(d_k,p)]$ which divides ord[Z($SL(d_k,p)$)] which divides p-1.

Therefore since p divides ord($SL(d_k,p)$), p divides ord($D \cap SL(d_k,p)$) and hence ord(D).

Thus p divides the order of the image of $B \cap Aut(H_k)$ in $Aut(H_k/\Phi(H_k))$ and hence p divides ord(B). By Theorem 3.55, B=Aut(G).

Theorem 3.57: If G is not elementary abelian of order 4 and if Aut(G) is supersolvable but not nilpotent, with ord(Aut(G))=$\lambda\mu$ where $(\lambda,\mu)=1$, $\mu \neq 1 \neq \lambda$ and p divides λ, then Aut(G) has a normal subgroup of order λ and does not have a normal subgroup of order μ.

Proof: By Corollary 3.7, $S_{n_1 d}(s_1)$ is a p-Sylow subgroup of Aut(G). By Theorems 3.52 and 3.10 and Corollary 3.6 and Theorem 2.5, $Aut(G)/S_{n_1 d}(s_1)$ is abelian. Hence Aut(G) has a normal subgroup of order λ.

Every non-trivial subgroup of Aut(G), being supersolvable, has a normal Sylow subgroup. A normal subgroup of order μ would have a normal Sylow subgroup which would be a normal Sylow subgroup of Aut(G). Since Aut(G) is not nilpotent, G is not cyclic and $(p-1)^2$ divides ord(Aut(G)). Hence Theorem 3.38 implies that the only normal Sylow subgroup of Aut(G) is the p-Sylow subgroup. Since p does not divide μ, Aut(G) can not have a normal subgroup of order μ.

CHAPTER IV

THE AUTOMORPHISM GROUP OF AN ABELIAN GROUP

Throughout this chapter G is an abelian group and, for p a prime, G_p is the p-Sylow subgroup of G. So G is the direct product of the G_p's for p dividing ord(G).

4.1 Necessary and Sufficient Conditions for the Nilpotence, Supersolvability or Solvability of Aut(G).

The theorems of this section result from the fact that the properties of nilpotence, supersolvability and solvability are preserved by subgroups and direct products of groups possessing them as well as corresponding results from Chapter III which will be listed following the theorem.

Theorem 4.1: Aut(G) is nilpotent if, and only if, G_p is cyclic for $p>2$ and G_2 is multicyclic. (Corollary 3.39).

Theorem 4.2: Aut(G) is supersolvable if, and only if, for each p, G_p is multicyclic or elementary abelian of order 4 (Theorem 3.52).

Theorem 4.3: Aut(G) is solvable if, and only if, G_p is multicyclic for $p>3$ and $\omega(G_p)\leq 2$ (cf. Definition 3.46) for $p\leq 3$ (Corollaries 3.44 and 3.47).

4.2 Normal Hall Subgroups of Aut(G).

This section ends with a theorem giving necessary and sufficient conditions for Aut(G) to have a normal Hall subgroup of a given order.

Theorem 4.4: For q a prime, Aut(G) has a normal q-Sylow subgroup if, and only if, G_q is multicyclic and when q divides ord(Aut(G_p)), with $p \neq q$, either G_p is cyclic or $q=3$, $p=2$ and G_p is elementary abelian of order 4.

Proof: Aut(G) has a normal q-Sylow subgroup if, and only if, Aut(G_p) has a normal q-Sylow subgroup for each prime p. By Theorem 3.10, G_q is multicyclic if, and only if, Aut(G_q) has a normal q-Sylow subgroup. By Theorems 3.56 and 3.57, if $p \neq q$ and q divides ord(Aut(G_p)), Aut(G_p) has a normal q-Sylow subgroup if, and only if, Aut(G_p) is nilpotent or G_p is elementary abelian of order 4, which gives $p=2$, $q=3$. For $p>2$, by Corollary 3.39, Aut(G_p) is nilpotent if, and only if, G_p is cyclic. Also by Corollary 3.39, Aut(G_2) is nilpotent if, and only if, G_2 is multicyclic, in which case there is no $q \neq 2$ which divides ord(Aut(G_2)).

Corollary 4.5: Aut(G) has a normal 2-Sylow subgroup if, and only if, Aut(G) is nilpotent, in which case Aut(G) is abelian if, and only if, G_2 is cyclic.

Theorem 4.6: For q a prime, Aut(G) has a normal q-complement if, and only if, G_q is cyclic (or q=2 and G_q is multicyclic or elementary abelian of order 4) and whenever q divides ord(Aut(G_p)) with $q \neq p$, G_p is multicyclic.

Proof: Aut(G) has a normal q-complement if, and only if, Aut(G_p) has a normal q-complement for each prime p. By Theorems 3.56 and 3.57, Aut(G_q) has a normal q-complement if, and only if, Aut(G_q) is nilpotent or G_q is elementary abelian of order 4. By Corollary 3.39, Aut(G_q) is nilpotent if, and only if, G_q is cyclic or q=2 and G_q is multicyclic. For $q \neq p$ and q dividing ord(Aut(G_p)), Theorems 3.56 and 3.57 and the fact that GL(2,2) does not have a normal 2-Sylow subgroup imply that Aut(G_p) has a normal q-complement if, and only if, Aut(G_p) is supersolvable and G_p is not elementary abelian of order 4. By Theorem 3.52, this is the case if, and only if, G_p is multicyclic.

Corollary 4.7: Aut(G) has a normal 2-complement if, and only if, Aut(G) is supersolvable.

Corollary 4.8: If q>3 is a prime and if G_q is cyclic and Aut(G) is supersolvable, then Aut(G) has a normal q-complement.

Proof: The result follows from Theorems 4.6 and 3.52 and the fact that q does not divide the order of GL(2,2).

Corollary 4.9: If Aut(G) has a normal 3-complement, then Aut(G) is supersolvable.

Proof: Suppose Aut(G) has a normal 3-complement. Then, by Theorem 4.6, $Aut(G_3)$ is abelian. By Theorem 3.52, if, for $p\neq 3$, there is an $Aut(G_p)$ which is not supersolvable, then G_p is not multicyclic and hence $(p-1)p(p+1)$ divides $ord(Aut(G_p))$. Therefore 3 divides $ord(Aut(G_p))$ and, by Theorem 4.6, G_p is multicyclic which is a contradiction. Hence $Aut(G_p)$ is supersolvable for all p which divide ord(G) and thus Aut(G) is supersolvable.

Theorem 4.10: If $ord(Aut(G))=\lambda\mu$ where $(\lambda,\mu)=1$ and if $ord(Aut(G_p))=\lambda_p\mu_p$ where λ_p divides λ and μ_p divides μ for each prime p, then Aut(G) has a normal subgroup of order λ if, and only if, for each prime p dividing ord(G); (1) G_p is multicyclic when p divides λ_p and $\mu_p\neq 1$ and (2) when p divides μ_p and $\lambda_p\neq 1$, G_p is cyclic or $p=2$, $\lambda_p=3$ and G_p is elementary abelian of order 4.

Proof: Aut(G) has a normal subgroup of order λ if, and only if, $Aut(G_p)$ has a normal subgroup of order λ_p for each p dividing ord(G).

Suppose $Aut(G_p)$ has a normal subgroup of order λ_p for each p dividing ord(G). Let p be a prime which divides ord(G). If p divides λ_p and $\mu_p\neq 1$, then by Theorem 3.56, $Aut(G_p)$ is supersolvable. By Theorem 3.52 and since GL(2,2)

has no normal 2-Sylow subgroup, G_p is multicyclic. If p divides μ_p and $\lambda_p \neq 1$, then, by Theorems 3.56 and 3.57, $\mathrm{Aut}(G_p)$ is nilpotent or G_p is elementary abelian of order 4. If $\mathrm{Aut}(G_p)$ is nilpotent, then, by Corollary 3.39, G_p is cyclic or p=2 and G_p is multicyclic, the latter contradicting $\lambda_p \neq 1$.

Suppose (1) and (2) hold for each prime p dividing ord(G). Lep p be a prime dividing ord(G).

If p does not divide $\lambda_p \mu_p$, then G_p is cyclic of order p and hence $\mathrm{Aut}(G_p)$ has a normal subgroup of order λ_p.

If $\mu_p = 1$, then $\mathrm{Aut}(G_p)$ is a normal subgroup of $\mathrm{Aut}(G_p)$ of order λ_p.

If p divides λ_p and $\mu_p \neq 1$, then, by (1), G_p is multicyclic. By Theorem 3.52, $\mathrm{Aut}(G_p)$ is supersolvable. Since G_p is not elementary abelian of order 4, $\mathrm{Aut}(G_p)$ has a normal subgroup of order λ_p, by Theorem 3.57.

If $\lambda_p = 1$, then 1 is a normal subgroup of $\mathrm{Aut}(G_p)$ of order λ_p.

If p divides μ_p and $\lambda_p \neq 1$, then, by (2), G_p is cyclic or p=2, $\lambda_p = 3$ and G_p is elementary abelian of order 4. In either case, $\mathrm{Aut}(G_p)$ has a normal subgroup of order λ_p.

INDEX OF NOTATION

I. Relations:

$\subseteq$	Is a subset of
$\subset$	Is a proper subset of
$\leq$	Is a subgroup of or is less than or equal to
$<$	Is a proper subgroup of or is less than
$\triangleleft$	Is a normal subgroup of
$\in$	Is an element of
$\equiv$	Is congruent to

II. Operations:

G^{α}	Image of the set G under the mapping α
g^{α}	Image of the element g under the mapping α
G/H	Factor group
x	Direct product of groups
[G:H]	Index of H in G
< >	Subgroup generated by
{ }	Set whose members are
{x|P}	Set of all x such that P is true
ord(G)	Number of elements in G
|G|	ord(G)
ord(g)	Order of the element g
(m,n)	Greatest common divisor of the integers m and n
exp(G)	Exponent of the group G
$\alpha\|_H$	Restriction of the mapping α to the set H

III. Groups and Sets:

$Aut(G)$	Automorphism group of G
$Z(G)$	Center of G
$C_G(H)$	Centralizer of H in G
$N_G(H)$	Normalizer of H in G
$\Phi(G)$	Frattini subgroup of G
$Fit(G)$	Fitting subgroup of G
$Sol(G)$	Maximal normal solvable subgroup of G
G'	Commutator subgroup of G
$\Omega_k(G)$	Subgroup, of the p-group G, generated by the elements whose p^k power is 1
$\mathcal{V}_k(G)$	Subgroup, of the p-group G, generated by the p^k powers of elements of G
$[H,A]$	Subgroup generated by all $h^{-1}h^{\alpha}$ where $h \in H \leq G$ and $\alpha \in A \leq Aut(G)$
$[H,A^{n+1}]$	$[[H,A^n],A]$
s	Chain of subgroups of the group G; $s: G=G_0>G_1>\cdots>G_t=1$
$S_o(s)$	$\{\alpha \in Aut(G) \mid G_i^{\alpha}=G_i$ for $i=1,2,\ldots,t\}$
$S_i(s)$	$\{\alpha \in S_{i-1} \mid g^{-1}g^{\alpha} \in G_i$ for all $g \in G_{i-1}\}$
ker α	Kernel of the homomorphism α
$\emptyset$	Empty set
A-B	Set of all x in A and not in B
GL(n,p)	General linear group of nonsingular nxn matrices over the field of order p

SL(n,p)	Special linear group of nxn unimodular matrices over the field of order p
PSL(n,p)	SL(n,p)/Z(SL(n,p))
G_p	p-Sylow subgroup of the abelian group G

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