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ABSTRACT

LIMITS: A MASTERY LEARNING APPROACH TO A UNIT ON LIMITS OF SEQUENCES AND FUNCTIONS IN A PRECALCULUS COURSE AND ACHIEVEMENT IN FIRST SEMESTER CALCULUS

by

Douglas William Nance

The teaching of limits to calculus students is a task faced by most college instructors. This study was an attempt to find a method of presenting limits to students in such a way that they would be prepared to work with a formal definition of limit of a function in a first semester calculus course. Specifically, a unit on limits of sequences and limits of functions was taught in four sections of Precalculus Mathematics at Central Michigan University. This unit included definitions of both limit of a sequence and limit of a function.

Two sections that received the unit on limits were taught using a mastery learning strategy. This consisted of (1) stating the instructional objectives, (2) presenting the unit of instruction, (3) administering a formative evaluation, (4) utilizing instructional alternatives, and (5) administering a summative evaluation. Advocates of this approach maintain that generally three fourths of the students can achieve at the same high level that only one fourth of the students usually achieve. This "achievement level" is referred to as the

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mastery level or criterion. Ordinarily, this level is between eighty and eighty-five percent or equivalently a criterion that would allow a student to receive a grade of A or B in a regular classroom situation. Students who attain this level are said to have "mastered" the material.

A unique aspect of this study was to identify students who attained the mastery level on the summative evaluation but not on the formative evaluation. These students were given the label "delayed mastery" and their progress in calculus the following semester was analyzed to determine carry-over effects of the mastery learning strategy.

Two sections of Precalculus students studied the same unit on limits but received instruction via an expository presentation. Five other Precalculus sections were used as a control group and received no instruction regarding limits. Performance of students from all nine sections was analyzed in the Calculus I course the semester following the presentation of the unit on limits to Precalculus students.

Treatment effects on the affective domain elements were determined in two ways. One was by interviewing randomly selected students from the Precalculus sections. The other was by analyzing responses to a questionnaire completed by Treatment and Control students enrolled in Calculus I. In addition to these analyses, enrollment figures in Precalculus and Calculus I for the years 1971, 1972, and 1973 were examined to see if the Treatment affected the percentage of Precalculus students enrolling in Calculus I.

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The effect of the treatment on achievement in Calculus I was analyzed at two stages of the course. Data from the test covering limits was used first. Student performance was measured on (1) the questions involving limits and (2) the total test scores. The semester percentage for each student provided the second set of data.

Findings of the study include the following:

1. The mastery learning strategy resulted in sixty-five percent of the students attaining the mastery level of eighty percent compared to thirty-four percent of the students attaining that level in the expository treatment group.
2. There was no significant difference in achievement in Calculus I between delayed mastery students and other mastery students.
3. There was no significant difference in enrollment in Calculus I between students in the Treatment sections and students in the Control sections.
4. There was no significant difference in achievement in Calculus I between students in the Treatment sections and students in the Control sections.

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LIMITS: A MASTERY LEARNING APPROACH TO A UNIT ON LIMITS OF SEQUENCES AND FUNCTIONS IN A PRECALCULUS COURSE AND ACHIEVEMENT IN FIRST SEMESTER CALCULUS

by

Douglas William Nance

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Secondary Education

1974

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ACKNOWLEDGMENTS

I would like to thank the following individuals who helped make this dissertation possible:

Professor John Wagner, advisor and Committee Chairman, for his advice and guidance during the last five years.

Professors Joseph Adney, Julian Brandou, and William Fitzgerald, doctoral committee members, for their advice and encouragement.

Professor William E. Lakey for his continued support of the research project.

Last in order but first in thought, I would like to thank my wife and typist, Helen M. Nance. Her patience, encouragement, and perseverance have enabled me to complete this project.

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CHAPTER I

INTRODUCTION

This study investigated a method of presenting the topic of limits to students. Specifically, a unit on limits of sequences and limits of functions was taught in four sections of Precalculus Mathematics at Central Michigan University (CMU). This differs from the current practice at CMU of teaching limits* only in the calculus course. This approach is consistent with G. Rising's[1] belief in presenting limits prior to calculus. It also gave a longer introduction to the definition of limit of a function and its subsequent use than is currently done at CMU.

Need for Study

The concept of "limit" is an essential part of a beginning calculus course. C.B. Allendoerfer said, "The essential idea in calculus is that of limit..."[2] Support for the importance of limits is given by the fact that several basic calculus topics are defined in terms of them. In particular, continuity of functions, differentiation of functions, and integration of functions all use limits in their definitions.

The central role of this concept makes it apparent that mathematicians

* The word limit(s) is imprecisely used and refers to the more specific topics of "limits of sequences" and "limits of functions". Since these are closely related but not always taught in the same course, the general term "limit(s)" will be used to mean either or both, whichever is more appropriate for the context.

and mathematics educators should be concerned about finding the most effective methods of presenting this topic to the beginning and prospective calculus student. The need for this concern was emphasized by R.G. Long in 1968 when he included in his "Report of the Cornell Conference", under topics for research, the following item: "In freshman calculus, what is the mechanism involved in acquisition of the limit concept?"[3]

There are differing opinions as to when, where, and how limits should be taught. E.W. Ferguson,[4] D.W. Hight,[5] M.F. Hubley,[6] and G.R. Rising[7] have supported the teaching of calculus (which includes limits) in the secondary schools. On the other hand, C.B. Allendoerfer,[8] A. Beninati,[9] and J.H. Neelley[10] are opposed to introduction at that level. Another difference of opinion involves the amount of formal logical procedures used in presenting the concept of limits. One approach uses a minimum of formal logic and avoids a precise definition, but attempts to give students an intuitive idea of the limit concept. Another approach utilizes a precise definition of limit and subsequent formal logical development leading to the proof of theorems involving limits. Since the definition of limits frequently employs the Greek symbols "ϵ" (epsilon) and "δ" (delta), this is referred to as the (ϵ, δ) or rigorous approach.

The intuitive and rigorous approaches exist as opposite ends of a spectrum and most calculus instructors would place themselves somewhere between these extremes. J.R. Anderson,[11] for example, believes in an intuitive approach with an (ϵ, δ) treatment postponed until advanced calculus.

Others such as C.B. Allendoerfer[12] and R.P. Boas, Jr.[13] believe in starting with an intuitive approach and leading up to an (ϵ, δ) treatment in a beginning course.

Mastery Learning

Another feature of the study was that a mastery learning strategy was employed to teach two of the sections involved in the experiment. Historically speaking, major mastery learning attempts date to the early 1900's. J.H. Block says, "As early as the 1920's there were at least two major attempts to produce mastery in students' learning."[14] Mastery learning consists of stating objectives, presenting the unit of instruction, administering a "formative"* evaluation, providing feedback from this evaluation, utilizing alternative instructional techniques, and then administering a "summative"* evaluation. The effectiveness of this strategy is indicated by J.H. Block in the following statement, "In general, three fourths of the students learning under mastery conditions have achieved to the same high standards as the top one-fourth learning under conventional, group-based instructional conditions."[15] A description of the aspects of mastery learning is given in Chapter II.

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* These terms are defined on pages 19-20 of this document.

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Delayed Mastery

Some students will be identified as "delayed-mastery" for the purposes of this study. These will be the students in the mastery treatment group that attain the mastery level (80 percent) on the summative evaluation but do not reach this level on the formative evaluation. The identification of these students is a unique aspect of this study. Subsequent achievement of this group of students will be analyzed to determine what effect the mastery learning approach in Precalculus had on their performance in Calculus I.

Level of Material

Limit problems can be of many levels of difficulty. Since mastery learning techniques have proven to be effective for lower level (Computation) problems (Chapter II), an additional aspect of this study was to categorize the material to be mastered by the treatment groups at the Comprehension, Application, or Analysis levels in the cognitive domain of J.W. Wilson's model for school learning. Part of the analyses will then be to see if mastery learning can be effectively employed in the learning of higher level items. A discussion of the components of this model are given in Chapter II.

Discussion

Nine sections of precalculus students at Central Michigan University constituted the population for this experiment. Each section contained approximately thirty students. Five of these sections were used for control. Of the remaining four, two were administered a section on limits of sequences and limits of functions via a classroom lecture method. The remaining two sections were administered the same unit of material via a mastery learning strategy.

Several reasons were considered when selecting the groups for treatment or control. Both mastery learning sections were taught by the experimenter. The other two treatment sections were taught by Dr. Douglas D. Smith who was the only other staff member at CMU teaching two sections of Precalculus Mathematics during the semester the experiment was conducted. The other five sections were used as a control group. Since these were taught by five different individuals, it was decided these would be the ones most appropriate for control since any other assignment of treatment groups would have meant at least three instructors working with treatment groups.

The results of this study involved both the affective (interest, attitude, etc.) and cognitive (achievement) domains. The affective domain results were checked by interviews, questionnaires, enrollments, and retention figures. The cognitive domain results were obtained from the calculus course given the semester following the teaching of the treatment units. Results of the test covering the section on limits were analyzed and achievement in the entire course was examined.

Statement of Hypotheses

The hypotheses to be studied are:

1. A mastery learning strategy, employed on a unit of limits of sequences and limits of functions, will result in no significant difference between the proportion of students achieving the mastery level in a mastery learning situation and the proportion of students achieving the mastery level in an expository classroom situation.

2. A unit on limits in Precalculus will not affect attitudes

   (a) prior to taking Calculus I

   (b) during Calculus I.

   The analysis of this hypothesis will be qualitative and will consist of reporting results of interviews and questionnaires completed by students involved in the study.

3. There will be no significant difference between the percentage of students enrolling and remaining in Calculus I after having had a unit on limits and the percentage of students from the Control group enrolling and remaining in Calculus I.

4. There will be no significant difference in achievement in Calculus I between students in the treatment groups that achieve mastery and those that did not achieve mastery.

5. There will be no significant difference in achievement in Calculus I between delayed mastery students and other mastery students in the treatment groups.

6. An introduction to limits prior to Calculus I will result in no significant difference in achievement

   (a) on the work with limits in Calculus I

   (b) in the first semester of Calculus I.

   This hypothesis was tested by examining test scores and semester averages of students involved in the study who enrolled in Calculus I.

Hypotheses 1, 3, 4, 5, and 6 were tested at the significance level of $\alpha = .05$.

Organization of the Dissertation

The remainder of the dissertation is divided into four chapters. Chapter II contains a review of relevant literature. Material involving when, where, and how limits are taught is examined. Explanations of and references to mastery learning and learning models are included. Chapter III describes the procedures used during the experiment. Included therein is the experimental model. Chapter IV contains the findings of the study and Chapter V gives the conclusions and implications.

FOOTNOTES -- CHAPTER I

[1]Rising, Gerald R. "Some Comments on Teaching of the Calculus in Secondary Schools", The American Mathematical Monthly, March, 1961, 68:287-290.

[2]Allendoerfer, C.B. "The Case Against Calculus", The Mathematics Teacher, Nov. 1963, 56:482-485, p. 484.

[3]Long, R.G. "Report of the Cornell Conference on Mathematics Education", 1968.

[4]Ferguson, Eugene W. "Calculus in the High School", The Mathematics Teacher, Oct. 1960, 53:451-453.

[5]Hight, Donald W. "The Limit Concept in the Education of Teachers", American Mathematical Monthly, Vol. 70, 1963, pp. 203-205.

[6]Hubley, Martin F. and Charles W. Maclay, "An Experiment in High School Calculus", The Mathematics Teacher, Nov. 1970, pp. 609-612.

[7]Rising, Gerald R. "Some Comments on Teaching of the Calculus in Secondary Schools", The American Mathematical Monthly, March 1961, 68:287-290.

[8]Allendoerfer, C.B. "The Case Against Calculus", The Mathematics Teacher, Nov. 1963, 56:482-485.

[9]Beninati, Albert, "It's time to take a closer look at high school calculus", The Mathematics Teacher, Jan. 1966, pp. 29-30.

[10]Neelley, J.H. "A Generation of High School Calculus", The American Mathematical Monthly, Dec. 1961, 68:1004-1005.

[11]Anderson, John Robert, "A Comparison of Student Performance in a One Year Freshman College Calculus Course Resulting from Two Different Methods of Instruction", Unpublished Ph.D. Dissertation, Purdue University, 1970.

[12]Allendoerfer, C.B. "The Case Against Calculus", The Mathematics Teacher, Nov. 1963, 56:482-485.

[13]Boas, R.P., Jr. "Calculus as an Experimental Science", American Mathematical Monthly, Vol. 78, June-July, 1971, pp. 664-667.

[14]Block, James H. (ed.) Mastery Learning: Theory and Practice, Holt, Rinehart and Winston, Inc., New York, 1971, p. 3.

[15]Ibid.

CHAPTER II

REVIEW OF RESEARCH

Limits

A college level course in beginning calculus is concerned with the concepts of limits of functions, continuity of functions, differentiation of functions, and integration of functions. Limits of functions is perhaps the most important of these since the others are defined in terms of them. R.V. Lynch referred to limit as "...the central concept of calculus..."[1] These thoughts were supported by D. Kleppner and N. Ramsey when they said, "The idea of limit...is at the heart of calculus..."[2] and "Once you have a real feeling for what is meant by a limit you will be able to grasp the ideas of differential calculus quite readily."[3] Further support for the importance of limits was given by D.W. Hight when he said, "It is the introduction of limits that allows one to proceed from elementary mathematics to a study of calculus and all its ramifications."[4]

There are several approaches to presenting the concept of limits to students of calculus. These range from a brief discussion of the concept with some examples (frequently called the "intuitionist" approach) to a development based on precise definition of limit and subsequent logically developed proofs and applications of theorems. The question of how much rigor should be applied in developing the concept of limits has been discussed by several people. C.B. Allendoerfer has said, "All too often we begin the course with an off-hand reference to limits as something too hard for the students to really understand..."[5] He continues with, "There are those, however, who begin the course with a brief, but full dress, treatment of limits using the epsilon-delta technique. This almost universally is wasted on the class, for they are confronted with a difficult new idea without an intuitive preparation."[6]

A study by J.R. Anderson[7] compared two approaches to presenting limits; one used the (ϵ, δ) definition and subsequent development while the other used an intuitive development with the (ϵ, δ) definition coming in advanced calculus. The second method was found to be significantly better with respect to achievement but there was no change in attitude as a result of method. Included in the concluding remarks of this study was, "Evidently neither method could make the (ϵ, δ) notation and proof any more palatable for the students."[8] T.H. McGannon, in a more general study,[9] examined a rigorous versus intuitive approach to calculus. Among the results of the analyses were the following: there is no difference between the rigorous method and the intuitive method of teaching calculus with respect to achievement of (a) a proficiency in manipulative skills, (b) an understanding of the fundamental concepts, and (c) an ability to make practical applications of the principles learned.

The fundamental nature of the limit concept in calculus and the lack of agreement regarding its presentation have resulted in attempts to teach limits prior to calculus. Gerald Rising supports this idea and recommends having such a unit taught in a high school course. He says, "Students can spend a substantial amount of time on two topics which most college teachers will agree deserve more time than they are able to allot to them. These two topics are limits and the definition of the derivative. Rather than gloss over these topics superficially in a lesson or two, as is so often done in college courses, the secondary program offers the possibility of spending two or three weeks on each topic."[10] He continues by saying, "By solid teaching of these topics [limit, derivative] the secondary teacher can help remove what I believe to be two of the toughest hurdles to mathematics students in the college program."[11]

Rising is supported by D.W. Hight who says, "The teaching of the concept of a limit should begin in the secondary schools..."[12] and "The greatest gap between the secondary school mathematics program and that of the colleges seems to be due to the present treatment of topics that involve limits."[13] Hight's belief that limits should be taught in secondary schools is a result of a stronger belief that limits should be taught prior to calculus. He says, "...it is important that students have a good understanding of the concepts of limit prior to entering calculus..."[14]; "...the students need a gradual introduction to the limit concept in order to grow in mathematical maturity and gain a readiness for calculus."[15]; and, "If he is unacquainted with the limit concept, he can hardly be prepared for calculus when the definitions of both the derivative and the integral involves limits."[16]

A concern for acquisition of the limit concept and success in calculus prompted the author to initiate this study which investigated some of the aforementioned variables. Currently at Central Michigan University, limits are taught only in the calculus course and are developed using a rigorous (ϵ, δ) approach. Many of the calculus students did not have a section on limits in their high school mathematics classes; consequently they were receiving instruction regarding limits that, according to Allendoerfer, is "...almost universally wasted on the class..."[17]

The ideas of Rising, Hight, and others were implemented in this study by including a unit on limits of sequences and limits of functions in four sections of CMU's Precalculus course. This unit included more than an intuitive presentation of the concept of limit. It contained precise definitions (involving [ϵ, δ] notation) and subsequent development of several theorems regarding limits. This approach is supported by the work of D.T. Coon[18] who, in a study involving the intuitive concept of limit possessed by precalculus students, found there was no significant correlation between intuitive knowledge of limit and achievement in calculus.

One aspect of this study was the use of a special instructional strategy in two of the treatment sections. This strategy consisted of teaching the unit on limits using a "mastery learning" approach. Various aspects of this approach are analyzed in the next section.

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Mastery Learning

Mastery learning is an educational strategy based on the premise that students can learn almost all of the material presented to them if certain variables are identified and controlled. Benjamin S. Bloom said, "Most students, perhaps over 90 percent, can master what teachers have to teach them, and it is the task of instruction to find the means which will enable students to master the subject under consideration. A basic task is to determine what is meant by mastery of the subject and to search for methods and materials which will enable the largest proportion of students to attain such mastery."[19]

The variables affecting learning were identified by John B. Carroll[20] as (1) aptitude for particular kinds of learning, (2) quality of instruction, (3) ability to understand instruction, (4) perseverance, and (5) time allowed for learning. Carroll believes that, "...if the students are normally distributed with respect to aptitude, but the kind and quality of instruction and the amount of time available for learning are made appropriate to the characteristics and needs of each student, the majority of students may be expected to achieve mastery of the subject."[21]

Several mastery learning strategies have been devised which attempt to control the variables described by Carroll. Bloom, when discussing what his group at Chicago had been doing, said, "Our approach has been to supplement regular group instruction by using diagnostic procedures and alternative instructional methods and materials in such a way as to bring a large proportion of the students to a predetermined standard of achievement. In this approach, we have tried to bring most of the students to mastery levels of achievement within the regular term, semester, or period of calendar time in which the course is usually taught."[22] Samuel T. Mayo[23] outlined a more descriptive procedure when he summarized a national meeting that dealt with the topic of mastery learning. He indicated a mastery learning strategy should include the following:

(1) Inform students about course expectations, even lesson expectations or unit expectations, so that they view learning as a cooperative rather than as a competitive enterprise.

(2) Set standards of mastery in advance; use prevailing standards or set new ones and assign grades in terms of performance rather than relative ranking.

(3) Use short diagnostic progress tests for each unit of instruction.

(4) Prescribe additional learning for those who do not demonstrate initial mastery.

(5) Attempt to provide additional time for learning for those persons who seem to need it.

The success of mastery learning strategies has been documented in several studies. Peter W. Airasian used a mastery strategy in a graduate-level course in test theory. According to his summary of the study, the method used produced successful results. He said, "The main result was striking. Whereas in the previous year only 30 percent of the students received an A grade, 80 percent of the sample achieved at or above the previous year's A grade score on a parallel exam and thus received A's."[24]

Kenneth M. Collins employed a mastery strategy in the teaching of four college mathematics courses for freshmen. Concerning the results of his study, he said, "In the modern algebra courses, 75 percent of the mastery compared to only 30 percent of the non-mastery students achieved the mastery criterion of an A or B grade. The calculus classes' results were similar: 65 percent of the mastery compared to 40 percent of the nonmastery students achieved the criterion."[25]

The results of other studies using mastery strategies are summarized in the following: (1) Fred S. Keller[26] had 65 percent to 70 percent of the class receive A's or B's; (2) Mildred E. Kersh[27] had 75 percent of a class achieve mastery as compared to 19 percent that achieved mastery in a control class; (3) Hogwan Kim[28] had results that indicated 74 percent of the experimental group compared to only 40 percent of the control group attained the mastery level; and (4) Samuel T. Mayo, Ruth C. Hunt, and Fred Tremmel[29] found when working with students in an introductory statistics course that 65 percent of the mastery learning students received a grade of A in contrast with 5 percent of the comparison group.

Mastery learning strategies accomplish more than having a high percentage of the students achieve at levels formerly attained by only a small percentage of the students. There are several affective consequences of such strateg

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strategies. For example, J.H. Block says, "In a system with few rewards, a student may not be rewarded no matter how well he learns so long as others learn better. If this situation occurs repeatedly, he is likely to eventually stop trying to learn well. In a system of unlimited rewards, by contrast, a student who learns well can be rewarded even though others may learn still better. Successful and rewarding learning experiences are likely, in turn, to kindle a desire to continued learning excellence."[30] Benjamin S. Bloom,[31] when discussing mastery learning, lists the following affective consequences: (1) students develop more interest for the courses they master, (2) students develop a more positive self concept, (3) mastery learning can develop a lifelong interest in learning. In addition to these consequences, Bloom says, "Mastery learning can be one of the more powerful sources of mental health."[32]

The success of mastery learning strategies induced the author to use such a strategy in one of the treatment sections of this experiment. It was based on the outline presented by S.T. Mayo. In accordance with item 1 of his outline, terminal objectives (unit expectations) were selected prior to starting the experiment. The difficulty of such a selection is documented by K. Collins when he says, "Specification of objectives is perhaps the most difficult variable to properly prepare."[33]

The next step in preparing a mastery strategy is to select standards of mastery. With regard to this, Bloom says, "While absolute standards carefully worked out for a subject are recommended, they are often difficult to set. One meth

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set. One method might be to use standards derived from previous experience in a particular course. For example, grades for one year might be based on grading standards arrived at the previous year if parallel examinations are used."[34] Along these same lines, Block says, "There are no hard and fast objective rules for setting mastery grading standards."[35] He further says, "A...standard setting method is to transfer existent grading standards set for the subject under non-mastery learning conditions to the courses taught under mastery conditions. Standards from previous or concurrent non-mastery teachings of the subject can be used. Generally, scores which earned students learning under non-mastery conditions A's and B's seem to be useful mastery grading standards."[36] Block supports Bloom by saying, "The empirical work to date suggests that if students learn 80 to 85 percent of the skills in each unit, then they are likely to exhibit maximal positive cognitive and affective development as measured at the subject's completion. This work also suggests that encouraging or requiring students to learn all or nearly all (90 to 95 percent) of each unit, besides being an unrealistic expectation in terms of student and teacher time and effort (Bormuth, 1969), may have marked negative consequences for student interest in and attitudes toward the learning (Block, 1970; Sherman, 1967)."[37]

The classroom instructional techniques for a mastery learning situation do not have to differ from what a teacher would ordinarily employ. With regard to this, Bloom says, "In the work we have done, we have attempted to have the teacher teach the course in much the same way as previously.

That is, the particular materials and methods of instruction in the current year should be about the same as in previous years."[38] It is the belief of Bloom and others of his group that this is the only way mastery learning can be effectively spread. He feels that if extensive retraining of teachers is required, then the necessary effort will not be expended by the teachers.

The next step in formulating a mastery learning strategy is the creation of diagnostic progress tests. These have been labeled "formative evaluations" by Michael Scriven.[39] According to Airasian, "Formative evaluations seek to identify learning weaknesses prior to the completion of instruction on a course segment--a unit, a chapter, or a lesson. The aim is to foster learning mastery by providing data which can direct subsequent or corrective teaching and learning."[40] Another description of the purpose of formative evaluations is given by Bloom when he says, "For those students who have thoroughly mastered the unit, the formative tests should reinforce the learning and assure the student that his present mode of learning and approach to study is adequate...For students who lack mastery of a particular unit, the formative tests should reveal the particular points of difficulty--the specific questions they answer incorrectly and the particular ideas, skills, and processes they still need to work on."[41]

The formative evaluations should be followed by diagnostic reports for each student. Learning difficulties indicated on these reports are to be corrected by instructional alternatives. A variety of alternatives will provide for some of the learning variables given by Carroll (page 14). Block lists the followin

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(1) Reteaching

(2) Small group problem sessions

(3) Individual tutoring

(4) Alternative learning materials

(a) Alternative textbooks

(b) Workbook and programmed instruction

(c) Audio-visual materials

(d) Academic games and puzzles

Bloom strongly advocates the small group problem sessions as evidenced by his remarks, "The best procedure we have found thus far is to have small groups of students...review the results of their formative evaluation tests and to help each other overcome the difficulties identified on these tests."[43] and, "Where learning can be turned into a cooperative process with everyone likely to gain from the process, small group learning procedures can be very effective."[44]

The last phase of a mastery learning strategy is administration of the "summative evaluation". This is the phrase used by Scriven[45] for the test given to determine how well the students have "mastered" the objectives specified earlier in the unit of instruction.

Summary: The mastery learning strategy used in one of the treatment groups of this experiment is based on common aspects of the various strategies previously mentioned and incorporated suggestions made concerning the compo

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components of such strategies. A schematic diagram is given in Figure 3.2.

Classification of Objectives

A successful mastery learning strategy requires specification of educational objectives for the material being taught. Since one of the purposes of this study was to foster better "understanding" of the limit concept, it was not sufficient to specify the educational objectives; it was also necessary to classify them according to the type of learning involved. According to Benjamin S. Bloom, "...a teacher, in classifying the goals of a teaching unit, may find that they all fall within the taxonomy category of recalling or remembering knowledge. Looking at the taxonomy categories may suggest to him that, for example, he could include some goals dealing with the application of this knowledge and with the analysis of the situations in which the knowledge is used."[46]

The taxonomy used for classifying objectives in this study is James W. Wilson's model.[47] The basic categories in the cognitive domain are: (A) Computation, (B) Comprehension, (C) Application, and (D) Analysis. A result of the use of Wilson's taxonomy was the inclusion of objectives at the Comprehension, Application, and Analysis levels of behavior. A copy of this taxonomy is contained on page 34.

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FOOTNOTES -- CHAPTER II

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CHAPTER III

PROCEDURES

Population and experimental design

The population for this study consisted of all students in nine sections of Precalculus Mathematics at Central Michigan University during the Fall Semester of the 1973-74 Academic Year. The study involved two treatment groups and a control group. Treatment A consisted of teaching a unit on limits of sequences and limits of functions to two of these sections using a mastery learning strategy as described on page 29. Treatment B consisted of teaching the same unit to two other sections but without employing a mastery learning approach. The remaining five sections were used as a control group and received no instruction regarding limits. The experimental design is given in Figure 3.1.

<table>
<tr><td colspan="3">Treatment A
n = 60</td><td colspan="2">Treatment B
n = 61</td><td>Control
n = 152</td></tr>
<tr><td colspan="2">Mastery
n = 39</td><td rowspan="2">Non-mastery
(NNM)
n = 21</td><td rowspan="2">Mastery
(SM)
n = 21</td><td rowspan="2">Non-mastery
(SNM)
n = 40</td><td rowspan="2">(C)</td></tr>
<tr><td>Early mastery
(NEM)
n = 5</td><td>Delayed mastery
(NDM)
n = 34</td></tr>
</table>

FIGURE 3.1

Population and Experimental Design

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Persons Assisting in the Study

Three persons assisted in some aspect of this study. Professor William E. Lakey, who currently teaches the Calculus I classes at CMU, reviewed all materials used in Treatments A and B, helped select the terminal behaviors for Treatment A (page 28), and conducted some interviews. Professor Douglas D. Smith taught the two sections that received Treatment B. He also reviewed the instructional materials and assisted in the construction of the formative evaluations. The third person was Dave Rajala, an undergraduate student, who was used as a resource person during the small group sessions between the formative and summative evaluations for the students in Treatment A.

Mastery Learning Strategy

The mastery learning strategy used in Treatment A differed in some respects from the strategy described by S.T. Mayo in Chapter II. One difference was that the strategy was employed on a unit of instruction rather than the entire course. Consequently, only one formative evaluation was administered. Secondly, students were not informed of all terminal objectives prior to the start of the unit because of the terminology involved. They were, however, informed of the objectives as soon as the required notation and terminology were introduced. For example, objective 4 (page 28) was stated after the definition for limit of a sequence was given. A third variation was that the iten

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that the items to be mastered would be classified in the Comprehension, Application, or Analysis levels of James W. Wilson's learning model.[1]

Figure 3.2 is a schematic representation of the instructional strategies used in the treatment sections of this study. Students in Treatments a and B had to record a score of at least 80 percent, as suggested by J.H. Block,[2] to achieve mastery of the unit.

Formulation of Instructional Objectives: Treatments A and B

The objectives for the unit of study in Treatments A and B were established by Professor Lakey and the author. At the conclusion of studying the unit on limits, the students were expected to be able to do the following:

(1) Given a sequence (function) that does not have a limit, indicate what part of the definition it fails to satisfy.

(2) Give an example of a sequence (function) that does not have a limit and include explanation.

(3) Given a real number, be able to produce a nonconstant sequence (function) whose limit is that real number and include explanation.

(4) When working with sequences, find the associated N_ϵ for a particular ϵ.

(5) When working with functions, find the associated δ_ϵ for a particular ϵ.

(6) Determine limits of sequences using the results of theorems.

(7) Determine limits of functions using the results of theorems.

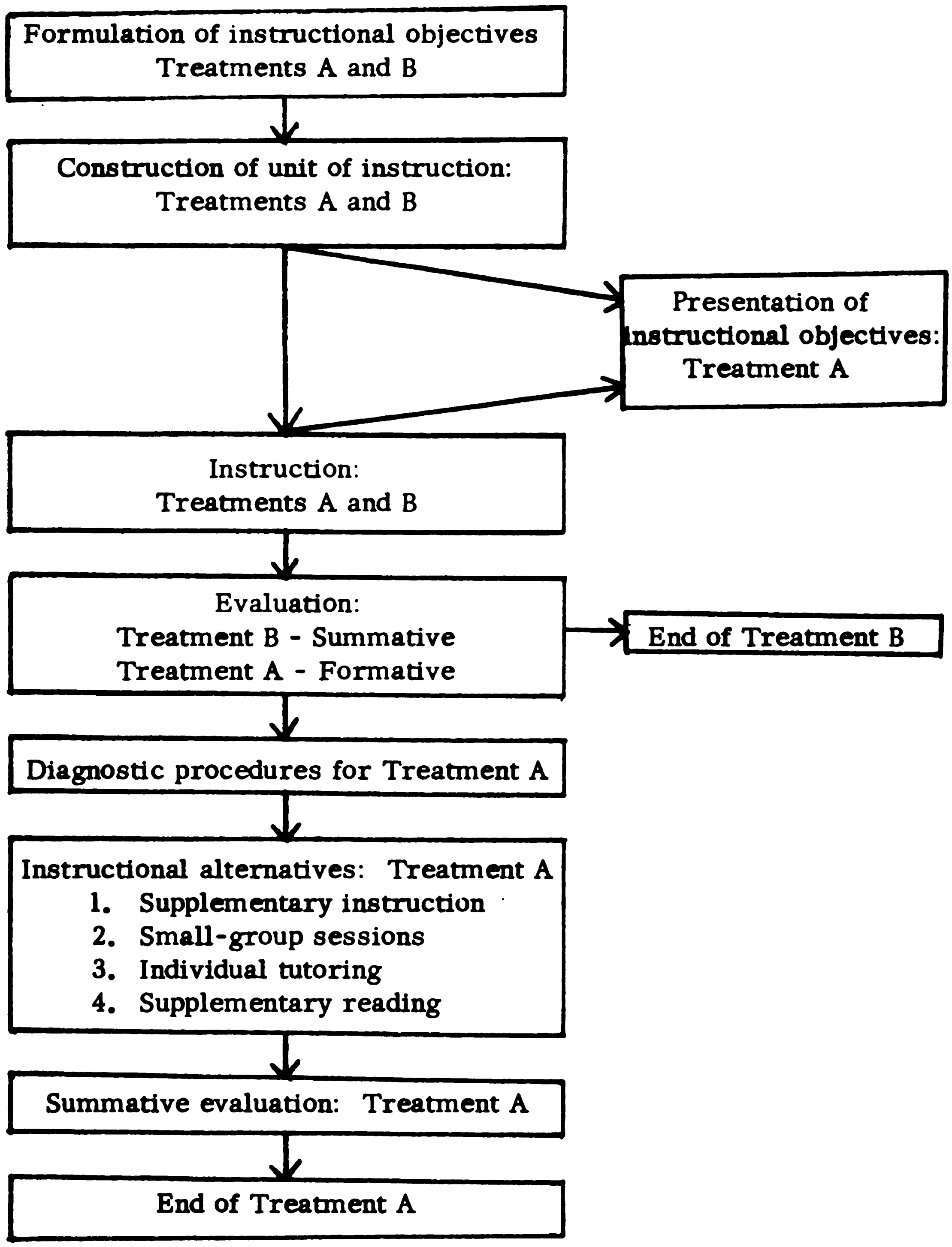

FIGURE 3.2

Instructional Strategies for Treatment Sections

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(8) Prove a given sequence has a particular limit using the definition for limit of a sequence.

(9) Prove a given function has a limit at a particular value of the independent variable using the definition for limit of a function.

Objectives six and seven are classified in the application level of Wilson's model. Satisfactory performance would require appropriate use of previously stated theorems to determine limits of sequences (functions). For example,

$$\lim_{x\to 0} \frac{3x^2+2}{x^2-1} = \frac{\lim_{x\to 0}(3x^2+2)}{\lim_{x\to 0}(x^2-1)} \quad \text{limit of quotient of functions}$$

$$= \frac{\lim_{x\to 0} 3x^2 + \lim_{x\to 0} 2}{\lim_{x\to 0} x^2 - \lim_{x\to 0}(-1)} \quad \text{limit of sums of functions}$$

These can now be simplified by previous work and other limit theorems to obtain

$$\lim_{x\to 0} \frac{3x^2+2}{x^2-1} = -2$$

Construction of Unit of Instruction: Treatments A and B

The unit of instruction for Treatments A and B was compiled by the author. Ideas, order of presentation, and problems were obtained from the following: (1) Limits and Continuity by W.K. Smith;[3] (2) Limits: A Transition to Calculus by O.L. Buchanan, Jr.;[4] (3) Sequences by K.E. O'Brien;[5]

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(4) Calculus for College Students by M.H. Protter and C.B. Morrey, Jr.;[6] and (5) The Advanced Calculus of One Variable by D.R. Lick.[7] A copy of the instructional unit is located in Appendix I.

Instruction: Treatments A and B

The experimental unit for Treatments A and B was started during the ninth week of a fifteen-week semester. It began by having the students consider several examples of sequences, some of which did not have limits. They then attempted to define when a sequence has a limit. These attempts were refined by the use of counterexamples. Eventually the definition was given as follows:

> Definition: A sequence $\{a_n\}$ has a limit L if for each $\epsilon > 0$, there is a positive integer N such that if $n > N$, then $|a_n - L| < \epsilon$.
>
> Notation: $\lim\limits_{n\to\infty} a_n = L$

To better understand the ϵ - N relationship, students were asked to find an N that would "work" for a particular value of ϵ and a particular sequence. The smallest value of N that satisfied the definition for a given ϵ was referred to as the "N associated with the given ϵ" and was denoted N_ϵ. After finding several N_ϵ's, the students examined a proof of the statement $\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$ that utilized the definition given above.

The students next considered more complicated sequences. For example, they were asked to evaluate $\lim\limits_{n\to\infty} \frac{3n^2 + n}{n^2}$. The difficulty of finding limits for these sequences made them realize the necessity for theorems that would allow them to evaluate such limits. Six theorems concerning limits of sequences were then stated, two of which were proven by using the definition. (See pages 95 and 96 of Appendix I).

Limits of functions were studied next. Examples were used to illustrate notation and the intuitive concept. Students were then asked to formulate a definition for the limit of a function at a particular value of the independent variable. Eventually the definition was given by the instructor as follows:

Definition: The function $f(x)$ has limit L as x approaches a if for each $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Notation: $\lim_{x \to a} f(x) = L$

After several examples and problems were studied, the need for theorems was again noted by considering limits of more complicated functions. Six theorems were then stated, three of which were proven. (See pages 105 to 107 of Appendix I) Students completed study of the experimental unit by evaluating the limits of functions at indicated values of the independent variable and attempting proofs of some results. A daily classroom record for Treatment A is available in Appendix II.

Quizzes: A study by Merrill, et. al.[8] has shown that specific review following incorrect responses makes students' learning increasingly efficient. This result was implemented in Treatment A by administering six instructional quizzes during the teaching of the unit. The specific review was then accomplished by working the problems immediately after the class had completed them and by providing written comments on the quiz papers that were returned to the students. These quizzes were not used as part of the mastery evaluation or grading procedure. They consisted of one problem each and the students were allowed at most ten minutes to work on them. If students were absent on the day a quiz was given, they were asked to work it outside of class and hand it in so appropriate comments could be made. These quizzes are listed in Appendix III.

Evaluation: Treatments A and B

The unit of instruction for Treatments A and B required nine instructional days. An evaluation was given on the tenth day. These were summative evaluations in Treatment B and formative evaluations in Treatment A. A seven-item test was used for the fifty-minute testing period. Parallel tests were randomly assigned to the treatment sections. Copies are available in Appendices IV and V.

Wilson's Model: Part of this study was to examine the mastery of items at specified levels of a learning model. James W. Wilson's model, given in Figure 3.3, was used and the items were tested at the Comprehension, Application, and Analysis levels. Figure 3.4 indicates the classification of items in the tests.

Grading the tests: The tests in Treatments A and B were graded by the

AFFECTIVE		COGNITIVE			
F.0	E.0	D.0	C.0	B.0	A.0

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COGNITIVE	A.0 Computation	A.1	Knowledge of specific facts
		A.2	Knowledge of terminology
		A.3	Ability to carry out algorithms
	B.0 Comprehension	B.1	Knowledge of concepts
		B.2	Knowledge of principles, rules, and generalizations
		B.3	Knowledge of mathematical structure
		B.4	Ability to transform problem elements from one mode to another
		B.5	Ability to follow a line of reasoning
		B.6	Ability to read and interpret a problem
	C.0 Application	C.1	Ability to solve routine problems
		C.2	Ability to make comparisons
		C.3	Ability to analyze data
		C.4	Ability to recognize patterns, isomorphisms, and symmetries
	D.0 Analysis	D.1	Ability to solve nonroutine problems
		D.2	Ability to discover relationships
		D.3	Ability to construct proofs
		D.4	Ability to criticize proofs
		D.5	Ability to formulate and validate generalizations
AFFECTIVE	E.0 Interests and Attitudes	E.1	Attitude
		E.2	Interest
		E.3	Motivation
		E.4	Anxiety
		E.5	Self-concept
	F.0 Appreciation	F.1	Extrinsic
		F.2	Intrinsic
		F.3	Operational

FIGURE 3.3

J.W. Wilson's Taxonomy

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LEVEL	TYPE OF ITEM
B.1	Given a sequence (function) that does not have a limit, indicate what part of the definition it fails to satisfy.
B.1	Give an example of a sequence (function) that does not have a limit and include explanation.
B.1	Given a real number, be able to produce a nonconstant sequence (function) whose limit is that real number and include explanation.
C.1	When working with sequences, find the associated N for a particular ϵ .
C.1	When working with functions, find the associated δ for a particular ϵ.
C.1	Determine limits of sequences using the results of theorems.
C.1	Determine limits of functions using the results of theorems.
D.3	Prove a given sequence has a particular limit using the definition for limit of a sequence.
D.3	Prove a given function has a limit at a particular value of the independent variable using the definition for limit of a function.

FIGURE 3.4

Classification of Test Items

author and Dave Rajala. One type of item on all four tests was scored before considering another type of item. Guidelines for assigning partial credit are given in Figure 3.5.

Type of Item (see page 28)	Partial Credit Assignment
1	Right idea, insufficient explanation - 9/11
2, 3	a. 6/12 for correct example but insufficient reasoning b. No credit if functions and sequences are interchanged
4, 5	5/15 for writing $\lvert A_n - L \rvert < \epsilon$
6, 7	a. Improper use of the quotient theorem: -3 b. Max. of 8/15 credit if all reasons are not stated
8, 9	8/20 for considering $\lvert f(x) - L \rvert < \epsilon$ or $\lvert A_n - L \rvert < \epsilon$

FIGURE 3.5

Guidelines for Assigning Partial Credit

Diagnostic Procedures for Treatment A

The tests given on the tenth day in Treatment A were used as formative evaluations. After they were graded, a diagnostic form was completed and returned to each student. This provided specific reference regarding the areas in which students needed additional work. A copy of this form is given in Figure 3.6.

According to the results of your test, you need more work in the following areas:

_____ a) Sequences that do not have limits

_____ b) Functions that do not have limits

_____ c) How to produce a sequence (function) with a given limit

_____ d) Finding an N_ϵ for a given ϵ

_____ e) Finding a δ_ϵ for a given ϵ

_____ f) Evaluating limits of sequences using results of theorems

_____ g) Evaluating limits of functions using results of theorems

_____ h) Proving a given sequence has a limit using the definition

_____ i) Proving a given function has a limit using the definition

Test results:

Problem	Score
1	_____
2	_____
3	_____
4	_____
5	_____
6	_____
7	_____

FIGURE 3.6

Diagnostic Forms for Treatment A

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Instructional Alternatives: Treatment A

Alternative learning strategies were made available to students in Treatment A after the formative evaluation was administered. The first of these was group lecture sessions covering material missed by the majority of students. These sessions were held on the first class day following the formative evaluations.

A second instructional alternative consisted of three two-hour small group sessions. For these sessions, the chairs in the classroom were separated into four areas as depicted in Figure 3.7.

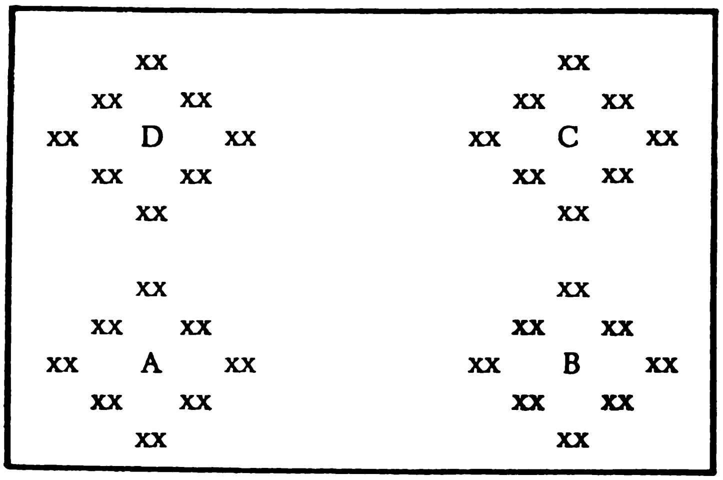

FIGURE 3.7

Room Arrangement for Small Group Sessions

Area A contained review material for items of type 1, 2, and 3 (see page 27); area B material for items of type 4 and 5; area C material for items of type 6 and 7; and area D material for items of type 8 and 9. Worksheets (Appendix

VII) were provided for each area and students could work in whichever area they chose. The author, Dave Rajala, and students who achieved at the mastery level on the formative evaluation were used as resource persons for these sessions. A record of attendance at all extra sessions in located in Appendix VIII.

Summative Evaluation

Students in Treatment A received a summative evaluation on the thirteenth class day following the introduction of the unit on limits. The scores on this evaluation were used as part of the course grade. A copy is contained in Appendix VI.

Delayed-Mastery Students

One aspect of this study was to identify "delayed-mastery" students in Treatment A. These were students who attained the mastery level (80%) on the second evaluation but who did not attain it on the first one. Figure 3.8 indicates the number so designated.

Analysis of Data

The data used to test the cognitive domain hypotheses (see page 6) was obtained from tests given in the Calculus I course at CMU during the second semester of the 1973-74 academic year. The first set of data came from the test given during the fifth week of the semester. Subsequent analyses exa

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Test	Section	Mastery	Non-mastery
1	8:00	2	24
1	9:00	3	32
2	8:00	17	9
2	9:00	22	13
Delayed mastery:		8:00	15
		9:00	19
	Total		34

FIGURE 3.8

Delayed Mastery Students

examined both the performance on items referring specifically to the study of limits and the performance on the total test. Copies of parallel forms of this test are available in Appendix IV. The second set of data was the semester averages for the course. A one-way analysis of variance using the six cells indicated in the experimental model of Figure 3.1 was utilized. The method of multiple comparisons was then used to identify differences.

Both qualitative and quantitative analyses were conducted in an attempt to determine the effect of the treatments on the affective domain. One qualitative analysis consisted of interviewing randomly selected (see Appendix X) students during the last two weeks of the Precalculus course. The interviews were conducted by the author and Professor Lakey. Figure 3.9 indicates the various subgr

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various subgroups, their respective sizes, and numbers of students selected for interviewing.

Group		Size	Number Selected For Interview	Number Interviewed
Nance Mastery	Early	5	2	2
	Delayed	34	5	5
Nance Non-mastery		21	5	5
Smith Mastery		21	5	3
Smith Non-mastery		40	5	5
Control		152	15	13

FIGURE 3.9

Number of Students Selected for Interview by Group

The fifteen selected from the control group were chosen by selecting three from each of the five sections. A copy of the questions asked during the interviews is available in Appendix XI .

A second qualitative analysis was conducted using the results of a questionnaire completed by students in Calculus I at the conclusion of studying the section on limits of functions. The responses to this questionnaire were analyzed using a chi square test statistic with several groupings of the cells in the experimental model. A copy of the questionnaire is available in

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Appendix XII.

The quantitative analyses for the affective domain consisted of examining the percentage of Fall Semester Precalculus students that enrolled in Calculus I the following semester and the percentage that completed Calculus I. A chi square test statistic was used with four cells and compared the number of students that enrolled in Calculus I to the number that did not enroll in Calculus I. The cells consisted of (1) Treatment A, (2) Treatment B, (3) control group, (4) Precalculus students from the Fall Semester of the Academic Year 1972-73, and (5) Precalculus students from the Fall Semester of the Academic Year 1971-72. A chi square test statistic was also used to analyze enrollment in Precalculus for the cells of the experimental model.

CHAPTER III -- FOOTNOTES

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CHAPTER IV

FINDINGS

This chapter contains findings of the study. These are reported according to the following sections: (1) post hoc test for homogeneity of variance and equality of mean, (2) mastery learning, (3) enrollment summaries, (4) affective domain elements, and (5) cognitive domain elements.

Post Hoc Test

A.C.T. scores for students in the nine sections of Precalculus involved in this study were checked for homogeneity of variance and equality of means to see if there were initial differences between sections. The Bartlett[1] test for homogeneity of variance fell within the acceptable range at the $\alpha = .05$ level. Subsequent analysis of variance for equality of means produced an F ratio that implied the acceptance of the hypothesis of equality of means between sections. Data used to produce these statistics are contained in Table 4.1.

Mastery Learning

Treatment A consisted of using a mastery learning approach to teach a section on limits to two of the four treatment sections. An outline is given on page 29 of Chapter III. The effectiveness of mastery learning was determined by an analysis of the formative and summative evaluation results for

Section	Sample Size	Mean	Standard Deviation
Treatment A	12	25.5	1.98
Treatment A	14	25.5	2.88
Treatment B	13	25.1	4.86
Treatment B	15	27.4	3.29
Control	12	24.1	3.09
Control	18	25.6	3.68
Control	11	26.5	4.70
Control	12	27.4	2.97
Control	12	26.9	2.50

Bartlett test statistic = 13.9 with 8 degrees of freedom
Rejection value: $\chi^2_{.05}(8) = 15.5$

Analysis of Variance Table

Source	Sum of Squares	DF	Mean Square	F Ratio
Between means	131.0261	8	16.3777	1.3784
Within	1306.9448	110	11.8813	
Total	1437.9664	118		

Rejection value: $F_{.05}(8,100) = 2.97$

TABLE 4.1

Analysis of A.C.T. Scores: Post Hoc Test

the four treatment sections of Treatment A and Treatment B. Treatment A formative evaluation item responses are given in Tables 4.2 and 4.3. Analysis of these responses determined the material retaught during the group lecture part of the alternative instructional methods. Both sections in this treatment group produced poor results on items of type two, four, and seven. Consequently, these items were discussed in class the following

Item* Type	1	2	3	4	6	7	9
Score	11 pts	12 pts	12 pts	15 pts	15 pts	15 pts	20 pts
0		12	8	13	5	17	3
1							
2				2			1
3						4	1
4	4						
5				1	1		
6		5	6	1			
7	1						
8					2		1
9	1						
10		1	2	1	1	1	1
11	18				2		
12		7	9	2	7	1	
13				4	1	1	
14							2
15				1	6	1	1
16							1
17							4
18							1
19							6
20							3
Average performance on item	87%	38%	55%	31%	64%	17%	66%

*This refers to the instructional objectives on page 28 of Chapter III.

TABLE 4.2

Formative Evaluation Item Results

Treatment A -- Test Form C

Item Type
Score
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Average performance on item

Item Type	1	2	3	4	6	7	9
Score	11 pts	12 pts	12 pts	15 pts	15 pts	15 pts	20 pts
0		22	7	14	11	25	3
1							
2				1		1	2
3			1		1	4	
4	1			1			
5		1		3	1	1	1
6	1	3	8	2			
7				1			
8	4				1		1
9	3	2	1				
10	4		1		1		
11	22						
12		7	17	1	7		2
13				5	6	1	
14				6	3		
15				1	4	3	3
16							4
17							1
18							1
19							9
20							8
Average performance on item	91%	30%	65%	41%	55%	15%	74%

TABLE 4.3

Formative Evaluation Item Results

Treatment A -- Test Form D

day. Table 4.4 contains scores obtained by students on the formative evaluation in Treatment A.

Early Mastery Scores	Nonmastery Scores										
87	78	72	66	62	59	55	50	43	38	27	24
83	77	71	64	60	58	54	49	41	37	27	23
82	77	71	64	60	57	54	48	40	32	26	11
81	75	71	64	60	57	51	46	40	31	26	11
80	73	70	63	60	56	50	45	39	28	25	4
	Mean = 52.2										
	Standard Deviation = 20.1361										

TABLE 4.4

Formative Evaluation Scores: Treatment A

These scores were recorded for the purpose of identifying early-mastery students.

The formative evaluation for students in Treatment A was followed by alternative learning procedures as indicated in Chapter III. A summative evaluation was administered at the conclusion of these procedures. Item responses and scores for this evaluation are contained in Tables 4.5 and 4.6. Thirty-nine students attained the mastery level, five of whom had been identified as early mastery by the formative evaluation.

Item Type	1	2	3	5	6	7	8
Score	11 pts	12 pts	12 pts	15 pts	15 pts	15 pts	20 pts
0		4	11	1	1	4	1
1							
2							
3	6	1		1		2	
4	1						
5					1	6	1
6	4	3	2				
7	1						
8	6			3	2	9	
9	13	1	3				
10	1		3		1	10	10
11	27	1	2		1		
12		49	38	2	8	7	2
13				13	7	2	
14					1	1	
15				39	37	18	11
16							
17							
18							6
19							2
20							26
Average performance on item	81%	89%	77%	91%	90%	68%	81%

TABLE 4.5

Summative Evaluation Item Results

Treatment A

Mastery Scores								Nonmastery Scores			
99	98	94	92	88	87	83	81	79	75	70	65
98	97	93	90	88	86	83	81	77	75	70	65
98	96	93	89	88	86	83	80	76	74	68	61
98	95	92	89	88	86	82	80	76	73	65	55
98	95	92	88	88	84	82		76	72	65	44

Mean = 82.5
Standard Deviation = 11.8937

TABLE 4.6

Summative Evaluation Scores: Treatment A

The summative evaluation for students in Treatment B was given the same day the formative evaluation was given in Treatment A. Item responses and scores on this evaluation are given in Tables 4.7, 4.8 and 4.9.

Summary: Thirty nine of the sixty students (65%) in Treatment A attained the mastery level while twenty one of sixty-one students (34%) in Treatment B attained that level. The average score on the summative evaluation in Treatment A was 82.5 compared to an average score of 63.9 in Treatment B.

Item Type	1	2	3	4	6	7	9
Score	11 pts	12 pts	12 pts	15 pts	15 pts	15 pts	20 pts
0	2	5	7	8	9	23	3
1							
2							
3				3			4
4							
5				1	1		2
6	4	4	1				
7					1		
8							3
9	4	2	1	1			
10		2	1	2	1		
11	22				1		
12		19	22	3	8	4	
13				2			
14				1			
15				11	11	5	
16							
17							
18							
19							
20							20
Average performance on item	86%	76%	75%	59%	58%	23%	69%

TABLE 4.7

Summative Evaluation Item Results

Treatment B -- Test Form A

Item Type	1	2	3	4	6	7	9
Score	11 pts	12 pts	12 pts	15 pts	15 pts	15 pts	20 pts
0	1	5	9	7	5	18	2
1							
2							
3	2			2			
4	1						
5	2			3		1	
6	1	2	1				
7							
8	6				1		
9	1		1	1	1		
10		2		1	4	1	
11	14	1			1	1	
12		18	17		8		
13				3			
14							
15				11	8	6	
16							
17							
18							
19							
20							26
Average performance on item	77%	78%	65%	58%	68%	30%	93%

TABLE 4.8

Summative Evaluation Item Results

Treatment B -- Test Form B

Mastery Scores				Nonmastery Scores							
100	98	85	84	79	70	67	60	54	45	40	34
100	94	85	83	78	70	67	59	48	44	40	28
100	92	85	80	77	70	65	55	47	44	39	27
100	88	85	80	76	69	64	55	47	43	38	22
100	85	85	80	74	68	61	54	46	40	38	19
			80								
Mean = 65.4											
Standard Deviation = 22.1505											

TABLE 4.9

Summative Evaluation Scores: Treatment B

The analyses in the remainder of this report involved grouping cells of the experimental model. Groupings used include the following:

(1) Treatment -- Nance Early Mastery (NEM), Nance Delayed Mastery (NDM), Nance Nonmastery (NNM), Smith Mastery (SM), Smith Nonmastery (SNM)

(2) Mastery -- NEM, NDM, and SM

(3) Nonmastery -- NNM and SNM

(4) Treatment A -- NEM, NDM, and NNM

(5) Treatment B -- SM and SNM

(6) Delayed mastery -- NDM

(7) Other mastery -- NEM and SM

Test statistics and other results were then analyzed for the following comparisons:

(1) Treatment versus Control

(2) Mastery versus Nonmastery versus Control

(3) Treatment A versus Treatment B versus Control

(4) Delayed mastery versus Other mastery versus Nonmastery versus Control

(5) All six cells of the experimental model

Enrollment

Enrollment figures were examined to see if the treatment had an effect on the number of students taking Calculus I. Analysis of these figures was accomplished by considering Precalculus and Calculus I enrollments for three years. All fall semester Precalculus and winter semester Calculus I sections were used for the Academic Years 1971-72 and 1972-73. Enrollment figures obtained from these years were compared with figures obtained from the nine Precalculus sections involved in the experiment. Table 4.10 contains this information.

Data from Table 4.10 were analyzed by using the chi square test statistic and a test for proportion. Six groupings were considered. The chi square statistic was used to test the null hypothesis of independence between classifications at the $\alpha = .05$ level of significance for five of these six groupings. The other grouping, Treatment versus Control, was tested with

Semester	Group	Number of students completing Precalculus	Number of students enrolling in Calculus I	Percent enrolled in Calculus I
Fall 1971	All sections	438	173	39
Fall 1972	All sections	363	169	47
Fall 1973	NEM	5	5	100
Fall 1973	NDM	34	13	38
Fall 1973	NNM	21	8	38
Fall 1973	SM	21	13	62
Fall 1973	SNM	40	15	38
Fall 1973	Control	152	85	56
Fall 1973	Treatment A	60	26	43
Fall 1973	Treatment B	61	28	46
Fall 1973	Mastery	60	31	52
Fall 1973	Nonmastery	62	23	38
Fall 1973	Other mastery	26	18	69
Fall 1973	Treatment	121	54	45

TABLE 4.10

Enrollment Summaries

a Z test statistic for equality of proportion with the alternative hypothesis of proportion of Treatment students enrolled in Calculus I is not equal to the proportion of Control students enrolled in Calculus I. A summary of these tests and conclusions is contained in Table 4.11. Three of the groupings produced chi square values that implied the rejection of the hypothesis of independence between grouping and enrollment. Cell components of the value of the test statistic that produced the rejections are located in Appendix XV.

Grouping	Test statistic value	Rejection value	Conclusion
1971, 1972, 1973 Control, 1973 Treatment	$\chi^2 = 12.6$	$\chi^2_{.05}(3) = 7.8$	Reject the null hypothesis
NEM, NDM, NNM, SM, SNM, C	$\chi^2 = 14.1$	$\chi^2_{.05}(5) = 11.1$	Reject the null hypothesis
Treatment A, Treatment B Control	$\chi^2 = 3.3$	$\chi^2_{.05}(2) = 5.99$	Do not reject the null hypothesis
Mastery, Nonmastery, Control	$\chi^2 = 5.4$	$\chi^2_{.05}(2) = 5.99$	Do not reject the null hypothesis
Delayed mastery, Other mastery, Nonmastery, Control	$\chi^2 = 10.7$	$\chi^2_{.05}(3) = 7.8$	Reject the null hypothesis
Treatment, Control	$Z = 1.86$	$Z \quad 1.96$	Do not reject $H_0: P_1 = P_2$

TABLE 4.11

Enrollment Analysis Results

Summary:

Control group

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test was used.

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Withdrawal

study withdrew

these were in t

Summary: The statistics of Appendix XV indicate that the Fall 1973 Control group had significantly more students enroll in Calculus I than did students from the previous two years. Consequently, the treatment effect on enrollment was determined by considering groupings within the year 1973. This analysis indicated that more Other mastery students than expected enrolled in Calculus I while enrollment was less than expected for Delayed mastery and Nonmastery students.

The Z test statistic was used to test equality of enrollment proportion between the Treatment group and the Control group. The Z value of 1.86 was not sufficient to reject the hypothesis of equality of proportion for a two-sided test at the $\alpha = .05$ level. For the produced Z value, it would require an α level of .0628 to reject the null hypothesis. The two-tailed test was used in this case because the original hypothesis was that the treatment effect would increase enrollment. Student comments during the presentation of the instructional unit in the Treatment group made it apparent, however, that enrollment might decrease as a result of the treatment. Therefore, rather than reverse the inequality of the hypothesis, a two-tailed test was used. A statistically significant difference would have been produced if a one-tailed test had been used with the proportions interchanged from the original conjecture.

Withdrawals from Calculus I: Nine of the 139 students involved in this study withdrew from Calculus I before the end of the semester. Two of these were in the Nance nonmastery group, one in the Smith nonmastery group, and s

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group, and six in the Control group. These results indicate that the treatment did not have an effect of the number of withdrawals from Calculus I.

Affective Domain

Interviews: Part of the treatment effect on the affective domain was determined by analyzing responses obtained from interviewing students in the Precalculus sections. A discussion of the selection of these students and the interview procedures is given in Chapter III. The interviews required approximately ten minutes apiece to conduct and were primarily used to investigate two elements in the affective domain: (1) the existence or lack of anxieties regarding the first semester calculus course, and (2) attitude toward the first semester calculus course. Results of the interviews are given in Table 4.12 and a summary of these responses are contained in Figure 4.1.

The interviewers responded to the following two items at the completion of each interview:

(1) Briefly discuss your conception of student anxieties or lack of anxieties regarding Calculus I.

(2) What is your opinion of the student's attitude toward Calculus I?

Summaries of responses to these items are contained in Figure 4.2.

Item	6			7			8				9			10-11-12	13				14		
Grouping	AA	A	BA	P	NU	N	MH	H	S	E	P	NU	N		MH	H	E	S	AA	A	BA
NEM	2	0	0	0	2	0	0	0	0	2	2	0	0	0	0	2	0	0	2	0	0
NDM	3	0	2	2	3	0	0	3	1	1	4	0	1	1	2	0	2	0	4	0	0
NNM	0	2	3	0	1	4	0	2	2	1	1	2	2	1	2	1	0	0	1	2	0
SM	3	0	0	2	1	0	0	0	2	1	2	1	0	0	0	2	1	0	3	0	0
SNM	2	2	1	3	1	1	0	2	1	2	1	3	1	2	2	2	1	0	2	2	0
Control	7	5	1	7	3	3	0	8	2	3	not applicable			0	0	1	6	3	8	2	0
Treatment A	5	2	5	2	6	4	0	5	3	4	7	2	3	2	4	3	2	0	7	2	0
Treatment B	5	2	1	5	2	1	0	2	3	3	3	4	1	2	2	4	2	0	5	2	0
Treatment	10	4	6	7	8	5	0	7	6	7	10	6	4	4	6	7	4	0	12	4	0
Mastery	8	0	2	4	6	0	0	3	3	4	8	1	1	1	2	4	3	0	9	0	0
Nonmastery	2	4	4	3	2	5	0	4	3	3	2	5	3	3	4	3	1	0	3	4	0
Other mastery	5	0	0	2	3	0	0	0	2	3	4	1	0	0	0	4	1	0	5	0	0

TABLE 4.12

Interview Responses

Item
Six
Seven
Eight
Nine
Ten, Ele and Twel
Thirteen
Fourte

Item	Summary of Responses
Six	One of thirteen Control group students thought he was doing below average while six of twenty Treatment group students thought they were doing below average.
Seven	Ten of twelve students in Treatment A indicated a lack of positive attitudes regarding the Precalculus course, while twelve of twenty-one students in Treatment B and the Control group had positive attitudes.
Eight	Eight of thirteen students in the Control group thought Precalculus was harder than they expected compared to seven of twenty in the Treatment group.
Nine	Eight of ten students who achieved mastery in either treatment group exhibited positive attitudes towards the unit on limits, while only two out of ten students in the nonmastery group favored the unit.
Ten, Eleven, and Twelve	Four students in the treatment sections changed their minds about taking Calculus I. Of these four changes, one decided not to take calculus because of a curriculum change, one decided to take calculus because of an advisor's recommendation, and two decided against calculus because of their unsatisfactory performance in Precalculus. None of the thirteen interviewed in the Control group changed their minds regarding calculus.
Thirteen	Thirteen out of seventeen students from the treatment sections thought Calculus I would be harder than other math courses they had taken, while only one out of ten in the Control group had the same reaction.
Fourteen	All Delayed-mastery students interviewed from Treatment A expect to achieve at an above average (A or B) level in calculus.

FIGURE 4.1

Summary of Interview Responses

Group
NEM
NDM
NNM
SM
SNM
Contro

Qu

analyz

studen

Group	
NEM	These students thought Calculus I would be more difficult than Precalculus, but they were not overly concerned. They recognized the sequence as the natural order and had healthy, positive attitudes. Students interviewed in this category expected to receive A's or B's in calculus and did not exhibit anxiety.
NDM	These students expressed appreciation for the mastery learning approach. Positive attitudes and lack of anxieties concerning Calculus I characterized this group. They expected calculus to be more difficult than previous mathematics courses, but they thought they could do at least B level work.
NNM	The students interviewed thought Calculus I would be very difficult. Two of the five said they were "afraid" of calculus. There were negative attitudes toward Calculus I and the section on limits in Precalculus.
SM	These students had attitudes and reactions similar to those of the Nance early-mastery group. Their attitudes were positive and they did not exhibit anxiety toward Calculus I.
SNM	These students exhibited a negative attitude concerning Calculus I. They think it will be hard and are not looking forward to it. They did not object to the section on limits, however, because they thought it would help them.
Control	The students interviewed in this group had positive attitudes and no anxieties about taking Calculus I. Nine of the ten responses indicated that it would not be a very difficult course.

FIGURE 4.2

Analysis of Anxieties and Attitudes Toward Calculus I

<u>Questionnaire</u>: Treatment effect on the affective domain was partially analyzed by examining the results of a questionnaire completed by Calculus I students during the Winter Semester 1974 (Appendix XII). All but eight of the

one-hundred thirty-nine students involved responded to the questionnaire. Table 4.13 contains tables formed by considering the responses of each cell in the experimental model to the five questions. The cell sizes of these tables were insufficient to produce statistically valid results when six groups were crossed with five responses per question. Consequently, responses and cells of the experimental model were grouped to produce larger cell sizes. The most refined grouping which produced this was a three-level response scale rather than a five-level scale. This was accomplished by combining responses "a" and "b" and responses "d" and "e". The groupings of the cells of the experimental model were those indicated on page 53. Table 4.14 contains results produced by these groupings. A chi square test was used to analyze the results and significance was tested at the $\alpha = .05$ level. The chi square values on twelve of the twenty questions reported in Table 4.14 indicate a rejection of the hypothesis of independence between groups and responses. An analysis of the values contributing to these rejection figures is reported in Appendix XIV. A summary of these results is contained in Figure 4.3.

Cognitive Domain

Test on limits: A test covering limits was administered to the Calculus I students during the fifth week of the Winter Semester 1974. Five forms (A, B, C, D, E) of the test were used. Items 2, 3, and 4 were parallel on all forms while Item 1 was parallel on forms A, B, and C. The test forms

Q 1

Group	a	b	c	d	e
NEM	0	0	2	3	0
NDM	3	3	6	1	0
NNM	0	3	3	0	0
SM	0	2	6	5	0
SNM	1	6	7	1	0
Control	5	11	34	25	4

Q 2

Group	a	b	c	d	e
NEM	0	0	3	2	0
NDM	0	1	10	1	1
NNM	1	0	4	1	0
SM	0	2	5	3	3
SNM	0	3	10	2	0
Control	6	29	33	10	1

Q 3

Group	a	b	c	d	e
NEM	1	1	2	1	0
NDM	0	11	2	0	0
NNM	1	1	3	1	0
SM	2	5	3	3	0
SNM	3	8	4	0	0
Control	10	51	13	4	1

Q 4

Group	a	b	c	d	e
NEM	3	2	0	0	0
NDM	8	4	0	1	0
NNM	2	4	0	0	0
SM	6	7	0	0	0
SNM	8	7	0	0	0
Control	30	37	11	0	1

Q 5

Group	a	b	c	d	e
NEM	0	1	1	1	2
NDM	0	4	6	2	1
NNM	1	0	3	2	0
SM	0	5	1	3	4
SNM	0	8	3	3	1
Control	2	45	21	5	6

TABLE 4.13

Questionnaire Responses

Grouping	Degrees of Freedom	Chi Square Critical Value	Chi Square Value by Question	
Treatment A			Q 1 -- 6.02	
			Q 2 -- 15.44	X
Treatment B	4	9.4	Q 3 -- 4.30	
			Q 4 -- 7.44	
Control			Q 5 -- 15.92	X
			Q 1 -- 5.78	
Treatment	2	5.99	Q 2 -- 13.87	X
			Q 3 -- 3.25	
Control			Q 4 -- 6.97	X
			Q 5 -- 10.28	X
Mastery			Q 1 -- 10.57	X
			Q 2 -- 16.61*	X
Nonmastery	4	9.4	Q 3 -- 4.43	
			Q 4 -- 8.81*	
Control			Q 5 -- 11.58	X
Delayed mastery				
			Q 1 -- 17.57*	X
Other mastery	6	12.59	Q 2 -- 21.72*	X
			Q 3 -- 12.62*	X
Nonmastery			Q 4 -- 8.82*	
			Q 5 -- 18.53	X
Control				

*Fewer than 80% of the cells had five or more responses.
X -- Reject the null hypothesis of independence.

TABLE 4.14

Questionnaire Data

Grouping	Question	Results
Delayed mastery Other mastery Nonmastery Control	1	Compared to Other mastery students, Delayed mastery students found Precalculus harder than expected.
	2	Delayed mastery students found calculus to be about as difficult as expected while Other mastery students found calculus easier than expected.
	3	Delayed mastery students found the study of limits more difficult than did the Other mastery students.
	5	Delayed mastery students' responses on this item were similar to Nonmastery students rather than Other mastery.
Treatment A Treatment B Control	2	Significantly fewer Treatment A students thought Calculus I was harder than expected.
	5	Treatment A students had fewer responses in the "frustrating" category while Treatment B students had fewer responses in the "less frustrating" category.
Treatment Control	2	Treatment group students had significantly fewer responses in the "harder" category while Control students had significantly more in the same category.
	4	Significantly fewer Treatment students studied sixteen or more hours than did those from Control.
	5	The treatment effect significantly reduced frustration when studying limits.
Mastery Nonmastery Control	1	Nonmastery students reported Precalculus was harder than expected.
	2	Mastery students found calculus easier than did the Nonmastery students.
	5	Mastery students found the study of limits less frustrating than did the Nonmastery students.

FIGURE 4.3

Summary of Questionnaire Results

differed only in that A, B, and C had a proof about limit of a function, whereas forms D and E had a proof about derivative of a function. Other items on the tests did not involve limits (see Appendix IX). Item and test scores are given in Appendix XVI.

Scores on Item 1 (twenty points possible) of forms A, B, and C were such that analysis of variance was not possible on the group means because the Bartlett test statistic for this item had a value of 28.1 while the critical value at the $\alpha = .05$ level of significance was 9.5. Responses were subsequently categorized into two levels: (1) those with scores less than eighteen, and (2) those with scores of eighteen, nineteen, and twenty. The number of responses in these categories by groupings of cells of the experimental model are given in Table 4.15.

Grouping	Score		Mean
	0-17	18-20	
NEM	0	1	20.0
NDM	1	8	18.8
NNM	2	3	15.0
SM	0	6	19.7
SNM	0	6	19.7
Control	5	38	18.9
Treatment A	3	12	17.6
Treatment B	0	12	19.7
Mastery	1	15	19.2
Nonmastery	2	9	17.6
Other mastery	0	7	19.7
Treatment	3	24	18.5

TABLE 4.15

Calculus I Test Item 1 -- Forms A, B, and C

Scores on Item 2 for all five forms of the calculus test also produced group means that did not permit analysis of variance. The Bartlett test statistic for this item was 14.2 while the critical value was 11.07. As before, the responses were categorized and reported by various groupings. This information is contained in Table 4.16.

Grouping	Score		Mean
	0-17	18-20	
NEM	1	4	19.0
NDM	3	10	18.2
NNM	5	2	15.6
SM	2	11	19.3
SNM	4	10	18.0
Control	28	52	17.2
Treatment A	9	16	17.6
Treatment B	6	21	18.6
Mastery	6	26	18.8
Nonmastery	9	12	17.2
Other mastery	3	15	19.2
Treatment	15	37	18.5

TABLE 4.16

Calculus I Test Item 2 -- Forms A, B, C, D, E

Each of the five test forms had three parallel items involving limits. Question two involved finding the limit of a quotient of functions with reasons given for each step. Questions three and four required students to produce examples of functions which met certain conditions and to include reasoning. The total possible points for these three items was thirty three. Analysis of these items was achieved by dividing the total scores into two categories and using a chi square test statistic. The categories were (1) scores from zero through twenty nine, and (2) scores from thirty through thirty three. Table 4.17 contains the responses by category and Table 4.18 contains results obtained from analyzing these scores.

Grouping	Score		Mean
	0-29	30-33	
NEM	2	3	30.0
NDM	5	8	28.5
NNM	6	1	24.9
SM	3	10	31.0
SNM	6	8	28.6
Control	48	32	27.2
Treatment A	13	12	27.8
Treatment B	9	18	29.8
Mastery	10	21	29.8
Nonmastery	12	9	27.4
Other mastery	5	13	30.7
Treatment	22	30	28.8

TABLE 4.17

Responses by Cells to Limit Items 2, 3, and 4 of the Calculus I Test

Grouping	Degrees of freedom	Chi square test statistic	Chi square critical value $\alpha = .05$
NEM, NDM, NNM, SM, SNM, C	5	11.30	11.07
Treatment-Control	1	3.96	3.84
Mastery-Nonmastery-Control	2	7.0	5.99
Delayed mastery-Other mastery-Nonmastery-Control	3	7.3	7.8
Treatment A-Treatment B-Control	2	7.2	5.99

TABLE 4.18

Analysis of Student Performance on Limit Items in Calculus I Test

Chi square contingency tables for grouping of cells of the experimental model are contained in Appendix XVII.

The data of Table 4.18 indicate that the Treatment group performed significantly better than the control group on the limit items. Further examination of this data and the material in Appendix XVII indicate that most of the achievement difference was due to students in Treatment B. Treatment A students performed as expected while fewer Treatment B students than expected scored low (below thirty) and more than expected scored high (thirty and above).

The last analysis made from data obtained from the Calculus I test was an examination of total scores. Means for groupings of cells of the experimental model are given in Table 4.19.

Grouping	Size	Mean	Standard Deviation
NEM	5	91.4	5.9
NDM	13	83.4	14.1
NNM	7	67.4	13.1
SM	13	91.8	6.3
SNM	14	81.4	14.2
Control	80	83.0	14.5
Treatment A	25	80.5	14.3
Treatment B	27	86.4	11.9
Mastery	31	88.2	10.6
Nonmastery	21	76.7	14.7
Other mastery	18	91.7	5.9
Treatment	52	83.6	13.7

TABLE 4.19

Group Means and Standard Deviations for Calculus I Test Scores

The Bartlett test statistic for homogeneity of variance of this data is 12.7 while $\chi^2_{.05}(5) = 11.07$. However, the F ratio for an analysis of variance of these means is 3.33 and $F_{.05}(5,120) = 2.29$. Consequently, one can be reasonably sure that the analysis of variance result is statistically significant. The analysis of variance table in Table 4.20 indicates a rejection of the hypothesis of equality of means.

Source	Sum of Squares	Degrees of Freedom	Mean Square	F Ratio
Between means	3081.8	5	616.4	3.3
Within	23324.4	126	185.1	
Total	26406.2	131		

TABLE 4.20

One-Way Analysis of Variance for Calculus I Test Scores

Due to unequal sample sizes, a Scheffe[2] multiple comparison test was used to identify differences between means. Table 4.21 contains the results of these comparisons. The null hypothesis of equality of means was rejected only for the comparisons of Nonmastery versus Mastery and Nonmastery versus Other mastery. All other comparisons, in particular Treatment versus Control, do not reject the hypothesis of equality of means.

Semester average in Calculus I: The second analysis of the treatment effect on the cognitive domain was conducted by examining students' semester averages in Calculus I (Appendix XVI). The average for each student was obtained by equally weighted percentage scores from the following three

Comparison	Confidence Interval	Conclusion*
Treatment A - Treatment B	-19.2 — 7.5	Do not reject
Treatment A - Control	-13.5 — 8.9	Do not reject
Treatment B - Control	- 6.7 — 13.8	Do not reject
Mastery - Nonmastery	.4 — 28.5	Reject
Mastery - Control	- 4.7 — 16.3	Do not reject
Nonmastery - Control	-20.5 — 3.2	Do not reject
Delayed mastery - Other mastery	-25.8 — 9.4	Do not reject
Delayed mastery - Nonmastery	- 7.6 — 25.6	Do not reject
Delayed mastery - Control	-13.4 — 14.1	Do not reject
Other mastery - Nonmastery	1.0 — 33.3	Reject
Other mastery - Control	- 4.6 — 21.7	Do not reject
Treatment - Control	- 8.6 — 8.6	Do not reject

*Rejection of the null hypothesis of equality of means occurs when the confidence interval does not include zero.

TABLE 4.21

Scheffe Multiple Comparison Test for Equality of Means
Calculus I Test

sources: (1) ten quizzes, (2) three one-hour examinations, and (3) a final examination. Averages for groupings of cells of the experimental model are given in Table 4.22. Means were analyzed by a one-way analysis of variance. The Bartlett test statistic was 11.15 while the rejection value was $\chi^2_{.05}(5) = 11.07$. However, the F ratio for analysis of variance was 5.57 compared to $F_{.05}(5,120) = 2.29$. As before, this allows us to reject the null hypothesis of equality of means between cells. The analysis of variance table is given in Table 4.23. Results of the Scheffe multiple comparison test are contained in Table 4.24. The only comparisons that produced a statistically significant difference of means were those that involved the

Grouping	Size	Mean	Standard Deviation
NEM	5	85.0	11.0
NDM	13	71.9	17.3
NNM	6	58.0	17.4
SM	13	80.0	12.4
SNM	14	74.1	9.6
Control	79	79.9	9.9
Treatment A	24	71.2	17.8
Treatment B	27	76.9	11.0
Mastery	31	77.4	14.7
Nonmastery	20	69.3	13.8
Other mastery	18	81.4	11.6
Treatment	51	74.0	14.9

TABLE 4.22

Semester Averages by Groupings of Cells of the Experimental Model

Source	Sum of Squares	Degrees of Freedom	Mean Square	F Ratio
Between means	3652	5	730.4	5.57
Within	16268	124	131.2	
Total	19920	129		

TABLE 4.23

Analysis of Variance Table for Semester Averages by Cell

Comparison	Confidence Interval		Conclusion*
Treatment A - Treatment B	-16.8	6.0	Do not reject
Treatment A - Control	-17.9	1.4	Do not reject
Treatment B - Control	-11.5	5.8	Do not reject
Mastery - Nonmastery	.7	25.1	Reject
Mastery - Control	-9.8	7.9	Do not reject
Nonmastery - Control	-24.2	3.4	Reject
Delayed mastery - Other mastery	-25.4	4.2	Do not reject
Delayed mastery - Nonmastery	-8.5	20.2	Do not reject
Delayed mastery - Control	-19.6	3.6	Do not reject
Other mastery - Nonmastery	2.5	30.3	Reject
Other mastery - Control	-8.5	13.7	Do not reject
Treatment - Control	-13.5	1.3	Do not reject

*Rejection of the null hypothesis of equality of means occurs when the confidence interval does not include zero.

TABLE 4.24

Scheffe Multiple Comparison Test for Equality of Means
Total Percentage Grade for Calculus I

Nonmastery students.

Summary: Two implications arise from the data of Table 4.22 and 4.24. The first is that the average score for Delayed mastery students (71.9) is much closer to that for Nonmastery students (69.3) than it is to that for Other mastery students (81.4). Although the difference between Delayed mastery and Other mastery is not statistically significant, it is an indication that Delayed mastery students tend to perform more like Nonmastery students than Other mastery.

The second implication is that the Treatment students scored 5.9 percentage points lower than the Control students. This is not statistically significant, but it suggests further investigation into the feasibility of taking time from the Precalculus course to introduce special topics.

FOOTNOTES -- CHAPTER IV

[1] Dixon, Wilfrid J. and Frank J. Massey, Jr. Introduction to Statistical Analysis. McGraw Hill, New York, 1969, p. 308.

[2] Ibid., p. 167.

CHAPTER V

CONCLUSIONS AND IMPLICATIONS

Included in this chapter are (1) a summary of the procedures of the study, (2) findings of the study, (3) conclusions based upon analysis of the data, (4) a discussion of selected components of the study, and (5) implications for further research.

Summary of Procedures

This study investigated the effects of teaching a unit on limits to precalculus students at Central Michigan University. Nine sections of approximately thirty students each constituted the population. Two sections received Treatment A, two received Treatment B, and five were used for a Control group. Treatment A consisted of using a mastery learning strategy to teach the unit; Treatment B consisted of presenting the same unit in a regular* classroom situation; and sections in the Control group received no instruction regarding limits.

The mastery learning strategy used in Treatment A consisted of (1) formulating instructional objectives, (2) presenting the unit of instruction, (3) administering a formative evaluation, (4) providing each student with diagnostic results, (5) using alternative instructional procedures, and

*"Regular" means no special strategy or technique was used. The instructor taught the way he normally would present any other material. This consisted mainly of lecturing and answering questions.

(6) administering a summative evaluation. Students who achieved at or above the mastery level (80%) on the summative evaluation but not on the formative evaluation were identified as "delayed-mastery" while those who attained that level on the formative evaluation were identified as "early-mastery .

Treatment effects on the affective domain elements were determined in two ways. One was by interviewing randomly selected students from the Precalculus sections. The other was by analyzing responses to a questionnaire completed by Treatment and Control students enrolled in Calculus I. In addition to these analyses, enrollment figures in Precalculus and Calculus I for the years 1971, 1972, and 1973 were examined to see if the Treatment affected the percentage of Precalculus students enrolling in Calculus I.

The effect of the treatment on achievement in Calculus I was analyzed at two stages of the course. Data from the test covering limits was used first. Student performance was measured on (1) the questions involving limits and (2) the total test scores. The semester percentage for each student provided the second set of data. For the purpose of these analyses, students in Treatment A were identified as Nance early mastery, Nance delayed mastery, or Nance nonmastery; and students in Treatment B were identified as Smith mastery or Smith nonmastery.

Results

The results of this study will be reported according to the organization implied in Chapters III and IV. First the mastery learning results will be given and then the treatment effects regarding Calculus I will be divided into enrollment, affective domain, and cognitive domain results.

Mastery Learning: The null hypothesis of H_0: the proportion of students achieving mastery in Treatment A (P_A) equals the proportion of students achieving mastery in Treatment B (P_B) was tested using a Z test statistic with a significance level of $\alpha = .05$. The alternative hypothesis was H_a: $P_A > P_B$. Thirty nine of sixty Treatment A students achieved mastery compared to twenty one of sixty one in Treatment B. The Z test statistic for these proportions is $Z = 3.33$ where the null hypothesis is rejected for values of $Z > 1.645$ when $\alpha = .05$. Therefore, the alternative hypothesis of $P_A > P_B$ was accepted.

Enrollment: (A) The null hypothesis of H_0: the proportion of Treatment group students (P_T) enrolling in Calculus I equals the proportion of Control group students (P_C) enrolling in Calculus I was tested using a Z test statistic with a significance level of $\alpha = .05$. The alternative hypothesis was H_a: $P_T \neq P_C$.

Sixty students in Treatment A, sixty one in Treatment B, and 152 in the Control group completed Precalculus. Of these students, twenty six, twenty eight, and eighty five respectively enrolled in Calculus I. This produced percentage enrollment figures of fifty-six percent for the Control group and forty-five percent for the Treatment group. The Z value was 1.86 where the rejection value at the $\alpha = .05$ level for a two-sided test was 1.96. Therefore, the null hypothesis of equality of proportions between Treatment and Control groups was accepted. (Comment: The hypothesis concerning enrollments was that enrollment percentage would be increased as a result of the treatment. The computed enrollments, however, indicated that eleven percent fewer students enrolled in Calculus I from the Treatment group. Consequently, a two-sided test was used with the alternative hypothesis H_a: $P_T \neq P_C$ rather than a one-sided test with the alternative hypothesis H_a: $P_T > P_C$. If the hypothesis had been that the treatment would decrease enrollment, then a one-sided test with alternative hypothesis H_a: $P_T < P_C$ would have produced a statistically significant difference in enrollment.)

(B) The percentage of students remaining in Calculus I was also not affected by the treatment. Fifty one of fifty four (94%) Treatment group students completed Calculus I compared to seventy nine of eighty five (93%) from the Control group. The Z test statistic for equality of proportions between these groups is $Z = -.25$ where the rejection value for $\alpha = .05$ is $Z > 1.645$.

Affective Domain: Treatment effect on the affective domain was analyzed by considering student responses to interviews and a questionnaire. The hypothesis as stated in Chapter I is: a unit on limits in Precalculus will improve attitudes (1) prior to taking Calculus I and (2) during Calculus I. The findings with respect to this hypothesis are reported below.

(A) Interviews conducted at the end of the Precalculus course indicated the following results:

1. Treatment group students thought they were doing poorly in

Precalculus.

2. Treatment group students thought Calculus I would be very difficult as opposed to Control group students who thought it would not be harder than other courses.

3. The treatment effect caused negative attitudes and anxieties toward Calculus I for the Nonmastery students. Otherwise, Mastery and Control students had positive attitudes and no apparent anxieties toward calculus.

(B) Questionnaire responses obtained during the third week of the Calculus I course are given in Chapter IV. These responses were analyzed using a chi square test statistic for each question on the questionnaire. Significance was tested at the $\alpha = .05$ level. The following two results of this analysis deserve special mention.

1. Significantly more than expected Control group students thought Calculus I was "harder than expected" compared to significantly fewer than expected Treatment group students who responded to the same item.

2. Treatment group students experienced significantly less frustration when studying limits than did the Control students.

Cognitive Domain: Treatment effect on achievement was examined by considering the Calculus I test covering limits and the final percentage score for Calculus I. Analysis of scores on the calculus test covering limits was accomplished by using a null hypothesis of H_0: there is no significant difference in achievement for cells of the experimental model. This was tested by using a one-way analysis of variance. When the items covering limits were tested, the Bartlett test statistic of 20.7 was too high to permit use of analysis of variance. Consequently, scores were categorized and a chi square test statistic was used. There was a significant difference at the $\alpha = .05$ level between the achievement of Treatment and Control students. Subsequent analysis of the chi square partial values indicated that the Treatment group students performed better than expected.

Analysis of the total test scores also used the null hypothesis of H_0: there is no significant difference in achievement for cells of the experimental model. In this case, the Bartlett test statistic was within the acceptable range and a one-way analysis of variance produced an F test statistic of F = 3.3 where the rejection value was $F_{.05}(5,120) = 2.29$. Consequently, the null hypothesis of equality of means was rejected. Subsequent multiple comparisons were made using a Scheffe test and indicated that the only statistically significant differences were between (1) the Nonmastery and Mastery groups and (2) the Nonmastery and Other mastery groups.

Semester averages in Calculus I were also analyzed based on the null hypothesis H_0: there is no significant difference in achievement for cells of the experimental model. A one-way analysis of variance* produced an F test statistic of F = 5.57 where the rejection value was $F_{.05}(5,120) = 2.29$.

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*The Bartlett test statistic for homogeneity of variance was 11.15 while the rejection value was $\chi^2_{.05}(5) = 11.07$. However, the F ratio was large enough to reject the null hypothesis and perform multiple comparisons.

Multiple comparisons then produced statistically significant differences between the following: (1) Mastery and Nonmastery, (2) Nonmastery and Control, and (3) Nonmastery and Other mastery. All other comparisons produced no significant difference.

Delayed Mastery: A unique aspect of this study was the identification of delayed mastery students in Treatment A. An analysis of their subsequent behavior was obtained by constructing a profile of this group compared to other cells of the experimental model. This information is contained in Figure 5.1.

Item	Group Most Closely Resembled
Interview responses	Other mastery
Questionnaire responses	Smith nonmastery
Enrollment in Calculus I	Nonmastery
First calculus test:	
(1) limit items	(1) Smith nonmastery
(2) total score	(2) Smith nonmastery
Semester percentage in Calculus I	Smith nonmastery

FIGURE 5.1

Profile of Delayed Mastery Students

The purpose of this identification and subsequent profile was to examine what carry-over effect the alternative instructional procedures and additional time spent studying limits would have on these students. Figure 5.1 indicates that the Delayed mastery group responses to all evaluation items used during the Calculus I course were similar to those of Nonmastery students in either

Treatment A or Treatment B. Consequently, the alternative instruction and extra time seem to have had little carry-over effect on the Delayed mastery students.

Conclusions

1. Reject the hypothesis that a mastery learning strategy, employed on a unit of limits of sequences and limits of functions, will result in no significant difference between the proportion of students achieving the mastery level in a mastery learning situation and the proportion of students achieving the mastery level in an expository classroom situation.
2. Do not reject the hypothesis that a unit on limits in Precalculus will not affect attitudes
   (a) prior to taking Calculus I
   (b) during Calculus I.
3. Do not reject the hypothesis that there will be no significant difference between the percentage of students enrolling and remaining in Calculus I after having had a unit on limits and the percentage of students from the Control group enrolling and remaining in Calculus I.
4. Do not reject the hypothesis that there will be no significant difference in achievement in Calculus I between students in the treatment groups that achieve mastery and those that did not achieve mastery.

5. Do not reject the hypothesis that there will be no significant difference in achievement in Calculus I between delayed mastery students and other mastery students in the treatment groups.
6. Reject the hypothesis that an introduction to limits prior to calculus will result in no significant difference in achievement on the work with limits in calculus.
7. Do not reject the hypothesis that an introduction to limits prior to calculus will result in no significant difference in achievement in the first semester of calculus.

Implications for Education

Instructional Unit: Approximately one-half of the time spent studying limits was used for the study of limits of sequences. Since the Calculus I course at Central Michigan University does not cover this topic, the time devoted to this may have been more effectively used in additional study of limits of functions. A second point is that notation was a problem for some of the students. The mathematical phrase $n > N$ tended to be confusing. Perhaps a change to $n > M$ would correct this in future work.

Mastery Learning: The mastery learning strategy of Treatment A was effective as indicated in the conclusions on page 83. However, several problems occurred which should be considered for future work. First, the time spent for alternative instructional techniques may have produced the same results in an expository situation. Related to the alternative instruction problem is the question of what to do with the early mastery students between the formative and summative evaluations. In this study, they were used as resource people during the small-group alternative instruction sessions. They might not enjoy doing this if an entire course were to be taught via a mastery strategy.

A second problem is that the difference between early mastery and delayed mastery students may not be an accurate indication of their abilities. Students in this study were informed that the formative evaluation would not count on their grade and that there would be a summative evaluation several days later. These reasons may have resulted in students being identified as delayed mastery when they could have achieved at the mastery level if only one evaluation were given. Closely related to this problem is that of how many evaluations should be given. Would those students who scored between sixty and eighty on one summative evaluation socre above eighty on a subsequent evaluation? How many examinations are required to maximize class progress and still account for individual differences?

A third area of concern is that both Treatment A and Treatment B resulted in less time being spent on the topics ordinarily studied in Precalculus. Since the semester average in Calculus I of Treatment students was 5.9 percentage points lower than the Control group, this may mean that prior knowledge of limits has less effect on success in Calculus I than the knowledge of other topics in Precalculus.

Sources of Variation: Sources of variation affecting the results of this study include the following:

1. Treatment A students spent three more days studying limits than did Treatment B students.
2. Both Treatment A and Treatment B students spent less time studying the other Precalculus topics than did the Control group students.
3. Five different instructors taught the Control sections.
4. Treatment A and Treatment B sections were taught by different instructors.
5. Treatment A sections met at 8:00 a.m. and 9:00 a.m. while Treatment B sections met at noon and 1:00 p.m.
6. Calculus I was taught at two different times: 10:00 a.m. and 2:00 p.m.
7. Students in Calculus I had nine different recitation instructors.
8. Students were not randomly selected for this study. However, a post hoc test of A.C.T. scores indicated that the sections were evenly matched for homogeneity of variance and equality of means.

Other comments: Statistical analyses of affective domain elements are difficult to obtain. In this study, it was not possible to show that the Treatment effect changed attitudes toward Calculus I. It was possible, however, to show that a statistically significant percentage of students from the Treatment group felt they experienced less frustration when studying limits in Calculus I than did the students from the Control group.

The next comment concerns the Calculus I test that produced data for part of the analysis in the cognitive domain. The limit items on these tests had such high averages that analysis of variance was not possible. If these items had been better able to distinguish between ability levels, Treatment effects could have been more thoroughly examined.

The last comment has social and economic implications. Although the Calculus I enrollments for Treatment and Control students were not significantly different ($\alpha = .05$), it is true that enrollment of Treatment students was 11% less than that of Control students. If some of the Treatment students did not take Calculus I because the unit on limits made them realize that Calculus was not what they expected, they might have enrolled in Calculus I and then dropped the course had it not been for the unit on limits. These students have thus saved some of their tuition and some time that would have been spent in Calculus I. On the other hand, enrollment figures are of primary importance in these days of careful auditing of student credit hours. Conceivably a mathematics department could suffer economic consequences if the Treatment of this study were extended and the enrollment percentages remained the same. An examination of this problem is suggested as a topic for further research.

Suggestions for Further Research

Following are listed related questions that deserve further consideration:

1. Do the carry-over effects of a mastery learning strategy differ from the carry-over effects of an expository strategy? This study has

shown there was no significant difference in achievement in Calculus I between Delayed mastery and Other mastery students. However, the profile of Figure 5.1 indicates that the Delayed mastery students most closely resemble Nonmastery students when one considers the analyses of this study. Perhaps subsequent research could more carefully define and control parameters relating to Delayed mastery students.

2. Would a greater variety of alternative instructional techniques produce even better results for mastery learning? (Audio-visual tapes, more effective use of the reading list, tutoring, etc.)
3. Would mastery learning be effective if all instructional objectives were tested at the Analysis level of Wilson's taxonomy? In this study, mastery of most items was tested at lower levels of difficulty. Results of the mastery strategy indicate it might be as effective if all items were tested at the highest level of the taxonomy.
4. What effect would a different Treatment (no sequences, no (ϵ, δ) notation, longer intuitive approach, etc.) concerning limits have on achievement in calculus?
5. How does a student's prior conception of the difficulty of calculus affect his achievement in calculus?
6. Does a unit on limits in Precalculus cause a <u>decrease</u> in enrollment in calculus? One result of this study was that Treatment group students had an eleven percent lower enrollment rate in Calculus I

than did the Control group. This was not statistically significant at the $\alpha = .05$ level, but it was substantial enough to warrant a replication of the study with particular emphasis paid to enrollment.

7. Does a unit on limits in Precalculus result in poorer achievement in calculus? The Treatment group students of this study had an achievement mean 5.9 percent lower than the Control group. This was not statistically significant at the $\alpha = .05$ level, but it is a justification for replication of the study.

The comments of this section suggest a replication of the study. Attention should be directed toward detecting a decrease in enrollment and achievement as a result of Treatment.

APPENDICES

APPENDIX I: Instructional Unit

LIMITS OF SEQUENCES

Examine several sequences and plot them on a number line.

Examples: 1, 1/2, 1/3, 1/4,...
2, 4, 8, 16,...
1, - 1/3, 1/9, -1/27,...
Others

Find the 7th, 8th, or 9th terms of each of the above sequences.

Find the nth term of each of the above sequences.

NOTATION: $a_1, a_2, a_3, \ldots, a_n, \ldots$ or $\left\{a_n\right\}_{n=1}$

Example: $\left\{\frac{1}{2n}\right\}_{n=1} = 1/2, 1/4, 1/6, \ldots, 1/2n, \ldots$

Type of sequences:

DEFINITION: An alternating sequence is a sequence whose terms are alternately positive and negative.

Example: $1, -1/3, 1/9, -1/27, \ldots = \left\{\frac{(-1)^{n+1}}{3^{n-1}}\right\}_{n=1}$

DEFINITION: A constant sequence is a sequence $\left\{a_n\right\}_{n=1}$ such that $a_n = K$ for some constant K.

Example: $3, 3, 3, \ldots = \left\{3\right\}_{n=1}$

Use the above examples to examine "getting close to" (plot on the number line).

1, 1/2, 1/3, 1/4, ... $\longrightarrow$ 0

2, 4, 8, 16, ... $\longrightarrow$?

1, -1/3, 1/9, -1/27, ... $\longrightarrow$?

2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, ... $\longrightarrow$?

3, 3, 3, ... $\longrightarrow$?

ASSIGNMENT #1

1. On the number line, plot several terms of the following sequences with given n^{th} terms:

   a) $a_n = \frac{1}{2n}$

   b) $a_n = \frac{2n}{n+1}$

   c) $\left\{(-2)^n\right\}_{n=1}$

   d) $\left\{2^n\right\}_{n=1}$

   e) $\left\{-(2^n)\right\}_{n=1}$

   f) $\left\{\frac{1}{n^2+2}\right\}_{n=1}$

   g) $\left\{1 + \frac{1}{(-1)^n n}\right\}_{n=1}$

2. What number, if any, do the above sequences "get close to"?

3. Find sequences (nonconstant) that "get close to" the following:

   a) 1

   b) -2

   c) "a", where a is any number

If the terms of a sequence "get close to" some number L, we say that the <u>limit of the sequence</u> $\{a_n\}$ is L. Since "getting close to" is not precise enough to convey the concept of limit of a sequence, we use the following definition:

DEFINITION: A sequence $\{a_n\}$ has limit L if for each $\epsilon > 0$, there is a positive integer N such that if $n > N$, then $|a_n - L| < \epsilon$.

NOTATION: $\lim\limits_{n \to \infty} a_n = L$

To illustrate this definition, consider the following examples.

Example 1:

Let $\{a_n\} = \left\{\frac{1}{n+1}\right\}$ and $L = 0$.

If $\epsilon = .01$, then an N is 99 (actually, any integer greater than 99 would also serve as N). Then for $n > 99$, $\left|\frac{1}{n+1} - 0\right| < .01$.

If $\epsilon = .00001$, then $N = 99{,}999$.

Note here that if $n > 99{,}999$, then $n+1 > 100{,}000$ and $\frac{1}{n+1} < \frac{1}{100{,}000} = \epsilon$.

But $\frac{1}{n+1} = \left|\frac{1}{n+1} - 0\right|$, hence $|a_n - L| < \epsilon$ for $n > N$ and the definition is satisfied.

Since the choice of N depends on the choice of ϵ, we usually write N_ϵ rather than N as above.

Example 2:

$\{a_n\}_{n=1}$ where $a_n = \frac{(-1)^n}{3^n}$.

For $\epsilon = .10$, find $N_{.10}$ such that for $n > N_{.10}$ $|a_n - 0| < \epsilon$.

Try $N_{.10} = 3$, then $a_n = \frac{1}{3^n}$ and for $n > 3$, $\frac{1}{3^n} < .10$.

Do the same for $\epsilon = .001$.

Using the definition of limit of a sequence, it is possible to prove that some sequences have limits.

Example: Prove $\lim_{n \to \infty} \frac{1}{n+1} = 0$.

Proof: Let $\epsilon > 0$ be given. We then need to find N_ϵ such that

$$\left|\frac{1}{n+1} - 0\right| < \epsilon \text{ when } n > N$$

But $\left|\frac{1}{n+1}\right| < \epsilon \Leftrightarrow \frac{1}{n+1} < \epsilon \Leftrightarrow \frac{1}{\epsilon} < n+1 \Leftrightarrow n > \frac{1}{\epsilon} - 1.$

Hence for a given $\epsilon > 0$, let $N_\epsilon > \frac{1}{\epsilon} - 1$ then for $n > N_\epsilon > \frac{1}{\epsilon} - 1$,

we have $n+1 > \frac{1}{\epsilon} \Leftrightarrow \frac{1}{n+1} < \epsilon \Leftrightarrow |a_n - 0| < \epsilon$.

$\therefore \lim_{n \to \infty} a_n = 0$

To further illustrate this proof, let $\epsilon = .001$.

Then the associated N_ϵ could be $\frac{1}{.001} - 1 = 1000 - 1 = 999$.

Hence for $n > 999$, $\frac{1}{n+1} < .001$.

Example: Prove $\lim_{n \to \infty} (1 + \frac{(-1)^n}{n}) = 1$

Let $\epsilon > 0$ be given. Then $|a_n - 1| = \left|1 + \frac{(-1)^n}{n} - 1\right| = \frac{1}{n}$.

Hence find N_ϵ such that $\frac{1}{n} < \epsilon$ for $n > N_\epsilon$.

This can be accomplished by letting $N_\epsilon > \frac{1}{\epsilon}$

Then $n > N_\epsilon \Rightarrow \frac{1}{\epsilon} < n \Rightarrow \frac{1}{n} < \epsilon$.

ASSIGNMENT #2

1. Prove $\lim\limits_{n\to\infty} \frac{1}{n+2} = 0$

2. Explain why the following do not have limits:

   a) 2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4,...

   b) $a_n = \begin{cases} \frac{2}{n+1}, & n \text{ odd} \\ \frac{n-2}{n}, & n \text{ even}, \ n \geq 2 \end{cases}$

   c) $\{2n\}_{n=1}$

3. Let $S = \left\{\frac{1}{n}\right\}_{n=1}$ and $\epsilon = .001$. Find the first (smallest) $N_{.001}$ such that $\left|\frac{1}{n}\right| < .001$.

4. Do the same as in problem number three for $\left\{\frac{1}{n^2}\right\}_{n=1}$

   What do you conclude about $\lim\limits_{n\to\infty} \frac{1}{n^2}$?

Although the limits of some sequences are attainable by the definition, consider a problem such as

$$\lim_{n\to\infty} \frac{n^2 + 5n + 6}{(n+2)(n+3)^2} \qquad \text{or} \qquad \lim_{n\to\infty} \frac{5n + 2}{n}$$

If we apply the definition here, the work becomes tedious.

To facilitate the finding of such limits, we will now prove some basic results which can then be applied to simplify some of the more difficult problems.

1. The limit of a constant sequence is that constant.

i.e. $\lim_{n \to \infty} K = K$

Proof: Let $\epsilon > 0$ be given and consider the expression $|a_n - L| < \epsilon$. In this case, $a_n = K$ and $L = K$, hence $|a_n - L| = |K - K| = 0$. We now have the problem of selecting N_ϵ such that for $n > N_\epsilon$, $|a_n - K| < \epsilon$. However, we have already shown that $|a_n - K| = 0$, hence any N_ϵ we choose will have the desired property, In particular, let N_ϵ equal, say, 100. Then for $n > 100$, $|a_n - K| < \epsilon$ and the definition is satisfied. Therefore, $\lim_{n \to \infty} K = K$.

2. If $\lim_{n \to \infty} a_n = L_1$ and $\lim_{n \to \infty} b_n = L_2$, then $\lim_{n \to \infty} (a_n + b_n) = L_1 + L_2$.

Proof: Let $\epsilon > 0$ be given. Then there exists N_1 such that for $n > N_1$, $|a_n - L_1| < \frac{\epsilon}{2}$. (note that $\frac{\epsilon}{2} > 0$ and $\lim_{n \to \infty} a_n = L_1$ by hypothesis). Similarly, there exists N_2 such that $|b_n - L_2| < \frac{\epsilon}{2}$ for $n > N_2$. Now consider $|(a_n + b_n) - (L_1 + L_2)|$. By the triangle in equality, $|(a_n + b_n) - L_1 + L_2| = |(a_n - L_1) + (b_n - L_2)| \leq |a_n - L_1| + |b_n - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $n > \max\{N_1, N_2\}$.

To illustrate how these theorems can be used, consider the following problems:

a) Evaluate $\lim_{n \to \infty} \frac{5n + 2}{n}$: $\lim_{n \to \infty} \frac{5n + 2}{n} = \lim_{n \to \infty} (\frac{5n}{n} + \frac{2}{n}) = \lim_{n \to \infty} (5 + \frac{2}{n}) = \lim_{n \to \infty} 5 + \lim_{n \to \infty} \frac{2}{n} = 5 + 0 = 5.$

b) Evaluate $\lim_{n\to\infty} \frac{3n^2 + n}{n^2} = \lim_{n\to\infty} \frac{3n^2}{n^2} + \frac{n}{n^2} = \lim_{n\to\infty} (3 + \frac{1}{n}) =$

$\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{1}{n} = 3 + 0 = 3.$

There are several other theorems concerning limits of sequences that will now be stated and illustrated but not proven.

3. If $\lim_{n\to\infty} a_n = L_1$, then $\lim_{n\to\infty} Ka_n = KL_1$ where K is a constant.

   Example: $\lim_{n\to\infty} \frac{5}{n+1} = 5 \lim_{n\to\infty} \frac{1}{n+1} = 5 \cdot 0 = 0.$

4. If $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$, then $\lim_{n\to\infty} a_n b_n = L_1 L_2$.

   Example: $\lim_{n\to\infty} \frac{(n+2)}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n+2}{n+2} =$

   $\lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} \frac{n+2}{n+2} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} 1 = 0 \cdot 1 = 0.$

5. If $\lim_{n\to\infty} a_n = L$ and $b_n = a_n$ for all n, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. At first glance it may appear that this theorem doesn't really say anything. However, it is very important in that it allows us to "reduce" expressions. Actually, we have used this theorem intuitively in several of the previous examples.

   Example: a) $\lim_{n\to\infty} \frac{2n(n+2)}{n(n+2)} = \lim_{n\to\infty} 2 = 2.$

   b) $\lim_{n\to\infty} \frac{n^2 + 5n + 6}{2(n+2)(n+3)} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$

6. If $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2 \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$.

Example: $$\lim_{n\to\infty} \frac{n^2+n+1}{3n^2+2n} = \lim_{n\to\infty} \frac{n^2+n+1}{3n^2+2n} \cdot \frac{n^2}{n^2} =$$

$$\lim_{n\to\infty} \frac{n^2+n+1}{n^2} \cdot \frac{n^2}{3n^2+2n} = \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^2}}{\frac{3n^2+2n}{n^2}} =$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3+\frac{2}{n}} = \frac{\lim_{n\to\infty}(1+\frac{1}{n}+\frac{1}{n^2})}{\lim_{n\to\infty}(3+\frac{2}{n})} = \frac{1}{3}$$

ASSIGNMENT #3

1. Evaluate the limits of the following sequences. Give a reason for each step.

a) $\left\{\frac{6n+13}{13n}\right\}_{n=1}$

b) $\left\{\frac{n}{n+1}+\frac{n}{2}\right\}_{n=1}$

c) $\left\{\left(\frac{2n+1}{n}\right)\left(5+\frac{7}{n}\right)\right\}_{n=1}$

d) $\left\{\frac{2n+1}{3n+4}\right\}_{n=1}$

e) $\left\{\frac{n}{n+1}\right\}_{n=1}$

f) $\left\{\frac{n^2+n}{n^2-2}\right\}_{n=1}$

g) $\left\{\frac{n^2}{n^3+4}\right\}_{n=1}$

2. Prove theorem three.

3. a) Let $a_{n \; n=1}$ be a sequence such that $\lim_{n \to \infty} a_n = 0$. Prove that

$\lim_{n \to \infty} a_n = 0$.

b) Prove the converse of part (a). (i.e., if $\lim_{n \to \infty} a_n = 0$, then

$\lim_{n \to \infty} a_n = 0$)

4. Prove that if $\lim_{n \to \infty} a_n = L$, then $\lim_{n \to \infty} (-a_n) = -L$.

Do this both by using the definition of limit of a sequence and by using the results of some of the theorems.

LIMITS OF FUNCTIONS

We now turn our attention to examining limits of functions. Let us first consider several examples.

1. Let $f(x) = x + 2$ and suppose x approaches 2 $(x \to 2)$.

   What is f(x) approaching? $(f(x) \to ?)$

   It is easily seen here that as x "gets close to" two, f(x) "gets close to" four.

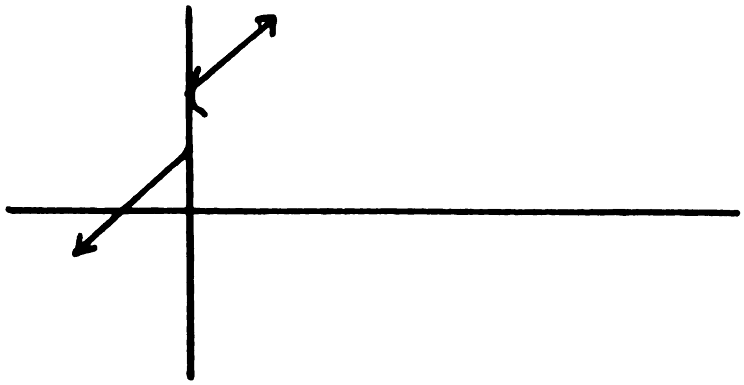
2. Let $f(x) = x^2$ and suppose $x \to -1$.

   Then $f(x) \to$ _________?

3. As a slightly different example, consider

$$f(x) = \begin{cases} x + 1 & x \leq 0 \\ x + 2 & x > 0 \end{cases}$$

When graphed, this would look like the following:



Let us now consider the same questions as on the previous examples.

Let $x \rightarrow 0$. Then $f(x) \rightarrow$ __________?

After some thought, we realize that there is a need to be more specific about what we mean by $x \rightarrow 0$ and $f(x) \rightarrow$ ________. It is this need for precision that will eventually lead to the definition for limit of a function.

In the meantime, example three illustrates the importance of knowing <u>how</u> x approaches the number we are considering (0 in this case). For example, let x assume the values 1/2, 1/4, 1/6, 1/8,... Then the corresponding values of f(x) would be 2 + 1/2, 2 + 1/4, 2 + 1/6, 2 + 1/8,... From this it can be seen that $f(x) \rightarrow 2$.

On the other hand, let x assume the values -1/2, -1/4, -1/6, -1/8,... The corresponding values for f(x) are 1 - 1/2, 1 - 1/4, 1 - 1/6, 1 - 1/8,... Accordingly, we would say that $f(x) \rightarrow 1$.

This function serves as an example that does not have a limit as $x \to 0$. That is because there are two possible values that f(x) is approaching rather than one. We indicate this by saying that as x approaches 0 from the right ($x \to 0^+$), $f(x) \to 2$; and as x approaches 0 from the left ($x \to 0^-$), $f(x) \to 1$.

In examples 1 and 2, it made no difference how x approached the stated number (we usually say "x approached a"). When this is the case, we write $\lim_{x \to a} f(x) = L$ where L is the number that f(x) is approaching. Hence, in example 1 we would write $\lim_{x \to 2} (x + 2) = 4$ and in example 2 we would write $\lim_{x \to -1} x^2 = 1$.

An adaptation of this notation could also be used for example 3. We would have $\lim_{x \to 0^-} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = 2$.

As you might expect, the idea of limit of a function is closely related to the idea of limit of a sequence. The main difference is that in finding the limit of a function it is necessary to know what value the dependent variable is approaching. For example, $\lim_{x \to 2} (x + 2) = 4$ but $\lim_{x \to 3} (x + 2) = 5$.

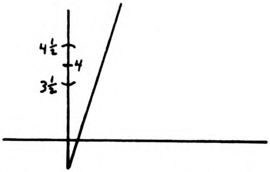
As before, the idea of "getting close to" is not sufficient for a definition. Therefore, the formal definition for limit of a function as x approaches a is:

The function f(x) has limit L as x approaches a if for each $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that if $0 < |x - a| < \delta_\epsilon$, then $|f(x) - L| < \epsilon$.

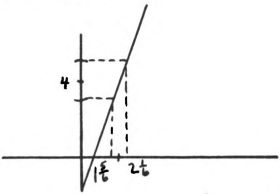
Intuitively, this says that if an interval is placed around L on the f(x) axis, then you can find an interval around "a" on the x axis such that f(x) is in the interval around L for all $x \neq a$ in the interval about a.

Example: Let f(x) = 3x - 2 and $x \rightarrow 2$. Then L = 4.

To see how this satisfies the definition, let ϵ = 1/2 and consider the interval (4 - 1/2, 4 + 1/2) about 4 on the y axis.



We now find an interval about 2 on the x axis that satisfies the definition.



for ϵ = 1/2, we see that $\delta_{1/2}$ = 1/6 and a suitable interval would be (2 - 1/6, 2 + 1/6).

For ϵ = 1/10, find a δ_ϵ that satisfies the definition of limit of a function.

We now show that the interval about "a" depends on the interval about L for the function f(x) = 3x - 2 with a = 2 and L = 4.

If $|f(x) - L| < \epsilon$, then $|(3x - 2) - 4| < \epsilon \Leftrightarrow$

$|3x - 6| < \epsilon \Leftrightarrow \quad |3(x - 2)| < \epsilon \Leftrightarrow$

$3\,|x - 2| < \epsilon \Leftrightarrow \quad |x - 2| < \epsilon/3 \Leftrightarrow$

$2 - \epsilon/3 < x < 2 + \epsilon/3$

Hence, for any given $\epsilon > 0$ and the associated interval $(L - \epsilon, L + \epsilon)$ about L on the y axis, we can use the interval $(2 - \epsilon/3, 2 + \epsilon/3)$ about two on the x axis to satisfy the definition. We say let $\delta = \epsilon/3$.

Then $0 < |x - 2| < \delta = \epsilon/3 \Rightarrow |f(x) - 4| < \epsilon$.

This illustrates the method of proving that a particular function has a particular limit as x approaches some value.

ASSIGNMENT #4

1. In the following problems, let x assume several values "close to" the specified number a, and examine the corresponding values for f(x). In each problem, make a conjecture about $\lim_{x \to a} f(x)$.

   a) $f(x) = x + 3, \; x \to 1$

   b) $f(x) = x^2 + 1, \; x \to 2$

   c) $f(x) = -4x + 5, \; x \to 0$

   d) $f(x) = x^2 - x + 1, \; x \to 1$

2. In each of the following problems, for the indicated intervals about the respective values of L, find an interval about a on the x axis such that if x is in the interval about a, then f(x) is in the interval about L.

a) $f(x) = x + 3$, $L = 4$, $a = 1$, $(4 - 1/2,\ 4 + 1/2)$ ($\epsilon = 1/2$)

b) $f(x) = x + 3$, $L = 4$, $a = 1$, $(4 - 1/100,\ 4 + 1/100)$ ($\epsilon = .01$)

c) $f(x) = x^2 + 1$, $L = 5$, $a = 2$, $(5 - 1/3,\ 5 + 1/3)$ ($\epsilon = 1/3$)

d) $f(x) = -4x + 5$, $L = 5$, $a = 0$, $(5 - 1/3,\ 5 + 1/3)$ ($\epsilon = 1/3$)

e) $f(x) = x + 3$, $L = 4$, $a = 1$, $(4 - \epsilon,\ 4 + \epsilon)$

f) $f(x) = 3x + 1$, $L = 7$, $a = 2$, $(7 - \epsilon,\ 7 + \epsilon)$

3. Using the definition of limit of a function, prove the following:

a) $\lim_{x \to a} x = a$

b) $\lim_{x \to 2} (2x + 1) = 5$

c) $\lim_{x \to a} (mx + b) = ma + b$, $m \neq 0$

There are several important parts of the definition of limit of a function that should be noted. First, we see that the definition implies that x must approach "a" from both the right and the left. This is because it says $|x - a| < \delta$ which is equivalent to $-\delta < x - a < \delta \Leftrightarrow a - \delta < x < a + \delta$. Hence one must consider values of x on either side of a.

Example: Explain why

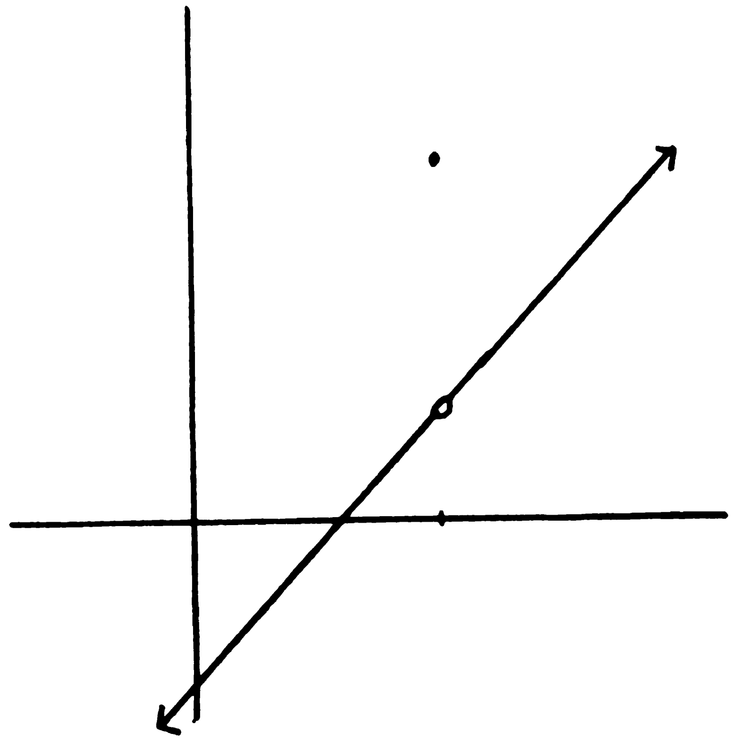
$$f(x) = \begin{cases} x + 1, & x > 0 \\ x, & x \leq 0 \end{cases}$$

does not have a limit as $x \to 0$.

Another consequence of the definition of limit of a function is that we do not consider what happens when $x = a$. This is excluded by the part that says $0 < |x - a|$. The reason for this is illustrated by the following example.

Consider

$$f(x) = \begin{cases} x - 3, & x \neq 5 \\ 6 & , x = 5 \end{cases}$$



As $x \longrightarrow a$, what happens to f(x)?

After trying a few values, is it clear that $f(x) \longrightarrow 2$? Since this is consistent with our earlier ideas of limit of a function, we say

$$\lim_{x \to 5} f(x) = 2$$

even if in this case $f(5) \neq 2$. (Note: a concept to be studied later, continuity of functions, will help explain functions of this sort.)

Notice that if we do not insist that $x \neq a$ in our definition, then for an interval about L = 2 where $\epsilon = 1/2$, it would not be possible to find a corresponding interval about a = 5 that satisfies the definition because we could then use

x = 5 and f(x) = 6 would not be in the interval about 2.

ASSIGNMENT #5

Tell whether or not the following functions have limits at the indicated values of x. For those that do not, indicate what part of the definition they fail to satisfy. (Sketch)

1. $f(x) = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$; a = 0

2. $f(x) = \begin{cases} x^2 + 1, & x < 1 \\ x + 1, & x > 1 \end{cases}$; a = 1

3. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 10, & x = 2 \end{cases}$; a = 2

4. $f(x) = \begin{cases} x + 1, & x \leq 3 \\ -2x + 1, & x > 3 \end{cases}$; a = 3

Limit Theorems for Functions:

As we did with sequences, we now state and prove several theorems that enable us to evaluate the limits of various functions.

Theorem 1. Let f(x) = K be a constant function. Then $\lim_{x \to a} f(x) = K$ for any real number a.

Proof: Let $\epsilon > 0$ be given and consider f(x) - K. Since f(x) = K, f(x) - K = 0 = 0, we have that f(x) - L $< \epsilon$ no matter what value of δ is used. Therefore, $0 < |x - a| < \delta$ (say $= 1$) implies $|f(x) - L| < \epsilon$ and the theorem is proven.

Theorem 2. Let f(x) and g(x) be functions such that $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$. Then $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$.

Proof: Let $\epsilon > 0$ be given and consider $|(f(x) + g(x)) - (L_1 + L_2)|$. As before, we have by the triangle inequality $|(f(x) + g(x)) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)| \leq |f(x) - L_1| + |g(x) - L_2|$. But since $\lim_{x \to a} f(x) = L_1$, there is a δ_1 such that $0 < |x - a| < \delta_1$ implies $|f(x) - L_1| < \epsilon/2$ and since $\lim_{x \to a} g(x) = L_2$, there is a δ_2 such that $0 < |x - a| < \delta_2$ implies $|g(x) - L_2| < \epsilon/2$. Hence for $\delta = \min\{\delta_2, \delta_1\}$, $0 < |x - a| < \delta$ implies both $|f(x) - L_1| < \epsilon/2$ and $|g(x) - L_2| < \epsilon/2$. Therefore $|(f(x) + g(x)) - (L_1 + L_2)| \leq |f(x) - L_1| + |g(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon$.

This then satisfies the definition and we have $\lim_{x \to a} (f(x) + g(x)) = L_1 + L_2$.

Example: Let $f(x) = 3x^2$ and $g(x) = 5$. Then $\lim_{x \to 2} f(x) = 12$ and $\lim_{x \to 2} g(x) = 5$ and $\lim_{x \to 2} (3x^2 + 5) = \lim_{x \to 2} 3x^2 + \lim_{x \to 2} 5 = 12 + 5 = 17$.

Theorem 3. Let f(x) be a function such that $\lim_{x \to a} f(x) = L$ and let K be a non-zero constant. Then $\lim_{x \to a} Kf(x) = K \cdot L$.

Proof: Let $\epsilon > 0$ be given and consider $|Kf(x) - KL|$. By properties of absolute value, we have $|Kf(x) - KL| = |K||f(x) - L|$. Then

$|Kf(x) - KL| < \epsilon \Leftrightarrow |K||f(x) - L| < \epsilon \Leftrightarrow |f(x) - L| < \epsilon/|K|$.

But since $\lim_{x \to a} f(x) = L$, we know there is a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon/|K|$. Therefore for $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon/|K| \Rightarrow |K||f(x) - L| < \epsilon \Rightarrow |Kf(x) - KL| < \epsilon$. This then satisfies the definition and we have $\lim_{x \to a} Kf(x) = K \cdot L$.

Example: Let $f(x) = 3x^2 + 5$ and suppose $\lim_{x \to 2} x^2 = 4$. Then $\lim_{x \to 2} (3x^2 + 5)$

$$= \lim_{x \to 2} 3x^2 + \lim_{x \to 2} 5 = 3 \lim_{x \to 2} x^2 + \lim_{x \to 2} 5 = 3 \cdot 4 + 5 = 17.$$

There are several other theorems involving limits of functions that will now be stated but not proven.

Theorem 4. Let $f(x)$ and $g(x)$ be functions such that $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$. Then $\lim_{x \to a} f(x)\, g(x) = L_1 L_2$.

Theorem 5. Let $f(x)$ and $g(x)$ be functions such that $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2 \neq 0$. Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.

Theorem 6. Let $f(x)$ be a function such that $\lim_{x \to a} f(x) = L \quad 0$. Then

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}.$$

<u>Theorem 7.</u> Let f(x) and g(x) be functions such that f(x) = g(x) for all x except x = a. Then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided these limits exist.

Example: (Theorem 7) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3) = 6.$

The following examples illustrate how the previous theorems can be used to simplify and evaluate the limits of several functions.

Example: $\lim_{x \to 3} \frac{x^2}{3x - 2} = \frac{\lim_{x \to 3} x^2}{\lim_{x \to 3} (3x - 2)}$ (limit of quotient)

$= \frac{(\lim_{x \to 3} x)(\lim_{x \to 3} x)}{\lim_{x \to 3} (3x - 2)}$ (limit of product)

$= \frac{3 \cdot 3}{3 \cdot 3 - 2}$ (previous problems)

$= \frac{9}{7}$

Example: (Indicate the reason(s) for each step)

$$\lim_{x \to +2} \sqrt{\frac{x^2 + x - 6}{x + 2}} = \lim_{x \to +2} \sqrt{\frac{(x + 3)(x + 2)}{x - 2}}$$

$$= \sqrt{\lim_{x \to +2} \frac{(x + 3)(x - 2)}{x - 2}}$$

$$= \sqrt{\lim_{x \to +2} (x + 3)}$$

$$= \sqrt{5}$$

ASSIGNMENT #6

1. Find the limits of the following functions. Give reason(s) for each of your steps.

   a) $\lim_{x \to 3} (x^2 - 3x + 5)$

   b) $\lim_{x \to 4} \frac{2x^2 - 6}{x^3 + 5}$

   c) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

   d) $\lim_{x \to 2} \sqrt{\frac{2x + 5}{3x - 2}}$

   e) $\lim_{x \to 2} \sqrt{\frac{x^3 - 8}{x^2 - 4}}$

   f) $\lim_{x \to 0} \frac{\sqrt{3 + x} - \sqrt{3}}{x}$

2. Let $f(x) = \frac{1}{x}$. Prove that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$ by

   a) theorems from this section.

   b) using the definition of limit of a function.

3. Let $f(x)$ be a function. Prove that $\lim_{x \to a} f(x) = 0$ iff $\lim_{x \to a} |f(x)| = 0$.

4. Let $f(x)$ be a function such that $\lim_{x \to 2} f(x) = L$.

   Prove that $\lim_{x \to 2} |f(x)| = |L|$.

5. Give an example of a function f(x) that does <u>not</u> have a limit as $x \to 3$.

6. Give an example of a function f(x) that does <u>not</u> have a limit as $x \to 3$ but is such that $\lim_{x \to 3} f(x) = 1$.

APPENDIX II: Daily Classroom Record for Treatment A

Wednesday, October 17

About thirty minutes of the fifty minute period was spent on the new unit. Students were made aware of the following groups involved in the study: (a) mastery treatment, (b) regular treatment, and (c) control. The students were also informed of impending interviews at the end of the semester. During the general discussion, students were informed that they would still cover the core of material in the regular Precalculus course.

The remaining instructional time permitted a discussion of notation, definitions, and examples preceding Assignment One. Assignment One was then given for Thursday.

Thursday, October 18

Problems associated with the assignment were discussed. Quiz number one was given.

The definition of page three is difficult to comprehend. Intuitively, the students seem to have no difficulty with the concept of limit of a sequence. There is some confusion between "n" and "N". To alleviate this, a proof should be carefully checked tomorrow.

Friday, October 19

Assignment Two was reviewed. Quiz 2a was given and 2b was assigned to be turned in on the following Tuesday.

Two theorems were proven. They were:

(1) $\lim_{n \to \infty} K = K$, where K is a constant

(2) If $\{a_n\}_{n=1}$ and $\{b_n\}_{n=1}$ are sequences such that $\lim_{n \to \infty} a_n = L_1$ and $\lim_{n \to \infty} b_n = L_2$, then $\lim_{n \to \infty} (a_n + b_n) = L_1 + L_2$

There really was not enough time today. The second theorem was finished right at the end of the period. Consequently, there was no time for illustration and application of the results.

Absence was a real problem today. Sixteen of the total of sixty were not in class. This will make working with the definition difficult for those not in attendance since this was our first good look at the proof of a theorem using the definition.

Tuesday, October 23

A brief review of previously presented limit material was given. Limit theorems three through six on page six were presented and illustrated. Proofs of these were omitted. Several examples illustrating the appropriate use of these theorems were given. Assignment Three was made for Wednesday.

Wednesday, October 24

Questions concerning Assignment Three were answered. Because of the difficulty encountered with problems three and four of this assignment, problem 3a was worked in class and then students were asked to turn in either 3b or 4 on Thursday. Quiz number three was given in class.

The last fifteen to twenty minutes of class time was spent discussing the intuitive concept of limit of a function.

Thursday, October 25

The first section on limits of functions was finished. The definition and appropriate notation were covered in this section. Assignment Four was given for Friday.

Friday, October 26

Questions from Assignment Four were answered. Students have less trouble working with the definition this time than they did before.

Quiz number four was given. Absence is still a problem; eighteen students were missing today. Section number two on limits of functions was finished and Assignment Five was given for Tuesday.

Tuesday, October 30

Questions were answered about the assignment. The last section was completed today. As before, only two theorems were proven in class. They were:

(1) Let $f(x) = K$ be a constant function. Then $\lim_{x \to a} f(x) = K$ for any real number a.

(2) Let $f(x)$ and $g(x)$ be real-valued functions such that $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$. Then $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$.

The rest were stated and illustrated. Assignment Six was given for Wednesday and quiz number five was assigned to be turned in Wednesday.

Wednesday, October 31

Questions about the assignment were answered. Quiz number six was given. The rest of the period was used as a general review for Thursday's test. The students are aware that this test is a formative evaluation.

Instructions to the class about testing were reviewed as follows:

(1) You may take the retest regardless of your score.
(2) The tests will be returned with diagnostic sheets on Friday, the first alternative instruction session.
(3) Other alternative instruction sessions will be held on Monday at 8:00 a.m., 9:00 a.m., and 7:00 p.m. to 9:00 p.m. Tuesday's regular class period will also be used for alternative instruction.
(4) The summative evaluation will be given on Wednesday, Nov. 7.

Frustration seemed to be a big factor today.

Thursday, November 1

A formative evaluation was administered to both sections.

APPENDIX III: Quizzes

1. Write the fifth term of the sequence

$$\left\{\frac{(-1)^n}{n^2+1}\right\}_{n=1}$$

2. (a) For the sequence $\left\{\frac{1}{2n}\right\}_{n=1}$ and $\epsilon = \frac{1}{50}$, find N such that

$$\left|\frac{1}{2n}\right| < \frac{1}{50} \quad \text{for } n > N.$$

(b) Prove $\lim_{n \to \infty} \frac{1}{2n} = 0$ using the definition for limit of a function.

3. Evaluate the limit of the following sequence. State a reason for each step.

$$\left\{\frac{2n^2 + 3n - 5}{3n^2 - n + 4}\right\}_{n=1}$$

4. Let $f(x) = 2x + 1$ and let $(7 - \frac{1}{10}, 7 + \frac{1}{10})$ be an interval about 7 on the f(x) axis. Find the associated interval $(3 - \delta, 3 + \delta)$ about 3 on the x axis such that if x is in $(3 - \delta, 3 + \delta)$, then f(x) is in $(7 - \frac{1}{10}, 7 + \frac{1}{10})$.

5. Prove $\lim_{x \to 3} (2x + 1) = 7$ using the definition of limit of a function.

6. Find $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$. State a reason for each step.

APPENDIX IV: Summative Evaluations - Treatment B

MATH 120 Name ________________

LIMITS - A

1. (11 pts.) Explain why the following sequence does not have a limit. (Hint: Plot the terms on the number line).

$$\left\{(-1)^n + \frac{1}{n}\right\}_{n=1}$$

2. (12 pts.) Give an example of a nonconstant sequence whose limit is -2. Include reasoning.

3. (12 pts.) Give an example of a function that does not have a limit as $x \to 1$. Include reasoning.

4. (15 pts.) For the following sequence, find the smallest $N_{\frac{1}{100}}$ such that $|a_n - 3| < \frac{1}{100}$ for $n > N_{\frac{1}{100}}$. Show your work.

$$\left\{\frac{3n+2}{n}\right\}_{n=1}$$

5. (15 pts.) Evaluate $\lim_{n \to \infty} \frac{2n^2 - 4}{3n^2 + 5n + 1}$. Give the reasons for each step.

6. (15 pts.) Evaluate $\lim_{x \to 0} \frac{x^2 + 5x}{x^2 - 2x}$. Give the reasons for each step.

7. (20 pts.) Prove, using the definition of limit of a function, that

$\lim_{x \to 4} (3x - 2) = 10.$

MATH 120 Name ________________

LIMITS - B

1. (11 pts.) Explain why the following sequence does not have a limit. (Hint: Plot the terms on the number line.)

$$\left\{(-1)^n - \frac{1}{n}\right\}_{n=1}$$

2. (12 pts.) Give an example of a nonconstant sequence whose limit is -1. Include reasoning.

3. (12 pts.) Give an example of a function that does not have a limit as $x \to 2$. Include reasoning.

4. (15 pts.) For the following sequence, find the smallest $N_{\frac{1}{100}}$ such that $\left|a_n - 3\right| < \frac{1}{100}$ for $n > N_{\frac{1}{100}}$. Show your work.

$$\left\{\frac{6n+1}{2n}\right\}_{n=1}$$

5. (15 pts.) Evaluate $\lim\limits_{n \to \infty} \dfrac{3n^3 - 2n^2 + 8}{4n^3 + n - 5}$. Give the reasons for each step.

6. (15 pts.) Evaluate $\lim\limits_{x \to 0} \dfrac{-x^2 + 6x}{2x^2 + x}$. Give the reasons for each step.

7. (20 pts.) Prove, using the definition of limit of a function, that

$\lim\limits_{x \to 3} (4x - 1) = 11.$

APPENDIX V: Formative Evaluation - Treatment A

MATH 120 Name ________________

LIMITS - C

1. (11 pts.) Explain why the following sequence does not have a limit. (Hint: Plot the terms on the number line.)

$$\left\{(-2)^n + \frac{1}{n}\right\}_{n=1}$$

2. (12 pts.) Give an example of a nonconstant sequence whose limit is -3. Include reasoning.

3. (12 pts.) Give an example of a function that does not have a limit as $x \to 3$. Include reasoning.

4. (15 pts.) For the following sequence, find the smallest $N_{\frac{1}{100}}$ such that $\left|a_n - 3\right| < \frac{1}{100}$ for $n > N_{\frac{1}{100}}$. Show your work.

$$\left\{\frac{3n - 3}{n}\right\}_{n=1}$$

5. (15 pts.) Evaluate $\lim_{n \to \infty} \frac{-4n^2 + n + 16}{2n^2 - 5}$. Give the reasons for each step.

6. (15 pts.) Evaluate $\lim_{x \to 0} \frac{x^3 + 3x^2 - 2x}{2x^3 - 5x}$. Give the reasons for each step.

7. (20 pts.) Prove, using the definition of limit of a function, that

$$\lim_{x \to 1} (7x - 5) = 2.$$

MATH 120 Name ______________

LIMITS - D

1. (11 pts.) Explain why the following sequence does not have a limit. (Hint: Plot the terms on the number line.)

$$\left\{(-2)^n - \frac{1}{n}\right\}_{n=1}$$

2. (12 pts.) Give an example of a nonconstant sequence whose limit is -5. Include reasoning.

3. (12 pts.) Give an example of a function that does not have a limit as $x \to 4$. Include reasoning.

4. (15 pts.) For the following sequence, find the smallest $N_{\frac{1}{100}}$ such that $\left|a_n - 3\right| < \frac{1}{100}$ for $n > N_{\frac{1}{100}}$. Show your work.

$$\left\{\frac{9n+1}{3n}\right\}_{n=1}$$

5. (15 pts.) Evaluate $\lim\limits_{n \to \infty} \frac{6n^3 + 10n - 1}{7n^3 + 4n^2 - 2n}$. Give the reasons for each step.

6. (15 pts.) Evaluate $\lim\limits_{x \to 0} \frac{3x^2 + 2x}{-2n^2 + 5x}$. Give the reasons for each step.

7. (20 pts.) Prove, using the definition of limit of a function, that

$$\lim_{x \to 6} (4x + 5) = 29.$$

APPENDIX VI: Summative Evaluation - Treatment A

MATH 120 -- Limits Name ________________

1. (a) (12 pts.) Give an example of a nonconstant function f(x) such that $\lim_{x \to 2} f(x) = 5$. Include explanation.

   (b) (12 pts.) Give an example of a sequence of positive terms that does not have a limit. Include explanation.

   (c) (11 pts.) Explain why the following function does not have a limit as $x \to -1$.

$$f(x) = \begin{cases} x + 1, & x \leq -1 \\ 3, & x > -1 \end{cases}$$

2. (15 pts.) For the following function, find $\delta_{\frac{1}{1000}}$ such that

$$|f(x) - 13| < \frac{1}{1000} \text{ when } 0 < |x - 6| < \delta_{\frac{1}{1000}} .$$

   $f(x) = 2x + 1, \ a = 6, \ L = 13.$

3. (15 pts.) Evaluate $\lim_{n \to \infty} \frac{-2n^2 + n - 3}{4n^2 - 5}$. Indicate reasons for each step.

4. (15 pts.) Evaluate $\lim_{x \to 2} \frac{x^3 - 4x}{x^3 + x^2 - 6x}$. Indicate reasons for each step.

5. (20 pts.) Prove, using the definition of limit of a sequence, that

$$\lim_{n \to \infty} \frac{n - 3n^2}{n^2} = -3.$$

APPENDIX VII: Instructional Alternative Worksheets

WORKSHEET 1

Explain why the following do not have limits:

1. $\left\{(-1)^n\right\}_{n=1}$

2. $\left\{(-2^n)\right\}_{n=1}$

3. $\left\{3^n\right\}_{n=1}$

4. $\left\{-2^n + \frac{1}{n}\right\}_{n=1}$

5. $f(x) = \begin{cases} 2, & x \leq 5 \\ 3, & x > 5 \end{cases}; \quad a = 5$

6. $f(x) = \begin{cases} x+1, & x \leq -3 \\ 2x, & x > -3 \end{cases}; \quad a = -3$

7. $f(x) = \begin{cases} \frac{1}{x-2}, & x \neq 2 \\ 6, & x = 2 \end{cases}; \quad a = 2$

Find a sequence(s) with the following limits:

1. 3

2. -2

3. 0

WORKSHEET 2

In the following problems, for the given ϵ, find the corresponding N_ϵ or δ_ϵ.

1. $\epsilon = \frac{1}{10}, \quad \left\{\frac{2n+1}{n}\right\}_{n=1} \quad L = 2$

2. $\epsilon = \frac{1}{1000}, \quad \left\{\frac{-3n+4}{2n}\right\}_{n=1} \quad L = -\frac{3}{2}$

3. $\epsilon = \frac{1}{100}, \quad f(x) = 3x+1, \ a = 2, \ L = 7$

4. $\epsilon = \frac{1}{50}, \quad f(x) = -2x, \ a = 0, \ L = 0$

5. $\epsilon = \frac{1}{200}, \quad f(x) = -3x+1, \ a = 2, \ L = -5$

WORKSHEET 3

Evaluate the limits of the following:

1. $\left\{\dfrac{3n^3 - 2n^2}{-2n^3 + n - 1}\right\}_{n=1}$

2. $f(x) = \dfrac{x^2 - 9}{x - 3}$, $a = 3$

3. $f(x) = \dfrac{4x^4 + 3x^3 - x^2}{2n^4 - 3x^2}$, $a = 0$

4. $\left\{\dfrac{2n^2 + 1}{-4n^3 + 2n - 5}\right\}_{n=1}$

5. $\left\{\dfrac{n + 1}{n}\right\}_{n=1}$

6. $f(x) = \dfrac{x^2 + 6x + 9}{x^2 + 5x + 6}$, $a = -3$

7. $f(x) = \dfrac{x^2 + 6x + 9}{x^2 + 5x + 6}$, $a = 0$

WORKSHEET 4

Using the definition of limit, prove the following:

1. $\lim_{n \to \infty} \frac{n+1}{n} = 1$

2. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$

3. $\lim_{n \to \infty} \frac{-3n + 4}{2n} = -\frac{3}{2}$

4. $\lim_{x \to 2} 4x - 5 = 3$

5. $\lim_{x \to 0} -2x = 0$

6. $\lim_{n \to \infty} \frac{2n+1}{n} = 2$

7. $\lim_{n \to \infty} \frac{1}{3n} = 0$

APPENDIX VIII: Attendance Summary for Alternative Instruction Sessions

Session	Date	Time	Number Attending
Group Lecture	Friday	8:00 a.m.	22
	Friday	9:00 a.m.	31
Small group sessions	Monday	8:00 a.m.	10
	Monday	9:00 a.m.	23
	Monday	7:00 p.m.	29
	Tuesday	8:00 a.m.	24
	Tuesday	9:00 a.m.	27

TABLE A.1

Attendance Summary for Alternative Instruction Sessions

APPENDIX IX: Calculus Tests

MATH 202: Test IA

1. Prove: $\lim_{x \to c} (ax + b) = ac + b$

2. Find $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 - 4x + 3}$. Use only one step each time a limit concept is used and give the reason for that step.

For problems 3, 4, and 5, give an example of a function f such that:

3. f has a right hand lim at 2, f has a left hand lim at 2, but $\lim_{x \to 2} f$ does not exist.

4. $\lim_{x \to a} f$ exists and is positive for all real numbers, a.

5. $\lim_{x \to a} f$ exists and $= f'(a)$. (Give both f and a)

6. (a) Give the definition of : "f is continuous at a if..."

   (b) Tell where f is continuous and where not continuous if

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

7. (a) $(3x^3 - 2x^2 + x - 4 - \frac{2}{x})' =$

   (b) $((x^2 - 3x) \cdot (x^3 + 5))' =$

   (c) $\frac{x^3 + 4x}{2x + 7} =$

   (d) $((x^2 - 3x + 4)^{10})' =$

MATH 202: Test 1B

1. Prove: $\lim_{x \to c} (ax + b) = ac + b$

2. Give one reason for each step using a limit concept and evaluate

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 - 4x + 3}$$

For problems 3, 4, and 5, give an example of a function f such that:

3. $\lim_{x \to a} f$ exists and is positive for all real numbers, a.

4. $\lim_{x \to 2} +f$ exists, $\lim_{x \to 2} -f$ exists, but $\lim_{x \to 2} f$ does not exist

5. $\lim_{x \to a} f$ exists and $= f'(a)$. (Give both f and a)

6. (a) Define: "f is continuous at a if..."

   (b) Tell where f is continuous and where not continuous and give reasons for your conclusions if...

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

7. (a) $(4x^3 - x^2 - 2x + 5 - \frac{3}{x})' =$

   (b) $((x^2 + 2x) \cdot (x^3 - 7))' =$

   (c) $\frac{x^3 - 3x}{4x + 2}' =$

   (d) $((x^2 - 5x + 1)^{12})' =$

MTH 202: Test 1C

1. Prove: $\lim_{x \to c} (ax + b) = ac + b$

2. Give one reason for each step using the limit concept and evaluate:

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 - 4x + 3}$$

For problems 3, 4, and 5, give an example of a function f such that:

3. $\lim_{x \to 3^+} f$ exists, $\lim_{x \to 3^-} f$ exists, but $\lim_{x \to 3} f$ does not exist

4. $\lim_{x \to a} f$ exists and is positive for all real numbers, a

5. $\lim_{x \to a} f$ exists and $= f'(a)$ (Give both f and a)

6. (a) Define: "f is continuous at a if..."

   (b) Tell where f is and is not continuous and give reasons for your conclusions if

$$f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1} & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

7. (a) $(5x^3 - 3x^2 + x - 1 + \frac{6}{x})' =$

   (b) $((x^2 + 9x) \cdot (x^3 - 4))' =$

   (c) $\left(\frac{x^3 - 6x}{2x + 1}\right)' =$

   (d) $((x^2 - 8x - 8)^{15})' =$

MTH 202: Test 1D

1. Prove: If n is a negative interger, then

$$(x^n)' = nx^{n-1}$$

2. $\lim_{x \to 4} \frac{x^2 - 7x + 12}{x^2 - 5x + 4} =$

Evaluate using one step each time a limit concept is used. Give that concept.

For problems 3, 4, and 5, give an example of a function f such that:

3. $\lim_{x \to 1^+} f$ exists, $\lim_{x \to 1^-} f$ exists, $\lim_{x \to 1} f$ does not exist, and $f(x) \neq GI(x)$

4. $\lim_{x \to a} f$ exists and is negative for all real a

5. $\lim_{x \to a} f$ exists and $= f'(a)$ (Give both f and a)

6. (a) Define: the derivative of a function, f, at x.

(b) Use the definition of derivative to find $f'(x)$ if $f(x) = \sqrt{x}$

(c) Evaluate $\lim_{t \to 0} \frac{(x + t)^{1/5} - x^{1/5}}{t}$

7. (a) $(3x^2 + 2x + 5 + \frac{2}{x})' =$

(b) $((x^2 - 3x) \cdot (3x^3 + 5))' =$

(c) $\left(\frac{x^2 + 2}{3x + 5}\right)' =$

(d) $((4x^2 - 3x + 7)^{10})' =$

MTH 202: Test 1E

1. Prove: If n is a negative integer, then

$$(x^n)' = nx^{n-1}$$

2. Evaluate using one step each time a limit concept is used. Give the reason to find

$$\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 6x + 5}$$

For problems 3, 4, and 5, give an example of a function f such that:

3. $\lim_{x \to 4^+} f$ exists, $\lim_{x \to 4^-} f$ exists, $\lim_{x \to 4} f$ does not exist, and f(x) is not the GI function

4. $\lim_{x \to a} f$ exists and is negative for all real a

5. $\lim_{x \to a} f$ exists and $= f'(a)$ (Give both f and a)

6. (a) Define the derivative of a function, f, at x

(b) Use this definition to find $f'(x)$ if $f(x) = \frac{1}{x}$

(c) Evaluate $\lim_{w \to 0} \frac{\sqrt[3]{x + w} - \sqrt[3]{x}}{w}$

7. (a) $(4x^2 - 3x + 4 + \frac{1}{x})' =$

(b) $((x^2 + 5x) \cdot (2x^3 + 7))' =$

(c) $\left(\frac{x^2 - 7}{9x + 4}\right)' =$

(d) $((7x^2 - 8x + 5)^{12})' =$

APPENDIX X: Description of Random Selection Procedure for Interviewees

After deciding upon the number of students to be interviewed in each group, the following procedure was utilized:

(1) the students in each group were numbered according to their alphabetical listing.

(2) a page and a column in the random number tables was selected by a random number.

(3) from the randomly selected starting place, the column was read down until the desired number of digits were located (for example, if twenty-one students were in the list and five were needed for interviews, one would continue down the column until locating five digits (numbers) less than or equal to twenty one).

(4) using these numbers, the corresponding names in the list were selected for interviews.

APPENDIX XI: Interview Questions

NAME ____________________________ DATE ______________________

GROUP ___________________________

1. Status at C.M.U. ______________________________

2. High School attended ________________________________

   Size A B C D Other ________________

3. Class rank _______________________

4. Mathematics courses taken prior to Precalculus

5. Why did you take Precalculus?

6. How are you doing in Precalculus?

7. How do you like Precalculus?

8. Is Precalculus harder or easier than you expected?________In what ways?

9. (Where applicable) What is your reaction to the section on limits?

10. Do you presently plan on taking Calculus I? Why or why not?

11. Did you previously plan on taking Calculus I?

12. If not included in number 10 above, indicate reasons for differences if they exist in number 10 and number 11.

13. How difficult do you think Calculus I will (would) be compared to other mathematics courses you have taken?

14. How do you think you will do in Calculus I? (Where applicable)

Interviewers Reaction

Briefly discuss your conception of student anxieties or lack of anxieties regarding Calculus I.

What is your opinion of the student's attitude toward Calculus I?

APPENDIX XII: Questionnaire Given to Calculus Students

Name ________________________________

Name of Precalculus instructor (if any) ______________________________

Semester Precalculus was taken (if any) ______________________________

In the following, please check the appropriate response:

1. Compared to what I expected, Precalculus was:

    _____a. much harder _____b. harder _____c. about the same
    _____d. easier _____e. much easier

2. So far, compared to what I expected, Calculus I is:

    _____a. much harder _____b. harder _____c. about the same
    _____d. easier _____e. much easier

3. Compared to other material you have studied, limits are:

    _____a. most difficult _____b. somewhat hard _____c. about average
    _____d. easy _____e. very easy

4. I have spent the following number of hours per week (outside of class) studying limits so far this semester:

    ____a. 0-5 ____b. 6-10 ____c. 11-15 ____d. 16-20 ____e. Other (please specify)

In some instances, one can distinguish between degree of difficulty of a subject and the level of frustration encountered when studying the subject. With regard to this, the following question is an attempt to discover how frustrating the study of limits is rather than how hard it is.

5. I find the study of limits this semester:

    _____a. very frustrating _____b. more frustrating than average
    _____c. average _____d. less frustrating than average
    _____e. no frustration

APPENDIX XIII: Interview Responses Tabulated for Various Groupings of Cells of the Experimental Model

Item	6			7			8				9			10-11-12	13				14		
Grouping	AA	A	BA	P	NU	N	MH	H	S	E	P	NU	N		MH	H	E	S	AA	A	BA
NEM	2	0	0	0	2	0	0	0	0	2	2	0	0	0	0	2	0	0	2	0	0
NDM	3	0	2	2	3	0	0	3	1	1	4	0	1	1	2	0	2	0	4	0	0
NNM	0	2	3	0	1	4	0	2	2	1	1	2	2	1	2	1	0	0	1	2	0
SM	3	0	0	2	1	0	0	0	2	1	2	1	0	0	0	2	1	0	3	0	0
SNM	2	2	1	3	1	1	0	2	1	2	1	3	1	2	2	2	1	0	2	2	0
Control	7	5	1	7	3	5	0	8	2	3	not applicable			0	0	1	6	3	8	2	0
Treatment A	5	2	5	2	6	4	0	5	3	4	7	2	3	2	4	3	2	0	7	2	0
Treatment B	5	2	1	5	2	1	0	2	3	3	3	4	1	2	2	4	2	0	5	2	0
Control	7	5	1	7	3	3	0	8	2	3	not applicable			0	0	1	6	3	8	2	0
Treatment	10	4	6	7	8	5	0	7	6	7	10	6	4	4	6	7	4	0	12	4	0
Control	7	5	1	7	3	3	0	8	2	3	not applicable			0	0	1	6	3	8	2	0
Mastery	8	0	2	4	6	0	0	3	3	4	8	1	1	1	2	4	3	0	9	0	0
Nonmastery	2	4	4	3	2	5	0	4	3	3	2	5	3	3	4	3	1	0	3	4	0
Control	7	5	1	7	3	3	0	8	2	3	not applicable			0	0	1	6	3	8	2	0
Delayed mastery	3	0	2	2	3	0	0	3	1	1	4	0	1	1	2	0	2	0	4	0	0
Other mastery	5	0	0	2	3	0	0	0	2	3	4	1	0	0	0	4	1	0	5	0	0
Nonmastery	2	4	4	3	2	5	0	4	3	3	2	5	3	3	4	3	1	0	3	4	0
Control	7	5	1	7	3	3	0	8	2	3	not applicable			0	0	1	6	3	8	2	0

TABLE A.2

Interview Responses

APPENDIX XIV: Cell Components* of Chi Square Values

for Questionnaire Items

Grouping: Treatment A -- Treatment B -- Control
Critical value (α = .05): 9.4

Q 2 (χ^2 = 15.44)

Grouping	a and b	c	d and e
TR-A	2 7.9 4.1	17 11.9 2.2	5 4.4 .1
TR-B	5 8.9 1.8	15 13.9 .1	8 5.1 1.6
Control	35 25.3 3.7	33 39.2 1.0	11 14.5 .8

Q 5 (χ^2 = 15.92)

Grouping	a and b	c	d and e
TR-A	6 12.1 3.1	10 6.4 2.0	8 5.5 1.1
TR-B	13 14.1 .1	4 7.5 1.6	11 6.4 3.3
Control	47 39.8 1.3	21 21.1 0.0	11 18.1 2.8

*The numbers in each cell are observed values, expected values, and chi square partial values.

TABLE A.3

Chi Square Values for Questionnaire Items

Grouping: Treatment -- Control
Critical value (α = .05): 5.99

Q 2 (χ^2 = 13.87)

Grouping	a and b	c	d and e
Treatment	7 16.7 5.6	32 25.8 1.5	13 9.5 1.3
Control	35 25.3 3.7	33 39.2 1.3	11 14.5 .8

Q 4 (χ^2 = 6.97)

Grouping	a and b	c	d and e
Treatment	27 22.6 0.8	24 24.1 0.0	1 5.2 3.4
Control	30 34.4 0.6	37 36.8 0.0	12 7.8 2.3

Q 5 (χ^2 = 10.28)

Grouping	a and b	c	d and e
Treatment	19 26.2 2.0	14 13.9 0.2	19 11.9 4.2
Control	47 40.0 1.3	21 21.1 0.0	11 18.1 2.8

TABLE A.3 (continued)

Grouping: Mastery -- Nonmastery -- Control
Critical value ($\alpha = .05$): 9.4

Q 1 ($\chi^2 = 10.57$)

Grouping	a and b	c	d and e
Mastery	8 8.0 0.0	14 13.7 0.0	9 9.2 0.5
Non-mastery	10 5.4 3.9	10 9.3 .1	1 6.2 4.4
Control	16 20.5 1.0	34 35.0 0.0	29 23.5 1.3

Q 2 ($\chi^2 = 16.61$)

Grouping	a and b	c	d and e
Mastery	3 10.0 4.9	18 15.4 0.4	10 5.7 3.2
Non-mastery	4 6.7 1.1	14 10.4 1.2	3 3.8 0.2
Control	35 25.3 3.7	33 39.2 1.0	11 14.5 0.8

Q 5 ($\chi^2 = 11.58$)

Grouping	a and b	c	d and e
Mastery	10 15.6 2.0	8 8.3 0.0	13 7.1 4.9
Non-mastery	9 10.6 0.2	6 5.6 0.0	6 4.8 0.3
Control	47 39.8 1.3	21 21.1 0.0	11 18.1 2.8

TABLE A.3 (continued)

Grouping: Delayed mastery -- Other mastery -- Nonmastery -- Control
Critical value (α = .05): 12.59

Q 1 (χ^2 = 17.57)

Grouping	a and b	c	d and e
Delayed mastery	6 3.7 1.6	6 5.7 0.0	1 3.7 2.1
Other mastery	2 4.7 1.5	8 7.9 0.0	8 5.3 1.4
Non-mastery	10 5.5 3.7	10 9.3 0.1	1 6.2 4.4
Control	16 20.5 1.0	34 35.0 0.0	29 23.5 1.3

Q 2 (χ^2 = 21.72)

Grouping	a and b	c	d and e
Delayed mastery	1 4.2 2.4	10 6.4 2.0	2 2.4 0.0
Other mastery	2 5.8 2.4	8 8.9 0.1	8 3.9 6.7
Non-mastery	4 6.7 1.1	14 10.4 1.2	3 3.8 0.2
Control	35 25.3 3.7	33 39.2 1.0	11 14.5 0.8

Q 3 (χ^2 = 12.62)

Grouping	a and b	c	d and e
Delayed mastery	0 1.7 1.7	11 7.6 1.5	2 3.7 0.8
Other mastery	3 2.3 0.2	6 10.5 1.9	9 5.1 3.0
Non-mastery	4 2.7 0.6	9 12.3 0.9	8 5.9 0.7
Control	10 10.3 0.0	51 46.4 0.5	18 22.3 0.8

Q 5 (χ^2 = 18.53)

Grouping	a and b	c	d and e
Delayed mastery	4 6.5 0.3	6 3.5 1.0	3 3.0 0.0
Other mastery	6 9.0 1.0	2 4.8 1.6	10 4.1 8.5
Non-mastery	9 10.6 0.2	6 5.6 0.0	6 4.8 0.3
Control	47 39.8 1.3	21 21.1 0.0	11 18.1 2.8

TABLE A.3 (continued)

APPENDIX XV: Cell Components* of Chi Square Values for Enrollment Analysis

Group	Fall 1971	Fall 1972	Fall 1973 Treatment	Fall 1973 Control
Number enrolled in Calculus I	173 196.0 2.7	169 162 0.3	54 54 0.0	85 68 4.0
Number not enrolled in Calculus I	265 242 2.1	194 200 0.2	67 67 0.0	67 85 3.0

Group	Nance Early Mastery	Nance Delayed Mastery	Nance Nonmastery	Smith Mastery	Smith Nonmastery	Control
Number enrolled in Calculus I	5 2.5 2.4	13 17.2 1.0	8 11.1 .9	13 10.6 .5	15 20.2 1.3	85 77.6 .7
Number not enrolled in Calculus I	0 2.5 2.5	21 16.8 1.0	13 10.9 .4	8 10.4 .5	25 20.0 1.4	68 75.4 .7

Group	Delayed Mastery	Other Mastery	Nonmastery	Control
Number enrolled in Calculus I	13 17.2 1.0	18 13.2 1.7	23 30.9 2.0	85 77.6 .7
Number not enrolled in Calculus I	21 16.8 1.0	8 12.4 1.6	38 30.1 2.0	68 75.4 .7

*The numbers in each cell are observed values, expected values, and chi square partial values.

TABLE A.4

Chi Square Values for Enrollment Analysis

APPENDIX XVI: Calculus I Test Results and Final Percentages

Group	Student Number	Item Number 1	2	3	4	Test Score	Final Percent
NEM	5		18	5	7	91	76
NEM	6		20	2	6	84	72
NEM	15		20	5	7	88	87
NEM	19	20	17	4	7	95	91
NEM	25		20	6	6	99	99
NDM	1		20	6	7	100	93
NDM	2	20	18	5	7	90	84
NDM	7		20	5	7	98	83
NDM	10		20	4	6	73	60
NDM	11		18	3	4	80	72
NDM	13	18	8	1	5	54	31
NDM	14	18	20	5	7	86	78
NDM	16	20	17	0	6	83	75
NDM	17	20	15	6	4	81	84
NDM	18	20	20	6	7	98	73
NDM	22	19	20	5	6	74	50
NDM	23	14	20	6	3	67	63
NDM	26	20	20	6	7	100	89
NNM	3		17	2	6	73	77
NNM	4	10	14	1	7	56	44
NNM	8		20	5	7	91	81
NNM	9	20	13	4	6	61	W
NNM	12	20	12	3	4	57	40
NNM	20	5	13	5	6	58	58
NNM	21	20	20	2	7	76	48
NNM	24						W
SM	27	20	20	6	6	95	87
SM	31		20	6	5	98	87
SM	34	20	20	6	6	97	77
SM	36	20	20	6	6	99	88
SM	37	20	14	6	7	92	85
SM	39		20	6	5	77	78
SM	40		20	6	7	93	94
SM	42		20	4	5	93	88
SM	43	20	20	6	7	96	91
SM	44		20	6	7	91	67
SM	47		20	6	6	88	49
SM	50		20	6	4	82	68
SM	53	18	17	6	5	92	81

TABLE A.5

Calculus I Test Results and Final Percentages

Group	Student Number	Item Number 1	2	3	4	Test Score	Final Percent
SNM	28		20	4	7	86	81
SNM	29		20	4	4	80	75
SNM	30		20	3	5	87	73
SNM	32		20	4	6	75	74
SNM	33	20	12	4	7	86	78
SNM	35		20	4	6	83	78
SNM	38	20	15	6	6	82	48
SNM	41	20	20	6	7	98	83
SNM	45						73
SNM	46	20	20	5	6	91	63
SNM	48	20	8	0	6	39	W
SNM	49		20	6	7	90	89
SNM	51		17	6	6	69	73
SNM	52		20	5	7	91	72
SNM	54	18	20	4	7	82	78
C	55		20	4	0	69	73
C	56		18	4	6	67	59
C	57	20	18	6	3	93	W
C	58		6	6	5	74	75
C	59	20	20	5	5	93	82
C	60	20	18	3	5	88	67
C	61	20	20	5	7	99	92
C	62		20	3	6	96	91
C	63	20	18	2	6	86	83
C	64		18	5	5	83	76
C	65						85
C	66		10	4	4	75	69
C	67		17	5	2	90	89
C	68		16	4	7	67	61
C	69	20	14	5	4	80	70
C	70	20	10	6	7	88	92
C	71	18	18	5	5	80	75
C	72	20	20	5	7	94	95
C	73	20	20	5	1	83	78
C	74	20	15	5	5	79	75
C	75		20	4	7	94	95
C	76	18	20	3	4	75	68
C	77		6	0	0	61	82
C	78		20	4	6	69	W
C	79	20	18	6	6	97	87
C	80		17	5	2	72	82

TABLE A.5 (continued)

Group	Student Number	Item Number 1	2	3	4	Test Score	Final Percent
C	81	19	12	4	6	60	65
C	82		20	3	6	89	82
C	83	20	17	6	5	91	85
C	84	20	20	6	6	84	90
C	85	20	18	6	6	92	88
C	86	20	20	6	6	91	83
C	87		12	0	2	25	W
C	88	20	18	6	6	93	80
C	89	20	20	6	6	99	90
C	90	20	20	6	5	96	88
C	91	20	20	6	7	96	86
C	92	20	20	4	6	94	76
C	93	20	18	5	7	83	71
C	94		18	6	7	94	85
C	95	20	20	6	6	87	79
C	96	20	17	6	6	90	73
C	97	20	17	0	6	90	97
C	98		20	5	5	87	82
C	99	20	20	2	6	93	87
C	100	16	18	2	1	56	67
C	101	18	20	6	6	89	74
C	102						W
C	103	15	15	4	6	78	69
C	104						W
C	105						77
C	106		20	6	7	100	95
C	107		10	3	5	80	67
C	108	20	20	6	7	98	95
C	109	20	20	3	2	77	61
C	110		20	6	5	98	83
C	111	20	17	3	7	83	80
C	112		20	6	7	100	98
C	113	20	20	6	3	86	81
C	114		8	5	5	57	60
C	115		18	6	1	78	85
C	116	20	20	4	6	86	78
C	117		4	3	3	69	77
C	118		14	3	5	71	67
C	119	14	14	5	5	72	76
C	120		20	5	7	95	83

TABLE A. 5 (continued)

Group	Student Number	Item Number 1	2	3	4	Test Score	Final Percent
C	121		20	6	7	96	91
C	122	20	16	5	7	86	81
C	123		20	6	3	96	92
C	124		20	6	4	88	85
C	125		16	3	7	68	59
C	126		17	0	2	59	64
C	127	18	16	6	7	94	81
C	128		20	6	6	97	86
C	129		17	5	5	85	82
C	130		20	4	6	88	96
C	131		20	5	5	91	86
C	132	2	17	6	6	49	W
C	133		20	6	7	100	91
C	134		0	4	7	37	70
C	135	17	20	2	5	85	75
C	136		20	6	6	93	85
C	137		17	3	0	79	71
C	138						75
C	139		20	3	5	89	80

TABLE A.5 (continued)

APPENDIX XVII: Contingency Tables* for Chi Square Test Statistics Computed for the Sum of Items Two, Three, and Four the Calculus I Test

Grouping	Score 0-29	Score 30-33	
NEM	2 2.7 .2	3 2.4 .2	5 .0382
NDM	5 6.9 .5	8 6.2 .5	13 .0992
NNM	6 3.7 1.4	1 3.3 1.6	7 .0534
SM	3 6.9 2.2	10 6.2 2.3	13 .0992
SNM	6 7.5 .3	8 6.6 .3	14 .1069
Control	48 42.7 .7	32 37.9 .9	80 .6107
Total	70	62	132

$\chi^2 = 11.3$**

$\chi^2_{.05}(5) = 11.07$

Grouping	Score 0-29	Score 30-33	
Treatment	22 27.6 1.126	30 24.4 1.274	52 .3939
Control	48 42.4 .732	32 37.6 .828	80 .6061
Total	70	62	132

$\chi^2 = 3.960$

$\chi^2_{.05}(1) = 3.841$

*Each cell contains observed values, expected values, and chi square partial sums.

**Only sixty-seven percent of the cells have expected values exceeding five.

TABLE A.6

Chi Square Values for Calculus I Test

Grouping	Score 0-29	Score 30-33	
Mastery	10 16.4 2.5	21 14.6 2.8	31 .2348
Non-mastery	12 11.1 .1	9 9.9 .1	21 .1591
Control	48 42.4 .7	32 37.6 .8	80 .6061
Total	70	62	132

$\chi^2 = 7$

$\chi^2_{.05}(2) = 5.99$

Grouping	Score 0-29	Score 30-33	
Delayed mastery	5 6.9 .5	8 6.1 .6	13 .0985
Other mastery	5 9.5 2.1	13 8.5 2.4	18 .1364
Non-mastery	12 11.1 .1	9 9.9 .1	21 .1591
Control	48 42.4 .7	32 37.6 .8	80 .6061
Total	70	62	132

$\chi^2 = 7.3$

$\chi^2_{.05}(3) = 7.8$

Grouping	Score 0-29	Score 30-33	
Treatment A	13 13.258 0.0	12 11.724 0.0	25 .1894
Treatment B	9 15.379 2.646	20 13.621 2.987	29 .2197
Control	48 42.7 .7	32 37.9 .9	80 .6107
Total	70	62	132

$\chi^2 = 7.233$

$\chi^2_{.05}(2) = 5.99$

TABLE A.6 (continued)

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