This is to certify that the

thesis entitled

On the Homology of
Local Cohen-Macaulay Rings

presented by

Alan B. Evans

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

W. T. Sledd
Major professor

Date July 25, 1979

ON THE HOMOLOGY OF LOCAL COHEN-MACAULAY RINGS

By

Alan B. Evans

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1979

ABSTRACT

ON THE HOMOLOGY OF LOCAL COHEN-MACAULAY RINGS

By

Alan B. Evans

Let (A,m,k) be a local Noetherian ring. The Poincaré series of A is the formal power series

$$P_A^k(z) = \sum_{i=0}^{\infty} \dim_k(\mathrm{Tor}_i^A(k,k))z^i.$$

A well-known conjecture is that $P_A^k(z)$ is a rational function of z.

In this paper, several formulas giving P_A^k are derived in the case of A Cohen-Macaulay, that is dimension(A) = depth(A). The embedding dimension of a Cohen-Macaulay ring A is less than or equal to $e(A) + \dim(A) - 1$, where $e(A)$ is the multiplicity. When equality holds, P_A^k is shown to be rational in Chapter I.

A perfect ideal I of A is an almost complete intersection when I is minimally generated by grade(I) $+ 1$ elements. In Chapter II, the homology of the Koszul complex of a regular local ring modulo certain almost complete intersections of grade three is computed. From this, a formulá for P_A^k is obtained. Furthermore, if I

is a perfect ideal of the regular local ring A such that I/I^2 has a free direct summand (as A/I-module) of rank equal to $\text{grade}(I) + 2$, $P^k_{A/I}$ is computed.

In the last chapter, examples are given which show that a composition of Golod homomorphisms need not be a Golod homomorphism and that it does not seem possible to characterize the class of Cohen-Macaulay rings using the deviations, $\epsilon_i(A)$.

Dedicated To

Rachael

ACKNOWLEDGMENTS

This paper was written under the guidance of Professor Wei-Eihn Kuan. I would like to thank Professor Kuan for his constant encouragement and Professor R.C. Cowsik for several helpful discussions during its preparation. My special thanks and appreciation go to Mary Reynolds for a superb job of typing the manuscript.

TABLE OF CONTENTS

CHAPTER 0

INTRODUCTION

Let (A,m,k) be a local (Noetherian) ring with maximal ideal m and residue field $k = A/m$, and let M be a finite A-module. A <u>projective resolution</u> of M over A is a (perhaps infinite) exact sequence of finite projective A-modules

$$(1) \qquad \cdots \to P_{i+1} \xrightarrow{d_{i+1}} P_i \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0.$$

See [9], p.75. M is said to have <u>projective dimension</u> <u>n over A</u> if there is a projective resolution

$$(2) \qquad 0 \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

of length n, but none shorter. The projective dimension of M is infinite if no finite resolution exists. Since A is local, the modules P_i may be assumed free [33], 3.G. Moreover, k serves as a test module for the projective dimension of M over A (abbreviated hereafter as $\mathrm{pd}_A(M)$). Precisely $\mathrm{pd}_A(M) \leq n$ if and only if $\mathrm{Tor}^A_{n+1}(M,k) = 0$, where $\mathrm{Tor}^A_i(M,k)$, the ith left derived functor of $\otimes$ [8], p.107, is computed as the ith homology of the complex obtained by tensoring the resolution (1) with k.

A free resolution of M

$$(3)\qquad \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to M \to 0$$

is <u>minimal</u> if $d_i(F_i) \subseteq mF_{i-1}$ for $i \geq 1$. In this case, $\mathrm{Tor}_i^A(M,k) \cong F_i \otimes_A k$ [33], 18.E. It can be proved that a minimal resolution always exists [33], Ex. 3, p.113. As a consequence, each $\mathrm{Tor}_i^A(M,k)$ is a finite vector space over k. The <u>ith Betti number</u> of M is defined then to be $b_i = \dim_k(\mathrm{Tor}_i^A(M,k))$. The formal power series $P_A^M(z) = \sum_{i=0}^{\infty} b_i z^i$ is known as the <u>Poincaré series</u> of M. With this notation, $\mathrm{pd}_A(M) < \infty$ if and only if $P_A^M(z)$ is a polynomial. It has been conjectured that P_A^M is in fact a rational function of z for all M.

The homological properties of the ring A are connected with problems in geometry. The <u>dimension</u> of A is the length of the largest proper chain of prime ideals $m = \mathfrak{p}_n \supset \mathfrak{p}_{n-1} \supset \cdots \supset \mathfrak{p}_1 \supset \mathfrak{p}_0$ [33], p.71. A is called a <u>regular</u> local ring if m can be generated by elements $x_1,\ldots,x_d$, where d = dimension of A. Regular local rings correspond roughly to non-singular points on an algebraic variety [34], p.343. For a local ring (A,m,k) of dimension n, the following are equivalent

(a) A is regular

(b) $\mathrm{pd}_A(k) = n$

(c) $\mathrm{pd}_A(M) \leq n$ for all finite M

(d) P_A^k is a polynomial

(e) $P_A^k(z) = (1+z)^n$.

[33], Theorems 41, 42 and 45.

Closest to regular local rings from a homological standpoint are the complete intersections. A sequence of elements $x_1,\ldots,x_r \in m$ are a <u>regular sequence</u> if for $1 \leq i \leq r$, x_i is not a zero-divisor on $A/(x_1,\ldots,x_{i-1})$. A <u>local complete intersection</u> is a local ring which can be written as a homomorphic image of a regular local ring modulo a regular sequence. The homological characterization is

(a) A a complete intersection

if and only if

(b) $P_A^k(z) = \dfrac{(1+z)^n}{(1-z^2)^m}$, $n,m \geq 0$ integers.

See [46]. Most progress on the problem of the rationality of P_A^k since Tate's paper [46] has been based on the elegant result of Golod [20]. There, the Poincaré series is related to homology operations on $H(K)$ known as Massey products. K denotes the Koszul complex over A, whose underlying graded algebra is the exterior algebra on a minimal generating set for m [33], 18.D. Regular local rings and complete intersections may also be classified using the Koszul complex. Namely, A is regular if and only if $H_1(K) = 0$ [24], Theorem 1.4.13 and A is a complete intersection if and only if $H(K)$ is the exterior algebra over $H_1(K)$ [24], Theorem 3.5.2.

Of frequent use is the fact that P_A^k has a (unique) representation as an infinite product

$$P_A^k(z) = \prod_{i=0}^{\infty} \frac{(1+z^{2i+1})^{\epsilon_{2i}}}{(1-z^{2i+2})^{\epsilon_{2i+1}}} .$$

This was first proved by Assmus using the Hopf algebra structure on $\mathrm{Tor}^A(k,k)$ [2]. The exponents $\epsilon_i = \epsilon_i(A)$ are non-negative integers and are called the <u>deviations</u> of A. It turns out that $\epsilon_0(A) = \dim_k(m/m^2)$, the so-called embedding dimension. Also, if A is a homomorphic image of a regular local ring (R,n), with $A \cong R/\mathfrak{A}$ and $\mathfrak{A} \subseteq n^2$, then $\epsilon_1(A) = \dim_k(\mathfrak{A}/n\mathfrak{A}) = \dim_k(H_1(K))$, where K is the Koszul complex [24], Lemma 1.4.15 and Prop. 3.3.4.

The general rationality problem for an A-module M has been reduced to the case $M = k$, A of dimension zero by Ghione and Gulliksen [19].

In the present paper, three formulas for P_A^k are obtained. The results deal with local rings which are Cohen-Macaulay, a concept which will be explained shortly.

Suppose M is a finite A-module. Then $x_1,\ldots,x_r \in m$ form a <u>regular sequence on M</u> if for $1 \leq i \leq r$, x_i is not a zero-divisor on $M/(x_1M+\ldots+x_{i-1}M)$. The <u>depth</u> of M, written $\mathrm{depth}_A(M)$, is the length of the longest regular sequence on M [33], 15.C. The <u>dimension</u> of M is the dimension of the ring $A/\mathrm{ann}(M)$, where $\mathrm{ann}(M) = \{x \in A \mid xM = 0\}$ is the <u>annihilator</u> of M. In general, the depth of M is at most equal to the dimension of M [33], Theorem 27, and M is defined to be <u>Cohen-Macaulay</u> when depth(M) = dimension(M). A itself is a <u>Cohen-Macaulay ring</u> if it is Cohen-Macaulay as A-module [33], 16.A. The hierarchy of the types of local rings introduced thus far is

regular $\Rightarrow$ complete intersection $\Rightarrow$ Cohen-Macaulay.

The notation used is standard. The minimal number of generators of the finite A-module M will be denoted by $\mu(M)$. If $\mathfrak{q} \subseteq \mathfrak{m}$ is an open ideal ($\mathfrak{m}^n \subseteq \mathfrak{q}$ for some n), then $e(A,\mathfrak{q})$ = the multiplicity [48], p.294 of $\mathfrak{q}$, with $e(A)$ short for $e(A,\mathfrak{m})$. Let $d_A(M)$ = dimension of M and $d(A)$ = dimension of A itself. For an ideal $I \subseteq \mathfrak{m}$, let $ht(I)$ be the height of I, the infimum of the lengths of saturated chains of prime ideals $\mathfrak{p}_n \supset \mathfrak{p}_{n-1} \supset \ldots \supset \mathfrak{p}_0$, $\mathfrak{p}_n$ a minimal prime containing I [33], p.71. If N is an A-module of finite length [3], p.77, then $\ell(N)$ = length of N.

For an ideal $I \subseteq \mathfrak{m}$, the <u>grade</u> of I is the length of the longest regular sequence on A which is contained in I. Given any ideal, grade$(I) \leq ht(I)$, with equality whenever A is Cohen-Macaulay [18], 11.15. Later, a distinction will be made between the grade of I and depth$_A(I)$, the depth of I as A-module. See [18], 21.7. Finally, a <u>system of parameters</u> for A is a sequence $x_1,\ldots,x_d$, $d = d(A)$, of elements from $\mathfrak{m}$ such that $\mathfrak{m}^n \subseteq (x_1,\ldots,x_d)$ for some n. An alternate characterization of the Cohen-Macaulay property is the requirement that every system of parameters for A form a regular sequence [18], 11.15.

When the superscript is omitted, P_A will be understood to mean P_A^k, and will be called the Poincaré series of A.

CHAPTER I

COHEN-MACAULAY RINGS OF MAXIMAL EMBEDDING DIMENSION

Let (A,m,k) be a local, Cohen-Macaulay ring. Under the assumption that k is infinite, Abhyankar showed that $\epsilon_0(A) \leq e(A) + d(A) - 1$ [1]. This restriction on k is not important for the study of the Poincaré series. Define $A^* = A[X]_{m[X]}$, the localization of the polynomial ring over A in one variable at the prime ideal $m[X] = \{f = \sum_i a_i X^i \mid a_i \in m\}$. By passing from A to A^*, the residue field may be assumed infinite [38], Ch. IV. From [24], Prop. 1.9.8 and Lemma 3.1.2, it follows that $\epsilon_i(A^*) = \epsilon_i(A)$ for $i \geq 1$, and $\epsilon_0(A^*) = \epsilon_0(A)$, by a result of Lech [29], Lemma 2, p.75. Thus from the infinite product representation, $P_A = P_{A^*}$.

Now, $\epsilon_0(A) = e(A) + d(A) - 1$ if and only if $(x_1,\ldots,x_d)m = m^2$ for some $x_1,\ldots,x_d \in m$, $d = d(A)$ [41], Theorem 1. In the case $d = 1$, m is said to be <u>stable</u> [31], and the rationality of the Poincaré series has been established in [13]. The following is an extension of this result to higher dimensions.

<u>Theorem 1.1</u>. Let (A,m,k) be a local Cohen-Macaulay ring of dimension $d \geq 1$ with $\varepsilon_0(A) = e(A) + d(A) - 1$. Then

$$P_A(z) = \frac{(1+z)^d}{1-(\varepsilon_0(A)-d)z} .$$

<u>Proof</u>: As mentioned earlier, the case of interest is $d \geq 2$. Assume $(x_1,\ldots,x_d)m = m^2$. First, note that $x_1,\ldots,x_d$ form a system of parameters for A, because

$$\begin{aligned}\sqrt{(x_1,\ldots,x_d)m} &= \sqrt{(x_1,\ldots,x_d)} \cap \sqrt{m} \\ &= \sqrt{(x_1,\ldots,x_d)} = \sqrt{m^2} = m\end{aligned}$$

[3], p.9. Next, at least one of the x_i must lie in $m \setminus m^2$. Otherwise, $m^3 \supseteq (x_1,\ldots,x_d)m = m^2$, contradicting Krull's intersection theorem [33], Cor. 2, p.69. Now, for $1 \leq i \leq d-1$, the dimension of $A/(x_1,\ldots,x_i)$ is $d-i$ [33], 12.K. Moreover, $(\bar{x}_{i+1},\ldots,\bar{x}_d)\bar{m} = (\bar{m})^2$, where the bar denotes the residue class in $A/(x_1,\ldots,x_i)$. Because $x_1,\ldots,x_d$ form a regular sequence [18], 11.15, $A/(x_1,\ldots,x_i)$ is again Cohen-Macaulay [33], p.104. By induction on d, $\bar{x}_{i+1}$ lies in $\bar{m} \setminus (\bar{m})^2$, hence $x_{i+1} \in m \setminus m^2$. Therefore all of $x_1,\ldots,x_d$ lie outside m^2.

Let $\bar{A} = A/(x_2,\ldots,x_d)$. Then $(\bar{x}_1)\bar{m} = (\bar{m})^2$, so that $\bar{m}$ is stable. Because $x_1,\ldots,x_d$ form a regular sequence in $m \setminus m^2$, $P_A(z) = (1+z)^{d-1} P_{\bar{A}}(z)$ [24], Cor. 3.4.2. But since $\bar{m}$ is stable, $P_{\bar{A}}(z) = \dfrac{1+z}{1-(\varepsilon_0(\bar{A})-1)z}$ [13].

Furthermore, reducing modulo each x_i decreases the embedding dimension by one. To see this, note that $x_1,\ldots,x_d$ can be extended to a minimal generating set for m. Suppose that $\sum_{i=1}^{d} c_i x_i = \sum_{j=1}^{n} y_j z_i \in m^2$. Since $(x_1,\ldots,x_d)m = m^2$, $\sum_{i=1}^{d} c_i x_i = \sum_{i=1}^{d} w_i x_i$ with $w_i \in m$. If some $c_i \notin m$, then from $x_i(c_i - w_i) = \sum_{j\neq i} w_j x_j$ it follows that

$$c_i x_i (1 - c_i^{-1} w_i) = \sum_{j\neq i} w_j x_j, \quad \text{and so}$$

$$x_i = c_i^{-1}(1 - c_i^{-1} w_i)^{-1} \sum_{j\neq i} w_j x_j ,$$

contradicting the fact that $x_1,\ldots,x_d$ form a regular sequence. Thus $c_i \in m$ for $i = 1,\ldots,d$ and $x_1,\ldots,x_d$ remain linear independent in m/m^2. Then for $i = 1,\ldots,d$, $x_i \notin m^2$ modulo $(x_1,\ldots,x_{i-1})$. Now using [18], 1.32 $d-1$ times, $\varepsilon_0(\bar{A}) = \varepsilon_0(A) - (d-1)$. Hence

$$P_A(z) = \frac{(1+z)^d}{1-(\varepsilon_0(A)-d)z} ,$$

as claimed.

<u>Remark 1.2.1</u>. A is Cohen-Macaulay if and only if A^* is Cohen-Macaulay [11], Theorem 2. Moreover, the dimension and multiplicity are invariant under the passage from A to A^* [32], Prop. p.277. As noted

earlier, $\varepsilon_0(A) = \varepsilon_0(A^*)$, so A^* has maximal embedding dimension whenever A does.

Remark 1.2.2. If $\varepsilon_0(A) = e(A) + d(A) - 1$, with $\varepsilon_0(A) - d(A) \geq 2$, then A is not a complete intersection, as is easily seen by comparing P_A with the Poincaré series $\frac{(1-z)^n}{(1-z^2)^m}$ of a complete intersection.

Before proceeding, a brief review of projective varieties is in order. Let k be an algebraically closed field of characteristic zero. Projective m-space over k, written $\mathbb{P}_k^m$, is defined to be the set of all points $(x_0,x_1,\ldots,x_m) \neq (0,0,\ldots,0)$, $x_i \in k$, modulo the equivalence relation $(x_0,x_1,\ldots,x_m) \sim (\lambda x_0,\lambda x_1,\ldots,\lambda x_m)$, $\lambda \in k \setminus \{0\}$. A projective variety in $\mathbb{P}_k^m$ is the locus of zeros of a finite set of homogeneous polynomials $f_1,\ldots,f_r \in S = k[X_0,X_1,\ldots,X_m]$ such that $f_1,\ldots,f_r$ generate a prime ideal in S. Associated to each projective variety $X \subseteq \mathbb{P}_k^m$ is its homogeneous coordinate ring, $S/I(X)$, where $I(X)$ consists of all forms in S which vanish identically on X. An ideal I of S is graded if I can be generated by forms. Then projective subvarieties of X correspond to graded prime ideals containing $I(X)$, with the exception of the irrelevant maximal ideal $(X_0,X_1,\ldots,X_m)$, which contains every graded ideal. If one ignores the grading and considers the locus of zeros of $I(X)$ in affine $m+1$ space $\mathbb{A}_k^{m+1}$, one has the cone associated to X, $C(X)$. $C(X)$ is an affine variety

through the origin in $\mathbb{A}_k^{m+1}$ which has dimension one greater than the dimension of X. Furthermore, many geometric properties of the vertex of $C(X)$ are closely linked to the geometry of X back in $\mathbb{P}_k^m$.

Conversely, if R is a graded k-algebra of finite type, there is a geometric object $\text{Proj}(R)$ associated with R [25], p.76. In case R is the homogeneous coordinate domain of a projective variety X, then $\text{Proj}(R) \cong X$.

Let $R = \bigoplus_{i \geq 0} R_i$ be the homogeneous coordinate domain of a variety X of dimension d contained in $\mathbb{P}_k^m$. Then there is a polynomial P_X of degree d with rational coefficients called the <u>Hilbert polynomial</u> of X such that $P_X(n) = \dim_k(R_n)$ for sufficiently large n. The <u>degree of X</u>, written $\deg(X)$, is $d!$ times the leading coefficient of $P_X(n)$ [35], 6.25. The integer $\deg(X)$ tells the number of points in which "most" linear subspaces $L \subseteq \mathbb{P}_k^m$ of dimension $m-d$ meet X [35], Theorem 5.1. At the lower end, $p_a(X) = (-1)^d(P_X(0) - 1)$ is called the <u>arithmetic genus</u> of X and is an important geometric invariant [35], p.115.

Now, in order to produce a Cohen-Macaulay ring of maximal embedding dimension, the following fact will be useful.

<u>Lemma 1.3</u>. Let $X \subseteq \mathbb{P}_k^m$ be a projective variety of dimension d over an algebraically closed field k of characteristic zero. Let (A,m,k) be the local ring at the vertex of the cone $C(X)$. That is, A is the localization of the coordinate ring of X at its maximal ideal $(X_0,X_1,\ldots,X_m)$. Then $e(A) = \deg(X)$.

<u>Proof</u>: Let R be the homogeneous coordinate ring of X. Let $P_X(n)$ be the Hilbert polynomial of X and let $P_A(n)$ be the characteristic polynomial of A [48], Ch. VIII. Consider $\mathrm{Gr}_m(A) = \bigoplus_{i\geq 0} m^i/m^{i+1}$, the <u>associated graded ring</u> of A [33], 10.C. Since A is already graded, $\mathrm{Gr}_m(A) \cong R$, or geometrically, $\mathrm{Proj}(\mathrm{Gr}_m(A)) \cong X$. Therefore, $P_X(n) = \dim_k(m^n/m^{n+1})$ for large n. By definition, $P_A(n) = \sum_{i=0}^{n} \dim_k(m^i/m^{i+1})$ for large n [48], Ch. VIII, Theorem 19. Therefore, $P_X(n+1) = P_A(n+1) - P_A(n)$ for sufficiently large n. But $P_A(n+1) - P_A(n) = P_A'(n) + Q_1(n)$, where P_A' is the formal derivative of P_A with respect to n and $Q_1(n)$ is a polynomial of degree less than $\deg(P_A) - 1$. This means that $P_A(n)$ is the indefinite integral of $P_X(n)$, plus a polynomial $Q(n)$ such that $\deg(Q) < \deg(P_A)$. Now $P_A(n) = \frac{e(A)n^{d+1}}{(d+1)!}$ plus terms of lower degree [48], Ch. VIII, §10, and $P_X(n) = \frac{n^d \deg(X)}{d!}$ plus terms of lower degree [35], 6.25. Therefore, $\frac{e(A)n^{d+1}}{(d+1)!} = \int \frac{n^d \deg(X)}{d!}$, the indefinite integral with respect to n, which of course equals $\frac{n^{d+1}\deg(X)}{(d+1)!}$. Hence $e(A) = \deg(X)$.

<u>Example 1.4</u>. Let Y be the curve in $\mathbb{P}^2$ defined by the equation $X_0X_1 = X_2^2$. $\mathbb{P}^2$ is covered by affine open sets $D_0 = \{X_0 \neq 0\}$, $D_1 = \{X_1 \neq 0\}$, $D_2 = \{X_2 \neq 0\}$ [35], 2A, on which the equations of Y are
$\frac{X_1}{X_0} = (\frac{X_2}{X_0})^2$, $\frac{X_0}{X_1} = (\frac{X_2}{X_1})^2$, and $(\frac{X_0}{X_2})(\frac{X_1}{X_2}) = 1$, respectively. Therefore, Y is smooth by the Jacobian criterion [25], p.31. Consider the Segré embedding $X = S(Y \times \mathbb{P}^1) \subseteq S(\mathbb{P}^2 \times \mathbb{P}^1) \subseteq \mathbb{P}^5$ (see [35], 2B). Since Y is rational, [44], p.6, and since the arithmetic genus is a birational invariant [25], III, Ex. 5.3, $p_a(Y) = p_a(\mathbb{P}^1) = 0$. [25], I, Ex. 7.2. So the Hilbert polynomials are $P_Y(n) = 2n+1$, $P_{\mathbb{P}^1}(n) = n+1$. By a theorem of Seidenberg, $P_X(n) = P_Y(n) \cdot P_{\mathbb{P}^1}(n) = 2n^2 + 3n + 1$ [42], Theorem 2. Thus $\deg(X) = 4$ [35], 6.25. Furthermore, X is arithmetically Cohen-Macaulay [45], Cor. on p.374. That is, (A,m), the local ring at the vertex of the cone on X, is Cohen-Macaulay. A is then a Cohen-Macaulay ring of dimension three and embedding dimension six whose multiplicity is four, by Lemma 1.4. Therefore $P_A(z) = \frac{(1+z)^3}{1-3z}$, by Theorem 1.1. A does not seem to fit any previously known criteria for rationality.

Suppose (A,m,k) is a local ring. Then there is a natural ring homomorphism from Z, the ring of integers, to A which sends 1 to 1. The generator of the kernel of this map is called the <u>characteristic</u> of A. A is said to be <u>equicharacteristic</u> if $\mathrm{char}(A) = \mathrm{char}(k)$. Let A be an equicharacteristic zero, Cohen-Macaulay local ring of dimension one. Two conditions which are known to imply the stability of m are

(1) A saturated [31], Cor. 5.3

and (2) A seminormal [12], Theorem 1.

For higher dimensions, neither imply that A has maximal embedding dimension. In fact, the stronger hypothesis of normality does not suffice.

Example 1.5. Let k be a field, for simplicity algebraically closed of characteristic zero. Let $\{X_{ij}\}$, $i = 1,\ldots,s$; $j = 1,\ldots,r$ with $s < r$, be indeterminants and let $R = k[\{X_{ij}\}]$. Define I to be the ideal generated by the $s \times s$ minors of (X_{ij}) and let m be the ideal of $S = R/I$ generated by the cosets $\{X_{ij} + I\}$. Let $U = \{\bar{X}_{ij}\}$, $i > j$; $V = \{\bar{X}_{ij}\}$, $i < s - r + j$; $W = \{\bar{X}_{ij} - \bar{X}_{1+k,j+k}\}$, $j = 1,\ldots,r-s+1$, $k = 1,\ldots,s-1$. It is known that S_m is a Cohen-Macaulay [14], normal [27], Cor. 3, p.1024, local ring. Moreover, Eagon has shown that $X = U \cup V \cup W$ is a system of parameters for S_m. In fact, if Q is the ideal of S generated by X,

$$(*) \qquad (S/Q)_m \cong \frac{k[Y_1,\ldots,Y_{r-s+1}]}{(Y_1,\ldots,Y_{r-s+1})^s},$$

where the Y_i are new indeterminants [14]. Therefore, $(m \cdot S_m)^s \subseteq Q \cdot S_m$. Suppose that $\bar{X}_{ij}^s = \sum_{\ell=1}^{n} t_k z_k$ for $\bar{X}_{ij} \not\in X$, with $z_k \in X$, $t_k \in S_m$. The ideal I is graded, so S_m inherits a graded structure. Explicitly, if

$t_\ell = \frac{s_\ell}{u_\ell}$, $u_\ell \in S \setminus m$, letting $u = \prod_{\ell=1}^{n} u_\ell$ and clearing denominators, $u\bar{X}_{ij}^s = \sum_{\ell=1}^{n} (u'_\ell s_\ell) z_\ell$, with $u'_\ell \in S \setminus m$. But $\bar{X}_{ij}^s \in m^s$, therefore each $u'_\ell s_\ell \in m^{s-1}$, since the elements of X are all linear. Thus $s_\ell \in m^{s-1}$ for $\ell = 1,\ldots,n$. Hence $t_\ell \in (m \cdot S_m)^{s-1}$, $\ell = 1,\ldots,n$, which implies $(m \cdot S_m)^s \subseteq (Q \cdot S_m)(m \cdot S_m)^{s-1}$. Clearly $(Q \cdot S_m)(m \cdot S_m)^{s-1} \subseteq (m \cdot S_m)^s$, so $(m \cdot S_m)^s = (Q \cdot S_m)(m \cdot S_m)^{s-1}$. In other words, $Q \cdot S_m$ is a reduction of $m \cdot S_m$, which entails that $e(S_m, Q \cdot S_m) = e(S_m)$ [39], Theorem 5. But since $Q \cdot S_m$ is generated by a regular sequence, $e(S_m, Q \cdot S_m) = \ell((S/Q)m)$ [48], VIII, Theorem 23 and [33], Theorem 32. The isomorphism (*) yields then that

$$\begin{aligned} e(S_m, Q \cdot S_m) &= \ell((S/Q)m) \\ &= \sum_{p=1}^{s-1} (\text{the number of monomials of degree } p \text{ in } r-s+1 \text{ variables}) \\ &= \sum_{p=1}^{s-1} \binom{p+(r-s+1)-1}{(r-s+1)-1} \\ &= \sum_{p=1}^{s-1} \binom{p+r-s}{r-s} \end{aligned}$$

Now, $\varepsilon_0(S_m) = rs$, $\dim(S_m) = rs - r + s - 1$, so Abhyankar's inequality becomes $rs \leq rs - r + s - 2 + \sum_{p=1}^{s-1} \binom{p+r-s}{r-s}$. An easy computation shows that $\sum_{p=1}^{s-1} \binom{p+r-s}{r-s} \geq (s-1)(r-s+1)$, with strict inequality for $s \geq 3$. Thus $e(S_m)$ is too large for S_m to have maximal embedding dimension. Notice also that the Poincaré series of S_m can be computed directly using [5], Theorem 6.7 and a comparison with the formula of Theorem 1.1 above yields the same conclusion.

A local ring (A,m,k) is said to be Gorenstein if A is Cohen-Macaulay and if every ideal I generated by a system of parameters is irreducible. That is $I = J_1 \cap J_2$ implies $J_1 = I$ or $J_2 = I$. If $A \cong R/\mathfrak{A}$ with R regular, a more tractable characterization of Gorenstein rings is the requirement that A be Cohen-Macaulay and $\mathrm{Ext}_R^r(A,R) \cong A$ [5], Definition 8.1, where $r = d(R) - d(A)$. Equivalently, $\dim_k(\mathrm{Ext}_A^d(k,A)) = 1$, where $d = d(A)$ [26], p.13. The Example 1.5 is not Gorenstein [14]. However, no conditions that ever imply Gorenstein can imply $\epsilon_0(A) = e(A) + d(A) - 1$. Because, by Serre's codimension two argument [43], Prop. 5, an ideal I of height two in a regular local ring R such that R/I is Gorenstein must be generated by a regular sequence. In other words, R/I is a complete intersection, and by Remark 1.2.2 above, local rings of maximal embedding dimension are not complete intersections when the regularity defect, $\epsilon_0(A) - d(A)$ is greater than or equal to two. In particular, the Cohen-Macaulay property together with unique factorization does not imply A has maximal embedding dimension, because such rings which are quotients of regular local rings are indeed Gorenstein [36].

Let I be an ideal of the local ring A. An inequality due to Rees states that $\mathrm{grade}(I) \leq \mathrm{pd}_A(A/I)$ [33], 24.2. When equality holds, I is said to be perfect. If A is assumed regular, then I is perfect if and only if A/I is Cohen-Macaulay [18], 24.8. I (also A/I) is called an <u>almost complete intersection</u> if $\mu(I) = \operatorname{grade}(I)$.

As a final addition to the preceding remarks, it is possible to prove

<u>Theorem 1.7</u>. Let (A,m,k) be regular of dimension $d \geq 4$, with $I \subseteq m^2$ perfect of height two such that $\epsilon_0(\bar{A}) = e(\bar{A}) + d(\bar{A}) - 1$, where $\bar{A} = A/I$. Then $\bar{A}$ is an almost complete intersection.

<u>Proof</u>: I perfect of height two means that $\operatorname{pd}_A(\bar{A}) = 2$, so $\operatorname{pd}_A(I) = \operatorname{pd}_A(\bar{A}) - 1 = 1$ [18], 18.1. As noted earlier, $\bar{A}$ is not a complete intersection. Therefore, $P_{\bar{A}}(z) = \dfrac{P_A(z)}{1 - rz^2 - (r-1)z^3} = \dfrac{(1+z)^d}{1 - rz^2 - (r-1)z^3}$, where $r = \dim_k H_1(K(\bar{A})) = \mu(I)$, with $K(\bar{A})$ the Koszul complex over $\bar{A}$ [5], Theorem 7.1. But by Theorem 1.1, $P_{\bar{A}}(z) = \dfrac{(1+z)^{d-2}}{1-2z}$. Thus $(1+z)^2(1-2z) = 1 - rz^2 - (r-1)z^3$ and so $r = 3$. That is, $\bar{A}$ is an almost complete intersection.

<u>Remark</u>. As is well known, almost complete intersections are not Gorenstein [28].

CHAPTER II

TWO CHANGE OF RINGS THEOREMS FOR PERFECT IDEALS OF GRADE THREE

Suppose (A,m,k) is a regular local ring, I an ideal of A. Then A/I is Cohen-Macaulay if and only if I is perfect, that is $\mathrm{grade}(I) = \mathrm{pd}_A(A/I)$. Perfect ideals of grade one are of course well understood, since they are free [18], 18.1, and being of height one must be principal. Perfect ideals of grade two were classified by Burch [8], and found to be determinental. Recently, structure theorems for perfect ideals of grade three which are either Gorenstein or almost complete intersections have been proven by Buchsbaum and Eisenbud [7]. Using the Buchsbaum-Eisenbud structure theorems, Avramov has been able to obtain a formula for the Poincaré series of A/I, I a Gorenstein ideal of grade three [5]. (A perfect ideal is said to be <u>Gorenstein</u> when A/I is a Gorenstein ring). Very little is known about the structure of perfect ideals of grade ≥ 4, and this remains an active area of research.

In this chapter, two change of rings formulas for the Poincaré series of a regular local ring modulo a perfect ideal are obtained. The first is for certain almost complete intersections of grade three and is based on the characterization of such ideals by Buchsbaum and Eisenbud [7]. The second formula concerns perfect ideals of height ≥ 3 satisfying a regularity condition on the conormal module.

The standard definition of an almost complete intersection does not require the ideal to be perfect. An ideal $I \subseteq m$ with $ht(I) = grade(I) = r$ is an <u>almost complete intersection</u> if $\mu(I) = r+1$. However, for the purposes of this paper, almost complete intersections, (ACI for short), are assumed perfect. Also, it is assumed that $I \subseteq m^2$. Let $I \subseteq m^2$ be an ACI of grade three. Then $I = ((x_1,x_2,x_3) : J)$, where $J = (y_1,\ldots,y_n)$ is a Gorenstein ideal of grade three and $x_1,x_2,x_3 \in J$ form a maximal regular sequence in J [7], Theorem 5.3. For brevity, put $\underline{x} = (x_1,x_2,x_3)$. The generators of J are given generically as the Pfaffians of alternating matrices and there is a generic free resolution of A/J over A [5], 8.3 and [7], Section 3.

Precisely, there exists an $n \times n$ alternating matrix g with entries $z_{ij} \in m$ such that

$$P : 0 \to A \xrightarrow{d_3} A^n \xrightarrow{d_2} A^n \xrightarrow{d_1} A \to A/J \to 0$$

with differential defined by $d_1 = {}^t d_3 = (\ldots,(-1)^{i+1}Pf_i(g),\ldots)$, $d_2 = g$, furnishes a minimal A-free resolution of A/J. Here $Pf_i(g) = y_i$ denotes the Pfaffian of the $(n-1)\times(n-1)$ alternating matrix obtained from g by deleting the ith row and ith column. In addition, P admits a commutative, associative algebra structure compatible with d [5], 8.4. Let L_i, $i = 1,\ldots,n$, M_i, $i = 1,\ldots,n$ and N be bases for P_1, P_2 and P_3 respectively. The multiplication on P is $L_iL_j = \sum_{k=1}^{n} \sigma_{ijk}\mathrm{Pf}_{ijk}(g)M_k$, $L_iM_j = M_jL_i = \delta_{ij}N$, where for $i,j,k \in \{1,\ldots,n\}$, σ_{ijk} denotes the sign of the permutation $(i,j,k,\{1,\ldots,n\}\setminus\{i,j,k\})$ and Pf_{ijk} is obtained by deleting rows and columns i,j,k from g.

Let $K : 0 \to K_3 \to K_2 \to K_1 \to A \to A/\underline{x} \to 0$ be the Koszul resolution of $A/\underline{x}$, and write d_K for the differential on K, d_P for the differential on P. Then $K_1 \cong AT_1 \oplus AT_2 \oplus AT_3$, with $d_K(T_i) = x_i$. Since $\underline{x} \subseteq I$, there exist $X_1, X_2, X_3 \in P_1$ such that $d_P(X_i) = x_i$. Defining $\psi_1(T_i) = X_i$ and extending by $\psi_2(T_pT_q) = X_pX_q$, $\psi_3(T_1T_2T_3) = X_1X_2X_3$, gives a map $\psi : K \to P$ which is a homomorphism of differential graded algebras:

$$\begin{array}{ccccccccccccc}
P: 0 & \to & P_3 & \to & P_2 & \to & P_1 & \to & P_0 & \to & A/J & \to & 0 \\
 & & \uparrow\psi_3 & & \uparrow\psi_2 & & \uparrow\psi_1 & & \uparrow = & & & & \\
K: 0 & \to & K_3 & \to & K_2 & \to & K_1 & \to & K_0 & \to & A/I & \to & 0 \\
 & & & & & & & & \| & & & & \\
 & & & & & & & & A & & & &
\end{array}$$

The construction of ψ is sketched on page 472 of [7].

Now $d_P^2 N = 0$ implies the relations

$$(2.1) \qquad \sum_{i=1}^{n} (-1)^{i+1} y_i z_{ij} = 0, \quad \text{for } j = 1,\ldots,n.$$

Let $X_i = \sum_{j=1}^{n} b_{ij}L_j$. Computing, one has

$$(2.2)\qquad X_pX_q = (\sum_{j=1}^{n} b_{pj}L_j)(\sum_{k=1}^{n} b_{qk}L_k)$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} b_{pj}b_{qk}[\sum_{k=1}^{n} \sigma_{jk\ell}Pf_{jk\ell}(g)M_\ell]$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sigma_{jk\ell}b_{pj}b_{qk}Pf_{jk\ell}(g)M_\ell.$$

Because K is an exterior algebra, $T_p^2 = 0$, $p = 1,2,3$, which implies $X_p^2 = 0$, since ψ is an algebra homomorphism. So,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} b_{pj}b_{pk}[\sum_{k=1}^{n} \sigma_{jk\ell}Pf_{jk\ell}(g)M_\ell] = 0$$

which implies

$$(2.3)\qquad \sum_{j=1}^{n} \sum_{k=1}^{n} b_{pj}b_{pk}\sigma_{jk\ell}Pf_{jk\ell}(g) = 0, \quad \text{for} \quad \ell = 1,\dots,n$$

and $p = 1,2,3$. Also,

$$(2.4)\qquad \psi_3(T_1T_2T_3) = X_1X_2X_3$$
$$= (\sum_{j=1}^{n} b_{1j}L_j)(\sum_{k=1}^{n} b_{2k}L_k)(\sum_{\ell=1}^{n} b_{3\ell}L_\ell)$$
$$= (\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} b_{1j}b_{2k}b_{3\ell}\sigma_{jk\ell}Pf_{jk\ell}(g))N.$$

There are relations determined by the fact that ψ is a chain map. First, $d_P \circ \psi_1 = Id \circ d_K$ implies

$$(2.5)\qquad \sum_{j=1}^{n} (-1)^{j+1} b_{ij}y_j = x_i \quad \text{for} \quad i = 1,2,3.$$

Since $d_P \circ \psi_2 = \psi_1 \circ d_K$, $d_P(X_pX_q) = \psi_1(x_pT_q - x_qT_p)$, so

$$x_p\left(\sum_{\ell=1}^{n} b_{q\ell}L_\ell\right) - x_q\left(\sum_{\ell=1}^{n} b_{p\ell}L_\ell\right)$$

$$= d_P\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{pj}b_{qk}Pf_{jk\ell}(g)M_\ell\right)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{pj}b_{qk}Pf_{jk\ell}(g)\left(\sum_{i=1}^{n} z_{\ell i}L_i\right).$$

Therefore,

(2.6) $$x_p b_{qi} - x_q b_{pi}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{pj}b_{qk}Pf_{jk\ell}(g)z_{\ell i},$$

$$i = 1,\ldots,n.$$

Finally,

$$\psi_2 \circ d_K(T_1T_2T_3) = \psi_2(x_1T_2T_3 - x_2T_1T_3 + x_3T_1T_2)$$

$$= x_1\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{2j}b_{3k}Pf_{jk\ell}(g)M_\ell\right)$$

$$- x_2\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{1j}b_{3k}Pf_{jk\ell}(g)M_\ell\right)$$

$$+ x_3\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{1j}b_{2k}Pf_{jk\ell}(g)M_\ell\right)$$

$$= d_P\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{1j}b_{2k}b_{3\ell}Pf_{jk\ell}(g)N\right)$$

$$= \left(\sum_{i=1}^{n} (-1)^{i+1} y_iM_i\right)\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \sigma_{jk\ell}b_{1j}b_{2k}b_{3\ell}Pf_{jk\ell}(g)\right).$$

Therefore,

(2.7) $$x_1\left(\sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jki}b_{2j}b_{3k}Pf_{jki}(g)\right)$$

$$- x_2\left(\sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jki}b_{1j}b_{3k}Pf_{jki}(g)\right)$$

$$+ x_3\left(\sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jki}b_{1j}b_{2k}Pf_{jki}(g)\right)$$

$$= (-1)^{i+1} y_i \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sigma_{jk\ell} b_{1j} b_{2k} b_{3\ell} \mathrm{Pf}_{jk\ell}(g) \right)$$

for $i = 1,\ldots,n$.

Now, Peskine and Szpiro have shown that $F : 0 \to P_1^\vee \to (K_1 \oplus P_2)^\vee \to (K_2 \oplus P_3)^\vee \to K_3^\vee \to A/I \to 0$ is an A-free resolution of A/I [40], Prop. 2.6. The notation $(\ )^\vee$ is that of Peskine and Szpiro for the A-module dual. One obtains F by reducing $\bar{\Omega}$, the mapping cone of $\psi : K \to P$, modulo the subcomplex $K_0 \oplus 0 \to 0 \oplus P_0$ and then dualizing. Let Ω stand for the reduced mapping cone. Recall that the mapping cone of a map $\varphi : U \to V$ of complexes is the complex $\cdots \to U_i \oplus V_{i+1} \to \cdots$ endowed with the differential $d(a,b) = (-d_U(a), \varphi(a) + d_V(b))$. Note that $F_0 \cong A$, $F_1 \cong A^4$, $F_2 \cong A^{n+3}$, $F_3 \cong A^n$. Furthermore, according to Prop. 1.3 of [7], F possesses the structure of a commutative, associative, differential algebra. What follows is an explicit calculation of a multiplication table for F.

As in [7], define $S_2(F) = (F \otimes F)/M$, where M is the graded submodule of $F \otimes F$ generated by $\{f \otimes g - (-1)^{(\deg f)(\deg g)} g \otimes f \mid f,g \in F$ both homogeneous$\}$. $S_2(F)$ is a complex with $S_2(F)_k \cong \left(\sum_{\substack{i+j=k \\ i<j}} F_i \otimes F_j \right) + G_k$, where

$$G_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \overset{2}{\wedge} F_{k/2} & \text{if } k = 4n+2,\ n \geq 0 \\ S_2(F_{k/2}) & \text{if } k = 4n,\ n \geq 0. \end{cases}$$

By the comparison theorem, there is a map of complexes $\Phi : S_2(F) \to F$ extending the isomorphism $(A/I) \otimes (A/I) \to A/I$, and which is the identity on the subcomplex $A \otimes F \subseteq S_2(F)$. Define $f \cdot g = \Phi(\overline{f \otimes g})$, where $\overline{f \otimes g}$ is the image in $S_2(F)$ of $f \otimes g$.

As a basis for F_1, choose $A_1, A_2, A_3, A_4 \in F_1 = (K_2 \oplus P_3)^\vee$ such that

$$A_1(T_1T_2) = 1,\ A_1(T_1T_3) = A_1(T_2T_3) = A_1(N) = 0$$

$$A_2(T_1T_3) = 1,\ A_2(T_1T_2) = A_2(T_2T_3) = A_2(N) = 0$$

$$A_3(T_2T_3) = 1,\ A_3(T_1T_2) = A_3(T_1T_3) = A_3(N) = 0$$

$$A_4(N) = 1,\ A_4(T_1T_2) = A_4(T_1T_3) = A_4(T_2T_3) = 0.$$

As a basis for $F_2 = (K_1 \oplus P_2)^\vee$, define $B_i(T_j) = \delta_{ij}$, $i = 1,2,3$ and $j = 1,\ldots,n$; $C_i(M_j) = \delta_{ij}$, $1 \leq i,j \leq n$. Finally, in $F_3 = (0 \oplus P_1)^\vee$ choose $D_1,\ldots,D_n$ so that $D_i(L_j) = \delta_{ij}$, $1 \leq i,j \leq n$.

To determine Φ, the action of the differential of F on basis elements is needed. By definition of the mapping cone,

$$\begin{aligned}(dA_1)(T_1T_2T_3,0) &= A_1(-d_K(T_1T_2T_3),\psi(T_1T_2T_3)) \\ &= A_1(-(x_1T_2T_3 - x_2T_1T_3 + x_3T_1T_2),\psi_3(T_1T_2T_3)) \\ &= -x_3 \text{ by (2.4).}\end{aligned}$$

Similarly,

$$(dA_2)(T_1T_2T_3,0) = x_2 \quad \text{and} \quad (dA_3)(T_1T_2T_3,0) = -x_1.$$

Also

$$\begin{aligned}(dA_4)(T_1T_2T_3,0) &= A_4(-d_\Omega(T_1T_2T_3),\psi_3(T_1T_2T_3)) \\ &= A_4(-x_1T_2T_3 + x_2T_1T_3 - x_3T_1T_2, \\ &\qquad (\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} b_{1j}b_{2k}b_{3\ell}Pf_{jk\ell}(g))N).\end{aligned}$$

Therefore,

$$(2.8.1) \qquad dA_1 = -x_3, \quad dA_2 = x_2, \quad dA_3 = -x_1,$$

$$dA_4 = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} b_{1j}b_{2k}b_{3\ell}\sigma_{jk\ell}Pf_{jk\ell}(g).$$

Let $a = a_3T_1T_2 + a_2T_1T_3 + a_1T_2T_3 \in K_2$. Then

$$\begin{aligned}(dB_1)(a,bN) &= B_1d_\Omega(a,bN) = B_1(-d_Ka,d_PbN + \psi_2(a)) \\ &= B_1(-a_3x_1T_2 + a_3x_2T_1 - a_2x_1T_3 + a_2x_3T_1 \\ &\qquad - a_1x_2T_3 + a_1x_3T_2, \\ &\qquad b(\sum_{i=1}^{n} (-1)^{i+1} y_iM_i) + \psi_2(a)) \\ &= a_3x_2 + a_2x_3.\end{aligned}$$

Similarly,

$$(dB_2)(a,bN) = a_1x_3 - a_3x_1 \quad \text{and} \quad (dB_3)(a,bN) = -a_2x_1 - a_1x_2.$$

Therefore

$$(2.8.2) \qquad dB_1 = x_2A_1 + x_3A_2, \quad dB_2 = x_3A_3 - x_1A_1,$$

$$dB_3 = -x_1A_2 - x_2A_3.$$

Now,

$$\begin{aligned}(dC_i)(a,bN) &= C_i(-d_K a, d_P bN + \psi_2(a)) \\ &= C_i((a_2x_3 + a_3x_2)T_1 - (a_1x_2 + a_2x_1)T_3 \\ &\quad + (a_1x_3 - a_3x_1)T_2,\ b(\sum_{i=1}^{n} (-1)^{i+1}\ y_iM_i) \\ &\quad + a_3X_1X_2 + a_2X_1X_3 + a_1X_2X_3) \\ &= C_i((a_2x_3 + a_3x_2)T_1 - (a_1x_2 + a_2x_1)T_3 \\ &\quad + (a_1x_3 - a_3x_1)T_2,\ b(\sum_{i=1}^{n} (-1)^{i+1}\ y_iM_i) \\ &\quad + a_3(\sum_{\ell=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jk\ell}b_{1j}b_{2k}Pf_{jk\ell}(g)M_\ell) \\ &\quad + a_2(\sum_{\ell=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jk\ell}b_{1j}b_{3k}Pf_{jk\ell}(g)M_\ell) \\ &\quad + a_1(\sum_{\ell=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jk\ell}b_{2j}b_{3k}Pf_{jk\ell}(g)M_\ell)) \quad \text{by } (2.2)\end{aligned}$$

which equals

$$\begin{aligned}(-1)^{i+1}\ by_i &+ a_3(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki}b_{1j}b_{2k}Pf_{jki}(g)) \\ &+ a_2(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki}b_{1j}b_{3k}Pf_{jki}(g)) \\ &+ a_1(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki}b_{2j}b_{3k}Pf_{jki}(g)).\end{aligned}$$

So,

$$\begin{aligned}(2.8.3)\quad dC_i = (-1)^{i+1}\ y_iA_4 &+ (\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki}b_{1j}b_{2k}Pf_{jki}(g))A_1 \\ &+ (\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki}b_{1j}b_{3k}Pf_{jki}(g))A_2 \\ &+ (\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki}b_{2j}b_{3k}Pf_{jki}(g))A_3.\end{aligned}$$

Finally, let $(a,c) = (\sum_{k=1}^{3} a_iT_i, \sum_{k=1}^{n} c_iM_i) \in K_1 \oplus P_2$. Then

$$(dD_i)(a,c) = D_i d_\Omega(a,c) = D_i(-d_K a, d_P c + \psi_1(a))$$
$$= D_i(0, \sum_{k=1}^{n} \sum_{j=1}^{n} c_k z_{kj} L_j + \sum_{k=1}^{3} a_k X_k)$$
$$\text{(since } K_1 = 0 \text{ in } \Omega)$$
$$= D_i(0, \sum_{k=1}^{n} \sum_{j=1}^{n} c_k z_{kj} L_j + \sum_{k=1}^{3} \sum_{j=1}^{n} a_k b_{kj} L_j)$$
$$= D_i(0, \sum_{j=1}^{n} (\sum_{k=1}^{n} c_k z_{kj} + \sum_{k=1}^{3} a_k b_{kj}) L_j)$$
$$= \sum_{k=1}^{n} c_k z_{ki} + \sum_{k=1}^{3} a_k b_{ki}.$$

Therefore,

$$dD_i = \sum_{j=1}^{3} b_{ji} B_j + \sum_{j=1}^{n} z_{ji} C_j. \tag{2.8.4}$$

To compute the products in F, notice first that $\Phi_1(\overline{f \otimes 1}) = f$, $\Phi_1(\overline{1 \otimes g}) = g$. Thus $\Phi_1 d(\overline{A_1 \otimes A_2}) = \Phi_1(\overline{dA_1 \otimes A_2} - \overline{A_1 \otimes dA_2}) = -x_3A_2 - x_2A_1$, $\Phi_1 d(\overline{A_2 \otimes A_3}) = x_2A_3 + x_1A_2$, and $\Phi_1 d(\overline{A_1 \otimes A_3}) = -x_3A_3 + x_1A_1$. Hence

$$\Phi_2(\overline{A_1 \otimes A_2}) = A_1 \cdot A_2 = -B_1$$
$$\Phi_2(\overline{A_1 \otimes A_3}) = A_1 \cdot A_3 = -B_2$$
$$\Phi_2(\overline{A_2 \otimes A_3}) = A_2 \cdot A_3 = -B_3.$$

Clearly $A_i \cdot A_i = 0$ for $i = 1,2,3$.

Next, $d(-\sum_{i=1}^{n} b_{3i} C_i)$ by (2.8.3) equals

$$(-\sum_{i=1}^{n} b_{3i} y_i) A_4 - (\sum_{i=1}^{n} b_{3i} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{1j} b_{2k} Pf_{jki}(g)) A_1$$
$$- (\sum_{i=1}^{n} b_{3i} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{1j} b_{3k} Pf_{jki}(g)) A_2$$

$$- \left(\sum_{i=1}^{n} b_{3i} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{2j} b_{3k} Pf_{jki}(g) \right) A_3$$

$$= -x_3 A_4 - \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} b_{1j} b_{2k} b_{3\ell} Pf_{jk\ell}(g) \right) A_1$$

$$- \left(\sum_{j=1}^{n} b_{1j} \left(\sum_{k=1}^{n} \sum_{i=1}^{n} \sigma_{jki} b_{3k} b_{3i} Pf_{jki}(g) \right) A_2 \right.$$

$$- \left(\sum_{j=1}^{n} b_{2j} \left(\sum_{k=1}^{n} \sum_{i=1}^{n} \sigma_{jki} b_{3k} b_{3i} Pf_{jki}(g) \right) A_3 \right.$$

$$= -x_3 A_4 - \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} b_{1j} b_{2k} b_{3\ell} Pf_{jki}(g) \right) A_1 \quad \text{(use (2.3))}$$

$$= \Phi_1(d(\overline{A_1 \otimes A_4})) \quad \text{by (2.8.1).}$$

Therefore $A_1 \cdot A_4 = -\sum_{i=1}^{n} b_{3i} C_i$. Similarly, $A_2 \cdot A_4 = -\sum_{i=1}^{n} b_{2i} C_i$ and $A_3 \cdot A_4 = -\sum_{i=1}^{n} b_{1i} C_i$. Clearly $A_4 \cdot A_4 = 0$.

Progressing to products of higher degree,

$$\Phi_2 d(\overline{A_1 \otimes B_1}) = \Phi_2(\overline{dA_1 \otimes B_1}) - \Phi_2(\overline{A_1 \otimes dB_1})$$

$$= \Phi_2(-x_3 B_1) - \Phi_2(\overline{A_1 \otimes (x_2 A_1 + x_3 A_2)})$$

$$= -x_3 B_1 - x_2 A_1 \cdot A_1 + x_3 B_1 = 0.$$

$$\Phi_2 d(\overline{A_1 \otimes B_2}) = \Phi_2(\overline{dA_1 \otimes B_2}) - \Phi_2(\overline{A_1 \otimes dB_2})$$

$$= -x_3 B_2 - \Phi_2(\overline{A_1 \otimes (x_3 A_3 - x_1 A_1)})$$

$$= -x_3 B_2 - x_3 A_1 \cdot A_3 + x_1 A_1 \cdot A_1$$

$$= -x_3 B_2 + x_3 B_2 = 0.$$

$$\Phi_2 d(\overline{A_1 \otimes B_3}) = \Phi_2(\overline{dA_1 \otimes B_3}) - \Phi_2(\overline{A_1 \otimes dB_3})$$

$$= -x_3 B_3 - \Phi_2(\overline{A_1 \otimes (-x_1 A_2 - x_2 A_3)})$$

$$= -x_3 B_3 - x_1 B_1 - x_2 B_2.$$

But

$$d\Big(-\sum_{j=1}^{n}(-1)^{j+1}y_jD_j\Big) = -\sum_{j=1}^{n}(-1)^{j+1}y_j\Big(\sum_{i=1}^{3}b_{ij}B_i + \sum_{i=1}^{n}z_{ij}C_i\Big)$$

$$= \sum_{i=1}^{3}\Big(\sum_{j=1}^{n}(-1)^{j+1}y_jb_{ij}\Big)B_i$$

$$+ \sum_{i=1}^{n}\Big(\sum_{j=1}^{n}(-1)^{j+1}y_jz_{ij}\Big)C_i$$

$$= -\sum_{i=1}^{n}x_iB_i \quad \text{by (2.5) and (2.1)}$$

$$= \Phi_2 d(\overline{A_1 \otimes B_3}).$$

Therefore, $A_1 \cdot B_3 = -\sum_{i=1}^{n}(-1)^{i+1}y_iD_i$. Similar calculations yield $A_2 \cdot B_1 = A_2 \cdot B_3 = A_3 \cdot B_2 = A_3 \cdot B_3 = 0$ and $A_2 \cdot B_2 = \sum_{i=1}^{n}(-1)^{i+1}y_iD_i = -A_3 \cdot B_1$. Next,

$$\Phi_2 d(\overline{A_4 \otimes B_1}) = \Phi_2(\overline{dA_4 \otimes B_1} - \overline{A_4 \otimes dB_1})$$

$$= \Phi_2\Big(\Big(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}b_{1j}b_{2k}b_{3\ell}\sigma_{jk\ell}Pf_{jk\ell}(g)\Big)B_1\Big)$$

$$+ \Phi_2(\overline{(x_2A_1 - x_3A_2) \otimes A_4})$$

$$= \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}b_{1j}b_{2k}b_{3\ell}\sigma_{jk\ell}Pf_{jk\ell}(g)\Big)B_1$$

$$+ x_2A_1 \cdot A_4 - x_3A_2 \cdot A_4$$

$$= \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}b_{1j}b_{2k}b_{3\ell}\sigma_{jk\ell}Pf_{jk\ell}(g)\Big)B_1$$

$$- \sum_{i=1}^{n}(b_{3i}x_2 - x_3b_{2i})C_i$$

$$= \Big(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}b_{1j}b_{2k}b_{3\ell}\sigma_{jk\ell}Pf_{jk\ell}(g)\Big)B_1$$

$$- \sum_{\ell=1}^{n}\Big(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sigma_{jki}b_{2j}b_{3k}Pf_{jki}(g)z_{i\ell}\Big)C_i$$

by (2.6).

But,

$$d\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{2j}b_{3k}Pf_{jk\ell}(g)D_{\ell}\right)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{2j}b_{3k}Pf_{jk\ell}(g)(b_{1\ell}B_1 + b_{2\ell}B_2 + b_{3\ell}B_3)$$

$$- \sum_{\ell=1}^{n}\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sigma_{jki}b_{2j}b_{3k}Pf_{jki}(g)z_{i\ell}\right)C_i$$

$$= \left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{1\ell}b_{2j}b_{3k}Pf_{jk\ell}(g)\right)B_1$$

$$+ \sum_{k=1}^{n} b_{3k}\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{2\ell}b_{2j}Pf_{jk\ell}(g)\right)B_2$$

$$+ \sum_{j=1}^{n} b_{2j}\left(\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{3k}b_{3\ell}Pf_{jk\ell}(g)\right)B_3$$

$$- \sum_{\ell=1}^{n}\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n}\sigma_{jki}b_{2j}b_{3k}Pf_{jki}(g)z_{i\ell}\right)C_i$$

$$= \Phi_2 d(\overline{A_4 \otimes B_1}) \quad \text{by } (2.3).$$

Therefore

$$\Phi_3(\overline{A_4 \otimes B_1}) = A_4 \cdot B_1 = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{2j}b_{3k}Pf_{jk\ell}(g)D_{\ell}.$$

Similarly,

$$A_4 \cdot B_2 = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{1j}b_{3k}Pf_{jk\ell}(g)D_{\ell}$$

and

$$A_4 \cdot B_3 = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\sigma_{jk\ell}b_{1j}b_{2k}Pf_{jk\ell}(g)D_{\ell}.$$

Continuing,

$$\Phi_2 d(\overline{A_1 \otimes C_i}) = \Phi_2(\overline{dA_1 \otimes C_i}) - \Phi_2(\overline{A_1 \otimes C_i})$$

$$= x_3C_i + (-1)^{i+1}\, y_i\left(\sum_{\ell=1}^{n} b_{3\ell}C_{\ell}\right)$$

$$+ \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{1j} b_{3k} Pf_{jki}(g)\right) B_1$$

$$+ \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{2j} b_{3k} Pf_{jki}(g)\right) B_2,$$

since $A_1 \cdot A_1 = 0$.

However,

$$d\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g) D_j\right)$$

$$= \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{1j} b_{3k} Pf_{jki}(g)\right) B_1$$

$$+ \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{2j} b_{3k} Pf_{jki}(g)\right) B_2$$

$$+ \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g)\right)\left(\sum_{\ell=1}^{n} z_{\ell j} C_\ell\right) \quad \text{by (2.3).}$$

Also,

$$\sum_{\ell=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g) z_{\ell j} C_\ell$$

$$= \sum_{\ell=1}^{n} \sum_{k=1}^{n} b_{3k}\left(\sum_{j=1}^{n} \sigma_{kij}(g) z_{\ell j}\right) C_\ell$$

since

$$\sum_{j=1}^{n} \sigma_{kij} Pf_{kij}(g) z_{\ell j} = \begin{cases} (-1)^i\, y_i & \text{if } \ell = k \\ 0 & \text{if } \ell \neq k \end{cases}$$

and $\sum_{j=1}^{n} \sigma_{kij} Pf_{kij}(g) z_{ij} = (-1)^{k+1} y_k$, by the formula on page 443 of [5]. So by (2.5),

$$x_3 C_i + (-1)^i\, y_i\left(\sum_{\ell=1}^{n} b_{3\ell} C_\ell\right)$$

$$= \sum_{\ell=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g) z_{\ell j} C_\ell$$

and thus, $d\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g) D_j\right) = \Phi_2 d(\overline{A_1 \otimes C_i})$.

Therefore,

$$A_1 \cdot C_i = \Phi_3(\overline{A_1 \otimes C_i}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g) D_j .$$

Similarly,

$$A_2 \cdot C_i = \Phi_3(\overline{A_2 \otimes C_i}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{2k} Pf_{jki}(g) D_j$$

and

$$A_3 \cdot C_i = \Phi_3(\overline{A_3 \otimes C_i}) = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{1k} Pf_{jki}(g) D_j .$$

I have been unable to find an explicit formula for products of the form $A_4 \cdot C_i$. This turns out to be unnecessary for the proof of Theorem 2.11 below, since it will be shown that the corresponding products vanish in $F \otimes k$. Putting together the above results gives

Table 2.9

$$A_1 \cdot A_2 = -B_1 \qquad A_1 \cdot A_1 = A_2 \cdot A_2 = A_3 \cdot A_3 = 0$$

$$A_1 \cdot A_3 = -B_2 \qquad A_4 \cdot A_4 = 0$$

$$A_2 \cdot A_3 = -B_3$$

$$A_1 \cdot A_4 = -\sum_{i=1}^{n} b_{3i} C_i \quad A_2 \cdot A_4 = -\sum_{i=1}^{n} b_{2i} C_i \quad A_3 \cdot A_4 = -\sum_{i=1}^{n} b_{1i} C_i$$

$$A_1 \cdot B_1 = A_1 \cdot A_2 = 0 \qquad A_2 \cdot B_1 = A_2 \cdot B_3 = 0$$

$$A_1 \cdot B_3 = -\sum_{i=1}^{n} (-1)^{i+1} y_i D_i \quad A_2 \cdot B_2 = \sum_{i=1}^{n} (-1)^{i+1} y_i D_i$$

$$A_3 \cdot B_2 = A_3 \cdot B_3 = 0$$

$$A_3 \cdot B_1 = -\sum_{i=1}^{n} (-1)^{i+1} y_i D_i$$

Table 2.9 continued

$$A_4 \cdot B_1 = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sigma_{jk\ell} b_{2j} b_{3k} Pf_{jk\ell}(g) D_\ell$$

$$A_4 \cdot B_2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sigma_{jk\ell} b_{1j} b_{3k} Pf_{jk\ell}(g) D_\ell$$

$$A_4 \cdot B_3 = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sigma_{jk\ell} b_{1j} b_{2k} Pf_{jk\ell}(g) D_\ell$$

$$A_1 \cdot C_i = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{3k} Pf_{jki}(g) D_j$$

$$A_2 \cdot C_i = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{2k} Pf_{jki}(g) D_j$$

$$A_3 \cdot C_i = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jki} b_{1k} Pf_{jki}(g) D_j .$$

Before going farther, some terminology from the homological theory of local rings must be introduced. The general reference is the book of Gulliksen and Levin [24]. Let (R,m,k) be a local ring and let X be a differential graded algebra whose graded pieces are R-modules. X is <u>strictly skew-commutative</u> if $xy = (-1)^{pq}\, yx$ for $x \in X_p$, $y \in X_q$ and $x^2 = 0$ if $\deg x$ is odd. X is a <u>divided power algebra</u> if to every element $x \in X$ of even positive degree there is a sequence of elements $x^{(\ell)} \in X$, $\ell = 0,1,2,\ldots$ satisfying

(1) $x^{(0)} = 1$, $x^{(1)} = x$, $\deg x^{(\ell)} = \ell \cdot \deg x$

(2) $x^{(j)} x^{(\ell)} = (j,\ell) x^{(j+\ell)}$, where $(j,\ell) = \dfrac{(j+\ell)!}{j!\,\ell!}$

(3) $(x+y)^{(\ell)} = \sum_{i+j=\ell} x^{(i)} y^{(j)}$

(4) for $\ell \geq 2$

$$(xy)^{(\ell)} = \begin{cases} 0 & \text{if } \deg x \text{ and } \deg y \text{ are odd} \\ x^{\ell} y^{(\ell)} & \text{if } \deg x \text{ is even and } \deg y \\ & \text{is even and positive} \end{cases}$$

(5) $(x^{(j)})^{(\ell)} = [j,\ell]x^{(j\ell)}$ for $\ell \geq 0$, $j \geq 1$,

where $[j,\ell] = \dfrac{(j\ell)!}{\ell!(j!)^{\ell}}$.

An <u>(R)-algebra</u> is a strictly skew-commutative, differential graded algebra endowed with a system of divided powers which is compatible with the differential. An (R)-algebra X is assumed connected, that is $1 \cdot R = X_0$, where 1 is the identity of X, and will usually be augmented over k. If $f : X \to Y$ is an (R)-algebra homomorphism, let $F_q Y$ be the (R)-subalgebra of Y generated by $f(X)$ and all the elements of Y of degree less than or equal to q. $\{F_q Y\}_{q \geq 0}$ is called the <u>filtration associated with f</u>. A homomorphism $f : X \to Y$ is said to be a <u>free extension</u> when $Y \cong X \otimes (\bigotimes_{i \in I} R\langle S_i \rangle)$, where $R\langle S_i \rangle$ is the (R)-algebra formed by adjoining the variable S_i to the trivial (R)-algebra in the following fashion. If $\deg S_i$ is odd, then $R\langle S_i \rangle$ denotes the exterior algebra generated by S_i. If $\deg S_i$ is even, $R\langle S_i \rangle$ is the polynomial algebra in a countable number of variables $S_i = S_i^{(1)}, S_i^{(2)}, \ldots$ with relations $S_i^{(j)} S_i^{(\ell)} = \dfrac{(j+\ell)!}{j!\ell!} S^{(j+\ell)}$ for $j,\ell \geq 1$.

Let X be an augmented (R)-algebra. The pair (X^*,h) is an <u>acyclic closure</u> of X if

(1) $h : X \to X^*$ is a free acyclic extension and

(2) $B(X^*) \subseteq C(X^*) + mX^* + X$.

Here $C(X^*)$ denotes the decomposable elements of X^*. That is, $C(X^*)$ is the submodule generated by all elements xx', where x and x' have positive degree, together with all divided powers $y^{(\ell)}$, $\ell > 1$, y of even positive degree. It turns out that a free acyclic extension $X \to Y$ of the canonical augmented (R)-algebra $X : R \to R/m$ is an acyclic closure of X if and only if Y is a minimal resolution of k. Gulliksen has proven the existence of the acyclic closure and in particular, the existence of a minimal (R)-algebra resolution of k [22].

Returning to almost complete intersections, by making an additional assumption, it is possible to determine the homology of the Koszul complex with the help of Table 2.9. First,

Lemma 2.10. Let (A,m,k) be a regular local ring and let $I \subseteq m^2$ be a (perfect) almost complete intersection of grade three. Suppose further that $I = (x_1,x_2,x_3) : J$, $J = (y_1,\ldots,y_n)$ Gorenstein of grade three as above, and $x_i = \sum_{j=1}^{n} (-1)^{j+1} b_{ij}y_j$, with $b_{ij} \in m$. Then $\varepsilon_3(A/I) \geq n+3$.

Proof: The resolution $F \to A/I$ is minimal precisely when $b_{ij} \in m$ for all i,j. Thus $\mathrm{Tor}^A(A/I,k) \cong F \otimes_A k$. Call this algebra Λ. Then Λ inherits the following algebra structure from $F : A_1 \cdot A_2 = -B_1$, $A_1 \cdot A_3 = -B_2$,

$A_2 \cdot A_3 = -B_3$, with all other products zero, except perhaps those of the form $A_4 \cdot C_i$. Let $K = K(A/I) \cong K(A) \otimes_A (A/I)$ be the Koszul complex over A/I. Then there is an isomorphism $\varphi : \Lambda \to H(K)$ of differential graded algebras which preserves Massey products [5], p.401. First of all, $\dim H_1(K) = \dim \Lambda_1 = \epsilon_1(A/I) = \mu(I) = 4$. Let $X \to k$ be the augmented (A/I)-algebra given by K, and let $h : X \to X^*$ be the acyclic closure of X, that is, a minimal (A/I)-algebra resolution of k. Let $\{F_q X^*\}$ be the filtration associated with h. Define $z_1, z_2, z_3, z_4 \in Z_1(X)$ to be cycles of degree one such that $\bar{z}_i = \varphi(A_i)$, $i = 1,2,3,4$, where the bar indicates the residue class in $H(K)$. Adjoin variables S_1, S_2, S_3, S_4 of degree two to X with $dS_i = z_i$, $i = 1,2,3,4$, so that $X \langle S_i \rangle \subseteq F_3 X^*$. Then $d(z_i S_4) = (dz_i)S_4 - z_i dS_4 = -z_i z_4 = -\delta_i = -d\lambda_i$, $i = 1,2,3$. Define $V_i = -\lambda_i - z_i S_4$, $i = 1,2,3$. One has $dV_i = -\delta_i - d(z_i S_4) = 0$, so V_1, V_2 and V_3 are cycles of degree three in $F_3 X^*$. $\bar{V}_1, \bar{V}_2$ and $\bar{V}_3$ are linearly independent in $H_3(F_3 X^*) = \tilde{H}_3(F_3 X^*)$, the reduced homology, because $\bar{z}_1, \bar{z}_2$ and $\bar{z}_3$ are themselves linearly independent in $H(K)$. Moreover, $\bar{V}_1, \bar{V}_2, \bar{V}_3$ and $\varphi(D_i)$, $i = 1,\dots,n$ are linearly independent because $\varphi(D_i) \in X$ for $i = 1,\dots,n$ and S_4 is an indeterminant over X. Combining the homology classes $\bar{V}_i$ with $\varphi(D_i)$ gives $\dim(\tilde{H}_3(F_3 X^*) \otimes k) \geq n + 3$, and by [24], Lemma 3.1.2, $\epsilon_3(A/I) = \dim(\tilde{H}_3(F_3 X^*) \otimes k)$.

<u>Theorem 2.11</u>. Suppose (A,m,k) is a regular local ring, and let $I \subseteq m^2$ be a grade three ACI with $I = \underline{x} : J$,

$J = (y_1, \ldots, y_n)$ Gorenstein of grade three, where $x_i = \sum_{j=1}^{n} (-1)^{j+1} b_{ij} y_j$, $b_{ij} \in m$, as in Lemma 2.10. Then $H_1(K) \cdot H_2(K) = 0$, and there exists a basis $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ for $H_1(K)$ such that $\dim(H_1(K)^2) = 3$ and $\bar{z}_1 \cdot \bar{z}_4 = \bar{z}_2 \cdot \bar{z}_4 = \bar{z}_3 \cdot \bar{z}_4 = 0$.

Proof: Most of the calculations were done above. The dimension of $H_1(K)^2$ is the dimension of Λ_1^2 which is three since Λ_1^2 is spanned by B_1, B_2 and B_3. It remains to show that $H_1(K) \cdot H_2(K) = 0$. From a formula due to Levin, Sakuma and Okuyama [24], Theorem 3.3.4,

$$\varepsilon_3(A/I) = \dim(H_3(K)/H_1(K) \cdot H_2(K)) + \binom{\varepsilon_1}{2} - \dim(H_1(K)^2)$$

which is greater than or equal to $n+3$, by Lemma 2.10. But $\dim(H_1(K)^2) = \dim(\Lambda_1^2) = 3$, $\varepsilon_1 = 4$ and $\dim H_3(K) = \dim \Lambda_3 = n$, so

$$n + 3 \leq n - \dim(H_1(K) \cdot H_2(K)) + 3.$$

That is, $H_1(K) \cdot H_2(K) = 0$.

Remark. The purpose in determining the structure of $H(K)$ is to be able to compute the Poincaré series of A/I. Since $P_{A/I}$ is determined by $H(K)$ [4], Cor. 5.10, and since the structure of $H(K)$ obtained above is the same as that computed for almost complete intersections of embedding dimension three by Golod in Proposition 1 of [21], it would follow that

$$P^k_{A/I}(z) = \frac{(1+z)^d}{1 - z - 3z^2 + (3-n)z^3 - z^5}\ .$$

But this result could be obtained without knowing $H(K)$, by first reducing modulo a regular sequence of length $d-3$ in $m \setminus m^2$. Now the determination of $H(K)$ is of independent interest.

However, there exists a method of reduction to dimension zero which avoids the somewhat complicated computations above. Pick a regular sequence $x_1,\ldots,x_{d-3}$ of length $d-3$ in $m \setminus m^2$ which remains regular under reduction modulo I. Upon reducing modulo $x_1,\ldots,x_{d-3}$, the homology of the Koszul complex is invariant [6], Prop. 1. So it is not surprising that the structure determined in Theorem 2.11 coincides with that in [21].

Consider an arbitrary ideal $I \subseteq m$ of the regular local ring (A,m,k). As is well-known, A/I is a complete intersection if and only if the conormal module I/I^2 is free as A/I-module [16],[47]. As a final note in this Chapter, it is shown that local rings which are close to being complete intersections in the above sense have rational Poincaré series.

<u>Theorem 2.12</u>. Let (A,m,k) be a regular local ring of dimension d, $I \subseteq m^2$ perfect of height r which is not a complete intersection, such that I/I^2 has a free direct summand of rank $r-2$. Then $P^k_{A/I}$ is rational.

<u>Proof</u>: Suppose $I/I^2 \cong (A/I)^{r-2} \oplus N$. This means that there exist $x_1,\ldots,x_{r-2} \in I$ and an ideal J, $I \supset J \supset I^2$ with $(x_1,\ldots,x_{r+2}) + J = I$, $(x_1,\ldots,x_{r-2}) \cap J \subseteq I^2$. In fact, $x_1,\ldots,x_{r-2}$ can be chosen to be a regular sequence [47], Lemma 3. Let $A^* = A/(x_1,\ldots,x_{r-2})$, $I^* = I/(x_1,\ldots,x_{r-2})$. Then $I/x_1 I$, and hence $I/x_1 A$, has finite projective dimension over $A/x_1 A$, [47], Prop. 1.2 and Lemma 2. By the "triangle inequality", [18], 18.3, I^* has finite projective dimension over A^*, as remarked in [47]. Then by a change of rings theorem, [18], 18.7, since $\mathrm{pd}_{A^*}(I^*)$, and hence $\mathrm{pd}_{A^*}(A/I)$, is finite, $\mathrm{pd}_{A^*}(A/I) = \mathrm{pd}_A(A/I) - (r-2) = 2$, as I is perfect. Furthermore, $\mathrm{grade}_{A^*}(I^*) = \mathrm{grade}_A(I) - (r-2) = 2$ [24], Prop. 1.4.7, which equals $\mathrm{ht}_{A^*}(I^*)$. Therefore, I^* is perfect of height two in A^*. As a consequence, $\mathrm{pd}_{A^*}(I^*) = 1$ [18], 18.1, and it is known then that

$$P^k_{A/I}(z) = \frac{P^k_{A^*}(z)}{1 - nz^2 - (n-1)z^3}$$

where $n = \mu(I^*)$, by [5], Theorem 7.1. But $P^k_{A^*}(z) = (1-z^2)^{-(r-2)} P^k_A(z)$ [24], Cor. 3.4.2, which equals $(1-z^2)^{-(r-2)}(1+z)^d$. Thus

$$P^k_{A/I}(z) = \frac{(1+z)^d}{(1-z^2)^{r-2}(1 - nz^2 - (n-1)z^3)}$$

is rational.

CHAPTER III

SOME EXAMPLES

A local ring (A,m,k) is said to be a <u>Golod ring</u> if all the Massey operations on $H(K)$ vanish. See [24], Chapter 4. More generally, let $f:(A,m,k) \to (B,n,k)$ be a homomorphism of local rings over k. That is, a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \nwarrow \quad \swarrow & \\ & k & \end{array}$$

is given with f local. Following Avramov, [5], f is a <u>small homomorphism</u> if the induced map $f^*:\mathrm{Tor}^A(k,k) \to \mathrm{Tor}^B(k,k)$ is injective. A homomorphism $f:(A,m,k) \to (B,n,k)$ over k is then called a <u>Golod homomorphism</u> if equivalently

(1) f is small and $\mathrm{Tor}^A(B,k)$ has trivial Massey products

(2) $n \cdot I\,\mathrm{Tor}^A(B,k) = 0$ and
$$P_A/P_B = 1 - z(P_A^B(z) - 1),$$

where $I\,\mathrm{Tor}^A(B,k)$ denotes the kernel of the canonical augmentation $\mathrm{Tor}^A(B,k) \to k$. Since the definition of the Massey products is quite lengthy and condition (2) is the one which will actually be used below, the interested reader is again referred to [24] or to [4]. Notice that whenever $f : A \to B$ is surjective, $n \cdot I\ \mathrm{Tor}^A(B,k)$ automatically is zero.

It is easy to see that a composition of small homomorphisms is small, [5], Lemma 3.8. The next example shows that a composition of Golod homomorphisms need not be a Golod homomorphism, a fact which does not seem to have appeared in the literature.

<u>Example 3.1</u>. Let (A,m,k) be a regular local ring of dimension d. Let $I = (I_0,x)$, I a perfect ideal of grade two, x a non-zero divisor on A/I_0, with $\mu(I_0) = n$. Then $\mathrm{pd}_A(A/I_0) = 2$ by definition. Let

$$F : 0 \to A^{n-1} \to A^n \to A \to A/I_0 \to 0$$
$$G : 0 \to A \xrightarrow{x} A \to A/xA \to 0$$

be minimal resolutions over A of A/I_0 and x/xA, respectively. The fact that the ranks in F are equal to n and $n-1$ is seen by tensoring with the quotient field of A, a flat extension. Consider the product complex

$$F \otimes G : 0 \to A^{n-1} \to A^{2n-1} \to A^{n+1} \to A \to A/I \to 0.$$

Since the differential on $F \otimes G$ is defined by

$$d(f \otimes g) = df \otimes g + (-1)^{\deg f} f \otimes dg,\ d(F \otimes G) \subseteq m(F \otimes G),$$

as F and G were minimal to start with. Moreover, $H_i(F \otimes G) = \mathrm{Tor}_i^A(A/I_0,A/xA) = 0$ for $i > 1$, since

$\mathrm{pd}_A(A/xA) = 1$ [33], 18.C, Lemma 6. But $H_1(F \otimes G) = \mathrm{Tor}_1^A(A/I_0, A/xA)$ can be computed as $H_1(G \otimes A/I_0)$, which is zero since x was chosen to be a non-zero divisor on A/I_0. Therefore, $F \otimes G$ is a minimal resolution of A/I. So the Betti numbers of A/I as A-module are $b_0 = 1$, $b_1 = n+1$, $b_2 = 2n-1$, $b_3 = n-1$, $b_4 = b_5 = \ldots = 0$, and $P_A^{A/I}(z) = 1 + (n+1)z + (2n-1)z^2 + (n-1)z^3$.

Now, $P^k_{A/I_0}(z) = \dfrac{(1+z)^d}{1 - nz^2 - (n-1)z^3}$, by [5], Theorem 7.1. Also,

$$(1) \quad P^k_{A/I} = \frac{P_{A/I_0}(z)}{1-z^2} = \frac{(1+z)^d}{(1-nz^2-(n-1)z^3)(1-z^2)}$$

if $x \in m^2$ modulo I_0, and

$$(2) \quad P^k_{A/I} = \frac{P^k_{A/I_0}(z)}{1+z} = \frac{(1+z)^d}{(1-nz^2-(n-1)z^3)(1+z)}$$

for $x \in m \setminus m^2$ modulo I_0, because x is a non-zero divisor on A/I_0, [24], Cor. 3.4.2.

Suppose $A \to A/I$ were a Golod homomorphism. Then by definition,

$$(3) \quad P^k_{A/I}(z) = \frac{P^k_A(z)}{1 - z(P_A^{A/I}(z) - 1)}$$

$$= \frac{(1+z)^d}{1-(n+1)z^2-(2n-1)z^3-(n-1)z^4}\,.$$

In case (1), $P^k_{A/I}(z) \neq \dfrac{(1+z)^d}{1-(n+1)z^2-(2n-1)z^3-(n-1)z^4}$,

since the denominators have different degrees. In case (2), a simple multiplication shows that

$$P^k_{A/I}(z) \neq \frac{(1+z)^d}{1-(n+1)z^2-(2n-1)z^3-(n-1)z^4}\ , \quad \text{as required}$$

by (3). Thus the composition $A \to A/I_0 \to A/I$ is not a Golod homomorphism, whereas both $A \to A/I_0$ and $A/I_0 \to A/I$ are [5], Theorem 7.1 and [30], Theorem 3.7.

Remark. This example was first considered by Buchsbaum and Eisenbud in [7], albeit for a different purpose.

In [17], Fröberg exhibited examples of two local Artin rings, one Gorenstein the other not, with the same Poincaré series. Thus P^k_A, or equivalently the deviations $\varepsilon_i(A)$, cannot be used to characterize the class of Gorenstein local rings, something which is of course possible for regular rings and complete intersections. In the next two examples, the deviations are computed for two local rings, one Cohen-Macaulay the other not, which shows that such a homological characterization does not seem possible in this case either.

Example 3.2. Let $R = k[[X,Y,Z]]$, $A = R/\mathfrak{A}$, where $\mathfrak{A} = (XY,YZ)$. Now, $\dim(A) = 2$ and it is not hard to see that $\operatorname{depth}(A) = \operatorname{depth}_R(A) \leq 1$. Since $\operatorname{depth}_R(A) \leq \dim_R(A) = \dim(A) = 2 < \operatorname{depth}_R(R)$, from the exact sequence of R-modules $0 \to \mathfrak{A} \to R \to A \to 0$ it follows that $\operatorname{depth}_R(A) = \operatorname{depth}_R(\mathfrak{A}) - 1$ by [18], p.237. So $\operatorname{depth}_R(A) \leq 1$ if and only if $\operatorname{depth}_R(\mathfrak{A}) \leq 2$. But $\operatorname{depth}_R(\mathfrak{A}) \neq 3$, the maximum possible value, since $\mathfrak{A}$ would then be free as

R-module [33], p.113. Therefore, A is not Cohen-Macaulay. This can also be seen geometrically, since the variety $k[X,Y,Z]/(XY,YZ)$ has an embedded component of dimension one through the origin. See [33], Theorem 30. However, A is a Golod ring with $P_A(z) = \frac{(1+z)^2}{1-z-z^2}$ [24], Theorem 4.3.4 (A is clearly not a complete intersection). Once P_A is known to be rational, the deviations can be computed using the α-invariants of Castillon and Micali [10]. Let $g(z) = P_A(-z)$. Then the α-invariants are the coefficients of the formal power series $\frac{g'(z)}{g(z)}$, and one has the formula

$$\epsilon_{m-1} = \frac{(-1)^m}{m} \sum_{d|m} \mu(\tfrac{m}{d})\alpha_d,$$

where μ here denotes the Möbius function. For the example under consideration, $\frac{g'(z)}{g(z)} = \frac{z^2-4}{(1+z-z^2)(1-z)}$ which is $\frac{-3}{1-z} + \frac{1}{\gamma_1 - z} + \frac{1}{\gamma_2 - z}$, $\gamma_1 = \frac{1+\sqrt{5}}{2}$, $\gamma_2 = \frac{1-\sqrt{5}}{2}$, upon expanding by partial fractions. From this, $\alpha_1 = -3$, $\alpha_2 = 1$, $\alpha_3 = -6, \ldots, \alpha_n = -2 + (-1)^n(\gamma_1^2 + \gamma_2^2)$. Hence, $\epsilon_0(A) = 3$, $\epsilon_1(A) = \mu(\mathfrak{A}) = 2$, $\epsilon_2(A) = 1$, $\epsilon_3(A) = 1$, $\epsilon_4(A) = 2$, $\epsilon_5(A) = 3$, with the sequence monotone increasing from there on.

<u>Example 3.3</u>. Let $S = k[[X,Y]]$, $B = S/(X,Y)^2$. Since $\dim(B) = 0$, B is Cohen-Macaulay. B is however, not Gorenstein because $0 = (X^2,Y) \cap (X,Y^2)$. On the other hand, B is a Golod ring [24], Theorem 4.3.5. Thus $P_B(z) = \frac{(1+z)^2}{1-c_1z^2-c_2z^3}$, where $c_1 = \dim H_1(K)$,

$c_2 = \dim H_2(K)$, K being the Koszul complex over B [24], Cor. 4.2.4. Now $\dim H_1(K) = \mu((X,Y)^2) = 3$ and $\dim H_2(K) = 2$, since $H_2(K) \cong \operatorname{ann}(m)$, where m is the maximal ideal of B [24], Lemma 1.4.2. Therefore, $P_B(z) = \dfrac{(1+z)^2}{1-3z^2-2z^3} = \dfrac{1}{1-2z}$. P_B can be computed at least two other ways. B is a complete intersection modulo its socle [23], and B satisfies the hypotheses of Fröberg's Cor. 1, p.38 of [17]. Let $g(z) = P_B(-z) = \dfrac{1}{1+2z}$, so $\dfrac{g'(z)}{g(z)} = \dfrac{-2}{1+2z}$. Hence $\alpha_n = (-1)^{n+1}2^{n+1}$ and $\varepsilon_0(B) = 2$, $\varepsilon_1(B) = 3$, $\varepsilon_2(B) = 2$, $\varepsilon_3(B) = 3$, $\varepsilon_4(B) = 6$, with the sequence monotone increasing thereafter.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] S.S. Abhyankar, "Local rings of high embedding dimension," Amer. J. Math. 89 (1967), 1073-1077.

[2] E.F. Assmus, "On the homology of local rings," Ill. J. Math. 3 (1959), 187-199.

[3] M.F. Atiyah and I.G. Macdonald, "Introduction to Commutative Algebra," Addison-Wesley, Reading, Mass., 1969.

[4] L.L. Avramov, "On the Hopf algebra of a local ring," Izv. Akad. Nauk. SSSR Ser. Mat. 38 (1974), 253-277; Math. USSR - Izv. 8 (1974), 259-284.

[5] L.L. Avramov, "Small homomorphisms of local rings," J. Algebra 50 (1978), 400-453.

[6] L.L. Avramov and E.S. Golod, "On the homology of the Koszul complex of a local Gorenstein ring," Mat. Zametki 9 (1971), 53-58; Math. Notes 9 (1971), 30-32.

[7] D.A. Buchsbaum and D. Eisenbud, "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3," Amer. J. Math. 99 (1977), 447-485.

[8] L. Burch, "On ideals of finite homological dimension in local rings," Proc. Cambridge Philos. Soc. 64 (1968), 941-948.

[9] H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, N.J., 1956.

[10] J.-B. Castillon et A. Micali, "Sur les nombres de Betti d'un anneau local," C.R. Acad. Sci. Paris Ser. A 279 (1974), 837-840.

[11] U. Daepp and A. Evans, "A note on Buchsbaum rings and localization of graded domains," to appear in Canad. J. Math.

[12] E.D. Davis, "On the geometric interpretation of seminormality," Proc. Amer. Math. Soc. 68 (1978), 1-5.

[13] R.N. Draper and K.G. Fischer, "The homology of certain local rings," J. Algebra 39 (1976), 94-99.

[14] J.A. Eagon, "Examples of Cohen-Macaulay rings which are not Gorenstein," Math. Z. 109 (1969), 109-111.

[15] J.A. Eagon and D.G. Northcott, "Ideals defined by matrices and a certain complex associated with them," Proc. Royal Soc. Ser. A 269 (1962), 188-204.

[16] D. Ferrand, "Suite régulière et intersection complete," C.R. Acad. Sci. Paris Ser. A 264 (1967), 427-428.

[17] R. Fröberg, "Determination of a class of Poincaré Series," Math. Scand. 37 (1975), 29-39.

[18] A.V. Geramita and C. Small, "Introduction to Homological Methods in Commutative Rings," Queen's Papers in Pure and Applied Math. No. 43, Queen's Univ., Kingston, Ontario, 1976.

[19] F. Ghione and T.H. Gulliksen, "Some reduction formulas for the Poincaré series of modules," Rend. Sc. fis. mat. e nat. Lincei 58 (1975), 82-91.

[20] E.S. Golod, "On the homology of some local rings," Soviet Math. 3 (1962), 745-748.

[21] E.S. Golod, "On the homology of some local rings, Uspehki Mat. Nauk 33-5 (1978), 177-178 (in Russian).

[22] T.H. Gulliksen, "A proof of the existence of minimal R-algebra resolutions," Acta Math. 120 (1968), 53-57.

[23] T.H. Gulliksen, "Massey operations and the Poincaré series of certain local rings," J. Algebra 22 (1972), 223-232.

[24] T.H. Gulliksen and G. Levin, "Homology of Local Rings," Queen's Papers in Pure and Applied Math. No. 20, Queen's Univ., Kingston, Ontario, 1969.

[25] R. Hartshorne, "Algebraic Geometry," Grad. Texts in Math 52, Springer-Verlag, New York/Heidelberg/ Berlin, 1977.

[26] J. Herzog and E. Kunz, "Der kanonische Modul eines Cohen-Macaulay Rings," Lecture Notes in Math. No. 238, Springer-Verlag, Berlin/Heidelberg/ New York, 1971.

[27] M. Hochster and J.A. Eagon, "Cohen-Macaulay rings, invariant theory, and the generic perfection of determinental loci," Amer. J. Math. 93 (1971), 1020-1058.

[28] E. Kunz, "Almost complete intersections are not Gorenstein rings," J. Algebra 28 (1974), 111-115.

[29] C. Lech, "Inequalities related to certain couples of local rings," Acta Math. 112 (1964), 69-89.

[30] G. Levin, "Local rings and Golod homomorphisms," J. of Algebra 37 (1975), 266-289.

[31] J. Lipman, "Stable ideals and Arf rings," Amer. J. Math. 93 (1971), 649-685.

[32] E. Matlis, "The multiplicity and reduction number of a one-dimensional local ring," Proc. London Math. Soc. (3) 26 (1973), 273-288.

[33] H. Matsumura, "Commutative Algebra," W.A. Benjamin, New York, 1970.

[34] D. Mumford, "Introduction to Algebraic Geometry," Harvard Univ. mimeographed notes.

[35] D. Mumford, "Algebraic Geometry I - Complex Projective Varieties," Springer-Verlag, Berlin/Heidelberg/ New York, 1976.

[36] M.P. Murthy, "A note on factorial rings," Arch. Math. 55 (1964), 418-420.

[37] M. Nagata, "Local Rings," Interscience Tracts in Pure and Applied Math. 13, J. Wiley, New York, 1962.

[38] D.G. Northcott, "Ideal Theory," Cambridge Univ. Press, 1953.

[39] D.G. Northcott and D. Rees, "A note on reductions of ideals with an application to the generalized Hilbert functions," Proc. Cambridge Phil. Soc. 50 (1954), 353-359.

[40] C. Peskine et L. Szpiro, "Liaison des variétés algébriques I," Invent. Math. 26 (1974), 271-302.

[41] J. Sally, "On the associated graded ring of a local Cohen-Macaulay ring," J. Math. Kyoto Univ. 17-1 (1977), 19-21.

[42] A. Seidenberg, "The hyperplane sections of arithmetically normal varieties," Amer. J. Math. 94 (1972), 609-630.

[43] J.P. Serre, "Sur les modules projectifs," Séminaire Dubreil-Pisot, 1960/61, Fasc. 1.

[44] I.R. Shafarevich, "Basic Algebraic Geometry," Springer-Verlag, Berlin/Heidelberg/New York, 1974.

[45] J. Stückrad and W. Vogel, "On Segré products and applications," J. Algebra 54 (1978), 374-389.

[46] J. Tate, "Homology of Noetherian rings and local rings," Ill. J. Math 1 (1957), 14-25.

[47] W.V. Vasconcelos, "Ideals generated by R-sequences," J. Algebra 6 (1967), 309-316.

[48] O. Zariski and P. Samuel, "Commutative Algebra," Van Nostrand, 1960.