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SATURATIONS OF AN ANALYTIC RING OVER
AN ALGEBRAICALLY CLOSED FIELD

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# SATURATIONS OF AN ANALYTIC RING OVER AN ALGEBRAICALLY CLOSED FIELD

By

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ABSTRACT

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The objective of this thesis is to adapt the theory of saturation as developed by Oscar Zariski to the case of analytic rings. We show that some of the necessary conditions for an adequate description of equisingularity with the help of saturation are fulfilled in this particular case.

Let $k$ be an algebraically closed, complete and non-trivially valued field. Let $A$ be an equidimensional and reduced analytic ring over $k$. $A = k[\{x_1,\ldots,x_d\}][y_1,\ldots,y_m]$ where $x_1,\ldots,x_d$ is a system of parameters of $A$ and all $y_i$ are integral over the convergent power series ring $k[\{x_1,\ldots,x_d\}]$. $x_1,\ldots,x_d$ are called strongly separating if there exist $m$ monic polynomials $P_i(Z)$, $1 \leq i \leq m$, which are separable over the quotient field $k(\{x_1,\ldots,x_d\})$ and such that $P_i(y_i) = 0$. Systems of strongly separating parameters always exist. The saturation of $A$ with respect to a strongly separating system of

parameters is defined and is again an analytic ring over $k$; we denote it by $\tilde{A}_x$. We can associate analytic set germs $V_A$ and $V_{\tilde{A}_x}$ with $A$ and $\tilde{A}_x$ respectively. We show that $V_A$ and $V_{\tilde{A}_x}$ are topologically equivalent, that is, there are representatives for each of them which are homeomorphic. The total ring of quotients is $\Phi(A) = \Phi(A/p_1) \oplus \ldots \oplus \Phi(A/p_s)$ where the $p_i$'s are the minimal primes of $A$. Denote by $F_j^*$ the least Galois extension of $k(\{x_1,\ldots,x_d\})$ which contains $\Phi(A/p_j)$. If we assume further that $[F_j^* : k(\{x_1,\ldots,x_d\})]$ and $\mathrm{char}(k)$ are relatively prime for all $j$, $1 \leq j \leq s$, and that $(x_1,\ldots,x_d)A$ is a reduction of the maximal ideal in $A$ then the multiplicities of $A$ and $\tilde{A}_x$ are the same.

We denote the relative Lipschitz-saturation of $A$ by $A_x^*$. If $k$ and $A$ are as above, except that no separability conditions are needed, then $A_x^*$ is again an analytic ring over $k$ and $V_A$ and $V_{A_x^*}$ are topologically equivalent.

Meinen Eltern

Frieda und Fritz Däpp-Beutler

gewidmet

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# CHAPTER 0

## INTRODUCTION

In a series of three papers published between 1965 and 1968, [23,24 and 25], Zariski started a theory with the aim of classifying the singularities of algebraic and algebroid varieties. In the last paper [25] he introduced the algebraic concept of the saturation of a local ring. His intention was to show that two points are equisingular if and only if their local rings have isomorphic saturations. In [25], p.985, Theorem 2.1 he succeeded in doing so for plane algebroid curves over algebraically closed fields of characteristic zero. He also obtained good results in the hypersurface case, Corollary 7.5 of [25], p.1019. In general, however, the question of equisingularity is still open. The definition itself is not agreed upon and several suggestions are competing with each other, see [29]. But the concept of saturation is also interesting from a purely ring theoretic point of view: it is a way to construct a new local ring from a given one. Zariski's second series of three papers, [26,27 and 28], which he published between 1971 and 1975, puts algebraic questions in the foreground. The relation between saturation and other ring theoretic concepts, like localization and completion, are explored.

In 1972, A. Seidenberg published a paper in which he applied the saturation to the complex analytic case. In particular, he proved that two complex analytic varieties whose saturations of the local rings at the origin are isomorphic, are locally homeomorphic at the origin, [22], Corollary, p.430. Similar results were already obtained by Zariski in [25], §5 and 6.

Our objective is to adapt the theory of saturation to the case of analytic rings of positive characteristic. The geometric objects which are associated with these rings are the analytic set germs. We show that the saturation of this class of rings is well defined and yields other local rings which are again analytic; Theorem (2.12). As in the complex case, we can show that analytic set germs which belong to two analytic rings with isomorphic saturations are topologically equivalent, Corollary (3.5). Under some restrictions the multiplicity of a ring is unchanged if we pass to the saturation, Theorem (4.3). These results can be considered as a minimum requirement for a successful algebraic tool which can help in classifying singularities of analytic set germs. This study is therefore a preliminary test for saturations of analytic rings. However, if it will eventually prove adequate has still to be seen. While pursuing this main goal some results concerning analytic rings and separability questions have been obtained which may be of interest on their own, see Chapter III. Also at the end, Chapter V, we include a brief discussion of the situation, if the relative Lipschitz-saturation of Pham-Teissier as defined in [12] is taken instead of the one by Zariski.

Terminology and notation not defined explicitly in the text follow that used by Zariski and Samuel in their books [30].

# CHAPTER I

# PRELIMINARIES

## 1.1 Analytic Rings

A field $k$ is said to be valued if there is a map $\| : k \to \mathbb{R}^+$ satisfying the following three conditions:

1) $|a| = 0$ if and only if $a = 0$

2) $|a+b| \leq |a| + |b|$ for all $a,b \in k$

3) $|ab| = |a||b|$ for all $a,b \in k$.

This value defines a metric on $k$ by setting $d(a,b) = |a-b|$ if $a,b \in k$. The valuation is said to be non-trivial if there is an element $a \in k$ such that $|a| \neq 0,1$. It is complete if the induced topology is complete. If $\text{char}(k) > 0$ and the valuation is non-trivial then it is non-archimedean and we can replace condition 2) by

2*) $|a+b| \leq \max\{|a|,|b|\}$ for all $a,b \in k$.

$k[[X_1,\ldots,X_n]]$ denotes the ring of formal power series in $n$ variables over the field $k$. $f(X_1,\ldots,X_n) = \sum f_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_n^{i_n} \in k[[X_1,\ldots,X_n]]$ is said to be convergent if there exists a neighborhood $U$ of $0$ in $k^n$ such that to every $(a_1,\ldots,a_n) \in U$ there are elements $(A_1,\ldots,A_n) \in (\mathbb{R}^+)^n$ and $D \in \mathbb{R}^+$ such that

$|a_1| < A_1, \ldots, |a_n| < A_n$ and $|f_{i_1,\ldots,i_n}| A_1^{i_1} \ldots A_n^{i_n} \leq D$ for all $(i_1,\ldots,i_n) \in (\mathbb{Z}^+)^n$. All convergent power series form a subring of $k[[X_1,\ldots,X_n]]$. We denote it by $k[\{X_1,\ldots,X_n\}]$, it is called the convergent power series ring in $n$ variables.

By an analytic ring over $k$ we mean a k-algebra which is the k-homomorphic image of some convergent power series ring with coefficients in $k$. A local ring $A$ is called k-analytic if it contains a subring $B$ such that $B$ is an analytic ring over $k$ and $A$ is a finite B-module. If $k$ is algebraically closed then a ring $A$ is k-analytic if and only if it is an analytic ring over $k$, Corollary 1.5, p.30 of [2].

We will need the following, later on:

<u>Lemma (1.1)</u>. Let $k$ be an algebraically closed, complete valued field. Let $A$ be an integral domain containing $k[\{x_1,\ldots,x_d\}]$ as subring where the $x_i$'s are analytically independent. If $A$ is a finite module over $k[\{x_1,\ldots,x_d\}]$, then $A$ is an analytic ring over $k$.

<u>Proof</u>: In view of the above remark it is enough to show that $A$ is local. By Hensel's Lemma as stated in (12.2), p.95 of [1], $k[\{x_1,\ldots,x_d\}]$ is Henselian. Theorem (43.12), p.183 of [14] gives now the required conclusion.

We will need a slightly stronger form of the normalization theorem for convergent power series rings than can usually be found. (Compare e.g., Theorem (45.5), on p.193 of [14].)

Theorem (1.2). Let $k$ be an algebraically closed, complete and non-trivially valued field and $A$ a local ring of dimension $d$.

If $A$ is an analytic ring over $k$ and $x_1,\dots,x_d$ is any system of parameters of $A$ then $k[\{x_1,\dots,x_d\}] \subseteq A$, $k[\{x_1,\dots,x_d\}]$ is k-isomorphic to a convergent power series ring in $d$ variables $k[\{X_1,\dots,X_d\}]$ and $A$ is a finite $k[\{x_1,\dots,x_d\}]$-module.

Conversely, if $k[\{x_1,\dots,x_d\}] \subseteq A$, $A$ is a finite $k[\{x_1,\dots,x_d\}]$-module and $\dim(k[\{x_1,\dots,x_d\}]) = d$, then $A$ is an analytic ring over $k$ and $x_1,\dots,x_d$ is a system of parameters of $A$.

Proof: If $x_1,\dots,x_d$ is a system of parameters then they are analytically independent. Otherwise there exists $F(Z_1,\dots,Z_d) \in k[\{Z_1,\dots,Z_d\}]$, a convergent power series ring in $d$ variables, such that $F(x_1,\dots,x_d) = 0$. We can apply a k-automorphism $\varphi$ to $k[\{Z_1,\dots,Z_d\}]$ such that $\varphi F$ is regular in $Z_d$. See Lemma 3, p.147 of [30]. (Notice that the automorphism given there on the formal power series ring restricts to one on the convergent power series ring). By the Weierstrass Preparation Theorem we have $\varphi F = E(Z_1,\dots,Z_d)(Z_d^s + R_{s-1}(Z_1,\dots,Z_{d-1})Z_d^{s-1} + \dots)$ where $E$ is a unit and the $R_i$'s are non-units. Hence $\varphi F(x_1,\dots,x_d) = 0$ implies $x_d^s + R_{s-1}(x_1,\dots,x_{d-1})x_d^{s-1} + \dots = 0$. This implies that $x_d^s \in (x_1,\dots,x_{d-1})A$ making $x_1,\dots,x_{d-1}$ into a system of parameters and thus leading to a contradiction. It is now clear that $k[\{x_1,\dots,x_d\}]$ is k-isomorphic to $k[\{X_1,\dots,X_d\}]$.

Next we show that $A$ is a finite module over $k[\{x_1,\dots,x_d\}]$. We can write $A = k[\{x_1,\dots,x_d,z_1,\dots,z_m\}]$. Since $x_1,\dots,x_d$ is a system of parameters we get $z_m^n \in (x_1,\dots,x_d)A$ for some $n$. Hence $f(x_1,\dots,x_d,z_1,\dots,z_m)$ $= z_m^n - \sum_{i=1}^{d} f_i(x_1,\dots,x_d,z_1,\dots,z_m)x_i = 0$. If we replace in the above series $x_j$ by $X_j$ and $z_j$ by $Z_j$, where capital letters denote indeterminates and set $F(X_1,\dots,X_d,Z_1,\dots,Z_m)$ $= Z_m^n - \sum_{i=1}^{d} f_i(X_1,\dots,X_d,Z_1,\dots,Z_m)X_i$ then $F$ is $Z_m$-regular. By the Weierstrass Preparation Theorem we can write an arbitrary $G \in k[\{X_1,\dots,X_d,Z_1,\dots,Z_m\}]$ as $G = U \cdot F + \sum_{i=1}^{n-1} R_i(X_1,\dots,X_d,Z_1,\dots,Z_{m-1})Z_m^i$. Therefore, if $G(x_1,\dots,x_d,z_1,\dots,z_m) \in A$ then $G = \sum_{i=1}^{n-1} R_i(x_1,\dots,x_d,$ $z_1,\dots,z_{m-1})z_m^i$, showing that $A$ is a finite module over $k[\{x_1,\dots,x_d,z_1,\dots,z_{m-1}\}]$. Inductively we see that $A$ is finite over $k[\{x_1,\dots,x_d\}]$.

For the converse we need only show that $x_1,\dots,x_d$ is a system of parameters of $A$. Let $p$ be an associated prime of $(x_1,\dots,x_d)A$. Then

$p \cap k[\{x_1,\ldots,x_d\}] = (x_1,\ldots,x_d)k[\{x_1,\ldots,x_d\}]$. From Corollary 5.8, p.61 of [3] we conclude that $p$ is the maximal ideal of $A$. This implies that $(x_1,\ldots,x_d)A$ is p-primary and hence is generated by a system of parameters.

Whenever we have a complete, non-trivial and algebraically closed field $k$ and a ring $A$ which is an analytic ring over $k$ then the above theorem allows us to write $A = k[\{x_1,\ldots,x_d\}][y_1,\ldots,y_s]$ where $x_1,\ldots,x_d$ is any system of parameters of $A$. This is the standard representation of such a ring.

## 1.2 Analytic Set Germs

Again, let $k$ be a valued and algebraically closed field. A set $V \subseteq k^n$ is called analytic at $a \in k^n$ if there is a neighborhood $U$ of $a$ such that $U \cap V$ is the set of zeroes of finitely many functions which are analytic on $V$. If $U_\alpha$ is an open neighborhood of zero in $k^n$ and $V_\alpha$ is a set which is analytic at each point of $U_\alpha$ then an equivalence relation is defined as follows: $(V_\alpha,U_\alpha) \sim (V_\beta,U_\beta)$ if and only if there is a third pair $(V_\gamma,U_\gamma)$ with $U_\gamma \subseteq U_\alpha \cap U_\beta$ and $V_\alpha \cap U_\gamma = V_\beta \cap U_\gamma = V_\gamma$. The equivalence class $(\overline{V_\alpha,U_\alpha})$ is called an analytic set germ and is often only denoted by $\bar{V}$ or $V$.

Two germs $\bar{V}_1$ and $\bar{V}_2$ are topologically equivalent if there are representatives $V_1$ and $V_2$ and continuous maps $\varphi : V_1 \to V_2$ and $\psi : V_2 \to V_1$ such that the compositions $\psi\varphi : V_1 \to V_1$ and $\varphi\psi : V_2 \to V_2$ are the identity maps. In addition, if the two maps $\varphi$ and $\psi$ are analytic, then we say $\bar{V}_1$ and $\bar{V}_2$ are analytically equivalent. For more details compare [7], where these concepts are developed in the case $k = \mathbb{C}$.

We associate an analytic set germ $V$ with an analytic ring over $k$, $A$, as follows: If $A = k[\{X_1,\ldots,X_n\}]/\mathfrak{A}$ and $F_1,\ldots,F_s$ generate $\mathfrak{A}$, then there is a neighborhood $U$ of $0$ in $k^n$ on which $F_1,\ldots,F_s$ converge. Let $W$ be the set of the common zeroes of $F_1,\ldots,F_s$ in $U$. $V$ is the equivalence class to which $(W,U)$ belongs. As in the algebraic case, $V$ does not depend on the particular generators we took. Also, the radical of $\mathfrak{A}$ gives rise to the same set germ. We will therefore usually assume that $\mathfrak{A}$ is a radical ideal. We will show that the analytic set germ does not depend on the particular representation of $A$.

<u>Theorem (1.3)</u>. Let $k$ be an algebraically closed, complete and non-trivially valued field . If $k[\{X_1,\ldots,X_n\}]/\mathfrak{A}$ and $k[\{Y_1,\ldots,Y_m\}]/\mathfrak{B}$ are k-isomorphic then their associated analytic set germs are analytically equivalent.

<u>Proof</u>: Choose generators, $\mathfrak{A} = (\alpha_1,\ldots,\alpha_s)$ and $\mathfrak{B} = (\beta_1,\ldots,\beta_t)$. We set $A = k[\{X_1,\ldots,X_n\}]/\mathfrak{A}$ and $B = k[\{Y_1,\ldots,Y_m\}]/\mathfrak{B}$. We denote the k-isomorphisms by $f : A \to B$ and $g = f^{-1}$. By $x_i$ we denote $X_i$ modulo $\mathfrak{A}$

and by $y_i$, $Y_i$ modulo $\mathfrak{B}$. Let $\Omega_j(X) \in k[\{X_1,\ldots,X_n\}]$ such that $\bar{\Omega}_j(x) = g(y_j)$ for $1 \leq j \leq m$, where $\bar{\Omega}_j$ is $\Omega_j$ modulo $\mathfrak{A}$. Let $\Lambda_k(Y) \in k[\{Y_1,\ldots,Y_m\}]$ such that $\bar{\Lambda}_k(y) = f(x_k)$ for $1 \leq k \leq n$. Here $\bar{\Lambda}_k$ is $\Lambda_k$ modulo $\mathfrak{B}$. For some set $U_0 \subseteq k^m$ we define $f^{\#} : U_0 \to k^n$ by $f^{\#}(b_1,\ldots,b_m) = (\Lambda_1(b),\ldots,\Lambda_n(b))$ and for an appropriate $V_0 \subseteq k^n$, $g^{\#} : V_0 \to k^m$ by $g^{\#}(a_1,\ldots,a_n) = (\Omega_1(a),\ldots,\Omega_m(a))$.

Further define $H_k(X) = \Lambda_k(\Omega_1(X),\ldots,\Omega_m(X)) - X_k$ for $1 \leq k \leq n$ and $Q_j(Y) = \Omega_j(\Lambda_1(Y),\ldots,\Lambda_n(Y)) - Y_j$ for $1 \leq j \leq m$. We claim that $H_k(X) \in \mathfrak{A}$ and $Q_j(Y) \in \mathfrak{B}$ for all $j$. For, we have $\bar{H}_k(x) = \bar{\Lambda}_k(\bar{\Omega}_1(x),\ldots,\bar{\Omega}_m(x)) - x_k =$ $\bar{\Lambda}_k(g(y_1),\ldots,g(y_m)) - x_k = g(\bar{\Lambda}_k(y)) - x_k) = gf(x_k) - x_k = 0$. Hence $H_k(X) \in \mathfrak{A}$. In the same way we prove that $Q_j(Y) \in \mathfrak{B}$. Hence we can write $H_k(X) = \sum_{\ell=1}^{s} \alpha_{k\ell}\alpha_\ell$ where $\alpha_{k\ell} \in k[\{X_1,\ldots,X_n\}]$ and $Q_j(Y) = \sum_{\ell=1}^{t} \beta_{j\ell}\beta_\ell$ where $\beta_{j\ell} \in k[\{Y_1,\ldots,Y_m\}]$.

Let $U_1$ be a neighborhood of zero in $k^n$ such that $\alpha_{k\ell}, \alpha_\ell$ and $\Omega_j$ converge on it for $1 \leq k \leq n$, $1 \leq \ell \leq s$, and $1 \leq j \leq m$. Let $V_1$ be a neighborhood of zero in $k^m$ such that $\beta_{j\ell}, \beta_\ell$ and $\Lambda_k$ converge on it for $1 \leq j \leq m$, $1 \leq \ell \leq t$, and $1 \leq k \leq n$. Let $V_A$ be the analytic set defined in $U_1$ by the functions $\alpha_1,\ldots,\alpha_t$. Let $V_B$ be the analytic set defined in $V_1$ by the functions $\beta_1,\ldots,\beta_s$. Let $U_2 = g^{\#-1}V_1$. Then $U_3 = U_1 \cap U_2$ is a neighborhood of zero in $k^n$ and $g^{\#} : V_A \cap U_3 \to k^n$ is now well defined.

We have: $f^{\#}g^{\#}(a) = a$ for all $a = (a_1,\ldots,a_n) \in V_A \cap U_3$. For, $f^{\#}g^{\#}(a) - a = f^{\#}(\Omega_1(a),\ldots,\Omega_m(a)) - a =$ $(\Lambda_1(\Omega(a)),\ldots,\Lambda_n(\Omega(a))) - a = (H_1(a),\ldots,H_n(a)) = 0$. (Subtraction is meant as a vector space operation in $k^n$.) Consequently $g^{\#}$ is one-to-one on $V_A \cap U_3$. Let $V_2 = f^{\#-1}U_1$. $V_3 = V_1 \cap V_2$ is a neighborhood of zero in $k^m$.

We claim that $g^{\#} : U_3 \cap V_A \to V_3 \cap V_A$ is a homeomorphism. The only thing left to show is that $g^{\#}(U_3 \cap V_A) \subseteq V_3 \cap V_B$. We have $\bar{\beta}_\ell(\bar{\Omega}_1(x),\ldots,\bar{\Omega}_m(x)) = \bar{\beta}_\ell(g(y_1),\ldots,g(y_m)) =$ $g(\bar{\beta}_\ell(y_1,\ldots,y_m)) = 0$ for $1 \leq \ell \leq t$. Hence $\beta_\ell(\Omega_1(X),\ldots,\Omega_m(X)) \in \mathfrak{A}$ for $1 \leq \ell \leq t$. If $a \in U_3 \cap V_A$ then $\beta_i(g^{\#}(a)) = \beta_i(\Omega_1(a),\ldots,\Omega_m(a)) = 0$ for $1 \leq \ell \leq t$, hence $g^{\#}(a) \in V_B$. From above we have $a = f^{\#}g^{\#}(a)$, hence $g^{\#} \in V_2$. Since $a \in U_2 = g^{\#-1}V_1$, also $g^{\#}(a) \in V_1$. This establishes the claim.

$g^{\#}$ is analytic on $U_3$ and $f^{\#}$ is analytic on $V_3$. This shows that the germs $V_A$ and $V_B$ are analytically equivalent.

The following lemma and its corollary give some information about the dimension of the ambient space of an analytic set germ associated with a given analytic ring over $k$.

Lemma (1.4). Let $k$ be a complete, non-trivially valued and algebraically closed field. Let $A$ be an analytic ring over $k$ and $y_1,\dots,y_s$ a set of generators for the maximal ideal in $A$. Then $A = k[\{y_1,\dots,y_s\}]$.

Proof: Clearly $k[\{y_1,\dots,y_s\}] \subseteq A$ and therefore $A = k[\{y_1,\dots,y_s,x_1,\dots,x_n\}]$ with maximal ideal $A(y_1,\dots,y_s,x_1,\dots,x_n) = A(y_1,\dots,y_s)$. Hence $x_1 = \sum_{\ell=1}^{s} A_\ell(y,x)x_\ell$ where $A_\ell(y,x) \in A$. Set $F = X_1 - \sum_{\ell=1}^{s} A_\ell(Y,X)X_\ell$. If $G(Y,X)$ is any convergent power series then by Weierstrass' Preparation Theorem there are convergent series $U(Y,X)$ and $R_0(Y,X_2,\dots,X_n)$ such that $G(Y,X) = U(Y,X)F(Y,X) + R_0(Y,X_2,\dots,X_n)$. Hence $G(y,x) = R_0(y,x_2,\dots,x_n)$ showing that $A = k[\{y,x_2,\dots,x_n\}]$. After applying this reduction $n$ times we see that $A = k[\{y_1,\dots,y_s\}]$.

Corollary (1.5). Let $k$ be a complete, non-trivially valued and algebraically closed field. Let $A$ be an analytic ring over $k$ of embedding dimension $n$. We have associated analytic set germs in $k^\ell$ for all $\ell \geq n$.

## 1.3 Saturation

In [25] Zariski gives a definition of saturation which we are going to use and which we will recall here, adapted to the special case of analytic rings over a field $k$.

$k$ is an algebraically closed, complete and non-trivially valued field. $A = k[\{x_1,\ldots,x_d\}][y_1,\ldots,y_n]$ is an analytic ring over $k$ with $x_1,\ldots,x_d$ a system of parameters. $\Phi(A)$ denotes the total ring of fractions of $A$, $k(\{x_1,\ldots,x_d\})$ is the field of fractions of $k[\{x_1,\ldots,x_d\}]$. The saturation $\tilde{A}_x$ of $A$ with respect to $x_1,\ldots,x_d$ will only be defined if the following five conditions hold (b,c and e are trivially satisfied):

a) $A$ is reduced.

b) $\Phi(A)$ is Noetherian and hence $\Phi(A) = F_1 \oplus \ldots \oplus F_s$ is a direct sum of fields.

c) $k(\{x_1,\ldots,x_d\})$ contains the element 1 of $\Phi(A)$.

d) Let $\varepsilon_i$ be the identity of $F_i$ in $\Phi(A)$. Then $F_i$ is a finite separable extension of $\varepsilon_i k(\{x_1,\ldots,x_d\})$ for all $i$, $1 \leq i \leq s$.

e) $A$ is integral over $R = A \cap k(\{x_1,\ldots,x_d\})$.

We will need some notation and a preliminary definition. We denote by $\Omega$ the algebraic closure of $k(\{x_1,\ldots,x_d\})$. If $y$ and $z$ are two elements of $\Phi(A)$ then we say that $y$ dominates $z$ if for any two $k(\{x_1,\ldots,x_d\})$-homomorphisms $\psi_1$ and $\psi_2$ of $\Phi(A)$ into $\Omega$ the following is true: If $\psi_1(z) \neq \psi_2(z)$ then $[\psi_1(y) - \psi_2(y)]/[\psi_1(z) - \psi_2(z)]$ is integral over $R$ and if $\psi_1(z) = \psi_2(z)$ then $\psi_1(y) = \psi_2(y)$.

A ring $A$ is said to be saturated if every element of its integral closure $\bar{A}$ (in its total ring of fractions) which dominates an element of $A$ is contained in $A$. The intersection of all saturated rings lying between $A$ and $\bar{A}$ is called the saturation of $A$ and is denoted by $\tilde{A}_x$. Apparently it depends on the system of parameters we have chosen.

Recall that if we have two rings $A \subseteq B$ then the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ defined by $f(p) = p \cap A$ is called radicial if the following two conditions are satisfied:

(1) for every prime ideal $p$ in $A$ there is at most one prime in $B$ which lies over $p$.

(2) If $p \in \mathrm{Spec}(A)$, $q \in \mathrm{Spec}(B)$ and $q$ lies over $p$ then $\Phi(B/q)$ is a purely inseparable extension of $\Phi(A/p)$.

Zariski's Theorem 4.1 [25], p.997 shows that the morphism $\mathrm{Spec}(\tilde{A}_x) \to \mathrm{Spec}(A)$ is radicial. We will make extensive use of this fact.

## CHAPTER II

## SATURATION OF AN ANALYTIC RING

In this section we will show that we can find good systems of parameters so that it is possible to define the saturation. The saturation is then again an analytic ring over the same field as the original ring.

<u>Lemma (2.1)</u>. Let $k$ be a valued field which is perfect and let $A$ be a reduced analytic ring over $k$. The integral closure $\bar{A}$ of $A$ in its total ring of quotients $\Phi(A)$ is a finite $A$-module.

$\Phi(A)$ is the direct sum of fields $\Phi(A) = F_1 \oplus \dots \oplus F_s$ and if $\epsilon_i$ is the identity of $F_i$ as an element in $\Phi(A)$, then $\bar{A}$ is the direct sum of the integral closures of the $A\epsilon_i$'s in the $F_i$'s.

<u>Proof</u>: Since $A$ is a finite module over a convergent power series ring we have by [14], (45.6) on p.194 that $A$ is a Weierstrass ring and hence is in particular pseudo-geometric. That is, if $p \in \mathrm{Spec}(A)$ then the integral closure of $A/p$ in its field of quotients is a finite $A/p$-module. The lemma follows now from (19.23), p.167 of [1].

The total ring of quotients of a Noetherian ring, and hence of a ring $A$ which is reduced and analytic over $k$ can be described more precisely. Namely $\Phi(A) = \Phi(A/p_1) \oplus \ldots \oplus \Phi(A/p_s)$ where $p_1, \ldots, p_s$ denote the minimal primes of $A$.

We introduce the following definition:

Definition (2.2). Suppose $A = k[\{x_1, \ldots, x_d\}][y_1, \ldots, y_m]$ where $x_1, \ldots, x_d$ is a system of parameters for $A$ and the $y_1, \ldots, y_m$ are integral over $k[\{x_1, \ldots, x_d\}]$. The system of parameters $x_1, \ldots, x_d$ is said to be strongly separating if there exist $m$ monic polynomials $P_i(Z)$ in $k[\{x_1, \ldots, x_d\}][Z]$ such that $P_i(y_i) = 0$ for $1 \leq i \leq m$ and which are separable considered as polynomials over the field $k(\{x_1, \ldots, x_d\})$.

An analytic ring over $k$ which has a strongly separating system of parameters is called strongly separable.

A ring $A$ is called equidimensional if $\dim(A) = \dim(A/p)$ for all associated primes $p$ of the zero-ideal in $A$.

Lemma (2.3). Let $k$ be an algebraically closed, complete and non-trivially valued field. Let $A$ be a reduced and equidimensional analytic ring over $k$ and $\Phi(A) = F_1 \oplus \ldots \oplus F_s$ its total ring of quotients. $\epsilon_i$ denotes the unit of $F_i$ in $\Phi(A)$. If $x_1, \ldots, x_d$ is a strongly separating system of parameters of $A$ then $F_i$ is a finite algebraic and separable extension of $\epsilon_i k(\{x_1,\ldots,x_d\})$ for $1 \le i \le s$.

<u>Proof</u>: We first consider the case where $A$ is a domain. We have then the following commutative diagram where all maps are the obvious inclusions.

$$\begin{tikzcd} & k(\{x_1,\ldots,x_d\})(y_1,\ldots,y_m) \arrow[dl, dash, "\eta"'] \arrow[dr, dash] & \\ k(\{x_1,\ldots,x_d\}) \arrow[dr, dash] & & k[\{x_1,\ldots,x_d\}][y_1,\ldots,y_m] \arrow[dl, dash] \\ & k[\{x_1,\ldots,x_d\}] & \end{tikzcd}$$

Clearly $k(\{x_1,\ldots,x_d\})(y_1,\ldots,y_m) = \Phi(k[\{x_1,\ldots,x_d\}][y_1,\ldots,y_m])$ $= \Phi(A)$. The monic irreducible polynomial of $y_i$ over the field $k(\{x_1,\ldots,x_d\})$ divides $P_i$ which is separable. Hence, $\eta$ is separable.

We can now look at the general case and denote the minimal primes of $A$ by $p_1,\ldots,p_s$. We claim that $k[\{x_1,\ldots,x_d\}] \cap p_i = (0)$ for $1 \le i \le s$. To see this we notice that for each $p_i$ we have a chain of prime ideals $p_i \subset q_1 \subset \ldots \subset q_d$ of length $d+1$ in $A$. Contracting this chain to $k[\{x_1,\ldots,x_d\}]$ we get another proper chain in this ring of length $d+1$, hence $p_i \cap k[\{x_1,\ldots,x_d\}] = (0)$. This establishes the claim.

We define the maps in the diagram beneath as follows: $f_1(\sum a_{i_1\ldots i_m} Y_1^{i_1} \ldots Y_m^{i_m} + p_i) = \sum a_{i_1\ldots i_m} \bar{Y}_1^{i_1} \ldots \bar{Y}_m^{i_m}$. Since $k[\{x_1,\ldots,x_d\}] \cap p_i = (0)$ this is an isomorphism. $f_2$ is defined analogously. $g_1(a) = (a+p_1,\ldots,a+p_s)$. $g_2$ is defined in the same way. $e$ and $h$ are the natural embeddings. It is clear that all subdiagrams commute, except for diagram $D$.

$$\begin{tikzcd}
 & & & & \Phi(A) & \\
 & k(\{x\})(\bar{y}) \arrow[rr, phantom, "\overset{f_2}{\cong}"] & & F_i \arrow[ur, "h"] & & \\
k(\{x\}) \arrow[ur, "e"] & k[\{x\}][\bar{y}] \arrow[u] \arrow[rr, phantom, "\overset{f_1}{\cong}"] & & A/p_i \arrow[u, "D"] & & k(\{x\}) \arrow[uul, "g_2"'] \\
 & & & A \arrow[u] \arrow[uuur, "g_1"] & & \\
 & & & k[\{x\}] \arrow[u] \arrow[uulll] \arrow[uull] \arrow[uurr] & &
\end{tikzcd}$$

$x$ stands for $x_1,\ldots,x_d$ and $y$ for $y_1,\ldots,y_m$. $\bar{y}$ denotes $y+p_i$.

If $P_i(Z)$ is an integral relation for $y_i$ over $k[\{x_1,\ldots,x_d\}]$ then it is also one for $\bar{y}_i$ over the same ring. The special case treated first shows now that $F_i$ is a finite separable algebraic extension of $k(\{x_1,\ldots,x_d\})$. The lemma will be proven if we show that $hf_2e(k(\{x\})) = \varepsilon_i g_2(k(\{x\})) : hf_2e(a) = h(a+p_i) = (0,\ldots,a+p_i,\ldots,0) = \varepsilon_i(a+p_1,\ldots,a+p_s) = \varepsilon_i g_2(a)$.

Our next goal is to show that all analytic rings over $k$ are strongly separable. To achieve this we will have to introduce some notation and to quote some theorems from [21].

Suppose $A$ is an analytic ring over $k$. A k-derivation $\delta : A \to M$ is called finite if $M$ is a finite A-module. The pair $(D_k(A), d_k)$, where $D_k(A)$ is a finite A-module and $d_k : A \to D_k(A)$ is a derivation, is called the universally finite derivation of $A$ if the following holds: For every finite module $M$ and every k-derivation $\delta : A \to M$ there exists a unique A-homomorphism $h$ which makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[rr, dash, "\delta"] \arrow[dr, "d_k"'] & & M \\ & D_k(A) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

In case $d_k : A \to D_k(A)$ exists, it is determined uniquely up to A-isomorphism. $D_k(A)$ is called the universally finite module of k-differentials of $A$.

<u>Theorem (2.4)</u> (Sheja-Storch) [21], p.146, (2.6). If $A$ is an analytic ring over $k$ then $D_k(A)$ exists.

<u>Definition (2.5)</u> [21], p.149. Suppose $A$ is an analytic integral domain of dimension $d$. A system of parameters $x_1,\ldots,x_d$ in $A$ is called separating if the quotient field of $A$ is separable algebraic over $k(\{x_1,\ldots,x_d\})$.

Note that separating and strongly separating agree for analytic domains over $k$.

If $A$ is a domain and $M$ is an A-module then $\operatorname{rank}_A(M) = \dim_{\Phi(A)}(M \otimes_A \Phi(A))$.

<u>Theorem (2.6)</u> (Scheja-Storch) [21], p.149, (4.1). Suppose $A$ is an analytic integral domain of dimension $d$. Then

1) $\operatorname{rank}_A(D_k(A)) \geq d$.
2) $\operatorname{rank}_A(D_k(A)) = d$ if and only if $A$ contains a separating system of parameters.
3) A system of parameters $x_1,\ldots,x_d$ of $A$ is separating if and only if $D_k(A)/(Adx_1+\ldots+Adx_d)$ is a torsion module.

<u>Definition (2.7)</u> [21], p.150. Suppose $A$ is an analytic ring over $k$ and $p \in \operatorname{Spec}(A)$. $p$ is said to be separable if and only if $\operatorname{rank}_{A/p}(D_k(A/p)) = \dim(A/p)$.

$A$ is said to be separable if $A$ is reduced and each minimal prime of $A$ is separable.

<u>Theorem (2.8)</u> (Scheja-Storch) [21], (4.2), p.150. If $k$ is perfect then every reduced analytic ring over $k$ is separable.

For our purposes we need strong separability. However, the following theorem shows that the two conditions imply each other.

<u>Theorem (2.9)</u>. Let $k$ be an algebraically closed complete and non-trivially valued field. Let $A$ be an equidimensional and reduced analytic ring over $k$. Then $A$ is separable if and only if it is strongly separable.

<u>Proof</u>: Of course we can assume that $\mathrm{char}(k) = p > 0$.

It follows easily from Lemma (2.3) and Theorem (2.6) that strongly separable implies separable.

Now we assume that $A$ is separable. Let $p_1,\dots,p_s$ be the minimal primes, $\pi_i : A \to A/p_i$ the projections and $d_i : A/p_i \to D_k(A/p_i)$ the universally finite derivations. By [21], p.149, Zusatz zu (4.1) we get $d = \dim(A) = \dim(A/p_i) = \mathrm{rank}_{A/p_i}(d_i(\pi_i m))$. We can now apply Hilfssatz (7.2) in [21], p.157, to find $x_1,\dots,x_d \in m$ such that for all $i$ $\{\pi_i(x_1),\dots,\pi_i(x_d)\}$ is a system of parameters and a $d_i$-free set, that means $\{d_i\pi_i(x_1),\dots,d_i\pi_i(x_d)\}$ is linearly independent over $A/p_i$. Let $y_1,\dots,y_d$ be any system of parameters of $A$. Then $y_1^p,\dots,y_d^p$ is a system of parameters, too. Now let $z_i = y_i^p(1+x_i)$; then

$z_1,\dots,z_d$ is a system of parameters since $1+x_i$ are units. $d_j[\pi_j(y_j^p(1+x_i))] = (\pi_j y_j)^p\, d_j\pi_j(x_i)$ and $\pi_j(y_i) \neq 0$ for otherwise $\dim(A/p_j) \leq \dim(A/y_iA) = d-1$ by [6], (10.5), p.73. Hence $\{\pi_j(z_1),\dots,\pi_j(z_d)\}$ is $d_j$-free for $1 \leq j \leq s$.

Consider the following sequence:

$$0 \to \sum_i (A/p_j)d_j\pi_j(z_j) \to D_k(A/p_j) \to D_k(A/p_j)/\sum_i (A/p_j)d_j\pi_j(z_i) \to 0$$

Since $\Phi(A/p_j)$ is flat over $A/p_j$ we get:

$$0 \to (\sum_i (A/p_j)d_j\pi_j(z_i)) \otimes \Phi(A/p_j) \to D_k(A/p_j) \otimes \Phi(A/p_j)$$
$$\to (D_k(A/p_j)/\sum_i (A/p_j)d_j\pi_j(z_i)) \otimes_{A/p_j} \Phi(A/p_j) \to 0$$

Since the ranks of the first two terms are equal, the rank of the last one is zero, that means $D_k(A/p_j)/\sum_i (A/p_j)d_j\pi_j(z_i)$ is torsion. It follows now from (2.6) that $\pi_j(z_1),\dots,\pi_j(z_d)$ are separating systems of parameters for all $j$, $1 \leq j \leq s$.

We write now $A = k[\{z_1,\dots,z_d\}][y_1,\dots,y_m]$ where $z_1,\dots,z_d$ is the system of parameters defined above. For clarity we set $y = y_i$ where $i$ is any of the indices $1,\dots,m$. We denote by $\mathrm{Ass}_A(0)$ the set of all associated primes of the ideal $(0)$ in $A$. Let $p \in \mathrm{Ass}_A(0)$ with $y \notin p$ and let $P(Z) = Z^n + a_{n-1}Z^{n-1} + \dots + a_0 \in k[\{z_1,\dots,z_d\}][Z]$ such that $P(y) = 0$. Since $k[\{z_1,\dots,z_d\}] \cap p = (0)$ we have a canonical inclusion $k[\{z_1,\dots,z_d\}] \xrightarrow{I} A/p$ and we denote the image of $a \in k[\{z_1,\dots,z_d\}]$ in $A/p$ by $\bar{a}$. Let

$\bar{P}(Z) = Z^n + \bar{a}_{n-1}Z^{n-1} + \ldots + \bar{a}_0 = \bar{P}_1(Z) \ldots \bar{P}_k(Z)$ where $\bar{P}_i(Z)$ denote the irreducible factors of $\bar{P}(Z)$ as polynomials over $\pi k[\{z_1,\ldots,z_d\}]$ which is a unique factorization domain. By a corollary to Gauss Lemma, see [8], p.147, Lemma 3, the $\bar{P}_i(Z)$'s are irreducible over $\pi k(\{z_1,\ldots,z_d\})$. Since also $\bar{P}(\bar{y}) = 0$ (here $\bar{y} = \pi(y)$ where $\pi : A \to A/p$ is the canonical surjection), we get $\bar{P}_i(Z) = \mathrm{Irr}(\bar{y}, \pi k(\{z_1,\ldots,z_d\}))$, the irreducible monic polynomial, for some $i$, $1 \leq i \leq k$. By choice of notation we may assume that $i = 1$. Since $\Phi(A/p)$ is separable over $\pi k[\{z_1,\ldots,z_d\}]$ we have that $\bar{P}_1(Z)$ is separable. If $\bar{P}_i(Z) = Z^m + \bar{b}_{m-1}Z^{m-1} + \ldots + \bar{b}_0$ then we set $Q_p(Z) = Z^m + b_{m-1}Z^{m-1} + \ldots + b_0$ where $b_i$ denotes the unique preimage of $\bar{b}_i$ under the map $\pi$. Since $\bar{P}_i$ is separable so is $Q_p$ and $Q_p(y) \in p$.

If $t$ is the number of primes belonging to zero and containing $y$ then set

$$F_y(Z) = Z^t \prod_{\substack{p \in \mathrm{Ass}_A(0) \\ y \notin p}} Q_p(Z).$$

$F_y$ is a monic polynomial over $k[\{z_1,\ldots,z_d\}]$ which is separable since each irreducible factor is. Also $F_y(0) = 0$ since

$$F_y(y) = y^t \prod Q_p(y) \in \prod_{p \in \mathrm{Ass}_A(0)} p \subseteq \bigcap_{p \in \mathrm{Ass}_A(0)} p = (0).$$

This shows that $z_1,\ldots,z_d$ is a strongly separating system of parameters.

Corollary (2.10). If $k$ and $A$ are as in Theorem (2.9) then $A$ is strongly separable.

Proof: Immediate consequence of (2.8) and (2.9).

The following example serves two purposes. First, it shows that not every system of parameters is strongly separating. It also shows that a system of parameters may be separating for one summand of the total ring of fractions but not for another one.

Example (2.11). Let $k$ be an algebraically closed, complete and non-trivially valued field of characteristic $p > 2$. Then $k[\{X_1,X_2\}]$ is a regular local unique factorization domain of dimension 2. $X_1^2 - X_2$ and $X_1^p - X_2$ are two prime elements in this ring. Let $P = (X_1^2 - X_2)k[\{X_1,X_2\}]$, $Q = (X_1^p - X_2)k[\{X_1,X_2\}]$ and $I = P \cap Q$. The example we want to consider is $A = k[\{X_1,X_2\}]/I$. If we set $p = P/I$ and $q = Q/I$ then $\mathrm{Ass}_A(0) = \{p,q\}$ and $A$ is reduced. $A/p \cong k[\{X_1\}]$ and $A/q \cong k[\{X_1\}]$. Hence $\dim(A/p) = \dim(A/q) = \dim(A) = 1$ and $A$ is equidimensional. $m = (x_1,x_2)A$ is the maximal ideal of $A$ and $x_1$ is a parameter since $m^2 \subseteq x_1A$. We can now write $A = k[\{x_1\}][Y]/(Y^2 - (x_1^2 + x_1^p)Y + x_1^{p+2})$. Since $P_Y(Z) = Z^2 - (x_1^2 + x_1^p)Z + x_1^{p+2} = (Z - x_1^2)(Z - x_1^p)$ is separable over $k(\{x_1\})$, $x_1$ is a strongly separating parameter. $x_2$ is a parameter, too: $m^{p+2} \subseteq x_2A$. Therefore $A = k[\{x_2\}][Y]/(Y^{p+2} - x_2Y^p - x_2Y^2 + x_2^2)$, $A/p = k[\{x_2\}][Y]/(Y^2 - x_2) = k[\{x_2\}]$ and $A/q = k[\{x_2\}][Y]/(Y^p - x_2)$. We see that

$\Phi(A/p)$ is separable and $\Phi(A/q)$ is not separable over $k(\{x_2\})$.

We can now state and prove our desired result:

Theorem (2.12). Let $k$ be an algebraically closed, complete and non-trivially valued field. Let $A$ be an equidimensional and reduced analytic ring over $k$. Then there exists a system of parameters $x_1,\ldots,x_d$ of $A$ such that the saturation of $A$ with respect to this system exists. For each system of parameters for which it exists the saturation is again an analytic ring over $k$ of dimension $d$.

Proof: We first have to check conditions a through e mentioned in Section 1.3: a) is part of the assumptions; b) follows from (2.1); c) is obvious. To satisfy d) we have to choose a strongly separating system of parameters. That we can find such a system follows from Corollary (2.10). Lemma (2.3) shows now that d) is fulfilled. For e): $k[\{x_1,\ldots,x_d\}] \subseteq A \cap k(\{x_1,\ldots,x_d\}) = R$ and that $A$ is integral over $R$ follows from Theorem (1.2).

Now suppose that $x_1,\ldots,x_d$ is a system of parameters for which the saturation is defined. We denote it by $\tilde{A}_x$ and $\bar{A}$ stands for the integral closure of $A$ in $\Phi(A)$. From Lemma (2.1) and the fact that $A$ is Noetherian we conclude that $\tilde{A}_x$ is a finite $A$ module. From Theorem (1.2) it follows that $\tilde{A}_x$ is finite over $k[\{x_1,\ldots,x_d\}]$.

Since $k[\{x_1,\ldots,x_d\}]$ is integrally closed in its quotient field we have $R = A \cap k(\{x_1,\ldots,x_d\})$ and it follows from [25], (4.1), p.997 that $\mathrm{Spec}(\tilde{A}_x) \to \mathrm{Spec}(A)$ is radicial. Hence $\tilde{A}_x$ is a local ring and $\dim(\tilde{A}_x) = \dim(A) = d$. By the second part of Theorem (1.2) we get that $\tilde{A}_x$ is an analytic ring over $k$.

Corollary (2.13). Let $k$ and $A$ be as in Theorem (2.12). If $x_1,\ldots,x_d$ is a system of parameters for which the saturation exists then it is strongly separating.

Proof: The second half of the proof to Theorem (2.9) shows that Zariski's condition d) implies that the system of parameters is strongly separating.

# CHAPTER III

## TOPOLOGICAL RELATION BETWEEN RING AND SATURATION

As we have shown the saturation of an analytic ring over $k$ is again an analytic ring over $k$, provided one takes a strongly separating system of parameters. As explained in Section 1.2 one can associate analytic set germs with both rings. The requirement that the two analytic set germs are topologically equivalent can be considered as a minimum requirement for an adequate definition of equisingularity. The purpose of this section is to show that this requirement is fulfilled.

Suppose we have two analytic rings over $k$, $A$ and $A'$, $A \subseteq A'$ and $A'$ is finite over $A$, say $A' = A[y_1,\ldots,y_m]$. If $A = k[\{x_1,\ldots,x_n\}]$ then there is an associated analytic set germ $V_A$ in $k^n$. $A' = k[\{x_1,\ldots,x_n,y_1,\ldots,y_m\}]$ gives then rise to a set germ $V_{A'}$ in $k^{n+m}$. In this situation we say that $V_{A'}$ lies over $V_A$.

If $D \in A$ then we will write $D(x_1,\ldots,x_n)$ for a representation of $D$ in $k[\{x_1,\ldots,x_n\}]$. $D(X_1,\ldots,X_n)$ is then the power series which has the same coefficients as $D(x_1,\ldots,x_n)$ but has the ring elements $x_i$ replaced by the indeterminates $X_i$. If $a = (a_1,\ldots,a_n) \in k^n$ then $D(a)$ simply means $D(X_1,\ldots,X_n)$ evaluated at $X_i = a_i$.

<u>Lemma (3.1)</u>. Let $k$ be an algebraically closed, complete and non-trivially valued field of characteristic $p > 0$. Let $A \subseteq A'$ be two analytic rings over $k$ such that $A'$ is a finite A-module. Further assume that there are $D \in A$, $D \neq 0$ and $\alpha \in \mathbb{N}$ such that $Da^{p^\alpha} \in A$ for all $a \in A'$. Then the analytic set germs $V_A$ and $V_{A'}$, where $V_{A'}$ lies over $V_A$, have representatives $(V_A,U)$ and $(V_{A'},U')$ such that above every $a \in V_A$ with $D(a) \neq 0$ there lies one and only one point of $V_{A'}$.

<u>Proof</u>: We first prove the uniqueness: If $A' = A[y_1,\ldots,y_m]$ then we have $Dy_i^{p^\alpha} = g_i \in A$ for $1 \leq i \leq m$. We take a set of defining functions for $V_{A'}$ and include among them the $m$ functions $D(X)Y_i^{p^\alpha} - g_i(X)$. $D(X)$ and $g_i(X)$ are defined as explained previous to the statement of the theorem. Let $a = (a_1,\ldots,a_n) \in V_A$ with $D(a) \neq 0$. Let $b$ and $c$ be two different points above $a$ in $V_{A'}$, $b = (a_1,\ldots,a_n,b_1,\ldots,b_m)$ and $c = (a_1,\ldots,a_n,c_1,\ldots,c_m)$. Since $b$ and $c$ are in $V_{A'}$ we have $D(a)b_i^{p^\alpha} - g_i(a) = 0$ or $b_i^{p^\alpha} - \dfrac{g_i(a)}{D(a)} = 0$ for $1 \leq i \leq m$. In the same way $c_i^{p^\alpha} - \dfrac{g_i(a)}{D(a)} = 0$ for $1 \leq i \leq m$. Hence $b_i$ and $c_i$ are

both solutions of the equation $Z^{p^\alpha} - r = 0$. Since $k$ is algebraically closed there is $s \in k$ such that $s^{p^\alpha} = r$ and therefore $Z^{p^\alpha} - r = (Z-s)^{p^\alpha} = 0$. Hence the equation has only one solution and we conclude that $b_i = c_i$ for $1 \leq i \leq m$, which shows that $b = c$.

It remains to show the existence. If $A' = k[\{X_1,\ldots,X_n,Y_1,\ldots,Y_m\}]/\mathfrak{B}$ let $F_1(X,Y),\ldots,F_s(X,Y)$ be a set of generators for $\mathfrak{B}$. Since $A'$ is a finite $A$-module we may assume that $F_i(X,Y) \in k[\{X\}][Y]$. Take $\rho$ big enough such that

$$[D(X)]^\rho [F_i(X,[g_1(X)/D(X)]^{p^{-\alpha}},\ldots,[g_m(X)/D(X)]^{p^{-\alpha}})]^{p^\alpha}$$

$$= G_i(X) \in k[\{X\}] \quad \text{for} \quad 1 \leq i \leq s.$$

We let $U'_\epsilon = \{(b_1,\ldots,b_{n+m}) \in k^{n+m} \mid |b_i| < \epsilon\}$. We choose $\epsilon$ small enough such that all $F_i(X,Y)$, $1 \leq i \leq s$ converge on $U'_\epsilon$ and consider $(V_{A'},U'_\epsilon)$. Now let $\omega > 0$ such that $\omega \leq \epsilon$, $[g_i(a)/D(a)]^{p^{-\alpha}} < \epsilon$ for all $i$, $1 \leq i \leq m$ whenever $a \in U_\omega = \{(a_1,\ldots,a_n) \in k^n \mid |a_i| < \omega\}$ and such that all $G_i(X)$, $1 \leq i \leq s$ and $D(X)$ are convergent on $U_\omega$. We can now include the $G_i(X)$ among the generators for an analytic set $(V_A,U_\omega)$. If $a \in V_A$ and $D(a) \neq 0$ then we let $b = (a,[g_1(a)/D(a)]^{p^{-\alpha}},\ldots,[g_m(a)/D(a)]^{p^{-\alpha}})$. $b \in U_\epsilon$ and $[F_i(b)]^{p^\alpha} = G_i(a)/D(a)^\rho = 0$ for $1 \leq i \leq s$. Hence $F_i(b) = 0$ which implies that $b \in (V_{A'},U_\epsilon)$.

We now state two lemmas which are well known in the complex case. We will point out at the end of this section (see Theorem (3.6)) why the usual short proofs will not work in our case. We think it is justified to repeat the proofs given by Kneser [10] here because they are not well known and are given only for the case $k = \mathbb{C}$. But the reader will see that the particular field is not of importance.

Lemma (3.2). Let $k$ be a valued field. Let $1 \leq m \leq n$ and $\alpha_1,\ldots,\alpha_n \in k$ ordered such that $|\alpha_1| \leq \ldots \leq |\alpha_n|$. Then there is a non-negative real valued function $\varphi_m$ on $(\mathbb{R}^+)^m$ such that the following two conditions are satisfied:

(1) $|\alpha_m| \leq \varphi_m(t_1,\ldots,t_m)$ where $t_i = |s_{n-i+1}(\alpha_1,\ldots,\alpha_n)|$ and $s_j$ denotes the j-th elementary symmetric polynomial.

(2) $\varphi_m(x_1,\ldots,x_k) \to 0$ as $x_1 \to 0,\ldots,x_k \to 0$.

Proof: Kneser [10], pp.102-104. We do induction on m. If $m = 1$ then we set $\varphi_1(x) = \sqrt[n]{x}$. (2) is clearly satisfied. To see that (1) holds, consider $\varphi_1(t_1) = \sqrt[n]{|\alpha_1 \ldots \alpha_n|} \geq \sqrt[n]{|\alpha_1|^n} = |\alpha_1|$.

We assume now that we have functions $\varphi_1,\ldots,\varphi_k$ each of them satisfying (1) and (2). Suppose that $\alpha_{k+1} = 0$. We set $\varphi_{k+1} = 0$. We can therefore assume that $\alpha_{k+1} \neq 0$. Then we have $\alpha_i \neq 0$ for all $i \geq k+1$. Setting $r = \min\{k,n-k\}$ we get

(3) $$s_{n-k}(\alpha_1,\ldots,\alpha_n) = (\alpha_{k+1}\ldots\alpha_n)[1+ \sum_{i=1}^{r} s_i(\alpha_1,\ldots,\alpha_k)s_i(\alpha_{k+1}^{-1},\ldots,\alpha_n^{-1})].$$

We notice the following inequalities:

(4) $$|s_i(\alpha_1,\ldots,\alpha_k)| \leq |\alpha_1|\ldots|\alpha_i| +\ldots+ |\alpha_{k-i}|\ldots|\alpha_k|$$
$$\leq \varphi_1(t_1)\ldots\varphi_i(t_1,\ldots,t_i)+\ldots+\varphi_{k-i}(t_1,\ldots,t_{k-i})\ldots$$
$$\varphi_k(t_1,\ldots,t_k) = s_i(\varphi_1(t_1),\ldots,\varphi_k(t_1,\ldots,t_k))$$

for $i \leq k$.

Also:

(5) $$|\alpha_{k+1}|^{n-k} \leq |\alpha_{k+1}\ldots\alpha_n|$$

(6) $$|s_i(\alpha_{k+1}^{-1},\ldots,\alpha_n^{-1})| \leq \binom{n-k}{i}|\alpha_{k+1}|^{-i} \quad \text{for all}$$

$i \leq n-k$.

Using (4) through (6) in (3) we get:

$$t_{k+1} = |s_{n-k}(\alpha_1,\ldots,\alpha_n)| \geq |\alpha_{k+1}|^{n-k}[1 - |\sum_{i=1}^{r} s_i(\alpha_1,\ldots,\alpha_k)s_i(\alpha_{k+1}^{-1},\ldots,\alpha_n^{-1})|]$$
$$\geq |\alpha_{k+1}|^{n-k}[1 - \sum_{i=1}^{r} \binom{n-k}{i}|\alpha_{k+1}|^{-i}\, s_i(\varphi_1(t_1),\ldots,\varphi_k(t_1,\ldots,t_k))].$$

We write this inequality as

(7) $$\sum_{i=1}^{r} \binom{n-k}{i}|\alpha_{k+1}|^{-i}\, s_i(\varphi_1(t_1),\ldots,\varphi_k(t_1,\ldots,t_k)) + t_{k+1}|\alpha_{k+1}|^{k-n} \geq 1; \quad r = \min\{k,n-k\}.$$

Let $L_x(Z) = x_{k+1}Z^{n-k} + \sum_{i=1}^{r} \binom{n-k}{i} s_i(\varphi_1(x_1),\ldots,\varphi_k(x_1,\ldots,x_k))Z^i$ where $(x_1,\ldots,x_{k+1}) \in (\mathbb{R}^+)^{k+1}$. If $(x_1,\ldots,x_{k+1}) \neq 0$ then $L(Z) \to \infty$ as $Z \to \infty$ and $L(Z) \to 0$ as $Z \to 0$. Let

$Z_x$ be the smallest positive root of the equation $L_x(Z) = 1$. We define now

$$\varphi_{k+1}(x_1,\ldots,x_{k+1}) = \begin{cases} Z_x^{-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

From (7) we get $|\alpha_{k+1}|^{-1} \geq Z_t$, hence $|\alpha_{k+1}| \leq \varphi_{k+1}(t_1,\ldots,t_{k+1})$ that means (1) is satisfied. It is easy to see that condition (2) is fulfilled, too.

Lemma (3.3). Let $k$ be an algebraically closed field, $f_1(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and $f_2(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_0$, $a_i, b_i \in k$. Denote the roots of $f_2$ by $x_1,\ldots,x_n$ and suppose $c$ is a root of $f_1$ of multiplicity $m$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|a_i - b_i| < \delta$, $0 \leq i \leq n-1$, then $|x_1 - c| < \varepsilon,\ldots,|x_m - c| < \varepsilon$ after appropriate enumeration of the roots of $f_2$.

Proof: If $c \in k$ we define

$$g_1(y) = f_1(y+c) = y^n + \alpha_{n-1}y^{n-1} + \ldots + \alpha_1 y$$

and

$$g_2(y) = f_2(y+c) = y^n + \beta_{n-1}y^{n-1} + \ldots + \beta_0$$

$$g_1(y) = (y+c)^n + \ldots + a_0 = \sum_{k=0}^{n} ( \sum_{\ell=k}^{n} \binom{\ell}{\ell - k} a_\ell c^{\ell-k}] y^k$$

$$= \sum_{k=0}^{n} [ \sum_{\ell=k}^{n} \binom{\ell}{k} a_\ell c^{\ell-k}] y^k .$$

(Note that if $m \in \mathbb{N}$ and $a \in k$ then we write $ma$ to abbreviate the field operation $\sum_{i=1}^{m} a$. In particular, expressions of the form $\binom{m}{n}a$ have to be understood in this sense, that is, with $\binom{m}{n} \in \mathbb{N}$.) Similar $g_2(y) = \sum_{k=0}^{n} [\sum_{\ell=k}^{n} \binom{\ell}{k} b_\ell c^{\ell-k}] y^k$. Hence the new coefficients are $\alpha_k = \sum_{\ell=k}^{n} \binom{\ell}{k} c^{\ell-k} a_\ell$ and $\beta_k = \sum_{\ell=k}^{n} \binom{\ell}{k} c^{\ell-k} b_\ell$. $|\alpha_k - \beta_k| = |\sum_{\ell=k}^{n} \binom{\ell}{k} c^{\ell-k}(a_\ell - b_\ell)| \le \sum_{\ell=k}^{n} \binom{\ell}{k} |c|^{\ell-k} |a_\ell - b_\ell| \le n(n!)c^*|a_{\ell^*} - b_{\ell^*}|$ where $c^* = \max\{1, |c|^{n-k}\}$ and $\ell^*$ is such that $|a_{\ell^*} - b_{\ell^*}| = \max_{0 \le i \le n-1} \{|a_i - b_i|\}$.

Suppose now that $c$ is a root of multiplicity $m$ of $f_1(x)$, then $\alpha_0 = \ldots = \alpha_{m-1} = 0$. Denote by $\gamma_1, \ldots, \gamma_n$ the roots of $g_2(y)$ ordered such that $|\gamma_1| \le \ldots \le |\gamma_n|$. Using the notation from Lemma (3.2) we have $t_i = |s_{n-i+1}(\gamma_1, \ldots, \gamma_n)| = |\beta_{i-1}|$. By Lemma (3.2) there is a function $\varphi_m$ such that $|\gamma_1| \le \ldots \le |\gamma_m| \le \varphi_m(t_1, \ldots, t_m)$. By the above inequalities we have $t_i = |\beta_{i-1}| \le n \cdot (n!)c^*|a_{\ell^*} - b_{\ell^*}|$, $1 \le i \le m$. Since $\varphi_m(t_1, \ldots, t_m) \to 0$ as $t_1, \ldots, t_m \to 0$ and $|x_i - c| = |\gamma_i + c - c| = |\gamma_i| \le \varphi_m(t_1, \ldots, t_m)$ for $1 \le i \le m$, the lemma is proven.

We will now combine the lemmas to prove the following theorem:

<u>Theorem (3.4)</u>. Let $A = k[\{x_1, \ldots, x_n\}]$ and $A' = k[\{x_1, \ldots, x_n, y_1, \ldots, y_m\}]$ where $k$ is an algebraically closed complete and non-trivially valued field. Suppose that

1) $A'$ is a finite $A$-module and $A \subseteq A'$.

2) $A'$ is reduced.

3) $A'$ is a radicial extension of $A$.

Let $V_A$ and $V_{A'}$ be associated analytic set germs in $k^n$ and $k^{n+m}$ respectively. Then there are representatives $(V_A, U)$ and $(V_{A'}, U')$ such that the projection $\pi : k^{n+m} \to k^n$ induces a homeomorphism on the analytic sets.

<u>Proof</u>: If $\mathrm{char}(k) = 0$ then $k = \mathbb{C}$ and the theorem is identical to Theorem 9 of [22], p.429. Hence we assume throughout the proof that $\mathrm{char}(k) > 0$.

Let $P_1, \ldots, P_s$ be the minimal primes of $A'$. Since $A'$ is reduced $P_1 \cap \ldots \cap P_s = (0)$ is an irredundant primary decomposition. Let $p_i = P_i \cap A$, then $(0) = p_1 \cap \ldots \cap p_s$. Suppose we could leave out one of the primes, say $p_1$. Then $p_1 \supset p_2 \cap \ldots \cap p_s$ and we have $p_1 \supseteq p_i$ for some $i$, $2 \leq i \leq s$, say $p_1 \supseteq p_2$. By the going up theorem, [3], (5.11), p.62, there is a prime $Q$ in $A'$ such that $P_2 \subseteq Q$ and $Q \cap A = p_1$. Since the extension is radicial we have $Q = P_1$ and hence $P_2 \subseteq P_1$ which is a contradiction. This shows that $p_1, \ldots, p_s$ are exactly the minimal primes of $A$ and $A$ is therefore also reduced. We have now $\Phi(A) = \Phi(A/p_1) \oplus \ldots \oplus \Phi(A/p_s)$ and $\Phi(A') = \Phi(A'/P_1) \oplus \ldots \oplus \Phi(A'/P_s)$. The map $(a+p_i)/(b+p_i) \to (a+P_i)/(b+P_i)$ determines a natural embedding of $\Phi(A)$ in $\Phi(A')$.

Next we show that there are analytic sets $(V_A, U)$ and $(V_{A'}, U')$ such that above every point of $V_A$ there lies exactly one point of $V_{A'}$. Hence the projection is a bijection between $(V_A, U)$ and $(V_{A'} \cap (U \times k^n), U' \cap (U \times k^n))$. We will show this by induction on the dimension $d$ of $A$. Suppose $d = 0$, then $(V_A, U)$ and $(V_{A'}, U')$ both contain only the origin and the statement follows trivially. Let us now assume that $d > 0$ and that the existence of two sets lying above each other in the required way is established for all smaller dimensions. Since $A \subseteq A' \subseteq \Phi(A')$ we can write $A' = \sum_1^t Aa_i$ where $a_i \in \Phi(A')$. Since the extension is radicial, each $\Phi(A'/P_i)$ is purely inseparable over $\Phi(A/p_i)$ and hence there is an $\alpha \in \mathbb{N}$ such that $a_i^{p^\alpha} = r_i/s_i \in \Phi(A)$, $r_i, s_i \in A$, for all $i$, $1 \leq i \leq t$. Let $D = \prod_1^t s_i \in A$. Then $D \neq 0$ and $Da^{p^\alpha} \in A$ for all $a \in A'$. By Lemma (3.1) we can find $(V_A, U)$ and $(V_{A'}, U')$ such that there is exactly one element in $V_{A'}$ above each element $a \in V_A$ if $D(a) \neq 0$. We consider the analytic subset of $V_A$ on which $D$ vanishes. Let $I' = \mathrm{rad}(D \cdot A')$, $I = I' \cap A$, $\bar{A} = A/I$ and $\overline{A'} = A'/I'$. Clearly, (1), $\overline{A'}$ is a finite $\bar{A}$-module and, (2), $\overline{A'}$ is reduced. But also (3) holds, the extension $\bar{A} \subseteq \overline{A'}$ is radicial. For, suppose $p \in \mathrm{Spec}(\bar{A})$, then it corresponds to some $\tilde{p} \in \mathrm{Spec}(A)$ with $I \subseteq \tilde{p}$. If $P$ and $Q$ are primes in $\overline{A'}$, $P \cap \bar{A} = Q \cap \bar{A} = p$ then consider the corresponding primes $\tilde{P}$ and $\tilde{Q}$ in $A'$. Now $\tilde{P} \cap A = \tilde{Q} \cap A = \tilde{p}$.

This contradicts the radiciality of $A'$ over $A$, hence there is at most one prime above $p$ in $\overline{A'}$. Further we have $\bar{A}/p \cong A/\tilde{p}$ and $\overline{A'}/P \cong A'/\tilde{P}$. Since $A/\tilde{p} \to A'/\tilde{P}$ is purely inseparable so is $\bar{A}/p \to \overline{A'}/P$. In conclusion we have that $\operatorname{Spec}(\overline{A'}) \to \operatorname{Spec}(\bar{A})$ is radicial. Since $D$ is not a zero divisor in $A'$ we have $\dim \overline{A'} < \dim A'$. By the induction hypothesis we have two sets $(V_{\bar{A}}, W)$ and $(V_{\overline{A'}}, W')$ which lie above each other in the required way and therefore do the sets $(V_A, U \cap W)$ and $(V_{A'}, U' \cap W')$.

It remains to show that the projection restricted to the analytic sets is a topological map. Clearly it is continuous, so the only thing left to show is that $(x_1,\dots,x_n) \to (x_1,\dots,x_n,y_1,\dots,y_m)$ is continuous.

Let $(x_1^{(i)},\dots,x_n^{(i)})$ be a sequence in $V_A$ which converges to $(x_1^{(0)},\dots,x_n^{(0)})$. Denote by $(x_1^{(i)},\dots,x_n^{(i)}, y_1^{(i)},\dots,y_m^{(i)})$ the corresponding sequence in $V_{A'}$. Let $\mathfrak{A}$ be a set of generating functions for $V_{A'}$ which includes the integral relations of the generators for $A'$ over $A$. We denote by $F_k(x,Z) = Z^{q_k} + \dots + f_{k0}(x)$ the integral polynomial for $y_k$ and hence $f_{kj}(x_1,\dots,x_n) \in k[\{x_1,\dots,x_n\}]$ for all $j$ and $k$. $a_{k1},\dots,a_{kq_k}$ are the $q_k$ roots of $F_k(x_1^{(0)},\dots,x_n^{(0)},Z) = 0$ and we set $a_{k1} = y_k^{(0)}$. Let $\varepsilon_\ell$ be a sequence of positive real numbers such that $\varepsilon_{\ell+1} < \varepsilon_\ell$, $2\varepsilon_1 < |a_{ki} - a_{kj}|$ if $a_{ki} \neq a_{kj}$ and $\varepsilon_\ell \to 0$ as $\ell \to \infty$. We denote by

$z_{k1}^{(i)},\ldots,z_{kq_k}^{(i)}$ the solutions of $F_k(x^{(i)},Z) = 0$. Then $y_k^{(i)} = z_{k\,\Omega_k(i)}^{(i)}$, $1 \le \Omega_k(i) \le q_k$. Since the $f_{kj}$ are all continuous functions we can apply Lemma (3.3) $(\sum_{k=1}^{m} q_k)$-times to find $N_\ell$ such that whenever $i \ge N_\ell$ then we will have $|a_{kj} - z_{kj}^{(i)}| < \varepsilon_\ell$ for $1 \le k \le m$ and $1 \le j \le q_k$. The above statement requires a proper choice of indexing for the roots of $F_k = 0$ which can always be made. If we have $y_k^{(i)} = z_{k1}^{(i)}$ from some $\ell$ on, then it follows that $y_k^{(i)} \to y_k^{(0)}$ as required. Suppose this is not the case for at least one k. Since we have only a finite number of choices we would have a subsequence $i_r$ such that $\Omega_k(i_r) \neq 1$ for some k but $\Omega_k(i_r) = u_k$ = constant for all k. Since $F(x_1^{(i_r)},\ldots,y_m^{(i_r)}) = 0$ for all $F \in \mathfrak{A}$ and since all F are continuous we have $F(x_1^{(0)},\ldots,x_n^{(0)},a_{1u_1},\ldots,a_{mu_m}) = 0$ for all $F \in \mathfrak{A}$ with $a_{iu_i} \neq a_{i1}$ for at least one i, $1 \le i \le m$. Thus we would have a second element lying above $(x_1^{(0)},\ldots,x_n^{(0)})$ which contradicts our previous findings.

The main theorem of this section follows now easily.

<u>Theorem (3.5)</u>. Let k be an algebraically closed, complete and non-trivially valued field. Let A be an equidimensional reduced ring which is analytic over k. Let $x_1,\ldots,x_d$ be a strongly separating system of parameters of A and $\tilde{A}_x$ the saturation with respect to

this system. Then the two associated analytic set germs are topologically equivalent.

In fact, the homeomorphism can be induced by the natural projection of the ambient spaces if the representations of the rings are chosen so that the associated set germs lie above each other.

Proof: By Theorem (2.12) $\tilde{A}_x$ is analytic over $k$ and hence a finite $k[\{x\}]$-module. This shows that $\tilde{A}_x$ is finite over $A$. Since $\tilde{A}_x \subseteq \Phi(A)$ and $A$ is reduced, we conclude that $\tilde{A}_x$ is reduced. The fact that $\tilde{A}_x$ is radicial is proven in Theorem 4.1, [25], p.997. We can now apply (3.4) and get the second half of our theorem. Theorem (1.3) shows that the particular representation of the ring does not matter and therefore finishes up the proof.

Lemma (3.2) and the part in the proof to Theorem (3.4) which establishes the continuity of the map can be proven much more easily in the case when $k = \mathbb{C}$. The shorter proofs are based on the fact that every bounded sequence in $\mathbb{C}$ has a convergent subsequence. The following theorem shows that we do not have this fact available in our situation and that we can therefore not hope to adapt the usual proofs.

Recall that a space is called sequentially compact if and only if every sequence has a convergent subsequence.

<u>Theorem (3.6)</u>. Let $k$ be an algebraically closed, non-trivially valued field of positive characteristic and let $A_\alpha = \{x \in k \mid |x| \leq \alpha\}$, where $\alpha \in \mathbb{R}^+$. Then $A_\alpha$ is not sequentially compact.

<u>Proof</u>: Since $A_\alpha$ is metric it is paracompact; see [4], p.186, Theorem 5.3. By [9], p.162, E), part (d), $A_\alpha$ is sequentially compact if and only if it is countably compact. The latter is the case if and only if $A_\alpha$ is compact, [4], p.230, Corollary 3.4.

Now suppose $A_\alpha$ is sequentially compact and hence compact. Then $k$ is locally compact, since addition is continuous. Since $\mathrm{char}(k) > 0$ the valuation is non-archimedean and from Theorem 1 of [18], p.245, it follows that the valuation is discrete, that is $|k - \{0\}|$ is a cyclic subgroup of the positive real numbers. Say $|x|$ is a generator of this group. We can assume that $|x| > 1$. It is easy to see that $|x| = \min\{|y| > 1 \mid y \in k\}$. Since $k$ is algebraically closed there is $a \in k$ such that $a^2 = x$ and therefore $1 < |a| < |x|$. This contradiction shows that $A_\alpha$ cannot be sequentially compact.

# CHAPTER IV

## MULTIPLICITY

In order to be equisingular it is certainly a necessary condition for two analytic set germs that their local rings have the same multiplicity. One of the main theorems in [28], Theorem 4.1, p.455, states that under certain conditions the multiplicity is preserved by passing to the saturation. The proof of our corresponding theorem follows basically the one of Zariski. However, the conditions are somewhat different so that some of the details have to be changed. For this reason we include here the somewhat lengthy proof in full.

We will need some definitions and results which are due to Northcott and Rees.

<u>Definition (4.1)</u>. If $\mathfrak{A}$ and $\mathfrak{B}$ are ideals of a ring $R$ then $\mathfrak{B}$ is a reduction of $\mathfrak{A}$ if $\mathfrak{B} \subseteq \mathfrak{A}$ and $\mathfrak{B}\mathfrak{A}^r = \mathfrak{A}^{r+1}$ for at least one positive integer $r$.

<u>Theorem (4.2)</u> (Northcott-Rees, [17], p.357). Let $(A,m)$ be local with $\mathrm{char}(A) = \mathrm{char}(A/m)$. If $q$ is an $m$-primary ideal of $A$ and $(x_1,\ldots,x_d) = \mathfrak{A}$ is a parameter ideal which is a reduction of $q$ then their multiplicities are the same: $e(\mathfrak{A}) = e(q)$.

Now we come to the main theorem in this section.

Theorem (4.3). Suppose $k$ is an algebraically closed, complete and non-trivially valued field. Let $A$ be an equidimensional and reduced ring which is analytic over $k$. Suppose $x_1,\ldots,x_d$ is a strongly separating system of parameters and the ideal it generates is a reduction of the maximal ideal of $A$. Denote by $F_j^*$ the least Galois extension of $k(\{x_1,\ldots,x_d\})$ which contains $\Phi(A/p_j)$, where $\{p_1,\ldots,p_s\} = \mathrm{Ass}_A(0)$. Suppose that $\mathrm{char}(A)$ and $[F_j^* : k(\{x_1,\ldots,x_d\})]$ are relatively prime for all $j$, $1 \leq j \leq s$. Then $e(A) = e(\tilde{A}_x)$.

Proof: See [28], §4. We construct a sequence of rings $A_i = A_{i-1}[L_i]$, where $A_0 = A$ and $L_i$ contains all elements of $\bar{A}$ which dominate some element in $A_{i-1}$. By Lemma (2.1), $\bar{A}$ is a finitely generated Noetherian $A$-module and hence so is $\tilde{A}_x$. Thus, there is an integer $n$ such that $A_n = \tilde{A}_x$. Each $A_i$ is finitely generated over $A$ and thus over $k[\{x_1,\ldots,x_d\}]$. Since the latter ring is a Henselian domain, we see that $A_i$ is local ([14], (43.12), p.183). Hence each $A_i$ is analytic over $k$ with maximal ideal $M_i$. Clearly $\tilde{A}_{ix} = \tilde{A}_x$.

If $m$ is the maximal ideal in $k[\{x_1,\ldots,x_d\}]$ then

(1) $e(mA) = e(mA_i)$ for all $i \geq 0$.

We will show first that no element of $k[\{x_1,\ldots,x_d\}]$ is a zero divisor in $A_i$. Recall from the proof of Lemma (2.3) that $k[\{x_1,\ldots,x_d\}] \cap p = (0)$ for all primes $p$ in $A$

which belong to zero. If $r \in A$ and $r$ is not a zero divisor in $A$ then $r$ is not a zero divisor in $\Phi(A)$, for, otherwise there is an element $a/s$ with $a,s \in A$ and $(r/1)\cdot(a/s) = 0$. Then $rat = 0$ for some non-zero divisor $t$ in $A$. This is impossible. Consequently, if $a \in A_i$ is a zero divisor and $a \in A$, then $a \in p$ for some $p \in \mathrm{Ass}_A(0)$ and hence $a \notin k[\{x_1,\ldots,x_d\}]$. We can now apply the projection formula of [30], p.299, Corollary 1 and get:

$$\begin{aligned}[A_i/M_i : k[\{x_1,\ldots,x_d\}]/m]e(mA_i) &= [A_i : k[\{x_1,\ldots,x_d\}]e(m) \\ &= [A : k[\{x_1,\ldots,x_d\}]e(m) \\ &= [A/M : k[\{x_1,\ldots,x_d\}]/m]e(mA).\end{aligned}$$

This implies that $e(mA_i) = e(mA)$.

We will need a valuation theoretic characterization of a reduction: Let $F_j = \Phi(A/p_j)$ where $\{p_1,\ldots,p_s\} = \mathrm{Ass}_A(0)$. Then $\Phi(A) = F_1 \oplus\ldots\oplus F_s$ and we denote by $\pi_j$ the j-th projection. Let $S_j$ be the set of all non-trivial valuations of $F_j$ which are nonnegative on $A/p_j$. If $v \in S_j$ and $x \in A$ then we will write $v(x)$ for $v(\pi_j(x))$ in order to keep the notation simpler. If $v \in S_j$ and $\mathfrak{A}$ is an ideal in $A$ then $v(\mathfrak{A}) = \min\{v(x) \mid x \in \mathfrak{A}\}$. $\mathfrak{A}_j$ denotes the ideal in $A/p_j$ which is generated by $\pi_j\mathfrak{A}$. The derived complete ideal of $\mathfrak{A}$ is defined as $\mathfrak{A}' = \bigcap_{j=1}^{s} \left(\bigcap_{v\in S_j} \pi_j^{-1}\mathfrak{A}_j R_v\right)$ where $R_v$ denotes the valuation ring of $v$. $S = \bigcup_{j=1}^{s} S_j$.

<u>Claim</u>: If $\mathfrak{A}$ and $\mathfrak{B}$ are ideals of A, $\mathfrak{A}$ contains some non-zero divisor and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A}$ is a reduction of $\mathfrak{B}$ if and only if $v(\mathfrak{A}) = v(\mathfrak{B})$ for all $v \in S$.

First, assume that $\mathfrak{A}$ is a reduction of $\mathfrak{B}$. We set $\bar{\mathfrak{A}} = \{x \in A \mid x$ is integral over $\mathfrak{A}\}$. By [16], p.156, Theorem 3, $\mathfrak{B} \subseteq \bar{\mathfrak{A}}$. By [15], p.167, Theorem 1, we have $\mathfrak{B} \subseteq \mathfrak{A}'$. For arbitrary $v \in S_j$ and arbitrary $x \in \mathfrak{B}$ we have $x \in \mathfrak{A}'$ and in particular $x \in \pi_j^{-1}\, \mathfrak{A}_j R_v$. We can write $\pi_j(x) = \sum_{i=1}^{n} \alpha_i \pi_j(a_i)$ where $(a_1,\ldots,a_n) = \mathfrak{A}$ and $\alpha_i \in R_v$. Now $v(x) \geq \min\{v_1(\alpha_1\pi_j(a_1)),\ldots,v(\alpha_n\pi_j(a_n))\} = v(\alpha_\ell\pi_j(a_\ell)) \geq v(a_\ell)$ for some $\ell$, $1 \leq \ell \leq n$. Therefore $v(\mathfrak{A}) \leq v(\mathfrak{B})$. The other inequality is obvious and we get $v(\mathfrak{A}) = v(\mathfrak{B})$.

Now suppose that $v(\mathfrak{A}) = v(\mathfrak{B})$ for all $v \in S$. Let $b \in \mathfrak{B}$, $\pi_j(b) \neq 0$ and $v \in S_j$, then there is $a \in \mathfrak{A}$ such that $v(a) \leq v(b)$. In case $v(a) = v(b)$ then $v(a/b) = v(a) - v(b) = 0$ or $\pi_j(a)/\pi_j(b) = u \in R_v$ and $u^{-1} = \pi_j(b)/\pi_j(a) \in R_v$. Hence $\pi_j(b) = u^{-1}\pi_j(a) \in \mathfrak{A}_j R_v$. In case $v(b) > v(a)$, then $v(a+b) = v(a)$. We may assume $\pi_j(a+b) \neq 0$, then $v(a/a+b) = 0$, hence $\pi_j(a)/\pi_j(a+b) = u \in R_v$ and $\pi_j(a)u^{-1} = \pi_j(a+b) \in \mathfrak{A}_j R_v$. Since $\pi_j(a) \in \mathfrak{A}_j R_v$ we get as before $\pi_j(b) \in \mathfrak{A}_j R_v$. The same is true if $\pi_j(b) = 0$. Since $j$ and $v$ were arbitrary we see that $b \in \mathfrak{A}'$ and hence $\mathfrak{B} \subseteq \mathfrak{A}'$. By Theorem 1 of [15], p.167, $\mathfrak{B} \subseteq \bar{\mathfrak{A}}$ and by Theorem 3 of [16], p.156 we have that $\mathfrak{B}$ is a reduction of $\mathfrak{A}$ as required. This finishes the proof of the claim.

We will now use induction on the index $i$ to show that $(x_1,\dots,x_d)A_i$ is a reduction of $M_i$ for all $i \geq 0$. The case $i = 0$ is trivial. The fact that $x_1,\dots,x_d$ is a system of parameters in $A_i$ for all $i$ follows from [30], p.276, Theorem 15(d). We may now assume that $(x_1,\dots,x_d)A_i$ is a reduction of $M_i$ or, equivalently, that $v(mA_i) = v(M_i)$ for all $v \in S$. We have to show that this implies

(2) $v(mA_{i+1}) = v(M_{i+1})$ for all $v \in S$.

To prove (2) notice first that we clearly have $v(M_{i+1}) \leq v(mA_{i+1})$. We write $A_{i+1} = A_i[T]$ where we can assume that $T \subseteq M_{i+1}$. For, if $y \in T$, but $y \not\in M_{i+1}$ then replace it by $z = y - c$ where $c \in k$. It suffices now to show the following: For all $y \in T$ and for all $v \in S\cdot$ we have $v(y) \geq v(x_i)$ for all $i$, $1 \leq i \leq d$. If this were true and $a \in M_{i+1}$ then $a = \sum \alpha_i b_i$, $b_i \in T$ and $\alpha_i \in A_i$. $v(a) \geq \min\{v(\alpha_i b_i)\} = v(\alpha_\ell b_\ell) = v(\alpha_\ell) + v(b_\ell) \geq v(b_\ell)$ for some $\ell$. Hence $v(M_{i+1}) \geq v(T) \geq v(mA_{i+1})$.

Let $F_j^*$ be the least Galois extension of $K = k(\{x_1,\dots,x_d\})$ containing $F_j$. Denote by $S_j^*$ the set of all valuations of $F_j^*$ which are nonnegative on $R = k[\{x_1,\dots,x_d\}]$. Since $\pi_j A$ is integral over $R$ the elements of $S_j^*$ are nonnegative on $\pi_j A$, that means $v^*|_{F_j} \in S_j$ for all $v^* \in S_j^*$. By [5], Theorem 13.2, p.94 to every $v \in S_j$ there is an extension $v^* \in S_j^*$. Hence

$S_j^*$ consists exactly of the extensions of $v \in S_j$ to $F_j^*$. Hence we can also prove equivalently that $v^*(y) \geq v^*(m)$ for all $v^* \in S^*$ and all $y \in T$. Let $S^O$ be all valuations of $K$ which are nonnegative on $R$. We have $v^*|_K \in S^O$ if $v^* \in S_j^*$ and $S_j^*$ consists of all the extensions of $S^O$ to $F_j^*$. All extensions belonging to $v_O \in S^O$ form a complete set of conjugates under $G = \mathrm{Gal}(F_j^*|K)$. More precisely, if $v^*$ extends $v_O$ and $\tau \in G$ then $v^*\tau$ extends $v_O$ and if $v'$ extends $v_O$ then $v' = v^*\tau'$ for some $\tau' \in G$; see [30], p.28, Corollary 3.

(3) If $y \in T$ arbitrary and $v_O \in S^O$ arbitrary, then there exists at least one $v^* \in S_j^*$ which extends $v_O$ and $v^*(y) \geq v^*(m)$.

For the proof of this statement, assume that this is not the case, or $v^*(y) < v^*(m)$ for all $v^*$ extending $v_O$. Let $h = [F_j : K]$, $g = [F_j^* : K]$. Note that if $p = \mathrm{char}(A)$ then $(h,p) = 1 = (g,p)$ and $h|g$. $T_{F_j^*|K}(y) = \sum_{\tau \in G} y^\tau = (g/h)T_{F_j/K}(y)$. Let $y_O = (1/g)T_{F_j^*|K}(y) = (1/h)T_{F_j|K}(y) = (1/g)\sum_{\tau \in G} y^\tau$. (If $n \in \mathbb{Z}$ then $n = \sum_{i=1}^{n} 1$, $1 \in k$ and $(1/n) = n^{-1}$.)

We show that $y_O \in m$, the maximal ideal of $R$. Since $y$ is integral over $R$ there is a minimal polynomial $f(X) = X^n + c_{n-1}X^{n-1} + \ldots$ of $y$ over $R$. By [8], p.147, Lemma 3, $f(X)$ is irreducible over $K$ and hence $y_0 = c_{n-1} \in R$. Let $\bar{R}_j$ be the integral closure of $R$ in $F_j$. By the lying over theorem there is $p \in \mathrm{Spec}(\bar{R}_j)$ such that $p \cap \pi_j(A_{i+1}) = \pi_j M_{i+1}$. $(\pi_j A_{i+1})^\tau$ is integral over $R$ and we have $p \cap (\pi_j A_{i+1})^\tau = (\pi_j M_{i+1})^\tau$ by [3], (5.8), p.61. Using the same theorem, $R \cap p = m$. $\pi_j(y_0) \in p$ since $y \in M_{i+1}$. Hence $y_0 \in m$.

For all extensions $v^*$ of $v_0$ we have $v^*(y_0) \geq v^*(m)$, hence $v^*(y) < v^*(y_0)$, hence $v^*(y - y_0) = v^*(y)$ for all $v^*$ extending $v_0$. Therefore $v^*(\sum_{\tau \in G}(y^\tau - y)) = v^*(y)$ for all extensions $v^*$ of $v_0$. This implies that for every extension $v^*$ of $v_0$ there is an element $\tau \in G$ such that $v^*(y^\tau - y) \leq v^*(y)$. Since $y \in L_{i+1}$ it dominates some $z \in A_i$ and we have $v^*(y^\tau - y) \geq v^*(z^\tau - z)$ for all $v^*$ in $S_j^*$ and for all $\tau \in G$. We may assume that $z \in M_i$ (if not, replace by $z - c$ where $c + M_i = z + M_i$ and $c \in k$). Hence $v^*(y) \geq v^*(z^\tau - z)$ for all $v^*$ extending $v_0$ and for a suitable $\tau \in G$. Fix $v_0^*$ extending $v_0$ and $\tau_0 \in G$ such that $v_0^*(z^{\tau_0}) = \min_{\tau \in G}\{v_0^*(z^\tau)\}$. Set $v_1^* = v_0^*\tau_0$ then we have $v_1^*(z) = v_0^*\tau_0(z) = v_0^*(z^{\tau_0}) = \min_{\tau \in G}\{v_0^*(z^\tau)\} = \min_{\tau \in G}\{v_1^*\tau_0^{-1}(z^\tau)\} = \min_{\tau \in G}\{v_1^*(z^{\tau_0^{-1}\tau})\}$. $v_1^*(z) = \min_{\tau \in G}\{v_1^*(z^\tau)\}$. Hence $v_1^*(z^\tau - z) \geq v_1^*(z)$ for all $\tau \in G$. For some $\tau \in G$, $v_1^*(y) \geq v_1^*(z^\tau - z) \geq v_1^*(z)$. Hence $v_1^*(y) \geq v_1^*(z) \geq v_1^*(M_i) = v_1^*(m)$. The last equality comes from the induction hypothesis. This proves statement (3).

Now consider an arbitrary extension $v^*$ of $v_0$. $v^*(m) = v^*(M_i)$ and since $z \in M_i$ we have for all $v^*$ in $S_j^*$ and for all $\tau$ in $G$, $v^*(z^\tau) \geq v^*(m)$. Hence $v^*(z^\tau - z) \geq v^*(m)$. Since $y$ dominates $z$ we get $v^*(y^\tau - y) \geq v^*(m)$ for all $v^* \in S_j^*$ and $\tau \in G$. Using (3) we get $v_1^*(y^\tau) \geq \min\{v_1^*(z^\tau - z), v_1^*(y)\} \geq v_1^*(m)$. Since $m^\tau = m$ and $v_1^*\tau$ ranges over all extensions of $v_0$ we have $v^*(y) \geq v^*(m)$ for all $v^*$ extending $v_0$. This finishes the proof of (2).

The theorem follows now easily. Namely, by Theorem (4.2) and the assumption that $mA$ is a reduction of $M_0$, we have $e(A) = e(M_0) = e(mA)$. By (1), $e(mA) = e(mA_n)$. By (2), together with Theorem (4.2): $e(mA_n) = e(M_n) = e(A_n)$. Since $A_n = \tilde{A}_x$ we have $e(A) = e(\tilde{A}_x)$ as desired.

<u>Definition (4.4)</u>. A system of parameters $x_1, \ldots, x_d$ of a local ring $(A,m)$ is said to be transversal if $e((x_1, \ldots, x_d)A) = e(m)$.

<u>Corollary (4.5)</u>. Let $k$ be as in (4.3). Let $A$ be an analytic integral domain over $k$. Let $x_1, \ldots, x_d$ be a (strongly) separating and transversal system of parameters. Denote by $F^*$ the least Galois extension of $k(\{x_1, \ldots, x_d\})$ which contains $\Phi(A)$. Suppose that char(k) and $[F^* : k(\{x_1, \ldots, x_d\})]$ are relatively prime. Then $e(A) = e(\tilde{A}_x)$.

<u>Proof</u>: The statement follows from (4.3) if we can show that $(x_1,\ldots,x_d)A$ is a reduction of the maximal ideal in $A$. This follows from [20], p.16, Theorem 3.2, if we can show that all minimal primes in the completion of $A$ are of dimension $d = \dim(A)$. [14], p.188, Theorem (44.1) implies that $A$ is analytically irreducible, that is, the completion is a domain.

As shown in (2.10) we can always find a strongly separating system of parameters if $k$ is algebraically closed and $A$ is equidimensional. If $k$ is infinite (which is of course the case if $k$ is algebraically closed), then there is a system of parameters in $A$ which generates a reduction of the maximal ideal, [17], p.356, Corollary to Theorem 2. It is not known to us, under what conditions we can find a system of parameters which satisfies both conditions. As in the algebroid case, the question remains open, whether the multiplicity is preserved if the parameters do not generate a reduction of the maximal ideal (see the remark beneath Corollary 4.2 in [28], p.460). However, there definitely are non-trivial cases to which our theorem applies. To show this, is the purpose of the following example.

<u>Example (4.6)</u>. This is a ring satisfying all conditions of (4.5) and having non-trivial saturation. Let $k$ be an algebraically closed, non-trivially valued field with $\mathrm{char}(k) > 3$. We consider $A = k[\{X,Y\}]/(Y^3 + Y^2 + X^2 + 2XY)$. It is easy to check that $A$ is a domain and

$\dim(A) = 1$. We write $x$ and $y$ for $X$ and $Y$ modulo the relation. $m = (x,y)$ is the maximal ideal and $x$ is a system of parameters since $m^2 \subseteq (x)$. Hence $A = k[\{x\}][y]$ where $y^3 + y^2 + 2xy + x^2 = 0$. $(x)$ is a reduction of $m$ since $m^2(x) = m^3$. One also checks that it is (strongly) separating. $f(Z) = Z^3 + Z^2 + 2xZ + x^2$ is irreducible in $k[\{x\}][Z]$ and hence in $k(\{x\})[Z]$. Therefore $[\Phi(A) : k(\{x\})] = [k(\{x\})(y) : k(\{x\})] = \deg(f(Z)) = 3$. $F^*$ is the splitting field of $f(Z)$ over $k(\{x\})$, hence $\mathrm{char}(A)$ and $[F^* : k(\{x\})]$ are relatively prime. That $A$ is not saturated can be seen as follows: If it were saturated then $A$ were an Arf ring since $\dim(A) = 1$ and $A$ is Cohen-Macaulay, [11], p.682, Corollary 5.3. Then we would have $\dim_{A/m}(m/m^2) = e(A)$, by [11], p.661, Theorem 2.2. Since $(Y^3 + Y^2 + 2XY + X^2) \subseteq (X,Y)^2$ and $k[\{X,Y\}]$ is a regular local ring of dimension 2, we have $\dim_{A/m}(m/m^2) = 2$. To calculate the multiplicity of $A$ we use [30], p.299, Corollary 1 and get $e(xA) = 3$. Since $(x)$ is a reduction of $m = (x,y)$ we have $e(A) = 3$. This contradiction shows that $A$ is not saturated.

## CHAPTER V

## RELATIVE LIPSCHITZ-SATURATION

An alternative definition of saturation was developed by Pham and Teissier [19]. We repeat it here as given in [12], p.792.

Definition (5.1). Let $R$ be a ring and let $g : A \to B$ be a homomorphism of R-algebras. The Lipschitz-saturation $A^*_{B,R}$ of $A$ in $B$, relative to $R \to A \to B$ is the set $A^*_{B,R} = \{x \in B | x \otimes_R 1 - 1 \otimes_R x$ is integral over the kernal of the canonical map $B \otimes_R B \to B \otimes_A B\}$. (This kernal is generated by all the elements $g(a) \otimes 1 - 1 \otimes g(a)$, $a \in A$).

$A$ is saturated in $B$ relative to $R \to A \to B$ if $A^*_{B,R} = g(A)$.

For properties of this saturation, see [12]. In the case of an analytic ring $A = k[\{x_1,\dots,x_d\}][y_1,\dots,y_n]$ we have $k[\{x_1,\dots,x_d\}] \to A \to \bar{A}$, $\bar{A}$ is the integral closure of $A$ in $\Phi(A)$. We will just write $A^*_x$ for $A^*_{\bar{A},k[\{x\}]}$, where $x = (x_1,\dots,x_d)$.

<u>Theorem (5.2)</u>. Let $k$ be an algebraically closed, complete and non-trivially valued field, $A$ an equidimensional and reduced analytic ring over $k$, and $x_1,\ldots,x_d$ a strongly separating system of parameters of $A$. Then $\tilde{A}_x \subseteq A_x^*$.

<u>Proof</u>: The existence of $\tilde{A}_x$ is proven in Theorem (2.12). In the proof of Lemma (2.3) we have shown that under the present hypothesis, $[\bigcup_{p \in \mathrm{Ass}_A(0)} p] \cap k[\{x_1,\ldots,x_d\}] = (0)$. $k[\{x_1,\ldots,x_d\}]$ is integrally closed in its quotient field. Hence the conditions for Corollary (4.2) of [12], p.807 are fulfilled and we get the statement of the theorem.

Proposition (1.4) of [12], p.797 shows that $A \to A_x^*$ is a radicial extension. For this reason we can translate some of our earlier results directly to the case of the Lipschitz-saturation. Because of Theorem (5.2) these results are actually stronger. Notice also, that no separability conditions are required in this case.

<u>Corollary (5.3)</u>. Let $k$ be an algebraically closed, complete and non-trivially valued field. Let $A$ be an equidimensional and reduced analytic ring over $k$. The Lipschitz-saturation of $A$ with respect to any system of parameters exists, is again an analytic ring over $k$ and $\dim(A) = \dim(A_x^*)$.

Proof: The same as for Theorem (2.12) except that we use Proposition (1.4) of [12], p.797 instead of [25], (4.1).

Corollary (5.4). Let $k$ and $A$ be as in (5.3). If $V$ and $W$ are analytic set germs associated with $A$ and $A_x^*$ respectively, then $V$ and $W$ are topologically equivalent. For appropriate representations, the homeomorphism is induced by the natural projection of the ambient spaces.

Proof: The same proof as for (3.5). Again Proposition (1.4) of [12], p.797 replaces Theorem (4.1) of [25].

We do however not know if an analogous statement to Theorem (4.3) holds.

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