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thesis entitled

Bifurcation and Oscillation for Systems of Equations

presented by

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has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Date June 22, 1979

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# BIFURCATION AND OSCILLATION FOR SYSTEMS OF EQUATIONS

By

Beth Angela Barron

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1979

ABSTRACT

BIFURCATION AND OSCILLATION
FOR SYSTEMS OF EQUATIONS

By

Beth A. Barron

This thesis concerns the system of differential equations

$$\ddot{x} + g(x,y) = \mu_1 f_1(t) + \lambda_1 \dot{x}$$

$$\ddot{y} + h(x,y) = \mu_2 f_2(t) + \lambda_2 \dot{y}$$

when the parameters $\mu_1, \mu_2, \lambda_1, \lambda_2$ are small. It is assumed that all functions are smooth, that $f_1$ and $f_2$ are periodic of period $T$, and that for $(\mu_1, \mu_2, \lambda_1, \lambda_2) = 0$, there exists a periodic solution $(p(t), q(t))$ of least period $T$. We investigate the existence and bifurcation of T-periodic solutions near $(p(t+\alpha), q(t+\alpha))$ ($\alpha$ in $[0,T]$) when $(\mu_1, \mu_2, \lambda_1, \lambda_2)$ is near the origin.

In the first part of the thesis, we assume that the corresponding linearized homogeneous equation has a one-dimensional null space and in sectorial regions of a neighborhood of the origin in $\mathbb{R}^4$ obtain results on the existence and numbers of solutions which reduce to

$(p(t+\alpha),q(t+\alpha))$ $(\alpha \in [0,T])$ at the origin. In the second part we assume a two dimensional null space and that $\lambda_1 = \lambda_2 = 0$. We then describe regions in the $\mu_1$-$\mu_2$ plane in which there exist solutions which reduce to $(p(t+\alpha),q(t+\alpha))$ $(\alpha \in [0,T])$ at the origin.

**To my parents**

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor, Dr. Shui Nee Chow, for his guidance and concern. I am especially grateful for his enthusiasm, which has made the last few years so enjoyable.

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# CHAPTER I

# INTRODUCTION

Problems in nonlinear oscillation arise frequently in the study of mechanical and electrical systems. Such problems have been studied extensively, but due to their complexity, most of the knowledge of these problems is applicable only in specific cases.

Perhaps the best known example of nonlinear oscillation is the simple pendulum [ 9 ], for which the equation is

$$m\ell^2\ddot{x} + mg\ell \sin x = 0$$

where $m$ is the attached mass, $\ell$ the length, $x$ the angular displacement, and $g$ the acceleration of gravity. Many other examples are discussed by Andronov, Vitt, and Khaikin in <u>Theory of Oscillators</u> [ 1 ]. The general equation

$$\ddot{x} + g(x) = 0 \tag{1.1}$$

describes many such oscillatory motions which occur with no external forcing or damping. In practice damping, such as friction, and external forcing will be present, at least in small amounts. This may be represented by the equation

$$\ddot{x} + g(x) = \mu f(t) + \lambda\dot{x} \tag{1.2}$$

where $\lambda$ and $\mu$ are near zero. It is usually assumed that (1.1) has a periodic solution $x_0(t)$ of the same period as $f(t)$.

In recent papers, Loud [ 7 ] and Hale and Taboas [ 6 ] have considered this equation where $\lambda$ and $\mu$ vary independently. For $(\mu,\lambda)$ near the origin and in a sectorial region about a certain line in the $\mu$-$\lambda$ plane, Loud observes solutions near $x_0(t+\alpha)$ for small values of $\alpha$. He also observes that these solutions are discontinuous at the origin [ 7 ].

Hale and Taboas [ 6 ] use a different method to obtain bifurcation curves and a characterization of the number of solutions in an entire neighborhood of the origin in the $\mu$-$\lambda$ plane. It is important that these solutions do not remain close to $x_0(t+\alpha_0)$ for a particular value of $\alpha_0$ but rather to a family $\{x_0(t+\alpha)\}$ where $\alpha$ varies over a compact set. Thus the classical perturbation method cannot be used to investigate these solutions in an entire neighborhood of the origin. Also the solutions will not be continuous at the origin.

In this thesis, we examine the system of equations

$$\ddot{x} + g(x,y) = \mu_1 f_1(t) + \lambda_1 \dot{x}$$

$$\ddot{y} + h(x,y) = \mu_2 f_2(t) + \lambda_2 \dot{y}$$

which describes a coupled system of oscillators. In Chapter II, we assume that the homogeneous linear equation has a null space of dimension one. Our method is similar to that of Hale and Taboas [ 6 ]. We obtain results for

$(\mu_1, \mu_2, \lambda_1, \lambda_2)$ in a neighborhood of the origin in $\mathbb{R}^4$, where we allow the parameters to vary independently in the spirit of Chow, Hale, and Mallet-Paret [ 2,3 ]. Results for a single equation may be obtained as a special case of the main theorem of Chapter II.

In Chapter III, we consider the more complex case where the homogeneous linear equation has a two dimensional null space. This case is not meaningful for a single equation, but can be considered for a system of equations. We consider this case without damping, and find that for $(\mu_1, \mu_2)$ near 0, solutions near a known solution of the homogeneous equation occur only on manifolds or in certain narrow regions, which we describe precisely.

## CHAPTER II

## ONE DIMENSIONAL NULL SPACE

### §1. Introduction

In all that follows we will be considering the coupled system of second order, scalar equations

$$(1.1)\qquad \begin{aligned} \ddot{x} + g(x,y) &= 0 \\ \ddot{y} + h(x,y) &= 0 \end{aligned}$$

and certain perturbations of this system. We will assume that $g$, $h$ and all other functions introduced are as smooth as is required.

We also assume that $(p(t),q(t))$ is a periodic solution of (1.1) of least period $T > 0$; in particular,

$$(1.2)\qquad \begin{aligned} \ddot{p}(t) + g(p(t),q(t)) &= 0 \\ \ddot{q}(t) + h(p(t),q(t)) &= 0 \end{aligned}$$

hold for all $t$. We will consider the existence of T-periodic solutions near $(p,q)$ of the perturbed system

$$(1.3)\qquad \begin{aligned} \ddot{x} + h(x,y) &= \mu_1 f_1 + \lambda_1 \dot{x} \\ \ddot{y} + h(x,y) &= \mu_2 f_2 + \lambda_2 \dot{y} \end{aligned}$$

where $f_1(t)$ and $f_2(t)$ are T-periodic and $\mu_1, \mu_2, \lambda_1, \lambda_2$ are near zero. Whenever we refer to a solution, it will be understood that we mean a solution near $(p,q)$.

## §2. Change of Coordinates

System (1.3) can be rewritten in the form

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(x_1,y_1) + \mu_1 f_1 + \lambda_1 x_2 \\
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -h(x_1,y_1) + \mu_2 f_2 + \lambda_2 y_2 .
\end{aligned} \tag{2.1}
$$

The quadruple $(p,\dot{p},q,\dot{q},)$ is a solution to this system when $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 0$. Let $\Gamma = \{(p(t),\dot{p}(t),q(t),\dot{q}(t)): 0 \leq t < T\} \subset \mathbb{R}^4$. $\Gamma$ is the orbit generated by the periodic solution $(p,\dot{p},q,\dot{q})$ and is a Jordan curve. We are seeking periodic solutions whose initial values are near $\Gamma$, and thus our problem will be much easier if we introduce a new coordinate system in a "tubular" neighborhood of $\Gamma$. In Hale [ 5 ], a method is described for defining such a "moving orthonormal system along $\Gamma$." By this result, any T-periodic solution $(x_1(t),x_2(t),y_1(t),y_2(t))$ of (2.1) has a unique representation of the form

$$
\begin{aligned}
x_1(t) &= p(t+\alpha) + z(t) \\
x_2(t) &= \dot{p}(t+\alpha) + \dot{z}(t) \\
y_1(t) &= q(t+\alpha) + w(t) \\
y_2(t) &= \dot{q}(t+\alpha) + \dot{w}(t),
\end{aligned}
$$

where $0 \leq \alpha < T$ and

$$[z(0),\dot{z}(0),w(0),\dot{w}(0)][\dot{p}(\alpha),\ddot{p}(\alpha),\dot{q}(\alpha),\ddot{q}(\alpha)]^* = 0. \tag{2.2}$$

Substituting this in (1.3), we obtain the following equations:

$$\ddot{p}(t+\alpha) + \ddot{z}(t) + g(p(t+\alpha),q(t+\alpha)) + g_x(p(t+\alpha),q(t+\alpha))z(t)$$
$$+ g_y(p(t+\alpha),q(t+\alpha))w(t)$$
$$= G(t,z,w,\alpha) + \mu_1 f_1(t) + \lambda_1[\dot{p}(t+\alpha)+\dot{z}(t)]$$

$$\ddot{q}(t+\alpha) + \ddot{w}(t) + h(p(t+\alpha),q(t+\alpha)) + h_x(p(t+\alpha),q(t+\alpha))z(t)$$
$$+ h_y(p(t+\alpha),q(t+\alpha))w(t)$$
$$= H(t,z,w,\alpha) + \mu_2 f_2(t) + \lambda_2[\dot{q}(t+\alpha)+\dot{w}(t)].$$

$H$ and $G$ are smooth functions of $t,z,w,\alpha$ and are $T$-periodic in $t$. Moreover, $H$ and $G$ are $O(|w|+|z|)$ [as $|w| + |z| \to 0$] uniformly in $t$ and in $\alpha \in [0,T]$. We now use (1.2) to rewrite the system as follows:

(2.3)(a)
$$\ddot{z} + [g_x(p(t+\alpha),q(t+\alpha)),g_y(p(t+\alpha),q(t+\alpha))]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \varphi_1$$
$$w + [h_x(p(t+\alpha),q(t+\alpha)),h_y(p(t+\alpha),q(t+\alpha))]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \varphi_2,$$

where

(2.3)(b)
$$\varphi_1 = G(t,z,w,\alpha) + \mu_1 f_1(t) + \lambda_1\dot{p}(t+\alpha) + \lambda_1\dot{z}(t)$$
$$\varphi_2 = H(t,z,w,\alpha) + \mu_2 f_2(t) + \lambda_2\dot{q}(t+\alpha) + \lambda_2\dot{w}(t);$$

or equivalently

(2.4)
$$\begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -g_x & 0 & -g_y & 0 \\ 0 & 0 & 0 & 1 \\ -h_x & 0 & -h_y & 0 \end{bmatrix}_{(p(t+\alpha),q(t+\alpha))} \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varphi_1 \\ 0 \\ \varphi_2 \end{bmatrix},$$

where $\varphi_1, \varphi_2$ are as in (2.3)(b) with $\dot{z}(t), \dot{w}(t)$ replaced by $z_2(t), w_2(t)$ respectively.

Thus, because of the change of coordinates, it is sufficient to look for $\alpha$, $w(t)$ and $z(t)$ which satisfy (2.2) and either (2.3) or (2.4).

## §3. Liapunov-Schmidt Method and Scaling

We will next consider the corresponding homogeneous equations

$$\text{(3.1)}\quad \begin{aligned} &\ddot{z} + [g_x(p(t+\alpha),q(t+\alpha)), g_y(p(t+\alpha),q(t+\alpha))] \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = 0 \\ &\ddot{w} + [h_x(p(t+\alpha),q(t+\alpha)), h_y(p(t+\alpha),q(t+\alpha))] \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = 0 \end{aligned}$$

and

$$\text{(3.2)}\quad \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -g_x & 0 & -g_y & 0 \\ 0 & 0 & 0 & 1 \\ -h_x & 0 & -h_y & 0 \end{bmatrix}(p(t+\alpha),q(t+\alpha)) \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = 0$$

Differentiation of (1.2) with respect to $t$ shows that for all $\alpha$, $(\dot{p}(t+\alpha), \dot{q}(t+\alpha))$ is a T-periodic solution of (3.1).

We will need some additional notation. Let $\mathcal{C}_T$ be the space of smooth T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^4$ with the norm $\|\cdot\|$ given by $\|x\| = \sup_{0 \le t \le T} |x(t)|$. Let $\mathcal{N} \subseteq \mathcal{C}_T$ be the space of solutions of (3.2), and let $m$ be the dimension of $\mathcal{N}$.

Our procedure for arriving at necessary and sufficient conditions for an arbitrary function $(z_1, z_2, w_1, w_2)$ to be a solution of (2.4) will be essentially the same as that in Hale [ 5 ] (pages 262-266). We define the $4 \times m$ matrix $\Phi(t)$ to have as its columns a basis for $\mathcal{N}$, with the first column $[\dot{p}(t+\alpha), \ddot{p}(t+\alpha), \dot{q}(t+\alpha), \ddot{q}(t+\alpha)]^*$. Let $\psi(t)$ be an $m \times 4$ matrix whose rows form a basis for the solution space of the equation which is adjoint to (3.2). We define the projection operators $P$ and $Q$ on $\mathcal{P}_T$ as follows:

$$P\theta(\cdot) = \Phi(\cdot)[\int_0^T \Phi^*(t)\Phi(t)dt]^{-1}[\int_0^T \Phi(t)^*\theta(t)dt]$$

$$Q\theta(\cdot) = \psi^*(\cdot)[\int_0^T \psi(t)\psi^*(t)dt]^{-1}[\int_0^T \psi(t)\theta(t)dt].$$

$P$ is a projection of $\mathcal{P}_T$ onto $\mathcal{N}$ and $Q$ of $\mathcal{P}_T$ onto the space spanned by the rows of $\psi$.

The Fredholm alternative implies that for a fixed $\varphi = [0, \varphi_1, 0, \varphi_2]^* \in \mathcal{P}_T$, equation (2.4) has a solution if and only if $Q\varphi = 0$. Thus for $\varphi \in (I-Q)\mathcal{P}_T$ there exists a particular solution $v_0(t,\varphi) = [v_{01}(t,\varphi), \dot{v}_{01}(t,\varphi), v_{02}(t,\varphi), \dot{v}_{02}(t,\varphi)]^*$ of (2.3)(a). Since $(I-P)v_0(t,\varphi)$ is also a solution of (2.3)(a), we may assume with no loss of generality that $Pv_0(\cdot,\varphi) = 0$. The general solution of (2.3)(a) will be of the form

$$v(t) = \Phi(t)\beta + v_0(t,\varphi) \quad \text{for} \quad \beta \in \mathbb{R}^m.$$

In order to satisfy condition (2.2) it is necessary that

$$[\Phi(0)\beta + v_0(0,\varphi)]^*[\dot{p}(\alpha), \ddot{p}(\alpha), \dot{q}(\alpha), \ddot{q}(\alpha)]^* = 0.$$

If $m = 1$, this becomes

$$\beta|\Phi(0)|^2 + v_0(0,\varphi)^*\Phi(0) = 0,$$

which uniquely defines $\beta$ as a function of $\varphi$. In this case we define the operator $\mathcal{K}: (I-Q)\mathcal{P}_T \to \mathcal{P}_T$ by

$$(\mathcal{K}\varphi)(\cdot) = \Phi(\cdot)\beta(\varphi) + v_0(\cdot,\varphi).$$

If $m \neq 1$, $\beta$ cannot be uniquely defined. Thus for a given function $\beta: (I-Q)\mathcal{P}_T \to \mathbb{R}^m$ for which $\beta(0) = 0$, we define the operator $\mathcal{K}_\beta$ by

$$(3.3) \qquad (\mathcal{K}_\beta\varphi)(\cdot) = \Phi(\cdot)\beta(\varphi) + v_0(\cdot,\varphi).$$

We may now state the following result, the proof of which is in Hale [ 5 ].

<u>Lemma 3.1</u>. Let $P,Q$, and $\mathcal{K}_\beta$ be as defined above. Then (2.4) has a T-periodic solution $[z_1,z_2,w_1,w_2]^*$ which is 0 when $\varphi_1 = \varphi_2 = 0$ if and only if $[z_1,z_2,w_1,w_2]^*$ satisfies the equations

$$(3.4)(a) \qquad \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = \mathcal{K}_\beta(I-Q)\begin{bmatrix} 0 \\ \varphi_1 \\ 0 \\ \varphi_2 \end{bmatrix},$$

for some function $\beta: (I-Q)\mathcal{P}_T \to \mathbb{R}^m$ such that $\beta(0) = 0$ and

$$[\Phi(0)\beta(\varphi) + v_0(0,\varphi)]^*[\dot{p}(\alpha) \;\; \ddot{p}(\alpha) \;\; \dot{q}(\alpha) \;\; \ddot{q}(\alpha)]^* = 0$$

and

$$(3.4)(b) \qquad Q \begin{bmatrix} 0 \\ \varphi_1 \\ 0 \\ \varphi_2 \end{bmatrix} = 0,$$

where $\varphi_1$ and $\varphi_2$ are as defined in (2.3)(b).

In this chapter, we are concerned only with the case $m = 1$, in which $\beta$ and $\chi$ are uniquely defined. For any $\alpha_0 \in [0,T]$, the Implicit Function Theorem now guarantees the existence of $\tilde{z}(t,\mu_1,\mu_2,\lambda_1,\lambda_2,\alpha)$, $\tilde{w}(t,\mu_1,\mu_2,\lambda_1,\lambda_2,\alpha)$ which satisfy (3.4)(a) for $(\mu_1,\mu_2,\lambda_1,\lambda_2,\alpha)$ in a neighborhood in $\mathbb{R}^4 \times [0,T]$ of $[0,0,0,0,\alpha_0]^*$. For such $\alpha,\mu_1,\mu_2,\lambda_1,$ and $\lambda_2$, $\tilde{z}$ and $\tilde{w}$ will be the unique solution of (3.5)(a) in a neighborhood $U \subseteq \mathcal{P}_T$ of $[0,0]^*$; moreover $\tilde{z}$ and $\tilde{w}$ will depend smoothly on all parameters. In addition, $\tilde{z}$ and $\tilde{w}$ are $O(|\mu_1| + |\mu_2| + |\lambda_1| + |\lambda_2|)$ [as $(\mu_1,\mu_2,\lambda_1,\lambda_2) \to 0$] uniformly for $\alpha \in [0,T]$.

Therefore the problem is reduced to one of locating values of $\alpha$ for which the operator $Q$ and the functions $\varphi_1$ and $\varphi_2$, defined by replacing $z$ and $w$ by $\tilde{z}$ and $\tilde{w}$ in (2.3)(b), satisfy (3.4)(b). Using the definitions of $\varphi_1,\varphi_2$, and $Q$ and denoting by $[\dot{r}(t+\alpha),r(t+\alpha),\dot{s}(t+\alpha),s(t+\alpha)]$ the vector $\psi(t)$, we may rewrite (3.4)(b) as

$$(3.5) \qquad L(\alpha)\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + K\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} + E(\mu_1,\mu_2,\lambda_1,\lambda_2,\alpha) = 0$$

where $L(\alpha) = [\int_0^T r(t+\alpha)f_1(t)dt \quad \int_0^T s(t+\alpha)f_2(t)dt]$,

$$K = [\int_0^T r(t+\alpha)p(t+\alpha)dt \quad \int_0^T s(t+\alpha)q(t+\alpha)dt],$$

and $E(\mu_1,\mu_2,\lambda_1,\lambda_2,\alpha) =$

$$= \int_0^T \{r(t+\alpha)[\lambda_1\dot{\tilde{z}} + G(t,\tilde{z},\tilde{w},\alpha)] + s(t+\alpha)[\lambda_2\dot{\tilde{w}} + H(t,\tilde{z},\tilde{w},\alpha)]\}dt.$$

We note that since $G$ and $H$ are $o(|z| + |w|)$ and $\tilde{z},\tilde{w}$, and hence $\dot{\tilde{z}}$ and $\dot{\tilde{w}}$ are $O(|\mu_1| + |\mu_2| + |\lambda_1| + |\lambda_2|)$, $E$ must be $o(|\mu_1| + |\mu_2| + |\lambda_1| + |\lambda_2|)$.

We hope to solve equation (3.5) for $\alpha$ in terms of $\mu_1,\mu_2,\lambda_1,\lambda_2$ in a neighborhood of the origin in $\mathbb{R}^4$. However, the Implicit Function Theorem cannot be used with equation (3.5) in its present form. Therefore we shall make a change of scale on the variables $\mu_1,\mu_2,\lambda_1,\lambda_2$. In order to guarantee that our results on the scaled problem will be applicable to the original problem, we prove the following lemma.

<u>Lemma 3.2</u>. Let $F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ be smooth and $f(\alpha,\eta) = B(\alpha)\eta + E(\alpha,\eta)$ where $\eta \in \mathbb{R}^n, \alpha \in \mathbb{R}, B(\alpha)$ is an $m \times n$ matrix and $E(\alpha,\eta) = o(|\eta|)$ uniformly for $\alpha \in [0,T]$. Let $\eta = \epsilon\omega$ and $\bar{F}(\alpha,\omega,\epsilon) = \epsilon^{-1}F(\alpha,\epsilon\omega)$. Suppose there exists $\omega_0$ and an open set $U$ in $\mathbb{R}^n \times \mathbb{R}$ containing $(\omega_0,0)$ and $(-\omega_0,0)$ and a function $\tilde{\alpha}(\omega,\epsilon)$

defined on $U$ such that $\tilde{\alpha}(\omega_o,0) = \tilde{\alpha}(-\omega_o,0) = 0$ and $\tilde{\alpha}(\omega,\epsilon)$ is the unique solution of $\overline{F}(\tilde{\alpha}(\omega,\epsilon),\omega,\epsilon) = 0$ for $(\omega,\epsilon) \in U$. If $(\omega_1,\epsilon_1)$ and $(\omega_2,\epsilon_2) \in U$, $\epsilon_1 \neq 0$ and $\epsilon_1\omega_1 = \epsilon_2\omega_2$, then $\tilde{\alpha}(\omega_1,\epsilon_1) = \tilde{\alpha}(\omega_2,\epsilon_2)$.

<u>Proof</u>. Let $\overline{E}(\alpha,\omega,\epsilon) = \epsilon^{-2}E(\alpha,\epsilon\omega)$. Then the first of the following statements is true by hypothesis and the others are equivalent.

$$B(\tilde{\alpha}(\omega_1,\epsilon_1))\omega_1 + \epsilon_1\overline{E}(\tilde{\alpha}(\omega_1,\epsilon_1),\omega_1,\epsilon_1) = 0$$

$$B(\tilde{\alpha}(\omega_1,\epsilon_1))\epsilon_1\omega_1 + E(\tilde{\alpha}(\omega_1,\epsilon_1),\epsilon_1\omega_1) = 0$$

$$B(\tilde{\alpha}(\omega_1,\epsilon_1))\epsilon_2\omega_2 + E(\tilde{\alpha}(\omega_1,\epsilon_1),\epsilon_2\omega_2) = 0$$

$$B(\tilde{\alpha}(\omega_1,\epsilon_1))\omega_2 + \epsilon_2\overline{E}(\tilde{\alpha}(\omega_1,\epsilon_1),\omega_2,\epsilon_2) = 0$$

$$\overline{F}(\tilde{\alpha}(\omega_1,\epsilon_1),\omega_2,\epsilon_2) = 0$$

By our hypothesis, $\tilde{\alpha}(\omega_2,\epsilon_2)$ is the unique solution of $\overline{F}(\alpha,\omega_2,\epsilon_2) = 0$ and thus $\tilde{\alpha}(\omega_1,\epsilon_1) = \tilde{\alpha}(\omega_2,\epsilon_2)$. □

The import of this lemma is that we can define $\tilde{\alpha}(\eta)$ to be $\tilde{\alpha}(\omega,\epsilon)$ for any $(\omega,\epsilon)$ such that $\eta = \epsilon\omega \neq 0$, and $\tilde{\alpha}(\eta)$ will satisfy $F(\tilde{\alpha}(\eta),\eta) = 0$.

For the remainder of this chapter, we will consider the problem

$$(3.5) \qquad L(\alpha)\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + K\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} + E(\alpha,\mu_1,\mu_2,\lambda_1,\lambda_2) = 0$$

as already reformulated in this section with the change of scale suggested above.

We let $\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} = \epsilon\omega$ and $\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \epsilon\gamma$ where $\omega$ and $\gamma$ are in $\mathbb{R}^2$, and define

$\overline{E}(\alpha,\omega,\gamma,\epsilon) = \epsilon^{-2}E(\alpha,\epsilon\omega_1,\epsilon\omega_2,\epsilon\gamma_1,\epsilon\gamma_2)$. Then (3.5) may be rewritten as

$$(3.6) \qquad F(\alpha,\omega,\gamma,\epsilon) \equiv L(\alpha)\omega + K\gamma + \epsilon\overline{E}(\alpha,\omega,\gamma,\epsilon) = 0.$$

Because of Lemma 3.2, solving (3.6) for $\tilde{\alpha}(\omega,\gamma,\epsilon)$ is equivalent to solving (3.5) for $\tilde{\alpha}(\mu_1,\mu_2,\lambda_1,\lambda_2)$ in the appropriate neighborhoods.

## §4. Local Results for Some Specific Cases

Before stating a general result, we examine a couple of specific cases. For the remainder of this chapter, we will assume that $(\omega_0,\gamma_0)$ is a point in $\mathbb{R}^4$ of magnitude one, and that for all $\alpha$, $L(\alpha) \neq [0,0]$.

First we suppose there exists a point $(\alpha_0,\omega_0,\gamma_0)$ such that $L(\alpha_0)\omega_0 + K\gamma_0 = 0$ and $L'(\alpha_0)\omega_0 \neq 0$. In terms of $F$, this means that $F(\alpha_0,\omega_0,\gamma_0,0) = 0$ and $D_\alpha F(\alpha_0,\omega_0,\gamma_0,0) \neq 0$. Thus by the Implicit Function Theorem, there exists a unique function $\tilde{\alpha}(\omega,\gamma,\epsilon)$ defined in a neighborhood $V_{(\alpha_0,\omega_0,\gamma_0)}$ of $(\omega_0,\gamma_0,0)$ such that $\tilde{\alpha}(0,0,0) = \alpha_0$ and $F(\tilde{\alpha}(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) \equiv 0$ in $V_{(\alpha_0,\omega_0,\gamma_0)}$. We may take $V_{(\alpha_0,\omega_0,\gamma_0)}$ to be of the form $B\times(-\epsilon_0,\epsilon_0)$ where $B$ is a ball about $(\omega_0,\gamma_0)$ in $\mathbb{R}^4$. The value of $\epsilon_0$ will of course depend on $(\alpha_0,\omega_0,\gamma_0)$.

Next we suppose instead that $(\alpha_0,\omega_0,\gamma_0)$ is a point which satisfies the conditions $L(\alpha_0)\omega_0 + K\gamma_0 = 0$; $L'(\alpha_0)\omega_0 = 0$; $L''(\alpha_0)\omega_0 \neq 0$. We will assume $L''(\alpha_0)\omega_0 < 0$; the result will be similar if $L''(\alpha_0)\omega_0 > 0$. In terms of

$F$, we have

$$F(\alpha_0,\omega_0,\gamma_0,0) = 0; \; D_\alpha F(\alpha_0,\omega_0,\gamma_0,0) = 0; \; D_\alpha^2 F(\alpha_0,\omega_0,\gamma_0,0) < 0.$$

If we assume $F$ is a $C^\infty$ function, we may apply the Malgrange Preparation Theorem [ 4 ]. This theorem implies that in a neighborhood of $(\alpha_0,\omega_0,\gamma_0,0)$, $F$ may be expressed as

$$F(\alpha,\omega,\gamma,\epsilon) = q(\alpha,\omega,\gamma,\epsilon)[\alpha^2 + \alpha r_1(\omega,\gamma,\epsilon) + r_2(\omega,\gamma,\epsilon)]$$

for some $C^\infty$ functions $r_1, r_2$ and $q$ where $q(\alpha_0,\omega_0,\gamma_0,0) \neq 0$. For almost every $(\omega_0,\gamma_0,0)$, there will be either two values of $\alpha$ or no values of $\alpha$ for which $F(\alpha,\omega,\gamma,\epsilon) = 0$.

This is essentially the result in which we are interested, but in order to obtain more specific information, we will prove this by a more direct method. Moreover our proof will not require $F$ to be a $C^\infty$ function.

By the Implicit Function Theorem, there exists a neighborhood $V$ of $(\omega_0,\gamma_0,0)$ and a unique function $\overline{\alpha}(\omega,\gamma,\epsilon)$ defined on $V$ such that $\overline{\alpha}(\omega_0,\gamma_0,0) = \alpha_0$ and

$$D_\alpha F(\overline{\alpha}(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) = 0 \quad \text{for} \quad (\omega,\gamma,\epsilon) \in V.$$

We define $M: V \to R$ by

$$M(\omega,\gamma,\epsilon) = F(\overline{\alpha}(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon).$$

For fixed $(\omega,\gamma,\epsilon)$ in $V$, $M(\omega,\gamma,\epsilon)$ is a local maximum of $F(\alpha,\omega,\gamma,\epsilon)$ with respect to $\alpha$.

Let

$$\mathcal{M} = \{(\omega,\gamma,\epsilon) \in V\colon M(\omega,\gamma,\epsilon) = 0\}.$$

Since $M(\omega_0,\gamma_0,0) = F(\alpha_0,\omega_0,\gamma_0,0) = 0$, it follows that $(\omega_0,\gamma_0,0) \in \mathcal{M}$, and it is easily seen that $\mathcal{M}$ is a manifold of codimension one. We may assume $V$ is chosen sufficiently small for the following to hold:

(4.1) There exists an interval $I$ about $\alpha_0$ such that $F(\alpha,\omega,\gamma,\epsilon) < M(\omega,\gamma,\epsilon)$ for all $(\omega,\gamma,\epsilon) \in V$ and $\alpha \in I$; moreover if $\alpha \in I - \{\alpha_0\}$ and $(\omega,\gamma,\epsilon) \in V$, then $L(\alpha)\omega + M\gamma \neq 0$.

(4.2) $\mathcal{M}$ divides $V - \mathcal{M}$ into exactly two simply connected regions, $V_1$ and $V_2$, such that in $V_1$, $M(\omega,\gamma,\epsilon) < 0$ and in $V_2$, $M(\omega,\gamma,\epsilon) > 0$.

(4.3) $\bar{\alpha}(V)$, the image of $V$ under $\bar{\alpha}$, is contained in $I$, and the distance from $\bar{\alpha}(V)$ to the complement of $I$ is at least as large as $\sup\limits_{(\omega,\gamma,\epsilon)\in V} \sqrt{2\beta_0^{-1}M(\omega,\gamma,\epsilon)}$ where $\beta_0$ is a positive lower bound for $-D_\alpha^2 F(\alpha,\omega,\gamma,\epsilon)$ in $I \times V$.

(4.4) $V$ is of the form $B \times (-\epsilon_0,\epsilon_0)$ where $B$ is a ball about $(\omega_0,\gamma_0)$ in $\mathbb{R}^4$. The value of $\epsilon_0$ again depends on $(\alpha_0,\omega_0,\gamma_0)$.

Under these assumptions on $V$, we can prove the following theorem.

<u>Theorem 4.1</u>. Let $F\colon \mathbb{R} \times \mathbb{R}^5 \to \mathbb{R}$ be a $C^2$ function and $(\alpha_0,\omega_0,\gamma_0,0)$ a point for which

$F(\alpha_0, \omega_0, \gamma_0, 0) = 0$; $D_\alpha F(\alpha_0, \omega_0, \gamma_0, 0) = 0$; $D_\alpha^2 F(\alpha_0, \omega_0, \gamma_0, 0) < 0$. Let the interval $I$ and neighborhoods $V, V_1, V_2$ be as defined above and satisfy conditions (4.1) - (4.4). Then there exist exactly two functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ defined on $\overline{V}_2$ and distinct in $V_2$ such that for $i = 1,2$, $\tilde{\alpha}_i(\omega_0, \gamma_0, 0) = \alpha_0$ and $F(\tilde{\alpha}_i(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) = 0$, $(\omega,\gamma,\epsilon) \in \overline{V}_2$. If $(\omega,\gamma,\epsilon) \in V_2$, then $F(\alpha,\omega,\gamma,\epsilon) \neq 0$ for all $\alpha \in I$.

<u>Proof</u>. First let's suppose $(\omega,\gamma,\epsilon)$ is in $V_1$, and let $\alpha \in I$. In this case we know

$$F(\alpha,\omega,\gamma,\epsilon) < M(\omega,\gamma,\epsilon) < 0.$$

Therefore $F(\alpha,\omega,\gamma,\epsilon) < 0$ for all points $(\alpha,\omega,\gamma,\epsilon)$ in $I \times V$, and so there is no solution of $F(\alpha,\omega,\gamma,\epsilon) = 0$.

Next let's suppose $(\omega,\gamma,\epsilon)$ is an arbitrary point in $V_2$. Recall that $\beta_0$ is a positive lower bound for $-D_\alpha^2 F$ on $I \times V$. For any $\alpha \in I$ the following is true.

$$M(\omega,\gamma,\epsilon) - F(\alpha,\omega,\gamma,\epsilon)$$

$$= F(\overline{\alpha}(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) - F(\alpha,\omega,\gamma,\epsilon)$$

$$= \int_{\overline{\alpha}(\omega,\gamma,\epsilon)}^{\alpha} D_\alpha F(a,\omega,\gamma,\epsilon)da$$

$$= -\int_{\overline{\alpha}(\omega,\gamma,\epsilon)}^{\alpha} [D_\alpha F(\overline{\alpha}(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon)$$

$$+ \int_{\overline{\alpha}(\omega,\gamma,\epsilon)}^{a} D_2^\alpha F(b,\omega,\gamma,\epsilon)db]da =$$

$$= \int_{\bar{\alpha}(\omega,\gamma,\epsilon)}^{\alpha} \left[ \int_{\bar{\alpha}(\omega,\gamma,\epsilon)}^{\alpha} -D_2^{\alpha}(b,\omega,\gamma,\epsilon)\,db \right] da$$

$$\geq \int_{\bar{\alpha}(\omega,\gamma,\epsilon)}^{\alpha} \left[ \int_{\bar{\alpha}(\omega,\gamma,\epsilon)}^{a} \beta_0\,db \right] da$$

$$= [\alpha - \bar{\alpha}(\omega,\gamma,\epsilon)]^2 \beta_0/2 .$$

By hypothesis (4.3), $I$ contains values for which

$$[\alpha - \bar{\alpha}(\omega,\gamma,\epsilon)]\beta_0/2 > M(\omega,\gamma,\epsilon) .$$

Since

$$F(\alpha,\omega,\gamma,\epsilon) < M(\omega,\gamma,\epsilon) \quad \text{for all} \quad \alpha \in I,$$

it follows that there exist $\alpha_1, \alpha_2 \in I$ such that

$$\alpha_1 < \bar{\alpha}(\omega,\gamma,\epsilon) < \alpha_2 ; \; F(\alpha_i,\omega,\gamma,\epsilon) = 0, D_{\alpha}F(\alpha_i,\omega,\gamma,\epsilon) \neq 0 \quad \text{for} \quad i = 1,2.$$

Since $(\omega,\gamma,\epsilon)$ was an arbitrary point in $V_2$ and $D_{\alpha}F(\alpha_i,\omega,\gamma,\epsilon) \neq 0$, $i = 1,2$, the Implicit Function Theorem implies the existence of functions $\tilde{\alpha}_1(\omega,\gamma,\epsilon)$ and $\tilde{\alpha}_2(\omega,\gamma,\epsilon)$ defined on $V_2$ such that $F(\tilde{\alpha}_i(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) = 0$ for all $(\omega,\gamma,\epsilon)$ in $V$ and $i = 1,2$. Moreover,

$$\lim_{(\omega,\gamma,\epsilon)\to(\omega_1,\gamma_1,\epsilon_1)\in M} \tilde{\alpha}_i(\omega,\gamma,\epsilon) = \bar{\alpha}(\omega_1,\gamma_1,\epsilon_1), \quad i = 1,2.$$

Limits may now be used to define $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ on $\bar{V}_2$. $\Box$

All that we've done in this section has been for a neighborhood about a particular $(\alpha_0, \omega_0, \gamma_0)$ which satisfies $L(\alpha_0)\omega_0 + K\gamma_0 = 0$. For the remainder of this chapter, we will use the notation $V_{(\alpha_0, \omega_0, \gamma_0)}$ and $I_{\alpha_0}$ to refer to the neighborhood $V$ and interval $I$ described above.

We have assumed that $L''(\alpha_0)\omega_0 < 0$. If $L''(\alpha_0)\omega_0 > 0$, a similar argument shows that our results hold with $V_1$ and $V_2$ reversed.

## §5. The Undamped Problem

In this section we consider the special case in which there is no damping. This case is of interest both in its own right and also because the proof, while very similar to the one in the general case, is more easily visualized than in the higher dimensional case.

This case corresponds to letting $\gamma = 0$ in (3.6). Thus in this section we will assume that $\omega_0$ is a point in $R^2$ of magnitude one and consider the equation

$$F(\alpha,\omega,\epsilon) = L(\alpha)\omega + \epsilon\overline{E}(\alpha,\omega,\epsilon) = 0 \tag{5.1}$$

under the following hypotheses:

(5.2) $L(\alpha)$ is never $0$.

(5.3) The set $\Omega = \{(\alpha_0,\omega_0): L(\alpha_0)\omega_0 = 0$ and $L'(\alpha_0)\omega_0 = 0\}$ is finite and $L''(\alpha_0)\omega_0 \neq 0$ for all $(\alpha_0,\omega_0) \in \Omega$.

(5.4) For a given $\omega_0$, there is at most one $\alpha_0$ for which $(\alpha_0,\omega_0) \in \Omega$.

Each of these hypotheses is generic; moreover (5.4) is actually unnecessary as we shall see in the proof, but it simplifies the counting procedures.

Let $\Omega_\alpha = \{\alpha \in [0,T): (\alpha,\omega) \in \Omega$ for some $\alpha\}$. Hypothesis (5.3) implies $\Omega_\alpha$ is finite. Let $n$ be the cardinality of $\Omega_\alpha$. Since $(\alpha_0,\omega_0) \in \Omega$ if and only if $(\alpha_0,-\omega_0) \in \Omega$, it follows that $\Omega$ has $2n$ elements.

For a given $\omega_0$, there are two possibilities:

(5.5) $L(\alpha)\omega_0 \neq 0$ for all $\alpha$, or

(5.6) the set $S_{\omega_0} = \{\alpha : L(\alpha)\omega_0 = 0\}$ is finite and each $\alpha_0 \in S_{\omega_0}$ satisfies exactly one of the following:

(5.6)(a) $$L'(\alpha_0)\omega_0 \neq 0$$

(5.6)(b) $$(\alpha_0, \omega_0) \in \Omega$$

with at most one $\alpha_0 \in S_{\omega_0}$ satisfying (5.6)(b).

First we consider an $\omega_0$ which satisfies (5.5). Since $L(\alpha)\omega_0$ is never zero, the function $|F(\alpha,\omega_0,0)|$ achieves a positive minimum for some $\alpha \in [0,T]$. Therefore there exists a neighborhood $V_{\omega_0}$ of $(\omega_0,0)$ such that $F([0,T] \times V_{\omega_0})$ is bounded away from zero. We may take $V_{\omega_0}$ to be of the form $B_{\delta(\omega_0)}(\omega_0) \times (-\epsilon(\omega_0), \epsilon(\omega_0))$ where $B_{\delta(\omega_0)}(\omega_0)$ is a ball of radius $\delta(\omega_0)$ about $\omega_0$ in $\mathbb{R}^2$.

For points $(\alpha_0,\omega_0)$ which satisfy (5.6)(a) or (5.6)(b), we define $V_{(\alpha_0,\omega_0)}$ to be the projection of the $V_{(\alpha_0,\omega_0,0)}$ defined in the previous section into the space $\{\gamma = 0\}$. We may assume that the neighborhoods $V_{(\alpha_0,\omega_0)}$ about points $(\alpha_0,\omega_0)$ in $\Omega$ are disjoint, so that the function $M$, which has already been defined for each $V_{(\alpha_0,\omega_0)}$, will be well defined on $\bigcup_{(\alpha_0,\omega_0)\in\Omega} V_{(\alpha_0,\omega_0)}$. We may also assume that $V_{(\alpha_0,\omega_0)}$ and $V_{(\alpha_0,-\omega_0)}$ are symmetric with respect to the origin. Then for $\omega_0$ satisfying (5.6) we may define

$$V_{\omega_0} = \bigcap_{\alpha_0 \in S_{\omega_0}} V_{(\alpha_0,\omega_0)},$$

which will then also be of the form

$B_{\delta(\omega_0)}(\omega_0) \times (-\epsilon(\omega_0), \epsilon(\omega_0))$.

Now for each $\omega_0 \in \mathbb{R}^2$ with norm one we have defined a neighborhood $V_{\omega_0}$. The set $\{V_{\omega_0}\}$ forms an open covering of the set $\{|\omega| = 1\}$, and thus there is a finite subcovering, $\{V_{\omega_1}, V_{\omega_2}, \ldots V_{\omega_j}\}$. Let

$W = \{\omega_i | V_{\omega_i}$ is in the finite subcovering$\}$

$\cup \{\omega | (\alpha,\omega) \in \Omega$ for some $\alpha\}$. Furthermore, let

$$A = \{(\omega,\epsilon): 1-\delta_0 < |\omega| < 1+\delta_0; |\epsilon| < \epsilon_0\}$$

where

$$\epsilon_0 < \min\{\epsilon(\omega_0) | \omega_0 \in W\},$$

and $\delta_0$ is chosen small enough that $A$ will be contained in $\bigcup_{\omega \in W} V_{\omega_i}$. Note that $\epsilon_0$ and $\delta_0$ must be positive but can be chosen as small as desired. Thus no contradictions will arise if $\epsilon_0$ and $\delta_0$ must be decreased to satisfy an additional condition which will be specified later.

We will next state seven lemmas which outline our results for this problem. We will then summarize these results in a theorem, which will be proved by proving the seven lemmas.

Lemma 5.1. Let $(\alpha_i, \omega_i)$ be in $\Omega$ and let the map $M_i$ and the 2-manifold $\mathcal{M}_i$ passing through $(\omega_0, 0)$ be as defined in §4. (Actually this is a reduction of what was done in §4 to the case where $\gamma \equiv 0$.). Then each $\mathcal{M}_i$ intersects the boundary of A in a closed curve which is not contractable to a point in the boundary of A, and the 2n manifolds $\mathcal{M}_i$ divide $A - \bigcup_{i=1}^{2n} \mathcal{M}_i$ into 2n connected components.

<u>Lemma 5.2</u>. The number of solutions $\tilde{\alpha}\,(\omega,\epsilon)$ of (5.1) is constant in each component, and these $\tilde{\alpha}$'s can be defined smoothly on the entire component.

<u>Lemma 5.3</u>. The number of solutions changes by two as $(\omega,\epsilon)$ moves across $M_i$ from one component into another.

<u>Lemma 5.4</u>. Let $(\omega_1,\epsilon_1)$ and $(\omega_2,\epsilon_2)$ be two points in A such that $\epsilon_1\omega_1 = \epsilon_2\omega_2$ and $\epsilon_1 \neq 0$. If a solution $\tilde{\alpha}$ is defined for either point, it is defined for both, and $\tilde{\alpha}\,(\omega_1,\epsilon_1) = \tilde{\alpha}\,(\omega_2,\epsilon_2)$. If M is defined for both points, then $\epsilon_1 M\,(\omega_1,\epsilon_1) = \epsilon_2 M\,(\omega_2,\epsilon_2)$.

Let N be a ball of radius $\epsilon_0$ about the origin in $\mathbb{R}^2$, and define a map $\tilde{P}$ which maps A onto a neighborhood containing N by $\tilde{P}(\omega,\epsilon) = \epsilon\omega$. Let $C_i = \tilde{P}(\mathcal{M}_i)$. Then the following will be true.

<u>Lemma 5.5</u>. There are exactly n curves $C_i$ given by $C_i = \tilde{P}(\mathcal{M}_i)$, each continuous and passing through the origin.

<u>Lemma 5.6</u>. For each $(\alpha_i,\omega_i) \in \Omega$, exactly one of the curves $C_i$ is tangent to the line $\mu = t\omega_i$ $(t \in R)$ at the origin.

Lemma 5.7. The $n$ curves $C_i$ divide $N$ into exactly $2n$ regions. The number of solutions $\alpha$ of the equation

$$(5.7) \qquad L(\alpha)\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + E(\alpha,\mu_1,\mu_2) = 0$$

is constant in each region and is the same as the number in the diagonally opposite region and in the corresponding region of $A$.

The following Theorem and its Corollaries summarize the results and will be proved by proving Lemmas 5.1 - 5.7.

Theorem 5.8. Consider the problem

$$(5.8) \qquad \begin{aligned} \ddot{x} + g(x,y) &= \mu_1 f_1 \\ \ddot{y} + h(x,y) &= \mu_2 f_2 \end{aligned}$$

where $g$, $h$, $f_1$, and $f_2$ are smooth and $f_1$, and $f_2$ are T-periodic functions of $t$. Suppose that $(p(t),q(t))$ is a solution of (5.8) and that $(\dot{p}(t),\dot{q}(t))$ is the unique, up to constant multiples, T-periodic solution of

$$(5.9) \qquad \begin{aligned} \ddot{x} + g_x(p(t),q(t))x + g_y(p(t),q(t))y &= 0 \\ \ddot{y} + h_x(p(t),q(t))x + h_y(p(t),q(t))y &= 0. \end{aligned}$$

Let $L(\alpha) = [\int_0^T r(t+\alpha)f_1(t)dt \quad \int_0^T s(t+\alpha)f_2(t)dt]$ where $[r(t),s(t)]$ is the T-periodic solution of the adjoint equation. Suppose further that $L(\alpha)$ is never zero; the set $\Omega = \{(\alpha_0,\omega_0): L(\alpha_0)\omega_0 = 0$ and $L'(\alpha_0)\omega_0 = 0\}$ is finite (of cardinality $2n$) and $L''(\alpha_0)\omega_0 \neq 0$ for all

$(\alpha_0, \omega_0) \in \Omega$; and for a given $\omega_0$, there is at most one $\alpha_0$ for which $(\alpha_0, \omega_0) \in \Omega$. Then there exists a neighborhood N of the origin in $\mathbb{R}^2$ and exactly n curves $C_i$ which pass through the origin and divide N into 2n regions such that the number of T-periodic solutions of (5.8) is constant in each region and changes by two as $\mu$ crosses one of the $C_i$ from one region into another. For each $\omega_0$ such that $(\alpha_0, \omega_0) \in \Omega$, there is exactly one of the $C_i$ which is tangent to the line $\mu = t\omega_0$ $(t \in R)$ at the origin.

Proof. This theorem follows immediately from Lemmas 5.1 - 5.7, which will be proven in the next section. □

Under the same assumptions, the following corollaries are true and follow from the proofs in the next section.

Corollary 5.9. If the winding number of the closed curve $L(\alpha)$ $(0 \leq \alpha \leq T)$ in $R^2$ is nonzero, then for each $\mu \in N$, there exist at least two T-periodic solutions of (5.8).

Corollary 5.10. If $\Omega$ is empty, then there exist exactly two T-periodic solutions of (5.8) for each $\mu$ in $N - \{0\}$.

## §6. Proofs of Lemmas

In this section we will prove the seven lemmas stated in §5.

<u>Proof of Lemma 5.1</u>. Let $(\alpha_0, \omega_0) \in \Omega$, and consider the function $M(\omega, \epsilon)$ as defined in §4. The set $\{(\omega, \epsilon): M(\omega, \epsilon) = 0\}$ is the manifold $\mathcal{M}_i$ passing through $(\omega_0, 0)$. $D_\omega M(\omega_0, 0) = L(\alpha_0)$ and $D_\epsilon M(\omega_0, 0) = E(\omega_0, \alpha_0, 0)$ which is bounded. Thus the 3-vector $[L(\alpha_0), E(\omega_0, \alpha_0, 0)]$ is normal to $\mathcal{M}_i$. Now since $L(\alpha_0)$ is orthogonal to $\omega_0$, it follows that $\mathcal{M}_i$ intersects the curve $\{|\omega_0|=1, \epsilon=0\}$ transversally. Thus for $\epsilon_0$ and $\delta_0$ chosen sufficiently small, $\mathcal{M}_i$ is homeomorphic to a disc which intersects the boundary of A in a closed curve which is not contractable to a point in the boundary of A. For each point $(\alpha_i, \omega_i)$ in $\Omega$, the point $(\alpha_i, -\omega_i)$ is also in $\Omega$. Since $\Omega_\alpha$ has cardinality n, $\Omega$ has cardinality 2n. For each of the 2n values $(\alpha_i, \omega_i)$ in $\Omega$ there exists a manifold $\mathcal{M}_i$ of the form just described. We can choose $\epsilon_0$ and $\delta_0$ as small as necessary to insure that each of the manifolds intersect the boundary of A appropriately and that no two of the $\mathcal{M}_i$ intersect in A. If follows that the $\mathcal{M}_i$ will divide A into 2n connected regions.

Before continuing, let us prove the following lemma, which will be of use both here and in the next sections.

<u>Lemma 6.1</u>. Let $\mathcal{O}$ be an open connected subset of $\mathbb{R}^n$ and $F(\alpha, z)$ a $C^1$ function from $[0,T] \times \mathcal{O}$ into R which is T-periodic in $\alpha$. If

$$(6.1) \qquad |D_\alpha F(\alpha, z)| + |F(\alpha, z)| \neq 0 \quad \text{for} \quad (\alpha, z) \in [0,T] \times \mathcal{O},$$

then the number of solutions $\alpha(z)$ ($\alpha(z) + nT$ considered the same for all integers n) of $F(\alpha(z), z) = 0$ is constant in $\mathcal{O}$, and these functions may be defined as continuous functions of $z$ in $\mathcal{O}$. If $S = \{(\alpha,z) \in [0,T) \times \mathcal{O}: D_\alpha F(\alpha,z) = 0\}$, then the number of solutions $\alpha_i(z)$ for which $(\alpha_i(z)+nT,z) \notin S$ for all $z \in \mathcal{O}$ is constant.

<u>Proof</u>. By the Implicit Function Theorem, a solution at any point at which $D_\alpha F \neq 0$ is defined uniquely in a neighborhood of that point and may be continued into overlapping neighborhoods unless $D_\alpha F(\alpha(z),z)$ approaches zero at the edge of some neighborhood. Therefore $\alpha(z)$ may be defined continuously in any open subset of $\mathcal{O}$ in which $D_\alpha F(\alpha(z),z) \neq 0$, and the number of solutions may change only at a point for which $D_\alpha F(\alpha,z) = 0$. Thus the number of solutions for which $D_\alpha F(\alpha(z),z) \neq 0$ in $\mathcal{O}$ is constant. □

<u>Proof of Lemma 5.2</u>. This follows directly from Lemma 6.1 since each region satisfies condition (6.1) on $\mathcal{O}$. □

<u>Proof of Lemma 5.3</u>. Let $(z_0,\epsilon_0) \in \mathcal{M}_i$ and let $\alpha_0$ be such that $(\alpha_0,z_0,\epsilon_0)$ satisfies

$$F(\alpha_0,z_0,\epsilon_0) = 0; \quad D_\alpha F(\alpha_0,z_0,\epsilon_0) = 0.$$

We have seen in §4 that there are two solutions $\alpha(z)$ of $F(\alpha(z),z) = 0$, $\alpha(z_0) = \alpha_0$ on one side of $\mathcal{M}_i$ and none on the other. All other solutions satisfy the conditions of the lemma and so their number is constant.

Therefore the total number changes by two as $\mathcal{M}_i$ is crossed. □

<u>Proof of Lemma 5.4</u>. Let $(\omega_1,\epsilon_1)$ and $(\omega_2,\epsilon_2)$ be two points in A such that $\epsilon_1\omega_1 = \epsilon_2\omega_2$ and $\epsilon_1 \neq 0$.

First suppose that both $(\omega_1,\epsilon_1)$ and $(\omega_2,\epsilon_2)$ are in one of the neighborhoods $V_{(\alpha_0,\omega_0)}$ for which $(\alpha_0,\omega_0) \in \Omega$. By our results in §4, functions $\bar{\alpha}$ and M are defined on V, and Lemma 3.2 may be applied to the function $\bar{\alpha}$ on V. Then for i = 1 or 2,

$$\begin{aligned} M(\omega_i,\epsilon_i) &= L(\bar{\alpha}(\omega_i,\epsilon_i))\omega_i + \epsilon_i\bar{E}(\omega_i,\bar{\alpha}(\omega_i,\epsilon_i),\epsilon_i) \\ &= \epsilon_i^{-1}[L(\bar{\alpha}(\omega_i,\epsilon_i))\epsilon_i\omega_i + E(\epsilon_i\omega_i,\bar{\alpha}(\omega_i,\epsilon_i)) . \end{aligned}$$

Since $\bar{\alpha}(\omega_1,\epsilon_1) = \bar{\alpha}(\omega_2,\epsilon_2)$ by Lemma 3.2 and $\epsilon_1\omega_1 = \epsilon_2\omega_2$, it follows that $\epsilon_1 M(\omega_1,\epsilon_1) = \epsilon_2 M(\omega_2,\epsilon_2)$. If M is defined for both $(\omega_1,\epsilon_1)$ and $(\omega_2,\epsilon_2)$, and they are not in the same $V_{(\alpha_0,\omega_0)}$, then it must be that $(\omega_1,\epsilon_1) \in V_{(\alpha_0,\omega_0)}$ and $(\omega_2,\epsilon_2) \in V_{(\alpha_0,-\omega_0)}$. In this case we consider $\bar{\alpha}$ as defined on $V_{(\alpha_0,\omega_0)} \cup V_{(\alpha_0,-\omega_0)}$ and the remainder of the proof is identical.

Because of what we have just proved, all $(\omega_1,\epsilon_1)$ and $(\omega_2,\epsilon_2)$ for which $\epsilon_1\omega_1 = \epsilon_2\omega_2$ will be on the same or antipodal manifolds $\mathcal{M}_i$ or will be in the same or antipodal components of $A - \bigcup_{i=1}^{2n} \mathcal{M}_i$. Thus if $\tilde{\alpha}$ is defined for $(\omega_1,\epsilon_1)$, it is defined on the entire component containing $(\omega_1,\epsilon_1)$ and on the antipodal component, one of which contains $(\omega_2, \epsilon_2)$. Therefore, we may apply Lemma 3.2 and this completes the proof. □

Proof of Lemma 5.5. We have seen that if $\epsilon_1\omega_1 = \epsilon_2\omega_2$, then and $(\omega_1, \epsilon_1)$ is on one of the manifolds, $(\omega_2, \epsilon_2)$ is on the same or antipodal manifold. Thus each manifold is mapped by $\tilde{P}$ to a curve in $\mathbb{R}^2$. Moreover manifolds which are antipodal are mapped by $\tilde{P}$ to the same curve $C_i$. Thus there are exactly n curves $C_i$, each given by $C_i = \tilde{P}(\mathcal{M}_i)$. For each $\mathcal{M}_i$, there is some $\omega_0$ for which $(\omega_0, 0)$ is on $\mathcal{M}_i$ and so each $C_i$ passes through the origin. □

Proof of Lemma 5.6. Let $C_i = \tilde{\mathcal{P}}(\mathcal{M}_i)$ where $\mathcal{M}_i$ is the manifold passing through $(\omega_0, 0)$. Recall that $M_i(\omega, 0) = 0$ only for $\omega = \omega_0$, and thus $\lim\limits_{\substack{\epsilon \to 0 \\ (\omega,\epsilon) \in \mathcal{M}_i}} \omega = \omega_0$.

This implies that

$$\lim_{\substack{\mu \to 0 \\ \mu \in C_i}} (\mu/|\mu|) = \lim_{\substack{\epsilon \to 0 \\ (\omega,\epsilon) \in \mathcal{M}_i}} \tilde{P}(\omega,\epsilon)/|\tilde{P}(\omega,\epsilon)| = \pm\omega_0 .$$

Therefore $C_i$ is tangent to the line $\mu = t\omega_0$ $(t \in \mathbb{R})$ at the origin. □

Proof of Lemma 5.7. From the above description of the $C_i$, we know that they intersect only at the origin and there transversally, and that each intersects the boundary of N twice. Thus they must split N into exactly 2n regions. Corresponding to each solution

$\tilde{\alpha}(\omega,\epsilon)$ defined in A, we may define $\tilde{\alpha}$ on $N-\{0\}$ by

$$\tilde{\alpha}(\mu_1,\mu_2) = \tilde{\alpha}(\omega,\epsilon) \quad \text{where} \quad \epsilon\omega = \mu.$$

$\tilde{\alpha}$ is well defined on $N-\{0\}$ by Lemma 5.4 and is a solution of (5.7) by the way it is defined. Clearly there are no additional solutions of (5.7) in $N-\{0\}$, since a corresponding solution could be defined in A. Therefore a region in N has the same number of solutions as the corresponding region in A and in fact, diagonally opposite regions in N correspond to the same region in A, one for $\epsilon > 0$, the other for $\epsilon < 0$. Therefore diagonally opposite regions have the same number of solutions. □

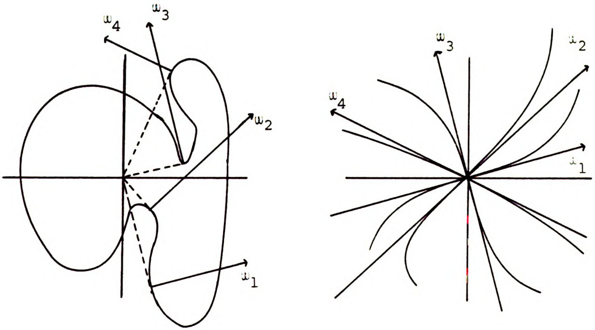


Figure 1. L(α) and the corresponding bifurcation curves.

## §7. The Theorem for Four Parameters

Next, let us consider a point $(\alpha_0, \omega_0, \gamma_0, 0)$ at which:

$$L(\alpha_0)\omega_0 + M\gamma_0 = 0$$

$$L'(\alpha_0)\omega_0 = 0$$

$$L''(\alpha_0)\omega_0 = 0$$

$$L'''(\alpha_0)\omega_0 \neq 0$$

Without loss of generality we may assume that $L'''(\alpha_0)\omega_0 < 0$. Recall that $|\omega_0, \gamma_0| = 1$. By arguments similar to those in §4 we can show that there is a unique function $\bar{\bar{\alpha}}(\omega, \gamma, \epsilon)$ such that $D_\alpha^2 F(\bar{\bar{\alpha}}(\omega, \gamma, \epsilon), \omega, \gamma, \epsilon) = 0$ for all $(\omega, \gamma, \epsilon)$ in a neighborhood $\tilde{V}$, of $(\omega_0, \gamma_0, 0)$ and $\bar{\bar{\alpha}}(\omega_0, \gamma_0, 0) = \alpha_0$. Define $\tilde{M}(\omega, \gamma, \epsilon) = D_\alpha F(\bar{\bar{\alpha}}(\omega, \gamma, \epsilon), \omega, \gamma, \epsilon)$, and $\tilde{\mathcal{M}} = \{(\omega, \gamma, \epsilon) \in \tilde{V} | \tilde{M}(\omega, \gamma, \epsilon) = 0\}$. Let $\tilde{V}$, $\tilde{V}_1$, and $\tilde{V}_2$ be analogous to $V$, $V_1$, and $V_2$ in §4. For $(\omega, \gamma, \epsilon) \in \tilde{V}_1$, there is no solution $\alpha$ to $D_\alpha F(\alpha, \omega, \gamma, \epsilon) = 0$ and for $(\omega, \gamma, \epsilon) \in \tilde{V}_2$, there are two solutions, $\bar{\alpha}_i(\omega, \gamma, \epsilon)$ such that $F_\alpha(\bar{\alpha}_i(\omega, \gamma, \epsilon), \omega, \gamma, \epsilon) = 0$ and $\bar{\alpha}_i(\omega_0, \gamma_0, 0) = \alpha_0$, $i = 1,2$. Next we define $M_i$ on $\tilde{V}_2 \cup \tilde{\mathcal{M}}$ by $M_i(\omega, \gamma, \epsilon) = F(\bar{\alpha}_i(\omega, \gamma, \epsilon), \omega, \gamma, \epsilon)$ and define $S_i = \{(\omega, \gamma, \epsilon) \in \tilde{V} \cup \tilde{\mathcal{M}} | M_i(\omega, \gamma, \epsilon) = 0\}$. Let $\mathcal{M} = S_1 \cup S_2$. We claim $\mathcal{M}$ is a 3-manifold containing $(\omega_0, \gamma_0, 0)$.

Referring again to the analogous argument in §4, we see that $\lim_{(\omega,\gamma,\epsilon) \to (\omega_2, \gamma_2, \epsilon_2) \in \tilde{M}} \overline{\alpha_i}(\omega, \gamma, \epsilon) = \bar{\bar{\alpha}}(\omega_2, \gamma_2, \epsilon_2)$; thus on $\tilde{\mathcal{M}}$, $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\bar{\alpha}}$. Therefore $S_1 \cap \tilde{\mathcal{M}} = S_2 \cap \tilde{\mathcal{M}} = S_1 \cap S_2 \cap \tilde{\mathcal{M}}$, which is nonempty since

$(\omega_0, \gamma_0, 0) \in S_1 \cap S_2 \cap \bar{\mathcal{M}}$. Let

$$G(\omega,\gamma,\epsilon) = \begin{bmatrix} F_\alpha(\bar{\bar{\alpha}}(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) \\ F(\bar{\alpha},(\omega,\gamma,\epsilon),\omega,\gamma,\epsilon) \end{bmatrix}$$

$$\frac{\partial G}{\partial(\omega,\gamma)} \quad \text{at} \quad (\omega_0,\gamma_0,0) = \begin{bmatrix} L'(\alpha_0) & 0 \\ L(\alpha_0) & K \end{bmatrix}$$

which has rank two unless $K = [0,0]$. We will consider this possibility later, so for now we may suppose $K \neq [0,0]$. Thus $S_1 \cap \tilde{\mathcal{M}}$, the zero set of $G$, has codimension two. Furthermore if $(\omega_1, \gamma_1, \epsilon_1)$ is a point such that $\bar{\alpha}_1(\omega_1,\gamma_1,\epsilon_1) = \bar{\alpha}_2(\omega_1,\gamma_1,\epsilon_1)$ then at $(\omega_1,\gamma_1,\epsilon_1)$ we also have

$$\frac{\partial\bar{\alpha}_1}{\partial(\omega,\gamma,\epsilon)}(\omega_1,\gamma_1,\epsilon_1) = \frac{\partial\bar{\alpha}_2}{\partial(\omega,\gamma,\epsilon)}(\omega_1,\gamma_1,\epsilon_1).$$

Since each $S_i$ is a manifold of codimension one, $\mathcal{M}$ is a manifold of codimension one in $\mathbb{R}^5$, that is, a 4-manifold.

As in §5, we may now show that there are two solutions $\tilde{\alpha}(\omega,\gamma,\epsilon)$ of $F(\alpha,\omega,\gamma,\epsilon) = 0$, $\tilde{\alpha}(\omega_0,\gamma_0,0) = \alpha_0$, on one side of $\mathcal{M}$, and none on the other.

In the above, we made the assumption that $K \neq [0,0]$. Let us now consider this case under the additional (generic) assumption that $L''(\alpha_0)\omega_0 \neq 0$ whenever $L(\alpha_0)\omega_0 = 0$ and $L'(\alpha_0)\omega_0 = 0$. This case now essentially reduces to the case in §5. For each curve $C_i$ in space $\{\lambda = 0\}$ defined in §5, we will now have a 3-manifold containing $C_i$ and tangent to the hyperplane $\mu = t\omega_0$ $(t \in \mathbb{R})$.

Finally, we will state a theorem concerning the bifurcation surfaces in the general case.

<u>Theorem 7.1</u>. Consider the problem

$$(7.1)\qquad \begin{aligned} \ddot{x} + g(x,y) &= \mu_1 f_1 + \lambda_1 \dot{x} \\ \ddot{y} + h(x,y) &= \mu_2 f_2 + \lambda_2 \dot{y} \end{aligned}$$

where $g$, $h$, $f_1$, and $f_2$ are smooth and $f_1$ and $f_2$ are T-periodic functions of $t$. Suppose that $(p(t),q(t))$ is a solution of (7.1) and that $(\dot{p}(t),\dot{q}(t))$ is the unique, up to constant multiples, T-periodic solution of

$$\begin{aligned} \ddot{x} + g_x(p(t),q(t))x + g_y(p(t),q(t))y &= 0 \\ \ddot{y} + h_x(p(t),q(t))x + h_y(p(t),q(t))y &= 0. \end{aligned}$$

Let $L(\alpha) = \left[\int_0^T r(t+\alpha) f_1(t)dt \quad \int_0^T s(t+\alpha) f_2(t)dt\right]$. Suppose also that $K$ has a null space of dimension one; $L(\alpha)$ is never zero; the set $\Omega = \{(\alpha_0,\omega_0,\gamma_0) : L(\alpha_0)\omega_0 + K\gamma_0 = 0$, $L'(\alpha_0)\omega_0 = 0$, and $L''(\alpha_0)\omega_0 = 0\}$ is finite and $L'''(\alpha_0)\omega_0 \neq 0$ for all $(\alpha_0,\omega_0,\gamma_0) \in \Omega$; for a given $(\omega_0,\gamma_0)$ there is at most one $\alpha_0$ for which $(\alpha_0,\omega_0,\gamma_0) \in \Omega$. Then there exists a neighborhood $N$ of the origin and a 3-surface $S \subseteq N$ (i.e. a "surface" of codimension 1) which is symmetric with respect to the origin and consists of the origin and the union of 3-manifolds, each of which passes through the origin. If $(\alpha_0,\omega_0,\gamma_0)$ satisfies $L(\alpha_0)\omega_0 + K\gamma_0 = 0$,

$$L'(\alpha_0)\omega_0 = 0, \quad \text{and} \quad |\omega_0|^2 + |\gamma_0|^2 = 1,$$

then the line $t(\omega_0, \gamma_0)$ is tangent to $S$ at the origin. The number of solutions of (7.1) is constant in each connected component of $N - S$ and changes by two as $S$ is crossed transversally.

<u>Proof</u>. We begin by defining
$C = \{(\omega_0, \gamma_0, 0), |\gamma_0|^2 < |\omega_0|^2 + |\gamma_0|^2 = 1, L(\alpha)\omega_0 + K\gamma_0 = 0$ and $L'(\alpha)\omega_0 = 0$ for some $\alpha\}$. $C$ is a 2-surface on $S^3 \times \{0\}$, and each point on $C$ is described by either the results in this section or the second case in §4. Thus through each point $(\omega_0, \gamma_0, 0) \in C$, there is a 4-manifold which divides a neighborhood $V$ of $(\omega_0, \gamma_0, 0)$ into two parts, one in which there are two solutions $\tilde{\alpha}(\omega, \gamma, \epsilon)$ of $F(\alpha, \omega, \gamma, \epsilon) = 0$, $\tilde{\alpha}(\omega_0, \gamma_0, 0) = \alpha_0$ (where $\alpha_0$ is the value for which $L(\alpha_0)\omega_0 + K\gamma_0 = 0$ and $L'(\alpha_0)\omega_0 = 0$) and the other in which there are none.

Now let $(\omega_0, \gamma_0)$ be an arbitrary point on $S^3$. Exactly one of the following is true:

(7.2) There is no value of $\alpha$ for which $L(\alpha)\omega_0 + K\gamma_0 = 0$.

(7.3) $S_{(\omega_0, \gamma_0)} = \{\alpha \mid L(\alpha)\omega_0 + K\gamma_0 = 0\}$ is nonempty and finite. Each $\alpha_i \in S_{(\omega_0, \gamma_0)}$ satisfies exactly one of the following.

(7.3)(a) $L'(\alpha_i)\omega_0 \neq 0$

(7.3)(b) $L'(\alpha_i)\omega_0 = 0$, $L''(\alpha_i)\omega_0 \neq 0$

(7.3)(c) $L'(\alpha_i)\omega_0 = 0$, $L''(\alpha_i)\omega_0 = 0$, $L'''(\alpha_i)\omega_0 \neq 0$

(7.4) $|\omega_0| = 0$ and $M\gamma_0 = 0$.

We have just discussed cases (7.3)(b) and (c) and we dealt with case (7.3)(a) in §4. We may define neighborhoods $V_{(\omega_0,\gamma_0)}$ of the points satisfying case 2 by intersections as in §6. Case (7) is analagous to the corresponding case (5.5) in §5, and in the same manner as there, we may define a neighborhood $V_{(\omega_0,\gamma_0)}$ of $(\omega_0,\gamma_0,0)$ in which (3.5) has no solution.

We now delete from $S^3 \times \mathbb{R}$ arbitrarily small balls $N_1$ and $N_2$ about $(0,\gamma_0,0)$ and $(0,-\gamma_0,0)$, where $(0,\gamma_0)$ satisfies (7.4). What remains, $S^3 \times \{0\} - (N_1 \cup N_2)$, is a compact set of points about which we have defined neighborhoods $V_{(\omega_0,\gamma_0)}$. We may cover $S^3 \times \{0\} - (N_1 \cup N_2)$ with a finite set of $V_{(\omega_0,\gamma_0)}$, in which we include a covering of $C - (N_1 \cup N_2)$. Let $\epsilon_0$ be such that if $A = \{(\omega,\gamma,\epsilon) \in \mathbb{R}^5 \colon d((\omega,\gamma,\epsilon),W) < \epsilon_0\}$; the finite covering also covers $A$. Let $\epsilon_0$ also be small enough to guarantee that each component of the set where $\tilde{C}$ intersects itself has an element for which $\epsilon = 0$ (That is $\tilde{C}$ intersects itself only when $C$ does). Because the functions $\bar{\alpha}(\omega,\gamma,\epsilon)$ which are the basis for defining the manifolds which comprise $\tilde{C}$ satisfy the conditions of Lemma II.2 where the $F$ in the lemma is $D_\alpha F$, the $\bar{\alpha}$ are defined continuously in a neighborhood of $C$ except at points at which $D_\alpha^2(\alpha,\omega,\gamma,\epsilon) = 0$. We have already shown that there is a single manifold at these points, and so $\tilde{C}$ is itself a manifold through $C$ except at points of intersection where it is locally the intersection of two manifolds.

Now let N be a ball about the origin in $\mathbb{R}^4$ of radius $\epsilon_0$, and let $P: A \to \mathbb{R}^4$ be defined by $P(\omega,\gamma,\epsilon) = (\epsilon\omega,\epsilon\gamma)$. P maps A onto a neighborhood of the origin which contains N, and P maps $\tilde{C}$ onto a 3-surface which we shall call S, and $P(N_1 \cup N_2)$ contains the intersection of the manifolds which make up S. We may again use Lemma 3.2 to define solutions $\tilde{\alpha}(\mu,\nu,\lambda_1,\lambda_2)$ in N. Moreover if $(\omega_0,\gamma_0,0) \in C$, $(\mu,\lambda) = t(\omega_0,\gamma_0)$ is tangent to S. The proofs are similar to those in §5 and §6.

## CHAPTER III

## TWO DIMENSIONAL NULL SPACE

### §1. Preliminaries

In the last chapter we assumed that the equation

$$(1.1)\quad \begin{aligned} \ddot{z} + [g_x(p(t+\alpha),q(t+\alpha)),g_y(p(t+\alpha),q(t+\alpha)]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} &= 0 \\ \ddot{w} + [h_x(p(t+\alpha),q(t+\alpha)),h_y(h(t+\alpha),q(t+\alpha)]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} &= 0 \end{aligned}$$

had a T-periodic solution $(\dot{p}(t+\alpha),\dot{q}(t+\alpha))$ which was unique up to constant multiples. In this chapter, we will assume that (1.1) has exactly two linearly independent T-periodic solutions, $(\dot{p}(t+\alpha),\dot{q}(t+\alpha))$ and $(\tilde{r}(t+\alpha),\tilde{s}(t+\alpha))$, and we will consider the equation

$$(1.2)\quad \begin{aligned} \ddot{x} + g(x,y) &= \mu_1 f_1 \\ \ddot{y} + h(x,y) &= \mu_2 f_2 \end{aligned}$$

where $\mu_1$ and $\mu_2$ are near zero.

We recall that by Lemma II.3.1, the conditions

$$(1.3)\,(a)\quad \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = \mathcal{K}_\beta(I-Q)\begin{bmatrix} 0 \\ \mu_1 f_1(\cdot) + G(\cdot,z,w,\alpha) \\ 0 \\ \mu_2 f_2(\cdot) + H(\cdot,z,w,\alpha) \end{bmatrix}$$

for some $\beta: (I-Q)\mathcal{P}_T \to \mathbb{R}^2$ such that

$$[\Phi(0)\beta(\varphi)+V_0(0,\varphi)][\dot{p}(\alpha)\ \ddot{p}(\alpha)\ \dot{q}(\alpha)\ \ddot{q}(\alpha)]^* = 0;\ \beta(0) = 0,$$

(1.3)(b)
$$Q\begin{bmatrix} 0 \\ \mu_1 f_1(\cdot)+G(\cdot,z,w,\alpha) \\ 0 \\ \mu_2 f_2(\cdot)+H(\cdot,z,w,\alpha) \end{bmatrix} = 0$$

are necessary and sufficient for (1.2) to have a T-periodic solution. Moreover such a solution must be of the form

$$x(t) = p(t+\alpha) + z(t)$$

$$y(t) = q(t+\alpha) + w(t)$$

where $(z,\dot{z},w,\dot{w})$ satisfy (1.3).

As in the previous chapter, the Implicit Function Theorem implies the existence of $\tilde{z}_\beta(t,\mu_1,\mu_2,\alpha)$ and $\tilde{w}_\beta(t,\mu_1,\mu_2,\alpha)$ which satisfy (1.3)(a) for $(\mu_1,\mu_2,\alpha)$ in a neighbornood of $[0,0,\alpha_0]^*$ in $\mathbb{R}^2 \times [0,T]$. Note that in this case, $\beta(\varphi)$ cannot be defined uniquely as in Chapter II, and so $\tilde{z}$ and $\tilde{w}$ will depend on $\beta$. This gives rise to a one-parameter family of solutions $\tilde{z}$ and $\tilde{w}$ of (1.3)(a). For now, we will assume that we have chosen a particular $\beta$ which satisfies

(1.4) $[\Phi(0)\beta(\varphi)+V_0(0,\varphi)]^*[\dot{p}(\alpha)\ddot{p}(\alpha)\dot{q}(\alpha)\ddot{q}(\alpha)]^* = 0;\ \beta(0) = 0.$

In §6, we will discuss the results if $\beta$ is allowed to vary. Let $\psi(t)$ be the matrix whose rows span the solution space of the adjoint homogeneous equation corresponding to (II.3.2). Let

$$\psi(t) = \begin{bmatrix} \dot{r}_1(t+\alpha) & r_1(t+\alpha) & \dot{s}_1(t+\alpha) & s_1(t+\alpha) \\ \dot{r}_2(t+\alpha) & r_2(t+\alpha) & \dot{s}_2(t+\alpha) & s_2(t+\alpha) \end{bmatrix}.$$

Then we may rewrite (1.2) as

$$L(\alpha)\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + E(\mu_1,\mu_2,\alpha) = 0 \tag{1.5}$$

where

$$L(\alpha) = \begin{bmatrix} \int_0^T r_1(t+\alpha)f_1(t)dt & \int_0^T s_1(t+\alpha)f_2(t)dt \\ \int_0^T r_2(t+\alpha)f_1(t)dt & \int_0^T s_2(t+\alpha)f_2(t)dt \end{bmatrix},$$

and

$$E(\mu_1,\mu_2,\alpha) = \begin{bmatrix} E_1(\mu_1,\mu_2,\alpha) \\ E_2(\mu_1,\mu_2,\alpha \end{bmatrix},$$

$$E_i(\mu_1,\mu_2,\alpha) = \int_0^T [r_i(t+\alpha)G(t,\tilde{z},\tilde{w},\alpha) + s_i(t+\alpha)H(t,\tilde{z},\tilde{w},\alpha)]dt$$

for $i = 1,2$.

## §2. Scaling

As in the previous chapter, we will want to use a scaling procedure, but the type of scaling used here will vary. The following lemma will guarantee that the results of the scaled problem may be used to solve the original.

Lemma 2.1. Let $F\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map given by $F(\alpha,r) = B(\alpha)r + E(\alpha,r)$ where $r \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, $B(\alpha)$ is a $2 \times 2$ matrix, and $E(\alpha,r) = O(|r|)$ as

$\eta \to 0$ uniformly for $\alpha \in [0,T]$. Let $\eta = [\epsilon\omega_1 \; \epsilon^{1+\lambda}\omega_2]^*$ and $\bar{F}(\alpha,\omega,\epsilon) = \bar{F}(\alpha,\omega_1,\omega_2,\epsilon) = \epsilon^{-1}F(\alpha,\epsilon\omega_1,\epsilon^{1+\lambda}\omega_2)$ for a fixed $\lambda \geq 0$. Suppose there exists $(\alpha_0,\omega_0)$ such that $F(\alpha_0,\omega_0,0) = 0$.

I. Suppose there exists a neighborhood $U$ in $\mathbb{R}^2$ of $(\omega_{20},0)$ and a unique pair of smooth functions, $\alpha(\omega_2,\epsilon)$ and $\omega_1(\omega_2,\epsilon)$, defined in $U$ such that

$$F(\alpha(\omega_2,\epsilon),\; \omega_1(\omega_2,\epsilon),\; \omega_2,\epsilon) = 0,$$

$$\omega_1(\omega_{2_0},0) = \omega_{1_0}, \quad \text{and} \quad \alpha(\omega_{2_0},0) = \alpha_0.$$

If $(\omega_2^{(1)},\epsilon_1)$ and $(\omega_2^{(2)},\epsilon_2)$ are both in $U$ and $\epsilon_1^{1+\lambda}\omega_2^{(1)} = \epsilon_2^{1+\lambda}\omega_2^{(2)}$, then $\epsilon_1\omega_1(\omega_2^{(1)},\epsilon_1) = \epsilon_2\omega_1(\omega_2^{(2)},\epsilon_2)$ and for $\epsilon_1,\epsilon_2 \neq 0$, $\alpha(\omega_2^{(1)},\epsilon_1) = \alpha(\omega_2^{(2)},\epsilon_2)$.

II. Suppose there exists a neighborhood $U$ in $\mathbb{R}^2$ of $(\omega_{10,})$ and a unique pair of smooth functions, $\alpha(\omega_1,\epsilon)$ and $\omega_2(\omega_1,\epsilon)$ defined in $U$ such that $\bar{F}(\alpha(\omega_1,\epsilon),\omega_1,\omega_2(\omega_1,\epsilon),\epsilon) = 0$, $\omega_2(\omega_{1_0},0) = \omega_{2_0}$ and $\alpha(\omega_{1_0},0) = \alpha_0$. If $(\omega_1^{(1)},\epsilon_1)$ and $(\omega_1^{(2)},\epsilon_2)$ are points in $U$ such that $\epsilon_1\omega_1^{(1)} = \epsilon_2\omega_1^{(2)}$, then $\epsilon_1^{1+\lambda}\omega_2(\omega_1^{(1)},\epsilon_1) = \epsilon_2^{1+\lambda}\omega_2(\omega_1^{(2)},\epsilon_2)$, and for $\epsilon_1,\epsilon_2 \neq 0$, $\alpha(\omega_1^{(1)},\epsilon_1) = \alpha(\omega_1^{(2)},\epsilon_2)$.

Proof. If $\epsilon_1 = \epsilon_2 = 0$, the result is clear, so let's assume $\epsilon_1,\epsilon_2 \neq 0$, and let the conditions of the lemma hold as in I. Then the first of the following statements is true by our hypotheses and the others are equivalent.

$$B(\alpha(\omega_2^{(1)},\epsilon_1))\begin{bmatrix}\omega_1(\omega_2^{(1)},\epsilon_1)\\ \epsilon_1^{\lambda}\omega_2^{(1)}\end{bmatrix}$$

$$+\ \epsilon_1 E(\omega_1(\omega_2^{(1)},\epsilon_1),\omega_2^{(1)},\alpha(\omega_2^{(1)},\epsilon_1),\epsilon_1) = 0$$

$$B(\alpha(\omega_2^{(1)},\epsilon_1))\begin{bmatrix}\epsilon_1\omega_1(\omega_2^{(1)},\epsilon_1)\\ \epsilon_1^{1+\lambda}\omega_2^{(1)}\end{bmatrix}$$

$$+\ \epsilon_1^2\bar{E}(\omega_1(\omega_2^{(1)},\epsilon_1),\omega_2^{(1)},\alpha(\omega_2^{(1)},\epsilon_1),\epsilon_1) = 0$$

$$B(\alpha(\omega_2^{(1)},\epsilon_1))\begin{bmatrix}\epsilon_2(\epsilon_2^{-1}\epsilon_1)\omega_1(\omega_2^{(1)},\epsilon_1)\\ \epsilon_2^{1+\lambda}\omega_2^{(2)}\end{bmatrix}$$

$$+\ E(\epsilon_2(\epsilon_2^{-1}\epsilon_1)(\omega_1(\omega_2^{(1)},\epsilon_1),\epsilon_2^{1+\lambda}\omega_2^{(2)},\alpha(\omega_2^{(1)},\epsilon_1)) = 0$$

$$B(\alpha(\omega_2^{(1)},\epsilon_1))\begin{bmatrix}\epsilon_2^{-1}\epsilon_1\omega_1(\omega_2^{(1)}\epsilon_1)\\ \epsilon_2^{1+\lambda}\omega_2^{(2)}\end{bmatrix}$$

$$+\ \epsilon_2\bar{E}(\epsilon_2^{-1}\epsilon_1\omega_1(\omega_2^{(1)},\epsilon_1),\epsilon_2^{1+\lambda}\omega_2^{(2)},\alpha(\omega_2^{(1)},\epsilon_1),\epsilon_2) = 0$$

Therefore

$$\alpha(\omega_2^{(2)},\epsilon_2) = \alpha(\omega_2^{(1)},\epsilon_1), \quad \text{and} \quad \epsilon_2^{-1}\epsilon_1\omega_1(\omega_2^{(1)},\epsilon_1) = \omega_1(\omega_2^{(2)},\epsilon_2)$$

since these are the unique functions defined in U and satisfying the last equation. An analagous proof shows II. □

§3. The Case for $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 \neq 0$

The $L(\alpha)$ we have defined is a $2 \times 2$ matrix which is a T-periodic function of $\alpha$. Let $\Omega = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ = the set of values of $\alpha$ for which $L(\alpha)$ is singular. We will assume that $L(\alpha) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $\alpha$, though it would not affect the local analysis for a particular $\alpha \in \Omega$ if $L(\alpha) = 0_{2\times 2}$ for a different $\alpha \in \Omega$.

In this section, we will consider values $\alpha_0 \in \Omega$ for which

$$(3.1) \qquad [JL(\alpha_0)]^*L'(\alpha_0)\omega_0 \neq 0$$

where $\omega_0$ is a unit eigenvector for $L(\alpha_0)$ and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In this case, we let $\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \epsilon \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \epsilon\omega$, and let $\bar{E}(\omega,\alpha,\epsilon) = \epsilon^{-2}E(\epsilon\omega_1, \epsilon\omega_2, \alpha)$. Our problem can then be reformulated as

$$(3.2) \qquad F(\omega,\alpha,\epsilon) = L(\alpha)\omega + \epsilon\bar{E}(\omega,\alpha,\epsilon) = 0.$$

For notational convenience, we will let

$$L(\alpha) = \begin{bmatrix} \ell_1(\alpha) & \ell_2(\alpha) \\ \ell_3(\alpha) & \ell_4(\alpha) \end{bmatrix}.$$

We have

$$F(\omega_0,\alpha_0,0) = 0 \quad \text{and} \quad D_\alpha F(\omega_0,\alpha_0,0) = L'(\alpha_0)\omega_0 \neq 0$$

by (3.1). Therefore by the Implicit Function Theorem, there exists a neighborhood $V$ of $(\omega_0, 0)$ and a unique function $\tilde{\alpha}(\omega,\epsilon)$ defined on $V$ such that

$\tilde{\alpha}(\omega_0,0) = \alpha_0$ and $F_i(\omega,\tilde{\alpha}(\omega,\epsilon),\epsilon) = 0$ for all $(\omega,\epsilon) \in V$ for either $i = 1$ or $i = 2$. Let's assume $i = 1$. (For $i = 2$, the same result is true by a similar argument.) Then for a given $\omega$, (3.2) will have a solution if and only if

$$F_2(\omega,\tilde{\alpha}(\omega,\epsilon),\epsilon) = 0.$$

We know that

$$F_2(\omega_0,\tilde{\alpha}(\omega_0,0),0) = 0$$

and that

$$D_\omega F_2 = D_\omega\tilde{\alpha}(\omega_0,0)[\ell_3'(\alpha_0) \;\; \ell_4'(\alpha_0)]\omega_0 + [\ell_3(\alpha_0) \;\; \ell_4(\alpha_0)]$$

Since $F_1(\omega,\tilde{\alpha}(\omega,\epsilon),\epsilon) = 0$ for all $(\omega,\epsilon) \in V$, it follows that

$$D_\omega\tilde{\alpha}(\omega_0,0)[\ell_1'(\alpha_0) \;\; \ell_2'(\alpha_0)]\omega_0 + [\ell_1(\alpha_0) \;\; \ell_2(\alpha_0)] = 0.$$

Thus

$$D_\omega\tilde{\alpha}(\omega_0,0) = -[\ell_1(\alpha_0) \;\; \ell_2(\alpha_0)]/[\ell_1(\alpha_0) \;\; \ell_2(\alpha_0)]\omega_0$$

which implies

$$D_\omega F_2 = -[\ell_1(\alpha_0) \;\; \ell_2(\alpha_0)]\{[\ell_3(\alpha_0) \;\; \ell_4(\alpha_0)]\omega_0/[\ell_1'(\alpha_0) \;\; \ell_2'(\alpha_0]\omega_0\}$$
$$+ [\ell_3(\alpha_0) \;\; \ell_4(\alpha_0)].$$

We have assumed $[\ell_1'(\alpha_0) \;\; \ell_2'(\alpha_0)]\omega_0 \neq 0$, and so $D_\omega F_2 = 0$ if and only if

$$-[\ell_1(\alpha_0),\ell_2(\alpha_0)][\ell_3'(\alpha_0),\ell_4'(\alpha_0)]\omega_0 + [\ell_3(\alpha_0) \;\; \ell_4(\alpha_0)]$$

$$[\ell_1'(\alpha_0) \quad \ell_2'(\alpha_0)]\omega_0 \neq 0,$$

which is equivalent to

$$\begin{bmatrix} \ell_3(\alpha_0) & -\ell_1(\alpha_0) \\ \ell_4(\alpha_0) & -\ell_2(\alpha_0) \end{bmatrix} \begin{bmatrix} \ell_1'(\alpha_0) & \ell_2'(\alpha_0) \\ \ell_3'(\alpha_0) & \ell_4'(\alpha_0) \end{bmatrix} \omega_0 \neq 0$$

or to

$$[J\ L(\alpha_0)]^* L'(\alpha_0)\omega_0 \neq 0.$$

This is our assumption and thus $D_\omega F_2 \neq 0$. Therefore, by the Implicit Function Theorem, we may solve uniquely for $\omega_1(\omega_2,\epsilon)$ in a neighborhood of $(\omega_{2_0},0)$ or for $\omega_2(\omega_1,\epsilon)$ in a neighborhood of $(\omega_{1_0},0)$ or for both to solve (3.2). Suppose we have defined $\omega_1(\omega_2,\epsilon)$. Then

$$\tilde{\alpha}(\omega_1,\omega_2,\epsilon) = \tilde{\alpha}(\omega_1(\omega_2,\epsilon),\omega_2,\epsilon) = \tilde{\alpha}(\omega_2,\epsilon)$$

is a function of $\omega_2$ and $\epsilon$, and since $\tilde{\alpha}(\omega_1,\omega_2,\epsilon)$ and $\omega_1(\omega_2,\epsilon)$ were uniquely defined, $\tilde{\alpha}(\omega_2,\epsilon)$ and $\omega_1(\omega_2,\epsilon)$ are the unique pair of functions satisfying

$$F(\omega_1(\omega_2,\epsilon),\omega_2,\tilde{\alpha}(\omega_2,\epsilon),\epsilon) = 0,\ \omega_1(\omega_{2_0},0) = \omega_{1_0}, \tilde{\alpha}(\omega_{2_0},0) = \alpha_0.$$

Therefore we can apply Lemma 2.1 with $\lambda = 0$ to guarantee that the functions

$$\mu_1(\mu_2) = \begin{cases} \epsilon\omega_1(\omega_2,\epsilon) & \text{where } \epsilon\omega_2 = \mu_2 \\ 0 \text{ if } & \mu_2 = 0 \end{cases}$$

and $\tilde{\alpha}(\mu_2) = \tilde{\alpha}(\omega_2,\epsilon)$ where $\epsilon\omega_2 = \mu_2$ are well defined in a neighborhood of origin.

A similar argument may be used to define $\mu_2(\mu_1)$ and $\tilde{\alpha}(\mu_1)$ in the case that $\omega_2(\omega_1,\epsilon)$ is defined such that

$F(\omega_1, \omega_2(\omega_1,\epsilon), \tilde{\alpha}(\omega_1,\omega_2(\omega_1,\epsilon),\epsilon),\epsilon) = 0$. In either case, we get a curve $C$ through the origin in the $\mu_1$-$\mu_2$ plane on which a solution $\tilde{\alpha}$ to (1.3) may be defined. Since

$$\lim_{\substack{\mu \to 0 \\ \mu \text{ on } C}} \frac{\mu_2}{\mu_1} = \lim_{\epsilon \to 0} \frac{\epsilon\omega_2}{\epsilon\omega_1(\omega_2,0)} = \frac{\omega_{2_0}}{\omega_{1_0}},$$

the vector $\omega_0$ is tangent to $C$ at the origin.

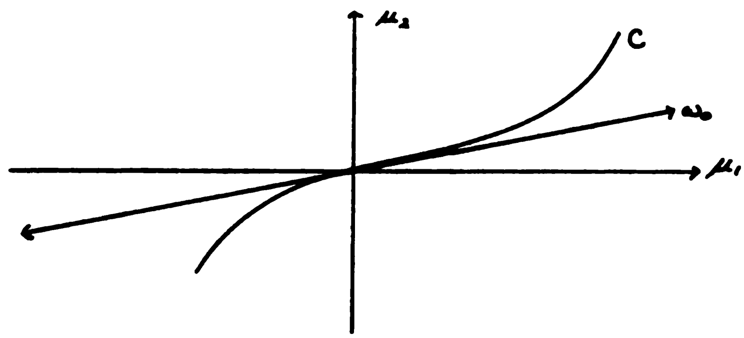


Figure 2. Curve on which solutions are defined.

## §4. The Case for $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0$.

In the previous section, we considered solutions of the system under the condition $L(\alpha_0)\omega_0 = 0$ and $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 \neq 0$. In this section, we suppose $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0$ and see what possibilities exist. Let $\omega_0 = \begin{bmatrix}\omega_{0_1}\\ \omega_{0_2}\end{bmatrix}$ be a unit eigenvector for $L(\alpha_0)$, and let $B$ be a $2 \times 2$ matrix such that $B\omega_0 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\det B = 1$. We note that

$$B^*J^*B = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b_2b_3 - b_1b_4 \\ b_1b_4 - b_2b_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J^*,$$

and so $B^*J^* = J^* B^{-1}$. Thus the following are equivalent

$$[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0$$

$$L(\alpha_0)^*J^*B^{-1}BL'(\alpha_0)B^{-1}B\omega_0 = 0$$

$$(B^{-1})^*L(\alpha_0)^*B^*J^*BL'(\alpha_0)B^{-1}B\omega_0 = 0$$

$$(JBL(\alpha_0)B^{-1})^*(BL'(\alpha_0)B^{-1})(B\omega_0) = 0.$$

Let $A(\alpha) = BL(\alpha)B^{-1} = \begin{bmatrix} a_1(\alpha) & a_2(\alpha) \\ a_3(\alpha) & a_4(\alpha) \end{bmatrix}$. We have

$A(\alpha_0)\begin{bmatrix}1\\0\end{bmatrix} = 0$, and so $a_1(\alpha_0) = a_3(\alpha_0) = 0$, and

$[JA(\alpha_0)]^*A'(\alpha_0)\begin{bmatrix}1\\0\end{bmatrix} = 0$ which implies

$$\begin{bmatrix} 0 & 0 \\ a_4(\alpha_0) & -a_2(\alpha_0) \end{bmatrix}\begin{bmatrix} a_1'(\alpha_0) & a_2'(\alpha_0) \\ a_3'(\alpha_00 & a_4'(\alpha_0) \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 0\,.$$

Thus $a_4(\alpha_0)a_1'(\alpha_0) - a_2(\alpha_0)a_3'(\alpha_0) = 0$. Since $B\omega_0 = \begin{bmatrix}1\\0\end{bmatrix}$,

$$b_1\omega_{0_1} + b_2\omega_{0_2} = 0 \quad \text{and}$$

$$b_3\omega_{0_1} + b_4\omega_{0_2} = 0, \quad \text{but}$$

$$b_1b_4 - b_2b_3 = 0$$

and so we get $b_4 = \omega_{1_0}$ and $b_3 = -\omega_{2_0}$. Moreover

$L(\alpha_0)\omega_0 = 0$, and so

$$\ell_1(\alpha_0)b_4 - \ell_2(\alpha_0)b_3 = 0$$

$$\ell_3(\alpha_0)b_4 - \ell_4(\alpha_0)b_3 = 0$$

It will be useful to specify as much as possible about the form of $A$. We know

$$A(\alpha_0) = \begin{bmatrix} b_1b_4\ell_1 + b_2b_4\ell_3 - b_1b_3\ell_2 - b_2b_3\ell_4 & -b_1b_2\ell_1 - b_2^2\ell_3 + b_1^2\ell_2 + b_1b_2\ell_4 \\ b_3b_4\ell_1 + b_4^2\ell_3 - b_3^2\ell_2 - b_3b_4 - \ell_4 & -b_2b_3\ell_1 - b_2b_4\ell_3 + b_1b_3\ell_2 + b_1b_4\ell_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b_1\ell_2 + b_1b_2(\ell_4 - \ell_1) - b_2^2\ell_3 \\ 0 & \ell_1 + \ell_4 \end{bmatrix}$$

where $\ell_i = \ell_i(\alpha_0)$.

Since $b_1$ and $b_2$ need satisfy only the condition $b_1b_4 - b_2b_3 = 1$, we may choose $b_1$ and $b_2$ such that

$$(4.1) \quad b_1^2\ell_2(\alpha_0) + b_1b_2(\ell_4(\alpha_0) - \ell_1(\alpha_0)) - b_2^2\ell_3(\alpha_0) = b \neq 0.$$

We also note that

$$a_1'(\alpha_0) = b_1[b_4\ell_1'(\alpha_0) + b_3\ell_2'(\alpha_0)] + b_2[b_4\ell_3'(\alpha_0) - b_3\ell_4'(\alpha_0)],$$

and so as long as

$$(4.2) \quad b_4\ell_1'(\alpha_0) + b_3\ell_2'(\alpha_0) \quad + \quad b_4\ell_3'(\alpha_0) - b_3\ell_4'(\alpha_0) \quad \neq 0,$$

we may choose $b_1$ and $b_2$ so that (4.1) holds and $a_1'(\alpha_0) \neq 0$. Condition (4.2) is the same as $L'(\alpha_0)\omega_0 \neq 0$. Furthermore $L(\alpha)\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + E(\mu_1,\mu_2,\alpha) = 0$ if and only if $b^{-1}L(\alpha)\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + b^{-1}E(\mu_1,\mu_2,\alpha) = 0$, so we may assume that

$$A(\alpha_0) = \begin{bmatrix} 0 & 1 \\ 0 & b^{-1}(\ell_1(\alpha_0) + \ell_4(\alpha_0)) \end{bmatrix}$$

by replacing $E$ with a multiple of $E$. Using this and the fact that $a_4(\alpha_0)a_1'(\alpha_0) - a_2(\alpha_0)a_3'(\alpha_0) = 0$, we arrive at the following form for $A(\alpha)$:

$$A(\alpha+\alpha_0) = \begin{bmatrix} a\alpha + \alpha^2 m_1(\alpha) & 1 + \alpha m_2(\alpha) \\ ac\alpha + \alpha^2 m_3(\alpha) & c + d\alpha + \alpha^2 m_4(\alpha) \end{bmatrix}$$

where $m_i(\alpha) = O(1)$ as $\alpha \to 0$.

We may now simplify the original problem as follows. The next four equations are equivalent

$$L(\alpha)\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + E(\mu_1,\mu_2,\alpha) = 0$$

$$L(\alpha)B^{-1}B\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + E(\mu_1,\mu_2,\alpha) = 0$$

$$b^{-1}BL(\alpha)B^{-1}B\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + b^{-1}BE(\mu_1,\mu_2,\alpha) = 0$$

$$A(\alpha)B\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + b^{-1}BE(\mu_1,\mu_2,\alpha) = 0$$

Moreover

$$A_{\alpha_0}(\tilde{\alpha}(\mu_1,\mu_2))B\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + b^{-1}BE_{\alpha_0}(\mu_1,\mu_2,\tilde{\alpha}(\mu_1,\mu_2)) = 0$$

if and only if

$$A(\tilde{\alpha}(\mu_1,\mu_2)+\alpha_0)B\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + b^{-1}E(\mu_1,\mu_2,\tilde{\alpha}(\mu_1,\mu_2)+\alpha_0) = 0$$

where

$$A_{\alpha_0}(\alpha) = A(\alpha+\alpha_0) \quad \text{and} \quad E_{\alpha_0}(\mu_1,\mu_2,\alpha) = E(\mu_1,\mu_2,\alpha+\alpha_0).$$

Therefore $\tilde{\alpha}(\mu_1,\mu_2)$ solves

$$A_{\alpha_0}(\alpha)B\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + b^{-1}BE_{\alpha_0}(\mu_1,\mu_2,\alpha) = 0$$

if and only if $\tilde{\alpha}(\mu_1,\mu_2) + \alpha_0$ solves

$$A(\alpha)B\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + b^{-1}BE(\mu_1,\mu_2,\alpha) = 0.$$

In the next section we will consider solutions of the problem (1.5) if $L(\alpha)$ has the form $\begin{bmatrix} a\alpha + \alpha^2 m_1(\alpha) & 1 + \alpha m_2(\alpha) \\ ac\alpha + \alpha^2 m_3(\alpha) & c + d\alpha + \alpha^2\ m_4(\alpha)\end{bmatrix}$. The preceding discussion shows that we will be able to obtain from this results for all cases in which $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0$.

## §5. Conditions for Existence of a Solution to the Reduced Problem

In this section, we consider the problem

(5.1) $$L(\alpha)\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + E(\mu_1,\mu_2,\alpha) = 0,$$

$$L(\alpha) = \begin{bmatrix} a\alpha + \alpha^2 m_1(\alpha) & 1 + \alpha m_2(\alpha) \\ ac\alpha + \alpha^2 m_3(\alpha) & c + d\alpha + \alpha^2 m_4(\alpha)\end{bmatrix}$$

$$a \neq 0$$

$$m_i(\alpha) = O(1), \quad i = 1, 4$$

$$E(\mu_1,\mu_2,\alpha) = o(|\mu_1| + |\mu_2|).$$

Since $E = o(|\mu_1| + |\mu_2|)$,

(5.2) $$\begin{aligned} E(\mu_1,\mu_2,\alpha) = {} & \frac{1}{2} D^2_{\mu_1} D(0,0,\alpha)\mu_1^2 + D_{\mu_1}D_{\mu_2}E(0,0,\alpha)\mu_1,\mu_2 \\ & + \frac{1}{2} D^2_{\mu_2} E(0,0,\alpha)\mu_2^2 + \frac{1}{6} D^3_{\mu_2} E(0,0,\alpha)\mu_1^3 \\ & + \frac{1}{2} D^2_{\mu_1} D_{\mu_2} E(0,0,\alpha)\mu_1^2\mu_2 \\ & + \frac{1}{2} D_{\mu_1} D_{\mu_2} E(0,0,\alpha)\mu_1,\mu_2^2 + \end{aligned}$$

$$+ \frac{1}{6} D^3_{\mu_2} D(0,0,\alpha)\mu_2^3$$

$$+ \frac{1}{24} D^4_{\mu_1} E(0,0,\alpha)\mu_1^4 + \ldots .$$

As before we wish to use a scaling procedure to convert (5.1) into an equation to which the Implicit Function Theorem may be applied, but a different change of variables is more useful in this case. We let $\begin{matrix}\mu_1 \\ \mu_2\end{matrix} = \begin{bmatrix}\epsilon\omega_1 \\ \epsilon\delta\omega_2\end{bmatrix}$ where $\delta = \epsilon^\lambda$ for $\lambda \geq 0$ and $\lambda$ will be chosen so that the scaled equation may possibly have solutions. (5.1) now becomes

$$\text{(5.3)} \qquad L(\alpha)\begin{bmatrix}\omega_1 \\ \delta\omega_2\end{bmatrix} + \epsilon\overline{E}(\omega,\alpha,\delta,\epsilon) = 0,$$

where

$$\epsilon^2\overline{E}(\omega,\alpha,\delta,\epsilon) = E(\epsilon\omega_1,\epsilon\delta\omega_2,\alpha).$$

Using (5.2), we see that

$$\begin{aligned}\epsilon^2\overline{E}(\omega,\alpha,\delta,\epsilon) = {} & \frac{1}{2} D^2_{\mu_1} E(0,0,\alpha)\epsilon^2\omega_1^2 + D_{\mu_1}D_{\mu_2}E(0,0,\alpha)\epsilon^2\delta\omega_1\omega_2 \\ & + \frac{1}{2} D^2_{\mu_2} E(0,0,\alpha)\epsilon^2\delta^2\omega_2^2 + \frac{1}{6} D^3_{\mu_1} E(0,0,\alpha)\epsilon^3\omega_1^3 \\ & + \frac{1}{2} D^2_{\mu_1} D_{\mu_2} E(0,0,\alpha)\epsilon^3\delta\omega_1^2\omega_2 \\ & + \frac{1}{2} D_{\mu_1} D^2_{\mu_2} E(0,0,\alpha)\epsilon^3\delta^2\omega_1\omega_2^2 \\ & + \frac{1}{6} D^3_{\mu_2} E(0,0,\alpha)\epsilon^3\delta^3\omega_2^3 \\ & + \frac{1}{24} D^4_{\mu_1} E(0,0,\alpha)\epsilon^4\omega_1^4 + \ldots .\end{aligned}$$

Thus $\epsilon\overline{E}(\omega,\alpha,\delta,\epsilon) = \epsilon A_{20} + \epsilon\delta A_{11} + \epsilon\delta^2 A_{02} + \epsilon^2 A_{30} + \epsilon^2\delta A_{21}$
$+ \epsilon^2\delta^2 A_{12} + \epsilon^2\delta^3 A_{03} + \epsilon^3 A_{40} + \ldots$ where

$A_{ij} = \begin{bmatrix} A_{ij}^{(1)} \\ A_{ij}^{(2)} \end{bmatrix}$ is a function of $\omega_1, \omega_2, \alpha$. We may now rewrite

(5.3) in the following form:

$$(5.4)(a)\qquad [(a\alpha\omega_1 + \delta\omega_2) + (\alpha^2 m_1(\alpha)\omega_1 + \alpha\delta m_2(\alpha)\omega_2)]$$
$$+ \epsilon[A_{20}^{(1)} + \delta A_{11}^{(1)} + \delta^2 A_{02}^{(1)}]$$
$$+ \epsilon^2[A_{30}^{(1)} + \delta A_{21}^{(1)} + \delta^2 A_{12}^{(1)} + \delta^3 A_{03}^{(1)}]$$
$$+ \epsilon^3[A_{40}^{(1)} + \ldots] + \ldots = 0$$

$$(5.4)(b)\qquad [(ac\alpha\omega_1 + c\delta\omega_2) + (\alpha^2 m_3(\alpha)\omega_1 + d\alpha\delta\omega_2) + \alpha^2\delta m_4(\alpha)\omega_2)]$$
$$+ \epsilon[A_{20}^{(2)} + \delta A_{11}^{(2)} + \delta^2 A_{02}^{(2)}]$$
$$+ \epsilon^2[A_{30}^{(2)} + \delta A_{21}^{(2)} + \delta^2 A_{12}^{(2)} + \delta^3 A_{03}^{(2)}]$$
$$+ \epsilon^3[A_{40}^{(2)} + \ldots] + \ldots = 0.$$

We will now apply the method of Newton's Polygon as described by W. D. MacMillan [ 8 ]. The slopes of the lines in the polygons represent values of $\lambda$ for which $\alpha = \epsilon^\lambda\overline{\alpha}(\epsilon)$ and $\delta = \epsilon^\lambda$ may possibly solve (5.4).

It is reasonable to hope that solutions of this type may exist. Equation (5.4) is solved by $\epsilon = 0$, $\alpha = 0$, and $\delta\omega_2 = 0$, and so if we solve for $\alpha$ and $\delta\omega_2$ as functions of $\epsilon$ in a neighborhood of zero, we expect $\alpha$ and $\delta\omega_2$ to have a power of $\epsilon$ as a factor. Let $\lambda$

be the highest power of $\epsilon$ contained in both. Then $\alpha = \epsilon^{\lambda}\bar{\alpha}(\epsilon)$ and $\delta\omega_2 = \epsilon^{\lambda}\omega_2(\epsilon)$. We could use the Implicit Function Theorem to verify this if the Jacobian matrix of equations (5.4) with respect to $\alpha$ and $(\delta\omega_2)$,

$$\begin{bmatrix} a\omega_1 & 1 \\ ac\omega_1 & c \end{bmatrix}$$

were nonsingular. Since it is not, we do not necessarily expect unique solutions of (5.4) in an entire neighborhood of zero, but solutions of the type described above may be possible under some conditions.

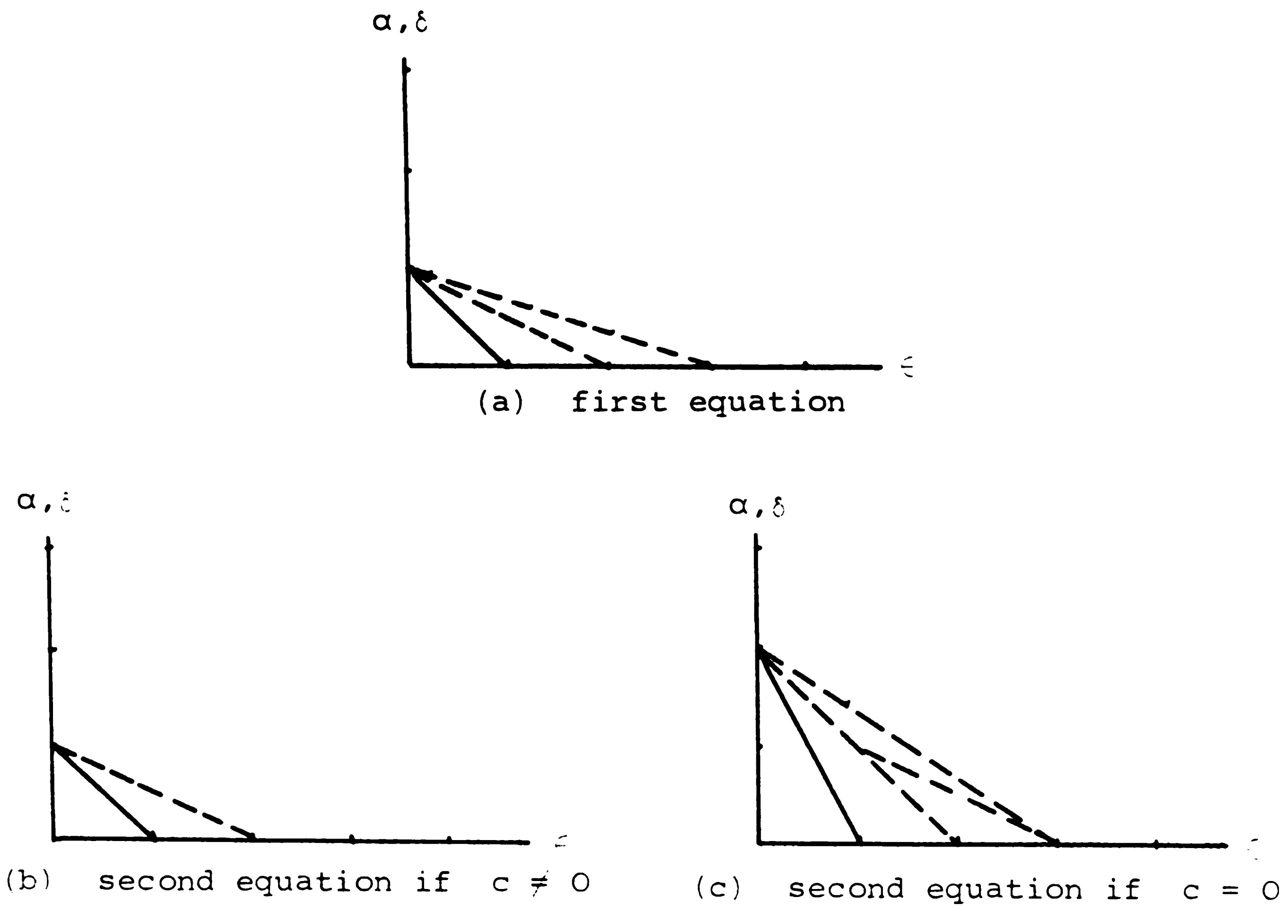


Figure 3. Newton's polygons for equations (5.4).

The solid lines in the diagram indicate possible values of $\lambda$ if $D_{\mu}^{2}E_1(0,\omega_0) \neq 0$ for (a) and $D_{\mu}^{2}E_2(0,\omega_0) \neq 0$ for (b) and (c). The dotted lines indicate possible values of $\lambda$ if these and perhaps other derivatives of $E$ vanish at $(0,\omega_0)$. If $c \neq 0$, $\lambda$ will equal one generically, but may be an integer greater than one if appropriate derivatives vanish as indicated above. If $c = 0$, $\lambda$ may equal one or one half generically, but may equal an integer multiple of one half if the appropriate derivatives vanish at $(0,\omega_0)$.

We will look only at the cases where $\beta = 1$ or $\beta = \frac{1}{2}$ are possible solutions. These will be the only possibilities with probability one, and the other cases could be dealt with in the same manner as the ones we will discuss.

First we consider the case $\lambda = 1$. We then have $\delta = \epsilon$ and $\alpha = \epsilon\bar{\alpha}$, and so equation (5.4) becomes

$$(5.5)(a) \quad [(a\bar{\alpha}\omega_1 + \omega_2) + \epsilon\bar{\alpha}^2 \cdot m_1(\epsilon\bar{\alpha})\omega_1 + \epsilon\bar{\alpha}m_2(\epsilon\bar{\alpha})\omega_2] + A_{20}^{(1)} + \epsilon[A_{11}^{(1)} + A_{30}^{(1)}] + \epsilon^2[A_{02}^{(1)} + A_{21}^{(1)} + A_{40}^{(1)}] + \ldots = 0$$

$$(5.5)(b) \quad [(ac\bar{\alpha}\omega_1 + c\omega_2) + (\epsilon\bar{\alpha}^2 m_3(\epsilon\bar{\alpha})\omega_1 + \epsilon\bar{\alpha}d\omega_2) + \epsilon^2\bar{\alpha}^2 m_4(\epsilon\bar{\alpha})\omega_2] + A_{20}^{(2)} + \epsilon[A_{11}^{(2)} + A_{30}^{(2)}] + \epsilon^2[A_{02}^{(2)} + A_{21}^{(2)} + A_{40}^{(2)}] + \ldots = 0$$

which we denote by $G_1(\bar{\alpha},\omega_1,\omega_2,\epsilon) = 0$. We are hoping for a solution for values of $\epsilon$ in a neighborhood of zero and so a necessary condition for a solution is the existence of $\omega_{2_0}$ and $\bar{\alpha}_0$ (Recall $\omega_{1_0} = 1$) such that $G_1(\bar{\alpha}_0, 1, \omega_{2_0}, 0) = 0$; i.e.

$$a\bar{\alpha}_0 + \omega_{2_0} + A_{20}^{(1)}(1, \omega_{2_0}, \bar{\alpha}_0) = 0$$

$$ac\bar{\alpha}_0 + c\omega_{2_0} + A_{20}^{(2)}(1, \omega_{2_0}, \bar{\alpha}_0) = 0$$

or

$$a\bar{\alpha}_0 + \omega_{2_0} + \frac{1}{2} D_{\mu_1}^2 E_1(0,0,0) = 0$$

$$ac\bar{\alpha}_0 + c\omega_{2_0} + \frac{1}{2} D_{\mu_1}^2 D_2(0,0,0) = 0.$$

This will be satisfied by any $\omega_{2_0}$ and $\bar{\alpha}_0 = -(2a)^{-1}(D_{\mu_1}^2 E_1(0,0,0) + \alpha\omega_{2_0})$ if and only if

$$\text{(5.6)} \qquad -cD_{\mu_1}^2 E_1(0,0,0) + D_{\mu_1}^2 E_2(0,0,0) = 0.$$

However in this case, the Jacobian matrix,

$$D_{(\omega_1, \omega_2, \alpha)} G(1, \omega_{2_0}, \bar{\alpha}_0, 0)$$

$$= \begin{bmatrix} a\bar{\alpha}_0 + E_{1\mu\mu}(0,0,0) & 1 & a \\ ac\bar{\alpha}_0 + E_{2\mu\mu}(0,0,0) & c & ac \end{bmatrix},$$

has rank 1, and so the existence of solutions to (5.5) will depend on the higher order terms.

Next we consider the case $\lambda = \frac{1}{2}$ which is possible when $c = 0$. In this case, $\delta = \epsilon^{1/2}$ and $\alpha = \epsilon^{1/2}\bar{\alpha}$, and equation (5.4) becomes

$$\text{(5.7)(a)} \quad [(a\bar{\alpha}\omega_1 + \omega_2) + \epsilon^{1/2}\bar{\alpha}^2 m_1(\epsilon^{1/2}\bar{\alpha})\omega_1 + \epsilon^{1/2}\bar{\alpha} m_2(\epsilon^{1/2}\bar{\alpha})\omega_2]$$

$$+ \epsilon^{1/2}[A_{20}^{(1)}] + \epsilon[A_{11}^{(1)}] + \epsilon^{3/2}[A_{02}^{(1)} + A_{30}^{(1)}] + \dots = 0$$

$$(5.7)(b) \quad [(\bar{\alpha}^2 m_3(\epsilon^{1/2}\bar{\alpha})\omega_1 + d\bar{\alpha}\omega_2) + \epsilon^{1/2}\bar{\alpha}^2 m_4(\epsilon^{1/2}\bar{\alpha})\omega_2]$$

$$+ A_{20}^{(2)} + \epsilon^{1/2} A_{11}^{(2)} + \epsilon[A_{02}^{(2)} + A_{30}^{(2)}] + \epsilon^{3/2} A_{21}^{(2)}$$

$$+ \ldots = 0$$

which we shall refer to as $G_2(\omega_1, \omega_2, \bar{\alpha}, \epsilon) = 0$. Again we recall that $\omega_{1_0} = 1$, and so a necessary condition is the existence of $\omega_{2_0}$ and $\bar{\alpha}_0$ such that $G_2(1, \omega_{2_0}, \bar{\alpha}_0, 0) = 0$, or

$$a\bar{\alpha}_0 + \omega_{2_0} = 0$$

$$\bar{\alpha}_0^2 m_3(0) + d\bar{\alpha}_0 \omega_{2_0} + A_{2_0}^{(2)} = 0$$

This will be satisfied by $\omega_{2_0} = -a\bar{\alpha}_0$ and an $\bar{\alpha}_0 \neq 0$ which satisfies $\bar{\alpha}_0^2(m_3(0) - ad) + \frac{1}{2} D_{\mu_1}^2 E_2(0,0,0) = 0$ if one exists. Thus a necessary condition for a solution of this type is that either

$$(5.8) \qquad -D_{\mu_1}^2 E_2(0,0,0)/2(m_3(0) - ad) > 0$$

$$(m_3(0) - ad \neq 0)$$

or

$$m_3(0) - ad = 0 \quad \text{and} \quad D_{\mu_1}^2 E_2(0,0,0) = 0.$$

The Jacobian matrix

$$D_{(\omega_1, \omega_2, \bar{\alpha})} G_3(1, \omega_{2_0}, \bar{\alpha}_0, 0)$$

$$= \begin{bmatrix} a\bar{\alpha}_0 & 1 & a \\ \bar{\alpha}_0^2 m_3(0) + E_{2_{\mu\mu}}(0,0,0) & \bar{\alpha}_0 d & 2\bar{\alpha}_0 m_3(0) + \alpha\omega_{2_0} \end{bmatrix}.$$

Note that the determinant of the submatrix consisting of columns two and three is $2\bar{\alpha}_0 m_3(0) + d\omega_{2_0} - a\bar{\alpha}_0 d$ $= 2\bar{\alpha}_0 m_3(0) - 2a\bar{\alpha}_0 d = 2\bar{\alpha}_0(m_3(0) - ad)$ and will be nonzero if $m_3(0) \neq ad$. The determinant of the submatrix consisting of columns one and three is

$2a\bar{\alpha}_0^2 m_3(0) + a\bar{\alpha}_0 d\omega_{2_0} - a\bar{\alpha}_0^2 m_3(0) - aD^2_{\mu_1} D_2(0,0,0)$

$= a[\bar{\alpha}_0^2 m_3(0) - a\bar{\alpha}_0^2 d - D^2_{\mu_1} E_2(0,0,0)]$

$= a\{\bar{\alpha}_0^2[m_3(0) - ad] - D^2_{\mu_1} E_2(0,0,0)$. This cannot be zero if $D^2_{\mu_1} E_2(0,0,0) \neq 0$ since $\bar{\alpha}_0$ must satisfy $\bar{\alpha}_0^2(m_3(0) - ad)$ $+ \frac{1}{2} D^2_{\mu_1} D_2(0,0,0) = 0$. Therefore

$$(m_3(0) - ad) \neq 0 \quad \text{and} \quad \frac{-D^2_{\mu_1} E_2(0,0,0)}{(m_3(0) - ad)} < 0$$

will be sufficient conditions for a solution. If $m_3(0) - ad = 0$, the existence of a solution depends on the higher order terms.

## §6. Statements of Results

In the previous sections, we have considered $\beta$ to be a fixed map from $(I-Q)\mathcal{P}_T$ to $\mathbb{R}^2$ which satisfied condition (1.4). In this section, we will discuss the result of allowing $\beta$ to vary. This will allow us to state theorems which have been proved for the most part in §3 or in §5.

Theorem 6.1. In the equation

$$\ddot{x} + g(x,y) = \mu_1 f_1$$
(6.1)
$$\ddot{y} + h(x,y) = \mu_2 f_2,$$

we assume that $g$, $h$, $f_1$, $f_2$ are smooth and $f_1$, $f_2$ are T-periodic functions of $t$.

Suppose that $(p(t),q(t))$ is a T-periodic solution of

$$\ddot{x} + g(x,y) = 0$$

$$\ddot{y} + h(x,y) = 0$$

and that there are exactly two linearly independent, T-periodic solutions of

$$\ddot{z} + [g_x(p(t+\alpha),q(t+\alpha)),g_y(p(t+\alpha),q(t+\alpha))]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = 0$$
(6.2)
$$\ddot{w} + [h_x(p(t+\alpha),q(t+\alpha)),h_y(p(t+\alpha),q(t+\alpha))]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = 0$$

Let $(r_1(t+\alpha),s_1(t+\alpha))$ and $(r_2(t+\alpha),s_2(t+\alpha))$ be the solutions to the adjoint equation, and let

$$L(\alpha) = \begin{bmatrix} \int_0^T r_1(t+\alpha)f_1(t)\,dt & \int_0^T s_1(t+\alpha)f_2(t)\,dt \\ \int_0^T r_2(t+\alpha)f_1(t)\,dt & \int_0^T s_2(t+\alpha)f_2(t)\,dt \end{bmatrix}.$$

Suppose that

(6.3) $L(\alpha_0)\omega_0 = 0;\ [JL(\alpha_0)]^*L'(\alpha_0)\omega_0 \neq 0;\ |\omega_0| = 1$

where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Then there exist two curves $C_1$ and $C_2$ in a neighborhood of the origin in $\mathbb{R}^2$ such that for $\mu$ between $C_1$ and $C_2$, there exists a solution to (6.1). $C_1$ and $C_2$ intersect at the origin where both are tangent to $\mu = t\omega_0$ $(t \in \mathbb{R})$.

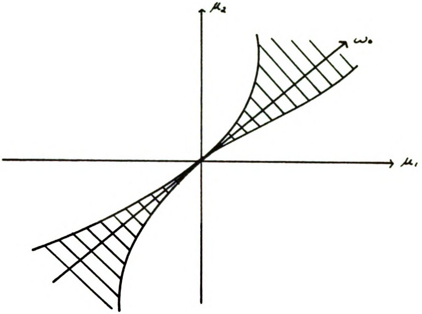


Figure 4. One region in which solutions exist.

<u>Proof</u>. $L(\alpha)$ is independent of $\beta$, and so for each $\beta$ which satisfies (1.4) we have already shown there exists a curve $C_\beta$ tangent to $\mu = t\omega_0$ $(t \in \mathbb{R})$ at the origin such that (6.1) has a solution for $\mu$ on $C_\beta$. It is clear from the definition of $\chi_\beta$ (II.3.3) that $\tilde{z}$, $\tilde{w}$, and thus $\alpha$ depend on $\beta$ smoothly. Thus $\bigcup_{\beta \text{ satisfies } (1.4)} C_\beta$ is a region of the form described

and so curves $C_1$ and $C_2$ which bound this region exist and are as described in the theorem. □

Such regions will exist for each $(\alpha_0, \omega_0)$ which satisfies (6.3) and so there may be several regions in which solutions exist.

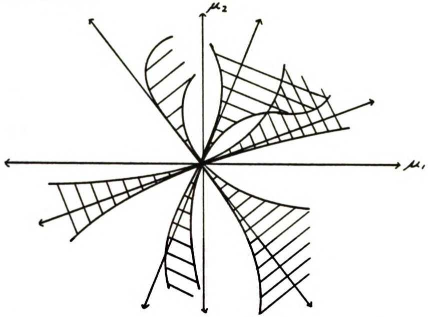


**Figure 5.** Solution regions when $[JL(\alpha_0)]^* L'(\alpha_0)\omega_0 \neq 0$.

Next, let's look at what was required for a solution under the conditions imposed in §4 and §5. All of the conditions depended on $L(\alpha)$ or on second or higher order derivatives of $E$. $L(\alpha)$ is independent of $\beta$, and so it remains to see how $E$ and its derivatives depend on $\beta$.

Recall that

$$(6.4)\quad E_i(\alpha,\mu_1,\mu_2) = \int_0^T r_i(t+\alpha)G(t,\tilde{z},\tilde{w},\alpha) + s_i(t+\alpha)H(t,\tilde{z},\tilde{w},\alpha)\,dt,$$

$i = 1,2,$ where $\tilde{z}, \tilde{w}$ are functions of $t, \mu_1, \mu_2$.

In fact,

$$(6.5)\quad \begin{bmatrix} \tilde{z}(\mu) \\ \tilde{w}(\mu) \end{bmatrix} = \Phi(0)\beta(\varphi) + V_0(t, I-Q\begin{bmatrix} \mu_1 f_1 + G(t,z(\mu),w(\mu),\alpha) \\ \mu_2 f_2 + H(t,z(\mu),w(\mu),\alpha) \end{bmatrix}).$$

From (6.5) we see that the derivatives of $\tilde{z}$ and $\tilde{w}$ depend on the derivatives of $\beta$ at $\mu = 0$. Therefore the derivatives of E depend on $\beta$ also. Therefore in considering the existence of solutions to (6.1) where $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0$, it is necessary to consider all functions $\beta(\varphi)$ which satisfy (1.4).

Thus we have the following theorems.

<u>Theorem 6.2</u>. For the equation

$$(6.1)\quad \begin{aligned} \ddot{x} + g(x,y) &= \mu_1 f_1 \\ \ddot{y} + h(x,y) &= \mu_2 f_2, \end{aligned}$$

we assume that g, h, $f_1$, $f_2$ are smooth and $f_1$, $f_2$ are T-periodic functions of t.

Suppose that (p(t),q(t)) is a T-periodic solution of

$$\begin{aligned} \ddot{x} + g(x,y) &= 0 \\ \ddot{y} + h(x,y) &= 0 \end{aligned}$$

and that there are exactly two linearly independent, T-periodic solutions of

$$(6.2)\quad \begin{aligned} \ddot{z} + [g_x(p(t+\alpha),q(t+\alpha)),g_y(p(t+\alpha),q(t+\alpha))]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} &= 0 \\ \ddot{w} + [h_x(p(t+\alpha),q(t+\alpha)),h_y(p(t+\alpha),q(t+\alpha))]\begin{bmatrix} z(t) \\ w(t) \end{bmatrix} &= 0. \end{aligned}$$

Let $(r_1(t+\alpha), s_1(t+\alpha))$ and $(r_2(t+\alpha), s_2(t+\alpha)$ be the solutions to the adjoint equation, and let

$$L(\alpha) = \begin{bmatrix} \int_0^T r_1(t+\alpha) f_1(t)\,dt & \int_0^T s_1(t+\alpha) f_2(t)\,dt \\ \int_0^T r_2(t+\alpha) f_1(t)\,dt & \int_0^T s_2(t+\alpha) f_2(t)\,dt \end{bmatrix}$$

and

$$E_i(\beta,\alpha,\mu_1,\mu_2) = \int_0^T r_i(t+\alpha) G(t,\tilde{z}_\beta,\tilde{w}_\beta,\alpha) + s_i(t+\alpha) H(t,\tilde{z}_\beta,\tilde{w}_\beta,\alpha)\,dt$$

where $\tilde{z}_\beta$, $\tilde{w}_\beta$ are functions of $t$, $\mu$ which satisfy

$$\begin{bmatrix} \tilde{z} \\ \dot{\tilde{z}} \\ \tilde{w} \\ \dot{\tilde{w}} \end{bmatrix} = \mathcal{K}_\beta (I - Q) \begin{bmatrix} 0 \\ \mu_1 f_1(\cdot) + G(\cdot, z, w, \alpha) \\ 0 \\ \mu_2 f_2(\cdot) + G(\cdot, z, w, \alpha) \end{bmatrix}$$

for a function $\beta\colon (I - Q)\mathcal{C}_T \to \mathbb{R}^2$ such that

(1.4) $[\Phi(0)\beta(\varphi) + V_0(0,\varphi)]^*[\dot{p}(\alpha)\ddot{p}(\alpha)\dot{q}(\alpha)\ddot{q}(\alpha)]^* = 0;\ \beta(0) = 0.$

(See §II.3.) Suppose $L(\alpha)$ has the form

$$L(\alpha) = \begin{bmatrix} a\alpha + \alpha^2 m_1(\alpha) & 1 + \alpha m_2(\alpha) \\ ac\alpha + \alpha^2 m_3(\alpha) & c + d\alpha + \alpha^2 m_4(\alpha) \end{bmatrix}$$

where $a \neq 0$ and for $i = 1 - 4$, $m_i(\alpha) = O(1)$ as $\alpha \to 0$.

Then equation (6.1) will have solutions only in regions which are bounded by curves given by $\mu_2 = c\mu_1^{n/2}$ + h.o.t. or $\mu_1 = c\mu_2^{2/n}$ + h.o.t. where where $n$ is an integer greater than one.

For such a region to exist where $n = 2$, it is necessary that

$$D^2_{\mu_1} E_2(\beta,0,0,0) = cD^2_{\mu_1} E_1(\beta,0,0,0).$$

for some $\beta$ which satisfies (1.4). This "region" may be just the curve which is its bound.

For $n = 1$, a necessary condition is that

$$c = 0, \quad m_3(0) - ad = 0 \quad \text{and} \quad D^2_{\mu_1} E_2(\beta,0,0,0) = 0$$

for $\beta$ satisfying (1.4), or $c = 0$,

$$m_3(0) - ad \neq 0 \quad \text{and} \quad D^2_{\mu_1} E_2(\beta,0,0,0)/(m_3(0) - ad) < 0$$

for $\beta$ satisfying (1.4). The second set of conditions is also sufficient.

<u>Proof</u>. This follows from §5 just as Theorem 6.1 followed from §3. □

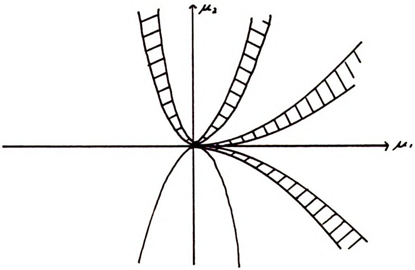


Figure 6. Solution regions when $[JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0$.

This theorem summarizes the results of §5; however it applies to very few cases. It is the results of §4 which allow us to apply this theorem to most problems for which

$$L(\alpha_0)\omega_0 = 0; \ [JL(\alpha_0)]^*L'(\alpha_0)\omega_0 = 0; \ |\omega_0| = 0$$

for some $(\alpha_0, \omega_0)$.

When the appropriate derivatives of $E$ vanish at zero, it is necessary to consider higher order terms in order to determine whether solutions may be possible. In that case, the method of §5 may be extended.

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